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Approximating the Bias and Variance of Chain Ladder Estimates Under a Compound Poisson Model

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Abstract¶

We consider the problem of estimating the outstanding claims produced by a homogeneous general insurance portfolio. The specific model considered in this paper is one where the number of claims in any loss period follows a Poisson distribution, settlement delays follow the same multinomial distribution, and settlements are single lump sums that are independent identically distributed random variables. Simulations using this model reveal that the development ratios and the outstanding claims estimates produced using the chain ladder method are positively biased. We obtain approximate formulas

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for the biases using Taylor series expansions of the random variables about their means. The same methods are used to obtain approximations for the variances and covariances of the projection ratios and the outstanding claims estimates. A simulation study reveals that our formulas are highly accurate.

Key words and phrases: *outstanding claims, reserving, stochastic run-off triangles, chain ladder moments*

1 Introduction

Suppose there are data available for n calendar accident years, with the calendar years labeled 0, 1, ..., $n-1$. We define the total claims paid in development year j of accident year i as S_{ij}, where $i, j = 0, 1, 2, \ldots, n-1$. Our aim is to estimate the outstanding claims at the end of calendar year $n-1$. The claim payments that are known to date form the upper triangle of the claims run-off as shown below.

Table 1
Claims Run-Off Triangle

Year of	Development Year (j)					
Origin (i)	0	1	2	$\cdots$	$n-2$	$n-1$
0	S_{00}	S_{01}	S_{02}	$\cdots$	$S_{0,n-2}$	$S_{0,n-1}$
1	S_{10}	S_{11}	S_{12}	$\cdots$	$S_{1,n-2}$	
$\vdots$	$\vdots$	$\vdots$				
$n-2$	$S_{n-2,0}$	$S_{n-2,1}$				
$n-1$	$S_{n-1,0}$					

For notational convenience we define X_{ab} as the sum of all run-off entries in the rectangle from cell (0,0) to cell (a,b) inclusive:

$$X_{ab} = \sum_{i=0}^{a} \sum_{j=0}^{b} S_{ij}.$$

In the standard application of the chain ladder method, the development ratios

$$m_{r/r-1} = \frac{X_{n-r-1,r}}{X_{n-r-1,r-1}} \tag{1}$$

are calculated for $r = 1, 2, \ldots, n-1$. If we define Y_i as the total claim payments observed to date for accident year i, and M_i as the product of all development ratios employed in the development of Y_i, then

$$Y_i = \sum_{j=0}^{n-i-1} S_{ij} \tag{2}$$

$$M_i = \prod_{k=n-i}^{n-1} m_{k/k-1}, \tag{3}$$

and the estimated outstanding claims for accident year i, OS_i, is

$$\mathrm{OS}_i = Y_i \times (M_i - 1). \tag{4}$$

The total of OS_i's over accident years with incomplete run-off (i.e., $i = 1, 2, \ldots, n-1$) gives the chain ladder's overall outstanding claims estimate.

Mack and Venter (2000), Renshaw and Verrall (1998), and other authors have noted that the chain ladder method was originally developed as a deterministic algorithm with no stochastic model underlying it. To estimate the bias and the prediction uncertainty (variance of outstanding claims) a stochastic model is essential. The conclusions reached depend on the model selected.

For example, Murphy (1994) adopted a regression approach to the chain ladder development process and concluded that the simple average development factor method and the weighted average development factor method are unbiased. Gogol (1995), however, has pointed out that "it is only because Murphy's models have unrealistic properties that it is possible to prove that the estimators are unbiased." Gogol demonstrated mathematically why there is positive bias. Mack (1993), on the other hand, assumed

$$\mathbb{E}\left[C_{i,r+1} \mid C_{i,k}, \quad k = 0, 1, \ldots, r\right] = \rho_r C_{i,r} \tag{5}$$

where $C_{i,j} = \sum_{k=0}^{j} S_{i,k}$, and ρ_r is a constant independent of i.

The assumption implied by equation (5) does not hold for our simple compound Poisson model. In our model the number of claims for a particular accident year is Poisson, the number of these claims settled in the various development years follows the multinomial distribution, and lump sum claim amounts are independent and identically distributed. Nor does it hold for the run-off of numbers of claims under the above Poisson/multinomial assumptions, for which the usual chain ladder calculation method is identical to the maximum likelihood estimation procedure for outstanding claims numbers.

The inapplicability of the Mack model to the collective model has been noted by Schiegl (2002), who points out that the expected cumulative claims by the end of development year $r+1$ should be proportional to the expected (as opposed to actual) cumulative claims at the end of the previous development year, i.e.,

$$\mathbb{E}\left[C_{i,r+1} | C_{i,k}, \quad k = 0, 1, \ldots, r\right] = \xi_r \mathbb{E}\left[C_{i,r}\right]$$

where ξ_r is a constant independent of i.

Stanard (1985) performed a number of simulations to investigate the bias and variance in outstanding claims estimates for various loss reserving methods including the chain ladder. His simulations assumed a relatively small random number of claims for each development year, a uniform distribution of accidents over the year, and an exponential distribution for claim reporting and for claim settlement. Stanard found that the chain ladder produced a substantial positive bias and, by considering the ratio of two random variables, proved there must be a bias.

Schiegl (2002) used simulation to investigate the safety loading required in conjunction with a chain ladder estimate of outstanding claims. For normalization purposes, she defined the relative bias as the expected value of the difference between the chain ladder and the simulated estimates of outstanding claims divided by the square root of the mean square error of the chain ladder estimate. She pointed out that because of correlations between the numerator and denominator, the sign of her relative bias may differ from the un-normalized expected difference between the chain ladder estimate and actual outstanding claims value. That is, if she had not divided by the square root of the estimated mean square error, she may have found a positive bias instead of a negative bias.

In practice, actuaries wish to know whether the chain ladder estimate tends to be higher or lower than the underlying value, as well as the variance of the various chain ladder estimates. These are the problems addressed in this paper. We note that Taylor (2002) has confirmed the positive bias that we demonstrate.

2 The Model

Consider a homogeneous general insurance portfolio described below:

1. The individual claim amounts are independent identically distributed random variables with first- and second-order moments about the origin of τ_1 and τ_2, respectively;
2. Each claim is settled as a single lump sum amount with no partial payment before settlement;
3. The total number of claims occurring in accident year i (including IBNR claims) is a Poisson random variable with mean λ_i; and
4. The probability that a claim is settled in development year j is p_j with n being sufficiently large so that $\sum_{j=0}^{n-1} p_j = 1$.

Let N_{ij} denote the number of claims settled for accident year i in development year j, then N_{ij} is a multinomial variable conditional on a Poisson variable. It follows, therefore, that N_{ij} is a Poisson random variable with expectation $\lambda_i p_j$, and the N_{ij}s are mutually independent for $i, j = 0, 1, \ldots, n-1$. Thus, the total claim payments for accident year i made in development year j, S_{ij}, has a compound Poisson distribution with mean $\lambda_i p_j \tau_1$ and variance $\lambda_i p_j \tau_2$. Furthermore, the run-off entries S_{ij} are mutually independent for $i, j = 0, 1, \ldots, n-1$.

For convenience, we define γ to be the ratio of the variance to the mean for each of the S_{ij}s. Under our model this ratio is

$$\gamma = \frac{\mathrm{Var}\left[S_{ij}\right]}{\mathbb{E}\left[S_{ij}\right]} = \frac{\lambda_i p_j \tau_2}{\lambda_i p_j \tau_1} = \frac{\tau_2}{\tau_1}, \tag{6}$$

which is independent of the accident and development years.

We further define the expected ultimate total claims cost for accident year i, α_i, as

$$\alpha_i = \mathbb{E}\left[\sum_{j=0}^{n-1} S_{ij}\right] = \sum_{j=0}^{n-1} \lambda_i p_j \tau_1 = \lambda_i \tau_1. \tag{7}$$

The following notation is used for convenience:

$$\theta_i = \sum_{u=0}^{i} \alpha_u \tag{8}$$

and

$$P_j = \sum_{u=0}^{j} p_u. \tag{9}$$

The source of the bias of chain ladder estimates under the compound Poisson model lies in the definition of the development ratios. Consider development ratios defined as the ratio of the expectations (as opposed to the ratio of observed actual values), i.e., as

$$m_{r/r-1} = \frac{\mathbb{E}\left[X_{n-r-1,r}\right]}{\mathbb{E}\left[X_{n-r-1,r-1}\right]}. \tag{10}$$

Noting that $\mathbb{E}\left[X_{ij}\right] = \theta_i P_j$ and $P_{n-1} = 1$, then

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{n-1} \mathrm{OS}_i\right] &= \mathbb{E}\left[\sum_{i=1}^{n-1} Y_i \times \left(\prod_{k=n-i}^{n-1} \frac{\theta_{n-k-1} P_k}{\theta_{n-k-1} P_{k-1}} - 1\right)\right] \\
&= \sum_{i=1}^{n-1} \alpha_i P_{n-i-1} \times \left(\prod_{k=n-i}^{n-1} \frac{P_k}{P_{k-1}} - 1\right) \\
&= \sum_{i=1}^{n-1} \alpha_i \left(1 - P_{n-i-1}\right),
\end{aligned}$$

which is exactly the expected amount of claims in the unobservable part of the run-off. Therefore using equation (10) for development ratios leads to unbiased estimates under our model. The chain ladder method, however, corresponds to equation (1) instead. As small as this distinction may seem, it introduces biases under our model.

3 The Bias and Variance of Development Ratios

For any rectangle of cells A of the run-off, define T_A to be the total claim payments observed in A, and define $\mu_A = \mathbb{E}[T_A]$. Consider a general development ratio m with rectangles A and B in the run-off defined such that $m = T_A/T_B$. As noted in Section 2, the unbiased ratio required to project the outstanding claims of the portfolio is μ_A/μ_B. The ratio m on the other hand, has expectation

$$\mathbb{E}[m] = \mathbb{E}\left[\frac{T_A}{T_B}\right] = \frac{\mu_A}{\mu_B}\mathbb{E}\left[\left(1 + \frac{T_A - \mu_A}{\mu_A}\right)\left(1 + \frac{T_B - \mu_B}{\mu_B}\right)^{-1}\right]. \tag{11}$$

Assuming the expression within the square brackets in equation (11) can be expanded as a series of the form $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$, and assuming μ_A and μ_B are sufficiently large so that all third and higher order terms are negligible leads to

$$\mathbb{E}[m] \approx \frac{\mu_A}{\mu_B}(1 + V_{BA}), \tag{12}$$

where for convenience we define the terms

$$K_{AB} = \frac{\mathbb{C}\mathrm{ov}[T_A, T_B]}{\mu_A \mu_B} \tag{13}$$

and

$$V_{BA} = K_{BB} - K_{AB}. \tag{14}$$

The quantity V_{BA} is termed the approximate proportional bias. From equations (12), (13), and (14) it is apparent that a stochastic run-off model yields a bias in chain ladder estimates under the compound Poisson model.

The following theorem is needed to assist in the development of our approximations. An illustration and proof of Theorem 1 is given in Appendix A.

Theorem 1. *Let G and H be rectangles of cells in the run-off, and let R be the smallest rectangle that includes all the cells of G and H. If the rows of R coincide with the rows of either G or H and the columns of R coincide with the columns of either G or H, then*

$$K_{GH} = \frac{\gamma}{\textit{Total Payments Expected in } R}$$

where γ is given in equation (6).

Because of the manner in which development ratios are calculated, $B \subset A$, so the smallest rectangle including all the cells of A and B is A itself; while, trivially, the smallest rectangle for B and B is B. Therefore applying Theorem 1 to equation (14), we deduce that the approximate proportional bias in m is

$$V_{BA} = \frac{\gamma}{\mu_B} - \frac{\gamma}{\mu_A}. \tag{15}$$

The direction of the bias depends on the relationship between μ_A and μ_B. For example, if negative incremental claims are allowed, then it is possible that $\mu_A < \mu_B$, and the bias in the chain ladder development ratios may be positive or negative. Under our compound Poisson model with nonnegative incremental claims, however, $\mu_A > \mu_B$; therefore, the development ratios used to project the cumulative sums of claim payments are positively biased, and the chain ladder approach will tend to overestimate outstanding claims liabilities.

The variance of the development ratio can be found as follows:

$$\mathbb{E}\left[m^2\right] = \left(\frac{\mu_A}{\mu_B}\right)^2 \mathbb{E}\left[\left(1 + \frac{T_A - \mu_A}{\mu_A}\right)^2 \left(1 + \frac{T_B - \mu_B}{\mu_B}\right)^{-2}\right].$$

Again, using a binomial expansion and neglecting the appropriate terms yields the approximation

$$\mathbb{E}\left[m^2\right] \approx \left(\frac{\mu_A}{\mu_B}\right)^2 (1 + K_{AA} + 3K_{BB} - 4K_{AB}).$$

As $B \subset A$, it follows from Theorem 1 that $K_{AA} = K_{AB}$. Subtracting $(\mathbb{E}[m])^2$ as approximated using equation (12) and ignoring third order terms, we conclude that

$$\mathbb{V}\mathrm{ar}[m] \approx \left(\frac{\mu_A}{\mu_B}\right)^2 V_{BA}. \tag{16}$$

4 Bias and Variance of Outstanding Claims

Two of the concerns in outstanding claims estimation are the bias and the variance of the overall estimate. The definition of the variance of the outstanding claims estimate requires some clarification. In our opinion there are three main variance measures that practitioners might consider:

1. The variance of the actual outstanding claims amount;
2. The variance of the outstanding claims estimate based upon particular estimates of the model parameters; and
3. The variance of the chain ladder outstanding claims estimate.

Many authors concentrate on the second measure. Given the subject of this paper, however, it is the third measure that is relevant and is the one used in this paper. Some preliminary results are now given.

4.1 Preliminary Results

4.1.1 Development Ratios Are Effectively Uncorrelated

Consider a 5×5 run-off triangle with rows and columns numbered from 0 to 4. The development ratios $m_{1/0}$ and $m_{3/2}$ are based on rectangles of cells A, B, C, and D defined such that $m_{1/0} = T_A/T_B$ and $m_{3/2} = T_C/T_D$. This implies $T_A = X_{3,1}$, $T_B = X_{3,0}$, $T_C = X_{1,3}$, and $T_D = X_{1,2}$. Working through the expansions as before and taking expectations, we find that

$$\mathbb{E}\left[m_{1/0}m_{3/2}\right] \approx \frac{\mu_A\mu_C}{\mu_B\mu_D}\left(1 + V_{BA} + V_{DC} + K_{AC} + K_{BD} - K_{CB} - K_{AD}\right). \tag{17}$$

From Theorem 1, $K_{AC} = K_{CB}$ and $K_{BD} = K_{AD}$, so that the subscripted K terms in equation (17) sum to zero. By inspection, the right side of equation (17) is equal to $\mathbb{E}\left[m_{1/0}\right]\mathbb{E}\left[m_{3/2}\right]$ as found using equation (12) and ignoring all third order and higher terms. We conclude that the covariance of the two development ratios is approximately zero. The same approach can be used to show that any two arbitrary development ratios are effectively uncorrelated.

4.1.2 Uncorrelated Accident Year Payments and Development Ratios

Within the same 5×5 run-off triangle as before, define C as the rectangle of cells relating to claim payments observed to date in respect of accident year 1, i.e., C contains $\{S_{ij}\}$ where $i = 1$ and $j = 0, 1, 2, 3$. Consider the development ratio $m_{2/1}$, with rectangles of cells A and B defined such that $m_{2/1} = T_A/T_B$. By inspection, T_A, T_B, and T_C share common run-off entries, so T_C is not independent of $m_{2/1}$. Adopting the same approach as before, we discover that

$$\mathbb{E}\left[T_C\, m_{2/1}\right] = \mathbb{E}\left[\frac{T_C T_A}{T_B}\right] \approx \frac{\mu_C\mu_A}{\mu_B}\left(1 + V_{BA} + K_{AC} - K_{BC}\right). \tag{18}$$

From Theorem 1, $K_{AC} = K_{BC}$. The right side of equation (18) is therefore equal to $\mathbb{E}\left[T_C\right]\mathbb{E}\left[m_{2/1}\right]$ as approximated by equation (12). So with covariance approximately zero, T_C and $m_{2/1}$ are effectively uncorrelated. The same conclusion is reached irrespective of the accident year chosen and the development ratio involving common run-off entries.

4.2 Bias, Variances, and Covariances

4.2.1 Product of Development Ratios

Consider three different development ratios, m_1, m_2, and m_3, defined as T_A/T_B, T_C/T_D, and T_E/T_F, respectively. Using the same techniques as before, we can show that

$$\mathbb{E}\left[m_1 m_2 m_3\right] \approx \frac{\mu_A \mu_C \mu_E}{\mu_B \mu_D \mu_F}\left(1 + V_{BA} + V_{DC} + V_{FE}\right) \tag{19}$$

$$\text{Proportional Bias} \approx V_{BA} + V_{DC} + V_{FE}$$

$$\mathbb{V}\text{ar}\left[m_1 m_2 m_3\right] \approx \left(\frac{\mu_A \mu_C \mu_E}{\mu_B \mu_D \mu_F}\right)^2 \left(V_{BA} + V_{DC} + V_{FE}\right). \tag{20}$$

In general, the proportional bias in the product of a set of development ratios is approximately the sum of the relevant V terms, and the variance is approximately the sum of the relevant V terms multiplied by the square of the product of the relevant unbiased ratios.

4.2.2 Accident Year Outstanding Claims Estimates

Let us use Y_i and M_i as defined in equations (2) and (3). Y_i is an unbiased estimator of the expected total claim payments to date, and as Y_i and M_i do not depend on any common run-off entries, they must be independent. According to equations (4) and (19), therefore, the actual bias (not the proportional bias) in OS_i is approximately the total claims expected in accident year i (that is, α_i) multiplied by the sum of the V terms relating to M_i.

Given the independence of Y_i and M_i, the variance of OS_i can be found as follows:

$$\mathbb{V}\text{ar}\left[(M_i - 1)Y_i\right] = \mathbb{V}\text{ar}\left[M_i\right]\mathbb{E}\left[Y_i^2\right] + \mathbb{V}\text{ar}\left[Y_i\right]\left(\mathbb{E}\left[M_i\right] - 1\right)^2. \tag{21}$$

4.2.3 Covariances Between Outstanding Claims Estimates for Different Accident Years

Consider Y_i, M_i, and OS_i as defined in equations (2), (3), and (4), and similarly define OS_q, Y_q, and M_q for a later accident year q. Under our model, Y_i and Y_q are independent. Because Y_q relates to a later accident year than Y_i, M_i is a factor of M_q. So let us write M_q as $M^* M_i$, where M^* and M_i do not contain any common development ratios and are effectively uncorrelated. The covariance of the two outstanding claims estimates is therefore:

$$\mathbb{C}\mathrm{ov}\left[\mathrm{OS}_i, \mathrm{OS}_q\right] = \mathbb{E}\left[(M_i - 1)Y_i(M^*M_i - 1)Y_q\right] \\ - \mathbb{E}\left[(M_i - 1)Y_i\right]\mathbb{E}\left[(M^*M_i - 1)Y_q\right]. \qquad (22)$$

Taking account of the independence of Y_i and Y_q, and the fact that all the other M and Y terms in equation (22) are effectively uncorrelated, we deduce that

$$\mathbb{C}\mathrm{ov}\left[\mathrm{OS}_i, \mathrm{OS}_q\right] \approx \mathbb{E}\left[Y_i\right]\mathbb{E}\left[Y_q\right]\mathbb{E}\left[M^*\right]\mathbb{V}\mathrm{ar}\left[M_i\right] \qquad (23)$$

for $i = 1, \ldots, q - 1$.

4.2.4 Variance of Overall Outstanding Claims Estimate

The overall outstanding claims estimate is the sum of the estimates for the individual accident years. Its variance is readily approximated from the variances of individual accident year estimates and the covariances of these estimates.

4.2.5 Non-Homogeneous Model of Claim Settlements

Thus far we have considered claim size patterns that are independent of notification delays. Given the strong assumption of independence of run-off entries, the above results will hold if it can be further assumed that claims at differing levels of severity are mutually independent.

For example, separate run-off triangles and sets of parameters for small, medium, and large claim sizes can be investigated, with the results of this paper applicable to each of these triangles. The items of interest can then be aggregated.

4.3 Practical Formulas

It is possible to simplify the results obtained so far in Section 4 for practical application by noting that the V term for the development ratio $m_{r/r-1}$ can be expressed as

$$V_r = \frac{\gamma}{\theta_{n-r-1}}\left(\frac{1}{P_{r-1}} - \frac{1}{P_r}\right). \qquad (24)$$

If we then define the (backwards) cumulative sum of the V terms as

$$V_r^C = \sum_{u=r}^{n-1} V_u, \tag{25}$$

we discover from equations (19) and (20) that

$$\mathbb{E}\left[m_{r/r-1} \times m_{r+1/r} \times \cdots \times m_{n-1/n-2}\right] \approx \frac{1}{P_{r-1}}\left(1 + V_r^C\right) \tag{26}$$

and

$$\mathbb{V}\mathrm{ar}\left[m_{r/r-1} \times m_{r+1/r} \times \cdots \times m_{n-1/n-2}\right] \approx \left(\frac{1}{P_{r-1}}\right)^2 V_r^C. \tag{27}$$

For $i = 1, \ldots, n-1$, the approximate bias of accident year i's outstanding claims estimate is

$$\mathrm{Bias}\left(\mathrm{OS}_i\right) \approx \alpha_i V_{n-i}^C. \tag{28}$$

Furthermore, using equations (26), (27), (21), and (23) and simplifying (details are given in Appendix B),

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left[\mathrm{OS}_i\right] \approx{}& \alpha_i^2 V_{n-i}^C + \alpha_i P_{n-i-1}\,\gamma\left(\frac{1 - P_{n-i-1}}{P_{n-i-1}}\right)^2 \\ &+ \alpha_i\,\gamma\left(\frac{3}{P_{n-i-1}} - 2\right) V_{n-i}^C\end{aligned} \tag{29}$$

and

$$\mathbb{C}\mathrm{ov}\left[\mathrm{OS}_i, \mathrm{OS}_q\right] \approx \alpha_i\,\alpha_q V_{n-i}^C \tag{30}$$

for $i = 1, \ldots, q-1$. Larger α parameters correspond to higher expected total claims. As the portfolios considered become larger, the V_r and V_r^C terms approach zero, as do the biases in individual development ratios and their products. This is due to the sums of α parameters that appear in the denominators of V terms.

Furthermore the approximate biases (equation (28)) are not linear in the total expected amounts of claims, as the V terms of equation (28) are multiplied by other α parameters. Nevertheless, we see that if the portfolio changes with all α's increasing by a common factor, then the approximate biases of the outstanding claims estimates do not change. This is not an intuitive result. We also see from equations (24) and (28) that the approximate biases are linear in γ. Taylor (2002) also details this result and finds numerical proof in a comparison of Exhibits I and II of Stanard (1985), whose simulation models roughly follow the restrictions of Section 2.

The covariances between different accident year claims estimates (equation (30)) grow in proportion to the portfolio size, as do the first two terms of the variance result for the accident year estimate (equation (29)). The last term in this variance approximation is similar to the bias approximations, as it does not grow in proportion to portfolio size.

5 Simulation Study

5.1 Simulation Results for Individual Development Ratios

One million simulations of a 5×5 run-off were used to test the results for individual development ratios. The assumed underlying claim number parameters and settlement proportions are shown in Table 2. Individual claim sizes were assumed to be exponentially distributed with a mean of 500. Therefore, $\tau_1 = 500$, $\tau_2 = 500{,}000$, and $\gamma = \tau_2/\tau_1 = 1{,}000$. This highly skewed distribution was used to stress test the results, as with lower skewness we might expect our formulas to produce better approximations.

Table 2
Expected Claim Numbers
And Proportions Settled in Successive Years

Accident Year (i)	Claim Frequency (λ_i)	Development Year (j)	Proportion Settled (p_j)
0	200	0	40%
1	300	1	30%
2	240	2	20%
3	360	3	5%
4	220	4	5%

The observed proportional biases and variances are compared in Table 3 with the approximate theoretical values. The proportional bias shown in Table 3 is the average simulated ratio less the unbiased ratio, expressed as a proportion of the unbiased ratio. It is clear that equations (15) and (16) produce reliable approximations.

Table 3
Proportional Biases and Variances Estimated by Simulation

RATIO	BIAS	VAR	NUM	DENOM	APBIAS	APVAR
$m_{1/0}$	0.0020	0.0061	385,000	220,000	0.0019	0.0060
$m_{2/1}$	0.0009	0.0014	333,000	259,000	0.0009	0.0014
$m_{3/2}$	0.0002	0.0003	237,500	225,000	0.0002	0.0003
$m_{4/3}$	0.0005	0.0006	100,000	95,000	0.0005	0.0006

Notes: RATIO = Development ratio; BIAS = Proportional bias of simulated ratios; VAR = Variance of simulated ratios; NUM = Expected numerator; DENOM = Expected denominator; APBIAS = Approximate proportional bias based on equation (15); and APVAR = Approximate variance based on equation (16).

While the approximations in Table 3 are consistent with the estimates obtained by simulation, the number of claims assumed was large. The simulations were therefore repeated with a tenfold decrease in the Poisson claim frequencies. Given the tiny size of the run-off, the approximations are surprisingly good (Table 4).

Table 4
Simulated Biases and Variances with Reduced Claim Frequencies

RATIO	BIAS	VARIANCE	APROXBIAS	APROXVAR
$m_{1/0}$	0.0206	0.0720	0.0195	0.0597
$m_{2/1}$	0.0093	0.0164	0.0086	0.0142
$m_{3/2}$	0.0026	0.0030	0.0023	0.0026
$m_{4/3}$	0.0064	0.0088	0.0053	0.0058

Notes: RATIO = Development ratio; BIAS = Proportional bias of simulated ratios; VARIANCE = Variance of simulated ratios; APROXBIAS = Approximate proportional bias based on equation (15); and APROXVAR = Approximate variance based on equation (16).

5.2 Simulation Results for Outstanding Claims Estimates

The same assumptions and simulations were used to determine the bias (Table 5) and second-order moments (Table 6) of the chain ladder outstanding claims estimates for each of the accident years. The comparisons with the approximate theoretical bias (Table 5) and second order moments (Table 7) are good. Calculation details for the theoretical formulas are given in Appendix C.

Table 5
Biases of Outstanding Claims Estimates

Accident Year	Bias of Simulated Estimates	Approximate Bias from Equation (28)	Expected Outstanding Claims	Relative Bias
1	81.84	78.95	7,500	1.09%
2	94.70	91.23	12,000	0.79%
3	301.78	291.28	54,000	0.56%
4	391.81	392.29	66,000	0.59%
Overall Result	870.13	853.75	139,500	0.62%

The variance of the overall outstanding claims estimate based on the simulation study is simply the sum of all the moments in Table 6, namely 4.362×10^8. This agrees closely with the approximate value obtained from Table 7 of 4.270×10^8. The discrepancy is about 2.1%.

As we know the parameters underlying the model, the variance of the true outstanding claims for all accident years (that is, the first variance measure mentioned in Section 4) can be evaluated quickly and easily: 1.395×10^8. The variance of the chain ladder estimate is around three times as great, reflecting the uncertainty introduced by the need to use parameters estimated by the chain ladder method.

Table 6
Covariance Matrix of
Simulated Outstanding Claims
Estimates by Accident Year $\times 10^{-6}$

	1	2	3	4
1	12.75	9.80	14.78	9.02
2	9.80	12.73	16.95	10.37
3	14.78	16.95	77.63	32.91
4	9.02	10.37	32.91	145.47

Table 7
Approximate Covariance Matrix of Outstanding Claims Estimates by Accident Year $\times 10^{-6}$

	1	2	3	4
1	12.33	9.47	14.21	8.68
2	9.47	12.40	16.42	10.04
3	14.21	16.42	76.24	32.04
4	8.68	10.04	32.04	144.31

The high accuracy with which the approximate theoretical variance estimates the true variance is due to some extent to the relatively high (but nevertheless realistic) assumed claim frequency. With lower claim frequencies, the errors in the approximate variances and covariances become more significant. With a tenfold decrease in claim frequencies, the error in the approximate variance of the overall outstanding claims estimate rises to 27.1% (4.543×10^7 compared with a simulated value of 5.819×10^7)—or a 13% error in the standard deviation. Even in this situation, with relatively few expected claims and a highly skewed claim size distribution, the approximations still provide reasonable indications of the degree of uncertainty in the chain ladder outstanding claims estimates.

In practice the underlying parameter values will be unknown. Estimating parameters from actual insurance data will introduce uncertainty and possibly biases in the parameter estimates, which in turn will affect the approximations of this paper. Such impacts are beyond the scope of this paper—it must be emphasized that the approximations are valid only if the parameter values are known in advance. For this reason we have performed a simulation study rather than applying the approximations to actual insurance data.

6 Concluding Remarks

The formulas we have derived allow accurate approximations to the biases introduced when the traditional chain ladder method is used to estimate the outstanding claims under a compound Poisson run-off, and accurate approximations to the variances and covariances of these estimates. Our analysis also reveals that under our simple stochastic model the development ratios of the chain ladder method are essentially uncorrelated, but are biased. Even in the ideal situation of a large portfolio with independent entries, outstanding claims estimates for different accident years are significantly correlated.

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Appendix A. Illustration and Proof of Theorem 1

Illustration

In the context of a 5×5 run-off with rows and columns numbered 0 to 4, consider rectangles G containing elements $\{S_{ij}\}$ $(i = 0, 1;\ j = 0, 1, 2, 3)$, and H containing elements $\{S_{ij}\}$ $(i = 0, 1, 2, 3, 4;\ j = 1, 2)$.

The smallest rectangle R incorporating all the elements of G and H is made up of the elements $\{S_{ij}\}$ $(i = 0, 1, 2, 3, 4;\ j = 0, 1, 2, 3)$ and the common component of G and H, say W, comprises the elements $\{S_{ij}\}$ $(i = 0, 1;\ j = 1, 2)$. We note that the rows of R coincide with the rows of H and that the columns of R coincide with those of G.

Recall the definition (given in Section 3) of T_A and μ_A for any rectangle of cells A. Because of the mutual independence of all cells of the run-off under our model, $\mathbb{C}\mathrm{ov}[T_G, T_H] = \mathbb{V}\mathrm{ar}[T_W]$, which by equation (6) and the assumptions of Section 2, yields

$$\mathbb{C}\mathrm{ov}[T_G, T_H] = \mathbb{V}\mathrm{ar}[T_W] = \sum_{i=0}^{1}\sum_{j=1}^{2} \alpha_i p_j \gamma.$$

The expectations of T_G and T_H are, respectively

$$\mu_G = \sum_{i=0}^{1}\sum_{j=0}^{3} \alpha_i p_j \quad \text{and} \quad \mu_H = \sum_{i=0}^{4}\sum_{j=1}^{2} \alpha_i p_j.$$

Therefore, K_{GH} (defined in equation (13)) is given by

$$K_{GH} = \frac{\gamma}{\sum_{i=0}^{4} \alpha_i \sum_{j=0}^{3} p_j},$$

where the denominator is the total claim payments expected in R.

Proof of Theorem 1

For any rectangle G, define θ_G as the sum of all the $\{\alpha_i\}$ values that relate to the cells of G within the run-off, and P_G as the sum of all the $\{p_j\}$ values that relate to to the cells of G. Then, with (i) G, H, and R defined as in the statement of the theorem; (ii) $W = G \cap H$; and (iii) using the fact that under the model all elements are independent,

$$\mu_G = \theta_G P_G;$$

$$\begin{aligned}\mu_H &= \theta_H P_H;\\ \mu_R &= \max(\theta_G, \theta_H) \times \max(P_G, P_H);\\ \mathbb{C}\mathrm{ov}[T_G, T_H] &= \mathbb{V}\mathrm{ar}[T_W] = \gamma \min(\theta_G, \theta_H) \min(P_G, P_H)\\ K_{GH} &= \frac{\gamma \min(\theta_G, \theta_H)\ \min(P_G, P_H)}{(\theta_G P_G)\ (\theta_H P_H)}.\end{aligned}$$

But

$$\begin{aligned}\theta_G \theta_H &= \min(\theta_G, \theta_H)\ \max(\theta_G, \theta_H);\\ P_G P_H &= \min(P_G, P_H)\ \max(P_G, P_H).\end{aligned}$$

It follows that

$$K_{GH} = \frac{\gamma}{\mu_R},$$

and the proof of the theorem is complete.

Appendix B. Derivation of Approximations

Variance Approximations

Recall the definition of M_i in equation (3) and the variance equation of equation (21). Substituting equations (26) and (27) in equation (21) yields

$$\begin{aligned}\mathbb{V}\mathrm{ar}[\mathrm{OS}_i] \approx{}& \frac{1}{P_{n-i-1}^2} V_{n-i}^C \left[(\alpha_i P_{n-i-1})^2 + (\alpha_i P_{n-i-1}\,\gamma)\right]\\ &+ (\alpha_i P_{n-i-1}\gamma)\left[\frac{1}{P_{n-i-1}}\left(1 + V_{n-i}^C\right) - 1\right]^2\\ ={}& \alpha_i^2 V_{n-i}^C + \frac{1}{P_{n-i-1}^2} V_{n-i}^C\ (\alpha_i P_{n-i-1}\,\gamma)\\ &+ (\alpha_i P_{n-i-1}\,\gamma)\left[\left(\frac{1 - P_{n-i-1}}{P_{n-i-1}}\right) + \frac{V_{n-i}^C}{P_{n-i-1}}\right]^2.\end{aligned}$$

Ignoring third order and higher terms in the expansion of the above expression and simplifying, we find that

$$\mathbb{V}\mathrm{ar}[\mathrm{OS}_i] \approx \alpha_i^2 V_{n-i}^C + \alpha_i P_{n-i-1}\gamma \left(\frac{1 - P_{n-i-1}}{P_{n-i-1}}\right)^2 + \alpha_i \gamma \left(\frac{3}{P_{n-i-1}} - 2\right) V_{n-i}^C.$$

Covariance Approximations

In the context of equation (23),

$$M^* = m_{n-q/n-q-1} \times \ldots \times m_{n-i-1/n-i-2}$$

for $i = 1, 2, \ldots, q-1$. Let V^* be the bias term corresponding to M^*. Now substituting equations (26) and (27) in equation (23) yields

$$\mathrm{Cov}\left[\mathrm{OS}_i, \mathrm{OS}_q\right] \approx \alpha_i P_{n-i-1} \alpha_q P_{n-q-1} \left(\frac{P_{n-i-1}}{P_{n-q-1}} (1 + V^*) \right) \times \frac{V^C_{n-i}}{P^2_{n-i-1}}$$

for $i = 1, 2, \ldots, q-1$. Ignoring third order terms in the expansion of the above expression, we arrive at the simplified result:

$$\mathrm{Cov}\left[\mathrm{OS}_i, \mathrm{OS}_q\right] \approx \alpha_i \alpha_q V^C_{n-i}$$

for $i = 1, 2, \ldots, q-1$.

Appendix C. Bias and Variance Approximations

Table C1 shows some of the details of the calculations behind the approximations displayed in Tables 5 and 7. The assumptions are the same as those used earlier in Table 2, and the claim size moments are $\tau_1 = 500$ and $\tau_2 = 500{,}000$. In this example, $\gamma = \tau_2/\tau_1 = 1{,}000$, and $n = 5$.

Table C1
Calculation Details

i	0	1	2	3	4	Source
λ_i	200	300	240	360	220	Table 2
α_i	100,000	150,000	120,000	180,000	110,000	Eqn (7)
θ_i	100,000	250,000	370,000	550,000	660,000	Eqn (8)
p_i	0.4	0.3	0.2	0.05	0.05	Table 2
P_i	0.4	0.7	0.9	0.95	1.00	Eqn (9)
V_i		0.0019	0.0009	0.0002	0.0005	Eqn (24)
V^C_{5-i}		0.0005	0.0008	0.0016	0.0036	Eqn (25)
OS_i bias		78.95	91.23	291.28	392.29	Eqn (28)

The approximate covariance matrix of the outstanding claims estimates by accident year $\times 10^{-6}$ (using equations (29) and (30)) is

$$\begin{pmatrix} 12.33 & 9.47 & 14.21 & 8.68 \\ 9.47 & 12.40 & 16.42 & 10.04 \\ 14.21 & 16.42 & 76.24 & 32.04 \\ 8.68 & 10.04 & 32.04 & 144.31 \end{pmatrix}.$$

E.g.: $\mathbb{C}\mathrm{ov}\,[OS_1, OS_2] = 150,000 \times 120,000 \times 0.0005263 = 9.47 \times 10^6$.