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PROPAGATION OF THERMO-MECHANICAL WAVES IN DEFORMING NONLINEAR VISCOELASTIC BODIES

by

Lili Zhang

A DISSERTATION

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Propagation of Thermo-Mechanical Waves in Deforming Nonlinear Viscoelastic Bodies

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University of Nebraska, 2013

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This work studies the propagation of thermo-mechanical disturbances in bodies made of viscoelastic materials that might already be loaded such that they are undergoing large inhomogeneous time varying deformations. In the process of this study we develop the general equations governing the thermo-mechanical motion of such disturbances and ones for internally constrained systems, provide the general structure of the solution, match the solution to existing results for the special case of time harmonic plane waves in elastic bodies and in viscoelastic bodies under constant homogenous loading, and consider some special applications.

The results of this work should have applications in the study of anisotropic and inhomogeneous bodies that are inhomogenously loaded with the possibility that these loads are time varying, and may become part of tools used for non-invasive and non-destructive testing of such bodies. Many common materials are anisotropic and inhomogeneous. These include most polymers, composites, soft and hard tissues, and all kinds of bio-mass. Many bodies are undergoing static or time varying inhomogeneous loading. Examples can vary from conditions that result in or from earthquakes and landslides to composites and live tissues functioning in loaded structures and bio-MEMS.

Some of the contributions of this dissertation are to introduce a full, thermodynamically consistent, nonlinear viscoelastic model to represent the material, to properly introduce thermo-mechanical coupling, to remove current limitations on the pre-deformation to be static, homogenous and around the equilibrium, to remove existing restrictions on the rate of loading of the perturbations, and to consider perturbations in the presence of material constraints.

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Words fail to capture the debt and gratitude I owe to my parents for their profound love and unconditional support. Thank you for encouraging me to pursue what I want to become and always making me feel cherished and loved. Just as the Chinese poem says "such kindness of warm sun, cannot be repaid by grass." I can never begin to repay my parents for everything they give me. I would like to dedicate my dissertation to my dearest mother.

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CHAPTER 1

Introduction

The aim of this dissertation is to study the propagation of infinitesimal thermo-mechanical time varying perturbations (waves) in a special class of nonlinear viscoelastic materials under static and time varying, homogeneous and inhomogeneous, pre-deformations. The study of the theory of infinitesimal thermo-mechanical perturbations on nonlinear viscoelastic materials and structures is of importance in characterizing the dynamic properties and addressing the stability issues of viscoelastic materials and structures under complex pre-loadings and pre-deformations.

1.1 Background on thermo-mechanical dynamic phenomena

Thermo-mechanical dynamic phenomena, for example, vibrations of mechanical structures and wave propagations in viscoelastic medium in the presence of pre-load and pre-deformation, play an essential part in many engineering fields: geophysics, civil engineering, seismology, underwater acoustics, non destructive control, bio-sensors, bio-actuators, aircraft and rocket stability, wave wind and earthquake loaded civil engineering structures, fatigue failure of turbines, settlement of railway tracks, non destructive testing, etc.

In civil engineering and geophysics, the problems of consolidation and tectonics involve earth masses that are initially under high stress. The folds and fractures in the sedimentary layers are the result of differential stress environment in the sediments. In the problems of foundation engineering, the influence of initial stress appears in a buoyancy effect, which amounts to floating a building on its foundation. The initial stress state inside the earth is mainly due to the pressure of overburden. Crustal rocks are always subjected to stresses [Garg, 2007]. A dynamical explanation of earthquake phenomena is required in earthquake research. In this context, the term dynamics implies a consideration of the initial stress with in the viscoelastic Earth that act to cause fault ruptures and ground displacements. However, most of the research does not take into account the effects of initial stress on the wave propagation in viscoelastic medium [Garg, 2007].

Intravascular ultrasound (IVUS), wavelet analysis of radio frequency intravascular ultrasound signals, and integrated backscatter intravascular ultrasound [Gorb and Walton, 2010] are medical imaging methodologies that detect blockage or narrowing of the vessels which may cause cardiovascular diseases [Rachev, 1997]. These techniques are in vivo invasive procedures performed through cardiac catheterization that produces detailed images of the interior walls of the artery to see blood vessels and allows one to detect an obstruction of the lumen of the artery. The catheter tip emits acoustic sound waves, usually in the 15-40 MHz range. The catheter also receives and conducts the return echo information out to the external computerized ultrasound equipment which constructs and displays a real time ultrasound image of a thin section of the blood vessel surrounding the catheter tip. However, many of these techniques are still under investigation and at present none of them can accurately identify a vulnerable plaque and how it is going to develop. Achieving satisfactory results by means of traditional techniques is very difficult due to the complex nature of the tissue characterization problem and the imaging characteristics [Gorb and Walton, 2010]. One of the main problems to solve is to understand the effect of the basic artery features, such as layered, anisotropic [Holzapfel et al., 2000], heterogeneous, pre-stressed [Fung and Liu, 1991], viscoelastic, on ultrasound wave propagation in the deformed body [Gorb and Walton, 2010].

In the general anisotropic and dissipative media, waves are attenuated differently in different directions. There has been a recent interest in the anisotropy of seismic attenuation as it may provide additional information about subsurface elastic properties. In particular, velocity and attenuation anisotropy have important implications for fracture characterization in exploration seismology Liu et al. and Zhu et al. [E.Liu et al., 2007, Zhu and Tsvankin, 2007].

In recent years, there has been significant interest in the soft materials with potential applications in bioMEMS for comfortable cancer detection and treatment Bashir[Bashir, 2004] and Bhushan [Bhushan, 2007]. A novel shear assay technique and micro patterned biomaterial surfaces can be used to characterize cell adhesion, viscoelastic properties, and prestress of the human asteosarcoma cells on biocompatible surfaces, in an effort to develop tools for characterizing cancer cell properties Kumar et al [Kumar and Weaver, 2009]. Therefore, some of the human organs exhibit viscoelastic properties along with initial stress and demands results for initially stressed viscoelastic medium.

Polydimethylsiloxane (PDMS)-based micropillars (or microcantilevers) have been used as bio-sensors for cellular force measurement to understand the fundamental physiological processes associated with cell growth, division, migration and apoptosis Zhao et al. [Zhao and Zhang, 2005] and Lin et al. [Lin et al., 2008]. They have also been considered for use as bio-actuators to drive both solid microstructures and fluids in a microchip without recourse to electrical power supply [Du et al., 2010]. The measurement accuracy of these sensitive devices depends on appropriate modeling to convert the micropillar deformations into the corresponding reaction forces.

Stress fibers in living cells behave as viscoelastic cables that are tensed through the action of actomyosin motors. It is required to quantify their retraction kinetics in situ, and to explore their contribution to overall mechanical stability of the cell and interconnected extra cellular matrix (ECM) [Kumar et al., 2006]. The interplay between the biophysical properties of the cell and ECM establishes a dynamic, mechanical reciprocity between the cell and the ECM in which the cell's ability to exert contractile stresses against the extracellular environment balances the elastic resistance of the ECM to that deformation. It has become clear that this force balance can regulate a surprisingly wide range of cellular properties that are all critical to tumorigenesis, including structure, motility, proliferation, and differentiation, see review by Kumar et al [Kumar and Weaver, 2009].

Damping augmentation is a common approach to vibration control in structures (vibroacoustic, vibration fatigue). Viscoelastic materials can be used to design efficient damping treatments. The mechanical properties of these materials however depend on frequency, but also on pre-stress and temperature. The study of viscoelastic medium under the effect of initial stress will help to design more efficient damping treatments. Vibration control in machines and structures for situations involving vibration excitations can be carried out by incorporating viscoelastic materials and applying the principle of vibratory energy dissipation due to damping as a result of deformation of viscoelastic materials [Nakra, 1998]. Recently, Rao [Rao, 2003] discussed recent applications of viscoelastic damping for noise control in automobiles and commercial airplanes.

Stress relaxation in prestressed laminates with viscoelastic matrices, together with creep deformation under constant rate loading or large changes in temperature can be used to develop a technique leading to improvement of damage and penetration resistance of laminated composite structures, such as army vehicles and their armor. Applications of fiber pre-stress can be explored in compressive pre-stressing of ceramic/FRp armor plates for improved resistance to projectile penetration [Garg, 2007].

1.2 Review on the study of viscoelastic solids under pre-deformation and pre-load

The theory developed in this dissertation permits studying a variety of phenomena in pre-deforming viscoelastic media. There is a large group of bodies in service that belong to the category of pre-deforming nonlinear viscoelastic bodies. One can mention, for example, the rolling resistance of automotive tires and elastomeric engine mounts for a passenger car [Lion and Kardelky, 2004]. Also, one might use this for studying the deformation of biological soft tissues such as tendons [Pioletti et al., 1998], and the study of the skin and possibly the brain, during multiple loading events, such as expected in impact or blast loading, see the recent review by Hoskins [Hoskins, 2007]. Other possible applications are in deformation ultrasound and magnetic resonance imaging, which are tools for non-destructive evaluation and medical imaging. In particular, this study, for example, could contribute to developing non-invasive methods to investigate properties during medical imaging [Sinkus et al., 2006]. One key issue to figure out in these problems is the effects of pre-loads and pre-deformations on the wave characteristics, such as is seen, for example, by Fatemi et al. [Fatemi et al., 2003] and Greenleaf et al. [Greenleaf et al., 2003], in which compression of soft tissues before imaging is shown to increase contrast and reduce de-correlation noise, but might also provide invaluable information on thermo-mechanical properties, for example during loading and growth.

There are relatively few studies dedicated to looking at thermo-mechanical waves superimposed on finite thermo-mechanical deformations in viscoelastic materials that consider both homogeneous and inhomogeneous pre-deformations, and temperature perturbations. Saccomandi [Saccomandi, 2005] derived the equations of small amplitude homogeneous and inhomogeneous plane shear waves in a homogeneous biaxially deformed Mooney-Rivin viscoelastic solids whose elastic part of the stress tensor is modeled through the use of the Mooney-Rivlin form of the stored energy function, and whose dissipative part is modeled as a Newtonian fluid. Destrade et al. [Destrade et al., 2009] and Quintanilla et al. [Quintanilla and Saccomandi, 2009] studied the propagation of small amplitude waves in a large statically deformed nonlinear viscoelastic solid and addressed both material and geometric stability. In this case, the viscoelastic material was characterized by a Cauchy stress tensor depending only on the left Cauchy Green deformation tensor and on the deformation rate tensor, as the author mentions, the constitutive model they use can account for classical effects like creep and recovery, but cannot describe stress relaxation. Rajagopal et al. [Rajagopal and Saccomandi, 2003] investigated shear wave propagations in a certain class of nonlinear viscoelastic materials which are modeled the same as in Destrade et al. [Destrade et al., 2009] and provided some exact solutions for initial data with compact and non-compact support. Garg [Garg, 2007] used Biot's theory to study the effect of initial homogeneous stress on phase velocity and attenuation of plane homogeneous waves in a general viscoelastic anisotropic medium, and showed that the attenuation is more sensitive to the initial stress as compared to the propagation velocities. Biot [Biot, 1965] presented the theory of elasticity and viscoelasticity of initially stressed solids and fluids including the applications to finite strains. Hayes et al. [Hayes, 1969, Hayes and Rivlin, 1972] discussed the propagation of a sinusoidal wave of small amplitude in an initially isotropic viscoelastic solid, which is subjected to a static pure homogeneous deformation, and they applied their results to the reflection-refraction problem at the plane interface between two half-spaces of such materials, subjected to different pure homogeneous deformations. They model the viscoelastic material as a Rivlin-Ericksen type of material which was detailed in Rivlin et al. [Rivlin and Wilmanski, 1987] for which the Cauchy stress at a material point at time t depends not only on the instantaneous value of the deformation gradient, but also on its time derivatives of various orders. Karnaukhov [Karnaukhov, 1977] presented a thermo-mechanical theory of small viscoelastic strains imposed on stable initially large strains for thermo-rheological materials, where the Cauchy stress is the partial derivative of the free energy with respect to the strain tensor and is represented by a single integral with the integrand depending on the history of the deformation. Rendek et al. [Rendek and Lion, 2010] and Lion et al. [Lion, 1998, Lion et al., 2009] investigated the dynamic behaviors of non-linear viscoelastic bodies under finite static pre-deformations superimposed by sinusoidal loads with small amplitudes. The storage and loss moduli were measured through dynamic mechanical analysis (DMA) under varying frequencies, pre-deformations and temperatures. The authors formulated the models for three-dimensional finite deformations by employing a finite linear viscoelasticity model, by using modified Mooney-Rivlin strain energy function, or a Neo-Hookean strain energy function with the viscosities depending on the deformation history. In order to derive the expressions for dynamic moduli, the authors linearised the constitutive equations with respect to the static pre-deformations. The identification process of the material constants based on the dynamic moduli was also provided.

Destrade [Destrade, 2000] considered the propagation of finite-amplitude linearly-polarized inhomogeneous transverse plane waves propagate in an incompressible Mooney-Rivlin rubber-like material maintained in a state of finite static homogeneous deformation. The author showed that such waves are possible when the directions of the normal to the planes of constant phase and of the normal to the planes of constant amplitude are orthogonal and conjugate with respect to the initial left Cauchy-Green strain tensor ellipsoid. Chadwick et al. [Chadwick et al., 1985] and Borejko [Borejko, 1987] considered the effects of the internal constraints on the properties of homogeneous and inhomogeneous waves in elastic media and developed the linearized dynamics of a non-heat-conducting elastic body subject to a general system of internal constraints and an arbitrary homogeneous pre-strain. Hayes and

Saccomandi [Hayes and Saccomandi, 2002] considered the propagation of finite amplitude plane transverse waves in a class of homogeneous isotropic incompressible viscoelastic solids subject to a homogeneous static deformation. The Cauchy stress in their paper was assumed as the sum of an elastic part and a dissipative viscoelastic part. The elastic part was of the form of the stress corresponding to a Mooney-Rivlin material, whereas the dissipative part was a linear combination of the first and second Rivlin-Ericksen tensors. The authors found that two finite amplitude transverse plane waves may propagate in every direction in the deformed body and they provide several exact solutions. Destrade and Saccomandi [Destrade and Saccomandi, 2004] provided new exact solutions for finite-amplitude, transverse, linearly polarized, inhomogeneous motions superposed upon a finite homogeneous static deformation. The viscoelastic body is modeled as a Mooney-Rivlin viscoelastic solid, whose constitutive equation consists of the sum of an elastic part (Mooney-Rivlin hyperelastic model) and a viscous part (Newtonian viscous fluid model). The authors found the conditions for the directions of polarization, of propagation, and of attenuation under which the waves may propagate. Specifically the authors studied solutions including traveling inhomogeneous finite-amplitude damped waves and standing damped waves.

Fosdick et al. [Fosdick and Yu, 1996] studied the thermodynamics and stability of a viscoelastic second grade solid where the rate effects are characterized by two microstructural coefficients in addition to the Newtonian viscosity. The authors showed the necessary and sufficient conditions for the material model to be compatible with thermodynamics and the free energy to be at a local minimum in equilibrium. In the paper the authors constructed a stability theorem for second grade solids subject to mechanically isolated motions and showed that the motion of the body relative to its center of mass dissipated in time. Fosdick et al. [Fosdick and Yu, 1998] studied the thermodynamics and stability of a history type viscoelastic solid with an exponentially decaying relaxation function. The authors constructed a stability theorem for a history type solid undergoing mechanically isolated motions by showing that the intrinsic motion of the body with respect to the center of mass is stable in the sense of Lyapunov.

1.3 Current limitations

In many of the cited studies the constitutive model used to describe the nonlinear viscoelastic material response is fairly special. For example, Mooney-Rivlin viscoelasticity, Newtonian viscosity, initial isotropy and incompressibility are common starting assumptions. In this dissertation we start with a fairly general constitutive modeling structure for the nonlinear viscoelastic response that allows both isotropic and anisotropic response. Wineman [Wineman, 2009] provided an overview of the current state of the subject of nonlinear viscoelastic solids. The review presented an introduction to the continuum theory of nonlinear viscoelastic solids, discussing the constitutive equations and illustrating their application to several problems of technical relevance by showing the solutions to boundary value problems that have appeared in the literature. The author discussed the formulation of constitutive equations for isotropic, transversely isotropic and orthotropic nonlinear viscoelastic solids and summarized some proposed constitutive equations for nonlinear viscoelastic solids, for example rate and differential type constitutive equations, Green-Rivlin multiple integral constitutive equations, finite linear viscoelasticity, Pipkin-Rogers constitutive theory, and quasi-linear viscoelasticity, with emphasis on nonlinear single integral finite linear viscoelastic and Pipkin-Rogers constitutive equations. The author presented the material symmetry restrictions on the proposed constitutive equations, like isotropy, transverse isotropy and orthotropy.

Other common limitations in many of the published articles are either the assumptions that the infinitesimal mechanical perturbations are superimposed only around stable thermodynamic equilibrium states, and/or the time rates of the infinitesimal perturbations are infinitesimal, not allowing the use of the results for states that largely deviate from thermodynamic equilibrium of the pre-deformations or when considering high frequency wave motions. The development provided here does not require making such assumptions.

1.4 Outline

For all the reasons given in the previous section, the synopsis of the present dissertation can be stated as follows.

In this dissertation, we look at time varying perturbations in a nonlinear viscoelastic material, with specific examples for wave motions in mind. The results are based on a general nonlinear thermo-viscoelastic constitutive modeling structure embedded into a thermodynamically consistent framework, which permits large deformations and also large deformation rates. The model is capable of capturing continuous material relaxation and for example, in the linear case, can specialize to a linear viscoelastic model with a continuous relaxation spectra. Using the proposed constitutive equations, we develop a superposition method to impose infinitesimal thermo-mechanical perturbations on a loading body. There are no specific equilibrium requirement on the thermodynamic state of the loading body, which has not been studied before. Earlier approaches to perturbing the pre-loaded body either were restricted to small deviations away from thermodynamic equilibrium of the pre-deformations, did not account for thermo-mechanical coupling effects, or controlled the time rate of the perturbation relative to the loading.

In **Chapter 2**, we present the notation and basics of nonlinear continuum mechanics, which includes kinematics and balance laws. We also develop the balance equations for thermo-mechanical perturbations superimposed on a deforming body. For finite material bodies, we study different types of boundary conditions for the perturbations. In order to study the propagation of discontinuities, we derive the perturbed jump conditions and consider two special cases.

Chapter 3 is devoted to presenting the finite deformation thermo-mechanically coupled elasticity theory and developing the perturbation equations for thermo-elastic solids undergoing large deformations. The propagation of different types of mechanical waves in an axis-symmetrically deformed thermo-elastic cylinder is also studied, which shows the potential application of the current work to load sensoring devices.

Chapter 4 presents the thermodynamically based viscoelastic constitutive equations used in this dissertation. For infinitesimal perturbations on a given changing base history, we develop the resulting constitutive model for the perturbations, and provide a general solution for the evolution of the associate internal variables. We consider the example of a homogeneous pre-deformation, present results for isothermal harmonic waves that are space attenuating and time damping, and also provide the associated special cases. In particular, we show the relation of this study to the work of Garg [Garg, 2007], Červený et al. [Červený and Pšenčík, 2005], and Biot [Biot, 1965].

Chapter 5 focuses on material constraints in elastic materials and in one-element visoelastic materials. The perturbations with material constraints are developed.

Chapter 6 discusses the propagation of mechanical waves in inhomogenously deforming viscoelastic bodies. We also look at the propagation of discontinuities (jumps) in deforming visoelastic bodies.

Chapter 7 summarizes and discusses the results, possible applications and future plans.

CHAPTER 2

Kinematics and balance laws for thermo-mechanical perturbations

2.1 Introduction

This chapter starts by introducing the notation through the presentation of nonlinear continuum mechanics. This includes discriptions of the kinematics and balance laws. Most of this development is based on Negahban [Negahban, 2012]. For more comprehensive treatment the reader is referred to the books on the subject by Truesdell and Noll [Truesdell and Noll, 1965] and Negahban [Negahban, 2012], among others.

Next, we consider the perturbation of loading history and in the process define the base, total and perturbation parts of a history, and present the balance laws for the perturbation terms and the boundary conditions for the perturbations. We also derive the perturbation equations for the jump conditions, which allow us to later study the propagation of surface discontinuity.

2.2 Kinematics

We consider a continuous material body with κ_o denoting the reference configuration and $\kappa(t)$ denoting the current configuration at time t. We let $\mathbf{X}$ denote the position vector of particles in the reference configuration and $\mathbf{x}$ denote the position in the current configuration. The motion of the body is given by the function $\mathbf{x}(\mathbf{X}, t)$, which maps each particle given by the position $\mathbf{X}$ in the reference configuration to its position in the current configuration at time t. The deformation gradient $\mathbf{F} = \mathbf{Grad}(\mathbf{x})$ is a second order tensor which relates $d\mathbf{X}$ and $d\mathbf{x}$ through the relation

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}, \tag{2.1}$$

where $\mathbf{Grad}(.)$ is the gradient relative to changes in position $\mathbf{X}$ in the reference configuration.

The relation of the deformation gradient to the nabla operator comes from the relation $d\mathbf{x} = d\mathbf{X}\nabla_{\mathbf{X}}(\mathbf{x})$, that gives the relation $\mathbf{F} = [\nabla_{\mathbf{X}}(\mathbf{x})]^T$, where the superscript "T" denotes the transpose. The displacement $\mathbf{u}(\mathbf{X}, t)$ of each particle is defined by

$$\mathbf{x}(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t). \tag{2.2}$$

The relation between the deformation gradient $\mathbf{F}$ and the displacement gradient $\mathbf{H}$ is given by $\mathbf{F} = \mathbf{I} + \mathbf{H}$, where $\mathbf{H} = \mathbf{Grad}(\mathbf{u}) = [\nabla_{\mathbf{X}}(\mathbf{u})]^T$, and has the property $d\mathbf{u} = \mathbf{H}d\mathbf{X}$. The left Cauchy stretch tensor is given as $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and the right Cauchy stretch tensor is given as $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

The velocity $\mathbf{v}$ of particle $\mathbf{X}$ is calculated from the motion of the body by taking a material time derivative, which holds $\mathbf{X}$ constant, to get

$$\mathbf{v}(t) = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}. \tag{2.3}$$

The velocity gradient is denoted by $\mathbf{L} = \mathbf{grad}(\mathbf{v})$ and provides the relation

$$d\mathbf{v} = \mathbf{L}d\mathbf{x}, \tag{2.4}$$

where $\mathbf{grad}(.)$ is the gradient relative to changes in position $\mathbf{x}$ in the current configuration. It follows that $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. The relation with the nabla operator comes from $d\mathbf{v} = d\mathbf{x}\nabla_{\mathbf{x}}(\mathbf{v})$ which provides $\mathbf{L} = [\nabla_{\mathbf{x}}(\mathbf{v})]^T$.

In a similar manner, we take the temperature to be given by a function $\theta(\mathbf{X}, t)$. Depending on the selected configuration, there are at least two commonly used temperature gradients. We denote these by $\mathbf{G}$ and $\mathbf{g}$ and define them through the relations

$$d\theta = \mathbf{G}d\mathbf{X} = \mathbf{g}d\mathbf{x}. \tag{2.5}$$

As can be seen, $\mathbf{G}$ is the gradient of temperature with respect to changes of position in the reference configuration and $\mathbf{g}$ is the gradient of temperature with respect to changes of position in the current configuration so that we have $\mathbf{G} = \mathbf{Grad}(\theta)$ and $\mathbf{g} = \mathbf{grad}(\theta)$, with the relation $\mathbf{G} = \mathbf{gF}$.

2.3 Balance laws

The prediction of material response requires the combination of several elements. In general, these elements include mathematical models describing the material's response characteristics (constitutive equations), specific conditions describing the initial state of the matter (initial conditions), conditions describing how the specific body is being influenced by its surrounding (boundary conditions), and laws describing how to combine these elements (balance laws). The focus of this section will be on the balance laws. These laws have a special place in the theory of material response since they are the same for all materials, in contrast to constitutive equations that are different for each material.

The five laws that we collectively call the balance laws include: the conservation of mass, the balance of linear momentum, the balance of angular momentum, the balance of work and energy and the entropy production inequality. Each of the balance laws is a general statement on how all materials will respond over time, and can be used to calculate the specific response of a particular material body only when augmented by constitutive models for the specific material, and specific initial and boundary conditions describing the initial state of the material body and the processing conditions.

2.3.1 Conservation of mass

Conservation of mass states that the mass in a body will not change if the particles in the body remain the same. This is mathematically written as

$$\rho J = \rho_o, \tag{2.6}$$

where ρ_o is material density in the reference configuration, ρ is material density in the current configuration, and J is the volume ratio of the current configuration relative to the reference configuration given by $J = det(\mathbf{F})$.

2.3.2 Balance of linear momentum

The law of balance of linear momentum states that the resultant of all applied forces on a material body is equal to the rate of change of linear momentum for that material body. This law is written as

$$\int_{S(t)} \mathbf{t}^{(n)} dS + \int_B \mathbf{b} dm = \frac{d}{dt}(\int_B \mathbf{v} dm), \tag{2.7}$$

where the first integral represents the resultant force due to traction on the surface of the body $S(t)$, the second integral represents the resultant body force, and the third integral represents the linear momentum of the body. In this expression $\mathbf{b}$ is the body force per unit mass of the body, $\mathbf{t}^{(n)}$ is the traction vector that can be replaced by the Cauchy stress tensor through the relation $\mathbf{t}^{(n)} = \mathbf{T}^T \hat{\mathbf{n}}$, and the integration over mass can be replaced by integration over volume through the relation $dm = \rho dV$.

In the current configuration, the differential form of this law can be obtained by assuming that the law must hold for each part of the body and that the arguments of the integrals are continuous for every subregion. This results in

$$div(\mathbf{T}^T) + \rho \mathbf{b} = \rho \mathbf{a}, \tag{2.8}$$

where the material time derivative of the velocity is replaced by the acceleration $\mathbf{a}$.

Since the domain of integration for the integral form of this balance law, given in (2.7), may be changing with time, it is sometimes desirable to work in the reference configuration, which has a domain that is unchanged with time. To do this we will use the relation for the total load applied on the surface of the body given as

$$\mathbf{P} = \int_S \mathbf{t}^{(n)} dS = \int_{S_o} \mathbf{t}_o^{(N)} dS_o, \tag{2.9}$$

where $\mathbf{P}$ is the total resultant load applied on the surface of the body, S is the surface of the segment in the current configuration and S_o is its associated surface in the reference configuration. The traction $\mathbf{t}_o^{(N)}$ is associated with the nominal stress $\mathbf{T}_o$ through the Cauchy relation $\mathbf{t}_o^{(N)} = \hat{\mathbf{N}}\mathbf{T}_o$.

Changing the volume integrals in (2.7) by using the relation $dV = JdV_o$ and introduction of the conservation of mass $\rho J = \rho_o$ yield the relation

$$\int_{S_o} \mathbf{t}_o^{(N)} dS_o + \int_{V_o} \mathbf{b}\rho_o dV_o = \int_{V_o} \mathbf{a}\rho_o dV_o. \tag{2.10}$$

This must hold not only for the entire body, but also for any segment of it. Therefore, if the functions in the arguments are continuous functions of position $\mathbf{X}$, then we obtain the alternate differential form of this balance law in the reference configuration as

$$Div(\mathbf{T}_o^T) + \rho_o \mathbf{b} = \rho_o \dot{\mathbf{v}}, \tag{2.11}$$

where $Div(.)$ is the divergence relative to changes in position $\mathbf{X}$ in the reference configuration, $\mathbf{T}_o$ is the nominal or engineering stress in the current configuration given in terms of the Cauchy stress $\mathbf{T}$ and deformation gradient as

$$\mathbf{T}_o = J\mathbf{F}^{-1}\mathbf{T}, \tag{2.12}$$

and $\mathbf{b}$ is the specific body force in the current configuration.

2.3.3 Balance of angular momentum

The law of balance of angular momentum states that the resultant moment applied on a body must equal the rate of change of angular momentum of that material body. This can be written as

$$\int_{S(t)} \mathbf{x} \times \mathbf{t}^{(n)} dS + \int_B \mathbf{x} \times \mathbf{b} dm = \frac{d}{dt}\left(\int_B \mathbf{x} \times \mathbf{v} dm\right), \tag{2.13}$$

where the first integral is the moment due to the traction on the surface of the body $S(t)$, the second integral is the moment due to body forces, and the third integral is the angular momentum of the body. In this expression, $\mathbf{x}$ represents the vector describing the position of the particle under consideration, either the position of the particle the load is applied on or the position of the particle for which angular momentum is to be calculated. This position must be measured relative to a point on an inertial reference frame.

Using the Cauchy relation $\mathbf{t}^{(n)} = \mathbf{T}^T \hat{\mathbf{n}}$ and the divergence and transport theorems and assuming the arguments of the integral is continuous result in the symmetry of the Cauchy stress $\mathbf{T} = \mathbf{T}^T$, which requires that $\mathbf{F}\mathbf{T}_o = \mathbf{T}_o^T \mathbf{F}^T$.

2.3.4 Balance of work and energy

The law of balance of work and energy states that the rate at which heat flows into a body plus the rate at which work is being done on that body is equal to the rate at which the kinetic plus internal energy of the body changes. The heat may be added to a body through the surface or directly to each particle, and the work can be done on a body by traction forces and by body forces.

The law of balance of work and energy can be written for a body as

$$-\int_{S(t)} \mathbf{q} \circ \hat{\mathbf{n}} dS + \int_B r dm + \int_{S(t)} \mathbf{t}^{(n)} \circ \mathbf{v} dS + \int_B \mathbf{b} \circ \mathbf{v} dm = \frac{d}{dt} \int_B \left(\frac{1}{2}\mathbf{v} \circ \mathbf{v} + e\right) dm, \tag{2.14}$$

where $\mathbf{q}$ is the heat flux vector, $q_n = \mathbf{q} \circ \hat{\mathbf{n}}$ is the net heat flux out of the body, r is the rate of heat addition per unit mass inside the body, and e is the internal energy per unit mass of the body known as the specific internal energy.

It follows, after the application of the divergence theorem, the transport theorem, the assumption of sufficient continuity in the variables and their derivatives, the balance of linear momentum, and standard arguments, that the differential form of the balance of work and energy can be written as

$$-div(\mathbf{q}) + \rho r + tr(\mathbf{TL}) = \rho \dot{e}. \tag{2.15}$$

The balance of energy can be written in terms of integrals over the shape of the body in the reference configuration and with the use of the engineering stress $\mathbf{T}_o$ and the "engineering" or "nominal heat flux" $\mathbf{q}_o$. Using the relation $dm = \rho_o dV_o$, the engineering traction vector $\mathbf{t}_o^{(N)}$ and heat flux vector $\mathbf{q}_o$, the balance of work and energy (2.14) can be written as

$$-\int_{S_o} \mathbf{q}_o \circ \hat{\mathbf{N}} dS_o + \int_{V_o} r\rho_o dV_o + \int_{S_o} \mathbf{t}_o^{(N)} \circ \mathbf{v} dS_o + \int_{V_o} \mathbf{b} \circ \mathbf{v} \rho_o dV_o = \frac{d}{dt} \int_{V_o} (\frac{1}{2}\mathbf{v} \circ \mathbf{v} + e)\rho_o dV_o. \tag{2.16}$$

Introduction of $\mathbf{t}_o^{(N)} = \mathbf{T}_o^T \hat{\mathbf{N}}$, application of the divergence theorem, and standard assumptions of continuity of the arguments result in

$$-Div(\mathbf{q}_o) + \rho_o r + tr(\mathbf{T}_o \dot{\mathbf{F}}) = \rho_o \dot{e}, \tag{2.17}$$

where r is the specific radiation in the current configuration, $\mathbf{q}_o$ is the nominal heat flux vector in the current configuration, which is related to the heat flux vector $\mathbf{q}$ in the current configuration through the relation $\mathbf{q}_o = J\mathbf{F}^{-1}\mathbf{q}$, and where e is the specific internal energy, given in terms of the free energy and entropy as $e = \psi + \theta\eta$.

2.3.5 The entropy production inequality

The entropy production inequality, also known as the second law of thermodynamics, states that the entropy in a material body of fixed mass increases at least as rapidly as entropy is added to the body through the addition of heat to the body, either by radiation directly into the body or by heat flow through the boundaries of the body.

The integral form of the entropy production inequality is given by

$$\int_B \frac{r}{\theta} dm - \int_{S(t)} \frac{1}{\theta} \mathbf{q} \circ \hat{\mathbf{n}} dS \leq \frac{d}{dt} \int_B \eta dm, \tag{2.18}$$

where η is the entropy per unit mass of the body, also known as the specific entropy.

Using the divergence and transport theorems, one can find the differential form of this law as

$$\rho\frac{r}{\theta} - div(\frac{1}{\theta}\mathbf{q}) \leq \rho\dot{\eta}. \tag{2.19}$$

We can introduce the relation $e = \psi + \eta\theta$ into the entropy production inequality, assume a strictly positive temperature scale and use the balance of energy to obtain the form known as the Clausius-Duhem inequality and written as

$$\rho\dot{\psi} - tr(\mathbf{TL}) + \rho\eta\dot{\theta} + \frac{1}{\theta}\mathbf{q}\circ\mathbf{g} \leq 0. \tag{2.20}$$

As was the case for the previous laws, we can directly transfer the integrals over the current configuration to integrals over the reference configuration. This process gives the entropy production inequality (2.18) as

$$\int_{V_o}\frac{r}{\theta}\rho_o dV_o - \int_{S_o}\frac{1}{\theta}\mathbf{q}_o\circ\hat{\mathbf{N}}dS_o \leq \frac{d}{dt}\int_{V_o}\eta\rho_o dV_o. \tag{2.21}$$

Following similar steps as above, we arrive at the differential form of this law given by

$$\rho_o\frac{r}{\theta} - Div(\frac{1}{\theta}\mathbf{q}_o) \leq \rho_o\dot{\eta}, \tag{2.22}$$

that must hold at every point in the reference configuration. In a similar process to that described above, one can introduce the free energy and arrive at a statement of the Clausius-Duhem inequality to be applied in the reference configuration. This form of the Clausius-Duhem inequality is given by

$$\rho_o\dot{\psi} - tr(\mathbf{T}_o\dot{\mathbf{F}}) + \rho_o\eta\dot{\theta} + \frac{1}{\theta}\mathbf{q}_o\circ\mathbf{Grad}(\theta) \leq 0. \tag{2.23}$$

2.4 Balance equations for small thermo-mechanical perturbations superimposed on a deforming body

We plan to study the response of a thermo-mechanically deforming body to superimposed infinitesimal thermo-mechanical perturbations. These perturbations may represent waves, for which we seek equations to describe. To construct these equations, we consider two thermo-mechanical loading histories: a *base history* and a *total history*. The total history is constructed from the base history by adding an infinitesimal thermo-mechanical perturbation. The response of the material to both the base history and the total history can be obtained from the constitutive equations, and both histories must satisfy the balance laws. In this section we obtain the general balance laws for the perturbations, their boundary conditions, and associated jump conditions. We do this by first defining the notation for describing the base, total and perturbation parts of a history.

2.4.1 Base and total history

In describing the process of perturbation we use two histories. We start with a base history on which we impose a perturbation to arrive at what we call the total history. We use a superscript "$*$" to describe variables associated with the total history, while we put no special marking for the variables evaluated for the base history.

The base history is assumed to start at an initial time t_i and is described by giving the history of the motion and temperature up to the current time t. This we write as

$$\mathcal{H}(t_i, t) = \{[\mathbf{x}(\mathbf{X}, \tau), \theta(\mathbf{X}, \tau)] | t_i < \tau < t\}, \tag{2.24}$$

where $\mathbf{x}$ and θ denote, respectively, the position vector and temperature of a material point for the base history. We take the term *intermediate configuration* to denote the configuration associate with the base history, and $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$ as the displacement vector for the base history.

The total history for the same time interval is described as

$$\mathcal{H}^*(t_i, t) = \{[\mathbf{x}^*(\mathbf{X}, \tau), \theta^*(\mathbf{X}, \tau)] | t_i < \tau < t\}, \tag{2.25}$$

where $\mathbf{x}^*$ and θ^* are, respectively, the position vector and temperature of a material point for the total history. We take the term *current configuration* to denote the configuration associate with the total history, and $\mathbf{u}^*(\mathbf{X}, t) = \mathbf{x}^*(\mathbf{X}, t) - \mathbf{X}$ as the displacement vector for the total history.

The difference between the value of a variable in the total history and the value of the same variable at the same time in the base history will be designated by adding "δ" to the left of the variable and will be the perturbation of the variable. For this notation we will have

$$\mathbf{u}^*(\mathbf{X}, t) = \mathbf{u}(\mathbf{X}, t) + \delta\mathbf{u}(\mathbf{X}, t), \tag{2.26}$$

where $\delta\mathbf{u}(\mathbf{X}, t)$ is the perturbation of the displacement vector. In a similar manner we can define the deformation gradients for the two histories by $\mathbf{F}(\mathbf{X}, t) = \mathbf{Grad}(\mathbf{x})$ and $\mathbf{F}^*(\mathbf{X}, t) = \mathbf{Grad}(\mathbf{x}^*)$ and write

$$\mathbf{F}^*(\mathbf{X}, t) = \mathbf{F}(\mathbf{X}, t) + \delta\mathbf{F}(\mathbf{X}, t), \tag{2.27}$$

where $\delta\mathbf{F}(\mathbf{X}, t)$ is the *perturbation of the deformation gradient.* Following a similar manner for the temperature and its gradient, we will write

$$\begin{aligned} \theta^*(\mathbf{X}, t) &= \theta(\mathbf{X}, t) + \delta\theta(\mathbf{X}, t), \\ \mathbf{G}^*(\mathbf{X}, t) &= \mathbf{G}(\mathbf{X}, t) + \delta\mathbf{G}(\mathbf{X}, t), \end{aligned} \tag{2.28}$$

where $\delta\theta(\mathbf{X}, t)$ is the perturbation of the temperature and $\delta\mathbf{G}(\mathbf{X}, t) = \mathbf{Grad}[\delta\theta(\mathbf{X}, t)]$ is the perturbation of the temperature gradient. Taking a time derivative gives the following relations between the associated derivatives of the two histories as

$$\begin{aligned} \dot{\mathbf{u}}^*(\mathbf{X}, t) &= \dot{\mathbf{u}}(\mathbf{X}, t) + \delta\dot{\mathbf{u}}(\mathbf{X}, t), \\ \dot{\mathbf{F}}^*(\mathbf{X}, t) &= \dot{\mathbf{F}}(\mathbf{X}, t) + \delta\dot{\mathbf{F}}(\mathbf{X}, t), \\ \dot{\theta}^*(\mathbf{X}, t) &= \dot{\theta}(\mathbf{X}, t) + \delta\dot{\theta}(\mathbf{X}, t), \\ \dot{\mathbf{G}}^*(\mathbf{X}, t) &= \dot{\mathbf{G}}(\mathbf{X}, t) + \delta\dot{\mathbf{G}}(\mathbf{X}, t). \end{aligned} \tag{2.29}$$

We define a *relative deformation gradient* $\check{\mathbf{F}}(\mathbf{X}, t)$, which maps the intermediate configuration to the current configuration, through

$$\mathbf{F}^*(\mathbf{X}, t) = \check{\mathbf{F}}(\mathbf{X}, t)\mathbf{F}(\mathbf{X}, t). \tag{2.30}$$

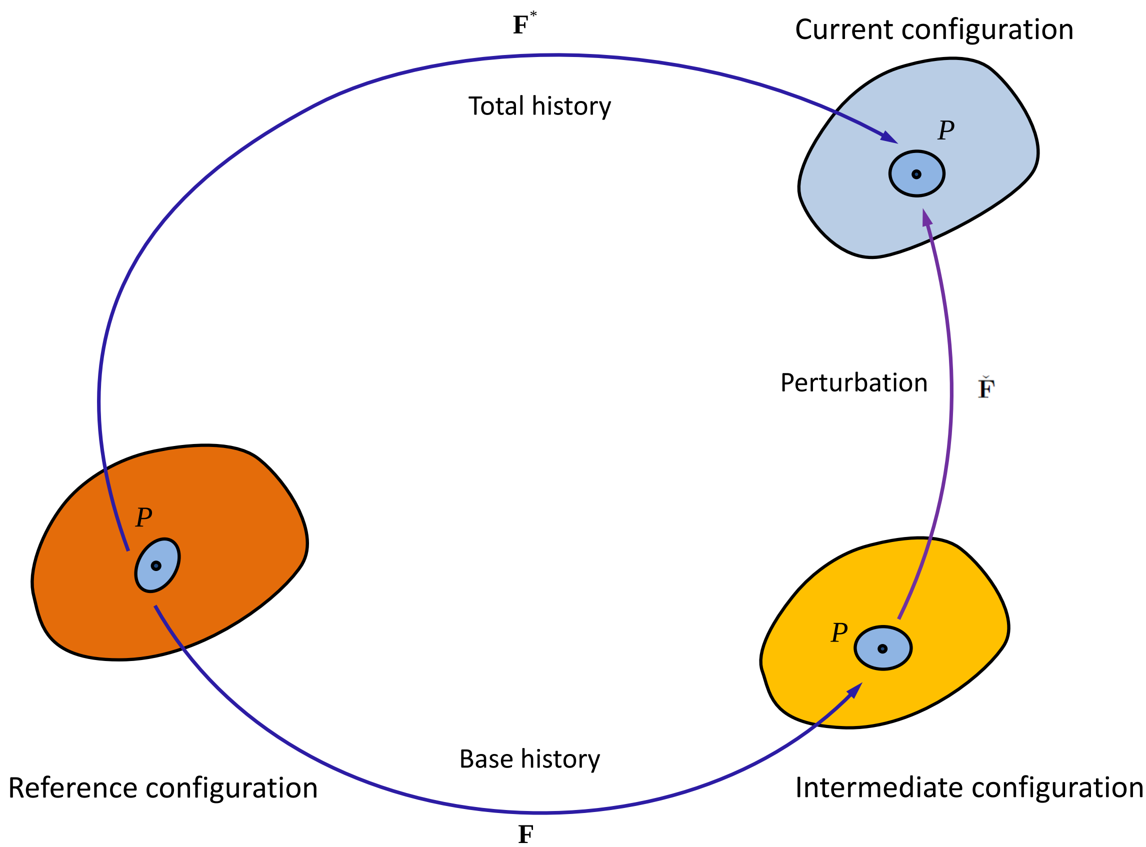


Figure 2.1: Deformation gradients in the base history and total history.

This is shown in Fig. 2.1.

The relation between the perturbation of the deformation gradient and the relative deformation gradient is given by

$$\delta\mathbf{F}(\mathbf{X}, t) = \check{\mathbf{H}}(\mathbf{X}, t)\mathbf{F}(\mathbf{X}, t), \tag{2.31}$$

where $\check{\mathbf{H}}(\mathbf{X}, t) = \check{\mathbf{F}}(\mathbf{X}, t) - \mathbf{I}$ is the displacement gradient associated with $\check{\mathbf{F}}(\mathbf{X}, t)$. The volume ratio J is given by $J^* = \check{J}J$, where $\check{J} = det(\check{\mathbf{F}})$ and $J^* = det(\mathbf{F}^*)$.

A proper perturbation $\check{\mathbf{H}}(\mathbf{X}, t)$ is assumed to be small in comparison to the identity tensor. This gives

$$\check{J} \approx 1 + \check{\varepsilon}_v, \tag{2.32}$$

where $\check{\varepsilon}_v = tr(\check{\boldsymbol{\epsilon}})$ is the infinitesimal volumetric strain of the relative infinitesimal strain associated with the relative deformation gradient $\check{\mathbf{H}}$, and given as

$$\check{\boldsymbol{\epsilon}} = \frac{1}{2}(\check{\mathbf{H}} + \check{\mathbf{H}}^T). \tag{2.33}$$

For an infinitesimal $\check{\mathbf{H}}$ we also know that

$$\check{\mathbf{F}}^{-1} \approx \mathbf{I} - \check{\mathbf{H}}. \tag{2.34}$$

2.4.2 Balance laws for the perturbation terms

The material responses for the base history $\mathcal{H}(t_i, t)$ and for the total history $\mathcal{H}^*(t_i, t)$ should each obey the balance laws. We will write these balance equations and with some manipulation show how to obtain equations for the thermo-mechanical perturbation.

For the base history we have the conservation of mass, balance of linear momentum, and balance of work and energy given by

$$\rho J = \rho_o, \tag{2.35}$$

$$Div(\mathbf{T}_o^T) + \rho_o \mathbf{b} = \rho_o \ddot{\mathbf{u}}, \tag{2.36}$$

$$-Div(\mathbf{q}_o) + \rho_o r + tr(\mathbf{T}_o \dot{\mathbf{F}}) = \rho_o \dot{e}, \tag{2.37}$$

and for the total history they are given by

$$\rho^* J^* = \rho_o, \tag{2.38}$$

$$Div(\mathbf{T}_o^{*T}) + \rho_o \mathbf{b}^* = \rho_o \ddot{\mathbf{u}}^*, \tag{2.39}$$

$$-Div(\mathbf{q}_o^*) + \rho_o r^* + tr(\mathbf{T}_o^* \dot{\mathbf{F}}^*) = \rho_o \dot{e}^*, \tag{2.40}$$

where $\mathbf{T}_o = J\mathbf{F}^{-1}\mathbf{T}$ and $\mathbf{T}_o^* = J^*\mathbf{F}^{*-1}\mathbf{T}^*$.

Dividing the equations of conservation of mass and subtracting the equations of balance of linear momentum and the balance of work and energy result in the general perturbation equations

$$\rho^* \check{J} = \rho, \tag{2.41}$$

$$Div(\delta \mathbf{T}_o^T) + \rho_o(\delta \mathbf{b}) = \rho_o \delta \ddot{\mathbf{u}}, \tag{2.42}$$

$$-Div(\delta \mathbf{q}_o) + \rho_o \delta r + tr(\mathbf{T}_o^* \dot{\mathbf{F}}^* - \mathbf{T}_o \dot{\mathbf{F}}) = \rho_o \delta \dot{e}. \tag{2.43}$$

These equations are general and hold for small or large perturbations.

We now look at the case of infinitesimal perturbations. For example, using the approximation $\check{J} \approx 1 + \check{\varepsilon}_v$ and ignoring higher order terms, the conservation of mass can be written as

$$\delta\rho = -\check{\varepsilon}_v \rho, \tag{2.44}$$

which only holds for infinitesimal perturbations. For the remaining balance laws we need to note the relation between the nominal and Cauchy stress, the associated heat fluxes, the working term, and the internal energy. For example, the perturbation of the nominal stress is given as

$$\begin{aligned} \delta \mathbf{T}_o =& \mathbf{T}_o^* - \mathbf{T}_o \\ =& J^* \mathbf{F}^{*-1} \mathbf{T}^* - J \mathbf{F}^{-1} \mathbf{T} \\ =& \check{J} J \mathbf{F}^{-1} \check{\mathbf{F}}^{-1} (\mathbf{T} + \delta \mathbf{T}) - J \mathbf{F}^{-1} \mathbf{T} \\ =& J \mathbf{F}^{-1} (\check{J} \check{\mathbf{F}}^{-1} - \mathbf{I}) \mathbf{T} + \check{J} J \mathbf{F}^{-1} \check{\mathbf{F}}^{-1} \delta \mathbf{T}. \end{aligned} \tag{2.45}$$

For an infinitesimal perturbation we also recall that $\check{\mathbf{F}}^{-1} \approx \mathbf{I} - \check{\mathbf{H}}$ so that

$$\begin{aligned} \mathbf{T}_o^* \approx& (1 + \check{\varepsilon}_v) J \mathbf{F}^{-1} (\mathbf{I} - \check{\mathbf{H}})(\mathbf{T} + \delta \mathbf{T}) \\ =& J \mathbf{F}^{-1} \mathbf{T} - J \mathbf{F}^{-1} \check{\mathbf{H}} \mathbf{T} + \check{\varepsilon}_v J \mathbf{F}^{-1} \mathbf{T} + J \mathbf{F}^{-1} \delta \mathbf{T}. \end{aligned} \tag{2.46}$$

Substituting this into the expression for $\delta \mathbf{T}_o$ and ignoring higher order terms one obtains

$$\delta \mathbf{T}_o \approx J \mathbf{F}^{-1} (\delta \mathbf{T} - \check{\mathbf{H}} \mathbf{T} + \check{\varepsilon}_v \mathbf{T}). \tag{2.47}$$

Substitution of this into the balance of linear momentum gives

$$Div[J(\delta\mathbf{T} - \mathbf{T}\check{\mathbf{H}}^T + \check{\varepsilon}_v\mathbf{T})\mathbf{F}^{-T}] + \rho_o(\delta\mathbf{b}) = \rho_o\delta\ddot{\mathbf{u}}. \tag{2.48}$$

In general, we know that for any arbitrary scalar function ϕ and second order tensor functions $\mathbf{A}$ and $\mathbf{B}$ we have the identities

$$Div(\phi\mathbf{A}) = \mathbf{A}\mathbf{Grad}(\phi) + \phi Div(\mathbf{A}), \tag{2.49}$$

$$Div(\mathbf{AB}) = \mathbf{Grad}(\mathbf{A}) : \mathbf{B} + \mathbf{A}Div(\mathbf{B}), \tag{2.50}$$

which are used to write the balance of linear momentum as

$$\begin{aligned}
&[\delta\mathbf{T} - \mathbf{T}\check{\mathbf{H}}^T + tr(\check{\mathbf{H}})\mathbf{T}][\mathbf{F}^{-T}\mathbf{Grad}(J) + JDiv(\mathbf{F}^{-T})] \\
&+ J\{[\mathbf{Grad}(\delta\mathbf{T}) - \mathbf{T}\mathbf{Grad}(\check{\mathbf{H}}^T) + \mathbf{T}\otimes\mathbf{Grad}(tr(\check{\mathbf{H}})) + tr(\check{\mathbf{H}})\mathbf{Grad}(\mathbf{T})] : \mathbf{F}^{-T} \\
&- \mathbf{Grad}(\mathbf{T}) : (\check{\mathbf{H}}^T\mathbf{F}^{-T})\} + \rho_o\delta\mathbf{b} = \rho_o\delta\ddot{\mathbf{u}},
\end{aligned} \tag{2.51}$$

where the form of $\delta\mathbf{T}$ is related to the constitutive model used and may be different for different material models.

In a similar way, the equation of balance of work and energy for the perturbation can be written as

$$\begin{aligned}
-Div[J\mathbf{F}^{-1}(\delta\mathbf{q} - \check{\mathbf{H}}\mathbf{q} + \check{\varepsilon}_v\mathbf{q})] + tr\{J\mathbf{F}^{-1}[(\mathbf{T} + \delta\mathbf{T} - \check{\mathbf{H}}\mathbf{T} + \check{\varepsilon}_v\mathbf{T})\dot{\check{\mathbf{H}}}\mathbf{F} \\
+ (\delta\mathbf{T} - \check{\mathbf{H}}\mathbf{T} + \check{\varepsilon}_v\mathbf{T} + \mathbf{T}\check{\mathbf{H}})\dot{\mathbf{F}}]\} + \rho_o\delta r \\
= \rho_o[\delta\dot{\psi} + \dot{\theta}\delta\eta + \eta\delta\dot{\theta} + (\delta\dot{\theta})(\delta\eta) + \theta\delta\dot{\eta} + \dot{\eta}\delta\theta + (\delta\theta)(\delta\dot{\eta})],
\end{aligned} \tag{2.52}$$

where $\delta r = r^* - r$. After some manipulation we arrive at

$$\begin{aligned}
&- Div[J\mathbf{F}^{-1}(\delta\mathbf{q} - \check{\mathbf{H}}\mathbf{q} + \check{\varepsilon}_v\mathbf{q})] + tr[(\mathbf{T}_o + \delta\mathbf{T}_o)\dot{\check{\mathbf{H}}}\mathbf{F}] + tr(\mathbf{T}_o\check{\mathbf{H}}\dot{\mathbf{F}}) \\
&+ tr(\delta\mathbf{T}_o\dot{\mathbf{F}}) + \rho_o\delta r = \rho_o[\delta\dot{\psi} + \dot{\theta}\delta\eta + \eta\delta\dot{\theta} + (\delta\dot{\theta})(\delta\eta) + \theta\delta\dot{\eta} + \dot{\eta}\delta\theta + (\delta\theta)(\delta\dot{\eta})],
\end{aligned} \tag{2.53}$$

where we use the standard assumptions for the perturbations as $\delta\mathbf{q} = \mathbf{q}^* - \mathbf{q}$, $\delta\psi = \psi^* - \psi$, and $\delta\eta = \eta^* - \eta$. In general, we know that for any arbitrary scalar function ϕ, vector function $\mathbf{u}$, and second order tensor function $\mathbf{A}$, we have the identities

$$Div(\phi\mathbf{u}) = \mathbf{Grad}(\phi) \cdot \mathbf{u} + \phi Div(\mathbf{u}), \tag{2.54}$$

$$Div(\mathbf{A}\mathbf{u}) = \mathbf{u} Div(\mathbf{A}^T) + \mathbf{A}^T : \mathbf{Grad}(\mathbf{u}), \tag{2.55}$$

which with the identities (2.49) and (2.50) can be used to write the equation for balance of work and energy for an inhomogeneous pre-loading as

$$\begin{aligned}
&- [\delta\mathbf{q} - \check{\mathbf{H}}\mathbf{q} + \check{\varepsilon}_v\mathbf{q}]Div(J\mathbf{F}^{-T}) - (J\mathbf{F}^{-T}) : [\mathbf{Grad}(\delta\mathbf{q}) - \mathbf{q}\mathbf{Grad}(\check{\mathbf{H}}^T) - \check{\mathbf{H}}\mathbf{Grad}(\mathbf{q}) \\
&+ \mathbf{q} \otimes \mathbf{Grad}(\check{\varepsilon}_v) + \check{\varepsilon}_v\mathbf{Grad}(\mathbf{q})] \\
&+ tr\{J\mathbf{F}^{-1}[(\mathbf{T} + \delta\mathbf{T} - \check{\mathbf{H}}\mathbf{T} + \check{\varepsilon}_v\mathbf{T})\dot{\check{\mathbf{H}}}\mathbf{F} + (\delta\mathbf{T} - \check{\mathbf{H}}\mathbf{T} + \check{\varepsilon}_v\mathbf{T} + \mathbf{T}\check{\mathbf{H}})\dot{\mathbf{F}}]\} \\
&+ \rho_o\delta r = \rho_o[\delta\dot{\psi} + \dot{\theta}\delta\eta + \eta\delta\dot{\theta} + (\delta\dot{\theta})(\delta\eta) + \theta\delta\dot{\eta} + \dot{\eta}\delta\theta + (\delta\theta)(\delta\dot{\eta})],
\end{aligned} \tag{2.56}$$

that after reorganization becomes

$$\begin{aligned}
&- [\delta\mathbf{q} - \check{\mathbf{H}}\mathbf{q} + \check{\varepsilon}_v\mathbf{q}]Div(J\mathbf{F}^{-T}) - (J\mathbf{F}^{-T}) : [\mathbf{Grad}(\delta\mathbf{q}) - \mathbf{q}\mathbf{Grad}(\check{\mathbf{H}}^T) - \check{\mathbf{H}}\mathbf{Grad}(\mathbf{q}) \\
&+ \mathbf{q} \otimes \mathbf{Grad}(\check{\varepsilon}_v) + \check{\varepsilon}_v\mathbf{Grad}(\mathbf{q})] \qquad (2.57) \\
&+ Jtr[(\mathbf{T} + \delta\mathbf{T} - \check{\mathbf{H}}\mathbf{T} + \check{\varepsilon}_v\mathbf{T})\dot{\check{\mathbf{H}}} + (\delta\mathbf{T} - \check{\mathbf{H}}\mathbf{T} + \check{\varepsilon}_v\mathbf{T} + \mathbf{T}\check{\mathbf{H}})\mathbf{L}] \\
&+ \rho_o\delta r = \rho_o[\delta\dot{\psi} + \dot{\theta}\delta\eta + \eta\delta\dot{\theta} + (\delta\dot{\theta})(\delta\eta) + \theta\delta\dot{\eta} + \dot{\eta}\delta\theta + (\delta\theta)(\delta\dot{\eta})].
\end{aligned} \tag{2.58}$$

From (2.51) and (2.57) we can clearly see the contributions of the time changing rates of the base history to the propagations of the superimposed thermo-mechanical perturbations. If we consider the base history as a static history, i.e. $\ddot{\mathbf{u}} = \mathbf{0}$ and write (2.51) in components form in Cartesian coordinates, we reduce (2.51) to the equation derived in Biot [Biot, 1965].

2.4.3 Boundary conditions for the perturbations

A finite body may be subjected to traction boundary conditions, displacement boundary conditions, temperature boundary conditions and heat flux boundary conditions. These would exist for the base history and for the total history, as is schematically shown in

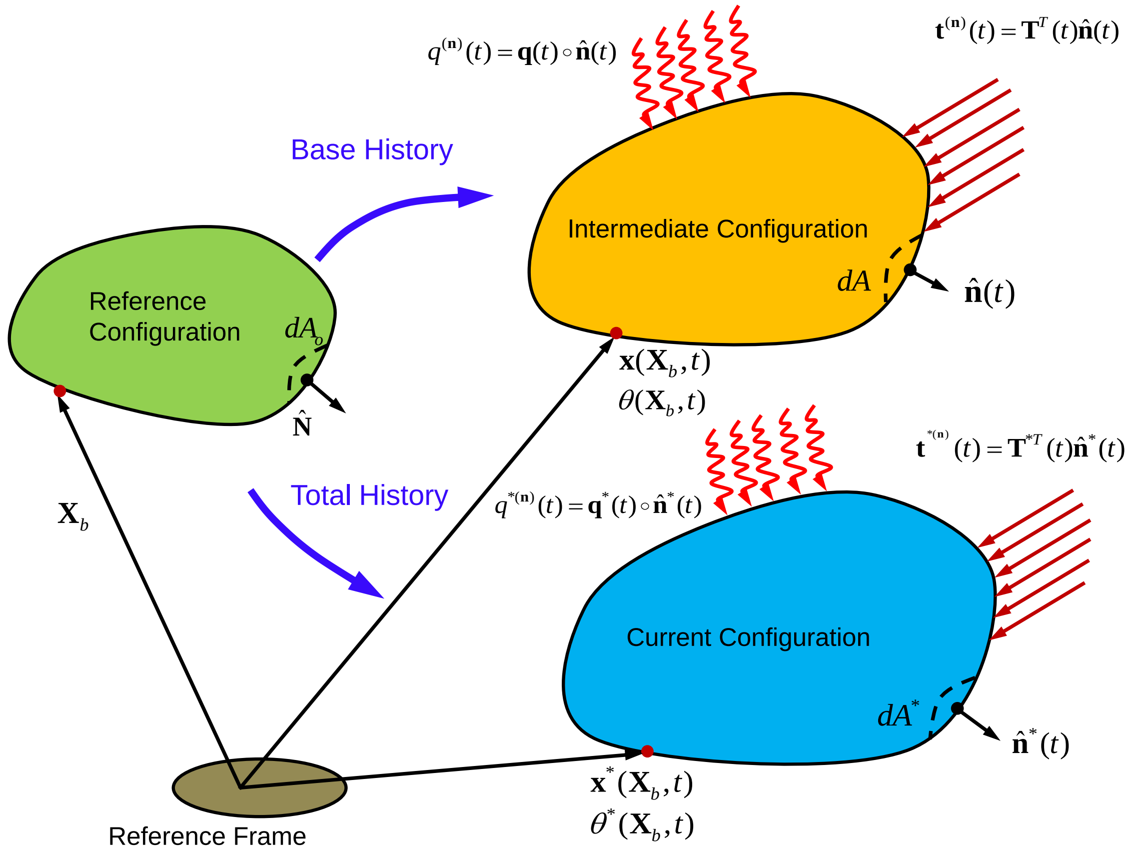


Figure 2.2: Perturbations of the traction, displacement, temperature and heat flux boundary conditions.

Figure 2.2. We use dA_o and $\hat{\mathbf{N}}$ to denote the differential surface area and the associated unit normal vector in the reference configuration, dA and $\hat{\mathbf{n}}$ to denote the differential surface area and the associated unit normal vector in the intermediate configuration, dA^* and $\hat{\mathbf{n}}^*$ to denote the differential surface area and the associated unit normal vector in the current configuration. $\mathbf{t}^{(\mathbf{n})}(t)$ is the traction boundary condition in the intermediate configuration, and $\mathbf{t}^{*(\mathbf{n}^*)}(t)$ is the traction boundary condition in the current configuration.

For the base history (intermediate configuration) we can show that

$$\hat{\mathbf{n}} = \det(\mathbf{F}) \frac{dA_o}{dA} \mathbf{F}^{-T} \hat{\mathbf{N}}, \tag{2.59}$$

where

$$\frac{dA_o}{dA} = \frac{1}{\sqrt{\det(\mathbf{C})\hat{\mathbf{N}} \circ (\mathbf{C}^{-1}\hat{\mathbf{N}})}}. \tag{2.60}$$

After substituting (2.60) into (2.59), we can get

$$\hat{\mathbf{n}} = \frac{\mathbf{F}^{-T}\hat{\mathbf{N}}}{\sqrt{\hat{\mathbf{N}} \circ (\mathbf{C}^{-1}\hat{\mathbf{N}})}}. \tag{2.61}$$

For the total history (current configuration) we can show that

$$\hat{\mathbf{n}}^* = \det(\mathbf{F}^*)\frac{dA_o}{dA^*}\mathbf{F}^{*-T}\hat{\mathbf{N}}, \tag{2.62}$$

where

$$\frac{dA_o}{dA^*} = \frac{1}{\sqrt{\det(\mathbf{C}^*)\hat{\mathbf{N}} \circ (\mathbf{C}^{*-1}\hat{\mathbf{N}})}}. \tag{2.63}$$

After substituting (2.63) into (2.62), we can get

$$\hat{\mathbf{n}}^* = \frac{\mathbf{F}^{*-T}\hat{\mathbf{N}}}{\sqrt{\hat{\mathbf{N}} \circ (\mathbf{C}^{*-1}\hat{\mathbf{N}})}}. \tag{2.64}$$

We can also relate the unite norms between the intermediate configuration and the current configuration through

$$\hat{\mathbf{n}}^* = \hat{\mathbf{n}} + \delta\hat{\mathbf{n}}, \tag{2.65}$$

where

$$\delta\hat{\mathbf{n}} = \{\frac{\hat{\mathbf{N}} \circ [\mathbf{F}^{-1}(\check{\mathbf{H}}^T + \check{\mathbf{H}})\mathbf{F}^{-T}\hat{\mathbf{N}}]}{2[\hat{\mathbf{N}} \circ (\mathbf{C}^{-1}\hat{\mathbf{N}})]^{\frac{3}{2}}}\mathbf{I} - \frac{\check{\mathbf{H}}^T}{\sqrt{\hat{\mathbf{N}} \circ (\mathbf{C}^{-1}\hat{\mathbf{N}})}}\}(\mathbf{F}^{-T}\hat{\mathbf{N}}). \tag{2.66}$$

We obtain the perturbation of the traction boundary condition between the two histories as

$$\mathbf{t}^{*(\mathbf{n}^*)} = \mathbf{t}^{(\mathbf{n})} + \delta\mathbf{t}^{(\mathbf{n})}, \tag{2.67}$$

where

$$\delta\mathbf{t}^{(\mathbf{n})} = \mathbf{T}^T\delta\hat{\mathbf{n}} + (\delta\mathbf{T}^T)\hat{\mathbf{n}}. \tag{2.68}$$

The displacement boundary condition can be prescribed for the material particles on the surface of the body. We use $\mathbf{X}_b$ to describe the position of the surface particles in the reference configuration, $\mathbf{x}(\mathbf{X}_b, t)$ to describe the position of the surface particles in the intermediate configuration, and $\mathbf{x}^*(\mathbf{X}_b, t)$ to describe the position of the surface particles in the current configuration. For the base history, the displacement boundary condition is given by

$$\mathbf{u}(\mathbf{X}_b, t) = \mathbf{x}(\mathbf{X}_b, t) - \mathbf{X}_b. \tag{2.69}$$

For the total history, the displacement boundary condition is given by

$$\mathbf{u}^*(\mathbf{X}_b, t) = \mathbf{x}^*(\mathbf{X}_b, t) - \mathbf{X}_b. \tag{2.70}$$

Therefore, the perturbation of the displacement boundary condition is given by

$$\delta\mathbf{u}(\mathbf{X}_b, t) = \delta\mathbf{x}(\mathbf{X}_b, t), \tag{2.71}$$

where

$$\begin{aligned} \delta\mathbf{u}(\mathbf{X}_b, t) &= \mathbf{u}^*(\mathbf{X}_b, t) - \mathbf{u}(\mathbf{X}_b, t), \\ \delta\mathbf{x}(\mathbf{X}_b, t) &= \mathbf{x}^*(\mathbf{X}_b, t) - \mathbf{x}(\mathbf{X}_b, t). \end{aligned} \tag{2.72}$$

The temperature boundary condition can be prescribed for the material particles on the surface of the body. For the base history, the boundary temperature can be written as $\theta(\mathbf{X}_b, t)$, and for the total history, the boundary temperature is given as $\theta^*(\mathbf{X}_b, t)$. We can calculate the perturbation of the boundary temperatures as $\delta\theta(\mathbf{X}_b, t) = \theta^*(\mathbf{X}_b, t) - \theta(\mathbf{X}_b, t)$.

We can introduce the heat flux through the surface of the body. We let $q^{(\mathbf{n})}$ represent the amount of heat that is flowing through the surface with normal $\hat{\mathbf{n}}$, measured per unit area and per unit time for the base history. This is given by

$$q^{(\mathbf{n})}(t) = \mathbf{q}(t) \circ \hat{\mathbf{n}}(t). \tag{2.73}$$

In a similar way we define $q^{*(\mathbf{n}^*)}$ as the heat flux passing through the surface with normal

$\hat{\mathbf{n}}^*$ for the total history. This is given by

$$q^{*(\mathbf{n}^*)}(t) = \mathbf{q}^*(t) \circ \hat{\mathbf{n}}^*(t). \tag{2.74}$$

The perturbation of the heat flux boundary condition is given by

$$q^{*(\mathbf{n}^*)} = q^{(\mathbf{n})} + \delta q^{(\mathbf{n})}, \tag{2.75}$$

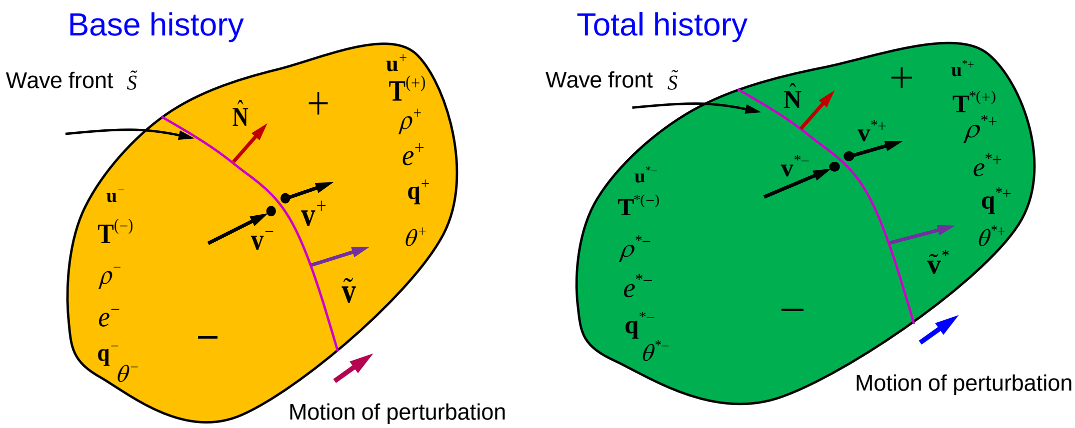
where

$$\delta q^{(\mathbf{n})} = \mathbf{q} \circ \delta\hat{\mathbf{n}} + (\delta\mathbf{q}) \circ \hat{\mathbf{n}}, \tag{2.76}$$

the expression for "$\delta\hat{\mathbf{n}}$" is given in (2.66).

2.4.4 Perturbed jump conditions

Up to this point we have assumed that all functions are sufficiently smooth and continuous over all the body so as to allow us to obtain the differential form of the balance laws. Now let us look at the case when the functions are not sufficiently continuous across a surface. As a general starting point, let us consider having a surface of discontinuity $\tilde{S}$ for the base history and its corresponding surface of discontinuity $\tilde{S}^*$ for the total history, as shown in Figure 2.3. For the general problem, we assume that there is a mapping connecting the two surfaces so that the relations given in the figure are meaningful, in spite of the fact that the domains on the two sides of the surface are materially different regions when comparing the base and total history. For our discussion, we will consider two special cases. First we consider the effect on the motion of the surface by perturbations on two sides of the surface, and the constraints between the possible perturbations. Next we consider the case where there is no jump in the base history, but the perturbation has a discontinuity. In general, for the base history the jump conditions associated with the conservation of mass, balance of linear momentum and balance of work and energy across any point of $\tilde{S}$ with normal $\hat{\mathbf{N}}$



$$
\begin{aligned}
&\mathbf{u}^{*+} = \mathbf{u}^+ + \delta\mathbf{u}^+, \quad \mathbf{u}^{*-} = \mathbf{u}^- + \delta\mathbf{u}^- \\
&\theta^{*+} = \theta^+ + \delta\theta^+, \quad \theta^{*-} = \theta^- + \delta\theta^- \\
&\mathbf{T}^{*T(+)} = \mathbf{T}^{T(+)} + \delta\mathbf{T}^{T(+)}, \quad \mathbf{T}^{*T(-)} = \mathbf{T}^{T(-)} + \delta\mathbf{T}^{T(-)} \\
&\rho^{*+} = \rho^+ + \delta\rho^+, \quad \rho^{*-} = \rho^- + \delta\rho^- \\
&\mathbf{v}^{*+} = \mathbf{v}^+ + \delta\mathbf{v}^+, \quad \mathbf{v}^{*-} = \mathbf{v}^- + \delta\mathbf{v}^- \\
&\tilde{\mathbf{v}}^* = \tilde{\mathbf{v}} + \delta\tilde{\mathbf{v}}, \quad \mathbf{q}^{*+} = \mathbf{q}^+ + \delta\mathbf{q}^+, \quad \mathbf{q}^{*-} = \mathbf{q}^- + \delta\mathbf{q}^- \\
&e^{*+} = e^+ + \delta e^+, \quad e^{*-} = e^- + \delta e^-
\end{aligned}
$$

$\delta\mathbf{u}^+,\ \delta\mathbf{u}^-,\ \delta\theta^+,\ \delta\theta^-$ are infinitesimal

Figure 2.3: Perturbation of the jump conditions.

are, respectively, given by

$$|[\rho(\mathbf{v} - \tilde{\mathbf{v}})]| \circ \hat{\mathbf{N}} = 0, \tag{2.77}$$

$$|[\rho\mathbf{v} \otimes (\mathbf{v} - \tilde{\mathbf{v}})]| \circ \hat{\mathbf{N}} = \left|\left[\mathbf{T}^T\right]\right| \circ \hat{\mathbf{N}}, \tag{2.78}$$

$$\left|\left[\rho(e - \frac{1}{2}\mathbf{v} \circ \mathbf{v})(\mathbf{v} - \tilde{\mathbf{v}})\right]\right| \circ \hat{\mathbf{N}} = \left|\left[\mathbf{v}\mathbf{T}^T - \mathbf{q}\right]\right| \circ \hat{\mathbf{N}}, \tag{2.79}$$

where we define the bracket "$|[.]|$" to describe the jump in the variable across the discontinuity such that, for example, $|[\mathbf{u}]| = \mathbf{u}^+ - \mathbf{u}^-$ for $\mathbf{u}^+$ being the value of $\mathbf{u}$ on the side that $\hat{\mathbf{N}}$ points into, and $\mathbf{u}^-$ being the value of $\mathbf{u}$ on the opposite side of $\tilde{S}$. In a similar manner, for the total history we will have the jump conditions

$$|[\rho^*(\mathbf{v}^* - \tilde{\mathbf{v}}^*)]| \circ \hat{\mathbf{N}}^* = 0, \tag{2.80}$$

$$|[\rho^*\mathbf{v}^* \otimes (\mathbf{v}^* - \tilde{\mathbf{v}}^*)]| \circ \hat{\mathbf{N}}^* = \left|\left[\mathbf{T}^{*T}\right]\right| \circ \hat{\mathbf{N}}^*, \tag{2.81}$$

$$\left|\left[\rho^*(e^* - \frac{1}{2}\mathbf{v}^* \circ \mathbf{v}^*)(\mathbf{v}^* - \tilde{\mathbf{v}}^*)\right]\right| \circ \hat{\mathbf{N}}^* = \left|\left[\mathbf{v}^*\mathbf{T}^{*T} - \mathbf{q}^*\right]\right| \circ \hat{\mathbf{N}}^*, \tag{2.82}$$

over the surface $\tilde{S}^*$ with normal $\hat{\mathbf{N}}^*$, and where $\mathbf{v}^* = \mathbf{v} + \delta\mathbf{v}$, $\tilde{\mathbf{v}}^* = \tilde{\mathbf{v}} + \delta\tilde{\mathbf{v}}$, etc.

As described, in the first case we look at the effect and the constraints resulting from perturbations in the continuous fields on the two sides of a discontinuity. That is, we select an instant that $\tilde{S} = \tilde{S}^*$ and $\hat{\mathbf{N}} = \hat{\mathbf{N}}^*$, and assume that the fileds on either side of the discontinuity are perturbed such that $\mathbf{u}^{*-} = \mathbf{u}^- + \delta\mathbf{u}^-$, $\mathbf{u}^{*+} = \mathbf{u}^+ + \delta\mathbf{u}^+$, $\theta^{*-} = \theta^- + \delta\theta^-$, $\theta^{*+} = \theta^+ + \delta\theta^+$, etc., as shown in the figure. Subtracting the two sets of jump conditions, while ignoring second and higher order terms, results in the equations

$$|[\rho\delta(\mathbf{v} - \tilde{\mathbf{v}}) + \delta\rho(\mathbf{v} - \tilde{\mathbf{v}})]| \circ \hat{\mathbf{N}} = 0, \tag{2.83}$$

$$|[\delta\rho\mathbf{v} \otimes (\mathbf{v} - \tilde{\mathbf{v}}) + \rho\delta\mathbf{v} \otimes (\mathbf{v} - \tilde{\mathbf{v}}) + \rho\mathbf{v} \otimes \delta(\mathbf{v} - \tilde{\mathbf{v}})]| \circ \hat{\mathbf{N}} = \left|\left[\delta\mathbf{T}^T\right]\right| \circ \hat{\mathbf{N}}, \tag{2.84}$$

$$\begin{aligned} &\left|\left[\delta\rho(e - \frac{1}{2}\mathbf{v} \circ \mathbf{v})(\mathbf{v} - \tilde{\mathbf{v}}) + \rho(\delta e - \mathbf{v} \circ \delta\mathbf{v})(\mathbf{v} - \tilde{\mathbf{v}}) + \rho(e - \frac{1}{2}\mathbf{v} \circ \mathbf{v})\delta(\mathbf{v} - \tilde{\mathbf{v}})\right]\right| \circ \hat{\mathbf{N}} \\ &\quad = \left|\left[\delta\mathbf{v}\mathbf{T}^T + \mathbf{v}\delta\mathbf{T}^T - \delta\mathbf{q}\right]\right| \circ \hat{\mathbf{N}}. \end{aligned} \tag{2.85}$$

Next, let us select to look at the case where there is no discontinuity in the response of the base history, but there is a discontinuity in the perturbation. In this case, for the base history we have $\tilde{\mathbf{v}} = \mathbf{0}$ and $\mathbf{u}^+ = \mathbf{u}^-$, $\theta^+ = \theta^-$, $\mathbf{v}^+ = \mathbf{v}^-$, etc. Once we introduce this into the jump conditions of the total history using $\mathbf{u}^{*-} = \mathbf{u}^- + \delta\mathbf{u}^-$, $\mathbf{u}^{*+} = \mathbf{u}^+ + \delta\mathbf{u}^+$, etc., we get the jump conditions for the perturbations as

$$\begin{aligned} &\{\rho\,|[\delta(\mathbf{v} - \tilde{\mathbf{v}})]| + |[\delta\rho]|\,\mathbf{v}\} \circ \hat{\mathbf{N}}^* = 0, \\ &\{|[\delta\rho]|\,\mathbf{v} \otimes \mathbf{v} + \rho\,|[\delta\mathbf{v}]| \otimes \mathbf{v} + \rho\mathbf{v} \otimes |[\delta(\mathbf{v} - \tilde{\mathbf{v}})]|\} \circ \hat{\mathbf{N}}^* = \left|\left[\delta\mathbf{T}^T\right]\right| \circ \hat{\mathbf{N}}^*, \\ &\left\{|[\delta\rho]|\,(e - \frac{1}{2}\mathbf{v} \circ \mathbf{v})\mathbf{v} + \rho(|[\delta e]| - \mathbf{v} \circ |[\delta\mathbf{v}]|)\mathbf{v} + \rho(e - \frac{1}{2}\mathbf{v} \circ \mathbf{v})\,|[\delta(\mathbf{v} - \tilde{\mathbf{v}})]|\right\} \circ \hat{\mathbf{N}}^* \\ &\quad = \left\{|[\delta\mathbf{v}]|\,\mathbf{T}^T + \mathbf{v}\left|\left[\delta\mathbf{T}^T\right]\right| - |[\delta\mathbf{q}]|\right\} \circ \hat{\mathbf{N}}^*. \end{aligned} \tag{2.86}$$

CHAPTER 3

Thermoelastic solids and wave propagation in thermoelastic materials

This chapter will focus on thermoelastic response and perturbation equations of the thermoelastic solids. We will start this chapter by defining a thermoelastic solid. As is the case in all of this dissertation, the development will be confined to first-gradient material models. That is, we focus on models that assume the responses at a material point are characterized by functions or functionals of position, temperature, and their first spatial gradients.

We will start by making the constitutive models behave as expected when superimposing rigid body motions. This is followed by a discussion of material symmetry. We then study the constraint of thermodynamics on the constitutive functions of thermoelasticity. As an introduction to the decomposition of the deformation into elastic, viscoelastic and thermal deformations, we look at the separation of the deformation gradient into elastic and thermal parts. Finite deformation thermo-mechanically coupled elasticity theory is presented and the constraints imposed on the constitutive model is derived for a thermodynamically consistent model. We then develop the perturbation equations of the thermoelastic constitutive model and consider the examples of infinitesimal wave propagation in a deformed thermoelastic body. In the last section, we show the potential applications of the current development to load sensors.

3.1 Thermoelastic solids

A characteristic of the elastic response is that the unloading response is very close to the loading response, sufficiently so that the difference can be ignored. We will consider the response of a thermodynamic body to be the values of its specific free energy ψ, specific entropy η, Cauchy stress $\mathbf{T}$, and heat flux vector $\mathbf{q}$. We will denote the response by $\mathcal{R}$ and consider it to be a function of the current conditions of the material point, given in terms of the position in the reference configuration $\mathbf{X}$, and current time t. We write this as

$$\mathcal{R}(\mathbf{X}, t) = \{\psi(\mathbf{X}, t), \eta(\mathbf{X}, t), \mathbf{T}(\mathbf{X}, t), \mathbf{q}(\mathbf{X}, t)\}. \tag{3.1}$$

We will select the independent variables to be the current position $\mathbf{x}$, temperature θ, and their first gradients with respect to space denoted, respectively, as $\mathbf{F}$ and $\mathbf{G}$. We use $\mathcal{U}$ to denote this variable set and write it for a given material point and at current time as

$$\mathcal{U}(\mathbf{X}, t) = \{\mathbf{x}(\mathbf{X}, t), \mathbf{F}(\mathbf{X}, t), \theta(\mathbf{X}, t), \mathbf{G}(\mathbf{X}, t)\}. \tag{3.2}$$

A thermoelastic material will be defined as one for which its thermodynamic response at each material point $\mathbf{X}$ can be written in terms of the current value of this variable set. We symbolically write this assumption as

$$\mathcal{R}(\mathbf{X}, t) = \mathcal{R}^{\dagger}[\mathbf{X}, \mathcal{U}(\mathbf{X}, t)], \tag{3.3}$$

where $\mathcal{R}^{\dagger}$ denotes the function that is used to evaluate $\mathcal{R}$. This assumption must be modified in the presence of material constraints such as incompressibility or inextensibility. The absence of time t as an explicit variable in the constitutive response function $\mathcal{R}^{\dagger}$ implies that time does not independently influence the response. This excludes all possibilities of explicit changes in the constitutive response with time such as aging or curing.

To simplify the presentation, we will frequently omit in the notation the dependence on $(\mathbf{X}, t)$. Therefore, we consider a thermoelastic material to be one that is characterized by the following explicit constitutive assumptions

$$
\begin{aligned}
\psi &= \psi^\dagger(\mathbf{X}, \mathbf{x}, \mathbf{F}, \theta, \mathbf{G}), \\
\eta &= \eta^\dagger(\mathbf{X}, \mathbf{x}, \mathbf{F}, \theta, \mathbf{G}), \\
\mathbf{T} &= \mathbf{T}^\dagger(\mathbf{X}, \mathbf{x}, \mathbf{F}, \theta, \mathbf{G}), \\
\mathbf{q} &= \mathbf{q}^\dagger(\mathbf{X}, \mathbf{x}, \mathbf{F}, \theta, \mathbf{G}),
\end{aligned}
\tag{3.4}
$$

where, the superscript "†" is used to distinguish between the material response on the left of the equality sign and its constitutive model on the right.

We will lay out a set of assumptions on how the material responds under specific conditions, and follow through to see what simplifications in the form of the constitutive response functions are implied by these assumptions.

3.1.1 The influence of pure rigid body motion on the constitutive response

We now consider the alteration of the current distortion of the body by adding a pure rigid body motion. It will be assumed that the temperature of each material particle stays the same (i.e., the temperature field rigidly translates and rotates the same as the body does). Let $\mathcal{U}$ designate the argument set before the rigid-body motion and let $\mathcal{U}^*$ designate the argument set after such a rigid body motion is imposed on the current conditions. Any rigid body motion can be written as

$$
\mathbf{x}^*(\mathbf{X}, t) = \mathbf{Q}(t)[\mathbf{x}(\mathbf{X}, t) - \mathbf{x}_o(t)] + \mathbf{x}_o(t), \tag{3.5}
$$

where $\mathbf{Q}$ is an orthogonal second-order tensor describing the rotation and $\mathbf{x}_o$ is the position of the point about which the body is rotated. For the motion to be a rigid body motion, both $\mathbf{Q}$ and $\mathbf{x}_o$ must only be functions of time (i.e., they are the same for all material points). Taking the gradient of both sides, it is clear that

$$
\mathbf{F}^* = \mathbf{Q}\mathbf{F}, \tag{3.6}
$$

where $\mathbf{F}$ is the deformation gradient associated with the motion $\mathbf{x}$ and $\mathbf{F}^*$ is the deformation gradient associated with the motion $\mathbf{x}^*$. Since the temperature field is assumed to transform rigidly with the body, the temperature of the material points will not change. That is,

$$\theta^*(\mathbf{X}, t) = \theta(\mathbf{X}, t). \tag{3.7}$$

It therefore follows that

$$\mathbf{G}^* = \mathbf{G}, \tag{3.8}$$

where $\mathbf{G}$ is the temperature gradient associated with the temperature field θ (i.e., $d\theta = \mathbf{G} \circ d\mathbf{X}$), and $\mathbf{G}^*$ is the temperature gradient associated with the temperature field θ^* (i.e., $d\theta^* = \mathbf{G}^* \circ d\mathbf{X}$). A plausible assumption is that such a rigid body motion will leave the free energy and entropy unchanged and will result in a rotation of the traction vector and the heat flux. That is, since the relative conditions of the material particles have only changed by a rigid body rotation, the free energy and entropy are unchanged and the heat flux and traction vector reorient to reflect the new relative location of the material particles. Any vector $\mathbf{u}$ connected to material points of the body will be transformed to $\mathbf{u}^* = \mathbf{Q}\mathbf{u}$ by such a rigid body rotation. Therefore, we will assume that both the traction vector and heat flux rotate in a similar manner. The combination of these assumptions makes us conclude that

$$\begin{aligned} \psi^* &= \psi, \\ \eta^* &= \eta, \\ \mathbf{t}^{*(n^*)} &= \mathbf{Q}\mathbf{t}^{(n)}, \\ \mathbf{q}^* &= \mathbf{Q}\mathbf{q}. \end{aligned} \tag{3.9}$$

These assumptions restrict the form of the constitutive functions. First, we note that a pure translation can always be selected equal to negative the current location, removing the current position from the list of arguments by replacing it with zero. We note that the normal vector $\mathbf{n}$ to a material surface before the rigid body rotation transforms to $\mathbf{n}^* = \mathbf{Q}\mathbf{n}$, that the traction $\mathbf{t}^{(n)}$ on the material surface with normal $\mathbf{n}$ before rotation is given by $\mathbf{t}^{(n)} = \mathbf{T}^T\mathbf{n}$, and that traction on the same material surface after the rotation is given by $\mathbf{t}^{*(n^*)} = \mathbf{T}^{*T}\mathbf{n}^*$. Combining these relations with the assumption on how the traction is to change results in

$$\mathbf{T}^* = \mathbf{QTQ}^T. \tag{3.10}$$

Therefore, given the initial response

$$\mathcal{R} = \{\psi, \eta, \mathbf{T}, \mathbf{q}\}, \tag{3.11}$$

then the response after superimposing a pure rigid body motion is given by

$$\mathcal{R}^* = \{\psi, \eta, \mathbf{QTQ}^T, \mathbf{Qq}\}. \tag{3.12}$$

Let us now impose these assumptions of how the material should respond onto the response functions. First we start with the specific free energy. The results will be similar for specific entropy. Next we will look at Cauchy stress and then the heat flux vector. If $\psi^* = \psi$ for every rigid body rotation, then the constitutive function for the specific free energy must be of a form such that

$$\psi^\dagger(\mathbf{X}, \mathbf{QF}, \theta, \mathbf{G}) = \psi^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) \tag{3.13}$$

for every orthogonal $\mathbf{Q}$. If $\mathbf{R}$ and $\mathbf{U}$ are, respectively, the orthogonal and right symmetric factors in the polar decomposition of $\mathbf{F} = \mathbf{RU}$, and if we take $\mathbf{Q} = \mathbf{R}^T$, then we can rewrite this as

$$\psi = \psi^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = \psi^\dagger(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}). \tag{3.14}$$

Therefore, the constitutive function for the specific free energy will yield the same value for every deformation gradient $\mathbf{F}$ as it would give for its associated $\mathbf{U}$. We now can conclude that the constitutive function for free energy cannot depend on the orthogonal part $\mathbf{R}$ of the deformation gradient. Since both the specific free energy and specific entropy have the same values after rigid body rotation, we can rewrite their constitutive functions in terms of functions of $\mathbf{U}$, as opposed to $\mathbf{F}$. That is,

$$\begin{aligned}\psi &= \bar{\psi}^{\dagger}(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}),\\ \eta &= \bar{\eta}^{\dagger}(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}).\end{aligned} \tag{3.15}$$

Since $\mathbf{U}$ is unaffected by rigid body rotations, these new forms for the constitutive functions automatically satisfy our assumptions, behaving as expected under rigid body rotations.

Now let us consider the implications of the restrictions on the constitutive function for Cauchy stress. The assumption $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$ imposes a constraint on the constitutive function for Cauchy stress. This can be written as

$$\mathbf{T}^{\dagger}(\mathbf{X}, \mathbf{Q}\mathbf{F}, \theta, \mathbf{G}) = \mathbf{Q}\mathbf{T}^{\dagger}(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G})\mathbf{Q}^T, \tag{3.16}$$

which must hold for all orthogonal tensors $\mathbf{Q}$. As above, taking $\mathbf{Q} = \mathbf{R}^T$ and rearranging the expressions to give

$$\mathbf{T} = \mathbf{T}^{\dagger}(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = \mathbf{R}\mathbf{T}^{\dagger}(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G})\mathbf{R}^T. \tag{3.17}$$

Defining the rotated stress $\mathbf{T}^R = \mathbf{R}^T\mathbf{T}\mathbf{R}$, it follows from the above relation that

$$\mathbf{T}^R = \mathbf{T}^{R\dagger}(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}), \tag{3.18}$$

where $\mathbf{T}^{R\dagger}$ is the function used to calculate the rotated stress. This $\mathbf{T}^{R\dagger}$ is actually the same function as the one used to calculate the Cauchy stress, but when inputting $\mathbf{U}$ in place of $\mathbf{F}$. Therefore, the rotated stress does not depend on the rigid body rotation. The Cauchy stress is obtained from the rotated stress by the relation $\mathbf{T} = \mathbf{R}\mathbf{T}^R\mathbf{R}^T$.

Finally, let us look at the simplification imposed on the constitutive equation for the heat flux. The condition $\mathbf{q}^* = \mathbf{Q}\mathbf{q}$ requires that

$$\mathbf{q}^{\dagger}(\mathbf{X}, \mathbf{Q}\mathbf{F}, \theta, \mathbf{G}) = \mathbf{Q}\mathbf{q}^{\dagger}(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}), \tag{3.19}$$

which must hold for all orthogonal $\mathbf{Q}$. Again, selecting $\mathbf{Q}$ equal to $\mathbf{R}^T$ yields

$$\mathbf{q} = \mathbf{q}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = \mathbf{R}\mathbf{q}^\dagger(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}). \tag{3.20}$$

Therefore, we can introduce the rotated heat flux $\mathbf{q}^R = \mathbf{R}^T\mathbf{q}$ that has a constitutive representation given by

$$\mathbf{q}^R = \mathbf{q}^{R\dagger}(\mathbf{X}, \mathbf{U}, \theta, \mathbf{G}). \tag{3.21}$$

The heat flux can be calculated by first calculating $\mathbf{q}^R$ using $\mathbf{U}$, and then using $\mathbf{q} = \mathbf{R}\mathbf{q}^R$.

The right Cauchy stretch is given by $\mathbf{C} = \mathbf{U}^2$ and can uniquely be inverted, because $\mathbf{U}$ is positive definite, to get $\mathbf{U} = \sqrt{\mathbf{C}}$. Similarly, the Green strain is $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ so that $\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I})$ and $\mathbf{U} = \sqrt{\mathbf{I} + 2\mathbf{E}}$, again uniquely defined. As a result, we may replace for $\mathbf{U}$ in any of the above constitutive functions and obtain new versions of these functions that depend on $\mathbf{C}$ or $\mathbf{E}$.

3.2 Material symmetry

Material symmetry normally refers to existing symmetry in the structure of the material. This is the symmetry that exists before we start the thermo-mechanical loading process. If present, symmetry in the material structure means that certain reorganizations of the structure are thermomechanically indistinguishable from each other. For example, for an initially isotropic material, we can rotate the material by any amount before cutting a testing specimen and will observe the exact same response and obtain the exact same results, irrespective of the test.

Material symmetry is described by mathematical mappings that reorganize the neighborhood of a point in the reference configuration. Fig. 3.1 shows a schematic of the idea of reorganizing a material's neighborhood. For first-gradient materials, these reorganizations can be represented by second-order transformation tensors, here denoted by $\mathbf{M}$ and shown in Fig. 3.1. Each material symmetry is characterized by a group of transformations $\mathbf{M}$ that take the neighborhood of a material point and reorganize it to materially equivalent neighborhoods (i.e., ones that are indistinguishable in their response from the original neighborhood). This group will be denoted by $\mathcal{G}$. As might be expected, the description of the transformation $\mathbf{M}$ is related closely to the reference configuration since these transformations (the $\mathbf{M}s$) represent reorganizations of the reference configuration. Let us look at how changing the reference configuration changes the representation of material symmetry. If $\mathbf{M}_\kappa$ is a symmetry described relative to configuration κ, and $\mathbf{M}_{\kappa^*}$ is the same symmetry described relative to configuration κ^*, then Noll's rule states that

$$\mathbf{M}_{\kappa^*} = \tilde{\mathbf{F}}\mathbf{M}_\kappa\tilde{\mathbf{F}}^{-1}, \tag{3.22}$$

where$\tilde{\mathbf{F}}$ is the deformation gradient mapping the neighborhood in κ to the associated neighborhood in κ^*. Since each member of the material symmetry group follows Noll's rule, we then can write the relation between the groups as

$$\mathcal{G}_{\kappa^*} = \tilde{\mathbf{F}}\mathcal{G}_\kappa\tilde{\mathbf{F}}^{-1}. \tag{3.23}$$

We note that Noll's rule refers to changing the reference configuration, but Noll's rule does not apply to actually deforming the material. The transformations in the material symmetry group need not be orthogonal. For example, an isotropic material normally is represented by the set of orthogonal transformations. This is actually a set of transformations with respect to a given reference configuration. If we choose to use another reference configuration, through Noll's rule we get a new set of transformations representing the same isotropy, but possibly some of these new transformations may not be orthogonal. This is because Noll's rule does not preserve orthogonality of the transformations.

With thermomechanical loading the material's symmetry can change. For example, by melting a crystal we can increase its symmetry, or by drawing an isotropic bar we might make it anisotropic. These types of changes are not what Noll's rule applies to. By melting the crystal or drawing the bar, we actually are subjecting the material to thermomechanical loading that is altering the state of the material, in the process creating a new material with possibly new symmetries. When changing the reference configuration, we simply are using a different, but related, mathematical formulation, without subjecting the actual material to any deformation whatsoever. Since we do not intend to make changes to the reference

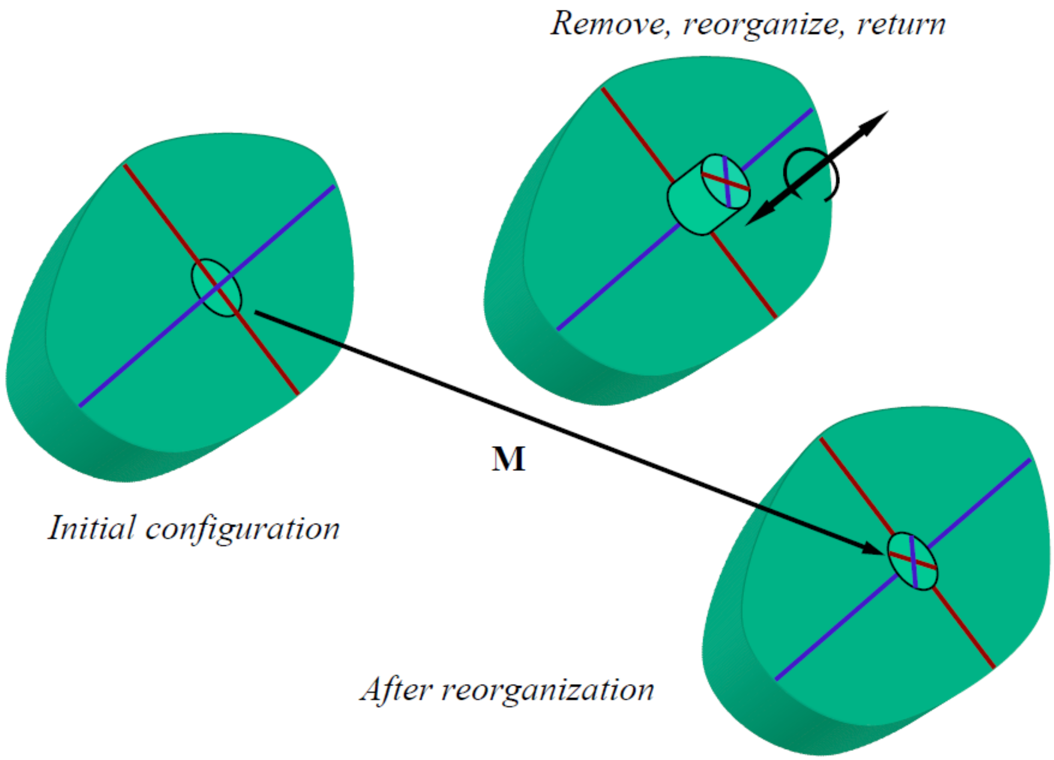


Figure 3.1: Schematic of a material symmetry reorganization of the neighborhood of a material point by Negahban [Negahban, 2012].

configuration, we will not use the required configuration subscript on the transformations **M**, always implied to be transformations of the reference configuration, which also normally will be the initial configuration.

Let us now consider the restriction imposed by material symmetry. Let us assume that t_o is the initial time from which we are to model the response of the material to further loading. For our first-gradient material, a loading history imposed on the state at t_o will be noted by $\mathcal{H}(t_o, t)$ and symbolically represented by

$$\mathcal{H}(t_o, t) = \overset{t}{\underset{s=t_o}{\mathcal{U}}}(s) = \{\mathbf{F}(s), \overset{t}{\underset{s=t_o}{\theta}}(s), \mathbf{G}(s)\}. \tag{3.24}$$

A history that is essentially identical to this, but for a reorganization of the neighborhood by material symmetry transformation **M** before the loading will be denoted by $\bar{\mathcal{H}}(t_o, t)$ and given by

$$\bar{\mathcal{H}}(t_o,t) = \overset{t}{\underset{s=t_o}{\bar{\mathcal{U}}(s)}} = \{\bar{\mathbf{F}}(s), \overset{t}{\underset{s=t_o}{\bar{\theta}(s)}}, \bar{\mathbf{G}}(s)\}$$
$$= \{\mathbf{F}(s)\mathbf{M}, \overset{t}{\underset{s=t_o}{\theta(s)}}, \mathbf{G}(s)\mathbf{M}\}. \tag{3.25}$$

The existing material symmetries at time t_o can be represented by a material symmetry group $\mathcal{G}(t_o)$. Each transformation $\mathbf{M}$ in $\mathcal{G}(t_o)$ represents a reorganization at time t_o that leaves the response identical for the two histories $\mathcal{H}(t_o,t)$ and $\bar{\mathcal{H}}(t_o,t)$, irrespective of history $\mathcal{H}(t_o,t)$ and the current time t, as shown in Fig. 3.2. Symbolically, we write this as

$$\mathcal{R}(t) = \bar{\mathcal{R}}(t) \tag{3.26}$$

for all pairs of histories $\mathcal{H}(t_o,t)$ and $\bar{\mathcal{H}}(t_o,t)$ related by each symmetry transformation $\mathbf{M}$ in the symmetry group of the material $\mathcal{G}(t_o)$.

3.3 Finite deformation thermoelastic model

In this section we will study thermoelasticity at large deformations, including large rigid body motions. We will do this in the context of the multiplicative decomposition of the deformation gradient into an elastic part, that is due to mechanical loading, and a thermal part, that is due to thermal expansion, given by

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^\theta, \tag{3.27}$$

where $\mathbf{F}^e$ is the elastic part, and $\mathbf{F}^\theta$ is the thermal part. For this separation to have a meaning, we need to define how to do it. We will do this by giving a method to calculate the thermal deformation gradient. We will assume the current value of $\mathbf{F}^\theta$ can be calculated from an initial value $\mathbf{F}^\theta_o$ and an evolution equation given as

$$\mathbf{L}^\theta = \boldsymbol{\alpha}\dot{\theta}, \tag{3.28}$$

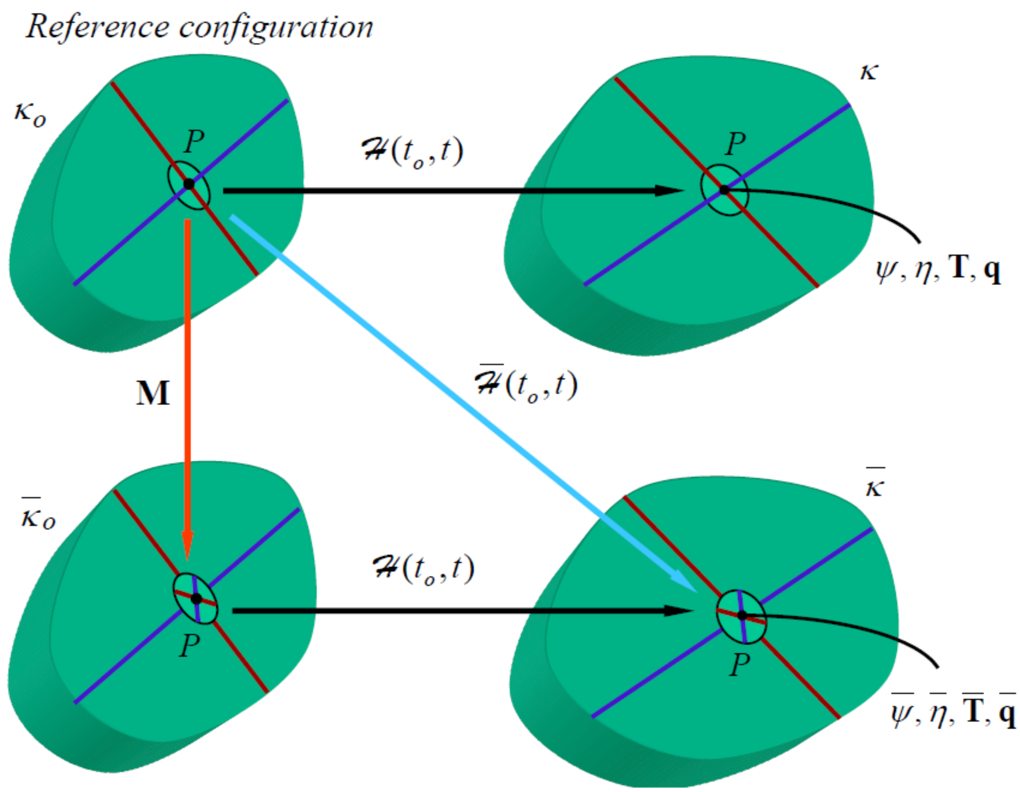


Figure 3.2: A deformation imposed before or after a material symmetry transformation $\mathbf{M}$ by Negahban [Negahban, 2012].

where $\boldsymbol{\alpha}$ is the coefficient of thermal expansion, a second order tensor function of $\mathbf{F}^e$, $\mathbf{F}^\theta$, and θ. We can use this evolution equation to integrate $\dot{\mathbf{F}}^\theta = \mathbf{L}^\theta \mathbf{F}^\theta$ to get $\mathbf{F}^\theta$ from $\mathbf{F}^\theta_o$.

The next assumption that we will make is that the free energy can be written as

$$\psi = \psi^\dagger[\mathbf{F}^e, \mathbf{F}^\theta, \mathbf{G}, \theta], \tag{3.29}$$

so that its derivative can be written as

$$\dot{\psi} = \partial_{\mathbf{F}^e}(\psi^\dagger) : \dot{\mathbf{F}}^e + \partial_{\mathbf{F}^\theta}(\psi^\dagger) : \dot{\mathbf{F}}^\theta + \partial_\theta(\psi^\dagger)\dot{\theta} + \partial_{\mathbf{G}}(\psi^\dagger) \circ \dot{\mathbf{G}}. \tag{3.30}$$

We now rewrite this in terms of $\dot{\mathbf{F}}$ by using $\mathbf{F} = \mathbf{F}^e\mathbf{F}^\theta$. We know that the derivative of $\mathbf{F}$ is

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e\mathbf{F}^\theta + \mathbf{F}^e\dot{\mathbf{F}}^\theta, \tag{3.31}$$

so that

$$\dot{\mathbf{F}}^e = \dot{\mathbf{F}}\mathbf{F}^{\theta-1} - \mathbf{F}^e\dot{\mathbf{F}}^\theta\mathbf{F}^{\theta-1}. \tag{3.32}$$

If we substitute this into $\dot{\psi}$, we get

$$\dot{\psi} = \partial_{\mathbf{F}^e}(\psi^\dagger) : (\dot{\mathbf{F}}\mathbf{F}^{\theta-1} - \mathbf{F}^e\dot{\mathbf{F}}^\theta\mathbf{F}^{\theta-1}) + \partial_{\mathbf{F}^\theta}(\psi^\dagger) : \dot{\mathbf{F}}^\theta + \partial_\theta(\psi^\dagger)\dot{\theta} + \partial_{\mathbf{G}}(\psi^\dagger) \circ \dot{\mathbf{G}}. \tag{3.33}$$

We now can substitute this into the Clausius-Duhem inequality (2.20) to get

$$[\mathbf{T}^{eT} - \mathbf{T}^T] : \mathbf{L} + [\rho\eta + \rho\partial_\theta(\psi^\dagger) - \Delta\mathbf{T}^{\theta T} : \boldsymbol{\alpha}]\dot{\theta} + [\rho\partial_{\mathbf{G}}(\psi^\dagger)] \circ \dot{\mathbf{G}} + [\frac{1}{\theta}\mathbf{q} \circ \mathbf{g}] \leq 0, \tag{3.34}$$

where we have used $\dot{\mathbf{F}}^\theta = \mathbf{L}^\theta\mathbf{F}^\theta = \boldsymbol{\alpha}\mathbf{F}^\theta\dot{\theta}$, and we have defined, respectively, the thermodynamic elastic and thermal stresses as

$$\mathbf{T}^{eT} \equiv \rho\partial_{\mathbf{F}^e}(\psi^\dagger)\mathbf{F}^{eT}, \tag{3.35}$$

$$\mathbf{T}^{\theta T} \equiv \rho\partial_{\mathbf{F}^\theta}(\psi^\dagger)\mathbf{F}^{\theta T}. \tag{3.36}$$

and the thermal overstress as

$$\Delta\mathbf{T}^\theta = \mathbf{F}^{e-1}\mathbf{T}^e\mathbf{F}^e - \mathbf{T}^\theta. \tag{3.37}$$

We assume that the entropy, Cauchy stress, and heat flux depend on the same argument set as we selected for the free energy. As a result, the terms in the square brackets are independent of the rates $\mathbf{L}$, $\dot{\theta}$, and $\dot{\mathbf{G}}$, which, given the assumption that this equation must hold for all rates, results in the relations

$$\begin{aligned}
&\mathbf{T}^{eT} - \mathbf{T}^{T} = \mathbf{0}, \qquad (3.38)\\
&\eta + \partial_\theta(\psi^\dagger) - \frac{1}{\rho}\Delta\mathbf{T}^{\theta T} : \boldsymbol{\alpha} = 0,\\
&\rho\partial_{\mathbf{G}}(\psi^\dagger) = \mathbf{0},\\
&\frac{1}{\theta}\mathbf{q}\circ\mathbf{g} \leq 0.
\end{aligned}$$

It follows that the free energy does not depend on the temperature gradient, so that

$$\psi = \psi^\dagger[\mathbf{F}^e, \mathbf{F}^\theta, \theta], \qquad (3.39)$$

and that

$$\mathbf{T}^{T} = \mathbf{T}^{eT} = \rho\partial_{\mathbf{F}^e}(\psi^\dagger)\mathbf{F}^{eT}, \qquad \eta = \frac{1}{\rho}\Delta\mathbf{T}^{\theta T} : \boldsymbol{\alpha} - \partial_\theta(\psi^\dagger), \qquad (3.40)$$

which indicate that the stress and the entropy do not depend on the temperature gradient.

Before we impose the effect of rigid body motions, we need to assume how the elastic and thermal parts of the deformation gradient change with rigid body motions. We know that as a result of a rigid body motion the deformation gradient $\mathbf{F}$ becomes $\mathbf{F}^* = \mathbf{QF}$. We will assume that the thermal deformation gradient does not change with rigid body motions, so that we have

$$\begin{aligned}
&\mathbf{F}^{e*} = \mathbf{QF}^e, \qquad (3.41)\\
&\mathbf{F}^{\theta*} = \mathbf{F}^\theta,
\end{aligned}$$

resulting in $\mathbf{F}^* = \mathbf{F}^{e*}\mathbf{F}^{\theta*} = \mathbf{QF}^e\mathbf{F}^\theta = \mathbf{QF}$. The assumption that the free energy does not change with rigid body motions requires that

$$\psi^\dagger(\mathbf{QF}^e, \mathbf{F}^\theta, \theta) = \psi^\dagger(\mathbf{F}^e, \mathbf{F}^\theta, \theta) \qquad (3.42)$$

for every orthogonal $\mathbf{Q}$. Since $\mathbf{G}$ does not change with rigid body motions, the assumption that the heat flux rotates by $\mathbf{Q}$ results in

$$\mathbf{q}^{\dagger}(\mathbf{Q}\mathbf{F}^{e}, \mathbf{F}^{\theta}, \theta, \mathbf{G}) = \mathbf{Q}\mathbf{q}^{\dagger}(\mathbf{F}^{e}, \mathbf{F}^{\theta}, \theta, \mathbf{G}) \tag{3.43}$$

for every orthogonal $\mathbf{Q}$. By the selection of $\mathbf{Q} = \mathbf{R}^{eT}$, we conclude that

$$\psi = \psi^{\dagger}(\mathbf{U}^{e}, \mathbf{F}^{\theta}, \theta), \tag{3.44}$$

and

$$\mathbf{q} = \mathbf{R}^{e}\mathbf{q}^{R^{e}} \tag{3.45}$$

for the rotated heat flux $\mathbf{q}^{R^{e}}$ with the model

$$\mathbf{q}^{R^{e}} = \mathbf{q}^{R^{e}\dagger}(\mathbf{U}^{e}, \mathbf{F}^{\theta}, \theta, \mathbf{G}). \tag{3.46}$$

The material symmetry constraint can be imposed once we assume how the elastic and thermal deformation gradients change. We assume that $\mathbf{F}$ changing to $\tilde{\mathbf{F}} = \mathbf{F}\mathbf{M}$ under the material symmetry transformation $\mathbf{M}$ results in the elastic and thermal parts transforming to

$$\begin{aligned} \tilde{\mathbf{F}}^{e} &= \mathbf{F}^{e}\mathbf{M}, \\ \tilde{\mathbf{F}}^{\theta} &= \mathbf{M}^{-1}\mathbf{F}^{\theta}\mathbf{M}, \end{aligned} \tag{3.47}$$

which gives $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}^{e}\tilde{\mathbf{F}}^{\theta} = \mathbf{F}^{e}\mathbf{M}\mathbf{M}^{-1}\mathbf{F}^{\theta}\mathbf{M} = \mathbf{F}\mathbf{M}$. This results in the condition on the coefficient of thermal expansion being

$$\tilde{\boldsymbol{\alpha}} = \mathbf{M}^{-1}\boldsymbol{\alpha}\mathbf{M}. \tag{3.48}$$

Assuming that the free energy and heat flux do not change because of a material symmetry reorganization before the deformation, we must have

$$\psi^\dagger[\mathbf{F}^e\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta\mathbf{M}, \theta] = \psi^\dagger[\mathbf{F}^e, \mathbf{F}^\theta, \theta], \tag{3.49}$$

$$\mathbf{q}^\dagger[\mathbf{F}^e\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta\mathbf{M}, \theta, \mathbf{M}^T\mathbf{G}] = \mathbf{q}^\dagger[\mathbf{F}^e, \mathbf{F}^\theta, \theta, \mathbf{G}]. \tag{3.50}$$

The condition on the constitutive model for the coefficient of thermal expansion is

$$\mathbf{M}^{-1}\boldsymbol{\alpha}^\dagger[\mathbf{F}^e, \mathbf{F}^\theta, \theta]\mathbf{M} = \boldsymbol{\alpha}^\dagger[\mathbf{F}^e\mathbf{M}, \mathbf{M}^{-1}\mathbf{F}^\theta\mathbf{M}, \theta]. \tag{3.51}$$

Combining the constraint imposed by rigid body motion and the constraint imposed by material symmetry, for the case of orthogonal $\mathbf{M}$, results in requirements such as

$$\psi^\dagger[\mathbf{M}^T\mathbf{C}^e\mathbf{M}, \mathbf{M}^T\mathbf{F}^\theta\mathbf{M}, \theta] = \psi^\dagger[\mathbf{C}^e, \mathbf{F}^\theta, \theta], \tag{3.52}$$

$$\mathbf{M}\mathbf{q}^{R^e\dagger}[\mathbf{M}^T\mathbf{C}^e\mathbf{M}, \mathbf{M}^T\mathbf{F}^\theta\mathbf{M}, \theta, \mathbf{M}^T\mathbf{G}] = \mathbf{q}^{R^e\dagger}(\mathbf{C}^e, \mathbf{F}^\theta, \theta, \mathbf{G}), \tag{3.53}$$

$$\boldsymbol{\alpha}^\dagger[\mathbf{C}^e, \mathbf{F}^\theta, \theta] = \mathbf{M}\boldsymbol{\alpha}^\dagger[\mathbf{M}^T\mathbf{C}^e\mathbf{M}, \mathbf{M}^T\mathbf{F}^\theta\mathbf{M}, \theta]\mathbf{M}^T. \tag{3.54}$$

3.4 Thermo-elastic constitutive models for the perturbations

In the previous section we developed the balance laws and their boundary and jump conditions for the perturbations. We will now evaluate the constitutive equations for these perturbations. We first start by decomposing the perturbation into elastic and thermal parts and then derive the perturbation term for the stress, followed by those for the free energy, entropy and heat flux. Finally, we evaluate the evolution rule for the perturbations of the internal parameter associated with thermal expansion and then integrate it to obtain a general solution.

3.4.1 Decomposition of the perturbation into elastic and thermal parts

In the constitutive model, both the base deformation and total deformation are each separated into elastic and thermal parts. The constitutive model provides evolution equations for the thermal parts of each and so in each case leaves the elastic part to be matched against the applied stress. As will be shown here, we can also set up the perturbation to have the same form.

For the base history, the deformation gradient $\mathbf{F}(t)$ is decomposed into the elastic deformation gradient $\mathbf{F}^e(t)$ and the thermal deformation gradient $\mathbf{F}^\theta(t)$ so that

$$\mathbf{F}(t) = \mathbf{F}^e(t)\mathbf{F}^\theta(t). \tag{3.55}$$

For the total history, the total deformation gradient $\mathbf{F}^*(t)$ is also decomposed into the elastic deformation gradient $\mathbf{F}^{e*}(t)$ and the thermal deformation gradient $\mathbf{F}^{\theta*}(t)$ so that

$$\mathbf{F}^*(t) = \mathbf{F}^{e*}(t)\mathbf{F}^{\theta*}(t). \tag{3.56}$$

The relations of the elastic and thermal parts for the base history and the total history are described in Fig. 3.3, where, $\tilde{\mathbf{F}}^e(t)$ and $\tilde{\mathbf{F}}^\theta(t)$ are the relative "increment" of elastic and thermal deformation gradients comparing each pair. The actual change in these deformation gradients from the base history to the total history is given by

$$\begin{aligned} \delta\mathbf{F}^e(t) &= \mathbf{F}^{e*}(t) - \mathbf{F}^e(t) = \tilde{\mathbf{F}}^e(t)\mathbf{F}^e(t) - \mathbf{F}^e(t) = \tilde{\mathbf{H}}^e(t)\mathbf{F}^e(t), \\ \delta\mathbf{F}^\theta(t) &= \mathbf{F}^{\theta*}(t) - \mathbf{F}^\theta(t) = \tilde{\mathbf{F}}^\theta(t)\mathbf{F}^\theta(t) - \mathbf{F}^\theta(t) = \tilde{\mathbf{H}}^\theta(t)\mathbf{F}^\theta(t), \end{aligned} \tag{3.57}$$

where $\tilde{\mathbf{H}}^e(t) = \tilde{\mathbf{F}}^e(t) - \mathbf{I}$ is the displacement gradient for the elastic perturbation and $\tilde{\mathbf{H}}^\theta(t) = \tilde{\mathbf{F}}^\theta(t) - \mathbf{I}$ is the displacement gradient for the thermal perturbation. As the perturbation is small, it can be shown that the increments represent small differences so that the deformation gradients are close to the identity $\mathbf{I}$. We can also define a direct separation of the perturbed deformation gradients $\check{\mathbf{F}}(t) = \check{\mathbf{F}}^e(t)\check{\mathbf{F}}^\theta(t)$ and obtain their relations with the "$\sim$" variables as shown in Fig. 3.3. This allows us to define a consistent set of relations given by

$$\begin{aligned} \check{\mathbf{F}}^e(t) &\equiv \tilde{\mathbf{F}}^e(t), \\ \check{\mathbf{F}}^\theta(t) &\equiv \mathbf{F}^e(t)\tilde{\mathbf{F}}^\theta(t)\mathbf{F}^{e-1}(t). \end{aligned} \tag{3.58}$$

We can introduce the displacement gradient for $\check{\mathbf{F}}^e(t) = \mathbf{I} + \check{\mathbf{H}}^e(t)$ and $\check{\mathbf{F}}^\theta(t) = \mathbf{I} + \check{\mathbf{H}}^\theta(t)$.

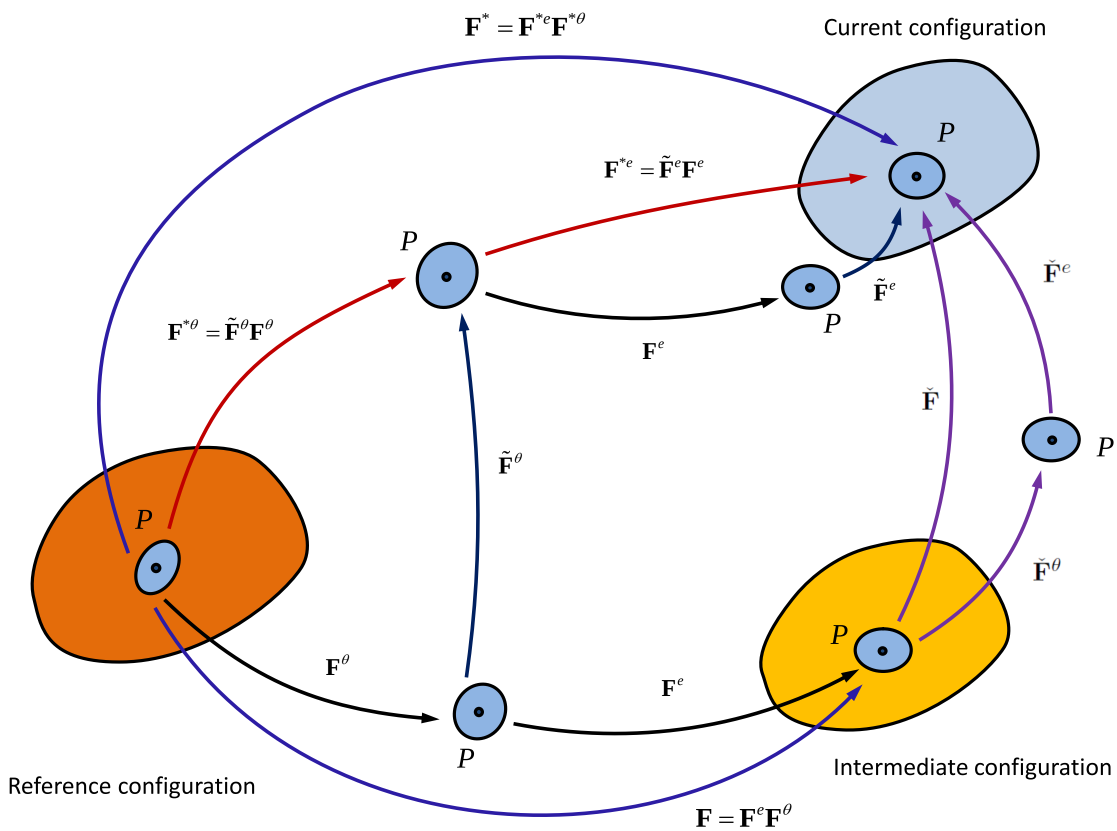


Figure 3.3: Decompositions of the perturbation into elastic and thermal parts.

Solving for these, we get

$$\check{\mathbf{H}}^e(t) = \tilde{\mathbf{H}}^e(t), \qquad (3.59)$$
$$\check{\mathbf{H}}^\theta(t) = \mathbf{F}^e(t)\tilde{\mathbf{H}}^\theta(t)\mathbf{F}^{e-1}(t).$$

It can be shown that $\check{\mathbf{F}}^e(t)$ and $\check{\mathbf{F}}^\theta(t)$ are each close to the identity $\mathbf{I}$, therefore, resulting in infinitesimal $\check{\mathbf{H}}^e(t)$ and $\check{\mathbf{H}}^\theta(t)$. It should be emphasized here that the perturbations of the kinematic variables obtained in (3.59) can be applied to time changing states, which obviously include as special cases the static base history. Therefore, we can reduce the incremental kinematic relations (3.59) to the case studied by [Biot, 1965] where the small mechanical perturbation is superimposed in the vicinity of the static pre-deformation by simply holding $\mathbf{F}^e(t)$ and $\mathbf{F}^\theta(t)$ both constant in time during the disturbances.

Since the perturbations imposed are infinitesimal, after eliminating the second and higher orders of the infinitesimal terms, we get the following approximations

$$\begin{aligned}
\check{\mathbf{F}}^{-1}(t) &\approx \mathbf{I} - \check{\mathbf{H}}(t), \qquad (3.60)\\
\check{\mathbf{F}}^{e-1}(t) &\approx \mathbf{I} - \check{\mathbf{H}}^{e}(t),\\
\check{\mathbf{F}}^{\theta-1}(t) &\approx \mathbf{I} - \check{\mathbf{H}}^{\theta}(t),\\
\check{\mathbf{H}}^{e}(t) &\approx \check{\mathbf{H}}(t) - \check{\mathbf{H}}^{\theta}(t).
\end{aligned}$$

3.4.2 Perturbation of the stress by a change in history

In this section we calculate the stress difference between the base and total history. Since the thermodynamic stresses are only a function of the state, minus the temperature gradient, given that the changes are small, we can calculate the change by using a Taylor series expansion.

We start this process by calculating the approximation to $\delta\mathbf{T}^{eT} = \mathbf{T}^{e*T} - \mathbf{T}^{eT}$. The difference between the thermodynamic elastic stresses from the base history and the total history is approximated by taking its derivative with respect to each variable and multiplying it by the change in the variable and finally adding the results to get

$$\delta\mathbf{T}^{eT} = \mathsf{E}^{e} : \delta\mathbf{F}^{e} + \mathsf{E}^{\theta} : \delta\mathbf{F}^{\theta} + \mathbf{E}^{\theta}\delta\theta, \qquad (3.61)$$

where the coefficients are associate tangent moduli with respect to the given variables and evaluated at the base history. They are defined as

$$\begin{aligned}
\mathsf{E}^{e} &\equiv \partial_{\mathbf{F}^{e}}(\mathbf{T}^{eT}), \qquad (3.62)\\
\mathsf{E}^{\theta} &= \partial_{\mathbf{F}^{\theta}}(\mathbf{T}^{eT}),\\
\mathbf{E}^{\theta} &= \partial_{\theta}(\mathbf{T}^{eT}).
\end{aligned}$$

We can now replace for the changes in the variables in terms of the "$\sim$" variables to get the thermodynamic stresses relative to the values at the same time in the base history from the equation

$$\mathbf{T}^{e*T} = \mathbf{T}^{eT} + \delta\mathbf{T}^{eT} \quad (3.63)$$
$$= \mathbf{T}^{eT} + \mathsf{E}^{e} : (\tilde{\mathbf{H}}^{e}\mathbf{F}^{e}) + \mathsf{E}^{\theta} : (\tilde{\mathbf{H}}^{\theta}\mathbf{F}^{\theta}) + \mathbf{E}^{\theta}\delta\theta.$$

Recalling the relation $\mathbf{T}^{eT} = \mathbf{T}^{T}$, the above equation provides the change in the Cauchy stress.

Taking account of the symmetry of the Cauchy stress $\delta\mathbf{T} = \delta\mathbf{T}^{T}$ or $\delta\mathbf{T}^{e} = \delta\mathbf{T}^{eT}$ and eliminating the elastic deformation gradient in favor of $\mathbf{F}$, this can be written as

$$\delta\mathbf{T} = (\mathsf{E}^{e}\mathbf{F}^{\theta-T}\mathbf{F}^{T}) : (\check{\mathbf{H}} - \check{\mathbf{H}}^{\theta}) + (\mathsf{E}^{\theta}\mathbf{F}^{T}) : (\mathbf{F}^{\theta}\mathbf{F}^{-1}\check{\mathbf{H}}^{\theta}) + (\delta\theta)\mathbf{E}^{\theta}. \quad (3.64)$$

3.4.3 Perturbations of the free energy, entropy and heat flux vector

The perturbations of the free energy, entropy and heat flux vector are obtained from steps similar to those described for the stress. We start first by defining the perturbation of the free energy, entropy and heat flux vector, respectively, as $\delta\psi = \psi^{*} - \psi$, $\delta\eta = \eta^{*} - \eta$, and $\delta\mathbf{q} = \mathbf{q}^{*} - \mathbf{q}$. Noting the dependence of these functions on the state variables, we obtain first order approximations for these perturbations given by

$$\delta\psi(t) = \partial_{\mathbf{F}^{e}}(\psi^{\dagger}) : \delta\mathbf{F}^{e} + \partial_{\mathbf{F}^{\theta}}(\psi^{\dagger}) : \delta\mathbf{F}^{\theta} + \partial_{\theta}(\psi^{\dagger})\delta\theta,$$
$$\delta\eta(t) = \partial_{\mathbf{F}^{e}}(\eta^{\dagger}) : \delta\mathbf{F}^{e} + \partial_{\mathbf{F}^{\theta}}(\eta^{\dagger}) : \delta\mathbf{F}^{\theta} + \partial_{\theta}(\eta^{\dagger})\delta\theta,$$
$$\delta\mathbf{q}(t) = \partial_{\mathbf{F}^{e}}(\mathbf{q}^{\dagger}) : \delta\mathbf{F}^{e} + \partial_{\mathbf{F}^{\theta}}(\mathbf{q}^{\dagger}) : \delta\mathbf{F}^{\theta} + \partial_{\mathbf{G}}(\mathbf{q}^{\dagger})\delta\mathbf{G} + \partial_{\theta}(\mathbf{q}^{\dagger})\delta\theta, \quad (3.65)$$

where $\delta\mathbf{G} = \mathbf{Grad}(\delta\theta)$ is the gradient of $\delta\theta$ with respect to changes in the reference configuration. It should be noted here that, from specific relations selected to satisfy the Clausius-Duhem inequality, we can directly obtain the forms for the entropy $\eta^{*}(t)$ and $\eta(t)$, and in a similar manner choose the possible constitutive functions for $\mathbf{q}^{*}$ and $\mathbf{q}$ taking account of $\frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0$ and $\frac{1}{\theta^{*}}\mathbf{q}^{*} \circ \mathbf{g}^{*} \leq 0$. For example, one simple choice of the heat flux in the isotropic case is the linear Fourier model considered in Lion and Reese et al. [Lion, 1997, Reese and Govindjee, 1997].

3.4.4 Evolution rule for the incremental perturbation of the internal parameter

In the constitutive equation for the thermo-elastic solid, there is one internal variable that must evolve based on the state of the material. This variable is the thermal deformation gradient. The evolution of this variable will be different for the base and the total histories, and so their difference will change based on how the two change. As such, the variables we have chosen to describe the perturbation between the internal variables of the two histories will have evolution equations that reflect the expected difference in the evolutions in the two histories. We will use the evolutions for the two histories to calculate in this section the evolution equation for the perturbation variables for the thermal deformation gradient.

We note that the relation between the velocity gradient and the deformation gradient rate is given by $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ and that similar rules are true for the associated internal variables so that for the base history we can write $\dot{\mathbf{F}}^{\theta} = \mathbf{L}^{\theta}\mathbf{F}^{\theta}$. Using this, we can reorganize the evolution equation given for the thermal deformation gradient during the base history and obtain

$$\dot{\mathbf{F}}^{\theta} = \dot{\theta}\boldsymbol{\alpha}\mathbf{F}^{\theta}. \tag{3.66}$$

In a similar fashion, the same evolution equation for the total history is given by

$$\dot{\mathbf{F}}^{\theta *} = \dot{\theta}^{*}\boldsymbol{\alpha}^{*}\mathbf{F}^{\theta *}. \tag{3.67}$$

The thermal expansion coefficient tensor $\boldsymbol{\alpha}(t)$ is also effected by the thermo-mechanical perturbation, and this material function for the base history and the total history is related by

$$\boldsymbol{\alpha}^{*}(t) = \boldsymbol{\alpha}(t) + \delta\boldsymbol{\alpha}(t), \tag{3.68}$$

where, $\boldsymbol{\alpha}^{*}(t)$ is the thermal expansion parameter in the total history, $\boldsymbol{\alpha}(t)$ is the thermal expansion parameter in the base history, and $\delta\boldsymbol{\alpha}(t)$ is the incremental thermal expansion parameter from the perturbation. Since $\boldsymbol{\alpha}(t)$ is assumed to be a function of the state of the material, we define the following coefficients which are the derivatives of $\boldsymbol{\alpha}$ with respect to its associate variables and given by

$$\begin{aligned} \mathbf{a}^e &\equiv \partial_{\mathbf{F}^e}(\boldsymbol{\alpha}^\dagger), \\ \mathbf{a}^\theta &\equiv \partial_{\mathbf{F}^\theta}(\boldsymbol{\alpha}^\dagger), \\ \boldsymbol{\alpha}^\theta &\equiv \partial_\theta(\boldsymbol{\alpha}^\dagger), \end{aligned} \tag{3.69}$$

where, $\mathbf{a}^e$ and $\mathbf{a}^\theta$ are fourth-order tensor functions, and $\boldsymbol{\alpha}^\theta$ is a second-order tensor function.

After substituting the increments of the kinematics variables given in (3.57), we obtain the increment $\delta\boldsymbol{\alpha}$ for the thermal expansion in terms of the "$\sim$" variables as

$$\delta\boldsymbol{\alpha} = \mathbf{a}^e : (\tilde{\mathbf{H}}^e\mathbf{F}^e) + \mathbf{a}^\theta : (\tilde{\mathbf{H}}^\theta\mathbf{F}^\theta) + \boldsymbol{\alpha}^\theta\delta\theta, \tag{3.70}$$

with the parameters $\mathbf{a}^e$, $\mathbf{a}^\theta$ and $\boldsymbol{\alpha}^\theta$ evaluated in the base history.

The relations for the time derivatives are given by

$$\dot{\theta}^*(t) = \dot{\theta}(t) + \delta\dot{\theta}(t), \tag{3.71}$$

and

$$\begin{aligned} \dot{\mathbf{F}}^{e*}(t) &= \dot{\tilde{\mathbf{F}}}^e(t)\mathbf{F}^e(t) + \tilde{\mathbf{F}}^e(t)\dot{\mathbf{F}}^e(t), \\ \dot{\mathbf{F}}^{\theta*}(t) &= \dot{\tilde{\mathbf{F}}}^\theta(t)\mathbf{F}^\theta(t) + \tilde{\mathbf{F}}^\theta(t)\dot{\mathbf{F}}^\theta(t). \end{aligned} \tag{3.72}$$

By using equations (3.71) and (3.72) and manipulating the two sets of evaluation equations (3.66) and (3.67) for the two histories, we get the effects of the perturbation on the thermal expansion through the following coupled first-order differential equation with the unknown incremental internal variable $\tilde{\mathbf{H}}^\theta(t)$ given by

$$\dot{\mathbf{F}}^\theta + \dot{\tilde{\mathbf{H}}}^\theta\mathbf{F}^\theta + \tilde{\mathbf{H}}^\theta\dot{\mathbf{F}}^\theta = (\dot{\theta} + \delta\dot{\theta})(\boldsymbol{\alpha} + \delta\boldsymbol{\alpha})(\mathbf{I} + \tilde{\mathbf{H}}^\theta)\mathbf{F}^\theta. \tag{3.73}$$

We use the relations between $\check{\mathbf{H}}^e$ and $\tilde{\mathbf{H}}^e$, and $\check{\mathbf{H}}^\theta$ and $\tilde{\mathbf{H}}^\theta$ given in (3.59), and simplify the notation in (3.73) by using $\boldsymbol{\Gamma}^\theta(t) \equiv \check{\mathbf{H}}^\theta(t)\mathbf{F}(t)$ to denote the unknown variable, and using $\delta\theta(t)$ and $\boldsymbol{\Gamma}(t) \equiv \check{\mathbf{H}}(t)\mathbf{F}(t)$ as the known values. From this, we get the following modified equation

$$
\begin{aligned}
&\dot{\mathbf{\Gamma}}^\theta - \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{\Gamma}^\theta \\
&\quad = \mathbf{F}\mathbf{F}^{\theta-1}\left[(\dot{\theta}+\delta\dot{\theta})\{\mathsf{a}^e : [(\mathbf{\Gamma}-\mathbf{\Gamma}^\theta)\mathbf{F}^{\theta-1}] + \mathsf{a}^\theta : (\mathbf{F}^\theta\mathbf{F}^{-1}\mathbf{\Gamma}^\theta) + \boldsymbol{\alpha}^\theta\delta\theta\} + (\delta\dot{\theta})\boldsymbol{\alpha}\right]\mathbf{F}^\theta \\
&\quad + (\delta\theta)\mathbf{F}\mathbf{F}^{\theta-1}\boldsymbol{\alpha}\mathbf{F}^\theta\mathbf{F}^{-1}\mathbf{\Gamma}^\theta,
\end{aligned}
\tag{3.74}
$$

which can be simplified to the form

$$\dot{\mathbf{\Gamma}}^\theta(t) = \mathsf{B}^\theta(t) : \mathbf{\Gamma}^\theta(t) + \mathsf{B}(t) : \mathbf{\Gamma}(t) + \boldsymbol{\theta}(t). \tag{3.75}$$

In this simplified form, $\mathsf{B}^\theta(t)$ and $\mathsf{B}(t)$ are fourth-order tensors and $\boldsymbol{\theta}(t)$ is a second-order tensor.

In order to solve the tensor form differential equation (3.75), we should represent the tensors in a curvilinear coordinate system and then solve the differential equations in the component form. For simplicity, here we choose an orthonormal base and write the corresponding component form in this base as

$$\dot{\Gamma}^\theta_{ij}(t) = \mathsf{B}^\theta_{ijkl}(t)\Gamma^\theta_{kl}(t) + \mathsf{B}_{ijkl}(t)\Gamma_{kl}(t) + \theta_{ij}(t). \tag{3.76}$$

To solve this system of first-order differential equations, we organize the unknowns into a one-dimensional array denoted by $\boldsymbol{\chi}$ that takes the form

$$\boldsymbol{\chi} \equiv (\Gamma^\theta_{11}, \Gamma^\theta_{12}, \Gamma^\theta_{13}, \Gamma^\theta_{21}, ...). \tag{3.77}$$

To do this, we introduce one transformation K_{ijk} which can transform the components of $\mathbf{\Gamma}^\theta$ into the one-dimensional array $\boldsymbol{\chi}$ through the relation

$$\chi_i = K_{ijk}\Gamma^\theta_{jk}, \tag{3.78}$$

where, the values of K_{ijk} are either zero or one, defined by the pattern of $\boldsymbol{\chi}$. The inverse transformation is given as

$$\Gamma^\theta_{ij} = K^{-1}_{ijk}\chi_k. \tag{3.79}$$

After substituting the transformation (3.79) into (3.76), and then substituting into the time derivative of (3.78), we get the first-order system of differential equations

$$\dot{\boldsymbol{\chi}}(t) = \mathbf{A}(t)\boldsymbol{\chi}(t) + \mathbf{f}(t), \tag{3.80}$$

where, the component of the coefficient matrix $A_{mn}(t)$ and the component of the inhomogeneous array $f_m(t)$ are given as

$$\begin{aligned} A_{mn}(t) &= K_{mij}\mathsf{B}^{\theta}_{ijkl}(t)K^{-1}_{kln}, \\ f_m(t) &= K_{mij}\mathsf{B}_{ijkl}(t)\Gamma_{kl}(t) + K_{mij}\theta_{ij}(t). \end{aligned} \tag{3.81}$$

From the existence and uniqueness theorem, there exists a unique solution, since the coefficient matrix $[A(t)]$ and the inhomogeneous array $[f(t)]$ are continuous. The general solution to the system (3.80) is provided by Myskis [Myskis, 1975] as

$$\boldsymbol{\chi}(t) = \int_{t_i}^{t} \mathbf{Y}(t,\tau)\mathbf{f}(\tau)d\tau + \mathbf{Y}(t,t_i)\boldsymbol{\chi}_i, \tag{3.82}$$

under the initial condition $\boldsymbol{\chi}_i = \boldsymbol{\chi}(t_i)$, and where,

$$\begin{aligned} \mathbf{Y}(t,t_i) =& \mathbf{I} + \int_{t_i}^{t} \mathbf{A}(\tau_1)d\tau_1 + \int_{t_i}^{t} \mathbf{A}(\tau_1)\int_{t_i}^{\tau_1} \mathbf{A}(\tau_2)d\tau_2 d\tau_1 \\ &+ \int_{t_i}^{t} \mathbf{A}(\tau_1)\int_{t_i}^{\tau_1} \mathbf{A}(\tau_2)\int_{t_i}^{\tau_2} \mathbf{A}(\tau_3)d\tau_3 d\tau_2 d\tau_1 + ... \end{aligned} \tag{3.83}$$

From the general solution for $\boldsymbol{\chi}(t)$ given in (3.82) and the inverse transformations from $\boldsymbol{\chi}(t)$ to $\boldsymbol{\Gamma}^{\theta}(t)$, we then can calculate the increment of the internal variable $\check{\mathbf{H}}^{\theta}(t)$.

3.5 Wave propagation in incompressible nonlinear elastic materials under simple shear deformation

Polydimethylsiloxane (PDMS) is modeled as an incompressible elastic material by Negahban [Negahban, 2012] with the free energy given by $\psi = \psi^{\dagger}(I_1^{*})$. The material function of PDMS is given by

$$\rho_o \frac{\partial \psi}{\partial I_1^*} = 15643(I_1^* - 3)^2 - 23730(I_1^* - 3) + 128638. \tag{3.84}$$

Let us consider PDMS under *simple shear*. Fig. 3.4 shows the in-plane deformation for simple shear. The out-of-plane deformation is assumed to be zero. The two coordinate systems are rectangular and mutually parallel, so that the coordinates are identical to those described for rectangular systems. The deformation associated with simple shear is written as

$$\begin{aligned} x_1 &= X_1 + \gamma X_2, \\ x_2 &= X_2, \\ x_3 &= X_3. \end{aligned} \tag{3.85}$$

The deformation gradient is given by

$$\mathbf{F} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + \gamma \mathbf{e}_1 \otimes \mathbf{e}_2. \tag{3.86}$$

The inverse of the deformation gradient is given by

$$\mathbf{F}^{-1} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 - \gamma \mathbf{e}_1 \otimes \mathbf{e}_2. \tag{3.87}$$

The right Cauchy stretch tensor is therefore given by

$$\mathbf{C} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}. \tag{3.88}$$

The first invariant is given by $I_1 = tr(\mathbf{C}) = 3 + \gamma^2$.

The left Cauchy stretch tensor is given by

$$\mathbf{B} = (1+\gamma^2)\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + \gamma \mathbf{e}_1 \otimes \mathbf{e}_2 + \gamma \mathbf{e}_2 \otimes \mathbf{e}_1. \tag{3.89}$$

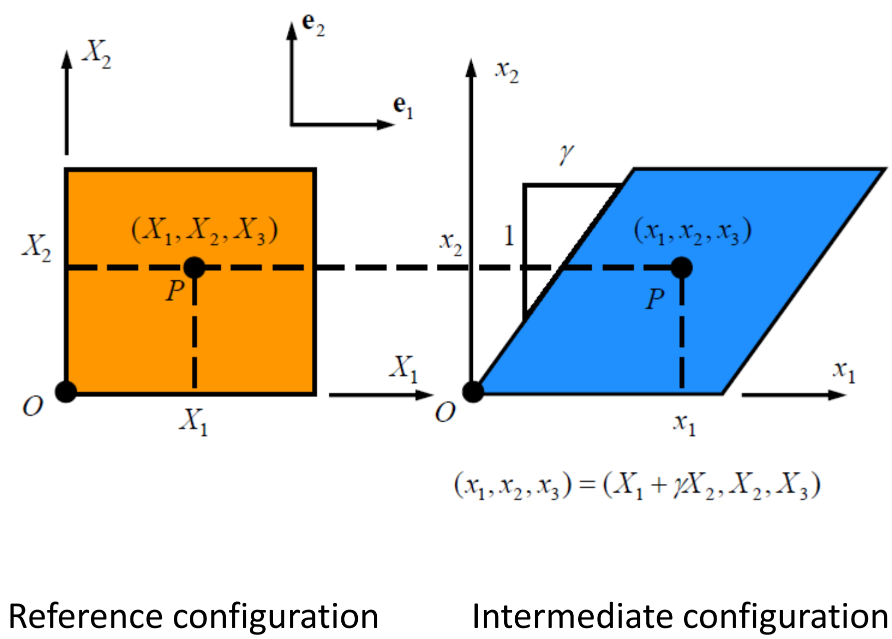


Figure 3.4: In-plane deformation for simple shear.

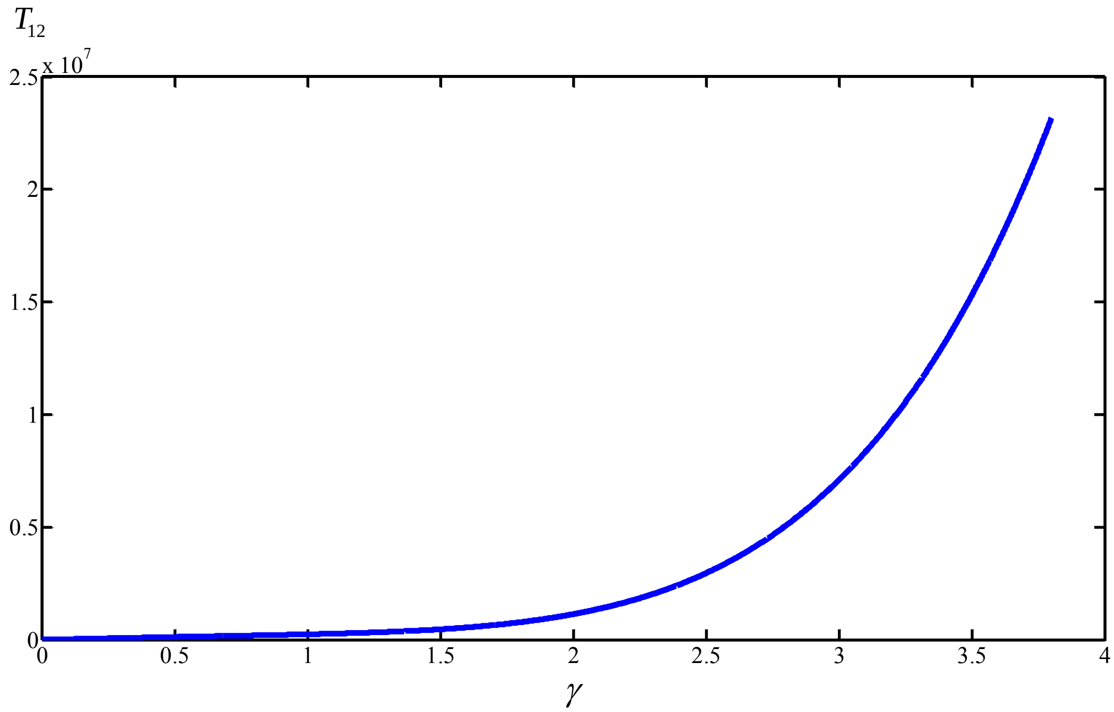


Figure 3.5: PDMS simple shear response.

The Cauchy stress for incompressible materials takes the form

$$\mathbf{T}^T = \mathbf{T}_E^T + p\mathbf{I}, \tag{3.90}$$

where,

$$\mathbf{T}_E^T = 2\rho_o \frac{\partial \psi}{\partial I_1^*}(\mathbf{B} - \frac{I_1}{3}\mathbf{I}), \tag{3.91}$$

and p is an indeterminate scalar due to the incompressibility.

The simple shear stress of PDMS is given by

$$T_{12} = 2(15643\gamma^5 - 23730\gamma^3 + 128638\gamma). \tag{3.92}$$

The response of PDMS under simple shear is shown in Fig. 3.5.

For incompressible materials the perturbation of Cauchy stress is given by

$$\delta\mathbf{T}^T = \mathsf{E}^e : \delta\mathbf{F} + \delta p\mathbf{I}. \tag{3.93}$$

The tangent modulus is defined as $\mathsf{E}^e \equiv \partial_{\mathbf{F}}(\mathbf{T}_E^T)$ and can be written in the rectangular coordinate system as

$$\begin{aligned}\mathsf{E}^e_{abcd} =& 2\rho_o\{\frac{\partial^2\psi}{\partial I_1^{*2}}[2(F_{cd} - \frac{I_1}{3}F_{dc}^{-1})](B_{ab} - \frac{I_1}{3}\delta_{ab}) \\ &+ \frac{\partial\psi}{\partial I_1^*}[F_{bd}\delta_{ac} + F_{ad}\delta_{bc} - \frac{2}{3}(F_{cd} - \frac{I_1}{3}F_{dc}^{-1})\delta_{ab}]\},\end{aligned} \tag{3.94}$$

where, for our PDMS model we have $\rho_o\frac{\partial^2\psi}{\partial I_1^{*2}} = 15643 \times 2(I_1^* - 3) - 23730$.

Since for small perturbation, we have $\check{J} \approx 1 + \check{\varepsilon}_v$ for

$$\begin{aligned}\check{\varepsilon}_v &= tr(\check{\boldsymbol{\varepsilon}}), \\ \check{\boldsymbol{\varepsilon}} &= \frac{1}{2}(\check{\mathbf{H}} + \check{\mathbf{H}}^T).\end{aligned} \tag{3.95}$$

The perturbation should satisfy the incompressibility constraint, i.e. $\check{J} = 1$. This requires

$$\check{\varepsilon}_v = 0. \tag{3.96}$$

For an incompressible material under a homogenous base deformation, the perturbation of balance of linear momentum is given by

$$[\mathbf{Grad}(\delta\mathbf{T}) - \mathbf{T}\mathbf{Grad}(\check{\mathbf{H}}^T)] : \mathbf{F}^{-T} = \rho_o\delta\ddot{\mathbf{u}}, \tag{3.97}$$

where,

$$\delta\mathbf{T} = (\mathsf{E}^e\mathbf{F}^T) : \check{\mathbf{H}} + \delta p\mathbf{I}. \tag{3.98}$$

Let us consider a wave motion given by

$$\delta\mathbf{u}(t) = \mathsf{u}\cos[\omega(t - \frac{\mathbf{n}\circ\mathbf{x}}{c})]\mathbf{d}. \tag{3.99}$$

The acceleration of this wave is

$$\delta\ddot{\mathbf{u}}(t) = -\mathsf{u}\omega^2 \cos\{\omega[t - \frac{\mathbf{n} \circ (\mathbf{X} + \mathbf{u})}{c}]\}\mathbf{d}. \tag{3.100}$$

The perturbation displacement gradient is given by

$$\check{\mathbf{H}} = \mathsf{u}\frac{\omega}{c} \sin[\omega(t - \frac{\mathbf{n} \circ \mathbf{X} + \mathbf{n} \circ \mathbf{u}}{c})]\mathbf{d} \otimes \mathbf{n}. \tag{3.101}$$

The incompressible perturbations should satisfy the balance of linear momentum equation given in (3.97), that is

$$\{(\mathsf{E}^e\mathbf{F}^T) : \mathbf{Grad}(\check{\mathbf{H}}) + \mathbf{I} \otimes \mathbf{Grad}(\delta p) - \mathbf{TGrad}(\check{\mathbf{H}}^T)\} : \mathbf{F}^{-T} = \rho_o \delta\ddot{\mathbf{u}}. \tag{3.102}$$

We write equation (3.102) in a rectangular coordinate system as

$$\begin{aligned} &[\mathbf{Grad}(\delta p)]_b F_{bi}^{-1} \\ &+ \mathsf{u}\omega^2 \cos[\omega(t - \frac{\mathbf{n} \circ (\mathbf{X} + \mathbf{u})}{c})] \left\{ T_{ij} n_a n_j \frac{1}{c^2} + \rho_o \delta_{ai} - \mathsf{E}^e_{ijad} F_{yd} n_j n_y \frac{1}{c^2} \right\} d_a = 0, \end{aligned} \tag{3.103}$$

where,

$$T_{ij} = 2\rho_o \frac{\partial \psi}{\partial I_1^*}(B_{ij} - \frac{I_1}{3}\delta_{ij}) + p\delta_{ij}. \tag{3.104}$$

We select the perturbation of the undetermined scalar δp in such a form that

$$[\mathbf{Grad}(\delta p)] = \mathsf{u}\omega^2 \cos[\omega(t - \frac{\mathbf{n} \circ (\mathbf{X} + \mathbf{u})}{c})]\mathbf{f}, \tag{3.105}$$

where, $\mathbf{f}$ is determined from the boundary conditions. Therefore equation (3.103) is written as

$$f_b F_{bi}^{-1} c^2 + (T_{i2} n_a - \mathsf{E}^e_{i2a2}) d_a + \rho_o c^2 \delta_{ai} d_a = 0. \tag{3.106}$$

Let us superimpose the wave given in (3.99) on the PDMS under the simple shear deformation described in Fig. 3.4. For the wave propagation direction $\mathbf{n} = \mathbf{e}_2$, there are two simple shear modes that satisfy the incompressibility constraint (3.96), i.e. in-plane shear wave and out-of-plane shear wave. For the in-plane shear wave $\mathbf{d} = \mathbf{e}_1$, the in-plane

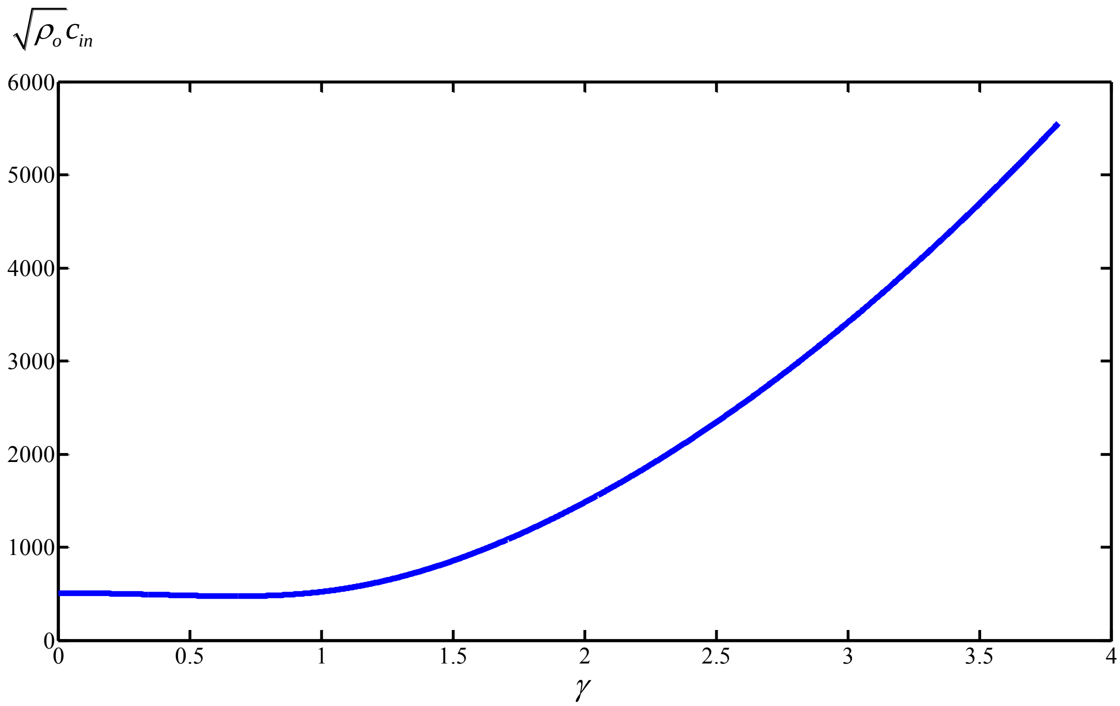


Figure 3.6: In-plane shear wave speed in PDMS under simple shear.

shear wave speed c_{in} is determined from

$$\rho_o c_{in}^2 = \mathsf{E}^e_{1212}, \tag{3.107}$$

where, by satisfying equation (3.106) we have

$$\begin{aligned} f_1 &= 0, \\ f_2 c^2 &= \mathsf{E}^e_{2212}, \\ f_3 &= 0. \end{aligned} \tag{3.108}$$

Fig. 3.6 shows the effect of the simple shear on the in-plane shear wave speed for the PDMS we considered.

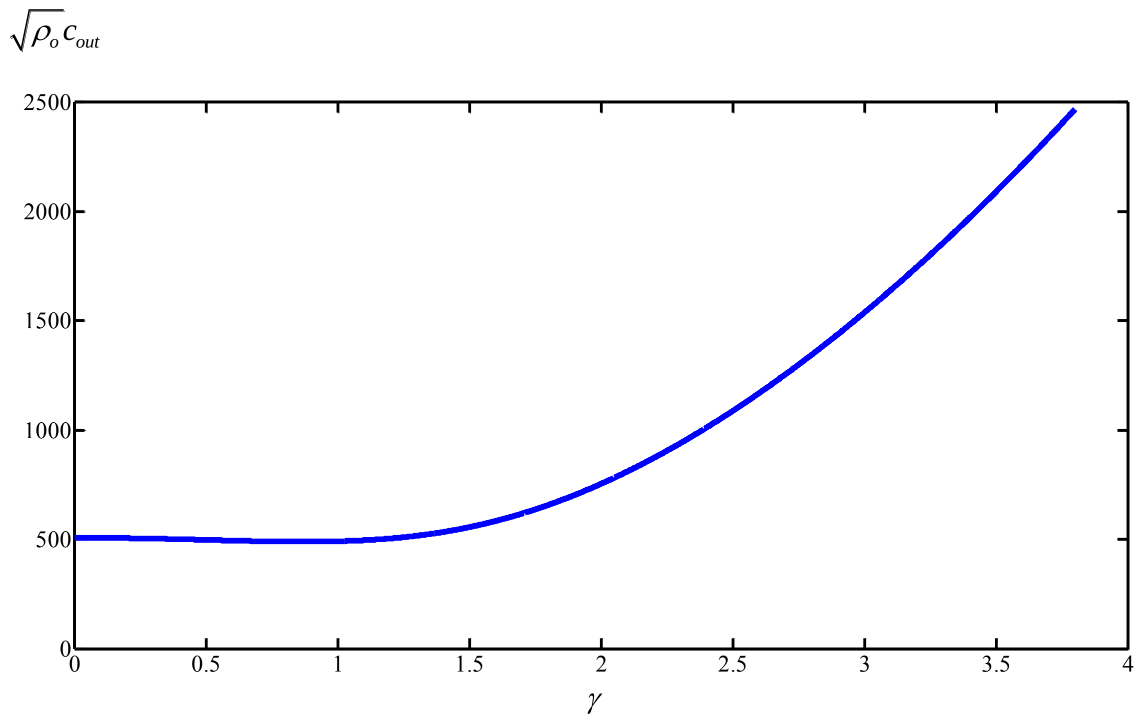


Figure 3.7: Out-of-plane shear wave speed in PDMS under simple shear.

For the out-of-plane shear wave $\mathbf{d} = \mathbf{e}_3$, the out-of-plane shear wave speed c_{out} is determined from

$$\rho_o c_{out}^2 = \mathsf{E}^e_{3232}, \tag{3.109}$$

where, by satisfying equation (3.106) we have

$$\begin{aligned} f_1 &= 0, \\ f_2 &= 0, \\ f_3 &= 0. \end{aligned} \tag{3.110}$$

Fig. 3.7 shows the effect of the simple shear on the out-of-plane shear wave speed for the PDMS we considered.

3.6 Wave propagation in axis-symmetrically deformed nonlinear elastic cylinder

In this section we study the axis-symmetric problem of wave propagation in a deformed nonlinear elastic cylinder under isothermal conditions. In particular, we look at a long cylinder deformed by a rigid central shaft and held at the outer surface, rigidly fixed, as shown in Fig. 3.8.

This is a special case of wave propagation in a general thermoelastic deformed body and will demonstrate how the results can be used. The material considered here is an initially isotropic nonlinear elastic material. The response functions of such materials are assumed to depend on three kinematical invariants I_1^*, I_2^* and I_3^*.

To keep the presentation simple, we further assume that the behavior of the material can be captured by two invariants I_1^* and I_3^*. As an example, we then look at the propagation of shear waves in polydimethylsiloxane (PDMS). PDMS is a typical nonlinear elastic material at room temperature and exhibits close to incompressible responses.

As shown in Fig. 3.8, we use (R, Θ, Z) to denote a cylindrical coordinate for the reference configuration and (r, θ, z) to denote a cylindrical coordinate for the current configuration. We use $(\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\Theta, \hat{\mathbf{e}}_Z)$ and $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$ to denote the unit vectors along the reference and current coordinates. The associated base vectors of the curvilinear coordinates are given by

$$\mathbf{e}_1 = \hat{\mathbf{e}}_R, \tag{3.111}$$

$$\mathbf{e}_2 = R\hat{\mathbf{e}}_\Theta,$$

$$\mathbf{e}_3 = \hat{\mathbf{e}}_Z, \tag{3.112}$$

and by

$$\mathbf{e}_1^* = \hat{\mathbf{e}}_r,$$

$$\mathbf{e}_2^* = r\hat{\mathbf{e}}_\theta,$$

$$\mathbf{e}_3^* = \hat{\mathbf{e}}_z. \tag{3.113}$$

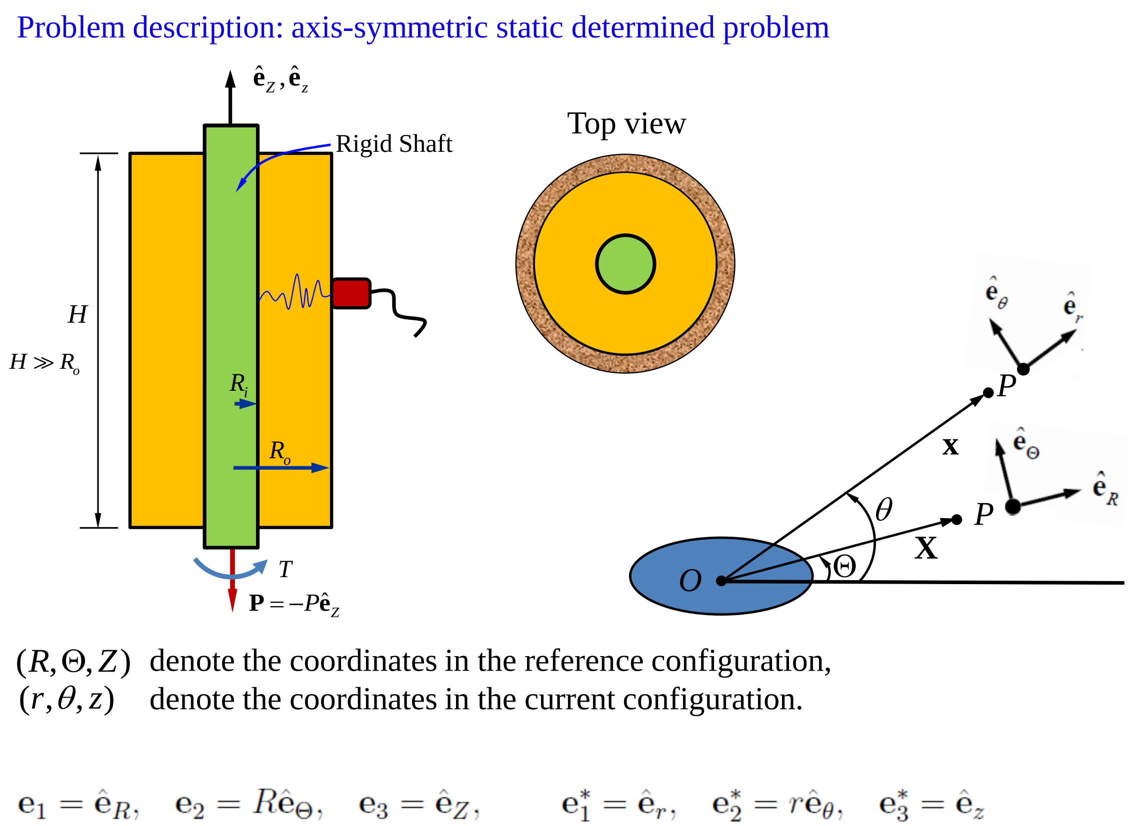


Figure 3.8: Propagation of waves in an axis-symmetric deformed thermo-elastic body.

For our body, we take a cylinder, in the reference configuration, with the height much longer than the outer radius such that $H \gg R_o$. We will further confine our attention to a response that does not depend on the axial coordinate Z, and we ignore the body force.

First, we will consider the application of an axial load $\mathbf{P} = -P\hat{\mathbf{e}}_Z$ through the rigid shaft. Next, we will look at a torque T applied through the rigid shaft. Three types of wave motion (longitudinal, in-plane shear and out-of-plane shear waves) will be, respectively, superimposed on these pre-deformations. The wave speeds will be obtained for each mode and will be shown to vary over space due to the inhomogeneity of the pre-deformations. In this process we will look at the propagation of shear waves in a PDMS cylinder under axial loads.

Due to the axis-symmetry and the long bar assumption, the deformations are independent of Θ and Z, and take the forms as

$$\begin{aligned}\theta &= \Theta + \bar{\theta}(R),\\ r &= R + \bar{r}(R),\\ z &= Z + \bar{z}(R),\end{aligned} \tag{3.114}$$

where, the specific functions of $\bar{\theta}(R)$, $\bar{r}(R)$ and $\bar{z}(R)$ are determined by the boundary conditions and the material properties.

3.6.1 Waves in an elastic cylinder under axial load

For an axial loaded cylinder we have

$$\begin{aligned}\theta &= \Theta,\\ r &= R + \bar{r}(R),\\ z &= Z + \bar{z}(R),\end{aligned} \tag{3.115}$$

and we get

$$\begin{aligned}\hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_Z,\\ \hat{\mathbf{e}}_r &= \hat{\mathbf{e}}_R,\\ \hat{\mathbf{e}}_\theta &= \hat{\mathbf{e}}_\Theta.\end{aligned} \tag{3.116}$$

The deformation gradient is calculated as

$$\mathbf{F} = (1 + \frac{\partial \bar{r}}{\partial R})\hat{\mathbf{e}}_R \otimes \hat{\mathbf{e}}_R + \frac{\partial \bar{z}}{\partial R}\hat{\mathbf{e}}_Z \otimes \hat{\mathbf{e}}_R + (1 + \frac{\bar{r}}{R})\hat{\mathbf{e}}_\Theta \otimes \hat{\mathbf{e}}_\Theta + \hat{\mathbf{e}}_Z \otimes \hat{\mathbf{e}}_Z. \tag{3.117}$$

The inverse of the deformation gradient is

$$\mathbf{F}^{-1} = \frac{1}{1 + \frac{\partial \bar{r}}{\partial R}}\hat{\mathbf{e}}_R \otimes \hat{\mathbf{e}}_R - \frac{\frac{\partial \bar{z}}{\partial R}}{1 + \frac{\partial \bar{r}}{\partial R}}\hat{\mathbf{e}}_Z \otimes \hat{\mathbf{e}}_R + \frac{R}{r}\hat{\mathbf{e}}_\Theta \otimes \hat{\mathbf{e}}_\Theta + \hat{\mathbf{e}}_Z \otimes \hat{\mathbf{e}}_Z. \tag{3.118}$$

The left Cauchy stretch is given by

$$\mathbf{B} = (1 + \frac{\partial \bar{r}}{\partial R})^2 \hat{\mathbf{e}}_R \otimes \hat{\mathbf{e}}_R + \frac{\partial \bar{z}}{\partial R}(1 + \frac{\partial \bar{r}}{\partial R})\hat{\mathbf{e}}_Z \otimes \hat{\mathbf{e}}_R + (1 + \frac{\partial \bar{r}}{\partial R})\frac{\partial \bar{z}}{\partial R}\hat{\mathbf{e}}_R \otimes \hat{\mathbf{e}}_Z + [(\frac{\partial \bar{z}}{\partial R})^2 + 1]\hat{\mathbf{e}}_Z \otimes \hat{\mathbf{e}}_Z + (1 + \frac{\bar{r}}{R})^2 \hat{\mathbf{e}}_\Theta \otimes \hat{\mathbf{e}}_\Theta. \tag{3.119}$$

The nominal stress takes the general form

$$\mathbf{T}_o = \begin{bmatrix} \hat{\mathbf{e}}_R & \hat{\mathbf{e}}_\Theta & \hat{\mathbf{e}}_Z \end{bmatrix} \begin{bmatrix} T_{o\_RR} & T_{o\_R\Theta} & T_{o\_RZ} \\ T_{o\_\Theta R} & T_{o\_\Theta\Theta} & T_{o\_\Theta Z} \\ T_{o\_ZR} & T_{o\_Z\Theta} & T_{o\_ZZ} \end{bmatrix} \otimes \begin{bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\Theta \\ \hat{\mathbf{e}}_Z \end{bmatrix}. \tag{3.120}$$

The axis-symmetry and the loading give the nominal stress for the base history in the form

$$\mathbf{T}_o = \begin{bmatrix} \hat{\mathbf{e}}_R & \hat{\mathbf{e}}_\Theta & \hat{\mathbf{e}}_Z \end{bmatrix} \begin{bmatrix} T_{o\_RR} & 0 & T_{o\_RZ} \\ 0 & T_{o\_\Theta\Theta} & 0 \\ T_{o\_ZR} & 0 & T_{o\_ZZ} \end{bmatrix} \otimes \begin{bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_\Theta \\ \hat{\mathbf{e}}_Z \end{bmatrix}. \tag{3.121}$$

The boundary conditions we consider for this problem are

$$\begin{aligned} \bar{r}(R_i) &= 0, \\ \bar{r}(R_o) &= 0, \\ \bar{z}(R_o) &= 0. \end{aligned} \tag{3.122}$$

The problem is statically determinate for the shear. This gives

$$T_{o\_RZ}(R) = \frac{P}{2\pi RH}. \tag{3.123}$$

As can be seen, we assume rigid supports on both the inner surface and outer surface of the cylinder. We also consider the outer surface to be stationary. Also, the shaft is assumed to uniformly distribute the load on the material.

The balance of linear momentum in the reference configuration is given by the following component equations

$$\frac{\partial T_{o\_RR}}{\partial R} + \frac{1}{R}\frac{\partial T_{o\_\Theta R}}{\partial \Theta} + \frac{\partial T_{o\_ZR}}{\partial Z} + \frac{1}{R}(T_{o\_RR} - T_{o\_\Theta\Theta}) = 0, \tag{3.124}$$
$$\frac{\partial T_{o\_R\Theta}}{\partial R} + \frac{1}{R}\frac{\partial T_{o\_\Theta\Theta}}{\partial \Theta} + \frac{\partial T_{o\_Z\Theta}}{\partial Z} + \frac{1}{R}(T_{o\_R\Theta} + T_{o\_\Theta R}) = 0,$$
$$\frac{\partial T_{o\_RZ}}{\partial R} + \frac{1}{R}\frac{\partial T_{o\_\Theta Z}}{\partial \Theta} + \frac{\partial T_{o\_ZZ}}{\partial Z} + \frac{1}{R}T_{o\_RZ} = 0.$$

The axis-symmetry simplifies the balance of linear momentum to

$$\frac{\partial T_{o\_RR}}{\partial R} + \frac{\partial T_{o\_ZR}}{\partial Z} + \frac{1}{R}(T_{o\_RR} - T_{o\_\Theta\Theta}) = 0,$$
$$\frac{1}{R}\frac{\partial T_{o\_\Theta\Theta}}{\partial \Theta} = 0,$$
$$\frac{\partial T_{o\_RZ}}{\partial R} + \frac{\partial T_{o\_ZZ}}{\partial Z} + \frac{1}{R}T_{o\_RZ} = 0, \tag{3.125}$$

where, the last two equations are automatically satisfied due to the axis-symmetry, the static determinacy of the shear and the long bar assumption.

The Cauchy stress has the form

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}}(\psi^\dagger)\mathbf{F}^T = \rho \sum_{n=1}^{3} \frac{\partial \psi^\dagger}{\partial I_i^*} \partial_{\mathbf{F}}(I_i^*)\mathbf{F}^T, \tag{3.126}$$

where, the invariants are

$$I_1 = tr(\mathbf{C}), \quad I_2 = tr(\mathbf{C}^2), \quad I_3 = tr(\mathbf{C}^3),$$
$$I_1^* = \frac{I_1}{J^{2/3}}, \quad I_2^* = \frac{I_2}{I_1^2}, \quad I_3^* = J = \det(\mathbf{F}), \tag{3.127}$$

and the derivatives with respect to the deformation gradient $\mathbf{F}$ are

$$\partial_{\mathbf{F}}(I_1^*) = \frac{2}{J^{2/3}}(\mathbf{F} - \frac{I_1}{3}\mathbf{F}^{-T}), \tag{3.128}$$
$$\partial_{\mathbf{F}}(I_2^*) = \frac{4}{I_1^2}\mathbf{F}(\mathbf{C} - \frac{I_2}{I_1}\mathbf{I}),$$
$$\partial_{\mathbf{F}}(I_3^*) = J\mathbf{F}^{-T}.$$

This gives the Cauchy stress in the form

$$\mathbf{T}^T = \rho[\frac{2}{J^{2/3}}\frac{\partial\psi}{\partial I_1^*}(\mathbf{B} - \frac{I_1}{3}\mathbf{I}) + \frac{4}{I_1^2}\frac{\partial\psi}{\partial I_2^*}(\mathbf{B}^2 - \frac{I_2}{I_1}\mathbf{B}) + J\frac{\partial\psi}{\partial I_3^*}\mathbf{I}]. \tag{3.129}$$

If we assume that the free energy takes the simpler form as $\psi = \psi^\dagger(I_1^*, I_3^*)$, we will have

$$\mathbf{T}^T = \rho[\frac{2}{J^{2/3}}\frac{\partial\psi}{\partial I_1^*}(\mathbf{B} - \frac{I_1}{3}\mathbf{I}) + J\frac{\partial\psi}{\partial I_3^*}\mathbf{I}]. \tag{3.130}$$

The nominal stress can be obtained from

$$\mathbf{T}_o = \det(\mathbf{F})\mathbf{F}^{-1}\mathbf{T}, \tag{3.131}$$

where, the components of the nominal stress are given by

$$\begin{aligned}
T_{o_RR} =&(1+\frac{\bar r}{R})\rho_{ref}[\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}(1+\frac{\partial\bar r}{\partial R})^2 - \frac{I_1}{3}\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*} + \frac{\partial\psi}{\partial I_3^*}],\\
T_{o_RZ} =&(1+\frac{\bar r}{R})\rho_{ref}\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}(1+\frac{\partial\bar r}{\partial R})\frac{\partial\bar z}{\partial R},\\
T_{o_ZR} =&-(1+\frac{\bar r}{R})\rho_{ref}\frac{\partial\bar z}{\partial R}[\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}(1+\frac{\partial\bar r}{\partial R})^2 - \frac{I_1}{3}\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*} + \frac{\partial\psi}{\partial I_3^*}]\\
&+(1+\frac{\partial\bar r}{\partial R})(1+\frac{\bar r}{R})\rho_{ref}\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}\frac{\partial\bar z}{\partial R}(1+\frac{\partial\bar r}{\partial R}),\\
T_{o_ZZ} =&-(1+\frac{\bar r}{R})\rho_{ref}\frac{\partial\bar z}{\partial R}\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}(1+\frac{\partial\bar r}{\partial R})\frac{\partial\bar z}{\partial R}\\
&+(1+\frac{\partial\bar r}{\partial R})(1+\frac{\bar r}{R})\rho_{ref}\{\frac{\partial\psi}{\partial I_3^*} - \frac{I_1}{3}\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*} + \frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}[(\frac{\partial\bar z}{\partial R})^2+1]\},\\
T_{o_\Theta\Theta} =&(1+\frac{\partial\bar r}{\partial R})(1+\frac{\bar r}{R})\rho_{ref}(\frac{R}{r})[\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}(1+\frac{\bar r}{R})^2 - \frac{I_1}{3}\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*} + \frac{\partial\psi}{\partial I_3^*}].
\end{aligned} \tag{3.132}$$

After considering $T_{o\_RZ}(R) = \frac{P}{2\pi H}R^{-1}$ from the static determinacy, we only have the first equation in (3.125) as

$$\rho_o(R+\bar r)\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}(1+\frac{\partial\bar r}{\partial R})\frac{\partial\bar z}{\partial R} = \frac{P}{2\pi H}. \tag{3.133}$$

Substituting the nominal stress given in (3.132) into the balance of linear momentum

(3.125) results in

$$\frac{\partial T_{o\_RR}}{\partial R}+\frac{1}{R}(1+\frac{\bar{r}}{R})\rho_o\{\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}(1+\frac{\partial\bar{r}}{\partial R})^2$$
$$-\frac{I_1}{3}\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}+\frac{\partial\psi}{\partial I_3^*}-(1+\frac{\partial\bar{r}}{\partial R})(\frac{R}{r})[\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}(1+\frac{\bar{r}}{R})^2-\frac{I_1}{3}\frac{2}{J^{5/3}}\frac{\partial\psi}{\partial I_1^*}+\frac{\partial\psi}{\partial I_3^*}]\}=0, \tag{3.134}$$

where,

$$J=\det(\mathbf{F})=(1+\frac{\partial\bar{r}}{\partial R})(1+\frac{\bar{r}}{R}),$$
$$I_1=(1+\frac{\partial\bar{r}}{\partial R})^2+(1+\frac{\bar{r}}{R})^2+(\frac{\partial\bar{z}}{\partial R})^2+1. \tag{3.135}$$

$\bar{r}(R)$ and $\bar{z}(R)$ are determined by solving equations given in (3.133) and (3.134) under the boundary conditions provided in (3.122).

For an incompressible material we have $J=1$ and therefore we have

$$\bar{r}=-\frac{d\bar{r}}{dR}(R+\bar{r}), \tag{3.136}$$

which results in $\bar{r}(R)=0$ under the boundary conditions $\bar{r}(R_i)=0$ and $\bar{r}(R_o)=0$.

For $\bar{r}(R)=0$ the invariant can be simplified to

$$I_1=3+(\frac{\partial\bar{z}}{\partial R})^2. \tag{3.137}$$

For an incompressible material the nonlinear elastic free energy is assumed to take a form as $\psi=\psi^\dagger(I_1^*)$, and therefore the Cauchy stress is given by

$$\mathbf{T}^T=2\rho_o\frac{\partial\psi}{\partial I_1^*}(\mathbf{B}-\frac{I_1}{3}\mathbf{I})+p\mathbf{I}, \tag{3.138}$$

where, p is an indeterminate scalar due to the incompressibility.

For an incompressible material the nominal stress is then given by

$$\mathbf{T}_o=\mathbf{F}^{-1}[2\rho_o\frac{\partial\psi}{\partial I_1^*}(\mathbf{B}-\frac{I_1}{3}\mathbf{I})+p\mathbf{I}]. \tag{3.139}$$

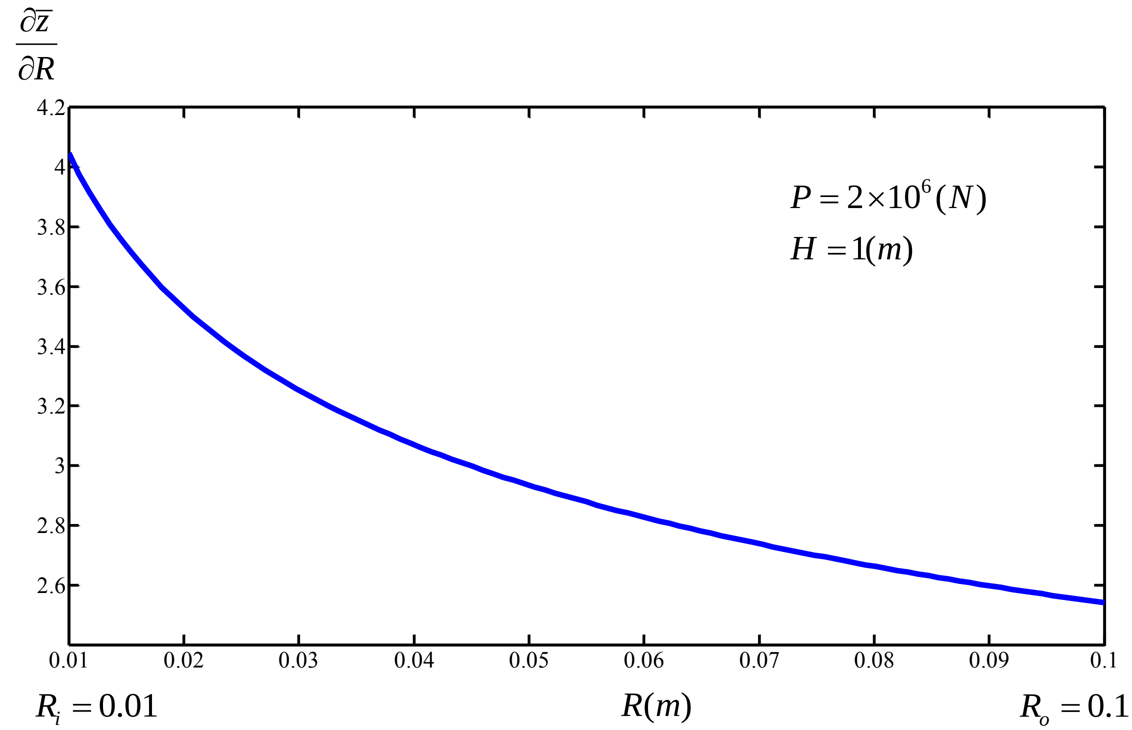


Figure 3.9: The shear deformation of a PDMS cylinder under axial load P.

Equation (3.123) then becomes

$$\frac{\partial \psi}{\partial I_1^*}\frac{\partial \bar{z}}{\partial R} = \frac{P}{4\pi H \rho_o R}. \tag{3.140}$$

For the incompressible elastic model for PDMS given in (3.84), equation (3.140) becomes

$$15643(\frac{\partial \bar{z}}{\partial R})^5 - 23730(\frac{\partial \bar{z}}{\partial R})^3 + 128638\frac{\partial \bar{z}}{\partial R} = \frac{P}{4\pi H R}, \tag{3.141}$$

where, $\frac{\partial \bar{z}}{\partial R}$ is determined for given P and R. For example, Fig. 3.9 shows the inhomogeneous shear deformations along the radius of the PDMS cylinder under axial load.

The superimposed wave propagates along the radial direction "$\mathbf{e}_r$" and can be described by

$$\delta\mathbf{u}(t) = \mathsf{u}\cos[\omega(t - \int_{\ell}\frac{d\ell}{c})]\mathbf{d}, \tag{3.142}$$

where,

$$\begin{aligned}
&\delta\mathbf{u}(t) = \mathbf{u}^{*}(t) - \mathbf{u},\\
&\mathbf{u} = \mathbf{x} - \mathbf{X},\\
&\mathbf{X} = R\mathbf{e}_R + Z\mathbf{e}_Z,\\
&\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z,
\end{aligned} \tag{3.143}$$

and where, $\mathbf{d}$ represents the unit dispalcement direction vector, c is the wave speed, and $\ell = \mathbf{x} \circ \mathbf{e}_r$.

The balance of linear momentum for the perturbation, given inhomogeneous base deformation, is given by

$$\begin{aligned}
&[(\delta\mathbf{T}) - \mathbf{T}\check{\mathbf{H}}^T][Div(\mathbf{F}^{-T})]\\
&+ [\mathbf{Grad}(\delta\mathbf{T}) - \mathbf{T}\mathbf{Grad}(\check{\mathbf{H}}^T)] : \mathbf{F}^{-T} - \mathbf{Grad}(\mathbf{T}) : (\check{\mathbf{H}}^T\mathbf{F}^{-T}) = \rho_o\delta\ddot{\mathbf{u}}.
\end{aligned} \tag{3.144}$$

For

$$\mathbf{F}^{-T} = \mathbf{e}_R \otimes \mathbf{e}_R - \frac{\partial\bar{z}}{\partial R}\mathbf{e}_R \otimes \mathbf{e}_Z + \mathbf{e}_\Theta \otimes \mathbf{e}_\Theta + \mathbf{e}_Z \otimes \mathbf{e}_Z, \tag{3.145}$$

we have

$$Div(\mathbf{F}^{-T}) = \mathbf{0}. \tag{3.146}$$

The perturbation of Cauchy stress is

$$\delta\mathbf{T} = (\mathsf{E}^e\mathbf{F}^T) : \check{\mathbf{H}} + \delta p\mathbf{I}, \tag{3.147}$$

and the tangent modulus is defined as

$$\mathsf{E}^e = \partial_{\mathbf{F}}(\mathbf{T}_E^T). \tag{3.148}$$

The acceleration of the wave is given by

$$\delta\ddot{\mathbf{u}}(t) = -\mathsf{u}\omega^2\cos[\omega(t - \int_\ell \frac{d\ell}{c})]\mathbf{d}. \tag{3.149}$$

The perturbation displacement gradient is given by

$$\begin{aligned}\check{\mathbf{H}} =& \mathsf{u}\sin[\omega(t - \int_\ell \frac{1}{c}d\ell)]\frac{\omega}{c}d_R\mathbf{e}_R\otimes\mathbf{e}_R \\ &- \frac{1}{R}\mathsf{u}\cos[\omega(t - \int_\ell \frac{1}{c}d\ell)]d_\Theta\mathbf{e}_R\otimes\mathbf{e}_\Theta \\ &+ \mathsf{u}\sin[\omega(t - \int_\ell \frac{1}{c}d\ell)]\frac{\omega}{c}d_\Theta\mathbf{e}_\Theta\otimes\mathbf{e}_R \\ &+ \frac{1}{R}\mathsf{u}\cos[\omega(t - \int_\ell \frac{1}{c}d\ell)]d_R\mathbf{e}_\Theta\otimes\mathbf{e}_\Theta \\ &+ \mathsf{u}\sin[\omega(t - \int_\ell \frac{1}{c}d\ell)]\frac{\omega}{c}d_Z\mathbf{e}_Z\otimes\mathbf{e}_R,\end{aligned} \tag{3.150}$$

where, $\mathbf{d} = d_R\mathbf{e}_R + d_\Theta\mathbf{e}_\Theta + d_Z\mathbf{e}_Z$, and

$$tr(\check{\mathbf{H}}) = \mathsf{u}\{\sin[\omega(t - \int_\ell \frac{1}{c}d\ell)]\frac{\omega}{c} + \frac{1}{R}\cos[\omega(t - \int_\ell \frac{1}{c}d\ell)]\}d_R. \tag{3.151}$$

The superimposed wave motion (3.142) should satisfy the balance of linear momentum for the incompressible perturbation terms given in (3.144). The component forms of these equations are written in the $\mathbf{e}_i^*$ system. For example, the Cauchy stress is given by

$$\begin{aligned}\mathbf{T} =& [2\rho_o\frac{\partial\psi}{\partial I_1^*}(1 - \frac{I_1}{3}) + p]\mathbf{e}_1^*\otimes\mathbf{e}_1^* \\ &+ 2\rho_o\frac{\partial\psi}{\partial I_1^*}\frac{\partial\bar{z}}{\partial R}\mathbf{e}_3^*\otimes\mathbf{e}_1^* \\ &+ 2\rho_o\frac{\partial\psi}{\partial I_1^*}\frac{\partial\bar{z}}{\partial R}\mathbf{e}_1^*\otimes\mathbf{e}_3^* \\ &+ \{2\rho_o\frac{\partial\psi}{\partial I_1^*}[(\frac{\partial\bar{z}}{\partial R})^2 + 1 - \frac{I_1}{3}] + p\}\mathbf{e}_3^*\otimes\mathbf{e}_3^* \\ &+ [2\rho_o\frac{\partial\psi}{\partial I_1^*}(1 - \frac{I_1}{3}) + p]\frac{1}{r^2}\mathbf{e}_2^*\otimes\mathbf{e}_2^*,\end{aligned} \tag{3.152}$$

the gradient of Cauchy stress is

$$
\begin{aligned}
\mathbf{Grad}(\mathbf{T}) =& \frac{\partial(T_{mn})}{\partial R}\mathbf{e}_m \otimes \mathbf{e}_n \otimes \mathbf{e}_1 + \frac{T_{1n}}{R^3}\mathbf{e}_2 \otimes \mathbf{e}_n \otimes \mathbf{e}_2 \\
&+ 2\frac{T_{22}}{R}\mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \frac{T_{22}}{R}\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \frac{T_{22}}{R}\mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \\
&+ \frac{T_{m1}}{R^3}\mathbf{e}_m \otimes \mathbf{e}_2 \otimes \mathbf{e}_2.
\end{aligned} \tag{3.153}
$$

The tangent modulus is given by $\mathsf{E}^e = \mathsf{E}_{xyab}\mathbf{e}^*_x \otimes \mathbf{e}^*_y \otimes \mathbf{e}^*_a \otimes \mathbf{e}^*_b$, where,

$$
\begin{aligned}
\mathsf{E}_{xyab} =& 2\rho_o\{2\frac{\partial^2\psi}{\partial I_1^{*2}}(F_{ab} - \frac{I_1}{3}F^{-1}_{ba})(B_{xy} - \frac{I_1}{3}I_{xy}) \\
&+ \frac{\partial\psi}{\partial I_1^*}[F_{yb}g^{-1}_{xa} + F_{xb}g^{-1}_{ya} - \frac{2}{3}(F_{ab} - \frac{I_1}{3}F^{-1}_{ba})I_{xy}]\}.
\end{aligned} \tag{3.154}
$$

The gradient of the perturbation of Cauchy stress is given by

$$
\mathbf{Grad}(\delta\mathbf{T}) = \frac{\partial(\mathsf{E}^e\mathbf{F}^T)}{\partial\alpha_i} : \check{\mathbf{H}} \otimes g^{-1}_{ij}\mathbf{e}_j + (\mathsf{E}^e\mathbf{F}^T) : \mathbf{Grad}(\check{\mathbf{H}}) + \mathbf{I} \otimes \mathbf{Grad}(\delta p). \tag{3.155}
$$

We also have

$$
\begin{aligned}
\mathbf{Grad}(\check{\mathbf{H}}^T) =& \frac{\partial(\check{H}_{nm})}{\partial R}\mathbf{e}_m \otimes \mathbf{e}_n \otimes \mathbf{e}_1 + \frac{\check{H}_{n1}}{R^3}\mathbf{e}_2 \otimes \mathbf{e}_n \otimes \mathbf{e}_2 \\
&+ \frac{\check{H}_{n2}}{R}\mathbf{e}_2 \otimes \mathbf{e}_n \otimes \mathbf{e}_1 - \frac{\check{H}_{n2}}{R}\mathbf{e}_1 \otimes \mathbf{e}_n \otimes \mathbf{e}_2 + \frac{\check{H}_{1m}}{R^3}\mathbf{e}_m \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \\
&+ \frac{\check{H}_{2m}}{R}\mathbf{e}_m \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \frac{\check{H}_{2m}}{R}\mathbf{e}_m \otimes \mathbf{e}_1 \otimes \mathbf{e}_2,
\end{aligned} \tag{3.156}
$$

and

$$
\begin{aligned}
\mathbf{Grad}(\check{\mathbf{H}}) =& \frac{\partial(\check{H}_{mn})}{\partial R}\mathbf{e}_m \otimes \mathbf{e}_n \otimes \mathbf{e}_1 + \frac{\check{H}_{1n}}{R^3}\mathbf{e}_2 \otimes \mathbf{e}_n \otimes \mathbf{e}_2 \\
&+ \frac{\check{H}_{2n}}{R}\mathbf{e}_2 \otimes \mathbf{e}_n \otimes \mathbf{e}_1 - \frac{\check{H}_{2n}}{R}\mathbf{e}_1 \otimes \mathbf{e}_n \otimes \mathbf{e}_2 + \frac{\check{H}_{m1}}{R^3}\mathbf{e}_m \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \\
&+ \frac{\check{H}_{m2}}{R}\mathbf{e}_m \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \frac{\check{H}_{m2}}{R}\mathbf{e}_m \otimes \mathbf{e}_1 \otimes \mathbf{e}_2.
\end{aligned} \tag{3.157}
$$

In this process we have used the relations

$$
\begin{aligned}
\frac{\partial(\mathbf{e}_1^*)}{\partial R} &= \mathbf{0}, \\
\frac{\partial(\mathbf{e}_1^*)}{\partial \Theta} &= \frac{\mathbf{e}_2}{R}, \\
\frac{\partial(\mathbf{e}_1^*)}{\partial Z} &= \mathbf{0}, \\
\frac{\partial(\mathbf{e}_2^*)}{\partial R} &= \frac{\mathbf{e}_2}{R}, \\
\frac{\partial(\mathbf{e}_2^*)}{\partial \Theta} &= -R\mathbf{e}_1, \\
\frac{\partial(\mathbf{e}_2^*)}{\partial Z} &= \mathbf{0}, \\
\frac{\partial(\mathbf{e}_3^*)}{\partial R} &= \mathbf{0}, \\
\frac{\partial(\mathbf{e}_3^*)}{\partial \Theta} &= \mathbf{0}, \\
\frac{\partial(\mathbf{e}_3^*)}{\partial Z} &= \mathbf{0}.
\end{aligned}
\tag{3.158}
$$

To satisfy the incompressible perturbation condition, i.e. $\check{\varepsilon}_v = 0$, this requires $d_R = 0$. We consider two types of wave propagating in the axially loaded incompressible cylinder: *in-plane shear wave* and *out-of-plane shear wave*. The wave speeds are evaluated by substituting the wave equations into equation (3.144) written in the $\mathbf{e}_i^*$ system.

For the in-plane shear wave, the unit displacement vector $\mathbf{d} = \frac{1}{r}\mathbf{e}_2^* = \mathbf{e}_\theta$. The acceleration of the wave (3.149) becomes

$$
\delta\ddot{\mathbf{u}}(t) = -\mathsf{u}\omega^2 \cos[\omega(t - \int_\ell \frac{d\ell}{c})]\frac{1}{r}\mathbf{e}_2^*, \tag{3.159}
$$

and the perturbation of the displacement gradient (3.150) becomes

$$
\begin{aligned}
\check{\mathbf{H}} = &- \mathsf{u}\frac{1}{r^2}\cos[\omega(t - \int_\ell \frac{1}{c}d\ell)]\mathbf{e}_1^* \otimes \mathbf{e}_2^* \\
&+ \mathsf{u}\frac{\omega}{c}\frac{1}{r}\sin[\omega(t - \int_\ell \frac{1}{c}d\ell)]\mathbf{e}_2^* \otimes \mathbf{e}_1^*.
\end{aligned}
\tag{3.160}
$$

The gradient of the perturbation displacement gradient (3.157) is simplified to

$$\begin{aligned}\mathbf{Grad}(\check{\mathbf{H}}) =&(\frac{\partial \check{H}_{12}}{\partial R} + \frac{\check{H}_{12}}{R})\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 + (\frac{\partial \check{H}_{21}}{\partial R} + \frac{\check{H}_{21}}{R})\mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \\ &+ (\frac{\check{H}_{12}}{R^3} + \frac{\check{H}_{21}}{R^3})\mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - (\frac{\check{H}_{21}}{R} + \frac{\check{H}_{12}}{R})\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2. \end{aligned} \tag{3.161}$$

We substitute the equations given in (3.159), (3.160) and (3.161) and other associated terms into the balance of linear momentum for perturbations (3.144), then we can calculate the in-plane shear wave speed. The in-plane shear wave speed is varying along the radial direction and given by

$$c = \sqrt{2\frac{\partial \psi}{\partial I_1^*}}. \tag{3.162}$$

The variation of the in-plane shear wave speed in PDMS under axial load is shown in Fig. 3.10.

For the out-of-plane shear wave, the unit displacement vector $\mathbf{d} = \mathbf{e}_3^* = \mathbf{e}_Z$. The acceleration of the wave (3.149) becomes

$$\delta \ddot{\mathbf{u}}(t) = -\mathsf{u}\omega^2 \cos[\omega(t - \int_\ell \frac{d\ell}{c})]\mathbf{e}_3^*, \tag{3.163}$$

and the perturbation of the displacement gradient (3.150) becomes

$$\check{\mathbf{H}} = \mathsf{u} \sin[\omega(t - \int_\ell \frac{1}{c} d\ell)]\frac{\omega}{c}\mathbf{e}_3^* \otimes \mathbf{e}_1^*. \tag{3.164}$$

The gradient of the perturbation displacement gradient (3.157) is simplified to

$$\mathbf{Grad}(\check{\mathbf{H}}) = \frac{\partial(\check{H}_{31})}{\partial R}\mathbf{e}_3 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{\check{H}_{31}}{R^3}\mathbf{e}_3 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2. \tag{3.165}$$

We substitute the equations given in (3.163), (3.164) and (3.165) and other associated terms into the balance of linear momentum for perturbations (3.144), then we can calculate the out-of-plane shear wave speed. The out-of-plane shear wave speed is varying along the

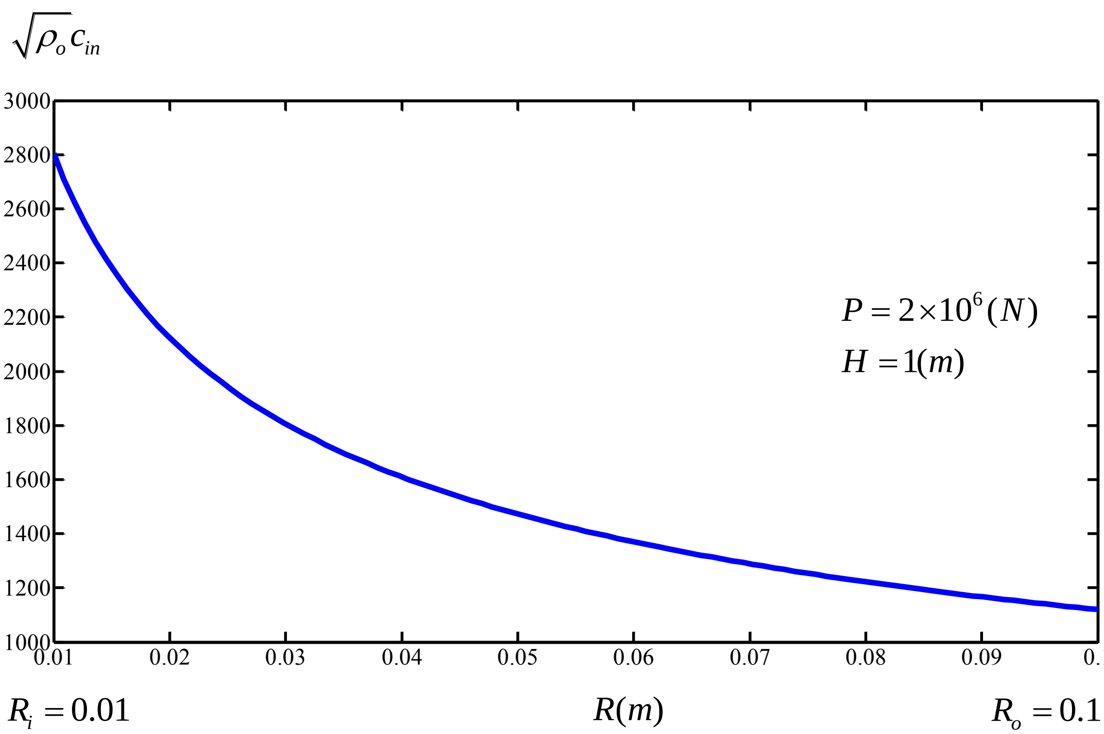


Figure 3.10: In-plane shear wave speed in a PDMS cylinder under axial load P.

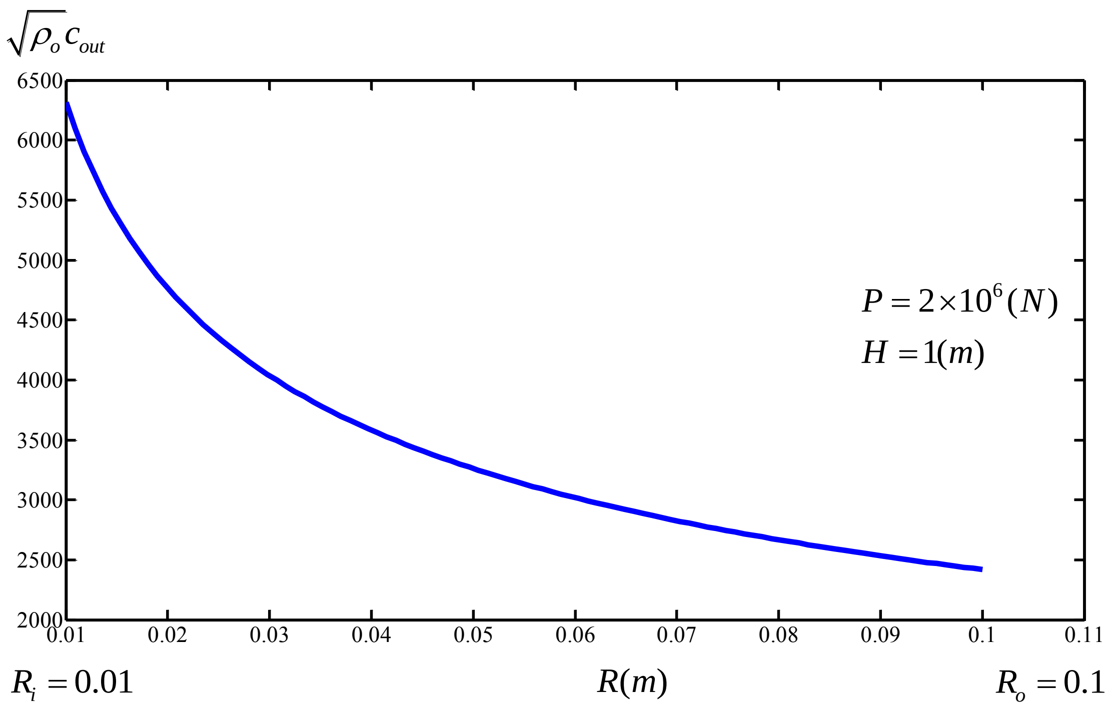


Figure 3.11: Out-of-plane shear wave speed in a PDMS cylinder under axial load P.

radial direction and given by

$$c = \sqrt{4\frac{\partial^2\psi}{\partial I_1^{*2}}(\frac{\partial \bar{z}}{\partial R})^2 + 2\frac{\partial \psi}{\partial I_1^*}}. \tag{3.166}$$

If no load is applied,

$$c = \sqrt{2\frac{\partial \psi}{\partial I_1^*}}. \tag{3.167}$$

The variation of the out-of-plane shear wave speed in PDMS under axial load is shown in Fig. 3.11.

The time of flight of the waves from R_o to R_i can be evaluated from

$$t = \int_{R_i}^{R_o} \frac{dR}{c}. \tag{3.168}$$

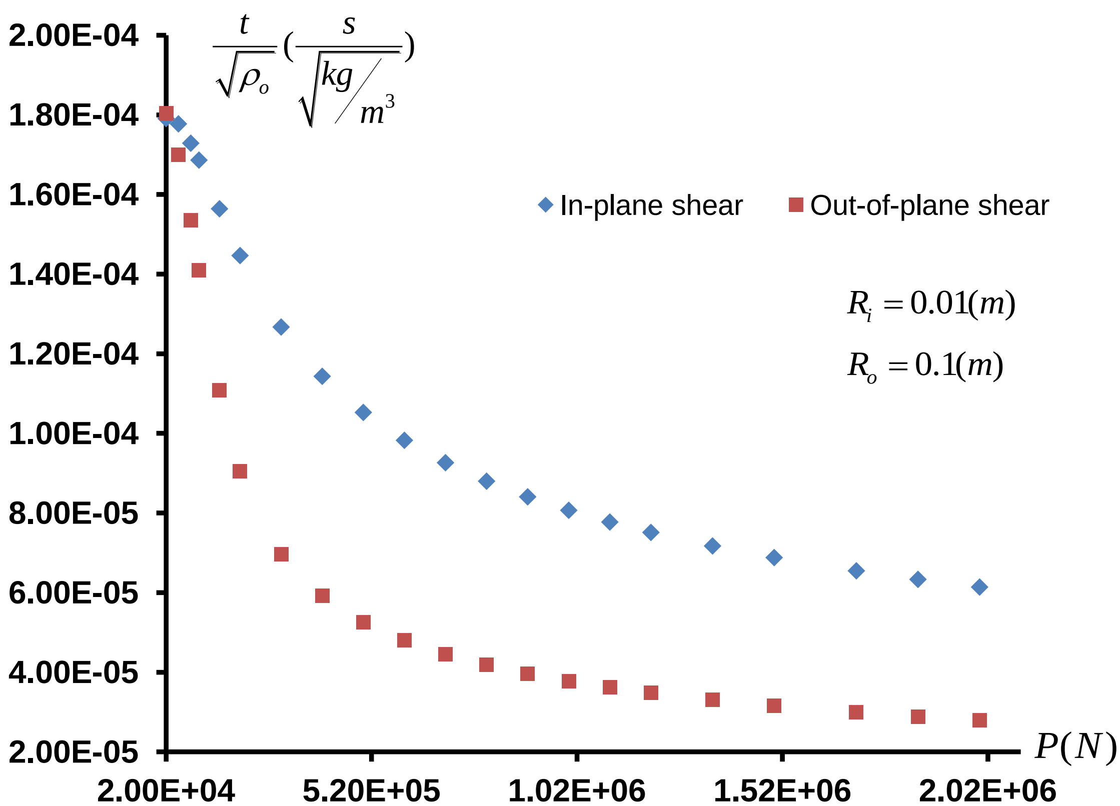


Figure 3.12: Time of flights of in-plane shear wave and out-of-plane shear wave in the PDMS cylinder under different axial loads.

Fig. 3.12 shows the time of flights of in-plane shear wave and out-of-plane shear wave in the PDMS cylinder under different axial loads.

The change of the time of flight Δt can be obtained from the change of the load applied ΔP by

$$\Delta t = -\Delta P \int_{R_i}^{R_o} c^{-2} \frac{\partial(c)}{\partial P} dR. \tag{3.169}$$

Load sensor (a forward application)

For an incompressible elastic material, if the nonlinear free energy $\psi(I_1^*)$ is known, from the measurement of the time of flight we can evaluate the applied load P. This can be used as a load sensor.

Considering $I_1^* - 3 = (\frac{\partial \bar{z}}{\partial R})^2$ and $I_3^* = J = 1$, and the relations between the material functions and the deformations given in (**??**), the deformation field $\bar{z}$ can be determined as a function of R and P.

After substituting the deformation function $\bar{z}(R,P)$ into the forms of time of flight of the in-plane shear wave and out-plane shear wave measured in experiments, the value of the load can be calculated. If both waves are considered, the accuracy of the measured load can be increased.

3.6.2 Waves in an elastic cylinder under torque

Let us look at another example, that is, torque T is applied through the rigid shaft as shown in Fig. 3.8. The deformation field under torque is described as

$$\begin{aligned} r &= R + \bar{r}(R), \\ \theta &= \Theta + \bar{\theta}(R), \\ z &= Z + \bar{z}(R). \end{aligned} \tag{3.170}$$

The relations between the cylindrical vectors in the reference configuration and in the current configuration are given by

$$\begin{aligned} \hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_Z, \\ \hat{\mathbf{e}}_r &= \cos(\bar{\theta})\hat{\mathbf{e}}_R + \sin(\bar{\theta})\hat{\mathbf{e}}_\Theta, \\ \hat{\mathbf{e}}_\theta &= \cos(\bar{\theta})\hat{\mathbf{e}}_\Theta - \sin(\bar{\theta})\hat{\mathbf{e}}_R. \end{aligned} \tag{3.171}$$

The deformation gradient is calculated as

$$\begin{aligned} \mathbf{F} =& [(1+\frac{\partial\bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})]\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_R \\ &+ [(1+\frac{\partial\bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta})]\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_R \\ &+ \frac{\partial\bar{z}}{\partial R}\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_R + (1+\frac{\bar{r}}{R})\cos(\bar{\theta})\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_\Theta - (1+\frac{\bar{r}}{R})\sin(\bar{\theta})\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_\Theta + \hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_Z. \end{aligned} \tag{3.172}$$

The left Cauchy stretch is given by

$$
\begin{aligned}
\mathbf{B} =&\{[(1+\frac{\partial\bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})]^2 + (1+\frac{\bar{r}}{R})^2\sin^2(\bar{\theta})\}\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_R \\
&+ \{[(1+\frac{\partial\bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta})][(1+\frac{\partial\bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})] \\
&- (1+\frac{\bar{r}}{R})^2\cos(\bar{\theta})\sin(\bar{\theta})\}\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_R \\
&+ \frac{\partial\bar{z}}{\partial R}[(1+\frac{\partial\bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})]\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_R \\
&+ \{[(1+\frac{\partial\bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})][(1+\frac{\partial\bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta})] \\
&- (1+\frac{\bar{r}}{R})^2\sin(\bar{\theta})\cos(\bar{\theta})\}\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_\Theta \\
&+ \{[(1+\frac{\partial\bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta})]^2 + (1+\frac{\bar{r}}{R})^2\cos^2(\bar{\theta})\}\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_\Theta \\
&+ \frac{\partial\bar{z}}{\partial R}[(1+\frac{\partial\bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta})]\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_\Theta \\
&+ [(1+\frac{\partial\bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})]\frac{\partial\bar{z}}{\partial R}\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_Z \\
&+ [(1+\frac{\partial\bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta})]\frac{\partial\bar{z}}{\partial R}\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_Z + [1+(\frac{\partial\bar{z}}{\partial R})^2]\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_Z. \quad (3.173)
\end{aligned}
$$

The inverse of the deformation gradient is

$$
\begin{aligned}
\mathbf{F}^{-1} =&\frac{1}{(1+\frac{\partial\bar{r}}{\partial R})}[\cos(\bar{\theta})\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_R + \sin(\bar{\theta})\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_\Theta - \frac{\partial\bar{z}}{\partial R}\cos(\bar{\theta})\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_R - \frac{\partial\bar{z}}{\partial R}\sin(\bar{\theta})\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_\Theta] \\
&- R[\frac{\partial\bar{\theta}}{\partial R}\frac{\cos(\bar{\theta})}{(1+\frac{\partial\bar{r}}{\partial R})} + \frac{\sin(\bar{\theta})}{r}]\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_R + R[\frac{\cos(\bar{\theta})}{r} - \frac{\partial\bar{\theta}}{\partial R}\frac{\sin(\bar{\theta})}{(1+\frac{\partial\bar{r}}{\partial R})}]\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_\Theta + \hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_Z.
\end{aligned}
\quad (3.174)
$$

The Cauchy stress takes the form as in (3.130), for this case, it is given by

$$
\begin{aligned}
\mathbf{T} =&T_{RR}\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_R + T_{\Theta R}\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_R + T_{ZR}\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_R + T_{R\Theta}\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_\Theta + T_{\Theta\Theta}\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_\Theta \\
&+ T_{Z\Theta}\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_\Theta + T_{RZ}\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_Z + T_{\Theta Z}\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_Z + T_{ZZ}\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_Z, \quad (3.175)
\end{aligned}
$$

where, $I_1 = tr(\mathbf{B})$, and the components are given by

$$
\begin{aligned}
T_{RR} =& \{[(1+\frac{\partial \bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\sin(\bar{\theta})]^2 + (1+\frac{\bar{r}}{R})^2\sin^2(\bar{\theta}) - \frac{I_1}{3}\}\rho_o\frac{2}{J^{5/3}}\frac{\partial \psi}{\partial I_1^*} \\
&+ \rho_o\frac{\partial \psi}{\partial I_3^*}, \\
T_{\Theta R} =& \{[(1+\frac{\partial \bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta})][(1+\frac{\partial \bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\sin(\bar{\theta})] \\
&- (1+\frac{\bar{r}}{R})^2\cos(\bar{\theta})\sin(\bar{\theta})\}\rho_o\frac{2}{J^{5/3}}\frac{\partial \psi}{\partial I_1^*}, \\
T_{ZR} =& \frac{\partial \bar{z}}{\partial R}[(1+\frac{\partial \bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\sin(\bar{\theta})]\rho_o\frac{2}{J^{5/3}}\frac{\partial \psi}{\partial I_1^*}, \\
T_{R\Theta} =& \{[(1+\frac{\partial \bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\sin(\bar{\theta})][(1+\frac{\partial \bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta})] \\
&- (1+\frac{\bar{r}}{R})^2\sin(\bar{\theta})\cos(\bar{\theta})\}\rho_o\frac{2}{J^{5/3}}\frac{\partial \psi}{\partial I_1^*}, \\
T_{\Theta\Theta} =& \{[(1+\frac{\partial \bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta})]^2 + (1+\frac{\bar{r}}{R})^2\cos^2(\bar{\theta}) - \frac{I_1}{3}\}\rho_o\frac{2}{J^{5/3}}\frac{\partial \psi}{\partial I_1^*} \\
&+ \rho_o\frac{\partial \psi}{\partial I_3^*}, \\
T_{Z\Theta} =& \frac{\partial \bar{z}}{\partial R}[(1+\frac{\partial \bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta})]\rho_o\frac{2}{J^{5/3}}\frac{\partial \psi}{\partial I_1^*}, \\
T_{RZ} =& [(1+\frac{\partial \bar{r}}{\partial R})\cos(\bar{\theta}) - (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\sin(\bar{\theta})]\frac{\partial \bar{z}}{\partial R}\rho_o\frac{2}{J^{5/3}}\frac{\partial \psi}{\partial I_1^*}, \\
T_{\Theta Z} =& [(1+\frac{\partial \bar{r}}{\partial R})\sin(\bar{\theta}) + (R+\bar{r})\frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta})]\frac{\partial \bar{z}}{\partial R}\rho_o\frac{2}{J^{5/3}}\frac{\partial \psi}{\partial I_1^*}, \\
T_{ZZ} =& [1+(\frac{\partial \bar{z}}{\partial R})^2 - \frac{I_1}{3}]\rho_o\frac{2}{J^{5/3}}\frac{\partial \psi}{\partial I_1^*} + \rho_o\frac{\partial \psi}{\partial I_3^*}.
\end{aligned}
\tag{3.176}
$$

The nominal stress defined as $\mathbf{T}_o = \det(\mathbf{F})\mathbf{F}^{-1}\mathbf{T}$ is given by

$$
\begin{aligned}
\mathbf{T}_o =&(1+\frac{\bar{r}}{R})[T_{RR}\cos(\bar{\theta}) + T_{\Theta R}\sin(\bar{\theta})]\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_R \\
&+ (R+\bar{r})\{T_{\Theta R}[\frac{1}{r}(1+\frac{\partial\bar{r}}{\partial R})\cos(\bar{\theta}) - \frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})] \\
&- T_{RR}[\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta}) + (1+\frac{\partial\bar{r}}{\partial R})\frac{1}{r}\sin(\bar{\theta})]\}\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_R \\
&+ (\frac{R+\bar{r}}{R})[(1+\frac{\partial\bar{r}}{\partial R})T_{ZR} - T_{RR}\frac{\partial\bar{z}}{\partial R}\cos(\bar{\theta}) - T_{\Theta R}\frac{\partial\bar{z}}{\partial R}\sin(\bar{\theta})]\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_R \\
&+ (\frac{R+\bar{r}}{R})[T_{R\Theta}\cos(\bar{\theta}) + T_{\Theta\Theta}\sin(\bar{\theta})]\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_\Theta \\
&+ (R+\bar{r})\{T_{\Theta\Theta}[\frac{1}{r}(1+\frac{\partial\bar{r}}{\partial R})\cos(\bar{\theta}) - \frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})] \\
&- T_{R\Theta}[\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta}) + (1+\frac{\partial\bar{r}}{\partial R})\frac{1}{r}\sin(\bar{\theta})]\}\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_\Theta \\
&+ (\frac{R+\bar{r}}{R})[(1+\frac{\partial\bar{r}}{\partial R})T_{Z\Theta} - T_{R\Theta}\frac{\partial\bar{z}}{\partial R}\cos(\bar{\theta}) - T_{\Theta\Theta}\frac{\partial\bar{z}}{\partial R}\sin(\bar{\theta})]\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_\Theta \\
&+ (\frac{R+\bar{r}}{R})[T_{RZ}\cos(\bar{\theta}) + T_{\Theta Z}\sin(\bar{\theta})]\hat{\mathbf{e}}_R\otimes\hat{\mathbf{e}}_Z \\
&+ (R+\bar{r})\{T_{\Theta Z}[\frac{1}{r}(1+\frac{\partial\bar{r}}{\partial R})\cos(\bar{\theta}) - \frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})] \\
&- T_{RZ}[\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta}) + \frac{1}{r}(1+\frac{\partial\bar{r}}{\partial R})\sin(\bar{\theta})]\}\hat{\mathbf{e}}_\Theta\otimes\hat{\mathbf{e}}_Z \\
&+ (\frac{R+\bar{r}}{R})[(1+\frac{\partial\bar{r}}{\partial R})T_{ZZ} - T_{\Theta Z}\frac{\partial\bar{z}}{\partial R}\sin(\bar{\theta}) - T_{RZ}\frac{\partial\bar{z}}{\partial R}\cos(\bar{\theta})]\hat{\mathbf{e}}_Z\otimes\hat{\mathbf{e}}_Z, \qquad (3.177)
\end{aligned}
$$

where, $J = \det(\mathbf{F}) = (1+\frac{\partial\bar{r}}{\partial R})(1+\frac{\bar{r}}{R})$.

Balance of linear momentum in the reference configuration is given as

$$
\begin{aligned}
&\frac{\partial T_{o\_RR}}{\partial R} + \frac{1}{R}\frac{\partial T_{o\_\Theta R}}{\partial\Theta} + \frac{\partial T_{o\_ZR}}{\partial Z} + \frac{1}{R}(T_{o\_RR} - T_{o\_\Theta\Theta}) = 0, \\
&\frac{\partial T_{o\_R\Theta}}{\partial R} + \frac{1}{R}\frac{\partial T_{o\_\Theta\Theta}}{\partial\Theta} + \frac{\partial T_{o\_Z\Theta}}{\partial Z} + \frac{1}{R}(T_{o\_R\Theta} + T_{o\_\Theta R}) = 0, \\
&\frac{\partial T_{o\_RZ}}{\partial R} + \frac{1}{R}\frac{\partial T_{o\_\Theta Z}}{\partial\Theta} + \frac{\partial T_{o\_ZZ}}{\partial Z} + \frac{1}{R}T_{o\_RZ} = 0. \qquad (3.178)
\end{aligned}
$$

Boundary conditions are given by

$$
\begin{aligned}
&\bar{r}(R_i) = 0,\\
&\bar{r}(R_o) = 0,\\
&\bar{\theta}(R_o) = 0,\\
&\bar{z}(R_i) = 0,\\
&\bar{z}(R_o) = 0,\\
&T_{o\_R\Theta}(R) = \frac{T}{2\pi R^2 H}.
\end{aligned} \tag{3.179}
$$

The axis-symmetry constrains the nominal stress to the form

$$
\begin{aligned}
&T_{o\_\Theta Z} = 0,\\
&T_{o\_RZ} = 0.
\end{aligned} \tag{3.180}
$$

The axis-symmetry constrains the balance of linear momentum in the reference configuration to the form

$$
\begin{aligned}
&\frac{\partial T_{o\_RR}}{\partial R} + \frac{1}{R}(T_{o\_RR} - T_{o\_\Theta\Theta}) = 0,\\
&\frac{\partial T_{o\_R\Theta}}{\partial R} + \frac{1}{R}(T_{o\_R\Theta} + T_{o\_\Theta R}) = 0.
\end{aligned} \tag{3.181}
$$

For incompressible materials i.e. $J = 1$, we have $r = R$. If we assume the deformation field is simple shear, i.e.

$$
\begin{aligned}
&\theta = \Theta + \bar{\theta}(R),\\
&z = Z,
\end{aligned} \tag{3.182}
$$

where, the boundary conditions (3.179) are automatically satisfied. For this deformation we have $I_1 = 3 + (\frac{\partial\bar{\theta}}{\partial R})^2 r^2$.

The balance of linear momentum for this simple shear field are

$$\frac{\partial T_{o\_RR}}{\partial R} + \frac{1}{R}(T_{o\_RR} - T_{o\_\Theta\Theta}) = 0, \tag{3.183}$$

and

$$\frac{\partial T_{o\_R\Theta}}{\partial R} + \frac{1}{R}(T_{o\_R\Theta} + T_{o\_\Theta R}) = 0, \tag{3.184}$$

where, after substituting $T_{o\_R\Theta} = \frac{T}{2\pi R^2 H}$ into equation (3.184), we get

$$T_{o\_\Theta R} = \frac{T}{2\pi R^2 H}, \tag{3.185}$$

where,

$$\begin{aligned} T_{o\_R\Theta} =& 2\rho_o \frac{\partial \psi}{\partial I_1^*}[R\frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta}) - (\frac{\partial \bar{\theta}}{\partial R})^2 \frac{R^2}{3}\sin(\bar{\theta})] + \rho_o \frac{\partial \psi}{\partial I_3^*}\sin(\bar{\theta}), \\ T_{o\_\Theta R} =& 2\rho_o \frac{\partial \psi}{\partial I_1^*}[R\frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta}) + (\frac{\partial \bar{\theta}}{\partial R})^3 \frac{R^3}{3}\cos(\bar{\theta}) + (\frac{\partial \bar{\theta}}{\partial R})^2 \frac{R^2}{3}\sin(\bar{\theta})] \\ & - \rho_o \frac{\partial \psi}{\partial I_3^*}[R\frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta}) + \sin(\bar{\theta})]. \end{aligned} \tag{3.186}$$

After manipulating equations we obtain the relations between the material functions and the deformation $\bar{\theta}(T, R)$ as

$$\begin{aligned} \frac{\partial \psi}{\partial I_1^*} &= \frac{T}{4\rho_o \pi R^3 H \frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta})}, \\ \frac{\partial \psi}{\partial I_3^*} &= \frac{T}{6\rho_o \pi R H \cos(\bar{\theta})}\frac{\partial \bar{\theta}}{\partial R}, \\ \frac{\partial^2 \psi}{\partial I_1^{*2}} &= -\frac{T}{8\rho_o \pi R^4 H (\frac{\partial^2 \bar{\theta}}{\partial R^2}R + \frac{\partial \bar{\theta}}{\partial R})\frac{\partial \bar{\theta}}{\partial R}\sin(\bar{\theta})} \\ & - \frac{T}{8\rho_o \pi R^5 H \cos(\bar{\theta})(\frac{\partial^2 \bar{\theta}}{\partial R^2}R + \frac{\partial \bar{\theta}}{\partial R})(\frac{\partial \bar{\theta}}{\partial R})^3}(\frac{\partial^2 \bar{\theta}}{\partial R^2}R + 3\frac{\partial \bar{\theta}}{\partial R}), \\ \frac{\partial^2 \psi}{\partial I_3^* \partial I_1^*} &= \frac{T}{6\rho_o \pi R^3 H \frac{\partial \bar{\theta}}{\partial R}\cos(\bar{\theta})} - \frac{T}{12\rho_o \pi R^2 H (\frac{\partial^2 \bar{\theta}}{\partial R^2}R + \frac{\partial \bar{\theta}}{\partial R})\sin(\bar{\theta})}\frac{\partial \bar{\theta}}{\partial R} \\ & - \frac{T}{12\rho_o \pi R^3 H \cos(\bar{\theta})(\frac{\partial^2 \bar{\theta}}{\partial R^2}R + \frac{\partial \bar{\theta}}{\partial R})\frac{\partial \bar{\theta}}{\partial R}}(\frac{\partial^2 \bar{\theta}}{\partial R^2}R + 3\frac{\partial \bar{\theta}}{\partial R}). \end{aligned} \tag{3.187}$$

The tangent modulus defined as $\mathsf{E}^e = \partial_{\mathbf{F}}(\mathbf{T}^{T\dagger})$ has the component form in the $\mathbf{e}_i^*$ base as

$$\begin{aligned}\mathsf{E}^{e*}_{ijcd} =&\rho_o\{-\frac{10}{3}F^{-1*}_{ba}g^*_{ac}g^*_{bd}\frac{\partial\psi}{\partial I^*_1}(B^*_{ij}-\frac{I_1}{3}I^*_{ij})+2[\frac{\partial^2\psi}{\partial I^{*2}_1}2(F^*_{ab}g^*_{ac}g^*_{bd}-\frac{I_1}{3}F^{-T*}_{ab}g^*_{ac}g^*_{bd})\\&+\frac{\partial^2\psi}{\partial I^*_1\partial I^*_3}F^{-1*}_{ba}g^*_{ac}g^*_{bd}](B^*_{ij}-\frac{I_1}{3}I^*_{ij})\\&+2\frac{\partial\psi}{\partial I^*_1}(\delta_{ic}\delta_{xd}F^*_{jy}g^*_{xy}+F^*_{ix}\delta_{jc}\delta_{yd}g^*_{xy}-\frac{2F^*_{ab}g^*_{ac}g^*_{bd}}{3}I^*_{ij})\\&+[\frac{\partial^2\psi}{\partial I^*_3\partial I^*_1}(F^*_{ab}g^*_{ac}g^*_{bd}-\frac{I_1}{3}F^{-1*}_{ba}g^*_{ac}g^*_{bd})+\frac{\partial^2\psi}{\partial I^{*2}_3}F^{-1*}_{ba}g^*_{ac}g^*_{bd}]I^*_{ij}\},\end{aligned}\tag{3.188}$$

where,

$$[g^*]=\begin{bmatrix}1&0&0\\0&r^2&0\\0&0&1\end{bmatrix},\tag{3.189}$$

$$[I^*]=[g^{*-1}]=\begin{bmatrix}1&0&0\\0&\frac{1}{r^2}&0\\0&0&1\end{bmatrix},\tag{3.190}$$

$$[F^*]=\begin{bmatrix}\cos(\bar\theta)&\frac{-\sin(\bar\theta)}{r}&0\\\frac{R\frac{\partial\bar\theta}{\partial R}\cos(\bar\theta)+\sin(\bar\theta)}{r}&\frac{\cos(\bar\theta)-R\frac{\partial\bar\theta}{\partial R}\sin(\bar\theta)}{r^2}&0\\0&0&1\end{bmatrix},\tag{3.191}$$

$$[F^{-1*}]=\begin{bmatrix}\cos(\bar\theta)-R\frac{\partial\bar\theta}{\partial R}\sin(\bar\theta)&\frac{\sin(\bar\theta)}{r}&0\\\frac{-[\sin(\bar\theta)+R\frac{\partial\bar\theta}{\partial R}\cos(\bar\theta)]}{r}&\frac{\cos(\bar\theta)}{r^2}&0\\0&0&1\end{bmatrix},\tag{3.192}$$

and

$$[B^*]=\begin{bmatrix}1&\frac{\partial\bar\theta}{\partial R}&0\\\frac{\partial\bar\theta}{\partial R}&\frac{1}{r^2}+(\frac{\partial\bar\theta}{\partial R})^2&0\\0&0&1\end{bmatrix}.\tag{3.193}$$

The superimposed displacement field $\delta\mathbf{u}(t)=\mathbf{u}^*(t)-\mathbf{u}$ is described as

$$\delta\mathbf{u}(t)=\mathsf{u}\cos[\omega(t-\int_\ell\frac{d\ell}{c})]\mathbf{d}=\mathsf{u}\cos[\omega(t-\int_\ell\frac{d\ell}{c})]d_i\mathbf{e}^*_i,\tag{3.194}$$

where, $\mathbf{d}$ represents the unit dispalcement direction vector, c is the wave speed varying along the radial direction, and ℓ is the wave path.

For the longitudinal wave $\mathbf{d} = d_1\mathbf{e}_1^* = d_1\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_r$, the wave speed c is varying along the radius and given by

$$\begin{aligned}c^2 = &\frac{\partial\psi}{\partial I_1^*}\{2\cos^2(\bar{\theta})\sin(\bar{\theta})R\frac{\partial\bar{\theta}}{\partial R}[\frac{2R^4}{9}(\frac{\partial\bar{\theta}}{\partial R})^2 - 1 - \frac{4R^2}{9}(\frac{\partial\bar{\theta}}{\partial R})^2 - \frac{R^2}{3}]\\
&+\cos(\bar{\theta})\sin^2(\bar{\theta})[\frac{2R^2}{3} - \frac{4R^6}{9}(\frac{\partial\bar{\theta}}{\partial R})^4 - \frac{8R^4}{9}(\frac{\partial\bar{\theta}}{\partial R})^2 + \frac{4R^2}{3}(\frac{\partial\bar{\theta}}{\partial R})^2 + \frac{4R^4}{9}(\frac{\partial\bar{\theta}}{\partial R})^4 + 2]\\
&-\sin^3(\bar{\theta})\frac{\partial\bar{\theta}}{\partial R}R^3[\frac{4R^2}{9}(\frac{\partial\bar{\theta}}{\partial R})^2 + \frac{8}{3}] + \cos^3(\bar{\theta})[\frac{4R^2}{9}(\frac{\partial\bar{\theta}}{\partial R})^2 + \frac{8}{3}] + 2R\frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})\}\\
&+\frac{\partial^2\psi}{\partial I_3^{*2}}[\cos(\bar{\theta}) - R\frac{\partial\bar{\theta}}{\partial R}\sin(\bar{\theta})][\cos^2(\bar{\theta}) + R\frac{\partial\bar{\theta}}{\partial R}\cos(\bar{\theta})\sin(\bar{\theta})(R^2-1) + R^2\sin^2(\bar{\theta})]\\
&+\frac{\partial^2\psi}{\partial I_1^{*2}}\{\cos^2(\bar{\theta})\sin(\bar{\theta})(\frac{\partial\bar{\theta}}{\partial R})^3R^3[\frac{4R^2}{9}(\frac{\partial\bar{\theta}}{\partial R})^2 - \frac{2}{3} - \frac{2R^4}{9}(\frac{\partial\bar{\theta}}{\partial R})^2 + \frac{2R^2}{3}]\\
&+\frac{2R^4}{9}(\frac{\partial\bar{\theta}}{\partial R})^4[R^3\frac{\partial\bar{\theta}}{\partial R}\sin^3(\bar{\theta}) - \cos^3(\bar{\theta})]\\
&+\sin^2(\bar{\theta})\cos(\bar{\theta})(\frac{\partial\bar{\theta}}{\partial R})^2R^2[\frac{2R^2}{3}(\frac{\partial\bar{\theta}}{\partial R})^2 - \frac{2R^4}{9}(\frac{\partial\bar{\theta}}{\partial R})^4\\
&+\frac{2R^6}{9}(\frac{\partial\bar{\theta}}{\partial R})^4 - \frac{8R^4}{9}(\frac{\partial\bar{\theta}}{\partial R})^2 + 4 - 4R^2]\}.\end{aligned} \tag{3.195}$$

If no load is applied, i.e. $\bar{\theta}(R) = 0$,

$$c^2 = \frac{\partial\psi}{\partial I_1^*}\frac{8}{3} + \frac{\partial^2\psi}{\partial I_3^{*2}}. \tag{3.196}$$

For the in-plane shear wave $\mathbf{d} = d_2\mathbf{e}_2^* = \hat{\mathbf{e}}_\theta$, the wave speed c is varying along the radius and given by

$$\begin{aligned}c^2 = &\frac{\partial\psi}{\partial I_1^*}\{2\cos(\bar{\theta})\sin^2(\bar{\theta})R^2[\frac{4}{3} - \frac{2R^2}{9}(\frac{\partial\bar{\theta}}{\partial R})^2(R^2-1) - \frac{R^2}{3}] + \frac{4R^5}{3}\sin^3(\bar{\theta})\frac{\partial\bar{\theta}}{\partial R}\\
&+2\cos^2(\bar{\theta})\sin(\bar{\theta})\frac{\partial\bar{\theta}}{\partial R}R^3(R^2 - \frac{1}{3}) + 2R^2\cos^3(\bar{\theta}) - 2\frac{\partial\bar{\theta}}{\partial R}R\sin(\bar{\theta})\}\\
&+\frac{\partial^2\psi}{\partial I_1^{*2}}(\frac{\partial\bar{\theta}}{\partial R})^2\{\sin^2(\bar{\theta})\cos(\bar{\theta})R^6[4 - \frac{2}{9}(\frac{\partial\bar{\theta}}{\partial R})^2 + \frac{2R^2}{9}(\frac{\partial\bar{\theta}}{\partial R})^2]\\
&-\frac{2R^7}{3}\frac{\partial\bar{\theta}}{\partial R}\sin^3(\bar{\theta}) + 4R^4\cos^3(\bar{\theta}) - \frac{2R^5}{3}\frac{\partial\bar{\theta}}{\partial R}\cos^2(\bar{\theta})\sin(\bar{\theta})\}\\
&+\frac{\partial^2\psi}{\partial I_3^{*2}}\sin^2(\bar{\theta})\cos(\bar{\theta})R^2(1-R^2).\end{aligned} \tag{3.197}$$

If no load is applied, then

$$c^2 = 2\frac{\partial \psi}{\partial I_1^*}R^2. \tag{3.198}$$

For the out-plane shear wave $\mathbf{d} = d_3\mathbf{e}_3^* = \mathbf{e}_3^*$, the wave speed c is varying along the radius and given by

$$c = \sqrt{2\frac{\partial \psi}{\partial I_1^*}\cos(\bar{\theta})}, \tag{3.199}$$

after considering equations (3.187), we obtain

$$c = \sqrt{\frac{T}{2\rho_o \pi R^3 H \frac{\partial \bar{\theta}}{\partial R}}}. \tag{3.200}$$

If no load is applied,

$$c = \sqrt{2\frac{\partial \psi}{\partial I_1^*}}. \tag{3.201}$$

The time of flight of the waves propagating from R_o to R_i is evaluated as

$$t = \int_{R_i}^{R_o} \frac{dR}{c}. \tag{3.202}$$

Torque sensor-forward application

If the nonlinear elastic free energy $\psi(I_1^*, I_3^*)$ is known, for the wave propagation in the simple shear deformation field under the unknown torques, from the measurement of the time of flights, we can evaluate the applied torque T, which can be used as the torque sensor.

Considering $I_1^* = 3 + (\frac{\partial \bar{\theta}}{\partial R})^2 R^2$ and $I_3^* = 1$, and the relations between the material functions and the deformations given in (3.187), the deformation field $\bar{\theta}$ can be determined as a function of R and T.

After substituting the deformation $\bar{\theta}(T, R)$ into the form of time of flights for the longitudinal wave, in-plane shear wave, and out-plane shear wave, that can be measured from experiments, then the value of the torque can be calculated.

CHAPTER 4

Thermo-viscoelastic solids and wave propagation in these materials

Viscoelastic materials are those which show stress relaxation at all magnitudes of loading, even at very small loads. Models normally describing viscoelasticity show rate-dependent response and even yielding-like behavior without the use of a yield function. As such, they normally are characterized as a different class of material response. Mechanical analogs to describe the phenomenon seen in viscoelasticity are constructed from springs and viscous dampers. For a nonlinear response, the components of these analogs can be nonlinear.

We start the chapter by looking at the nonlinear viscoelastic constitutive model which is constructed from a continuous series of standard linear solids put in parallel. In this process, we consider the effects of rigid body rotation and the constraints of material symmetry. We then derive the viscoelastic constitutive models for the thermo-mechanical perturbations. As a special example, we present the equations for perturbations on a homogeneous pre-deformation, and consider the examples of attenuating and non-attenuating plane harmonic waves.

4.1 Viscoelastic solids

The results in this section are developed for a thermodynamically based nonlinear viscoelastic material model. As will be seen, the model is of single integral form, and can be shown

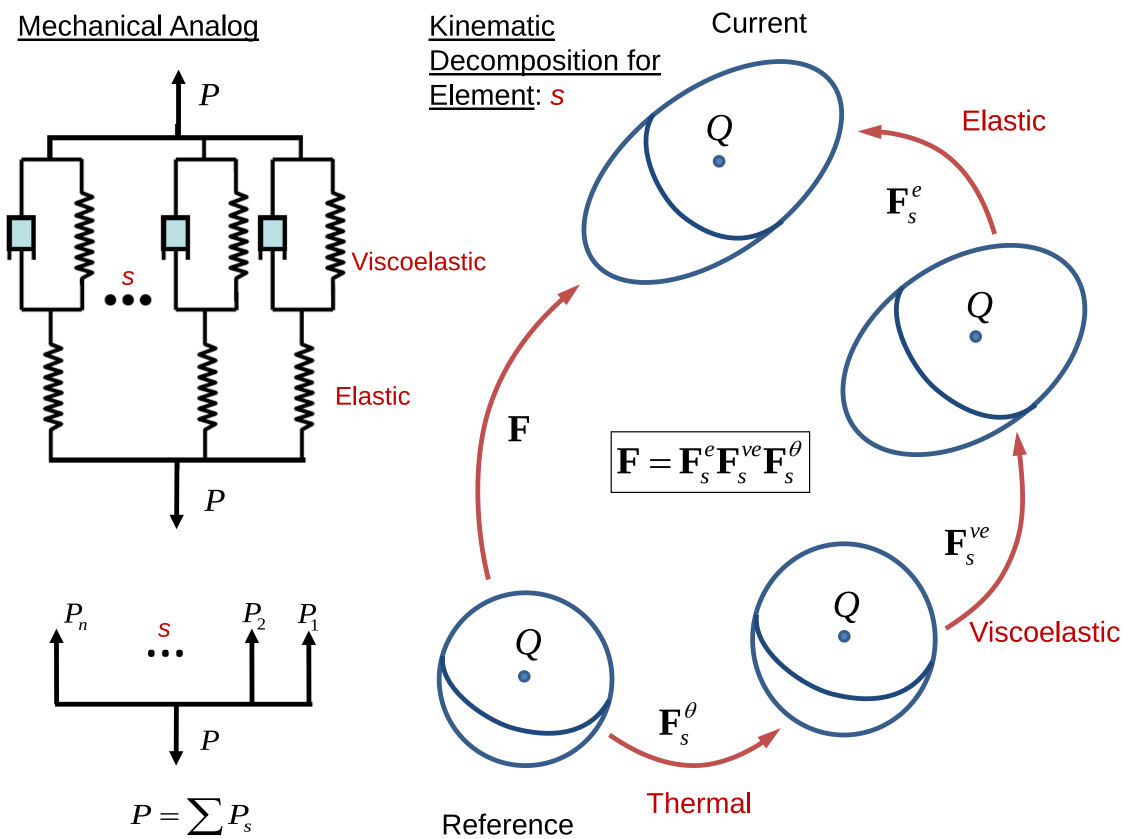


Figure 4.1: An infinite element analog constructed from elements in parallel that each separates into a thermal, viscoelastic, and elastic element in series.

to specialize at the limit of infinitesimal deformations to a general single integral linear thermodynamic viscoelastic response. The analog to the model we use is visualized by an infinite number of elements put in parallel as shown in Figure 4.1. We take s to denote a continuous variable used to parameterize these elements so that each s denotes a different element, and where $-\infty < s < \infty$. Each s is assumed to represent a nonlinear viscoelastic element, but for the purpose of understanding the model each can be visualized as the sum of a thermal, a nonlinear viscoelastic Kelvin-Voigt, and a nonlinear elastic element in series. As shown in Figure 4.1, the deformation gradient $\mathbf{F}$ for each element s is assumed to be composed of three parts given by

$$\mathbf{F} = \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}_s^{\theta}, \tag{4.1}$$

where $\mathbf{F}_s^e$ is the elastic deformation gradient, $\mathbf{F}_s^{ve}$ is the viscoelastic deformation gradient and $\mathbf{F}_s^{\theta}$ is the thermal deformation gradient, paralleling the idea proposed for the structure of each element, but in a nonlinear multiplicative decomposition. The multiplicative decomposition of the deformation gradient, originally suggested by Lee [Lee, 1969] in the context of elastoplasticity, has also been applied in nonlinear viscoelasticity by Sidoroff [Sidoroff, 1974] and others. As will be shown at the end of this section, the constitutive assumptions that follow lead to the fact that the decomposition $\mathbf{F} = \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}_s^{\theta}$ is uniquely related to the decomposition $\mathbf{F} = \mathbf{F}_s^e \overline{\mathbf{F}}_s^{\theta} \overline{\mathbf{F}}_s^{ve}$ for each given loading history. The parallel results for a theory based on the decomposition $\mathbf{F} = \mathbf{F}_s^e \overline{\mathbf{F}}_s^{\theta} \overline{\mathbf{F}}_s^{ve}$ are given in the appendix.

The response of each element s is assumed to be fully described by the values of the associated elastic, viscoelastic, and thermal deformation gradients plus the values of temperature and temperature gradient. As such, the model is assuming limited influence of elements on each other. One can define a state $\mathcal{S}_s$ for each element s as

$$\mathcal{S}_s \equiv [\mathbf{F}_s^e, \mathbf{F}_s^{ve}, \mathbf{F}_s^{\theta}, \theta, \mathbf{G}], \tag{4.2}$$

which fully describes the response of the element. As is commonly done, we start by assuming equal presence of state variables in all the response functions so that we take the response functions for the specific free energy ψ, the specific entropy η, the Cauchy stress $\mathbf{T}$ and the heat flux vector $\mathbf{q}$ given by the equations

$$\begin{aligned}
\psi_s(t) &\equiv \psi_s^{\dagger}[\mathbf{F}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^{\theta}(t), \theta(t), \mathbf{G}(t)], \\
\eta_s(t) &\equiv \eta_s^{\dagger}[\mathbf{F}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^{\theta}(t), \theta(t), \mathbf{G}(t)], \\
\mathbf{T}_s(t) &\equiv \mathbf{T}_s^{\dagger}[\mathbf{F}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^{\theta}(t), \theta(t), \mathbf{G}(t)], \\
\mathbf{q}_s(t) &\equiv \mathbf{q}_s^{\dagger}[\mathbf{F}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^{\theta}(t), \theta(t), \mathbf{G}(t)],
\end{aligned} \tag{4.3}$$

where the superscript "†" denotes the constitutive function used to calculate the specific quantity. We next assume the total response is the sum of the response in the elements to obtain the single integral models

$$
\begin{aligned}
\psi(t) &\equiv \int_{-\infty}^{\infty} \psi_s(t) ds, \\
\eta(t) &\equiv \int_{-\infty}^{\infty} \eta_s(t) ds, \\
\mathbf{T}(t) &\equiv \int_{-\infty}^{\infty} \mathbf{T}_s(t) ds, \\
\mathbf{q}(t) &\equiv \int_{-\infty}^{\infty} \mathbf{q}_s(t) ds.
\end{aligned} \tag{4.4}
$$

As in common, we require the material response to satisfy the second law of thermodynamics. This was given in the form of the Clausius-Duhem inequality in equation (2.20), that must hold for all admissible processes. For this equation we need the expression of the rate of change of the free energy. We can calculate this from the assumed constitutive form and write

$$
\dot{\psi}(t) = \int_{-\infty}^{\infty} \dot{\psi}_s(t) ds, \tag{4.5}
$$

where

$$
\dot{\psi}_s(t) = \partial_{\mathbf{F}_s^e}(\psi_s^\dagger) : \dot{\mathbf{F}}_s^e + \partial_{\mathbf{F}_s^{ve}}(\psi_s^\dagger) : \dot{\mathbf{F}}_s^{ve} + \partial_{\mathbf{F}_s^\theta}(\psi_s^\dagger) : \dot{\mathbf{F}}_s^\theta + \partial_\theta(\psi_s^\dagger)\dot{\theta} + \partial_{\mathbf{G}}(\psi_s^\dagger) \circ \dot{\mathbf{G}}. \tag{4.6}
$$

For convenience, we define, respectively, the thermodynamic elastic, back, and thermal stresses as

$$
\begin{aligned}
\mathbf{T}_s^{eT} &\equiv \rho \partial_{\mathbf{F}_s^e}(\psi_s^\dagger) \mathbf{F}_s^{eT}, \\
\mathbf{T}_s^{bT} &\equiv \rho \partial_{\mathbf{F}_s^{ve}}(\psi_s^\dagger) \mathbf{F}_s^{veT}, \\
\mathbf{T}_s^{\theta T} &\equiv \rho \partial_{\mathbf{F}_s^\theta}(\psi_s^\dagger) \mathbf{F}_s^{\theta T},
\end{aligned} \tag{4.7}
$$

and, respectively, define the over stresses from the back and thermal stresses as

$$
\Delta \mathbf{T}_s^b \equiv \mathbf{F}_s^{e-1} \mathbf{T}_s^e \mathbf{F}_s^e - \mathbf{T}_s^b, \tag{4.8}
$$

$$
\Delta \mathbf{T}_s^\theta \equiv \mathbf{F}_s^{ve-1} \mathbf{F}_s^{e-1} \mathbf{T}_s^e \mathbf{F}_s^e \mathbf{F}_s^{ve} - \mathbf{T}_s^\theta. \tag{4.9}
$$

We also use the identity

$$\mathbf{L}_s^e = \mathbf{L} - \mathbf{F}_s^e \mathbf{L}_s^{ve} \mathbf{F}_s^{e-1} - \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{L}_s^\theta \mathbf{F}_s^{ve-1} \mathbf{F}_s^{e-1}, \tag{4.10}$$

where $\mathbf{L}_s^e \equiv \dot{\mathbf{F}}_s^e \mathbf{F}_s^{e-1}$, $\mathbf{L}_s^{ve} \equiv \dot{\mathbf{F}}_s^{ve} \mathbf{F}_s^{ve-1}$, and $\mathbf{L}_s^\theta \equiv \dot{\mathbf{F}}_s^\theta \mathbf{F}_s^{\theta-1}$. Once we introduce all these into the Clausius-Duhem inequality, we arrive at

$$\begin{aligned}[\int_{-\infty}^{\infty} \mathbf{T}_s^{eT} ds - \mathbf{T}^T] : \mathbf{L} - \int_{-\infty}^{\infty} \Delta \mathbf{T}_s^{bT} : \mathbf{L}_s^{ve} ds \\ - \int_{-\infty}^{\infty} \Delta \mathbf{T}_s^{\theta T} : \mathbf{L}_s^\theta ds + \rho \dot{\theta} \int_{-\infty}^{\infty} [\eta_s + \partial_\theta(\psi_s^\dagger)] ds + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0,\end{aligned} \tag{4.11}$$

which must hold for all admissible processes. We add one additional assumption by taking the form of thermal expansion as

$$\mathbf{L}_s^\theta \equiv \boldsymbol{\alpha}_s \dot{\theta}, \tag{4.12}$$

where $\boldsymbol{\alpha}_s$, a function of the state $\mathcal{S}_s$, represents the coefficient of thermal expansion for element s. A sufficient, but not too restrictive, condition to satisfy the Clausius-Duhem inequality is then given by assuming each element is dissipative and that its free energy does not depend on the temperature gradient. This sufficient condition can be written as

$$\begin{aligned}&\mathbf{T} = \int_{-\infty}^{\infty} \mathbf{T}_s^e ds, \\ &\eta = -\int_{-\infty}^{\infty} [\partial_\theta(\psi_s^\dagger) - \frac{1}{\rho} \Delta \mathbf{T}_s^{\theta T} : \boldsymbol{\alpha}_s] ds, \\ &- \Delta \mathbf{T}_s^{bT} : \mathbf{L}_s^{ve} \leq 0, \\ &\frac{1}{\theta} \mathbf{q}_s \circ \mathbf{g} \leq 0.\end{aligned} \tag{4.13}$$

A fairly general viscoelastic model can be developed by taking for each element a nonlinear evolution equation (flow law) of the form

$$\mathbf{L}_s^{ve} \equiv \mathsf{C}_s : \Delta \mathbf{T}_s^{bT}, \tag{4.14}$$

where C_s is a fourth order coefficient function depending on the state $\mathcal{S}_s$ at the current time, excluding dependence on the temperature gradient. This is the form we consider for our study, which, from the Clausius-Duhem inequality, must always satisfy

$$\Delta \mathbf{T}_s^{bT} : \mathbf{C}_s : \Delta \mathbf{T}_s^{bT} \geq 0, \tag{4.15}$$

for every admissible over stress $\Delta \mathbf{T}_s^b$. In our development we also leave off the dependence on the temperature gradient for all variables other than the heat flux vector.

A simplification of the model for a finite number of elements can be easily obtained by changing the integrals to sums over a finite number of individual elements. Also, addition of a thermo-elastic element in parallel with the continuous system can easily express the equilibrium stress for systems that do not include a back stress in the elements (i.e., systems that have elements that resemble the Maxwell fluid).

There are some possible choices available in the literature for modeling the evolutions of viscoelastic flows. For example, for the case of isotropy, Lion [Lion, 1997] and Reese and Govindjee [Reese and Govindjee, 1998, Reese and Govindjee, 1997] proposed thermodynamical consistent constitutive models for finite deformation viscoelasticity that utilize nonlinear evolution laws by employing nonlinear viscosities which can be shown to be special forms of the viscoelastic flow coefficient function $\mathbf{C}_s$ we introduced above. Their models are based on an additive split of the Helmholtz free energy into an equilibrium and a non-equilibrium part and multiplicative decomposition of the deformation gradient $\mathbf{F}$ into an elastic part $\mathbf{F}_e$ and an inelastic part $\mathbf{F}_i$ as $\mathbf{F} = \mathbf{F}_e \mathbf{F}_i$, where $\mathbf{F}_i$ is the internal parameter to be determined from the evolution equations.

The assumptions we used to create the viscoelasticity theory have a special structure that relates the models formulated based on the decomposition $\mathbf{F} = \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}_s^\theta$ to models constructed based on the decomposition $\mathbf{F} = \mathbf{F}_s^e \overline{\mathbf{F}}_s^\theta \overline{\mathbf{F}}_s^{ve}$. To see this relation, consider having two theories that start from similar assumptions, but one is based on the former decomposition and the other is based on the latter. For each we would have a flow rule and thermal expansion rule. The flow rules for $\mathbf{L}_s^{ve}$ and $\overline{\mathbf{L}}_s^{ve}$ would each be given by the associated variables defining the state, but would not depend on the rates. The thermal expansions would be given by

$$\mathbf{L}_s^{\theta} = \boldsymbol{\alpha}_s \dot{\theta}, \tag{4.16}$$

$$\overline{\mathbf{L}}_s^{\theta} = \overline{\boldsymbol{\alpha}}_s \dot{\theta}, \tag{4.17}$$

where, again, $\boldsymbol{\alpha}_s$ and $\overline{\boldsymbol{\alpha}}_s$ would each be given by the associated variables defining the state, but would not depend on the rates. If we now consider the relation $\mathbf{F}_s^{ve}\mathbf{F}_s^{\theta} = \overline{\mathbf{F}}_s^{\theta}\overline{\mathbf{F}}_s^{ve}$ and take its time derivative, we would get

$$\dot{\mathbf{F}}_s^{ve}\mathbf{F}_s^{\theta} + \mathbf{F}_s^{ve}\dot{\mathbf{F}}_s^{\theta} = \dot{\overline{\mathbf{F}}}_s^{\theta}\overline{\mathbf{F}}_s^{ve} + \overline{\mathbf{F}}_s^{\theta}\dot{\overline{\mathbf{F}}}_s^{ve}, \tag{4.18}$$

which could be rewritten as

$$\mathbf{L}_s^{ve}\mathbf{F}_s^{ve}\mathbf{F}_s^{\theta} + \mathbf{F}_s^{ve}\mathbf{L}_s^{\theta}\mathbf{F}_s^{\theta} = \overline{\mathbf{L}}_s^{\theta}\overline{\mathbf{F}}_s^{\theta}\overline{\mathbf{F}}_s^{ve} + \overline{\mathbf{F}}_s^{\theta}\overline{\mathbf{L}}_s^{ve}\overline{\mathbf{F}}_s^{ve}. \tag{4.19}$$

We can now introduce the thermal expansion rules to get

$$\mathbf{L}_s^{ve}\mathbf{F}_s^{ve}\mathbf{F}_s^{\theta} + \mathbf{F}_s^{ve}\boldsymbol{\alpha}_s\mathbf{F}_s^{\theta}\dot{\theta} = \overline{\boldsymbol{\alpha}}_s\overline{\mathbf{F}}_s^{\theta}\overline{\mathbf{F}}_s^{ve}\dot{\theta} + \overline{\mathbf{F}}_s^{\theta}\overline{\mathbf{L}}_s^{ve}\overline{\mathbf{F}}_s^{ve}. \tag{4.20}$$

If we examine the terms, only the second and third terms depend on the rate of loading, which in this case is characterized by the temperature rate. If we assume the temperature rate can be selected arbitrarily, then the relation can only be satisfied if the first term equals the fourth term, and the second term equals the third term. This gives

$$\mathbf{L}_s^{ve}\mathbf{F}_s^{ve}\mathbf{F}_s^{\theta} = \overline{\mathbf{F}}_s^{\theta}\overline{\mathbf{L}}_s^{ve}\overline{\mathbf{F}}_s^{ve}, \tag{4.21}$$

$$\mathbf{F}_s^{ve}\boldsymbol{\alpha}_s\mathbf{F}_s^{\theta} = \overline{\boldsymbol{\alpha}}_s\overline{\mathbf{F}}_s^{\theta}\overline{\mathbf{F}}_s^{ve}, \tag{4.22}$$

and can be reorganized to get the relations

$$\overline{\boldsymbol{\alpha}}_s = \mathbf{F}_s^{ve}\boldsymbol{\alpha}_s\mathbf{F}_s^{ve-1}, \tag{4.23}$$

$$\overline{\mathbf{F}}_s^{\theta}\overline{\mathbf{L}}_s^{ve}\overline{\mathbf{F}}_s^{\theta-1} = \mathbf{L}_s^{ve}. \tag{4.24}$$

Examination of the relations brings us to the conclusion that if one constructs the thermal expansion and flow rule for one assumed decomposition, then the values of the thermal expansion and flow rule for the other decomposition are uniquely defined. Since we can integrate these equations for each given loading history to get the corresponding thermal and viscoelastic deformation gradients, this result states that for each loading history the two decompositions are uniquely related. Therefore, models based on one decomposition will be closely related to models based on the other, and would use state parameters that could be calculated, even though not algebraically, from each other.

4.2 Superposition of rigid body rotations

How the "internal parameters" $\mathbf{F}_s^{ve}$ and $\mathbf{F}_s^{\theta}$ are influenced by rigid body motions strongly influences how the resulting thermodynamic model behaves. In particular, it can be shown that certain selections result in expressions that are consistent with other imposed conditions, such as the symmetry of Cauchy stress that results from the balance of angular momentum. This section will look at one such selection and its influence on the resulting expressions.

Let us look at the expected effect of imposing rigid body rotations on the history. As can be seen in Fig. 4.2, pure rigid body rotations move points on the body on arcs of spheres defined by the particular point on the body, the center of rotation, and the specified rotation. Given the motion of a body as $\mathbf{x}(\mathbf{X}, t)$, a motion derived from this by superimposing rigid body rotations about $\mathbf{x}_o(s)$ can be written as

$$\mathbf{x}^*(\mathbf{X}, s) = \mathbf{Q}(s)[\mathbf{x}(\mathbf{X}, s) - \mathbf{x}_o(s)] + \mathbf{x}_o(s), \tag{4.25}$$

where $\mathbf{Q}(s)$ is an orthogonal tensor representing a rigid body rotation, and s takes values from the starting time t_o to the current time t. Consider a general first gradient material having a response $\mathcal{R}(\mathbf{X}, t)$ given by

$$\mathcal{R}(\mathbf{X}, t) = \{\psi(\mathbf{X}, t), \eta(\mathbf{X}, t), \mathbf{T}(\mathbf{X}, t), \mathbf{q}(\mathbf{X}, t)\}, \tag{4.26}$$

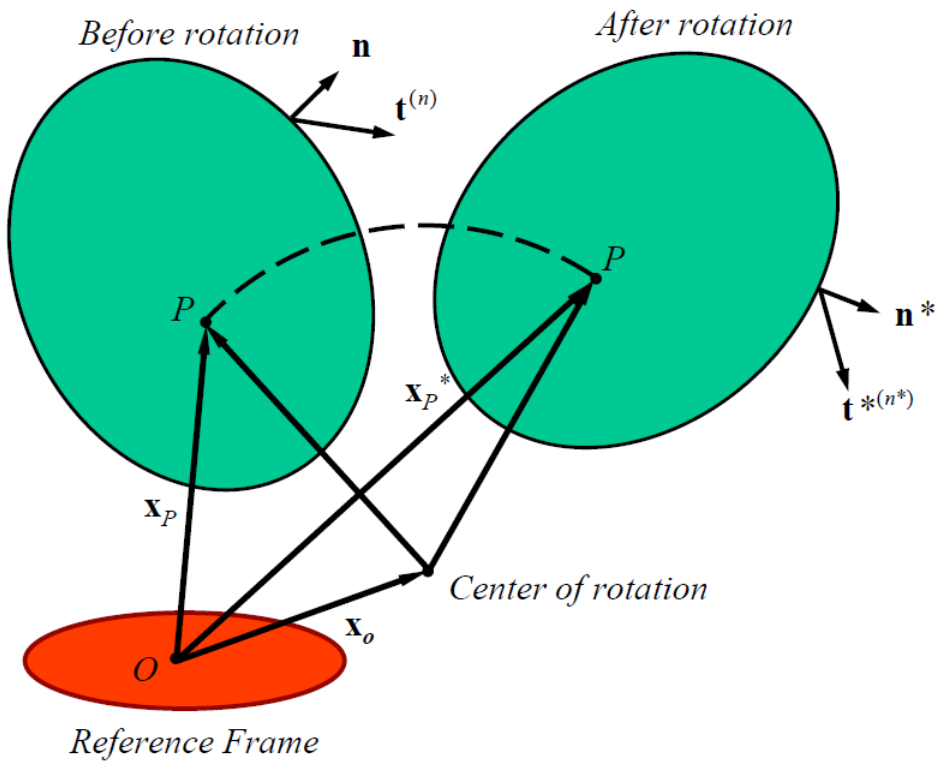


Figure 4.2: Rigid body rotation of the current configuration by Negahban [Negahban, 2012].

when subjected to the history described by $\mathbf{x}(\mathbf{X}, s)$ and the response $\mathcal{R}^*(\mathbf{X}, t)$ given by

$$\mathcal{R}^*(\mathbf{X}, t) = \{\psi^*(\mathbf{X}, t), \eta^*(\mathbf{X}, t), \mathbf{T}^*(\mathbf{X}, t), \mathbf{q}^*(\mathbf{X}, t)\}, \tag{4.27}$$

when subjected to the history described by $\mathbf{x}^*(\mathbf{X}, s)$. It would be expected that the rigid body rotations should not alter the value of the specific free energy and specific entropy. The influence of rigid body motions on the stress is normally dictated by the influence of rigid body motions on the traction. Normally, a rigid body motion is assumed to reorient the traction by the amount of the rigid body rotation. As shown in Fig. 4.3, if the body is rotated by a rigid body rotation given by the orthogonal transformation $\mathbf{Q}$, then any normal $\hat{\mathbf{n}}$ transforms to $\hat{\mathbf{n}}^* = \mathbf{Q}\hat{\mathbf{n}}$ and the traction on the surface with normal $\hat{\mathbf{n}}$ given by $\mathbf{t}^n$ changes to $\mathbf{t}^{*(n^*)} = \mathbf{Q}\mathbf{t}^{(n)}$. This can be shown to require that the Cauchy stress $\mathbf{T}$ changes to the Cauchy stress $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$. As for the traction, the heat flux vector is assumed to rotate by the amount of the rigid body rotation so that we can write

$$R^*(\mathbf{X}, t) = \{\psi(\mathbf{X}, t), \eta(\mathbf{X}, t), \mathbf{Q}(t)\mathbf{T}(\mathbf{X}, t)\mathbf{Q}^T(t), \mathbf{Q}(t)\mathbf{q}(\mathbf{X}, t)\}. \tag{4.28}$$

The general response functional $\mathcal{R}^\dagger$ for the first gradient material can be written as

$$\mathcal{R}(\mathbf{X}, t) = \mathcal{R}^\dagger \left[\mathbf{X}, \mathbf{x}(\mathbf{X}, s), \mathbf{F}(\mathbf{X}, \underset{s=t_o}{\overset{t}{s}}), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s) \right]. \tag{4.29}$$

This same response functional should give $\mathcal{R}^*(\mathbf{X}, t)$ when introducing into it the "$*$" history so that we should have

$$\mathcal{R}^*(\mathbf{X}, t) = \mathcal{R}^\dagger \left[\mathbf{X}, \mathbf{Q}(s)[\mathbf{x}(\mathbf{X}, s) - \mathbf{x}_c(s)] + \mathbf{x}_c(\underset{s=t_o}{\overset{t}{s}}), \mathbf{Q}(s)\mathbf{F}(\mathbf{X}, s), \theta(\mathbf{X}, s), \mathbf{G}(\mathbf{X}, s) \right]. \tag{4.30}$$

Assumptions of invariance to rigid body motions is given by the assumption that when the history of the deformation is transformed through $\mathbf{F}(t) \to \mathbf{F}^*(t) = \mathbf{Q}(t)\mathbf{F}(t)$, and the internal parameters are unaffected by rigid body motions superimposed and follow the rules

$$\begin{aligned} &\mathbf{F}_s^e(t) \to \mathbf{F}_s^{e*}(t) = \mathbf{Q}(t)\mathbf{F}_s^e(t), \\ &\mathbf{F}_s^{ve}(t) \to \mathbf{F}_s^{ve*}(t) = \mathbf{F}_s^{ve}(t), \\ &\mathbf{F}_s^\theta(t) \to \mathbf{F}_s^{\theta*}(t) = \mathbf{F}_s^\theta(t), \end{aligned} \tag{4.31}$$

which is consistent with the assumption $\mathbf{F}^* = \mathbf{F}_s^{e*}\mathbf{F}_s^{ve*}\mathbf{F}_s^{\theta*} = \mathbf{Q}\mathbf{F}$.

We postulate that the free energy is not affected by rigid body motions and follows the rule

$$\psi_s(t) \to \psi_s^*(t) = \psi_s(t), \tag{4.32}$$

which also gives the rule

$$\eta_s(t) \to \eta_s^*(t) = \eta_s(t). \tag{4.33}$$

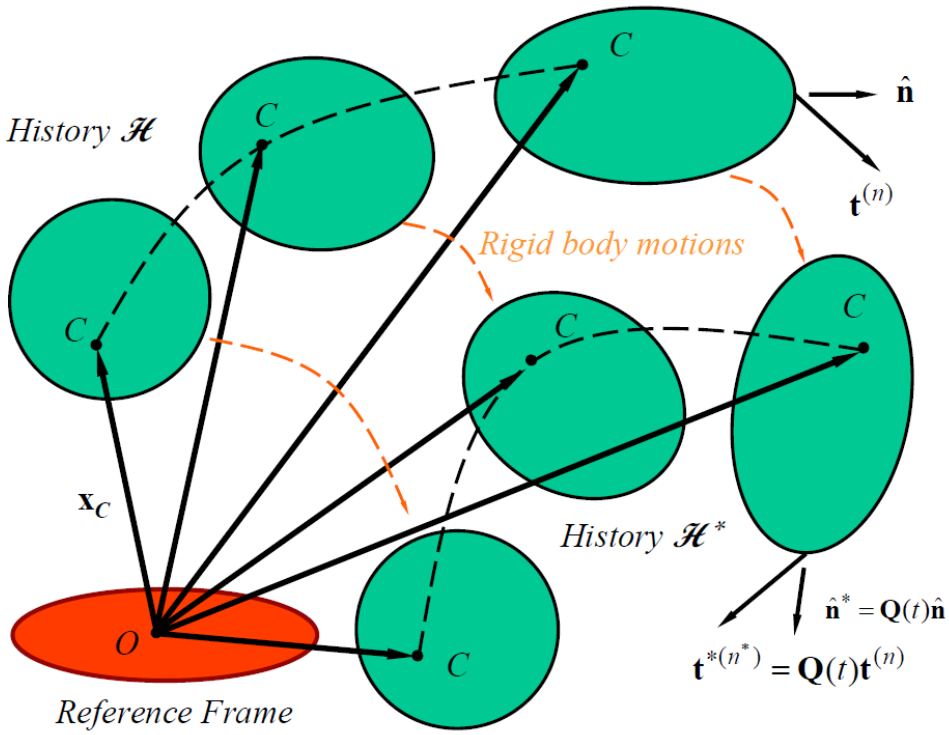


Figure 4.3: Two deformation histories that are identical but for the fact that each configuration in one is obtained by an arbitrary rigid body translation and rotation of the configuration of other is assumed to simply rotate the traction vector by the final amount of rotation by Negahban [Negahban, 2012].

The thermodynamic stresses in each element s have the properties

$$\begin{aligned} \mathbf{T}_s^e(t) &\rightarrow \mathbf{T}_s^{e*}(t) = \mathbf{Q}(t)\mathbf{T}_s^e(t)\mathbf{Q}^T(t), \\ \mathbf{T}_s^b(t) &\rightarrow \mathbf{T}_s^{b*}(t) = \mathbf{T}_s^b(t), \\ \mathbf{T}_s^\theta(t) &\rightarrow \mathbf{T}_s^{\theta*}(t) = \mathbf{T}_s^\theta(t), \end{aligned} \tag{4.34}$$

which result in the over-stresses in each element s having the properties

$$\begin{aligned} \Delta\mathbf{T}_s^b(t) &\rightarrow \Delta\mathbf{T}_s^{b*}(t) = \Delta\mathbf{T}_s^b(t), \\ \Delta\mathbf{T}_s^\theta(t) &\rightarrow \Delta\mathbf{T}_s^{\theta*}(t) = \Delta\mathbf{T}_s^\theta(t). \end{aligned} \tag{4.35}$$

The viscoelastic flow rule coefficient and the thermal expansion coefficient in each element s follow the rule

$$\begin{aligned} \mathbf{C}_s(t) &\rightarrow \mathbf{C}_s^*(t) = \mathbf{C}_s(t), \\ \boldsymbol{\alpha}_s(t) &\rightarrow \boldsymbol{\alpha}_s^*(t) = \boldsymbol{\alpha}_s(t). \end{aligned} \tag{4.36}$$

Since $\mathbf{Q}(s)$ is an arbitrary rotation, we can select $\mathbf{Q}(s) = \mathbf{R}_s^{eT}(s)$, where $\mathbf{R}_s^e$ is the orthogonal part in the polar decomposition of $\mathbf{F}_s^e(s) = \mathbf{R}_s^e(s)\mathbf{U}_s^e(s)$. This selection requires that the free energy in each element s has the property

$$\psi_s^\dagger[\mathbf{F}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)] = \psi_s^\dagger[\mathbf{U}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)], \tag{4.37}$$

and requires that the viscoelastic flow rule coefficient and the thermal expansion coefficient in each element s have the property

$$\begin{aligned} \mathbf{C}_s^\dagger[\mathbf{F}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)] &= \mathbf{C}_s^\dagger[\mathbf{U}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)], \\ \boldsymbol{\alpha}_s^\dagger[\mathbf{F}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)] &= \boldsymbol{\alpha}_s^\dagger[\mathbf{U}_s^e(t), \mathbf{F}_s^{ve}(t), \mathbf{F}_s^\theta(t), \theta(t)]. \end{aligned} \tag{4.38}$$

4.3 Material symmetry constraints

We next will look at imposing material symmetry on the constitutive functions. Symmetry in the material's response can substantially limit the possible forms of the constitutive equations and, therefore, simplify the process of characterizing the material response. The basic idea behind material symmetry is introduced in Chapter 3. In this section we will focus on imposing material symmetry in the context of the current viscoelastic constitutive model. We will let $\mathbf{M}$ denote a general transformation representing a symmetry of the material at a material point of consideration. As was discussed before, $\mathbf{M}$ is a transformation that represents a reorganization of the neighborhood of the given point in the reference configuration, and this transformation changes the neighborhood in a way that leaves it thermodynamically indistinguishable from before reorganization. A symmetry of the material represented by the transformation $\mathbf{M}$ allows us to change the history of deformation gradient $\mathbf{F}(t)$ to $\mathbf{F}(t)\mathbf{M}$ for any deformation gradient history without seeing a change in the response of the material at that point. We denote this change of history by a "$\sim$" over the symbol and write it as $\mathbf{F}(t) \to \tilde{\mathbf{F}}(t) = \mathbf{F}(t)\mathbf{M}$. To construct a consistent theory, we need to have the kinematic variables $\mathbf{F}_s^e$, $\mathbf{F}_s^{ve}$, and $\mathbf{F}_s^\theta$ change in a way that is consistent with $\mathbf{F}(t) \to \tilde{\mathbf{F}}(t) = \mathbf{F}(t)\mathbf{M}$. To create such a consistent relation, we will assume the internal parameters transform in response to this change by

$$\begin{aligned}
&\mathbf{F}_s^e(t) \to \tilde{\mathbf{F}}_s^e(t) = \mathbf{F}_s^e(t)\mathbf{M}, \\
&\mathbf{F}_s^{ve}(t) \to \tilde{\mathbf{F}}_s^{ve}(t) = \mathbf{M}^{-1}\mathbf{F}_s^{ve}(t)\mathbf{M}, \\
&\mathbf{F}_s^\theta(t) \to \tilde{\mathbf{F}}_s^\theta(t) = \mathbf{M}^{-1}\mathbf{F}_s^\theta(t)\mathbf{M},
\end{aligned} \tag{4.39}$$

which is consistent with the condition $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_s^e\tilde{\mathbf{F}}_s^{ve}\tilde{\mathbf{F}}_s^\theta = \mathbf{F}\mathbf{M}$. The assumption that the free energy of each element s is the same in value for a change in the deformation history described by $\mathbf{F} \to \mathbf{F}\mathbf{M}$ can be written as

$$\psi_s(t) \to \tilde{\psi}_s(t) = \psi_s(t), \tag{4.40}$$

and giving the rule for specific entropy

$$\eta_s(t) \to \tilde{\eta}_s(t) = \eta_s(t). \tag{4.41}$$

A consistent set of assumptions on how the thermodynamic stresses change are given by the rules

$$\begin{aligned}
&\mathbf{T}_s^e(t) \to \tilde{\mathbf{T}}_s^e(t) = \mathbf{T}_s^e(t), \\
&\mathbf{T}_s^b(t) \to \tilde{\mathbf{T}}_s^b(t) = \mathbf{M}^{-1}\mathbf{T}_s^b(t)\mathbf{M}, \\
&\mathbf{T}_s^\theta(t) \to \tilde{\mathbf{T}}_s^\theta(t) = \mathbf{M}^{-1}\mathbf{T}_s^\theta(t)\mathbf{M},
\end{aligned} \tag{4.42}$$

which result in the thermodynamic over-stresses change in a similar way

$$\Delta\mathbf{T}_s^b(t) \rightarrow \Delta\tilde{\mathbf{T}}_s^b(t) = \mathbf{M}^{-1}\Delta\mathbf{T}_s^b(t)\mathbf{M},$$
$$\Delta\mathbf{T}_s^\theta(t) \rightarrow \Delta\tilde{\mathbf{T}}_s^\theta(t) = \mathbf{M}^{-1}\Delta\mathbf{T}_s^\theta(t)\mathbf{M}. \quad (4.43)$$

The restriction imposed by material symmetry on the viscoelastic flow coefficient then becomes

$$\tilde{\mathsf{C}}_s : (\mathbf{M}^T\Delta\mathbf{T}_s^{bT}\mathbf{M}^{-T}) = \mathbf{M}^{-1}(\mathsf{C}_s : \Delta\mathbf{T}_s^{bT})\mathbf{M}, \quad (4.44)$$

and the constraint on the coefficient of thermal expansion becomes

$$\boldsymbol{\alpha}_s(t) \rightarrow \tilde{\boldsymbol{\alpha}}_s(t) = \mathbf{M}^{-1}\boldsymbol{\alpha}_s(t)\mathbf{M}. \quad (4.45)$$

A full thermodynamic multi-dimensional model would be obtained once we specify a function for the specific free energy, the viscoelastic velocity gradient, the thermal expansion coefficient, and the heat flux, keeping in mind that in all processes $\mathbf{L}_s^{ve}$ and $\mathbf{q}_s$ need to satisfy the thermodynamic constraints.

4.4 Constitutive models for the perturbations

In Chapter 2 we developed the balance laws and their boundary and jump conditions for the perturbations. We will now evaluate the constitutive equations for these perturbations. We first start by decomposing the perturbation into elastic, viscoelastic and thermal parts, then derive the perturbation term for the stress, followed by those for the free-energy, entropy, and heat flux. Finally, we evaluate the evolution rules for the perturbations of the internal parameters and then integrate them to obtain a general solution.

4.4.1 Decomposition of the perturbation into elastic, viscoelastic, and thermal parts

In the constitutive model both the base deformation and total deformation are each separated into elastic, viscoelastic, and thermal parts. The constitutive model provides evolution equations for the viscoelastic and thermal parts of each and so in each case leaves the elastic part to be matched against the applied stress. As will be shown here, we can also setup the perturbation to have the same form.

For the base history the deformation gradient $\mathbf{F}(t)$ is decomposed into the elastic deformation gradient $\mathbf{F}^e_s(t)$, the viscoelastic deformation gradient $\mathbf{F}^{ve}_s(t)$ and the thermal deformation gradient $\mathbf{F}^\theta_s(t)$ so that

$$\mathbf{F}(t) = \mathbf{F}^e_s(t)\mathbf{F}^{ve}_s(t)\mathbf{F}^\theta_s(t). \tag{4.46}$$

For the total history, the total deformation gradient $\mathbf{F}^*(t)$ is also decomposed into the elastic deformation gradient $\mathbf{F}^{*e}_s(t)$, the viscoelastic deformation gradient $\mathbf{F}^{*ve}_s(t)$ and the thermal deformation gradient $\mathbf{F}^{*\theta}_s(t)$ so that

$$\mathbf{F}^*(t) = \mathbf{F}^{*e}_s(t)\mathbf{F}^{*ve}_s(t)\mathbf{F}^{*\theta}_s(t). \tag{4.47}$$

The relations of the elastic, viscoelastic and thermal parts in the base history and in the total history are described in Figure 4.4, where, $\tilde{\mathbf{F}}^e_s(t)$, $\tilde{\mathbf{F}}^{ve}_s(t)$ and $\tilde{\mathbf{F}}^\theta_s(t)$ are the relative "increment" of elastic, viscoelastic and thermal deformation gradients comparing each pair. The actual change in these deformation gradients from the base history to the total history is given by

$$\begin{aligned}
\delta\mathbf{F}^e_s(t) &= \mathbf{F}^{*e}_s(t) - \mathbf{F}^e_s(t) = \tilde{\mathbf{F}}^e_s(t)\mathbf{F}^e_s(t) - \mathbf{F}^e_s(t) = \tilde{\mathbf{H}}^e_s(t)\mathbf{F}^e_s(t),\\
\delta\mathbf{F}^{ve}_s(t) &= \mathbf{F}^{*ve}_s(t) - \mathbf{F}^{ve}_s(t) = \tilde{\mathbf{F}}^{ve}_s(t)\mathbf{F}^{ve}_s(t) - \mathbf{F}^{ve}_s(t) = \tilde{\mathbf{H}}^{ve}_s(t)\mathbf{F}^{ve}_s(t),\\
\delta\mathbf{F}^\theta_s(t) &= \mathbf{F}^{*\theta}_s(t) - \mathbf{F}^\theta_s(t) = \tilde{\mathbf{F}}^\theta_s(t)\mathbf{F}^\theta_s(t) - \mathbf{F}^\theta_s(t) = \tilde{\mathbf{H}}^\theta_s(t)\mathbf{F}^\theta_s(t),
\end{aligned} \tag{4.48}$$

where $\tilde{\mathbf{H}}^e_s(t) = \tilde{\mathbf{F}}^e_s(t) - \mathbf{I}$ is the displacement gradient for the elastic perturbation, $\tilde{\mathbf{H}}^{ve}_s(t) = \tilde{\mathbf{F}}^{ve}_s(t) - \mathbf{I}$ is the displacement gradient for the viscoelastic perturbation, and $\tilde{\mathbf{H}}^\theta_s(t) = \tilde{\mathbf{F}}^\theta_s(t) - \mathbf{I}$ is the displacement gradient for the thermal perturbation. As the perturbation is small, it can be shown that the increments represent small differences so that the deformation gradients are close to $\mathbf{I}$. We can also define a direct separation of the perturbed deformation gradients $\check{\mathbf{F}}(t) = \check{\mathbf{F}}^e_s(t)\check{\mathbf{F}}^{ve}_s(t)\check{\mathbf{F}}^\theta_s(t)$ and obtain their relations with the "$\sim$" variables as shown

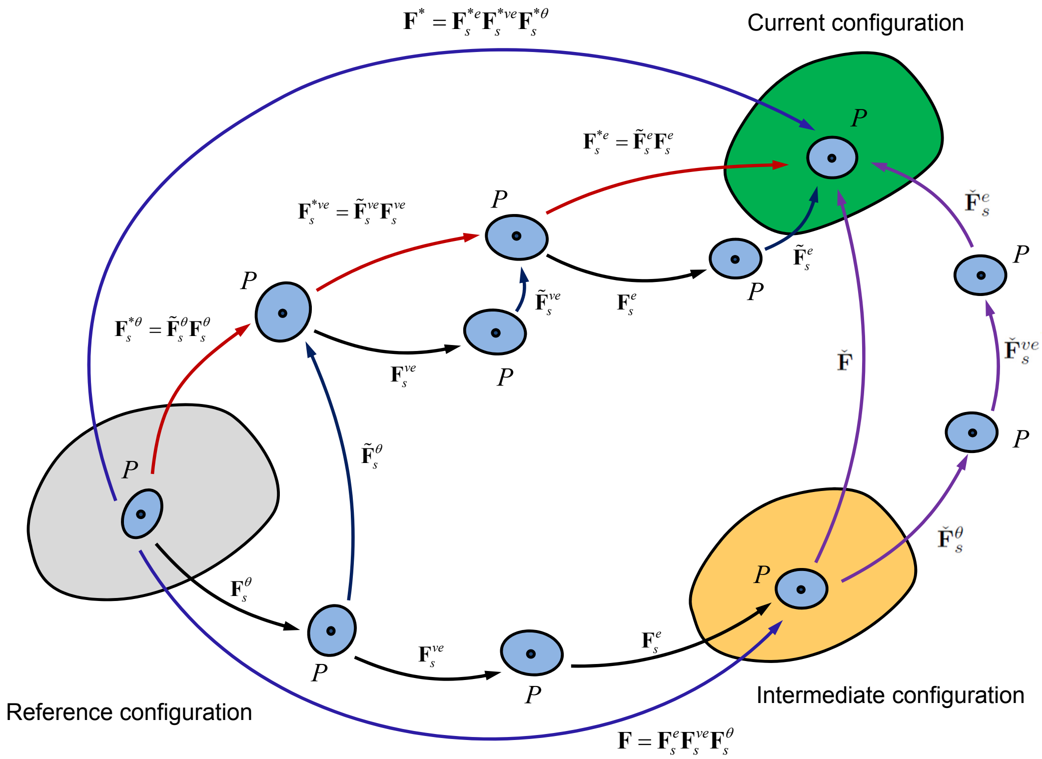


Figure 4.4: Decompositions of the perturbation into elastic, viscoelastic, and thermal parts [Zhang and Negahban, 2012].

in Figure 4.4. This allows us to define a consistent set of relations given by

$$
\begin{aligned}
&\check{\mathbf{F}}_s^{e}(t) \equiv \tilde{\mathbf{F}}_s^{e}(t), \\
&\check{\mathbf{F}}_s^{ve}(t) \equiv \mathbf{F}_s^{e}(t)\tilde{\mathbf{F}}_s^{ve}(t)\mathbf{F}_s^{e-1}(t), \\
&\check{\mathbf{F}}_s^{\theta}(t) \equiv \mathbf{F}_s^{e}(t)\mathbf{F}_s^{ve}(t)\tilde{\mathbf{F}}_s^{\theta}(t)\mathbf{F}_s^{ve-1}(t)\mathbf{F}_s^{e-1}(t).
\end{aligned}
\tag{4.49}
$$

We can introduce the displacement gradient for $\check{\mathbf{F}}_s^{e}(t) = \mathbf{I} + \check{\mathbf{H}}_s^{e}(t)$, $\check{\mathbf{F}}_s^{ve}(t) = \mathbf{I} + \check{\mathbf{H}}_s^{ve}(t)$, and

$\check{\mathbf{F}}_s^\theta(t) = \mathbf{I} + \check{\mathbf{H}}_s^\theta(t)$. Solving for these we get

$$\begin{aligned}
\check{\mathbf{H}}_s^e(t) &= \tilde{\mathbf{H}}_s^e(t), \\
\check{\mathbf{H}}_s^{ve}(t) &= \mathbf{F}_s^e(t)\tilde{\mathbf{H}}_s^{ve}(t)\mathbf{F}_s^{e-1}(t), \\
\check{\mathbf{H}}_s^\theta(t) &= \mathbf{F}_s^e(t)\mathbf{F}_s^{ve}(t)\tilde{\mathbf{H}}_s^\theta(t)\mathbf{F}_s^{ve-1}(t)\mathbf{F}_s^{e-1}(t).
\end{aligned} \tag{4.50}$$

It can be shown that $\check{\mathbf{F}}_s^e(t)$, $\check{\mathbf{F}}_s^{ve}(t)$ and $\check{\mathbf{F}}_s^\theta(t)$ are each close to $\mathbf{I}$, therefore resulting in infinitesimal $\check{\mathbf{H}}_s^e(t)$, $\check{\mathbf{H}}_s^{ve}(t)$ and $\check{\mathbf{H}}_s^\theta(t)$.

It should be emphasized here that the perturbations of the kinematic variables obtained in (4.50) can be applied to time changing states, which obviously include as special cases the thermodynamic equilibrium states. Therefore we can reduce the incremental kinematic relations (4.50) to the case studied by Lion [Lion, 1998] where the small mechanical perturbation was superimposed in the vicinity of the thermodynamic equilibrium state of the pre-deformation by simply holding $\mathbf{F}_s^e(t)$, $\mathbf{F}_s^{ve}(t)$, and $\mathbf{F}_s^\theta(t)$ all constant in time during the disturbances.

Since the perturbations imposed are infinitesimal, after eliminating the second and higher orders of the infinitesimal terms, we get the following approximations

$$\begin{aligned}
\check{\mathbf{F}}^{-1}(t) &\approx \mathbf{I} - \check{\mathbf{H}}(t), \\
\check{\mathbf{F}}_s^{e-1}(t) &\approx \mathbf{I} - \check{\mathbf{H}}_s^e(t), \\
\check{\mathbf{F}}_s^{ve-1}(t) &\approx \mathbf{I} - \check{\mathbf{H}}_s^{ve}(t), \\
\check{\mathbf{F}}_s^{\theta-1}(t) &\approx \mathbf{I} - \check{\mathbf{H}}_s^\theta(t), \\
\check{\mathbf{H}}_s^e(t) &\approx \check{\mathbf{H}}(t) - \check{\mathbf{H}}_s^{ve}(t) - \check{\mathbf{H}}_s^\theta(t).
\end{aligned} \tag{4.51}$$

4.4.2 Perturbation of the stress by a change in history

To calculate the stress differences between the base and total history, we start by calculating the stress differences between the components of each element and then integrate them. Since the thermodynamic stresses evaluated for each element are only a function of the state of the element, minus the temperature gradient, given that the changes are small, we can calculate the change by using a Taylor series expansion. After integrating over the element, this process gives the difference between the stresses for the base and total history.

We start this process by calculating for each element the approximation to $\delta\mathbf{T}_s^{eT}(t) = \mathbf{T}_s^{*eT}(t) - \mathbf{T}_s^{eT}(t)$ and $\delta\mathbf{T}_s^{bT}(t) = \mathbf{T}_s^{*bT}(t) - \mathbf{T}_s^{bT}(t)$. In each element the difference between the thermodynamic elastic stresses from the base history and the total history is approximated by taking its derivative with respect to each variable and multiplying it by the change in the variable and finally adding the results to get

$$\delta\mathbf{T}_s^{eT} = \mathsf{E}_s^{e} : \delta\mathbf{F}_s^{e} + \mathsf{E}_s^{ve} : \delta\mathbf{F}_s^{ve} + \mathsf{E}_s^{\theta} : \delta\mathbf{F}_s^{\theta} + \mathbf{E}_s^{\theta}\delta\theta, \tag{4.52}$$

where the coefficients are associate tangent moduli with respect to the given variables and evaluated at the base history. They are defined as

$$\begin{aligned}
\mathsf{E}_s^{e} &\equiv \partial_{\mathbf{F}_s^e}(\mathbf{T}_s^{eT\dagger}),\\
\mathsf{E}_s^{ve} &\equiv \partial_{\mathbf{F}_s^{ve}}(\mathbf{T}_s^{eT\dagger}),\\
\mathsf{E}_s^{\theta} &\equiv \partial_{\mathbf{F}_s^{\theta}}(\mathbf{T}_s^{eT\dagger}),\\
\mathbf{E}_s^{\theta} &\equiv \partial_{\theta}(\mathbf{T}_s^{eT\dagger}).
\end{aligned} \tag{4.53}$$

Similarly, we calculate the difference of the back stresses from

$$\delta\mathbf{T}_s^{bT} = \mathsf{E}_s^{be} : \delta\mathbf{F}_s^{e} + \mathsf{E}_s^{bve} : \delta\mathbf{F}_s^{ve} + \mathsf{E}_s^{b\theta} : \delta\mathbf{F}_s^{\theta} + \mathbf{E}_s^{b\theta}\delta\theta, \tag{4.54}$$

where,

$$\begin{aligned}
\mathsf{E}_s^{be} &\equiv \partial_{\mathbf{F}_s^e}(\mathbf{T}_s^{bT\dagger}),\\
\mathsf{E}_s^{bve} &\equiv \partial_{\mathbf{F}_s^{ve}}(\mathbf{T}_s^{bT\dagger}),\\
\mathsf{E}_s^{b\theta} &\equiv \partial_{\mathbf{F}_s^{\theta}}(\mathbf{T}_s^{bT\dagger}),\\
\mathbf{E}_s^{b\theta} &\equiv \partial_{\theta}(\mathbf{T}_s^{bT\dagger}).
\end{aligned} \tag{4.55}$$

We can now replace for the changes in the variables in terms of the "$\sim$" variables to get the total thermodynamic stresses in an element from its values at the same time in the base history from the equations

$$\begin{aligned}
\mathbf{T}_s^{*eT} &= \mathbf{T}_s^{eT} + \delta\mathbf{T}_s^{eT} \\
&= \mathbf{T}_s^{eT} + \mathsf{E}_s^{e} : (\tilde{\mathbf{H}}_s^e\mathbf{F}_s^e) + \mathsf{E}_s^{ve} : (\tilde{\mathbf{H}}_s^{ve}\mathbf{F}_s^{ve}) + \mathsf{E}_s^{\theta} : (\tilde{\mathbf{H}}_s^{\theta}\mathbf{F}_s^{\theta}) + \mathbf{E}_s^{\theta}\delta\theta, \\
\mathbf{T}_s^{*bT} &= \mathbf{T}_s^{bT} + \delta\mathbf{T}_s^{bT} \\
&= \mathbf{T}_s^{bT} + \mathsf{E}_s^{be} : (\tilde{\mathbf{H}}_s^e\mathbf{F}_s^e) + \mathsf{E}_s^{bve} : (\tilde{\mathbf{H}}_s^{ve}\mathbf{F}_s^{ve}) + \mathsf{E}_s^{b\theta} : (\tilde{\mathbf{H}}_s^{\theta}\mathbf{F}_s^{\theta}) + \mathbf{E}_s^{b\theta}\delta\theta.
\end{aligned} \tag{4.56}$$

Integration of the elastic part over s provides the change in the Cauchy stress as

$$\delta\mathbf{T}(t) = \int_{-\infty}^{\infty} \delta\mathbf{T}_s^e(t)ds. \tag{4.57}$$

Taking account of the symmetry of the Cauchy stress $\delta\mathbf{T}(t) = \delta\mathbf{T}^T(t)$ or $\delta\mathbf{T}_s^e = \delta\mathbf{T}_s^{eT}$ and eliminating the elastic deformation gradient in favor of $\mathbf{F}$, this can be written as

$$\begin{aligned}
\delta\mathbf{T} = &\int_{-\infty}^{\infty} [(\mathsf{E}_s^e\mathbf{F}_s^{ve-T}\mathbf{F}_s^{\theta-T}\mathbf{F}^T) : (\check{\mathbf{H}} - \check{\mathbf{H}}_s^{ve} - \check{\mathbf{H}}_s^{\theta}) + (\mathsf{E}_s^{ve}\mathbf{F}_s^{\theta-T}\mathbf{F}^T) : (\mathbf{F}_s^{ve}\mathbf{F}_s^{\theta}\mathbf{F}^{-1}\check{\mathbf{H}}_s^{ve})]ds \\
&+ \int_{-\infty}^{\infty} [(\mathsf{E}_s^{\theta}\mathbf{F}^T) : (\mathbf{F}_s^{\theta}\mathbf{F}^{-1}\check{\mathbf{H}}_s^{\theta}) + (\delta\theta)\mathbf{E}_s^{\theta}]ds.
\end{aligned} \tag{4.58}$$

4.4.3 Perturbations of the free energy, entropy and heat flux vector

The perturbations of the free energy, entropy and heat flux vector are obtained from steps similar to those described for the stress. We start first by defining the perturbation for each element s of the free energy, entropy and heat flux vector, respectively, as $\delta\psi_s = \psi_s^* - \psi_s$, $\delta\eta_s = \eta_s^* - \eta_s$, and $\delta\mathbf{q}_s = \mathbf{q}_s^* - \mathbf{q}_s$. Noting the dependence of these functions on the state variables, we obtain first order approximations for these perturbations given by

$$\begin{aligned}
\delta\psi_s(t) &= \partial_{\mathbf{F}_s^e}(\psi_s^\dagger) : \delta\mathbf{F}_s^e + \partial_{\mathbf{F}_s^{ve}}(\psi_s^\dagger) : \delta\mathbf{F}_s^{ve} + \partial_{\mathbf{F}_s^\theta}(\psi_s^\dagger) : \delta\mathbf{F}_s^\theta + \partial_\theta(\psi_s^\dagger)\delta\theta, \\
\delta\eta_s(t) &= \partial_{\mathbf{F}_s^e}(\eta_s^\dagger) : \delta\mathbf{F}_s^e + \partial_{\mathbf{F}_s^{ve}}(\eta_s^\dagger) : \delta\mathbf{F}_s^{ve} + \partial_{\mathbf{F}_s^\theta}(\eta_s^\dagger) : \delta\mathbf{F}_s^\theta + \partial_\theta(\eta_s^\dagger)\delta\theta, \\
\delta\mathbf{q}_s(t) &= \partial_{\mathbf{F}_s^e}(\mathbf{q}_s^\dagger) : \delta\mathbf{F}_s^e + \partial_{\mathbf{F}_s^{ve}}(\mathbf{q}_s^\dagger) : \delta\mathbf{F}_s^{ve} + \partial_{\mathbf{F}_s^\theta}(\mathbf{q}_s^\dagger) : \delta\mathbf{F}_s^\theta + \partial_{\mathbf{G}}(\mathbf{q}_s^\dagger)\delta\mathbf{G} + \partial_\theta(\mathbf{q}_s^\dagger)\delta\theta,
\end{aligned} \tag{4.59}$$

where $\delta\mathbf{G} = \mathbf{Grad}(\delta\theta)$ is the gradient of $\delta\theta$ with respect to motions in the reference configuration. It should be noted here that from the relations assumed so as to satisfy the Clausius-Duhem inequality, we can directly obtain the forms for the entropy $\eta_s^*(t)$ and $\eta_s(t)$, and choose the possible constitutive function for $\mathbf{q}_s^*$ and $\mathbf{q}_s$ taking account of $\frac{1}{\theta}\mathbf{q}_s \circ \mathbf{g} \leq 0$ and $\frac{1}{\theta^*}\mathbf{q}_s^* \circ \mathbf{g}^* \leq 0$. For example, one simple choice of the heat flux in the isotropic case is the linear Fourier model considered in Lion and Reese et al. [Lion, 1997, Reese and Govindjee, 1997].

From the assumptions in (4.4), we have the following incremental relations between the overall responses at a material point and the responses in each element as

$$\begin{aligned} \delta\psi &= \int_{-\infty}^{\infty} (\delta\psi_s) ds, \\ \delta\eta &= \int_{-\infty}^{\infty} (\delta\eta_s) ds, \\ \delta\mathbf{q} &= \int_{-\infty}^{\infty} (\delta\mathbf{q}_s) ds. \end{aligned} \tag{4.60}$$

4.4.4 Evolution rules for the incremental perturbation of the internal parameters

In the constitutive equation for each element there are two internal variables that must evolve based on the state of the element. These variables are the viscoelastic and thermal deformation gradients. The evolution of these variables will be different for the base and the total histories, and so their difference will change based on how the two change. As such, the variables we have chosen to describe the perturbation between the internal variables of the two histories will have evolution equations that reflect the expected difference in the evolutions in the two histories. We will use the evolutions for the two histories to calculate in this section the evolution equations for the perturbation variables for the viscoelastic and thermal deformation gradients. We note that the relation between the velocity gradient and the deformation gradient rate is given by $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, and that similar rules are true for the associated internal variables so that for the base history we can write $\dot{\mathbf{F}}_s^{ve} = \mathbf{L}_s^{ve}\mathbf{F}_s^{ve}$ and $\dot{\mathbf{F}}_s^{\theta} = \mathbf{L}_s^{\theta}\mathbf{F}_s^{\theta}$. Using this we can reorganize the evolution equations given for the viscoelastic and thermal deformation gradients and for each element, during the base history, we obtain

$$\dot{\mathbf{F}}_s^{ve} = [\mathbf{C}_s : (\mathbf{F}_s^{eT}\mathbf{T}_s^{eT}\mathbf{F}_s^{e-T} - \mathbf{T}_s^{bT})]\mathbf{F}_s^{ve},$$
$$\dot{\mathbf{F}}_s^{\theta} = \boldsymbol{\alpha}_s\mathbf{F}_s^{\theta}\dot{\theta}. \tag{4.61}$$

In a similar fashion, the same evolution equations for the total history are given by

$$\dot{\mathbf{F}}_s^{*ve} = [\mathbf{C}_s^* : (\mathbf{F}_s^{*eT}\mathbf{T}_s^{*eT}\mathbf{F}_s^{*e-T} - \mathbf{T}_s^{*bT})]\mathbf{F}_s^{*ve},$$
$$\dot{\mathbf{F}}_s^{*\theta} = \boldsymbol{\alpha}_s^*\mathbf{F}_s^{*\theta}\dot{\theta}^*. \tag{4.62}$$

The thermal expansion coefficient tensor $\boldsymbol{\alpha}_s(t)$ and the viscoelastic flow parameter $\mathbf{C}_s(t)$ in each element s are also effected by the thermo-mechanical perturbation, and these material functions for the base history and the total history are related by

$$\boldsymbol{\alpha}_s^*(t) = \boldsymbol{\alpha}_s(t) + \delta\boldsymbol{\alpha}_s(t),$$
$$\mathbf{C}_s^*(t) = \mathbf{C}_s(t) + \delta\mathbf{C}_s(t), \tag{4.63}$$

where $\boldsymbol{\alpha}_s^*(t)$ and $\mathbf{C}_s^*(t)$ are the thermal expansion parameter and the viscoelastic flow parameter in the total history, $\boldsymbol{\alpha}_s(t)$ and $\mathbf{C}_s(t)$ are the thermal expansion parameter and the viscoelastic flow parameter in the base history, and $\delta\boldsymbol{\alpha}_s(t)$ and $\delta\mathbf{C}_s(t)$ are the incremental thermal expansion parameter and the incremental viscoelastic flow parameter from the perturbation. Since $\boldsymbol{\alpha}_s(t)$ and $\mathbf{C}_s(t)$ are assumed to be functions of the state of the element, we define the following coefficents which are the derivatives of $\boldsymbol{\alpha}_s$ and $\mathbf{C}_s$ with respect to their associate variables and given by

$$\mathbf{a}_s^e \equiv \partial_{\mathbf{F}_s^e}(\boldsymbol{\alpha}_s^\dagger),$$
$$\mathbf{a}_s^{ve} \equiv \partial_{\mathbf{F}_s^{ve}}(\boldsymbol{\alpha}_s^\dagger),$$
$$\mathbf{a}_s^{\theta} \equiv \partial_{\mathbf{F}_s^{\theta}}(\boldsymbol{\alpha}_s^\dagger),$$
$$\boldsymbol{\alpha}_s^{\theta} \equiv \partial_{\theta}(\boldsymbol{\alpha}_s^\dagger), \tag{4.64}$$

and

$$\begin{aligned}
\mathbf{D}_s^e &\equiv \partial_{\mathbf{F}_s^e}(\mathbf{C}_s^\dagger),\\
\mathbf{D}_s^{ve} &\equiv \partial_{\mathbf{F}_s^{ve}}(\mathbf{C}_s^\dagger),\\
\mathbf{D}_s^\theta &\equiv \partial_{\mathbf{F}_s^\theta}(\mathbf{C}_s^\dagger),\\
\mathbf{C}_s^\theta &\equiv \partial_\theta(\mathbf{C}_s^\dagger),
\end{aligned} \tag{4.65}$$

where $\mathbf{a}_s^e$, $\mathbf{a}_s^{ve}$, $\mathbf{a}_s^\theta$ and $\mathbf{C}_s^\theta$ are fourth order tensor functions, $\boldsymbol{\alpha}_s^\theta$ is a second order tensor, $\mathbf{D}_s^e$, $\mathbf{D}_s^{ve}$ and $\mathbf{D}_s^\theta$ are sixth order tensor functions.

After substituting the increments of the kinematics variables given in (4.48), we obtain the increments $\delta\boldsymbol{\alpha}_s$ and $\delta\mathbf{C}_s$ for the thermal expansion and the viscoelastic flow in terms of the "$\sim$" variables as

$$\delta\boldsymbol{\alpha}_s = \mathbf{a}_s^e : (\tilde{\mathbf{H}}_s^e\mathbf{F}_s^e) + \mathbf{a}_s^{ve} : (\tilde{\mathbf{H}}_s^{ve}\mathbf{F}_s^{ve}) + \mathbf{a}_s^\theta : (\tilde{\mathbf{H}}_s^\theta\mathbf{F}_s^\theta) + \boldsymbol{\alpha}_s^\theta\delta\theta, \tag{4.66}$$

$$\delta\mathbf{C}_s = \mathbf{D}_s^e : (\tilde{\mathbf{H}}_s^e\mathbf{F}_s^e) + \mathbf{D}_s^{ve} : (\tilde{\mathbf{H}}_s^{ve}\mathbf{F}_s^{ve}) + \mathbf{D}_s^\theta : (\tilde{\mathbf{H}}_s^\theta\mathbf{F}_s^\theta) + \mathbf{C}_s^\theta\delta\theta, \tag{4.67}$$

with the parameters $\mathbf{a}_s^e$, $\mathbf{a}_s^{ve}$, $\mathbf{a}_s^\theta$, $\boldsymbol{\alpha}_s^\theta$, $\mathbf{D}_s^e$, $\mathbf{D}_s^{ve}$, $\mathbf{D}_s^\theta$ and $\mathbf{C}_s^\theta$ evaluated in the base history.

The relations for the time derivatives are given by

$$\dot{\theta}^*(t) = \dot{\theta}(t) + \delta\dot{\theta}(t), \tag{4.68}$$

and

$$\begin{aligned}
\dot{\mathbf{F}}_s^{*e}(t) &= \dot{\tilde{\mathbf{F}}}_s^e(t)\mathbf{F}_s^e(t) + \tilde{\mathbf{F}}_s^e(t)\dot{\mathbf{F}}_s^e(t),\\
\dot{\mathbf{F}}_s^{*ve}(t) &= \dot{\tilde{\mathbf{F}}}_s^{ve}(t)\mathbf{F}_s^{ve}(t) + \tilde{\mathbf{F}}_s^{ve}(t)\dot{\mathbf{F}}_s^{ve}(t),\\
\dot{\mathbf{F}}_s^{*\theta}(t) &= \dot{\tilde{\mathbf{F}}}_s^\theta(t)\mathbf{F}_s^\theta(t) + \tilde{\mathbf{F}}_s^\theta(t)\dot{\mathbf{F}}_s^\theta(t).
\end{aligned} \tag{4.69}$$

By using (4.68) and (4.69) and manipulating the two sets of evaluation equations (4.61) and (4.62) for the two histories, we get the effects of the perturbation on the flow rules through the following two coupled first order differential equations with the unknown incremental internal variables $\tilde{\mathbf{H}}_s^{ve}(t)$ and $\tilde{\mathbf{H}}_s^{\theta}(t)$ given by

$$\begin{aligned}
&\dot{\mathbf{F}}_s^{\theta} + \dot{\tilde{\mathbf{H}}}_s^{\theta}\mathbf{F}_s^{\theta} + \tilde{\mathbf{H}}_s^{\theta}\dot{\mathbf{F}}_s^{\theta} = (\dot{\theta} + \delta\dot{\theta})(\boldsymbol{\alpha}_s + \delta\boldsymbol{\alpha}_s)(\mathbf{I} + \tilde{\mathbf{H}}_s^{\theta})\mathbf{F}_s^{\theta}, \\
&\dot{\tilde{\mathbf{H}}}_s^{ve}\mathbf{F}_s^{ve} + \dot{\mathbf{F}}_s^{ve} + \tilde{\mathbf{H}}_s^{ve}\dot{\mathbf{F}}_s^{ve} \\
&= \{(\mathsf{C}_s + \delta\mathsf{C}_s) : [\mathbf{F}_s^{eT}(\mathbf{I} + \tilde{\mathbf{H}}_s^{eT})(\mathbf{T}_s^{eT} + \delta\mathbf{T}_s^{eT})(\mathbf{I} - \tilde{\mathbf{H}}_s^{eT})\mathbf{F}_s^{e-T} \\
&- (\mathbf{T}_s^{bT} + \delta\mathbf{T}_s^{bT})]\}(\mathbf{I} + \tilde{\mathbf{H}}_s^{ve})\mathbf{F}_s^{ve}.
\end{aligned} \tag{4.70}$$

We use the relations between $\check{\mathbf{H}}_s^{e}$ and $\tilde{\mathbf{H}}_s^{e}$, $\check{\mathbf{H}}_s^{ve}$ and $\tilde{\mathbf{H}}_s^{ve}$, and $\check{\mathbf{H}}_s^{\theta}$ and $\tilde{\mathbf{H}}_s^{\theta}$ given in (4.50), and simplify the notation in (4.70) by using $\boldsymbol{\Gamma}_s^{\theta}(t) \equiv \check{\mathbf{H}}_s^{\theta}(t)\mathbf{F}(t)$ and $\boldsymbol{\Gamma}_s^{ve}(t) \equiv \check{\mathbf{H}}_s^{ve}(t)\mathbf{F}(t)$ to denote the unknown values, and $\delta\theta(t)$ and $\boldsymbol{\Gamma}(t) \equiv \check{\mathbf{H}}(t)\mathbf{F}(t)$ as the known values. From this we get the following two modified equations. The first becomes

$$\begin{aligned}
\dot{\boldsymbol{\Gamma}}_s^{\theta} =& (\dot{\theta} + \delta\dot{\theta})\mathbf{F}\mathbf{F}_s^{\theta-1}[(\mathsf{a}_s^{e}\mathbf{F}_s^{ve-T}\mathbf{F}_s^{\theta-T}) : (\boldsymbol{\Gamma} - \boldsymbol{\Gamma}_s^{ve} - \boldsymbol{\Gamma}_s^{\theta}) \\
&+ (\mathsf{a}_s^{ve}\mathbf{F}_s^{\theta-T}) : (\mathbf{F}_s^{ve}\mathbf{F}_s^{\theta}\mathbf{F}^{-1}\boldsymbol{\Gamma}_s^{ve}) \\
&+ \mathsf{a}_s^{\theta} : (\mathbf{F}_s^{\theta}\mathbf{F}^{-1}\boldsymbol{\Gamma}_s^{\theta})]\mathbf{F}_s^{\theta} + [(\delta\dot{\theta})\mathbf{F}\mathbf{F}_s^{\theta-1}\boldsymbol{\alpha}_s\mathbf{F}_s^{\theta}\mathbf{F}^{-1} + \dot{\mathbf{F}}\mathbf{F}^{-1}]\boldsymbol{\Gamma}_s^{\theta} \\
&+ \mathbf{F}\mathbf{F}_s^{\theta-1}[(\dot{\theta} + \delta\dot{\theta})(\delta\theta)\boldsymbol{\alpha}_s^{\theta} + (\delta\dot{\theta})\boldsymbol{\alpha}_s]\mathbf{F}_s^{\theta},
\end{aligned} \tag{4.71}$$

which can be simplified to the form

$$\dot{\boldsymbol{\Gamma}}_s^{\theta}(t) = \mathsf{B}_s^{\theta}(t) : \boldsymbol{\Gamma}_s^{\theta}(t) + \mathsf{B}_s^{ve}(t) : \boldsymbol{\Gamma}_s^{ve}(t) + \mathsf{B}_s(t) : \boldsymbol{\Gamma}(t) + \boldsymbol{\theta}_s(t). \tag{4.72}$$

The second equation becomes

$$
\begin{aligned}
&\dot{\mathbf{\Gamma}}_s^{ve} + \dot{\theta}\mathbf{F}\mathbf{F}_s^{\theta-1}\boldsymbol{\alpha}_s\mathbf{F}_s^{\theta}\mathbf{F}^{-1}\mathbf{\Gamma}_s^{ve} - \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{\Gamma}_s^{ve} - \dot{\theta}\mathbf{\Gamma}_s^{ve}\mathbf{F}_s^{\theta-1}\boldsymbol{\alpha}_s\mathbf{F}_s^{\theta} \\
&= \mathbf{F}\mathbf{F}_s^{\theta-1}\mathbf{F}_s^{ve-1}\{\mathbf{C}_s : [\mathbf{F}_s^{ve-T}\mathbf{F}_s^{\theta-T}\mathbf{F}^{T}(\delta\mathbf{T}_s^{eT})\mathbf{F}^{-T}\mathbf{F}_s^{\theta T}\mathbf{F}_s^{veT} \\
&+ \mathbf{F}_s^{ve-T}\mathbf{F}_s^{\theta-T}(\mathbf{\Gamma}^{T} - \mathbf{\Gamma}_s^{veT} - \mathbf{\Gamma}_s^{\theta T})\mathbf{T}_s^{eT}\mathbf{F}^{-T}\mathbf{F}_s^{\theta T}\mathbf{F}_s^{veT} \\
&- \mathbf{F}_s^{ve-T}\mathbf{F}_s^{\theta-T}\mathbf{F}^{T}\mathbf{T}_s^{eT}\mathbf{F}^{-T}(\mathbf{\Gamma}^{T} - \mathbf{\Gamma}_s^{veT} - \mathbf{\Gamma}_s^{\theta T})\mathbf{F}^{-T}\mathbf{F}_s^{\theta T}\mathbf{F}_s^{veT} \\
&- \delta\mathbf{T}_s^{bT}]\}\mathbf{F}_s^{ve}\mathbf{F}_s^{\theta} + \mathbf{F}\mathbf{F}_s^{\theta-1}\mathbf{F}_s^{ve-1}\{[\mathbf{D}_s^{e} : ((\mathbf{\Gamma} - \mathbf{\Gamma}_s^{ve} - \mathbf{\Gamma}_s^{\theta})\mathbf{F}_s^{\theta-1}\mathbf{F}_s^{ve-1}) \\
&+ \mathbf{D}_s^{ve} : (\mathbf{F}_s^{ve}\mathbf{F}_s^{\theta}\mathbf{F}^{-1}\mathbf{\Gamma}_s^{ve}\mathbf{F}_s^{\theta-1}) \\
&+ \mathbf{D}_s^{\theta} : (\mathbf{F}_s^{\theta}\mathbf{F}^{-1}\mathbf{\Gamma}_s^{\theta}) + \mathbf{C}_s^{\theta}\delta\theta] : (\mathbf{F}_s^{eT}\mathbf{T}_s^{eT}\mathbf{F}_s^{e-T} - \mathbf{T}_s^{bT})\}\mathbf{F}_s^{ve}\mathbf{F}_s^{\theta},
\end{aligned}
\tag{4.73}
$$

which can be organized into the form

$$
\dot{\mathbf{\Gamma}}_s^{ve}(t) = \mathbf{Y}_s^{ve}(t) : \mathbf{\Gamma}_s^{ve}(t) + \mathbf{Y}_s^{\theta}(t) : \mathbf{\Gamma}_s^{\theta}(t) + \mathbf{Y}_s(t) : \mathbf{\Gamma}(t) + \boldsymbol{\eta}_s(t). \tag{4.74}
$$

In these two simplified forms, $\mathbf{B}_s^{\theta}(t)$, $\mathbf{B}_s^{ve}(t)$, $\mathbf{B}_s(t)$, $\mathbf{Y}_s^{ve}(t)$, $\mathbf{Y}_s^{\theta}(t)$ and $\mathbf{Y}_s(t)$ are fourth order tensors, $\boldsymbol{\theta}_s(t)$ and $\boldsymbol{\eta}_s(t)$ are second order tensors.

In order to solve the tensor form differential equations (4.72) and (4.74), we should represent the tensors in a curvilinear coordinate system and then solve the differential equations in the component form. For simplicity, here we choose an orthonormal base and write the corresponding component form in this base as

$$
\begin{aligned}
\dot{\Gamma}^{\theta}_{s\_ij}(t) &= \mathsf{B}^{\theta}_{s\_ijkl}(t)\Gamma^{\theta}_{s\_kl}(t) + \mathsf{B}^{ve}_{s\_ijkl}(t)\Gamma^{ve}_{s\_kl}(t) + \mathsf{B}_{s\_ijkl}(t)\Gamma_{kl}(t) + \theta_{s\_ij}(t), \\
\dot{\Gamma}^{ve}_{s\_ij}(t) &= \mathsf{Y}^{ve}_{s\_ijkl}(t)\Gamma^{ve}_{s\_kl}(t) + \mathsf{Y}^{\theta}_{s\_ijkl}(t)\Gamma^{\theta}_{s\_kl}(t) + \mathsf{Y}_{s\_ijkl}(t)\Gamma_{kl}(t) + \eta_{s\_ij}(t).
\end{aligned}
\tag{4.75}
$$

To solve this system of first order differential equations, we organize the unknows into a one-dimensional array denoted by $\boldsymbol{\chi}_s$ that takes the form

$$
\boldsymbol{\chi}_s \equiv (\Gamma^{ve}_{s\_11}, \Gamma^{ve}_{s\_12}, \Gamma^{ve}_{s\_13}, \Gamma^{ve}_{s\_21}, ..., \Gamma^{\theta}_{s\_11}, \Gamma^{\theta}_{s\_12}, \Gamma^{\theta}_{s\_13}, \Gamma^{\theta}_{s\_21}, ...). \tag{4.76}
$$

To do this, we introduce two transformations T_{ijk} and K_{ijk} which can transform the components of $\mathbf{\Gamma}_s^{ve}$ and $\mathbf{\Gamma}_s^{\theta}$ into the one-dimensional array $\boldsymbol{\chi}_s$ through the relation

$$\chi_{s\_i} = T_{ijk}\Gamma^{ve}_{s\_jk} + K_{ijk}\Gamma^{\theta}_{s\_jk}, \tag{4.77}$$

where $T_{ijk} = 0$ for $i = 10, ..., 18$ and $K_{ijk} = 0$ for $i = 1, ..., 9$, and the remaining values are either zero or one, defined by the pattern of $\boldsymbol{\chi}_s$. The inverse transformations are given as

$$\Gamma^{ve}_{s\_ij} = T^{-1}_{ijk}\chi_{s\_k}, \quad \Gamma^{\theta}_{s\_ij} = K^{-1}_{ijk}\chi_{s\_k}. \tag{4.78}$$

After substituting the transformations (4.78) into (4.75), and then substituting into the time derivative of (4.77), we get the first order system of differential equations

$$\dot{\boldsymbol{\chi}}_s(t) = \mathbf{A}_s(t)\boldsymbol{\chi}_s(t) + \mathbf{f}_s(t), \tag{4.79}$$

where, the component of the coefficient matrix $A_{s\_mn}(t)$ and the component of the inhomogeneous array $f_{s\_m}(t)$ are given as

$$\begin{aligned} A_{s\_mn}(t) =& T_{mij}\mathsf{Y}^{ve}_{s\_ijkl}(t)T^{-1}_{kln} + K_{mij}\mathsf{B}^{ve}_{s\_ijkl}(t)T^{-1}_{kln} + K_{mij}\mathsf{B}^{\theta}_{s\_ijkl}(t)K^{-1}_{kln} \\ &+ T_{mij}\mathsf{Y}^{\theta}_{s\_ijkl}(t)K^{-1}_{kln}, \end{aligned} \tag{4.80}$$

$$f_{s\_m}(t) = T_{mij}\mathsf{Y}_{s\_ijkl}(t)\Gamma_{kl}(t) + T_{mij}\eta_{s\_ij}(t) + K_{mij}\mathsf{B}_{s\_ijkl}(t)\Gamma_{kl}(t) + K_{mij}\theta_{s\_ij}(t). \tag{4.81}$$

From the existence and uniqueness theorem, there exists a unique solution, since the coefficient matrix $[A_s(t)]$ and the inhomogeneous array $[f_s(t)]$ are continuous. The general solution to the system (4.79) is provided by Myskis [Myskis, 1975] as

$$\boldsymbol{\chi}_s(t) = \int_{t_i}^{t} \mathbf{Y}_s(t,\tau)\mathbf{f}_s(\tau)d\tau + \mathbf{Y}_s(t,t_i)\boldsymbol{\chi}_{si}, \tag{4.82}$$

under the initial condition $\boldsymbol{\chi}_{si} = \boldsymbol{\chi}_s(t_i)$, and where,

$$\begin{aligned} \mathbf{Y}_s(t,t_i) =& \mathbf{I} + \int_{t_i}^{t} \mathbf{A}_s(\tau_1)d\tau_1 + \int_{t_i}^{t} \mathbf{A}_s(\tau_1)\int_{t_i}^{\tau_1} \mathbf{A}_s(\tau_2)d\tau_2 d\tau_1 \\ &+ \int_{t_i}^{t} \mathbf{A}_s(\tau_1)\int_{t_i}^{\tau_1} \mathbf{A}_s(\tau_2)\int_{t_i}^{\tau_2} \mathbf{A}_s(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \dots . \end{aligned} \tag{4.83}$$

From the general solution for $\boldsymbol{\chi}_s(t)$ given in (4.82) and the inverse transformations from $\boldsymbol{\chi}_s(t)$ to $\boldsymbol{\Gamma}_s^{ve}(t)$ and $\boldsymbol{\Gamma}_s^{\theta}(t)$, we then can calculate the increments of the internal variables $\check{\mathbf{H}}_s^{ve}(t)$ and $\check{\mathbf{H}}_s^{\theta}(t)$.

4.5 Homogeneous pre-deformation

As a special example, in this section we look at the equations for perturbations on a homogeneous pre-deformation, and consider the examples of attenuating and non-attenuating plane harmonic waves.

For a homogeneous pre-deformation the general equations for the balance laws given in (2.51) and (2.57) can be simplified by setting the divergence and gradients of the homogeneous terms equal to zero. For the balance of linear momentum this results in

$$[\mathbf{Grad}(\delta\mathbf{T}) - \mathbf{T}\mathbf{Grad}(\check{\mathbf{H}}^T)] : \mathbf{F}^{-T} + \mathbf{T}\mathbf{F}^{-T}\mathbf{Grad}(\check{\varepsilon}_v) + \rho\delta\mathbf{b} = \rho\delta\ddot{\mathbf{u}}. \tag{4.84}$$

The equation for balance of work and energy for a homogeneous pre-loading can be written as

$$\begin{aligned}
&-(J\mathbf{F}^{-T}) : [\mathbf{Grad}(\delta\mathbf{q}) - \mathbf{q}\mathbf{Grad}(\check{\mathbf{H}}^T) + \mathbf{q}\otimes\mathbf{Grad}(\check{\varepsilon}_v)] \\
&+ Jtr[(\mathbf{T} + \delta\mathbf{T} - \check{\mathbf{H}}\mathbf{T} + \check{\varepsilon}_v\mathbf{T})\dot{\check{\mathbf{H}}} + (\delta\mathbf{T} - \check{\mathbf{H}}\mathbf{T} + \check{\varepsilon}_v\mathbf{T} + \mathbf{T}\check{\mathbf{H}})\mathbf{L}] \\
&+ \rho_o\delta r = \rho_o[\delta\dot{\psi} + \dot{\theta}\delta\eta + \eta\delta\dot{\theta} + (\delta\dot{\theta})(\delta\eta) + \theta\delta\dot{\eta} + \dot{\eta}\delta\theta + (\delta\theta)(\delta\dot{\eta})].
\end{aligned} \tag{4.85}$$

These special balance equations for homogeneous pre-loadings can be further simplified by selecting additional assumptions on the base history and the perturbations. For example, one can study different aspects of the problem of propagations of specific types of perturbations, one can decouple the thermal effect from the mechanical loadings by controlling the temperature field, or one can assume that the base history is "static" compared to the perturbations. In particular, for example one can specify the form of the perturbations, like the case considered by Destrade et al. [Destrade et al., 2009] where incompressible two-dimensional waves were superimposed on a static deformed body by taking one component of $\delta\mathbf{u}$ equal to zero.

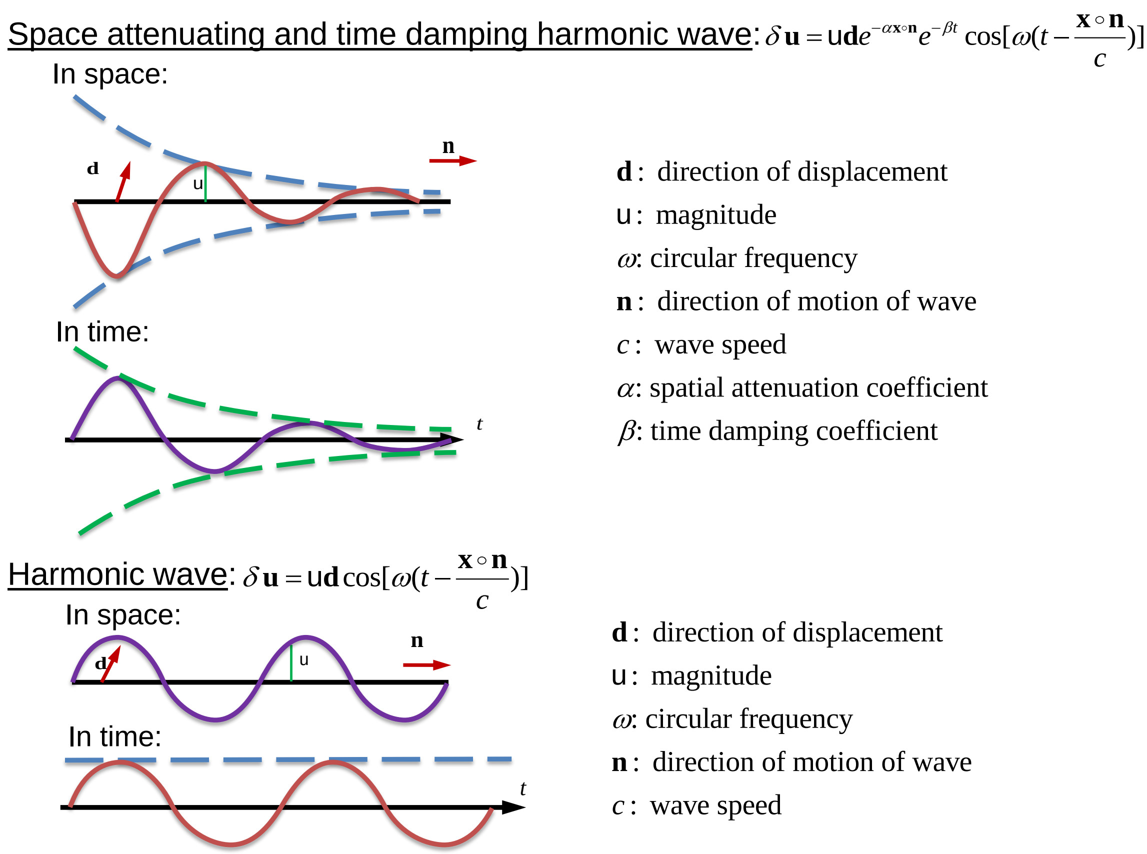


Figure 4.5: Space attenuating and time damping plane harmonic wave and a pure harmonic wave.

In the following we consider the case of isothermal plane harmonic waves in an isothermal static and homogeneously pre-deformed body, obtaining the general mechanical equations for such waves.

4.5.1 Isothermal plane waves propagating in a homogeneous pre-deformed body

The characteristics of propagation of time harmonic and time damping plane waves provide a natural tool for addressing the material stability of the deformed body. In this section we restrict our analysis to some simple perturbation types, find the relations between the material parameters, the pre-deformations and the superimposed wave properties, then reduce the present results to those provided in other studies. In particular, we assume that the loading rate of the base history is slow enough compared to the propagations of the superimposed waves, so that we can take the variables associated with the base history as constant in time.

The base and the total histories are described by

$$\mathcal{H}(t_i,t) = \{[\mathbf{x}(\mathbf{X},\tau),\theta_r]|t_i < \tau < t\},$$
$$\mathcal{H}^*(t_i,t) = \{[\mathbf{x}^*(\mathbf{X},\tau),\theta_r]|t_i < \tau < t\}, \tag{4.86}$$

where θ_r is the constant reference temperature for both loading histories, therefore, the two processes are kept isothermal.

Space attenuating and time damping plane wave in a homogeneous pre-deformed viscoelastic body

Let us look at a homogeneous plane wave which attenuates in space and decays in time and can be described by

$$\delta\mathbf{u}(\mathbf{x},t) = \mathsf{u}e^{-\alpha\mathbf{n}\cdot\mathbf{x}}e^{-\beta t}\cos[\omega(t - \frac{\mathbf{n}\cdot\mathbf{x}}{c})]\mathbf{d}, \tag{4.87}$$

where, as shown in Figure 4.5, u is the magnitude of the superimposed displacement vector, $\mathbf{d}$ is a unit vector along the displacement direction, α is the spatial attenuation coefficient, β is the time damping factor, $\mathbf{n}$ is a unit vector along the wave propagation direction, ω is the circular frequency, c is the wave speed, and $\mathbf{x}$ is the position vector in the intermediate configuration. It is clear to see that the amplitude variation has two parts, one is from the spatial attenuation and the other is from the time decay.

From the displacement equation of the imposed wave (4.87), we can calculate the perturbation displacement gradient $\check{\mathbf{H}}$ as

$$\begin{aligned}\check{\mathbf{H}} =& \mathsf{u}e^{-\beta t}e^{-\alpha\mathbf{n}\cdot(\mathbf{X}+\mathbf{u})}\{-\alpha\cos[\omega(t - \frac{\mathbf{n}\cdot\mathbf{X}+\mathbf{n}\cdot\mathbf{u}}{c})] \\ &+ (\frac{\omega}{c})\sin[\omega(t - \frac{\mathbf{n}\cdot\mathbf{X}+\mathbf{n}\cdot\mathbf{u}}{c})]\}\mathbf{d}\otimes\mathbf{n}.\end{aligned} \tag{4.88}$$

The acceleration of this imposed displacement is calculated from

$$\delta\ddot{\mathbf{u}}(t) = \mathsf{u}e^{-\alpha\mathbf{n}\cdot(\mathbf{X}+\mathbf{u})}e^{-\beta t}\{(\beta^2-\omega^2)\cos[\omega(t-\frac{\mathbf{n}\cdot(\mathbf{X}+\mathbf{u})}{c})] + 2\beta\omega\sin[\omega(t-\frac{\mathbf{n}\cdot(\mathbf{X}+\mathbf{u})}{c})]\}\mathbf{d}. \tag{4.89}$$

We substitute the perturbation displacement gradient into (4.79) to get the evolution of the viscoelastic deformation for this mechanical perturbation, and keeping in mind that $\mathbf{F}_s^\theta = \mathbf{I}$ and $\tilde{\mathbf{F}}_s^\theta = \mathbf{I}$ since the temperature field is kept constant, the solution to this specific problem can be obtained from the general solution given in (4.82) as

$$\boldsymbol{\chi}_s(t) = \int_{t_i}^{t} e^{(t-\tau)\mathbf{A}_s}\mathbf{f}_s(\tau)d\tau + e^{(t-t_i)\mathbf{A}_s}\boldsymbol{\chi}_{si}, \tag{4.90}$$

under the initial condition

$$\boldsymbol{\chi}_{si} = \boldsymbol{\chi}_s(t_i) = \mathbf{0}. \tag{4.91}$$

We then use the inverse transformations from $\boldsymbol{\chi}_s(t)$ to $\boldsymbol{\Gamma}_s^{ve}(t)$, and calculate $\check{\mathbf{H}}_s^{ve}$ by using $\check{\mathbf{H}}_s^{ve} = \boldsymbol{\Gamma}_s^{ve}\mathbf{F}^{-1}$.

The material response to this superimposed infinitesimal mechanical wave should satisfy the balance of linear momentum for the homogeneous preloading equation. We substitute the acceleration term (4.89) and $\check{\mathbf{H}}_s^{ve}$ into equation (4.84). This must hold over any time t and over space. We equate the coefficients of the "*sin*" and "*cos*" terms, so that for an arbitrary wave propagation direction $\mathbf{n}$, we get two eigenvalue problems to solve as

$$(\mathbf{K}-h\mathbf{I})\mathbf{d} = \mathbf{0},$$
$$(\mathbf{R}-\lambda\mathbf{I})\mathbf{d} = \mathbf{0}, \tag{4.92}$$

where the components of the matrices $\mathbf{K}$ and $\mathbf{R}$ are given by

$$K_{iu} = \int_0^{(t-t_i)} G_{iu}\sin(\omega\xi)d\xi,$$
$$R_{iu} = \int_0^{(t-t_i)} G_{iu}\cos(\omega\xi)d\xi + JF_{xm}n_jn_x\int_{-\infty}^{\infty}\mathsf{E}^e_{s\_ijul}F^{ve-1}_{s\_ml}ds, \tag{4.93}$$

for

$$G_{iu} = JT_{yzw}F_{hv}n_jn_h\int_{-\infty}^{\infty}(F_{ab}^{-1}T_{bln}^{-1}\mathsf{E}_{s\_ijkl}^{ve}F_{s\_ka}^{ve} - T_{kmn}^{-1}\mathsf{E}_{s\_ijkl}^{e}F_{s\_ml}^{ve-1})\mathsf{Y}_{s\_zwuv}(e^{\xi\mathbf{A}_s})_{ny}e^{\beta\xi}ds. \tag{4.94}$$

In order to get nontrivial solutions for $\mathbf{d}$, the characteristic equations have to be zero, so that we must have

$$\begin{aligned}\det|\mathbf{K} - h\mathbf{I}| &= 0,\\ \det|\mathbf{R} - \lambda\mathbf{I}| &= 0.\end{aligned} \tag{4.95}$$

The eigenvalues h and λ are functions of wave speed, attenuation coefficient, time damping factor, and circular frequency, and given as

$$\begin{aligned}h &= 2\rho_o\frac{\omega c^2(c\alpha\beta^2 - c\alpha\omega^2 + \alpha^2c^2\beta - \omega^2\beta)}{(\alpha^2c^2+\omega^2)^2},\\ \lambda &= \rho_o\frac{c^2(\alpha^2c^2\beta^2 - \omega^2\beta^2 - \alpha^2c^2\omega^2 + \omega^4 - 4\omega^2\alpha c\beta)}{(\alpha^2c^2+\omega^2)^2}.\end{aligned} \tag{4.96}$$

Manipulation of the above equations results in an equation for "α" given as

$$\begin{aligned}h^2c^4\alpha^4 - 4hc^3\omega\lambda\alpha^3 + 2\omega^2c^2(2\lambda^2 - h^2 - 2\rho_oc^2\lambda)\alpha^2 + 4\omega^3ch(\lambda - 2\rho_oc^2)\alpha\\ + (h^2 - 4\rho_o^2c^4 + 4\rho_oc^2\lambda)\omega^4 = 0,\end{aligned} \tag{4.97}$$

and one for "β" given as

$$\beta = \frac{h(\alpha^2c^2 - \omega^2) - 2\omega\lambda\alpha c}{2\rho_o\omega c^2}. \tag{4.98}$$

Guz [Guz, 1999] investigated and formulated the stability criterion of the state of equilibrium for viscoelastic bodies as follows: *the state of equilibrium is regarded as stable if perturbations attenuate in time and is unstable if perturbations increase in time indefinitely.*

For the specific perturbation (4.87) we have considered, the material stability is ensured when there is no amplitude growth for a given phase, as indicated by Boulanger at al. [Boulanger and Hayes, 1993]. Namely, when $\alpha\mathbf{n}\cdot\mathbf{x} + \beta t > 0$ with the phase $\omega(t - \frac{\mathbf{n}\cdot\mathbf{x}}{c}) =$ *constant*. This gives the stability condition as $\alpha + \frac{\beta}{c} > 0$, where conditional stability is given by $\alpha + \frac{\beta}{c} = 0$ where the perturbations are propagating periodically. Since the wave speed "$c > 0$" for propagating waves, the stability condition constrains the possible values of "α" and "β." The stability condition that $\alpha + \frac{\beta}{c} > 0$, after substituting the representations of "α" and "β," results in bounds on the material parameters, the pre-deformations, the propagation direction and the circular frequency so as to ensure the stability of the equilibrium state of the pre-deformed materials.

The possible directions of particle displacement are given by the common real eigenvectors of the two equations in (4.92). The eigenvectors can be affected by the initial material symmetry and the anisotropy introduced by the pre-deformation. As a result, there might be cases under which there are no pure shear or pure longitudinal waves existing, unlike the special case of a linear isotropic viscoelastic material that always permits them.

Attenuating plane wave in a homogeneous pre-deformed viscoelastic body

Next let us take a look at a time harmonic plane attenuating wave propagating in a homogeneous pre-deformed viscoelastic body. To do this, we take $\beta = 0$ in equation (4.98) so that "α" should satisfy

$$h\alpha^2 c^2 - 2\omega\lambda\alpha c = h\omega^2, \tag{4.99}$$

and at the same time "α" should be a root of equation (4.97). The stability condition now reduces to $\alpha > 0$.

This type of attenuating plane wave is described by the displacement

$$\delta\mathbf{u}(t) = \mathsf{u}e^{-\alpha\mathbf{n}\cdot\mathbf{x}}\cos[\omega(t - \frac{\mathbf{n}\cdot\mathbf{x}}{c})]\mathbf{d}, \tag{4.100}$$

and results in simplified forms of the results given in the last section and one obtains two eigenvalue problems that are the same form as in (4.92). If we assume that the current time t is large enough compared to the initial time t_i that the material responses are in the steady state, the components of the matrices $\mathbf{K}$ and $\mathbf{R}$ are given by

$$\begin{aligned} K_{iu} &= \int_0^\infty G_{iu}\sin(\omega\xi)d\xi, \\ R_{iu} &= \int_0^\infty G_{iu}\cos(\omega\xi)d\xi + JF_{xm}n_jn_x\int_{-\infty}^\infty \mathsf{E}^e_{s\_ijul}F^{ve-1}_{s\_ml}ds, \end{aligned} \tag{4.101}$$

for

$$G_{iu} = JT_{yzw}F_{hv}n_jn_h\int_{-\infty}^\infty (F^{-1}_{ab}T^{-1}_{bln}\mathsf{E}^{ve}_{s\_ijkl}F^{ve}_{s\_ka} - T^{-1}_{kmn}\mathsf{E}^e_{s\_ijkl}F^{ve-1}_{s\_ml})\mathsf{Y}_{s\_zwuv}(e^{\xi\mathbf{A}_s})_{ny}ds. \tag{4.102}$$

The eigenvalue problems in (4.92) with the components given in (4.101) can be shown to be equivalent to the modified Christoffel equations obtained in Garg [Garg, 2007] for the propagation of plane homogeneous waves in a general viscoelastic anisotropic media under static homogeneous initial pre-deformation.

If there is no pre-deformation, i.e. $\mathbf{F}(t) = \mathbf{F}^{ve}(t) = \mathbf{I}$, the two matrices reduce to

$$\begin{aligned} K_{iu} &= \int_0^\infty G_{iu}\sin(\omega\xi)d\xi, \\ R_{iu} &= \int_0^\infty G_{iu}\cos(\omega\xi)d\xi + Jn_jn_m\int_{-\infty}^\infty \mathsf{E}^e_{s\_ijum}ds, \end{aligned} \tag{4.103}$$

for

$$G_{iu} = JT_{yzw}n_jn_v\int_{-\infty}^\infty (T^{-1}_{kln}\mathsf{E}^{ve}_{s\_ijkl} - T^{-1}_{kmn}\mathsf{E}^e_{s\_ijkm})\mathsf{Y}_{s\_zwuv}(e^{\xi\mathbf{A}_s})_{ny}ds. \tag{4.104}$$

The eigenvalue problems are reduced to plane attenuating waves propagating in a general linear viscoelastic anisotropic media (also called the modified Christoffel equations in Garg and Červený et al. [Garg, 2007, Červený and Pšenčík, 2005]).

Non-attenuating, time harmonic plane wave in a homogeneous pre-deformed body

Next let us look at the propagation of infinitesimal time harmonic, non-attenuating plane waves in a pre-deformed body. We construct this perturbation by taking "$\alpha = 0$" in equation (4.97) and "$\beta = 0$" in equation (4.98). As a result, we must satisfy the two simultaneous equations

$$h = 0,$$
$$\lambda = \rho_o c^2. \tag{4.105}$$

From the stability condition we note that this type of perturbation is conditionally stable.

In order to satisfy the first equation in (4.105), we can conclude that the "$\mathbf{K}(\omega)$" matrix has to be a **zero** matrix, since all of its eigenvalues are zeros, and since the "$\mathbf{K}(\omega)$" matrix and the "$\mathbf{G}(\xi)$" matrix are "*sin*" transforms of each other, we have to require $\mathbf{G}(\xi) = \mathbf{0}$ in equation (4.93), which will lead to the requirement

$$T_{yzw} F_{hv} F^{-1}_{ab} T^{-1}_{bln} n_j n_h \int_{-\infty}^{\infty} (\mathsf{E}^{ve}_{s\_ijkl} F^{ve}_{s\_ka}) \mathsf{Y}_{s\_zwuv} (e^{\xi \mathbf{A}_s})_{ny} ds$$
$$= T_{yzw} F_{hv} T^{-1}_{kmn} n_j n_h \int_{-\infty}^{\infty} (\mathsf{E}^{e}_{s\_ijkl} F^{ve-1}_{s\_ml}) \mathsf{Y}_{s\_zwuv} (e^{\xi \mathbf{A}_s})_{ny} ds, \tag{4.106}$$

for $0 < \xi$. We can expand the exponent using the identity

$$e^{[A]} = [I] + [A] + \frac{1}{2!}[A]^2 + \frac{1}{3!}[A]^3 + \dots . \tag{4.107}$$

This results in a polynomial series in ξ which must be satisfied for all ξ. One may satisfy this by forcing the coefficient of each power of ξ in the equation to go to zero.

Therefore, we obtain the conditions for time harmonic non-attenuating plane waves propagating in the pre-deformed body. From equation (4.106) we can show that even for viscoelastic materials, non-attenuating plane waves probably exist under certain pre-deformations and along certain propagation directions. It can be easily shown that a pure elastic material, where the viscoelastic flow coefficient $\mathsf{C}_s = \mathbf{0}$, will have $\mathsf{Y}_{s\_zwuv} = 0$ in equation (4.106) and automatically satisfy it as a special case.

The infinitesimal non-attenuating perturbation is given as

$$\delta \mathbf{u}(t) = \mathsf{u} \cos[\omega(t - \frac{\mathbf{n} \cdot \mathbf{x}}{c})] \mathbf{d}, \tag{4.108}$$

as shown in Figure 4.5. This results in the displacement gradient

$$\check{\mathbf{H}} = \mathsf{u}(\frac{\omega}{c})\sin[\omega(t - \frac{\mathbf{n}\cdot\mathbf{X} + \mathbf{n}\cdot\mathbf{u}}{c})]\mathbf{d}\otimes\mathbf{n}, \tag{4.109}$$

and the acceleration

$$\delta\ddot{\mathbf{u}}(t) = -\mathsf{u}\omega^2\cos\{\omega[t - \frac{\mathbf{n}\cdot(\mathbf{X}+\mathbf{u})}{c}]\}\mathbf{d}. \tag{4.110}$$

This results in the equation

$$(\mathbf{R} - \lambda\mathbf{I})\mathbf{d} = \mathbf{0}, \tag{4.111}$$

for the matrix $\mathbf{R}$ given by

$$R_{iu} = JF_{xm}n_jn_x\int_{-\infty}^{\infty}\mathsf{E}^{e}_{s\_ijul}F^{ve-1}_{s\_ml}ds. \tag{4.112}$$

For the nontrivial solutions for $\mathbf{d}$, the characteristic equation has to be zero. The eigenvalues are λ and given in (4.105), the eigenvectors of equation (4.111) are the directions of the partical displacements.

It can be shown that equation (4.111) is equivalent to the Christoffel equation derived by Garg [Garg, 2007] for a plane homogeneous wave propagating in a perfectly elastic anisotropic/isotropic medium under the effect of initial stresses. If we restrict our analysis to plane elastic longitudinal waves propagating in a homogeneous pre-deformed medium, equation (4.111) is reduced to the equation obtained in Biot [Biot, 1965] for studying one dimensional wave propagation along one of the principle axes of the homogeneous pre-deformation.

We should emphasize that the perturbation forms we have selected here are very simple and ideal. This is done so that we can decouple the thermal effect from the elastic responses of the material and we have studied it while holding the temperature field constant and homogeneous during the whole process. It will be of great interest and useful to study the fully coupled thermal-elastic responses to thermal-mechanical perturbations. In this case, we need to invoke the equation of balance of work and energy for the perturbation terms given in (2.57).

CHAPTER 5

Perturbations with material constraints

Material constraints change the results for a thermodynamically consistent constitutive model. Material constraints refer to internal restrictions in the material such as incompressibility, where the volume of the material cannot change, or inextensibility, where material lines along a particular direction cannot change their length. In general, material constraints are idealizations, reflecting the difficulty of initiating one mode of deformation, for example, relative to other modes.

This chapter starts by looking at material constraints described by relations between components of the loading: deformation gradient $\mathbf{F}$, temperature θ and temperature gradient $\mathbf{G}$. We also consider material constraints that depend not only on loading variables $\mathbf{F}$, θ and $\mathbf{G}$, but also on the internal parameters $\mathbf{F}^{ve}$ and/or $\mathbf{F}^{\theta}$. Next, we study the effect on the perturbation when we have material constraints.

5.1 Material constraints in elastic materials

Let us look at the material constraints in the form of relations between components of the loading $\mathcal{L} = (\mathbf{F}, \theta, \mathbf{G})$. For example, the constraint equation describing incompressibility is

$$det(\mathbf{F}) = constant, \tag{5.1}$$

since $J = det(\mathbf{F})$ is the volume ratio relative to the volume in the reference configuration. Obviously the constant is unity if the reference configuration is taken to be a configuration that the material actually takes, such as the initial configuration.

Letting f denote the constraint equation, the generic constraint equation can be written as $f(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = 0$. As can be seen, for each material point, f represents a relation between the deformation gradient, temperature, and temperature gradient. As we have already introduced the loading $\mathcal{L} = (\mathbf{F}, \theta, \mathbf{G})$, this constraint can also be written as $f(\mathbf{X}, \mathcal{L}) = 0$.

Even though one can have several simultaneous constraint conditions, we will focus first on a single scalar constraint. If more than one constraint condition exists, they need to be compatible in the sense that satisfying one constraint will not exclude the possibility of satisfying the others.

The existence of a material constraint in an elastic material also changes the characteristics of the constitutive response functions. Let p be a scalar that denotes the additional information needed to calculate the constitutive response functions. It will be assumed that all constitutive functions must depend on this additional variable. That is,

$$\begin{aligned} \psi &= \psi^\dagger(\mathbf{X}, \mathcal{L}, p), \\ \mathbf{T} &= \mathbf{T}^\dagger(\mathbf{X}, \mathcal{L}, p), \\ \eta &= \eta^\dagger(\mathbf{X}, \mathcal{L}, p), \\ \mathbf{q} &= \mathbf{q}^\dagger(\mathbf{X}, \mathcal{L}, p). \end{aligned} \tag{5.2}$$

The constraint restricts how the components of $\mathcal{L}$ can change. The relation between the rates of change of these variables with respect to time can be obtained from taking the time derivative of the constraint to get $\dot{f} = 0$. Expanding this gives

$$\partial_{\mathbf{F}}(f) : \dot{\mathbf{F}} + \partial_\theta(f)\dot{\theta} + \partial_{\mathbf{G}}(f) \circ \dot{\mathbf{G}} = 0. \tag{5.3}$$

As can be seen, this is a scalar relation between the thirteen components of $\dot{\mathcal{L}} = (\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{G}})$, and thus reduces the degrees of freedom from thirteen to twelve. As a result, we can no longer arbitrarily assign values to all thirteen components of $\dot{\mathcal{L}}$. To simplify the presentation we will write this equation as

$$\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}} = 0. \tag{5.4}$$

The constraint condition $\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}} = 0$ states that all admissible loading rates $\dot{\mathcal{L}}$ are "orthogonal" to $\partial_{\mathcal{L}}(f)$. That is, the projection of $\dot{\mathcal{L}}$ onto $\partial_{\mathcal{L}}(f)$ is zero. We can construct every arbitrary loading rate $\dot{\mathcal{L}}^{\diamondsuit}$ by adding to an admissible loading rate $\dot{\mathcal{L}}$ an appropriate loading rate along $\partial_{\mathcal{L}}(f)$, which can be written as

$$\dot{\mathcal{L}}^{\diamondsuit} = \dot{\mathcal{L}} + \alpha \partial_{\mathcal{L}}(f), \tag{5.5}$$

where α is a scalar factor which may be changed as needed. Using this relation, one can construct an admissible loading rate $\dot{\mathcal{L}}$ from any arbitrary loading rate $\dot{\mathcal{L}}^{\diamondsuit}$ by selecting α such that $\dot{\mathcal{L}} = \dot{\mathcal{L}}^{\diamondsuit} - \alpha \partial_{\mathcal{L}}(f)$ satisfies the constraint condition. To have $\dot{\mathcal{L}}$ satisfy the constraint condition $\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}} = 0$, we must, therefore, have

$$\partial_{\mathcal{L}}(f) \circ [\dot{\mathcal{L}}^{\diamondsuit} - \alpha \partial_{\mathcal{L}}(f)] = 0, \tag{5.6}$$

which results in an expression for α given by

$$\alpha = \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^{\diamondsuit}}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}. \tag{5.7}$$

We thus will have

$$\dot{\mathcal{L}} = \dot{\mathcal{L}}^{\diamondsuit} - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^{\diamondsuit}}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathcal{L}}(f), \tag{5.8}$$

which yields the relations

$$\begin{aligned} \dot{\mathbf{F}} &= \dot{\mathbf{F}}^{\diamondsuit} - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^{\diamondsuit}}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathbf{F}}(f), \\ \dot{\theta} &= \dot{\theta}^{\diamondsuit} - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^{\diamondsuit}}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\theta}(f), \\ \dot{\mathbf{G}} &= \dot{\mathbf{G}}^{\diamondsuit} - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^{\diamondsuit}}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathbf{G}}(f). \end{aligned} \tag{5.9}$$

The material time derivative of the free energy for an elastic material when we have a constraint is given by

$$\dot{\psi} = \partial_{\mathcal{L}}(\psi) \circ \dot{\mathcal{L}} + \partial_p(\psi)\dot{p}, \tag{5.10}$$

where $\dot{\mathcal{L}}$ is constrained to loading paths that are consistent with the constraint condition. One can write $\dot{\psi}$ in terms of an arbitrary loading rate $\dot{\mathcal{L}}^{\diamondsuit}$ using the above relation to get

$$\begin{aligned}\dot{\psi} &= \partial_{\mathcal{L}}(\psi) \circ [\dot{\mathcal{L}}^{\diamondsuit} - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^{\diamondsuit}}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}\partial_{\mathcal{L}}(f)] + \partial_p(\psi)\dot{p} \\ &= [\partial_{\mathcal{L}}(\psi) - \frac{\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}\partial_{\mathcal{L}}(f)] \circ \dot{\mathcal{L}}^{\diamondsuit} + \partial_p(\psi)\dot{p}.\end{aligned} \tag{5.11}$$

The Clausius-Duhem inequality is given as

$$\rho\dot{\psi} - tr(\mathbf{TL}) + \rho\eta\dot{\theta} + \frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \le 0, \tag{5.12}$$

which must hold for admissible $\mathcal{L}$ and $\dot{\mathcal{L}}$. After introducing the relations given above into this expression one will get

$$\begin{aligned}&\rho[\partial_{\mathcal{L}}(\psi) - \frac{\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}\partial_{\mathcal{L}}(f)] \circ \dot{\mathcal{L}}^{\diamondsuit} + \rho\partial_p(\psi)\dot{p} - (\mathbf{T}^T\mathbf{F}^{-T}) : [\dot{\mathbf{F}}^{\diamondsuit} - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^{\diamondsuit}}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}\partial_{\mathbf{F}}(f)] \\ &+ \rho\eta[\dot{\theta}^{\diamondsuit} - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^{\diamondsuit}}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}\partial_{\theta}(f)] + \frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \le 0.\end{aligned} \tag{5.13}$$

Reorganization of the terms yields

$$\begin{aligned}&\{\rho[\partial_{\mathcal{L}}(\psi) - \frac{\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}\partial_{\mathcal{L}}(f)] + \frac{(\mathbf{T}^T\mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}\partial_{\mathcal{L}}(f) \\ &- \rho\eta\frac{\partial_{\theta}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}\partial_{\mathcal{L}}(f)\} \circ \dot{\mathcal{L}}^{\diamondsuit} - (\mathbf{T}^T\mathbf{F}^{-T}) : \dot{\mathbf{F}}^{\diamondsuit} + \rho\eta\dot{\theta}^{\diamondsuit} + \rho\partial_p(\psi)\dot{p} + \frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \le 0,\end{aligned} \tag{5.14}$$

which must hold for every arbitrary $\dot{\mathcal{L}}^{\diamondsuit} = (\dot{\mathbf{F}}^{\diamondsuit}, \dot{\theta}^{\diamondsuit}, \dot{\mathbf{G}}^{\diamondsuit})$, and any arbitrary $\dot{p}$. We note that the system is linear in $\dot{\mathcal{L}}^{\diamondsuit}$ and $\dot{p}$, so that we can organize the equation into five terms, where the first term only contains $\dot{\mathbf{F}}^{\diamondsuit}$, the second term only contains $\dot{\theta}^{\diamondsuit}$, the third term only contains $\dot{\mathbf{G}}^{\diamondsuit}$, the fourth term only contains $\dot{p}$, and the fifth term is $(\mathbf{q} \circ \mathbf{g})/\theta$. The factor multiplying each rate is a function of $\mathbf{X}$, $\mathcal{L}$, and p, and independent of $\dot{\mathcal{L}}^{\diamondsuit}$ and $\dot{p}$. For the equation to hold for all values of $\dot{\mathcal{L}}^{\diamondsuit}$ and $\dot{p}$, the factors multiplying $\dot{\mathbf{F}}^{\diamondsuit}$, $\dot{\theta}^{\diamondsuit}$, $\dot{\mathbf{G}}^{\diamondsuit}$ and $\dot{p}$

each must be zero. The result of this process is the following five relations

$$\rho\partial_{\mathbf{F}}(\psi) - [\frac{\rho\partial_{\mathcal{L}}(\psi)\circ\partial_{\mathcal{L}}(f) - (\mathbf{T}^T\mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f) + \rho\eta\partial_{\theta}(f)}{\partial_{\mathcal{L}}(f)\circ\partial_{\mathcal{L}}(f)}]\partial_{\mathbf{F}}(f) - \mathbf{T}^T\mathbf{F}^{-T} = \mathbf{0}, \tag{5.15}$$
$$\rho\partial_{\theta}(\psi) - [\frac{\rho\partial_{\mathcal{L}}(\psi)\circ\partial_{\mathcal{L}}(f) - (\mathbf{T}^T\mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f) + \rho\eta\partial_{\theta}(f)}{\partial_{\mathcal{L}}(f)\circ\partial_{\mathcal{L}}(f)}]\partial_{\theta}(f) + \rho\eta = 0,$$
$$\rho\partial_{\mathbf{G}}(\psi) - [\frac{\rho\partial_{\mathcal{L}}(\psi)\circ\partial_{\mathcal{L}}(f) - (\mathbf{T}^T\mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f) + \rho\eta\partial_{\theta}(f)}{\partial_{\mathcal{L}}(f)\circ\partial_{\mathcal{L}}(f)}]\partial_{\mathbf{G}}(f) = \mathbf{0},$$
$$\partial_p(\psi) = 0,$$
$$\frac{1}{\theta}\mathbf{q}\circ\mathbf{g} \leq 0.$$

The fourth relation excludes the dependence of free energy on p, so that only the other constitutive functions may depend on it. Since we have not given any particular form or physical interpretation to p, other than that it is the quantity needed to fully determine the values of the responses, as shown in [Negahban, 2012], we can take p to be

$$p = -[\frac{\rho\partial_{\mathcal{L}}(\psi)\circ\partial_{\mathcal{L}}(f) - (\mathbf{T}^T\mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f) + \rho\eta\partial_{\theta}(f)}{\partial_{\mathcal{L}}(f)\circ\partial_{\mathcal{L}}(f)}]. \tag{5.16}$$

This will yield the following five relations

$$\mathbf{T}^T = \rho\partial_{\mathbf{F}}(\psi)\mathbf{F}^T + p\partial_{\mathbf{F}}(f)\mathbf{F}^T, \tag{5.17}$$
$$\eta = -\partial_{\theta}(\psi) - \frac{p}{\rho}\partial_{\theta}(f),$$
$$\partial_{\mathbf{G}}(\psi) = -\frac{p}{\rho}\partial_{\mathbf{G}}(f),$$
$$\partial_p(\psi) = 0,$$
$$\frac{1}{\theta}\mathbf{q}\circ\mathbf{g} \leq 0.$$

In addition, the constraint condition must also be satisfied so that

$$f(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = 0, \tag{5.18}$$

and note the fact that free energy does not depend on p.

For the case of an absolutely incompressible material given by the constraint f =

$det(\mathbf{F}) - 1 = 0$, we have

$$\partial_{\mathbf{F}}(f) = det(\mathbf{F})\mathbf{F}^{-T} = \mathbf{F}^{-T}, \tag{5.19}$$
$$\partial_{\theta}(f) = \partial_{\mathbf{G}}(f) = 0.$$

The response of such a material is given by

$$\mathbf{T}^T = \rho\partial_{\mathbf{F}}(\psi)\mathbf{F}^T + p\mathbf{I}, \tag{5.20}$$
$$\eta = -\partial_{\theta}(\psi),$$
$$\partial_{\mathbf{G}}(\psi) = \mathbf{0},$$
$$\frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0.$$

The first equation states that the Cauchy stress can be determined from the free energy only up to an unknown hydrostatic stress $p\mathbf{I}$. We physically interpret this to say that for an incompressible material the constraint makes it such that we can add any hydrostatic stress $p\mathbf{I}$ onto any state of stress without altering its shape. The second equation states that for this material the entropy is fully determined from the free energy, and is independent of p. The third equation states that the free energy is independent of the temperature gradient, and, therefore, so are the Cauchy stress and entropy. It should be noted that for the absolutely incompressible material the volume cannot change even with temperature. Therefore, this constraint does not even accommodate for thermal expansion. An isothermally incompressible material constraint can be constructed that accommodates thermal expansion by making the volume change fully determined by the temperature change, remaining constant if there is no temperature change.

The balance laws should always be satisfied and are given by

$$\rho J = \rho_o, \tag{5.21}$$
$$Div(\mathbf{T}_o^T) + \rho_o\mathbf{b} = \rho_o\ddot{\mathbf{u}},$$
$$-Div(\mathbf{q}_o) + \rho_o r + tr(\mathbf{T}_o\dot{\mathbf{F}}) = \rho_o\dot{e},$$

where,

$$\mathbf{T}_o^T = \rho_o \partial_{\mathbf{F}}(\psi) + pJ\partial_{\mathbf{F}}(f),$$
$$Div(\mathbf{T}_o^T) = Div[\rho_o \partial_{\mathbf{F}}(\psi)] + J\partial_{\mathbf{F}}(f)\mathbf{Grad}(p) + pDiv[J\partial_{\mathbf{F}}(f)]. \tag{5.22}$$

The material constraint $f(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = 0$ should also be held during the process, so that

$$\partial_{\mathbf{F}}(f) : \dot{\mathbf{F}} + \partial_\theta(f)\dot{\theta} + \partial_{\mathbf{G}}(f) \circ \dot{\mathbf{G}} = 0. \tag{5.23}$$

5.2 Material constraints in one-element visoelastic materials

We will consider the process of constructing the influence of the material constraints in materials that depend not only on the deformation gradient, temperature, and temperature gradient, but also on internal variables. This type of material and constraint is more general and allows us to study the influence of restrictions of the evolutions of the internal variables, in addition to the constraint on the components of loading. To do this, we construct a single-element viscoelastic model. The results will be similar for the general models with a continuous series of elements in parallel.

The general constraint is assumed to be given by a scalar constraint function C written as

$$C(\mathbf{F}, \theta, \mathbf{G}, \mathbf{F}^{ve}, \mathbf{F}^\theta) = 0. \tag{5.24}$$

This requires that

$$\partial_{\mathbf{F}}(C) : \dot{\mathbf{F}} + \partial_\theta(C)\dot{\theta} + \partial_{\mathbf{G}}(C) \circ \dot{\mathbf{G}} + \partial_{\mathbf{F}^{ve}}(C) : \dot{\mathbf{F}}^{ve} + \partial_{\mathbf{F}^\theta}(C) : \dot{\mathbf{F}}^\theta = 0. \tag{5.25}$$

As before, we assume the viscoelastic flow rule to be a function of the state of the material and the indeterminate scalar p, representing the added information needed to define the state, and written as

$$\mathbf{L}^{ve} = \mathbf{L}^{ve\dagger}(\mathbf{F}^e, \mathbf{F}^{ve}, \mathbf{F}^\theta, \theta, p). \tag{5.26}$$

We postulate the thermal expansion rule taking the form

$$\mathbf{L}^{\theta} = \boldsymbol{\alpha}\dot{\theta}, \tag{5.27}$$

where, the thermal expansion coefficient $\boldsymbol{\alpha}$ is a function of the state of the material and the indeterminate scalar p and written as

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}^{\dagger}[\mathbf{F}^{e}, \mathbf{F}^{ve}, \mathbf{F}^{\theta}, \theta, p]. \tag{5.28}$$

Substituting the evolution rules into (5.25) results in

$$\partial_{\mathbf{F}}(C) : \dot{\mathbf{F}} + [\partial_{\theta}(C) + \partial_{\mathbf{F}^{\theta}}(C) : (\boldsymbol{\alpha}\mathbf{F}^{\theta})]\dot{\theta} + \partial_{\mathbf{G}}(C) \circ \dot{\mathbf{G}} = -\partial_{\mathbf{F}^{ve}}(C) : (\mathbf{L}^{ve}\mathbf{F}^{ve}). \tag{5.29}$$

Using $\mathcal{U} \equiv [\partial_{\mathbf{F}}(C), [\partial_{\theta}(C) + \partial_{\mathbf{F}^{\theta}}(C) : (\boldsymbol{\alpha}\mathbf{F}^{\theta})], \partial_{\mathbf{G}}(C)]$, $\mathcal{L} \equiv [\mathbf{F}, \theta, \mathbf{G}]$, and $\dot{\mathcal{L}} \equiv [\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{G}}]$, we will write this as

$$\mathcal{U} \circ \dot{\mathcal{L}} = -\partial_{\mathbf{F}^{ve}}(C) : (\mathbf{L}^{ve}\mathbf{F}^{ve}). \tag{5.30}$$

Since $\dot{\mathcal{L}}$ represents a loading rate, the constraint requires that an admissible loading rate be such that its projection along $\mathcal{U}$ be "$-\partial_{\mathbf{F}^{ve}}(C) : (\mathbf{L}^{ve}\mathbf{F}^{ve})$." We can manipulate the above equation to get

$$\mathcal{U} \circ (\dot{\mathcal{L}} - \beta\mathcal{U}) = 0, \tag{5.31}$$

where,

$$\beta = -\frac{\partial_{\mathbf{F}^{ve}}(C) : (\mathbf{L}^{ve}\mathbf{F}^{ve})}{\mathcal{U} \circ \mathcal{U}}. \tag{5.32}$$

This being the only constraint, we can construct any arbitrary loading rate $\dot{\mathcal{L}}^{\diamondsuit}$ from an admissible loading rate $\dot{\mathcal{L}}$ by adding an arbitrary amount along $\mathcal{U}$ onto "$\dot{\mathcal{L}} - \beta\mathcal{U}$" to get

$$\dot{\mathcal{L}}^{\diamondsuit} = \dot{\mathcal{L}} - \beta\mathcal{U} + \alpha\mathcal{U}, \tag{5.33}$$

where α is an arbitrary scalar. Introducing this into the constraint equation requires that

$$\mathcal{U} \circ \dot{\mathcal{L}}^{\diamondsuit} - \alpha\mathcal{U} \circ \mathcal{U} = 0. \tag{5.34}$$

We can solve this for α to get

$$\alpha = \frac{\mathcal{U} \circ \dot{\mathcal{L}}^{\diamondsuit}}{\mathcal{U} \circ \mathcal{U}}. \tag{5.35}$$

Therefore, we can write

$$\dot{\mathcal{L}} = \dot{\mathcal{L}}^{\diamondsuit} + \beta\mathcal{U} - \frac{\mathcal{U} \circ \dot{\mathcal{L}}^{\diamondsuit}}{\mathcal{U} \circ \mathcal{U}}\mathcal{U}. \tag{5.36}$$

This can be written in "component form" with use of $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$ and $\mathbf{L}^{\diamondsuit} = \dot{\mathbf{F}}^{\diamondsuit}\mathbf{F}^{-1}$ as

$$\begin{aligned}
&\mathbf{L} = \mathbf{L}^{\diamondsuit} + (\beta - \frac{\mathcal{U} \circ \dot{\mathcal{L}}^{\diamondsuit}}{\mathcal{U} \circ \mathcal{U}})\partial_{\mathbf{F}}(C)\mathbf{F}^{-1}, \\
&\dot{\theta} = \dot{\theta}^{\diamondsuit} + (\beta - \frac{\mathcal{U} \circ \dot{\mathcal{L}}^{\diamondsuit}}{\mathcal{U} \circ \mathcal{U}})[\partial_{\theta}(C) + \partial_{\mathbf{F}^{\theta}}(C) : (\boldsymbol{\alpha}\mathbf{F}^{\theta})], \\
&\dot{\mathbf{G}} = \dot{\mathbf{G}}^{\diamondsuit} + (\beta - \frac{\mathcal{U} \circ \dot{\mathcal{L}}^{\diamondsuit}}{\mathcal{U} \circ \mathcal{U}})\partial_{\mathbf{G}}(C).
\end{aligned} \tag{5.37}$$

In a constraint material, we need more information to calculate the response than just the history of the deformation gradient and temperature. For example, we know the stress in an incompressible material only can be calculated up to a hydrostatic component, which is then matched to the applied stress, or selected such that it satisfies the balance laws and boundary conditions. Not knowing what the form of the indeterminacy is going to be, we add a scalar p to the response functions to represent the added information needed to calculate the response, and later show that its form can be evaluated knowing the constraint. For the single-element viscoelastic model we will assume the free energy is given by

$$\psi = \psi^{\dagger}[\mathbf{F}^{e}, \mathbf{F}^{ve}, \mathbf{F}^{\theta}, \mathbf{G}, \theta, p], \tag{5.38}$$

and will assume similar forms for the other response functions.

Putting this into the Clausius-Duhem inequality we get

$$\begin{aligned}
&- \Delta\mathbf{T}^{eT} : \mathbf{L} + [\rho\eta + \rho\partial_{\theta}(\psi) - \Delta\mathbf{T}^{\theta T} : \boldsymbol{\alpha}]\dot{\theta} \\
&+ \rho\partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}} + \rho\partial_{p}(\psi)\dot{p} - \Delta\mathbf{T}^{veT} : \mathbf{L}^{ve} + \frac{1}{\theta}\mathbf{q} \circ \mathbf{g} \leq 0,
\end{aligned} \tag{5.39}$$

where, we have defined the thermodynamic stresses as

$$
\begin{aligned}
\mathbf{T}^{eT} &\equiv \rho \partial_{\mathbf{F}^e}(\psi)\mathbf{F}^{eT}, \\
\mathbf{T}^{veT} &\equiv \rho \partial_{\mathbf{F}^{ve}}(\psi)\mathbf{F}^{veT}, \\
\mathbf{T}^{\theta T} &\equiv \rho \partial_{\mathbf{F}^\theta}(\psi)\mathbf{F}^{\theta T},
\end{aligned} \tag{5.40}
$$

and where, the thermodynamic overstresses are defined as

$$
\begin{aligned}
\Delta\mathbf{T}^{e} &\equiv \mathbf{T} - \mathbf{T}^{e}, \\
\Delta\mathbf{T}^{ve} &\equiv \mathbf{F}^{e-1}\mathbf{T}^{e}\mathbf{F}^{e} - \mathbf{T}^{ve}, \\
\Delta\mathbf{T}^{\theta} &\equiv \mathbf{F}^{ve-1}\mathbf{F}^{e-1}\mathbf{T}^{e}\mathbf{F}^{e}\mathbf{F}^{ve} - \mathbf{T}^{\theta}.
\end{aligned} \tag{5.41}
$$

Before we proceed, let us note that without a constraint $\Delta\mathbf{T}^e = \mathbf{0}$ since $\mathbf{T}$ becomes equal to $\mathbf{T}^e$, which is not the case here. If we now substitute for $\mathbf{L}$, $\dot{\theta}$ and $\dot{\mathbf{G}}$ from, respectively, equations (5.37), we get

$$
\begin{aligned}
&\left\{-\Delta\mathbf{T}^{eT} + \frac{[\partial_{\mathbf{F}}(C)\mathbf{F}^T]}{\mathcal{U}\circ\mathcal{U}}\varpi\right\} : \mathbf{L}^{\diamondsuit} \\
&+ \left\{\rho\eta + \rho\partial_\theta(\psi) - \Delta\mathbf{T}^{\theta T} : \boldsymbol{\alpha} + \frac{[\partial_\theta(C) + \partial_{\mathbf{F}^\theta}(C) : (\boldsymbol{\alpha}\mathbf{F}^\theta)]}{\mathcal{U}\circ\mathcal{U}}\varpi\right\}\dot{\theta}^{\diamondsuit} \\
&+ \left\{\rho\partial_{\mathbf{G}}(\psi) + \frac{[\partial_{\mathbf{G}}(C)]}{\mathcal{U}\circ\mathcal{U}}\varpi\right\}\circ\dot{\mathbf{G}}^{\diamondsuit} \\
&+ \rho\partial_p(\psi)\dot{p} - \beta\varpi - \Delta\mathbf{T}^{veT} : \mathbf{L}^{ve} + \frac{1}{\theta}\mathbf{q}\circ\mathbf{g} \leq 0,
\end{aligned} \tag{5.42}
$$

where,

$$
\begin{aligned}
\varpi =& \Delta\mathbf{T}^{eT} : [\partial_{\mathbf{F}}(C)\mathbf{F}^{-1}] - [\rho\eta + \rho\partial_\theta(\psi) - \Delta\mathbf{T}^{\theta T} : \boldsymbol{\alpha}][\partial_\theta(C) + \partial_{\mathbf{F}^\theta}(C) : (\boldsymbol{\alpha}\mathbf{F}^\theta)] \\
&- \rho[\partial_{\mathbf{G}}(\psi)\circ\partial_{\mathbf{G}}(C)].
\end{aligned} \tag{5.43}
$$

The development is very much identical to those given in the last section. To extract the equations, we note that the term "$-\beta\varpi - \Delta\mathbf{T}^{veT} : \mathbf{L}^{ve} + \frac{1}{\theta}\mathbf{q}\circ\mathbf{g}$" does not depend on the loading rate $\dot{\mathcal{L}}$, $\dot{\mathcal{L}}^{\diamondsuit}$, or $\dot{p}$. Since $\dot{\mathcal{L}}^{\diamondsuit}$ and $\dot{p}$ can be selected arbitrarily, we have to conclude, through arguments that are similar to those given before, that we must have

$$
\begin{aligned}
&\mathbf{T}^T = \rho\partial_{\mathbf{F}^e}(\psi)\mathbf{F}^{eT} + \frac{[\partial_{\mathbf{F}}(C)\mathbf{F}^T]}{\mathcal{U}\circ\mathcal{U}}\varpi, \qquad (5.44)\\
&\eta = -\partial_\theta(\psi) + \frac{1}{\rho}\Delta\mathbf{T}^{\theta T} : \boldsymbol{\alpha} - \frac{[\partial_\theta(C) + \partial_{\mathbf{F}^\theta}(C) : (\boldsymbol{\alpha}\mathbf{F}^\theta)]}{\rho\mathcal{U}\circ\mathcal{U}}\varpi,\\
&\partial_{\mathbf{G}}(\psi) = -\frac{[\partial_{\mathbf{G}}(C)]}{\rho\mathcal{U}\circ\mathcal{U}}\varpi,\\
&\partial_p(\psi) = 0,\\
&-\beta\varpi - \Delta\mathbf{T}^{veT} : \mathbf{L}^{ve} + \frac{1}{\theta}\mathbf{q}\circ\mathbf{g} \le 0.
\end{aligned}
$$

Examining the equations one immediately notes that the free energy cannot depend on the indeterminant parameter p. Further examination of the terms, noting that ψ does not depend on p, reveals that ϖ and $\boldsymbol{\alpha}$ are the only terms on the right-hand side in the expressions for stress and for entropy that can depend on p, which it inherits through the possible dependence of Cauchy stress and entropy on p. Also, one notes that the free energy cannot depend on the temperature gradient if the constraint does not, which represents the most common form of the constraint. We next assume that homogeneous temperature fields always are admissible and, therefore, conclude that

$$-\beta\varpi - \Delta\mathbf{T}^{veT} : \mathbf{L}^{ve} \le 0. \qquad (5.45)$$

One may note that in this model the parameter p can influence the viscoelastic flow and the thermal expansion. For example, the viscosity of an incompressible material can depend on pressure. The results for a general model with a continuous series of elements in parallel are similar.

5.3 The effect of material constraints on perturbations

In this section, we will look at how the material constraints influence the perturbation equations. We will investigate the effect on the perturbations from the general material constraint which is studied in the previous section and mathematically can be expressed as $C(\mathbf{F}, \theta, \mathbf{G}, \mathbf{F}^{ve}, \mathbf{F}^\theta) = 0$. We first look at how the incremental internal parameters will evolve under material constraint. Then we study the conditions on the admissible loading rates for the perturbations. Again, in the rest of this section, we construct the arguments for a single-element viscoelastic model. The results will be similar for the general model with a continuous series of elements in parallel.

5.3.1 Evolution of incremental internal parameters under a material constraint

As defined before, the base history is assumed to start at an initial time t_i and is described by giving the history of the motion and temperature up to the current time t. This is written as

$$\mathcal{H}(t_i, t) = \{[\mathbf{x}(\mathbf{X}, \tau), \theta(\mathbf{X}, \tau)] | t_i < \tau < t\}, \tag{5.46}$$

where $\mathbf{x}$ and θ denote, respectively, the position vector and temperature of a material point for the base history.

The total history for the same time interval is described as

$$\mathcal{H}^*(t_i, t) = \{[\mathbf{x}^*(\mathbf{X}, \tau), \theta^*(\mathbf{X}, \tau)] | t_i < \tau < t\}, \tag{5.47}$$

where $\mathbf{x}^*$ and θ^* are, respectively, the position vector and temperature of a material point for the total history. The constraint relation should always be satisfied during both of the loading histories. In the base history, the constraint equation can be described as

$$C(\mathbf{F}, \theta, \mathbf{G}, \mathbf{F}^{ve}, \mathbf{F}^{\theta}) = 0. \tag{5.48}$$

In the total history, the constraint equation can be written as

$$C(\mathbf{F}^*, \theta^*, \mathbf{G}^*, \mathbf{F}^{ve*}, \mathbf{F}^{\theta*}) = 0. \tag{5.49}$$

The relations of the kinematic variables between the base and total histories described in Chapter 4 are still true for the single-element viscoelastic model considered here. The deformation gradients of the two histories are related through

$$\delta\mathbf{F} = \mathbf{F}^* - \mathbf{F}, \tag{5.50}$$

where, $\delta\mathbf{F}$ is the perturbation of the deformation gradient. Following a similar notation for the temperature and its gradient, we will write

$$\begin{aligned} \theta^* &= \theta + \delta\theta, \\ \mathbf{G}^* &= \mathbf{G} + \delta\mathbf{G}, \end{aligned} \tag{5.51}$$

where, $\delta\theta$ is the perturbation of the temperature and $\delta\mathbf{G} = \mathbf{Grad}(\delta\theta)$ is the perturbation of the temperature gradient. Taking a time derivative gives the following relations between the associated derivatives of the two histories as

$$\begin{aligned} \dot{\mathbf{F}}^* &= \dot{\mathbf{F}} + \delta\dot{\mathbf{F}}, \\ \dot{\theta}^* &= \dot{\theta} + \delta\dot{\theta}, \\ \dot{\mathbf{G}}^* &= \dot{\mathbf{G}} + \delta\dot{\mathbf{G}}. \end{aligned} \tag{5.52}$$

We define a relative deformation gradient $\check{\mathbf{F}}$ which maps the intermediate configuration to the current configuration through

$$\mathbf{F}^* = \check{\mathbf{F}}\mathbf{F}. \tag{5.53}$$

The relation between the perturbation of the deformation gradient and the relative deformation gradient is given by

$$\delta\mathbf{F} = \check{\mathbf{H}}\mathbf{F}, \tag{5.54}$$

where $\check{\mathbf{H}} = \check{\mathbf{F}} - \mathbf{I}$ is the displacement gradient associated with $\check{\mathbf{F}}$.

The relations of the elastic, viscoelastic and thermal parts in the base history and in the total history are given by

$$\delta\mathbf{F}^e = \mathbf{F}^{e*} - \mathbf{F}^e = \tilde{\mathbf{F}}^e\mathbf{F}^e - \mathbf{F}^e = \tilde{\mathbf{H}}^e\mathbf{F}^e,$$
$$\delta\mathbf{F}^{ve} = \mathbf{F}^{ve*} - \mathbf{F}^{ve} = \tilde{\mathbf{F}}^{ve}\mathbf{F}^{ve} - \mathbf{F}^{ve} = \tilde{\mathbf{H}}^{ve}\mathbf{F}^{ve},$$
$$\delta\mathbf{F}^\theta = \mathbf{F}^{\theta*} - \mathbf{F}^\theta = \tilde{\mathbf{F}}^\theta\mathbf{F}^\theta - \mathbf{F}^\theta = \tilde{\mathbf{H}}^\theta\mathbf{F}^\theta, \tag{5.55}$$

where, $\tilde{\mathbf{F}}^e$, $\tilde{\mathbf{F}}^{ve}$ and $\tilde{\mathbf{F}}^\theta$ are the relative "increment" of elastic, viscoelastic and thermal deformation gradients comparing each pair, and $\tilde{\mathbf{H}}^e = \tilde{\mathbf{F}}^e - \mathbf{I}$ is the displacement gradient for the elastic perturbation, $\tilde{\mathbf{H}}^{ve} = \tilde{\mathbf{F}}^{ve} - \mathbf{I}$ is the displacement gradient for the viscoelastic perturbation, and $\tilde{\mathbf{H}}^\theta = \tilde{\mathbf{F}}^\theta - \mathbf{I}$ is the displacement gradient for the thermal perturbation. As the perturbation is small, it can be shown that the increments represent small differences so that the deformation gradients are close to the identity $\mathbf{I}$.

Using first order Taylor series expansion, the constraint equation in the total history can be expanded around the base history as

$$C(\mathbf{F}^*, \theta^*, \mathbf{G}^*, \mathbf{F}^{ve*}, \mathbf{F}^{\theta*}) = C(\mathbf{F}, \theta, \mathbf{G}, \mathbf{F}^{ve}, \mathbf{F}^\theta) \tag{5.56}$$
$$+ \partial_{\mathbf{F}}(C) : \delta\mathbf{F} + \partial_\theta(C)\delta\theta + \partial_{\mathbf{G}}(C) : \delta\mathbf{G} + \partial_{\mathbf{F}^{ve}}(C) : \delta\mathbf{F}^{ve} + \partial_{\mathbf{F}^\theta}(C) : \delta\mathbf{F}^\theta = 0.$$

Since the constraint is satisfied in the base history, this requires

$$\partial_{\mathbf{F}}(C) : \delta\mathbf{F} + \partial_\theta(C)\delta\theta + \partial_{\mathbf{G}}(C) : \delta\mathbf{G} + \partial_{\mathbf{F}^{ve}}(C) : \delta\mathbf{F}^{ve} + \partial_{\mathbf{F}^\theta}(C) : \delta\mathbf{F}^\theta = 0. \tag{5.57}$$

After substituting the kinematics relations (5.55), the above equation becomes

$$\partial_{\mathbf{F}}(C) : \delta\mathbf{F} + \partial_\theta(C)\delta\theta + \partial_{\mathbf{G}}(C) : \delta\mathbf{G} + [\partial_{\mathbf{F}^{ve}}(C)\mathbf{F}^{veT}] : \tilde{\mathbf{H}}^{ve} + [\partial_{\mathbf{F}^\theta}(C)\mathbf{F}^{\theta T}] : \tilde{\mathbf{H}}^\theta = 0, \tag{5.58}$$

which indicates the restriction on the perturbations $\delta\mathbf{F}$, $\delta\theta$, $\delta\mathbf{G}$ and the incremental internal variables $\tilde{\mathbf{H}}^{ve}$ and $\tilde{\mathbf{H}}^\theta$ from the material constraint. This relation should be satisfied all the time.

In the total history we assume the viscoelastic flow rule to be a function of the state of the material and the indeterminate scalar p^* and written as

$$\mathbf{L}^{ve*} = \mathbf{L}^{ve\dagger}(\mathbf{F}^{e*}, \mathbf{F}^{ve*}, \mathbf{F}^{\theta*}, \theta^*, p^*). \tag{5.59}$$

In the total history, the thermal expansion rule takes the form

$$\mathbf{L}^{\theta*} = \boldsymbol{\alpha}^* \dot{\theta}^*, \tag{5.60}$$

where, the thermal expansion coefficient $\boldsymbol{\alpha}^*$ is a function of the state of the material and the indeterminate scalar p^* and written as

$$\boldsymbol{\alpha}^* = \boldsymbol{\alpha}^\dagger[\mathbf{F}^{e*}, \mathbf{F}^{ve*}, \mathbf{F}^{\theta*}, \theta^*, p^*]. \tag{5.61}$$

The thermal expansion coefficient tensor $\boldsymbol{\alpha}$ and the viscoelastic flow $\mathbf{L}^{ve}$ are also effected by the thermo-mechanical perturbation, and these material functions for the base history and the total history are related by

$$\begin{aligned} \boldsymbol{\alpha}^* &= \boldsymbol{\alpha} + \delta\boldsymbol{\alpha}, \\ \mathbf{L}^{ve*} &= \mathbf{L}^{ve} + \delta\mathbf{L}^{ve}, \end{aligned} \tag{5.62}$$

where,

$$\begin{aligned} \delta\boldsymbol{\alpha} &= \partial_{\mathbf{F}^e}(\boldsymbol{\alpha}) : \delta\mathbf{F}^e + \partial_{\mathbf{F}^{ve}}(\boldsymbol{\alpha}) : \delta\mathbf{F}^{ve} + \partial_{\mathbf{F}^\theta}(\boldsymbol{\alpha}) : \delta\mathbf{F}^\theta + \partial_\theta(\boldsymbol{\alpha})\delta\theta + \partial_p(\boldsymbol{\alpha})\delta p, \\ \delta\mathbf{L}^{ve} &= \partial_{\mathbf{F}^e}(\mathbf{L}^{ve}) : \delta\mathbf{F}^e + \partial_{\mathbf{F}^{ve}}(\mathbf{L}^{ve}) : \delta\mathbf{F}^{ve} + \partial_{\mathbf{F}^\theta}(\mathbf{L}^{ve}) : \delta\mathbf{F}^\theta + \partial_\theta(\mathbf{L}^{ve})\delta\theta + \partial_p(\mathbf{L}^{ve})\delta p, \end{aligned} \tag{5.63}$$

and $\delta p = p^* - p$ denotes the perturbation on the indeterminate parameter and is assumed to be infinitesimal.

The evolution rules for the incremental perturbation of the internal variables $\tilde{\mathbf{H}}^\theta$ and

$\tilde{\mathbf{H}}^{ve}$ are given by

$$
\begin{aligned}
&\dot{\tilde{\mathbf{H}}}^{\theta}(\mathbf{I}-\tilde{\mathbf{H}}^{\theta}) + \tilde{\mathbf{H}}^{\theta}\dot{\mathbf{F}}^{\theta}\mathbf{F}^{\theta-1} - \dot{\mathbf{F}}^{\theta}\mathbf{F}^{\theta-1}\tilde{\mathbf{H}}^{\theta} = \boldsymbol{\alpha}\delta\dot{\theta}\\
&+ (\dot{\theta}+\delta\dot{\theta})\{[\partial_{\mathbf{F}^e}(\boldsymbol{\alpha})\mathbf{F}^{eT}] : \tilde{\mathbf{H}}^{e} + [\partial_{\mathbf{F}^{ve}}(\boldsymbol{\alpha})\mathbf{F}^{veT}] : \tilde{\mathbf{H}}^{ve} + [\partial_{\mathbf{F}^{\theta}}(\boldsymbol{\alpha})\mathbf{F}^{\theta T}] : \tilde{\mathbf{H}}^{\theta}\\
&+ \partial_{\theta}(\boldsymbol{\alpha})\delta\theta + \partial_{p}(\boldsymbol{\alpha})\delta p\},\\
&\dot{\tilde{\mathbf{H}}}^{ve}(\mathbf{I}-\tilde{\mathbf{H}}^{ve}) + \tilde{\mathbf{H}}^{ve}\dot{\mathbf{F}}^{ve}\mathbf{F}^{ve-1} - \dot{\mathbf{F}}^{ve}\mathbf{F}^{ve-1}\tilde{\mathbf{H}}^{ve} = [\partial_{\mathbf{F}^e}(\mathbf{L}^{ve})\mathbf{F}^{eT}] : \tilde{\mathbf{H}}^{e}\\
&+ [\partial_{\mathbf{F}^{ve}}(\mathbf{L}^{ve})\mathbf{F}^{veT}] : \tilde{\mathbf{H}}^{ve} + [\partial_{\mathbf{F}^{\theta}}(\mathbf{L}^{ve})\mathbf{F}^{\theta T}] : \tilde{\mathbf{H}}^{\theta} + \partial_{\theta}(\mathbf{L}^{ve})\delta\theta + \partial_{p}(\mathbf{L}^{ve})\delta p,
\end{aligned}
\tag{5.64}
$$

which is a system of first order differential equations in tensor form.

The procedures to solve equations in (5.64) are similar as those for solving equations in (4.70) and are not discussed in detail here.

5.3.2 Conditions on the admissible loading rates for the perturbations

Let us now look at the admissible loading rates, which satisfy the material constraint, of the perturbation superimposed on the base history. In the total history, the constraint equation is given by

$$C(\mathbf{F}^*, \theta^*, \mathbf{G}^*, \mathbf{F}^{ve*}, \mathbf{F}^{\theta*}) = 0. \tag{5.65}$$

This requires that

$$\partial_{\mathbf{F}^*}(C) : \dot{\mathbf{F}}^* + \partial_{\theta^*}(C)\dot{\theta}^* + \partial_{\mathbf{G}^*}(C) \circ \dot{\mathbf{G}}^* + \partial_{\mathbf{F}^{ve*}}(C) : \dot{\mathbf{F}}^{ve*} + \partial_{\mathbf{F}^{\theta*}}(C) : \dot{\mathbf{F}}^{\theta*} = 0. \tag{5.66}$$

Substituting the evolution rules (5.59) and (5.60) into (5.66) results in

$$\partial_{\mathbf{F}^*}(C) : \dot{\mathbf{F}}^* + [\partial_{\theta^*}(C) + \partial_{\mathbf{F}^{\theta*}}(C) : (\boldsymbol{\alpha}^*\mathbf{F}^{\theta*})]\dot{\theta}^* + \partial_{\mathbf{G}^*}(C) \circ \dot{\mathbf{G}}^* = -\partial_{\mathbf{F}^{ve*}}(C) : (\mathbf{L}^{ve*}\mathbf{F}^{ve*}), \tag{5.67}$$

where, in the base history, the corresponding required relation due to the constraint is

$$\partial_{\mathbf{F}}(C) : \dot{\mathbf{F}} + [\partial_{\theta}(C) + \partial_{\mathbf{F}^{\theta}}(C) : (\boldsymbol{\alpha}\mathbf{F}^{\theta})]\dot{\theta} + \partial_{\mathbf{G}}(C) \circ \dot{\mathbf{G}} = -\partial_{\mathbf{F}^{ve}}(C) : (\mathbf{L}^{ve}\mathbf{F}^{ve}). \tag{5.68}$$

Using first order Taylor series expansion on the functions evaluated in the total history around the base history results in

$$\begin{aligned}
&\partial_{\mathbf{F}^*}(C) = \partial_{\mathbf{F}}(C) + \delta[\partial_{\mathbf{F}}(C)],\\
&\partial_{\mathbf{F}^{ve*}}(C) = \partial_{\mathbf{F}^{ve}}(C) + \delta[\partial_{\mathbf{F}^{ve}}(C)],\\
&\partial_{\mathbf{F}^{\theta*}}(C) = \partial_{\mathbf{F}^{\theta}}(C) + \delta[\partial_{\mathbf{F}^{\theta}}(C)],\\
&\partial_{\theta^*}(C) = \partial_{\theta}(C) + \delta[\partial_{\theta}(C)],\\
&\partial_{\mathbf{G}^*}(C) = \partial_{\mathbf{G}}(C) + \delta[\partial_{\mathbf{G}}(C)],
\end{aligned} \tag{5.69}$$

where,

$$\begin{aligned}
\delta[\partial_{\mathbf{F}}(C)] =& \partial_{\mathbf{F}}[\partial_{\mathbf{F}}(C)] : \delta\mathbf{F} + \partial_{\theta}[\partial_{\mathbf{F}}(C)]\delta\theta + \partial_{\mathbf{G}}[\partial_{\mathbf{F}}(C)] \circ \delta\mathbf{G}\\
&+ \partial_{\mathbf{F}^{ve}}[\partial_{\mathbf{F}}(C)] : \delta\mathbf{F}^{ve} + \partial_{\mathbf{F}^{\theta}}[\partial_{\mathbf{F}}(C)] : \delta\mathbf{F}^{\theta},\\
\delta[\partial_{\mathbf{F}^{ve}}(C)] =& \partial_{\mathbf{F}}[\partial_{\mathbf{F}^{ve}}(C)] : \delta\mathbf{F} + \partial_{\theta}[\partial_{\mathbf{F}^{ve}}(C)]\delta\theta + \partial_{\mathbf{G}}[\partial_{\mathbf{F}^{ve}}(C)] \circ \delta\mathbf{G}\\
&+ \partial_{\mathbf{F}^{ve}}[\partial_{\mathbf{F}^{ve}}(C)] : \delta\mathbf{F}^{ve} + \partial_{\mathbf{F}^{\theta}}[\partial_{\mathbf{F}^{ve}}(C)] : \delta\mathbf{F}^{\theta},\\
\delta[\partial_{\mathbf{F}^{\theta}}(C)] =& \partial_{\mathbf{F}}[\partial_{\mathbf{F}^{\theta}}(C)] : \delta\mathbf{F} + \partial_{\theta}[\partial_{\mathbf{F}^{\theta}}(C)]\delta\theta + \partial_{\mathbf{G}}[\partial_{\mathbf{F}^{\theta}}(C)] \circ \delta\mathbf{G}\\
&+ \partial_{\mathbf{F}^{ve}}[\partial_{\mathbf{F}^{\theta}}(C)] : \delta\mathbf{F}^{ve} + \partial_{\mathbf{F}^{\theta}}[\partial_{\mathbf{F}^{\theta}}(C)] : \delta\mathbf{F}^{\theta},\\
\delta[\partial_{\theta}(C)] =& \partial_{\mathbf{F}}[\partial_{\theta}(C)] : \delta\mathbf{F} + \partial_{\theta}[\partial_{\theta}(C)]\delta\theta + \partial_{\mathbf{G}}[\partial_{\theta}(C)] \circ \delta\mathbf{G}\\
&+ \partial_{\mathbf{F}^{ve}}[\partial_{\theta}(C)] : \delta\mathbf{F}^{ve} + \partial_{\mathbf{F}^{\theta}}[\partial_{\theta}(C)] : \delta\mathbf{F}^{\theta},\\
\delta[\partial_{\mathbf{G}}(C)] =& \partial_{\mathbf{F}}[\partial_{\mathbf{G}}(C)] : \delta\mathbf{F} + \partial_{\theta}[\partial_{\mathbf{G}}(C)]\delta\theta + \partial_{\mathbf{G}}[\partial_{\mathbf{G}}(C)] \circ \delta\mathbf{G}\\
&+ \partial_{\mathbf{F}^{ve}}[\partial_{\mathbf{G}}(C)] : \delta\mathbf{F}^{ve} + \partial_{\mathbf{F}^{\theta}}[\partial_{\mathbf{G}}(C)] : \delta\mathbf{F}^{\theta}.
\end{aligned} \tag{5.70}$$

The relations of the loading rates between the base and total histories are given by

$$\begin{aligned}
&\dot{\mathbf{F}}^* = \dot{\mathbf{F}} + \delta\dot{\mathbf{F}},\\
&\dot{\theta}^* = \dot{\theta} + \delta\dot{\theta},\\
&\dot{\mathbf{G}}^* = \dot{\mathbf{G}} + \delta\dot{\mathbf{G}}.
\end{aligned} \tag{5.71}$$

After substituting (5.69), (5.62), (5.55) and (5.71) into (5.67), and omitting the second and higher order terms, and substracting (5.68), we obtain the condition on perturbations and rates of perturbations from the material constraint that is given by

$$
\begin{aligned}
&\{\partial_{\mathbf{F}}(C) + \delta[\partial_{\mathbf{F}}(C)]\} : (\delta\dot{\mathbf{F}}) \\
&+ \left\{\partial_{\theta}(C) + \delta[\partial_{\theta}(C)] + \partial_{\mathbf{F}^{\theta}}(C) : (\boldsymbol{\alpha}\delta\mathbf{F}^{\theta} + \boldsymbol{\alpha}\mathbf{F}^{\theta} + \delta\boldsymbol{\alpha}\mathbf{F}^{\theta}) + \delta[\partial_{\mathbf{F}^{\theta}}(C)] : (\boldsymbol{\alpha}\mathbf{F}^{\theta})\right\}(\delta\dot{\theta}) \\
&+ \{\partial_{\mathbf{G}}(C) + \delta[\partial_{\mathbf{G}}(C)]\} \circ (\delta\dot{\mathbf{G}}) \\
&+ \delta[\partial_{\mathbf{F}}(C)] : \dot{\mathbf{F}} + \left\{\delta[\partial_{\theta}(C)] + \partial_{\mathbf{F}^{\theta}}(C) : (\boldsymbol{\alpha}\delta\mathbf{F}^{\theta} + \delta\boldsymbol{\alpha}\mathbf{F}^{\theta}) + \delta[\partial_{\mathbf{F}^{\theta}}(C)] : (\boldsymbol{\alpha}\mathbf{F}^{\theta})\right\}\dot{\theta} \\
&+ \delta[\partial_{\mathbf{G}}(C)] \circ \dot{\mathbf{G}} = -\partial_{\mathbf{F}^{ve}}(C) : (\mathbf{L}^{ve}\delta\mathbf{F}^{ve} + \delta\mathbf{L}^{ve}\mathbf{F}^{ve}) - \delta[\partial_{\mathbf{F}^{ve}}(C)] : (\mathbf{L}^{ve}\mathbf{F}^{ve}).
\end{aligned}
\tag{5.72}
$$

CHAPTER 6

The propagation of mechanical waves and jumps in inhomogenously deforming viscoelastic bodies

In this chapter we will apply the general theories of thermo-mechanical perturbations developed in Chapter 2 and Chapter 4 to special problems involving fairly general mechanical wave propagating in inhomogenously deforming and relaxing viscoelastic bodies. The constitutive model for the viscoelastic material under consideration is presented in Chapter 4. To do this we first describe the superimposed wave equation, then substitute it into the balance laws for the perturbations, and next we obtain the relations between the characteristics of the wave propagation, the material parameters and the underlying deformations in the base history.

We also study the propagation of wave fronts (jumps) superimposed on homogeneously and inhomogeneously deforming viscoelastic bodies. In this process, we derive the conditions for the wave fronts (jumps) to split their directions during propagation.

6.1 The propagation of mechanical waves in inhomogeneously deformed or inhomogenously deforming viscoelastic bodies with moderate loading rates

This section will focus on the propagation of mechanical waves in viscoelastic bodies undergoing inhomogeneous large deformations. We will restrict our interest to static predeformations or to the base deformations with moderate loading rates compare to the wave. In other words, we assume that during the time interval of the wave propagation considered the changes in the base loading can be negligible.

We describe the mechanical wave equation in a curvilinear coordinate system. As we will see later in this chapter, the choice of using a curvilinear coordinate system will simplify the process of studying the propagation of curved waves. We will restrict our interest to isothermal loading, which means the temperature is kept at the reference temperature for both loading histories and we thus assume that there will be no thermal expansion.

The position vector of the material point in the base history (intermidiate configuration) is given by

$$\mathbf{x}(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t), \tag{6.1}$$

where, $\mathbf{u}(\mathbf{X}, t)$ is the displacement vector in the base history. The position vector of the material point in the total history (current configuration) is given by

$$\mathbf{x}^*(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}^*(\mathbf{X}, t), \tag{6.2}$$

where, $\mathbf{u}^*(\mathbf{X}, t)$ is the total displacement vector. We define $\delta\mathbf{u}(\mathbf{X}, t)$ as the superimposed displacement vector given by $\mathbf{u}^*(\mathbf{X}, t) = \mathbf{u}(\mathbf{X}, t) + \delta\mathbf{u}(\mathbf{X}, t)$.

As shown in Fig. 6.1, if we use the coordinate system α_i to describe the position of a material particle in the reference configuration, and use β_i to describe the position in the intermediate configuration, we can write differential displacements associated with the change in coordinates, respectively, as

$$d\mathbf{X} = d\alpha_i \mathbf{e}_i,$$
$$d\mathbf{x} = d\beta_i \tilde{\mathbf{e}}_i, \tag{6.3}$$

where, the base vectors in each system are defined, respectively, by

$$\mathbf{e}_i = \frac{\partial \mathbf{X}}{\partial \alpha_i},$$
$$\tilde{\mathbf{e}}_i = \frac{\partial \mathbf{x}}{\partial \beta_i}, \tag{6.4}$$

with the metric tensors of the two bases given, respectively, as

$$g_{ij} = \mathbf{e}_i \circ \mathbf{e}_j,$$
$$\tilde{g}_{ij} = \tilde{\mathbf{e}}_i \circ \tilde{\mathbf{e}}_j. \tag{6.5}$$

The deformation gradient for the base history is defined as

$$\mathbf{F} = \frac{\partial \beta_i}{\partial \alpha_j} g^{-1}_{jk} \tilde{\mathbf{e}}_i \otimes \mathbf{e}_k, \tag{6.6}$$

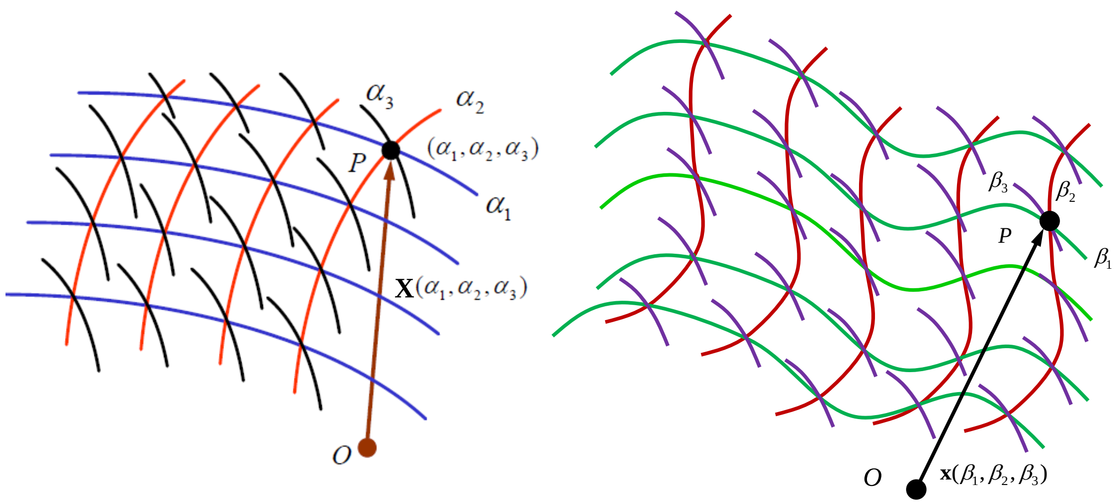
and the inverse of the deformation gradient is

$$\mathbf{F}^{-1} = \frac{\partial \alpha_i}{\partial \beta_j} \tilde{g}^{-1}_{jk} \mathbf{e}_i \otimes \tilde{\mathbf{e}}_k. \tag{6.7}$$

We model the wave motion based on that given in (4.87) for attenuating and damping harmonic plane waves. We thus describe the infinitesimal mechanical wave superimposed on an inhomogenously pre-deformed or inhomogenously deforming body with moderate rates as

$$\delta\mathbf{u}(\mathbf{x}, t) = \mathsf{u}(t) f^{\dagger}(H, t) g^{\dagger}(L, t) \mathbf{d}(L), \tag{6.8}$$

where, $\mathsf{u}(t)$ is the infinitesimal magnitude factor, $L(\beta_1, \beta_2, \beta_3)$ is the wave path in the intermediate configuration, the direction of the tangent line to the wave path L is the wave propagation direction, $\mathbf{d}(L)$ represents the dispalcement direction, $H(\beta_1, \beta_2, \beta_3)$ is the



Curvilinear coordinate system in RC

Curvilinear coordinate system in IC

Base vectors: $\mathbf{e}_i = \dfrac{\partial \mathbf{X}}{\partial \alpha_i}$

Base vectors: $\tilde{\mathbf{e}}_i = \dfrac{\partial \mathbf{x}}{\partial \beta_i}$

Figure 6.1: Curvilinear coordinate systems in the reference configuration and in the intermediate configuration.

attenuation path in the intermediate configuration, the direction of the tangent line to the attenuation path H is the wave attenuating direction, $f^\dagger(H,t)$ is the amplitude function factor, and $g^\dagger(L,t)$ is the phase function factor. From this wave equation, we define $\frac{\partial_H(f)}{f}$ as a measure of space attenuation, and $\frac{\partial_t(f)}{f}$ as a measure of time damping. We also define the phase speed $c(L,t)$ of this wave motion as

$$c(L,t) = -\frac{\partial_t[g^\dagger(L,t)]}{\partial_L[g^\dagger(L,t)]}. \tag{6.9}$$

From the symmetry property of second derivatives of $g^\dagger(L,t)$, we have the following relations between the partial derivatives of $g^\dagger(L,t)$ and $c(L,t)$ as

$$\begin{aligned}\partial_t[g^\dagger(L,t)]\partial_t[c(L,t)] + c(L,t)^2\partial_L[c(L,t)]\partial_L[g^\dagger(L,t)] + c(L,t)^3\partial_L\{\partial_L[g^\dagger(L,t)]\} \\ = \partial_t\{\partial_t[g^\dagger(L,t)]\}c(L,t).\end{aligned} \tag{6.10}$$

The similarity of this wave and the one used in Chapter 4 can be seen if we recall the space attenuating and time damping plane wave in a homogeneous pre-deformed viscoelastic body, given in equation (4.87), was written as

$$\delta\mathbf{u}(\mathbf{x},t) = \mathsf{u}e^{-\alpha\mathbf{n}\circ\mathbf{x}-\beta t}\cos[\omega(t - \frac{\mathbf{n}\circ\mathbf{x}}{c})]\mathbf{d}. \tag{6.11}$$

We thus see, for example, if we take $L = \mathbf{n}\circ\mathbf{x}$, $H = \mathbf{n}\circ\mathbf{x}$, $\mathbf{d}(L)$ as a constant, $f = e^{-\alpha\mathbf{n}\circ\mathbf{x}-\beta t}$, and $g = \cos[\omega(t - \frac{\mathbf{n}\circ\mathbf{x}}{c})]$, we specialize the general wave in (6.8) to the plane wave of Chapter 4. For this case, we can calculate the characteristic parameters of the plane wave, such as spatial attenuation coefficient α, time damping factor β, and wave speed c using the current definitions as

$$\begin{aligned}\frac{\partial_H(f)}{f} &= -\alpha, \\ \frac{\partial_t(f)}{f} &= -\beta, \\ -\frac{\partial_t(g)}{\partial_L(g)} &= c.\end{aligned} \tag{6.12}$$

The second order time derivatve of the superimposed displacement is given by

$$\delta\ddot{\mathbf{u}}(\mathbf{x},t) = [\ddot{\mathsf{u}}fg + \mathsf{u}\frac{\partial^2(f)}{\partial t^2}g + \mathsf{u}f\frac{\partial^2(g)}{\partial t^2} + 2\dot{\mathsf{u}}\frac{\partial(f)}{\partial t}g + 2\dot{\mathsf{u}}f\frac{\partial(g)}{\partial t} + 2\mathsf{u}\frac{\partial(f)}{\partial t}\frac{\partial(g)}{\partial t}]\mathbf{d}, \tag{6.13}$$

where, we have used

$$\begin{aligned} \dot{L}(\beta_1,\beta_2,\beta_3) &= 0, \\ \dot{H}(\beta_1,\beta_2,\beta_3) &= 0, \end{aligned} \tag{6.14}$$

based on the assumption that rates of the base deformation is moderate or zero,

$$\frac{\partial \beta_i}{\partial t} = 0. \tag{6.15}$$

These equations are based on the approximations that the wave propagation is much faster than the changing of the base loading, which indicates that the changes of the wave propagation path and the attenuation path during the wave event are negligible.

We introduce an arbitrary set of base vectors $\widehat{\mathbf{e}}_i$, and construct the relations with the curvilinear coordinate system used in the reference configuration as

$$\begin{aligned} \mathbf{e}_j &= P_{jk}\widehat{\mathbf{e}}_k, \\ \frac{\partial(\widehat{\mathbf{e}}_m)}{\partial \alpha_i} &= Q_{mi\_a}\widehat{\mathbf{e}}_a, \end{aligned} \tag{6.16}$$

where, P_{jk} and Q_{mi_a} are the transformation arrays. In this case, $\mathbf{d}(L)$ can be written in the $\widehat{\mathbf{e}}_i$ system as

$$\mathbf{d}(L) = \widehat{d}_i\widehat{\mathbf{e}}_i, \tag{6.17}$$

where, $\widehat{d}_i$ are the components of $\mathbf{d}(L)$ in the $\widehat{\mathbf{e}}_i$ system. The gradient of the superimposed wave motion written in the $\widehat{\mathbf{e}}_i$ system is

$$\begin{aligned} \mathbf{Grad}[\delta\mathbf{u}(\mathbf{x},t)] &= \frac{\partial[\delta\mathbf{u}(\mathbf{x},t)]}{\partial\alpha_i} \otimes g_{ij}^{-1}\mathbf{e}_j = \frac{\partial[\mathsf{u}(t)f^\dagger(H,t)g^\dagger(L,t)\widehat{d}_m\widehat{\mathbf{e}}_m]}{\partial\alpha_i} \otimes g_{ij}^{-1}\mathbf{e}_j \\ &= \mathsf{u}(t)\{[g\frac{\partial(f)}{\partial H}\frac{\partial(H)}{\partial\alpha_i} + f\frac{\partial(g)}{\partial L}\frac{\partial(L)}{\partial\alpha_i}]\widehat{d}_m + fg[\frac{\partial(\widehat{d}_m)}{\partial\alpha_i} + \widehat{d}_nQ_{ni\_m}]\}g_{ij}^{-1}P_{jk}\widehat{\mathbf{e}}_m \otimes \widehat{\mathbf{e}}_k, \end{aligned} \tag{6.18}$$

where,

$$\frac{\partial(H)}{\partial\alpha_i} = \frac{\partial(H)}{\partial\beta_1}\frac{\partial\beta_1}{\partial\alpha_i} + \frac{\partial(H)}{\partial\beta_2}\frac{\partial\beta_2}{\partial\alpha_i} + \frac{\partial(H)}{\partial\beta_3}\frac{\partial\beta_3}{\partial\alpha_i},$$
$$\frac{\partial(L)}{\partial\alpha_i} = \frac{\partial(L)}{\partial\beta_1}\frac{\partial\beta_1}{\partial\alpha_i} + \frac{\partial(L)}{\partial\beta_2}\frac{\partial\beta_2}{\partial\alpha_i} + \frac{\partial(L)}{\partial\beta_3}\frac{\partial\beta_3}{\partial\alpha_i},$$
$$\frac{\partial[\widehat{d}_m(L)]}{\partial\alpha_i} = \frac{\partial[\widehat{d}_m(L)]}{\partial L}\frac{\partial(L)}{\partial\alpha_i}. \tag{6.19}$$

The displacement gradient of the perturbation $\check{\mathbf{H}}$ is defined in Chapter 2 as

$$\check{\mathbf{H}} = [\mathbf{Grad}(\delta\mathbf{u})]\mathbf{F}^{-1}. \tag{6.20}$$

Since the temperature is kept at the reference temperature for both loading histories, there will be no thermal expansion. In order to evaluate the unknown internal variable $\boldsymbol{\Gamma}_s^{ve}(t) = \check{\mathbf{H}}_s^{ve}(t)\mathbf{F}(t)$ in the evolution equation (4.73) for the incremental viscoelastic deformation, we write the component form of the tensor equation in the $\widehat{\mathbf{e}}_i$ system as

$$\dot{\Gamma}^{ve}_{s\_ij}(t) = \mathsf{Y}^{ve}_{s\_ijkl}(t)\Gamma^{ve}_{s\_kl}(t) + \mathsf{Y}_{s\_ijkl}(t)\Gamma_{kl}(t), \tag{6.21}$$

where, $\boldsymbol{\Gamma} \equiv \check{\mathbf{H}}\mathbf{F} = [\mathbf{Grad}(\delta\mathbf{u})]$. We have used the following rules to extract the components of the tensors in equation (4.73).

Extracting the components of second order tensor $\mathbf{A} = A_{ij}\widehat{\mathbf{e}}_i \otimes \widehat{\mathbf{e}}_j$, fourth order tensor $\mathsf{C} = \mathsf{C}_{ijkl}\widehat{\mathbf{e}}_i \otimes \widehat{\mathbf{e}}_j \otimes \widehat{\mathbf{e}}_k \otimes \widehat{\mathbf{e}}_l$, and sixth order tensor $\mathsf{D} = \mathsf{D}_{ijklst}\widehat{\mathbf{e}}_i \otimes \widehat{\mathbf{e}}_j \otimes \widehat{\mathbf{e}}_k \otimes \widehat{\mathbf{e}}_l \otimes \widehat{\mathbf{e}}_s \otimes \widehat{\mathbf{e}}_t$, in the $\widehat{\mathbf{e}}_i$ system can be obtained by using the dual bases given by

$$\bar{\mathbf{e}}_1 = \frac{\widehat{\mathbf{e}}_2 \times \widehat{\mathbf{e}}_3}{(\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2) \circ \widehat{\mathbf{e}}_3},$$
$$\bar{\mathbf{e}}_2 = \frac{\widehat{\mathbf{e}}_3 \times \widehat{\mathbf{e}}_1}{(\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2) \circ \widehat{\mathbf{e}}_3},$$
$$\bar{\mathbf{e}}_3 = \frac{\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2}{(\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2) \circ \widehat{\mathbf{e}}_3}. \tag{6.22}$$

The components are then extracted using the standard rules

$$
\begin{aligned}
A_{ij} &= \bar{\mathbf{e}}_i \circ (\mathbf{A}\bar{\mathbf{e}}_j), \\
\mathsf{C}_{ijkl} &= \{[(\mathbf{C}\bar{\mathbf{e}}_l)\bar{\mathbf{e}}_k]\bar{\mathbf{e}}_j\} \circ \bar{\mathbf{e}}_i, \\
\mathsf{D}_{ijklst} &= \{\{\{[(\mathbf{D}\bar{\mathbf{e}}_t)\bar{\mathbf{e}}_s]\bar{\mathbf{e}}_l\}\bar{\mathbf{e}}_k\}\bar{\mathbf{e}}_j\} \circ \bar{\mathbf{e}}_i.
\end{aligned} \tag{6.23}
$$

To solve this system of first order differential equations in (6.21), we organize the unknows into a one-dimensional array denoted by $\boldsymbol{\chi}_s$ that takes the form

$$
\boldsymbol{\chi}_s \equiv (\Gamma^{ve}_{s\_11}, \Gamma^{ve}_{s\_12}, \Gamma^{ve}_{s\_13}, \Gamma^{ve}_{s\_21}, ...). \tag{6.24}
$$

To do this, we introduce transformation T_{ijk} which transforms the components of $\boldsymbol{\Gamma}^{ve}_s$ into the one-dimensional array $\boldsymbol{\chi}_s$ through the relation

$$
\chi_{s\_i} = T_{ijk}\Gamma^{ve}_{s\_jk}, \tag{6.25}
$$

where, $T_{ijk} = 0$ or 1 defined by the pattern of $\boldsymbol{\chi}_s$. The inverse transformation is given by

$$
\Gamma^{ve}_{s\_ij} = T^{-1}_{ijk}\chi_{s\_k}. \tag{6.26}
$$

After substituting the inverse transformation (6.26) into (6.21), and then substituting into the time derivative of (6.25), we get a system of first order differential equations

$$
\dot{\boldsymbol{\chi}}_s(t) = \mathbf{A}_s(t)\boldsymbol{\chi}_s(t) + \mathbf{f}_s(t), \tag{6.27}
$$

where, the components of the coefficient matrix $A_{s\_mn}(t)$ and the components of the inhomogeneous array $f_{s\_m}(t)$ are given as

$$
\begin{aligned}
A_{s\_mn}(t) &= T_{mij}\mathsf{Y}^{ve}_{s\_ijkl}(t)T^{-1}_{kln}, \\
f_{s\_m}(t) &= T_{mij}\mathsf{Y}_{s\_ijkl}(t)\Gamma_{kl}(t).
\end{aligned} \tag{6.28}
$$

We denote t_w as the starting time of the wave event, and assume that the duration "$t - t_w$"

is very small.

From the existence and uniqueness theorem, there exists a unique solution, since the coefficient matrix $[A_s(t)]$ and the inhomogeneous array $[f_s(t)]$ are continuous. The general solution to the system (6.27) is provided by Myskis [Myskis, 1975] as

$$\boldsymbol{\chi}_s(t) = \int_{t_w}^{t} \mathbf{Y}_s(t,\tau)\mathbf{f}_s(\tau)d\tau + \mathbf{Y}_s(t,t_w)\boldsymbol{\chi}_{sw}, \tag{6.29}$$

under the initial condition $\boldsymbol{\chi}_{sw} = \boldsymbol{\chi}_s(t_w)$, and where,

$$\begin{aligned}\mathbf{Y}_s(t,t_w) =& \mathbf{I} + \int_{t_w}^{t} \mathbf{A}_s(\tau_1)d\tau_1 + \int_{t_w}^{t} \mathbf{A}_s(\tau_1)\int_{t_w}^{\tau_1} \mathbf{A}_s(\tau_2)d\tau_2 d\tau_1 \\ &+ \int_{t_w}^{t} \mathbf{A}_s(\tau_1)\int_{t_w}^{\tau_1} \mathbf{A}_s(\tau_2)\int_{t_w}^{\tau_2} \mathbf{A}_s(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \ldots .\end{aligned} \tag{6.30}$$

From the general solution for $\boldsymbol{\chi}_s(t)$ given in (6.29) and using the inverse transformation from $\boldsymbol{\chi}_s(t)$ to $\boldsymbol{\Gamma}_s^{ve}(t)$, the evolution of incremental visoelastic flow $\check{\mathbf{H}}_s^{ve}(t)$ is obtained. We then can calculate the components of $\check{\mathbf{H}}_s^{ve}(t)$ in the $\widehat{\mathbf{e}}_i$ system from the following equation

$$\check{H}^{ve}_{s\_jk}\widehat{\mathbf{e}}_j \otimes \widehat{\mathbf{e}}_k = \Gamma^{ve}_{s\_jt}\widehat{F}\,{}^{-1}_{bk}\widehat{g}_{tb}\widehat{\mathbf{e}}_j \otimes \widehat{\mathbf{e}}_k. \tag{6.31}$$

In the $\widehat{\mathbf{e}}_i$ system, we can write $\mathbf{F}^{-1} = \widehat{F}\,{}^{-1}_{tk}\widehat{\mathbf{e}}_t \otimes \widehat{\mathbf{e}}_k$. Examining the solution for $\check{\mathbf{H}}_s^{ve}(t)$ reveals that it includes $\widehat{d}_m$ and $\frac{\partial(\widehat{d}_m)}{\partial\alpha_i}$ inherited from $\mathbf{Grad}[\delta\mathbf{u}(\mathbf{x},t)]$.

Since the base deformations are inhomogenous, the perturbation equation for balance of linear momentum takes the inhomogenous form given in (2.51), and can be represented as three scalar equations in the $\widehat{\mathbf{e}}_i$ system given by using (6.23) as

$$\begin{aligned}M_{ijk}\check{H}_{jk} + \int_{-\infty}^{\infty} M^{ve}_{s\_ijk}\check{H}^{ve}_{s\_jk}ds + N_{ijkl}[\mathbf{Grad}(\check{\mathbf{H}})]_{jkl} + \int_{-\infty}^{\infty} N^{ve}_{s\_ijkl}[\mathbf{Grad}(\check{\mathbf{H}}^{ve}_s)]_{jkl}ds \\ + N^t_{ijkl}[\mathbf{Grad}(\check{\mathbf{H}}^T)]_{jkl} + L_{ij}[\mathbf{Grad}(tr\check{\mathbf{H}})]_j + m_i = \rho_o\delta\ddot{u}_i,\end{aligned} \tag{6.32}$$

where, M_{ijk}, $M^{ve}_{s\_ijk}$, N_{ijkl}, $N^{ve}_{s\_ijkl}$, N^t_{ijkl}, L_{ij}, and m_i are material parameters that depend on the base history.

The terms that appear in equation (6.32) can be calculated as follows

$$\mathbf{Grad}(\check{\mathbf{H}}_s^{ve}) = \frac{\partial \check{\mathbf{H}}_s^{ve}}{\partial \alpha_i} \otimes g_{ix}^{-1}\mathbf{e}_x = \frac{\partial [\check{H}_{s\_jk}^{ve}\widehat{\mathbf{e}}_j \otimes \widehat{\mathbf{e}}_k]}{\partial \alpha_i} \otimes g_{ix}^{-1}\mathbf{e}_x \tag{6.33}$$
$$= [\frac{\partial \check{H}_{s\_jk}^{ve}}{\partial \alpha_i} + \check{H}_{s\_mk}^{ve} Q_{mi\_j} + \check{H}_{s\_jm}^{ve} Q_{mi\_k}] g_{ix}^{-1} P_{xy} \widehat{\mathbf{e}}_j \otimes \widehat{\mathbf{e}}_k \otimes \widehat{\mathbf{e}}_y,$$

$$\mathbf{Grad}(\check{\mathbf{H}}) = \mathbf{Grad}\{[\mathbf{Grad}(\delta\mathbf{u})]\mathbf{F}^{-1}\}$$
$$= \frac{\partial [\mathbf{Grad}(\delta\mathbf{u})]}{\partial \alpha_x}\mathbf{F}^{-1} \otimes g_{xj}^{-1}\mathbf{e}_j + [\mathbf{Grad}(\delta\mathbf{u})]\mathbf{Grad}(\mathbf{F}^{-1}) \tag{6.34}$$
$$= \frac{\partial [\mathbf{Grad}(\delta\mathbf{u})]}{\partial \alpha_x}\mathbf{F}^{-1} \otimes g_{xj}^{-1} P_{jt}\widehat{\mathbf{e}}_t + [\mathbf{Grad}(\delta\mathbf{u})]\mathbf{Grad}(\mathbf{F}^{-1}),$$

$$\mathbf{Grad}(\check{\mathbf{H}}^T) = \frac{\partial (\mathbf{F}^{-T})}{\partial \alpha_i}[\mathbf{Grad}(\delta\mathbf{u})]^T \otimes g_{ij}^{-1}\mathbf{e}_j + \mathbf{F}^{-T}\mathbf{Grad}\{[\mathbf{Grad}(\delta\mathbf{u})]^T\} \tag{6.35}$$
$$= g_{ij}^{-1} P_{jk} \frac{\partial (\mathbf{F}^{-T})}{\partial \alpha_i}[\mathbf{Grad}(\delta\mathbf{u})]^T \otimes \widehat{\mathbf{e}}_k + g_{ij}^{-1} P_{jk}\mathbf{F}^{-T}\frac{\partial [\mathbf{Grad}(\delta\mathbf{u})]^T}{\partial \alpha_i} \otimes \widehat{\mathbf{e}}_k,$$

and

$$\mathbf{Grad}(tr\check{\mathbf{H}}) = \frac{\partial (tr\check{\mathbf{H}})}{\partial \alpha_i} g_{ij}^{-1}\mathbf{e}_j = \frac{\partial (\check{H}_{xy}\widehat{g}_{xy})}{\partial \alpha_i} g_{ij}^{-1} P_{jk}\widehat{\mathbf{e}}_k. \tag{6.36}$$

The perturbation of Cauchy stress $\delta\mathbf{T}$ is given by

$$\delta\mathbf{T} = \int_{-\infty}^{\infty} [(\mathsf{E}_s^e \mathbf{F}_s^{ve-T}\mathbf{F}^T) : (\check{\mathbf{H}} - \check{\mathbf{H}}_s^{ve}) + (\mathsf{E}_s^{ve}\mathbf{F}^T) : (\mathbf{F}_s^{ve}\mathbf{F}^{-1}\check{\mathbf{H}}_s^{ve})]ds. \tag{6.37}$$

The gradient of the perturbed Cauchy stress is given by

$$\mathbf{Grad}(\delta\mathbf{T}) = \int_{-\infty}^{\infty} \{\frac{\partial (\mathsf{E}_s^e \mathbf{F}_s^{ve-T}\mathbf{F}^T)}{\partial \alpha_i} : (\check{\mathbf{H}} - \check{\mathbf{H}}_s^{ve}) \otimes g_{ij}^{-1}\mathbf{e}_j\}ds$$
$$+ \int_{-\infty}^{\infty} \{(\mathsf{E}_s^e \mathbf{F}_s^{ve-T}\mathbf{F}^T) : \mathbf{Grad}(\check{\mathbf{H}} - \check{\mathbf{H}}_s^{ve})\}ds$$
$$+ \int_{-\infty}^{\infty} \{\frac{\partial (\mathsf{E}_s^{ve}\mathbf{F}^T)}{\partial \alpha_i} : (\mathbf{F}_s^{ve}\mathbf{F}^{-1}\check{\mathbf{H}}_s^{ve}) \otimes g_{ij}^{-1}\mathbf{e}_j\}ds$$
$$+ \int_{-\infty}^{\infty} \{(\mathsf{E}_s^{ve}\mathbf{F}^T) : [\frac{\partial (\mathbf{F}_s^{ve}\mathbf{F}^{-1})}{\partial \alpha_i}\check{\mathbf{H}}_s^{ve} \otimes g_{ij}^{-1}\mathbf{e}_j + (\mathbf{F}_s^{ve}\mathbf{F}^{-1})\mathbf{Grad}(\check{\mathbf{H}}_s^{ve})]\}ds. \tag{6.38}$$

In this process, the first and second order partial derivatives of $f^\dagger(H,t)$, $g^\dagger(L,t)$, and $\widehat{d}_m(L)$ with respect to the coordinate α_i are required and given by

$$\begin{aligned}
\frac{\partial[f^\dagger(H,t)]}{\partial\alpha_i} &= \frac{\partial(f)}{\partial H}\frac{\partial(H)}{\partial\alpha_i}, \\
\frac{\partial^2[f^\dagger(H,t)]}{\partial\alpha_j\partial\alpha_i} &= \frac{\partial^2(f)}{\partial H^2}\frac{\partial(H)}{\partial\alpha_i}\frac{\partial(H)}{\partial\alpha_j} + \frac{\partial(f)}{\partial H}\frac{\partial^2(H)}{\partial\alpha_j\partial\alpha_i}, \\
\frac{\partial[g^\dagger(L,t)]}{\partial\alpha_i} &= \frac{\partial(g)}{\partial L}\frac{\partial(L)}{\partial\alpha_i}, \\
\frac{\partial^2[g^\dagger(L,t)]}{\partial\alpha_j\partial\alpha_i} &= \frac{\partial^2(g)}{\partial L^2}\frac{\partial(L)}{\partial\alpha_j}\frac{\partial(L)}{\partial\alpha_i} + \frac{\partial(g)}{\partial L}\frac{\partial^2(L)}{\partial\alpha_j\partial\alpha_i}, \\
\frac{\partial^2[\widehat{d}_m(L)]}{\partial\alpha_i\partial\alpha_j} &= \frac{\partial^2(\widehat{d}_m)}{\partial L^2}\frac{\partial(L)}{\partial\alpha_j}\frac{\partial(L)}{\partial\alpha_i} + \frac{\partial(\widehat{d}_m)}{\partial L}\frac{\partial^2(L)}{\partial\alpha_i\partial\alpha_j},
\end{aligned} \tag{6.39}$$

where,

$$\begin{aligned}
\frac{\partial^2(H)}{\partial\alpha_j\partial\alpha_i} =& \frac{\partial^2(H)}{(\partial\beta_1)^2}\frac{\partial\beta_1}{\partial\alpha_j}\frac{\partial\beta_1}{\partial\alpha_i} + \frac{\partial^2(H)}{\partial\beta_2\partial\beta_1}(\frac{\partial\beta_2}{\partial\alpha_j}\frac{\partial\beta_1}{\partial\alpha_i} + \frac{\partial\beta_1}{\partial\alpha_j}\frac{\partial\beta_2}{\partial\alpha_i}) \\
&+ \frac{\partial^2(H)}{\partial\beta_3\partial\beta_1}(\frac{\partial\beta_3}{\partial\alpha_j}\frac{\partial\beta_1}{\partial\alpha_i} + \frac{\partial\beta_1}{\partial\alpha_j}\frac{\partial\beta_3}{\partial\alpha_i}) \\
&+ \frac{\partial^2(H)}{(\partial\beta_2)^2}\frac{\partial\beta_2}{\partial\alpha_j}\frac{\partial\beta_2}{\partial\alpha_i} + \frac{\partial^2(H)}{\partial\beta_3\partial\beta_2}(\frac{\partial\beta_3}{\partial\alpha_j}\frac{\partial\beta_2}{\partial\alpha_i} + \frac{\partial\beta_2}{\partial\alpha_j}\frac{\partial\beta_3}{\partial\alpha_i}) + \frac{\partial^2(H)}{(\partial\beta_3)^2}\frac{\partial\beta_3}{\partial\alpha_j}\frac{\partial\beta_3}{\partial\alpha_i} \\
&+ \frac{\partial(H)}{\partial\beta_1}\frac{\partial^2\beta_1}{\partial\alpha_j\partial\alpha_i} + \frac{\partial(H)}{\partial\beta_2}\frac{\partial^2\beta_2}{\partial\alpha_j\partial\alpha_i} + \frac{\partial(H)}{\partial\beta_3}\frac{\partial^2\beta_3}{\partial\alpha_j\partial\alpha_i},
\end{aligned} \tag{6.40}$$

and

$$\begin{aligned}
\frac{\partial^2(L)}{\partial\alpha_j\partial\alpha_i} =& \frac{\partial^2(L)}{(\partial\beta_1)^2}\frac{\partial\beta_1}{\partial\alpha_j}\frac{\partial\beta_1}{\partial\alpha_i} + \frac{\partial^2(L)}{\partial\beta_2\partial\beta_1}(\frac{\partial\beta_2}{\partial\alpha_j}\frac{\partial\beta_1}{\partial\alpha_i} + \frac{\partial\beta_1}{\partial\alpha_j}\frac{\partial\beta_2}{\partial\alpha_i}) \\
&+ \frac{\partial^2(L)}{\partial\beta_3\partial\beta_1}(\frac{\partial\beta_3}{\partial\alpha_j}\frac{\partial\beta_1}{\partial\alpha_i} + \frac{\partial\beta_1}{\partial\alpha_j}\frac{\partial\beta_3}{\partial\alpha_i}) \\
&+ \frac{\partial^2(L)}{(\partial\beta_2)^2}\frac{\partial\beta_2}{\partial\alpha_j}\frac{\partial\beta_2}{\partial\alpha_i} + \frac{\partial^2(L)}{\partial\beta_3\partial\beta_2}(\frac{\partial\beta_3}{\partial\alpha_j}\frac{\partial\beta_2}{\partial\alpha_i} + \frac{\partial\beta_2}{\partial\alpha_j}\frac{\partial\beta_3}{\partial\alpha_i}) + \frac{\partial^2(L)}{(\partial\beta_3)^2}\frac{\partial\beta_3}{\partial\alpha_j}\frac{\partial\beta_3}{\partial\alpha_i} \\
&+ \frac{\partial(L)}{\partial\beta_1}\frac{\partial^2\beta_1}{\partial\alpha_j\partial\alpha_i} + \frac{\partial(L)}{\partial\beta_2}\frac{\partial^2\beta_2}{\partial\alpha_j\partial\alpha_i} + \frac{\partial(L)}{\partial\beta_3}\frac{\partial^2\beta_3}{\partial\alpha_j\partial\alpha_i}.
\end{aligned} \tag{6.41}$$

Substituting the above terms into the component form of the perturbation equation for balance of linear momentum (6.32), we obtain three scalar equations, which are second order partial differential equations regarding the amplitude factor $f^\dagger(H,t)$, the phase factor

$g^{\dagger}(L,t)$, and the dispalcement direction $\mathbf{d}(L)$ over space and time.

These three scalar equations allow us to formulate problems from different perspectives. For example,

(1) For a given base history, we specify the wave path L, the attenuation path H, and the forms for $f^{\dagger}(H,t)$ and $g^{\dagger}(L,t)$, we can solve for the displacement direction of the wave motion $\mathbf{d}(L)$.

(2) For a given base history, we specify the wave path L, the attenuation path H, and the displacement direction $\mathbf{d}(L)$ of the wave motion, we can get information about the possible forms of $f^{\dagger}(H,t)$ and $g^{\dagger}(L,t)$.

(3) For a given base history, we specify the forms for $f^{\dagger}(H,t)$, $g^{\dagger}(L,t)$ and the displacement direction $\mathbf{d}(L)$, we can obtain information about the possible wave path L and attenuation path H of the wave motion.

(4) We specify the wave path L and attenuation path H, and the forms for $f^{\dagger}(H,t)$, $g^{\dagger}(L,t)$, and the displacement direction $\mathbf{d}(L)$, we can get certain conditions on the base loading history to permit such wave motions.

(5) For a given base history, we specify the wave path L, the attenuation path H, the displacement direction $\mathbf{d}(L)$ of the wave motion, and the functionals of $f^{\dagger}(H,t)$ and $g^{\dagger}(L,t)$, we can calculate the characteristics of such wave motion, such as wave speed, attenuation coefficient, and damping parameter. We will look at this case in the next section.

6.2 The propagation of attenuating mechanical waves in an inhomogenously deformed viscoelastic body

Let us look at a special case. In this example we will specify the wave path L, the attenuation path H, the displacement direction of the wave motion $\mathbf{d}(L)$, and the forms for $f^{\dagger}(H,t)$ and $g^{\dagger}(L,t)$. We will obtain equations to solve for the wave propagation properties by following the procedures introduced in the previous section.

The path of the wave propagation "L" in an inhomogenously deformed body depends not only on the material properties but also on the inhomogeneity of the base deformation. Therefore, the path can be curved.

We describe the infinitesimal mechanical attenuating wave motion as

$$\delta\mathbf{u}(\mathbf{x},t) = \mathsf{u}e^{-\int_L \alpha(\ell)d\ell}\cos[\omega(t-\int_L \frac{d\ell}{c(\ell)})]\mathbf{d}(L), \tag{6.42}$$

where, the attenuation path H is, in this case, coincident with the wave path L. The direction of the tangent line to the wave path L is the wave propagation direction. The amplitude factor is $f^\dagger(L,t) = e^{-\int_L \alpha d\ell}$, and the phase factor is $g^\dagger(L,t) = \cos[\omega(t - \int_L \frac{d\ell}{c})]$. $\mathbf{d}(L)$ represents the displacement vector, $\alpha(L) = -\frac{\partial_L(f)}{f}$ represents the attenuation coefficent, $c(L) = -\frac{\partial_t[g^\dagger(L,t)]}{\partial_L[g^\dagger(L,t)]}$ represents the wave speed, and $d\ell$ represents the incremental segment along the wave path.

For simplicity, we select the $\widehat{\mathbf{e}}_i$ system such that $\widehat{\mathbf{e}}_i = \tilde{\mathbf{e}}_i$, that is, we represent the vectors and tensors in the intermediate configuration. For example, $\mathbf{d}(L) = d_i\widehat{\mathbf{e}}_i$ and $\mathbf{F}^{-1} = \widehat{F}^{\;-1}_{tk}\widehat{\mathbf{e}}_t \otimes \widehat{\mathbf{e}}_k$.

The superimposed displacement gradient can be calculated from $\check{\mathbf{H}} = [\mathbf{Grad}(\delta\mathbf{u})]\mathbf{F}^{-1}$, that is,

$$\check{\mathbf{H}} = \mathsf{u}e^{-\int_L \alpha d\ell}\{\cos[\omega(t-\int_L \frac{d\ell}{c})]\widehat{K}_{mn} + \sin[\omega(t-\int_L \frac{d\ell}{c})]\widehat{M}_{mn}\}(\widehat{\mathbf{e}}_m \otimes \widehat{\mathbf{e}}_n), \tag{6.43}$$

where,

$$\widehat{K}_{mn} = g^{-1}_{ij}P_{jg}\widehat{g}_{gt}\widehat{F}^{\;-1}_{tn}[\frac{\partial(d_m)}{\partial\alpha_i} + d_pQ_{pi\_m} - \alpha(L)\frac{\partial(L)}{\partial\alpha_i}d_m], \tag{6.44}$$
$$\widehat{M}_{mn} = \frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_i}d_m g^{-1}_{ij}P_{jg}\widehat{F}^{\;-1}_{tn}\widehat{g}_{gt}.$$

Substituting the superimposed displacement gradient (6.43) into the general solution (6.29) and taking the initial condition as $\boldsymbol{\chi}_{sw} = \mathbf{0}$ at $t = t_w$ result in the solution for the incremental internal variable $\check{H}^{ve}_{s\_jk}$ as

$$\check{H}^{ve}_{s\_jk} = \mathsf{u}e^{-\int_L \alpha d\ell}\{\cos[\omega(t-\int_L \frac{d\ell}{c})]h^{*vec}_{s\_jk} + \sin[\omega(t-\int_L \frac{d\ell}{c})]h^{*ves}_{s\_jk}\}, \tag{6.45}$$

where,

$$h^{*vec}_{s\_jk} = \Omega^c_{je}[\frac{\partial(d_k)}{\partial\alpha_e} + d_a Q_{ae\_k} - \alpha(L)\frac{\partial(L)}{\partial\alpha_e}d_k] - \Omega^s_{je}\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_e}d_k, \tag{6.46}$$
$$h^{*ves}_{s\_jk} = \Omega^c_{je}\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_e}d_k + \Omega^s_{je}[\frac{\partial(d_k)}{\partial\alpha_e} + d_a Q_{ae\_k} - \alpha(L)\frac{\partial(L)}{\partial\alpha_e}d_k],$$

and,

$$\mathsf{Z}_{jmne} = T^{-1}_{jtm}T_{npq}\widehat{F}^{-1}_{bk}\widehat{g}_{tb}\mathsf{Y}_{s\_pqkl}g^{-1}_{ef}P_{fl}, \tag{6.47}$$
$$\Omega^c_{je} = \mathsf{Z}_{jmne}\int_0^{t-t_w}\{[e^{\xi\mathbf{A}_s}]_{mn}\cos(\omega\xi)\}d\xi,$$
$$\Omega^s_{je} = \mathsf{Z}_{jmne}\int_0^{t-t_w}\{[e^{\xi\mathbf{A}_s}]_{mn}\sin(\omega\xi)d\xi\}.$$

Since the base deformations are inhomogenous, the perturbation equation for the balance of linear momentum takes the component form as given in (6.32) in the $\widehat{\mathbf{e}}_i$ system, and for this specific wave motion, the corresponding terms are

$$\begin{aligned}[\mathbf{Grad}(\check{\mathbf{H}})]_{jkl} =& \mathsf{u}e^{-\int_L \alpha d\ell}\{\cos[\omega(t-\int_L\frac{d\ell}{c})][-\alpha(L)\frac{\partial(L)}{\partial\alpha_i}\widehat{K}_{jk} + \frac{\partial(\widehat{K}_{jk})}{\partial\alpha_i} \\ &- \frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_i}\widehat{M}_{jk} + \widehat{K}_{dk}Q_{di\_j} + \widehat{K}_{jn}Q_{ni\_k}] \\ &+ \sin[\omega(t-\int_L\frac{d\ell}{c})][-\alpha(L)\frac{\partial(L)}{\partial\alpha_i}\widehat{M}_{jk} + \frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_i}\widehat{K}_{jk} + \frac{\partial(\widehat{M}_{jk})}{\partial\alpha_i} \\ &+ \widehat{M}_{dk}Q_{di\_j} + \widehat{M}_{jn}Q_{ni\_k}]\}P_{cl}g^{-1}_{ic},\end{aligned} \tag{6.48}$$

$$\begin{aligned}[\mathbf{Grad}(\check{\mathbf{H}}^T)]_{jkl} =& \mathsf{u}e^{-\int_L \alpha d\ell}\{\cos[\omega(t-\int_L\frac{d\ell}{c})][-\alpha(L)\frac{\partial(L)}{\partial\alpha_i}\widehat{K}_{kj} + \frac{\partial(\widehat{K}_{kj})}{\partial\alpha_i} \\ &- \frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_i}\widehat{M}_{kj} + \widehat{K}_{km}Q_{mi\_j} + \widehat{K}_{nj}Q_{ni\_k}] \\ &+ \sin[\omega(t-\int_L\frac{d\ell}{c})][-\alpha(L)\frac{\partial(L)}{\partial\alpha_i}\widehat{M}_{kj} + \frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_i}\widehat{K}_{kj} + \frac{\partial(\widehat{M}_{kj})}{\partial\alpha_i} \\ &+ \widehat{M}_{km}Q_{mi\_j} + \widehat{M}_{nj}Q_{ni\_k}]\}P_{xl}g^{-1}_{ix},\end{aligned} \tag{6.49}$$

$$
\begin{aligned}
\{\mathbf{Grad}[tr(\check{\mathbf{H}})]\}_j =& \mathsf{u}e^{-\int_L \alpha d\ell}\{\cos[\omega(t-\int_L \frac{d\ell}{c})][(-\alpha(L)\frac{\partial(L)}{\partial\alpha_i}\widehat{K}_{mn}+\frac{\partial(\widehat{K}_{mn})}{\partial\alpha_i} \\
&-\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_i}\widehat{M}_{mn})\widehat{g}_{mn}+\widehat{K}_{mn}\frac{\partial(\widehat{g}_{mn})}{\partial\alpha_i}] \\
&+\sin[\omega(t-\int_L \frac{d\ell}{c})][(\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_i}\widehat{K}_{mn}-\alpha(L)\frac{\partial(L)}{\partial\alpha_i}\widehat{M}_{mn}+\frac{\partial(\widehat{M}_{mn})}{\partial\alpha_i})\widehat{g}_{mn} \\
&+\widehat{M}_{mn}\frac{\partial(\widehat{g}_{mn})}{\partial\alpha_i}]\}g_{ix}^{-1}P_{xj},
\end{aligned}
\tag{6.50}
$$

and

$$
\begin{aligned}
[\mathbf{Grad}(\check{\mathbf{H}}_s^{ve})]_{jkl} =& \mathsf{u}e^{-\int_L \alpha d\ell}\{\cos[\omega(t-\int_L \frac{d\ell}{c})][-\alpha(L)\frac{\partial(L)}{\partial\alpha_i}h_{s\_jk}^{*vec}+\frac{\partial(h_{s\_jk}^{*vec})}{\partial\alpha_i} \\
&-\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_i}h_{s\_jk}^{*ves}+h_{s\_bk}^{*vec}Q_{bi\_j}+h_{s\_jb}^{*vec}Q_{bi\_k}] \\
&+\sin[\omega(t-\int_L \frac{d\ell}{c})][\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_i}h_{s\_jk}^{*vec}-\alpha(L)\frac{\partial(L)}{\partial\alpha_i}h_{s\_jk}^{*ves} \\
&+\frac{\partial(h_{s\_jk}^{*ves})}{\partial\alpha_i}+h_{s\_bk}^{*ves}Q_{bi\_j}+h_{s\_jb}^{*ves}Q_{bi\_k}]\}g_{ix}^{-1}P_{xl}.
\end{aligned}
\tag{6.51}
$$

Substituting the above terms into equation (6.32) and equating the coefficents of the "*sine*" and "*cosine*" terms, respectively, we obtain the following two sets of equations

$$
\begin{aligned}
&M_{ijk}\widehat{K}_{jk}+\int_{-\infty}^{\infty} M_{s\_ijk}^{ve}h_{s\_jk}^{*vec}ds \\
&+N_{ijkl}g_{zc}^{-1}P_{cl}[-\alpha(L)\frac{\partial(L)}{\partial\alpha_z}\widehat{K}_{jk}+\frac{\partial(\widehat{K}_{jk})}{\partial\alpha_z}-\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}\widehat{M}_{jk}+\widehat{K}_{dk}Q_{dz\_j}+\widehat{K}_{jn}Q_{nz\_k}] \\
&+g_{zx}^{-1}P_{xl}\int_{-\infty}^{\infty} N_{s\_ijkl}^{ve}[-\alpha(L)\frac{\partial(L)}{\partial\alpha_z}h_{s\_jk}^{*vec}+Q_{bz\_j}h_{s\_bk}^{*vec}+Q_{bz\_k}h_{s\_jb}^{*vec}+\frac{\partial(h_{s\_jk}^{*vec})}{\partial\alpha_z}-\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}h_{s\_jk}^{*ves}]ds \\
&+N_{ijkl}^{t}g_{zx}^{-1}P_{xl}[-\alpha(L)\frac{\partial(L)}{\partial\alpha_z}\widehat{K}_{kj}+\frac{\partial(\widehat{K}_{kj})}{\partial\alpha_z}-\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}\widehat{M}_{kj}+\widehat{K}_{km}Q_{mz\_j}+\widehat{K}_{nj}Q_{nz\_k}] \\
&+L_{ij}g_{zx}^{-1}P_{xj}\{[-\alpha(L)\frac{\partial(L)}{\partial\alpha_z}\widehat{K}_{mn}+\frac{\partial(\widehat{K}_{mn})}{\partial\alpha_z}-\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}\widehat{M}_{mn}]\widehat{g}_{mn}+\widehat{K}_{mn}\frac{\partial(\widehat{g}_{mn})}{\partial\alpha_z}\} \\
&=-\rho_o\omega^2 d_i,
\end{aligned}
\tag{6.52}
$$

and

$$M_{ijk}\widehat{M}_{jk} + \int_{-\infty}^{\infty} M^{ve}_{s\_ijk} h^{*ves}_{s\_jk} ds$$
$$+ N_{ijkl} g^{-1}_{zc} P_{cl}[-\alpha(L)\frac{\partial(L)}{\partial\alpha_z}\widehat{M}_{jk} + \frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}\widehat{K}_{jk} + \frac{\partial(\widehat{M}_{jk})}{\partial\alpha_z} + \widehat{M}_{dk}Q_{dz\_j} + \widehat{M}_{jn}Q_{nz\_k}]$$
$$+ g^{-1}_{zx} P_{xl}\int_{-\infty}^{\infty} N^{ve}_{s\_ijkl}[\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}h^{*vec}_{s\_jk} - \alpha(L)\frac{\partial(L)}{\partial\alpha_z}h^{*ves}_{s\_jk} + \frac{\partial(h^{*ves}_{s\_jk})}{\partial\alpha_z} + Q_{bz\_j}h^{*ves}_{s\_bk} + Q_{bz\_k}h^{*ves}_{s\_jb}]ds$$
$$+ N^{t}_{ijkl} g^{-1}_{zx} P_{xl}[-\alpha(L)\frac{\partial(L)}{\partial\alpha_z}\widehat{M}_{kj} + \frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}\widehat{K}_{kj} + \frac{\partial(\widehat{M}_{kj})}{\partial\alpha_z} + \widehat{M}_{km}Q_{mz\_j} + \widehat{M}_{nj}Q_{nz\_k}]$$
$$+ L_{ij} g^{-1}_{zx} P_{xj}\{[\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}\widehat{K}_{mn} - \alpha(L)\frac{\partial(L)}{\partial\alpha_z}\widehat{M}_{mn} + \frac{\partial(\widehat{M}_{mn})}{\partial\alpha_z}]\widehat{g}_{mn} + \widehat{M}_{mn}\frac{\partial(\widehat{g}_{mn})}{\partial\alpha_z}\} = 0. \tag{6.53}$$

After organizing these equations, we obtain the following two systems of equations in terms of the characteristics of the admissible wave mode, such as the wave speed $c(L)$, the first order derivative of wave speed $c'(L)$, the attenuation coefficient $\alpha(L)$, the first order derivative of attenuation coefficient $\alpha'(L)$, the displacement direction d_p, the first order derivative $d'_p = \frac{d[d_p(L)]}{dL}$, and the second order derivative $d''_p = \frac{d^2[d_p(L)]}{dL^2}$ given by

$$X_{ijk}\bar{K}_{jk} - J_{ijk}\bar{M}_{jk} + R_{ijkz}\frac{\partial(\bar{K}_{jk})}{\partial\alpha_z} + \int_{-\infty}^{\infty}[S_{s\_ijk}\Lambda^{c}_{s\_jk}]ds \tag{6.54}$$
$$+ g^{-1}_{zx}P_{xl}\int_{-\infty}^{\infty}[N^{ve}_{s\_ijkl}\frac{\partial(\Lambda^{c}_{s\_jk})}{\partial\alpha_z}]ds - g^{-1}_{zx}P_{xl}\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}\int_{-\infty}^{\infty}[N^{ve}_{s\_ijkl}\Lambda^{s}_{s\_jk}]ds = -\rho_o\omega^2 d_i,$$

and

$$X_{ijk}\bar{M}_{jk} + J_{ijk}\bar{K}_{jk} + R_{ijkz}\frac{\partial(\bar{M}_{jk})}{\partial\alpha_z} + \int_{-\infty}^{\infty}[S_{s\_ijk}\Lambda^{s}_{s\_jk}]ds \tag{6.55}$$
$$+ g^{-1}_{zx}P_{xl}\frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}\int_{-\infty}^{\infty}[N^{ve}_{s\_ijkl}\Lambda^{c}_{s\_jk}]ds + g^{-1}_{zx}P_{xl}\int_{-\infty}^{\infty}[N^{ve}_{s\_ijkl}\frac{\partial(\Lambda^{s}_{s\_jk})}{\partial\alpha_z}]ds = 0,$$

where,

$$\bar{K}_{jk} = \{G_{ik}[Q_{pi\_j} - \alpha(L)B_{ijp}]\}\, d_p + (G_{ik}B_{ijp})\, d'_p, \tag{6.56}$$
$$\bar{M}_{jk} = \left[\frac{\omega}{c(L)} G_{ik}B_{ijp}\right] d_p,$$
$$\Lambda^c_{s\_jk} = \left[\Omega^c_{je}Q_{pe\_k} - \alpha(L)\Omega^c_{je}B_{ekp} - \Omega^s_{je}B_{ekp}\frac{\omega}{c(L)}\right] d_p + \left(\Omega^c_{je}B_{ekp}\right) d'_p,$$
$$\Lambda^s_{s\_jk} = \left[\Omega^c_{je}B_{ekp}\frac{\omega}{c(L)} + \Omega^s_{je}Q_{pe\_k} - \alpha(L)\Omega^s_{je}B_{ekp}\right] d_p + \left(\Omega^s_{je}B_{ekp}\right) d'_p,$$

and

$$\frac{\partial(\bar{K}_{jk})}{\partial\alpha_z} = \left\{\frac{\partial\left(G_{ik}Q_{pi\_j}\right)}{\partial\alpha_z} - \left[\alpha'(L)\frac{\partial(L)}{\partial\alpha_z}G_{ik}B_{ijp} + \alpha(L)\frac{\partial\left(G_{ik}B_{ijp}\right)}{\partial\alpha_z}\right]\right\} d_p \tag{6.57}$$
$$+ \left\{G_{ik}[Q_{pi\_j} - \alpha(L)B_{ijp}]\frac{\partial(L)}{\partial\alpha_z} + \frac{\partial\left[G_{ik}\frac{\partial(L)}{\partial\alpha_i}\right]}{\partial\alpha_z}\delta_{jp}\right\} d'_p + \left[G_{ik}B_{ijp}\frac{\partial(L)}{\partial\alpha_z}\right] d''_p,$$
$$\frac{\partial(\bar{M}_{jk})}{\partial\alpha_z} = \frac{\omega}{c(L)}\left[\frac{\partial\left(G_{ik}B_{ijp}\right)}{\partial\alpha_z} - \frac{c'(L)}{c(L)}\frac{\partial(L)}{\partial\alpha_z}G_{ik}B_{ijp}\right] d_p + \left[G_{ik}B_{ijp}\frac{\partial(L)}{\partial\alpha_z}\right] d'_p,$$

$$\frac{\partial(\Lambda^c_{s\_jk})}{\partial\alpha_z} = \left[\frac{\partial\left(\Omega^c_{je}Q_{pe\_k}\right)}{\partial\alpha_z} - \alpha'(L)\frac{\partial(L)}{\partial\alpha_z}\Omega^c_{je}B_{ekp} - \alpha(L)\frac{\partial\left(\Omega^c_{je}B_{ekp}\right)}{\partial\alpha_z}\right]d_p \tag{6.58}$$
$$- \frac{\omega}{c(L)}\left[\frac{\partial\left(\Omega^s_{je}B_{ekp}\right)}{\partial\alpha_z} - \Omega^s_{je}B_{ekp}\frac{c'(L)}{c(L)}\frac{\partial(L)}{\partial\alpha_z}\right]d_p$$
$$+ \left[\left(\Omega^c_{je}Q_{pe\_k} - \alpha(L)\Omega^c_{je}B_{ekp} - \frac{\omega}{c(L)}\Omega^s_{je}B_{ekp}\right)\frac{\partial(L)}{\partial\alpha_z} + \frac{\partial\left(\Omega^c_{je}B_{ekp}\right)}{\partial\alpha_z}\right]d'_p$$
$$+ \left(\Omega^c_{je}B_{ekp}\right)\frac{\partial(L)}{\partial\alpha_z}d''_p,$$
$$\frac{\partial(\Lambda^s_{s\_jk})}{\partial\alpha_z} = \left[\frac{\partial\left(\Omega^c_{je}B_{ekp}\right)}{\partial\alpha_z}\frac{\omega}{c(L)} - \Omega^c_{je}B_{ekp}\frac{\partial(L)}{\partial\alpha_z}\frac{\omega}{c(L)^2}c'(L) + \frac{\partial\left(\Omega^s_{je}Q_{pe\_k}\right)}{\partial\alpha_z}\right]d_p$$
$$- \left[\alpha'(L)\frac{\partial(L)}{\partial\alpha_z}\Omega^s_{je}B_{ekp} + \alpha(L)\frac{\partial\left(\Omega^s_{je}B_{ekp}\right)}{\partial\alpha_z}\right]d_p$$
$$+ \left\{\left[\Omega^c_{je}B_{ekp}\frac{\omega}{c(L)} + \Omega^s_{je}Q_{pe\_k} - \alpha(L)\Omega^s_{je}B_{ekp}\right]\frac{\partial(L)}{\partial\alpha_z} + \frac{\partial\left(\Omega^s_{je}B_{ekp}\right)}{\partial\alpha_z}\right\}d'_p$$
$$+ \left(\Omega^s_{je}B_{ekp}\right)\frac{\partial(L)}{\partial\alpha_z}d''_p.$$

The components of the coefficient matrix in equations (6.54) and (6.55) are given by

$$X_{ijk} = W_{ijk} - \alpha(L)\frac{\partial(L)}{\partial\alpha_z}R_{ijkz}, \tag{6.59}$$
$$J_{ijk} = \frac{\omega}{c(L)}\frac{\partial(L)}{\partial\alpha_z}R_{ijkz},$$
$$R_{ijkz} = g^{-1}_{zc}P_{cl}(N_{ijkl} + L_{il}\widehat{g}_{jk} + N^t_{ikjl}),$$
$$W_{ijk} = M_{ijk} + g^{-1}_{zc}P_{cl}[(N_{iukl} + N^t_{ikul})Q_{jz\_u} + (N_{ijul} + N^t_{iujl})Q_{kz\_u} + L_{il}\frac{\partial(\widehat{g}_{jk})}{\partial\alpha_z}],$$
$$S_{s\_ijk} = O_{s\_ijk} - \alpha(L)g^{-1}_{zc}P_{cl}N^{ve}_{s\_ijkl}\frac{\partial(L)}{\partial\alpha_z},$$
$$O_{s\_ijk} = M^{ve}_{s\_ijk} + g^{-1}_{zc}P_{cl}(N^{ve}_{s\_iukl}Q_{jz\_u} + N^{ve}_{s\_ijul}Q_{kz\_u}),$$
$$G_{ik} = g^{-1}_{iu}P_{ug}\widehat{F}^{\,-1}_{tk}\widehat{g}_{gt},$$
$$B_{ekp} = \frac{\partial(L)}{\partial\alpha_e}\delta_{kp}.$$

The two systems of equations (6.54) and (6.55) should be simultaneously satisfied for the admissible wave mode.

6.3 The propagation of wave fronts (jumps) in deforming visoelastic bodies

Up to this point, we have assumed that all the base loadings and the perturbations are continuous and sufficiently smooth over all the body. Now we will look at the case when the perturbations are not sufficiently continuous, that is, in a volume the perturbation can be discontinuous over an entire surface, say $\tilde{S}$ as shown in Fig. 6.2, yet continuous at all other points of the volume.

In this section we study the propagation of mechanical wave fronts (jumps) in inhomogeneously and homogeneously deforming and relaxing viscoelastic bodies. We select to look at the case where there is no discontinuity in the response of the base history, but there is a discontinuity in the perturbation.

To do this, we first describe the superimposed jump (discontinuity) equations. Then we look at the jump conditions, which should be satisfied during the jump propagation. The perturbation equations for jump conditions are derived in detail in Chapter 2. Next we show the propagation characteristics of the jump in terms of the material properties and the time varying base loadings. As an example, we will look at how the longitudinal jump propagates in a triaxially deforming viscoelastic body.

6.3.1 Wave front (jump) equations

Let us look at how the wave fronts propagate in an inhomogeneously deforming and relaxing viscoelastic body. We will obtain the effects of the base deformation and the relaxation of the viscoelastic body on the propagation properties of the jump.

As shown in Fig. 6.2, we denote $\tilde{S} = \tilde{\mathbf{x}}[\tilde{\mathbf{x}}_o(\mathbf{X}, \tau), t]$ as the moving wave front (jump) surface at the current time t, with the initial wave front surface as $\tilde{S}_o = \tilde{\mathbf{x}}_o(\mathbf{X}, \tau)$. We use $\hat{\mathbf{N}}$ as the normal vector to the moving wave front surface, and use $\tilde{\mathbf{v}}[\tilde{\mathbf{x}}_o(\mathbf{X}, \tau), t]$ describing the wave front velocity. The "+" region is in front of the jump surface (base loading history)

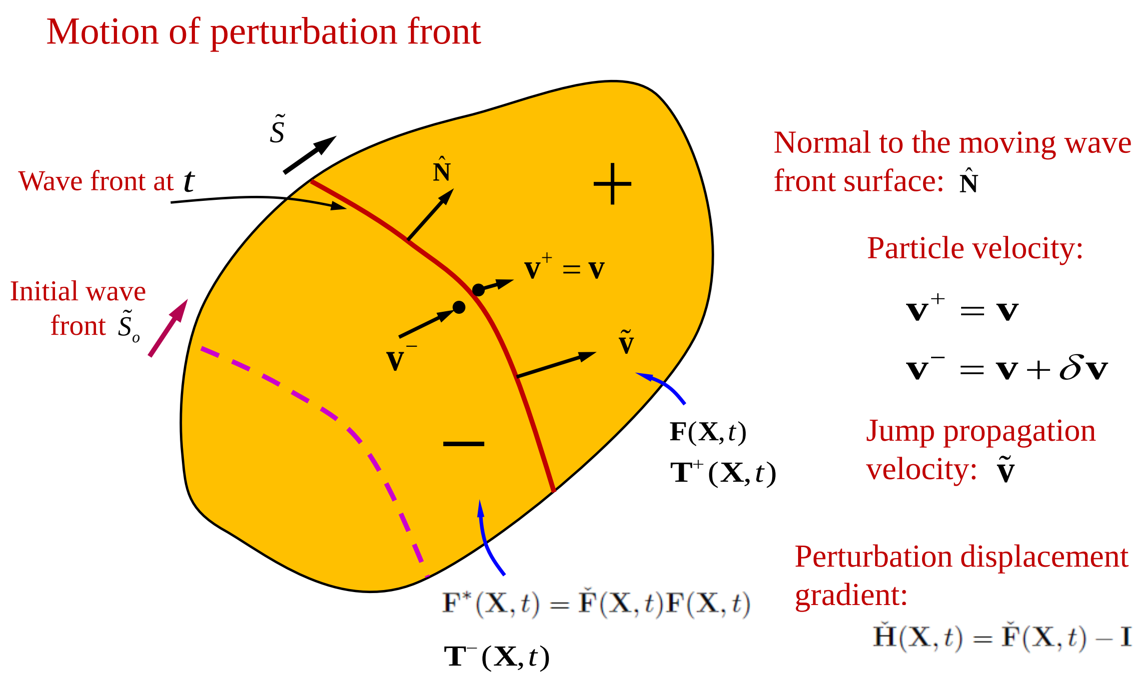


Figure 6.2: Propagation of the wave front (jump) in a deforming and relaxing viscoelastic body.

and the "−" region is behind the jump surface (total loading history). As before, we denote $\mathbf{u}(\mathbf{X}, t)$ as the displacement field in the base history (in front of the wave front surface) and $\delta\mathbf{u}(\mathbf{X}, t)$ as the superimosed displacement field. The material particle velocity in front of the wave front surface is $\mathbf{v}^+(\mathbf{X}, t) = \frac{d}{dt}[\mathbf{u}(\mathbf{X}, t)]$, and the material particle velocity behind the wave front surface as $\mathbf{v}^-(\mathbf{X}, t) = \frac{d}{dt}[\mathbf{u}(\mathbf{X}, t)] + \frac{d}{dt}[\delta\mathbf{u}(\mathbf{X}, t)]$.

As defined in Chapter 2 on the kinematic variables, the perturbation displacement gradient is $\check{\mathbf{H}}(\mathbf{X}, t)$, and the infinitesimal strain associated with the perturbation is given by

$$\check{\varepsilon}(\mathbf{X}, t) = \frac{1}{2}[\check{\mathbf{H}}(\mathbf{X}, t) + \check{\mathbf{H}}^T(\mathbf{X}, t)]. \tag{6.60}$$

The volumetric strain associated with the perturbation is

$$\check{\varepsilon}_v(\mathbf{X}, t) = tr[\check{\boldsymbol{\varepsilon}}(\mathbf{X}, t)]. \tag{6.61}$$

The jump of density across the wave front surface $\tilde{S}$ can be obtained from

$$\rho^-(\mathbf{X}, t) - \rho^+(\mathbf{X}, t) = -\check{\varepsilon}_v(\mathbf{X}, t)\rho^+(\mathbf{X}, t). \tag{6.62}$$

On the wave front surface $\tilde{S}$, the loading on the material particles is a jump (step loading), therefore, the interval of integration in the solution to the incremental viscoelastic flow (6.29) is $t - t_w = 0$, which results in $\check{\mathbf{H}}_s^{ve}(t) = \mathbf{0}$.

The perturbation of Cauchy stress $\delta\mathbf{T}^T$ for the material particles on the jump surface $\tilde{S}$ is defined as

$$\mathbf{T}^{T(+)}(\mathbf{X}, t) - \mathbf{T}^{T(-)}(\mathbf{X}, t) = -\delta\mathbf{T}^T(\mathbf{X}, t), \tag{6.63}$$

where, for the continuous viscoelastic elements in parallel, the change in the Cauchy stress is the integration of the elastic part over "s"

$$\delta\mathbf{T}^T(\mathbf{X}, t) = \int_{-\infty}^{\infty} \delta\mathbf{T}_s^{eT}(\mathbf{X}, t)ds, \tag{6.64}$$

and, after substituting $\check{\mathbf{H}}_s^{ve}(t) = \mathbf{0}$ into the perturbation of elastic stress in each element "s," we have

$$\delta\mathbf{T}_s^{eT}(\mathbf{X}, t) = \mathsf{E}_s^e(\mathbf{X}, t) : [\check{\mathbf{H}}(\mathbf{X}, t)\mathbf{F}(\mathbf{X}, t)\mathbf{F}_s^{ve-1}(\mathbf{X}, t)]. \tag{6.65}$$

The jump conditions are the relations that connect the values of the parameters on the two sides of a surface on which the parameters are discontinuous.

After manipulating the jump conditions for the perturbations in (2.86), we get the following perturbation equations for the jump conditions of conservation of mass and balance of linear momentum, respectively, as

$$\begin{aligned} &\left[\rho^+(\mathbf{v}^+ - \tilde{\mathbf{v}}) - \rho^-(\mathbf{v}^- - \tilde{\mathbf{v}})\right] \circ \hat{\mathbf{N}} = 0, \\ &\left[\rho^+\mathbf{v}^+ \otimes (\mathbf{v}^+ - \tilde{\mathbf{v}}) - \rho^-\mathbf{v}^- \otimes (\mathbf{v}^- - \tilde{\mathbf{v}})\right] \circ \hat{\mathbf{N}} = \left[\mathbf{T}^{T(+)} - \mathbf{T}^{T(-)}\right] \circ \hat{\mathbf{N}}. \end{aligned} \tag{6.66}$$

After substituting the perturbation equations for density (6.62) and for Cauchy stress (6.63) into the jump conditions (6.66), we obtain the following relations, that include the jump velocity $\tilde{\mathbf{v}}$ and the normal $\hat{\mathbf{N}}$ to the jump surface $\tilde{S}$, as

$$\tilde{\mathbf{v}}(\mathbf{X},t)\circ\hat{\mathbf{N}}(\mathbf{X},t)=[\mathbf{v}^{+}(\mathbf{X},t)\circ\hat{\mathbf{N}}(\mathbf{X},t)]+\frac{[\breve{\varepsilon}_v(\mathbf{X},t)-1][\delta\mathbf{v}(\mathbf{X},t)\circ\hat{\mathbf{N}}(\mathbf{X},t)]}{\breve{\varepsilon}_v(\mathbf{X},t)}, \tag{6.67}$$

and

$$\begin{aligned}\frac{\rho_o(\mathbf{X})}{J(\mathbf{X},t)\breve{\varepsilon}_v(\mathbf{X},t)}&[1-\breve{\varepsilon}_v(\mathbf{X},t)][\delta\mathbf{v}(\mathbf{X},t)\circ\hat{\mathbf{N}}(\mathbf{X},t)]\delta\mathbf{v}(\mathbf{X},t)\\&=\int_{-\infty}^{\infty}\{\mathbf{E}_s^e(\mathbf{X},t):[\check{\mathbf{H}}(\mathbf{X},t)\mathbf{F}(\mathbf{X},t)\mathbf{F}_s^{ve-1}(\mathbf{X},t)]\}ds\circ\hat{\mathbf{N}}(\mathbf{X},t),\end{aligned} \tag{6.68}$$

where, $\delta\mathbf{v}=\mathbf{v}^{-}-\mathbf{v}^{+}$.

The velocity of the wave front (jump) surface

Let us apply these perturbation equations of the jump conditions to calculate the velocity of the moving jump surface. For a given base loading history ("+" region in Fig. 6.2), we assume that the normal vector $\hat{\mathbf{N}}$ to the jump surface is known.

Taking a dot product with $\hat{\mathbf{N}}$ of both sides in equation (6.68), we obtain the following equation for $\delta\mathbf{v}(\mathbf{X},t)\circ\hat{\mathbf{N}}(\mathbf{X},t)$ given by

$$(\delta\mathbf{v}\circ\hat{\mathbf{N}})(\delta\mathbf{v}\circ\hat{\mathbf{N}})=\frac{\breve{\varepsilon}_v J}{\rho_o(1-\breve{\varepsilon}_v)}\hat{\mathbf{N}}\circ\left\{\int_{-\infty}^{\infty}[\mathbf{E}_s^e:(\check{\mathbf{H}}\mathbf{F}\mathbf{F}_s^{ve-1})]ds\circ\hat{\mathbf{N}}\right\}. \tag{6.69}$$

Substituting this into (6.67), we get the condition on the velocity of the moving wave front surface $\tilde{S}$ in terms of the responses to the undergoing base deformations and the perturbations as

$$\tilde{\mathbf{v}}\circ\hat{\mathbf{N}}=(\mathbf{v}^{+}\circ\hat{\mathbf{N}})\pm(\breve{\varepsilon}_v-1)\sqrt{\frac{J}{\rho_o\breve{\varepsilon}_v}\hat{\mathbf{N}}\circ\left\{\int_{-\infty}^{\infty}[\mathbf{E}_s^e:(\check{\mathbf{H}}\mathbf{F}\mathbf{F}_s^{ve-1})]ds\circ\hat{\mathbf{N}}\right\}}. \tag{6.70}$$

This shows the effects of the material properties, the base loadings and the perturbations on the velocity of the propagation of the jump.

The splitting of the wave front surface during propagation

Let us look at how the material properties, the base history and the perturbations influence the direction of motion of the jump surface. We will obtain the conditions that require the wave front to split its direction during propagation. We assume given the base loading history, and we assume that the jump velocity $\tilde{\mathbf{v}}$ and the normal to the jump surface $\hat{\mathbf{N}}$ are both unknown.

We can rewrite the perturbation equation (6.68) for jump condition as

$$\left\{\frac{\rho_o(1-\check{\varepsilon}_v)}{J\check{\varepsilon}_v}(\delta\mathbf{v}\otimes\delta\mathbf{v})-\int_{-\infty}^{\infty}[\mathsf{E}_s^e:(\check{\mathbf{H}}\mathbf{F}\mathbf{F}_s^{ve-1})]ds\right\}\circ\hat{\mathbf{N}}=\mathbf{0}, \tag{6.71}$$

where, $\hat{\mathbf{N}}(\mathbf{X},t)$ is the unknown unit vector, i.e. $\hat{\mathbf{N}}(\mathbf{X},t)\circ\hat{\mathbf{N}}(\mathbf{X},t)=1$.

We represent the tensors and vectors in an arbitrary base $\widehat{\mathbf{e}}_i$ as

$$\begin{aligned}
\hat{\mathbf{N}}(\mathbf{X},t) &= \hat{N}_z\widehat{\mathbf{e}}_z, \qquad (6.72)\\
\mathsf{E}_s^e(\mathbf{X},t) &= \mathsf{E}_{s-ijkl}^e\widehat{\mathbf{e}}_i\otimes\widehat{\mathbf{e}}_j\otimes\widehat{\mathbf{e}}_k\otimes\widehat{\mathbf{e}}_l,\\
\check{\mathbf{H}}(\mathbf{X},t) &= \check{H}_{xy}\widehat{\mathbf{e}}_x\otimes\widehat{\mathbf{e}}_y,\\
\mathbf{F}(\mathbf{X},t) &= F_{cd}\widehat{\mathbf{e}}_c\otimes\widehat{\mathbf{e}}_d,\\
\mathbf{F}_s^{ve-1}(\mathbf{X},t) &= F_{s-uv}^{ve-1}\widehat{\mathbf{e}}_u\otimes\widehat{\mathbf{e}}_v,\\
\delta\mathbf{v}(\mathbf{X},t) &= \delta v_w\widehat{\mathbf{e}}_w.
\end{aligned}$$

The component form of equation (6.71) in the $\widehat{\mathbf{e}}_i$ system is given by a set of homogeneous linear equations

$$K_{iz}\hat{N}_z=0, \tag{6.73}$$

where,

$$K_{iz}=\frac{\rho_o}{J\check{\varepsilon}_v}(1-\check{\varepsilon}_v)(\delta v_a)\widehat{g}_{az}(\delta v_i)-[\int_{-\infty}^{\infty}(\mathsf{E}_{s-ijkl}^eF_{s-uv}^{ve-1})ds]\check{H}_{xy}F_{cd}\widehat{g}_{yc}\widehat{g}_{du}\widehat{g}_{kx}\widehat{g}_{lv}\widehat{g}_{jz}. \tag{6.74}$$

The rank of a matrix is the largest order of any non-zero minor in the matrix. (The order of a minor is the side-length of the square sub-matrix of which it is the determinant.)

If the rank of the coefficient matrix $[K_{iz}]$ is equal to 3, there is a trivial solution to this homogeneous linear system.

If the rank of the coefficient matrix $[K_{iz}]$ is equal to 2, there is $3 - 2 = 1$ independent solution to this homogeneous linear system. In this case, the jump will propagate along one unique direction.

If the rank of the coefficient matrix $[K_{iz}]$ is equal to 1, there are $3 - 1 = 2$ independent solutions to this homogeneous linear system. In this case, the jump may split its direction during propagation. The possible normal vector $\hat{\mathbf{N}}(\mathbf{X}, t)$ is therefore the linear combination of the two independent solutions and may be not unique.

Once the solution of the normal vector $\hat{\mathbf{N}}(\mathbf{X}, t)$ is obtained, if the solution exists, we can substitute it into equation (6.67) or (6.70) to evaluate the jump velocity along the associated normal direction $\tilde{\mathbf{v}}(\mathbf{X}, t) \circ \hat{\mathbf{N}}(\mathbf{X}, t)$.

6.3.2 The propagation of a longitudinal jump in triaxially deforming viscoelastic bodies

Let us look at the propagation properties of the longitudinal jump in a triaxially deforming viscoelastic body as described in Fig. 6.3. Triaxial extension is a homogeneous deformation that takes a cube into a cuboid. Fig. 6.3 shows the reference and intermediate configurations for this deformation. As can be seen, X_i are used to define position in the reference configuration and rectangular coordinates x_i are used to describe position in the intermediate configuration. The two coordinates are selected such that they are parallel. The same orthonormal curvilinear base vectors are used for both coordinates. In summary, we will be using

$$
\begin{aligned}
&(\alpha_1, \alpha_2, \alpha_3) = (X_1, X_2, X_3), \\
&(\beta_1, \beta_2, \beta_3) = (x_1, x_2, x_3), \\
&\mathbf{e}_i = \tilde{\mathbf{e}}_i, \\
&g_{ij} = \tilde{g}_{ij} = \delta_{ij}.
\end{aligned}
\tag{6.75}
$$

For this set of coordinate systems, the deformation can be written as

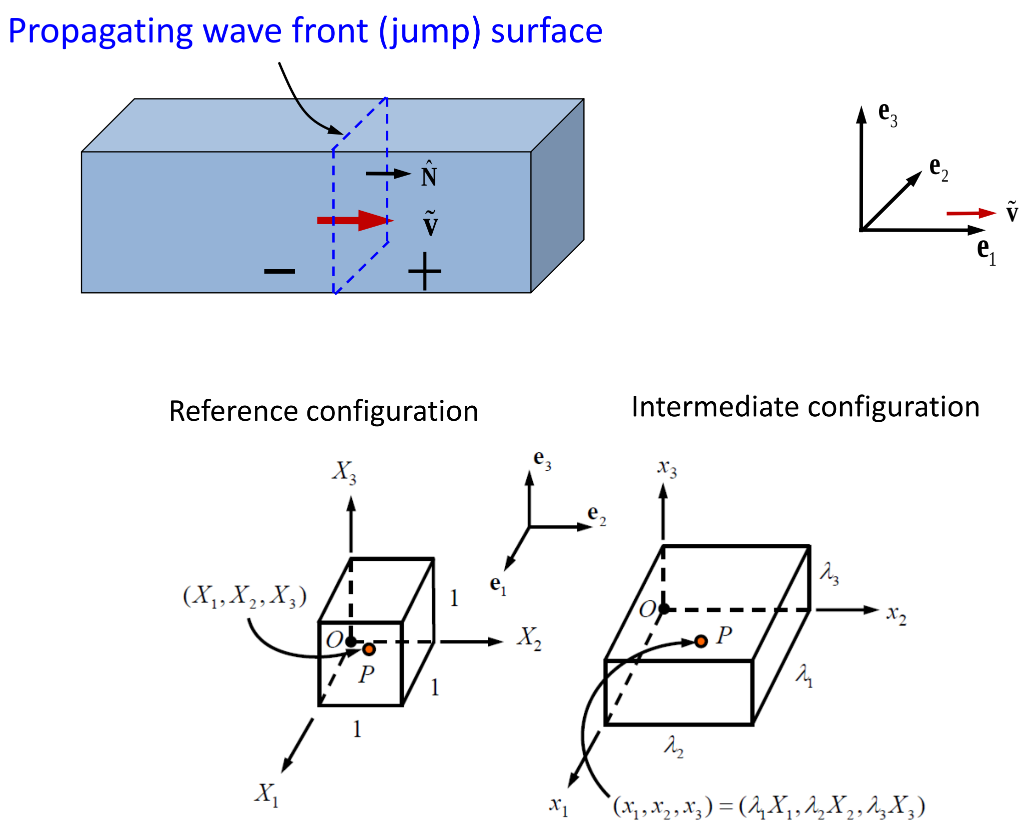


Figure 6.3: Propagation of a longitudinal jump in triaxially deforming viscoelastic bodies.

$$x_1 = \lambda_1(t)X_1, \tag{6.76}$$
$$x_2 = \lambda_2(t)X_2,$$
$$x_3 = \lambda_3(t)X_3,$$

where, $\lambda_i(t)$ are changing with time, each one is a stretch ratio associated with extension along one of the coordinate directions.

The deformation gradient is calculated as

$$\mathbf{F}(t) = \lambda_1(t)\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2(t)\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3(t)\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{6.77}$$

This results in $J = \lambda_1(t)\lambda_2(t)\lambda_3(t)$.

The viscoelastic deformation gradient in element s is then given by

$$\mathbf{F}_s^{ve}(t) = \lambda_{s\_1}^{ve}(t)\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_{s\_2}^{ve}(t)\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_{s\_3}^{ve}(t)\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{6.78}$$

and the elastic deformation gradient in element s is

$$\mathbf{F}_s^{e}(t) = \frac{\lambda_1(t)}{\lambda_{s\_1}^{ve}(t)}\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{\lambda_2(t)}{\lambda_{s\_2}^{ve}(t)}\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{\lambda_3(t)}{\lambda_{s\_3}^{ve}(t)}\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{6.79}$$

The elastic stress in element s takes the form

$$\mathbf{T}_s^{e}(t) = T_{s\_11}^{e}(t)\mathbf{e}_1 \otimes \mathbf{e}_1 + T_{s\_22}^{e}(t)\mathbf{e}_2 \otimes \mathbf{e}_2 + T_{s\_33}^{e}(t)\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{6.80}$$

We assume the normal to the wave front surface is along $\mathbf{e}_1$ direction, i.e. $\hat{\mathbf{N}} = \mathbf{e}_1$. The longitudinal jump equation is described by

$$\delta\mathbf{u}(\mathbf{X}, t) = \mathsf{u}(X_1, t)\mathbf{e}_1, \tag{6.81}$$

where, the superimposed displacement direction is along $\mathbf{e}_1$, and the magnitude of the jump displacement is $\mathsf{u}(X_1, t)$.

The gradient of the superimposed displacement is given by

$$\mathbf{Grad}[\delta\mathbf{u}(\mathbf{X},t)] = \frac{\partial[\mathsf{u}(X_1,t)]}{\partial X_1}\mathbf{e}_1 \otimes \mathbf{e}_1, \tag{6.82}$$

and the displacement gradient of the jump is

$$\check{\mathbf{H}} = \frac{\partial[\mathsf{u}(X_1,t)]}{\partial X_1}\lambda_1^{-1}\mathbf{e}_1 \otimes \mathbf{e}_1. \tag{6.83}$$

The infinitesimal strain associated with the perturbation is given by

$$\check{\boldsymbol{\varepsilon}} = \frac{\partial[\mathsf{u}(X_1,t)]}{\partial X_1}\lambda_1^{-1}\mathbf{e}_1 \otimes \mathbf{e}_1, \tag{6.84}$$

and the volumetric strain associated with this infinitesimal strain is

$$\check{\varepsilon}_v = tr[\check{\boldsymbol{\varepsilon}}] = \frac{\partial[u(X_1,t)]}{\partial X_1}\lambda_1^{-1}. \tag{6.85}$$

Substituting the base deformations and the perturbations described above into equation (6.70) results in the propagation velocity of the longitudinal jump given as

$$\begin{aligned}\tilde{v}_1(t) =& v_1^+(t) \\ &\pm [\frac{\partial[\mathsf{u}(X_1,t)]}{\partial X_1}\lambda_1(t)^{-1} - 1]\left[\sqrt{\frac{\lambda_2(t)\lambda_3(t)\lambda_1(t)^2}{\rho_o}\left\{\int_{-\infty}^{\infty}\{\mathsf{E}^e_{s-1111}(t)\lambda^{ve-1}_{s\_1}(t)\}ds\right\}}\right].\end{aligned} \tag{6.86}$$

If the jump is superimposed at long time after the base deformation becomes constant, there will be no relaxations in the base loading during propagation of the jump, we will have a constant jump velocity as

$$\tilde{v}_1 = \pm[\frac{\partial[\mathsf{u}(X_1,t)]}{\partial X_1}\lambda_1^{-1} - 1]\left[\sqrt{\frac{\lambda_2\lambda_3\lambda_1^2}{\rho_o}\left\{\int_{-\infty}^{\infty}\{\mathsf{E}^e_{s-1111}(\infty)\lambda^{ve-1}_{s\_1}(\infty)\}ds\right\}}\right], \tag{6.87}$$

where, $\mathsf{E}^e_{s-1111}(\infty)$ is the long term elastic modulus of element s, and $\lambda^{ve-1}_{s\_1}(\infty)$ is the long term viscoelastic deformation in element s.

CHAPTER 7

Conclusion and future work

7.1 Discussion and conclusion

This dissertation is primarily focused on studying the propagation properties of thermo-mechanical perturbations in deforming nonlinear materials, which include thermo-elastic solids and thermo-viscoelastic solids. The results are based on general nonlinear thermo-elastic and thermo-viscoelastic constitutive modeling structures embedded into a thermodynamically consistent framework, which permit large deformations and also large deformation rates under a broad range of temperatures. The significant contributions made in this process are listed below.

1. We considered two thermo-mechanical loading histories: a base history and a total history. The total history is constructed from the base history by adding an infinitesimal thermo-mechanical perturbation. We have derived the balance laws for the thermo-mechanical perturbations superimposed on a time varying base loading history. These perturbation equations are the same for all materials and therefore, are used throughout the entire dissertation for the thermo-elastic and visoelastic materials we used. We also developed the perturbation equations for jump conditions, which enable us to study how the thermo-mechanical discontinuities (jumps) propagate in deforming bodies.

2. We have studied the perturbation of nonlinear thermo-elastic solids and obtained

evolution equations for the incremental elastic and thermal deformations. We constructed the relations between thermo-mechanical perturbations, material properties and the undergoing base deformations. We showed how inhomogeneity in the base deformation effects the propagations of various isothermal waves.

3. In an effort to develop the theory for the propagation of thermo-mechanical perturbations in deforming and relaxing viscoelastic bodies, we worked with a general nonlinear thermo-viscoelastic constitutive model. This model is capable of capturing continuous material relaxation and, for example, in the linear case, can specialize to a general linear viscoelastic model with a continuous relaxation spectra. Using the proposed constitutive equations, we developed a superposition method to impose infinitesimal thermo-mechanical perturbations on a deforming body. There are no specific equilibrium requirement on the thermodynamic state of the loading body, a case that has not been studied before. Earlier approaches to perturbing the pre-deformed body were either restricted to small deviations away from thermodynamic equilibrium of the pre-deformations, did not account for thermo-mechanical coupling effects, or controlled the loading rate of the perturbations. In the process, we obtained the constitutive equations for the perturbations and solved for the evolutions of the incremental viscoelastic flow and thermal expansion resulting from the superimposed perturbations.

4. We considered perturbations with material constraints. This includes the constraints depending on loadings and the constraints depending on loadings and the internal variables. We also calculated the restrictions on the perturbations from the existing material constraints.

5. As examples, we considered isothermal waves in homogeneously preloaded materials and showed how the current results specialize to those published in the literature for simple and attenuating harmonic waves. We also looked at the propagations of mechanical waves and jumps in inhomogenously deforming viscoelastic bodies.

From these particular perspectives, we have addressed the objective of this dissertation, that is to describe how the thermo-mechanical perturbation in the forms of a wave or a jump propagates in deforming bodies.

In general, the equations enable us to study new types of wave motion and to investigate, for example, novel material modeling approaches and techniques based on testing under non-equilibrium conditions. Based on the present thermo-mechanical perturbation equations, many practical problems regarding material stability and geometric stability can also be investigate for time changing material bodies and structures which use model conditions that are much closer to reality.

7.2 Future work

We can extend the ideas on the propagation of perturbations in deforming bodies based on the results in this work.

(1). **Specifying the material model**

The current results are based on a fairly general thermo-dynamically consistent constitutive modeling structure for thermo-elastic and viscoelastic solids. This generality enables us to study a broad range of materials, such as polymers, bio-materials, composite materials, graded materials and soft materials. The characteristic responses of these materials include loading rate dependency, anisotropy and inhomogeneity. We can directly apply the results to incorporate their specific constitutive models and study the propagation properties of thermo-mechanical perturbations in such materials during deformations.

(2). **Studying the stability of the base history**

The current perturbation method can be used to investigate whether the thermo-mechanical base loading is stable for the selected material with or without material constraint. We can superimpose a perturbation in the vicinity of the base history and study how the perturbation evolves with time.

(3). **Studying the uniqueness and stability of the solution to the initial and boundary problem**

We can also use the perturbation approach to study the uniqueness and stability of the response of a material body with finite dimensions subject to initial and boundary conditions. We can apply the perturbations for the boundary conditions and see the effect on the response of the material particles inside the body due to the boundary perturbations. In particular, this method can be used to analyze the quality of results from the computational methods, for example, Finite Element Method.

(4). **Special propagation properties of the perturbation**

The current results allow us to design material properties and the base history in order to permit the perturbation propagating with the desired properties, such as the direction of propagation, wave modes, wave speed and attenuation property.

(5). **Characterizing material properties**

From the characteristics of the propagation of perturbations in a material undergoing various base histories, we can obtain certain information about the material properties, that can be used to characterize the material in addition to the conventional techniques.

(6). **Studying thermo-mechanical coupled waves**

The current development allows us to investigate the more complicated wave motions, such as the propagation of thermo-mechanical coupled waves. This can be useful for problems where the thermal properties are of interest.

CHAPTER 8

Appendix

If the reader would like to construct a theory based on the decomposition $\mathbf{F} = \mathbf{F}_s^e \overline{\mathbf{F}}_s^\theta \overline{\mathbf{F}}_s^{ve}$, as opposed to $\mathbf{F} = \mathbf{F}_s^e \mathbf{F}_s^{ve} \mathbf{F}_s^\theta$, the process is fairly similar. The response of each element s is assumed to be fully described by the values of the associated elastic, viscoelastic, and thermal deformation gradients plus the values of temperature and temperature gradient. As such, the model is assuming limited influence of elements on each other. For the alternate decomposition one can define a state $\overline{\mathcal{S}}_s$ for each element s as

$$\overline{\mathcal{S}}_s \equiv [\mathbf{F}_s^e, \overline{\mathbf{F}}_s^{ve}, \overline{\mathbf{F}}_s^\theta, \theta, \mathbf{G}], \tag{8.1}$$

which fully describes the response of the element.

In this case we define, respectively, the thermodynamic elastic, back, and thermal stresses as

$$\begin{aligned} \overline{\mathbf{T}}_s^{eT} &\equiv \rho \partial_{\mathbf{F}_s^e}(\psi_s^\dagger) \mathbf{F}_s^{eT}, \\ \overline{\mathbf{T}}_s^{bT} &\equiv \rho \partial_{\overline{\mathbf{F}}_s^{ve}}(\psi_s^\dagger) \overline{\mathbf{F}}_s^{veT}, \\ \overline{\mathbf{T}}_s^{\theta T} &\equiv \rho \partial_{\overline{\mathbf{F}}_s^\theta}(\psi_s^\dagger) \overline{\mathbf{F}}_s^{\theta T}, \end{aligned} \tag{8.2}$$

and, respectively, define the over stresses from the back and thermal stresses as

$$\Delta\overline{\mathbf{T}}_s^b \equiv \overline{\mathbf{F}}_s^{\theta-1}\overline{\mathbf{F}}_s^{e-1}\overline{\mathbf{T}}_s^e\overline{\mathbf{F}}_s^e\overline{\mathbf{F}}_s^\theta - \overline{\mathbf{T}}_s^b, \tag{8.3}$$

$$\Delta\overline{\mathbf{T}}_s^\theta \equiv \overline{\mathbf{F}}_s^{e-1}\overline{\mathbf{T}}_s^e\overline{\mathbf{F}}_s^e - \overline{\mathbf{T}}_s^\theta. \tag{8.4}$$

The associated tangent moduli with respect to the given variables and evaluated at the base history are defined as

$$\overline{\mathbf{E}}_s^e \equiv \partial_{\mathbf{F}_s^e}(\overline{\mathbf{T}}_s^{eT}), \tag{8.5}$$

$$\overline{\mathbf{E}}_s^{ve} \equiv \partial_{\overline{\mathbf{F}}_s^{ve}}(\overline{\mathbf{T}}_s^{eT}), \tag{8.6}$$

$$\overline{\mathbf{E}}_s^\theta \equiv \partial_{\overline{\mathbf{F}}_s^\theta}(\overline{\mathbf{T}}_s^{eT}), \tag{8.7}$$

$$\overline{\mathbf{E}}_s^\theta v\partial_\theta(\overline{\mathbf{T}}_s^{eT}), \tag{8.8}$$

and,

$$\overline{\mathbf{E}}_s^{be} \equiv \partial_{\mathbf{F}_s^e}(\overline{\mathbf{T}}_s^{bT}), \tag{8.9}$$

$$\overline{\mathbf{E}}_s^{bve} \equiv \partial_{\overline{\mathbf{F}}_s^{ve}}(\overline{\mathbf{T}}_s^{bT}), \tag{8.10}$$

$$\overline{\mathbf{E}}_s^{b\theta} \equiv \partial_{\overline{\mathbf{F}}_s^\theta}(\overline{\mathbf{T}}_s^{bT}), \tag{8.11}$$

$$\overline{\mathbf{E}}_s^{b\theta} \equiv \partial_\theta(\overline{\mathbf{T}}_s^{bT}). \tag{8.12}$$

We also use the identity

$$\mathbf{L} = \overline{\mathbf{L}}_s^e + \overline{\mathbf{F}}_s^e\overline{\mathbf{L}}_s^\theta\overline{\mathbf{F}}_s^{e-1} + \overline{\mathbf{F}}_s^e\overline{\mathbf{F}}_s^\theta\overline{\mathbf{L}}_s^{ve}\overline{\mathbf{F}}_s^{\theta-1}\overline{\mathbf{F}}_s^{e-1}, \tag{8.13}$$

where $\overline{\mathbf{L}}_s^e \equiv \dot{\overline{\mathbf{F}}}_s^e\overline{\mathbf{F}}_s^{e-1}$, $\overline{\mathbf{L}}_s^{ve} \equiv \dot{\overline{\mathbf{F}}}_s^{ve}\overline{\mathbf{F}}_s^{ve-1}$, and $\overline{\mathbf{L}}_s^\theta \equiv \dot{\overline{\mathbf{F}}}_s^\theta\overline{\mathbf{F}}_s^{\theta-1}$. In this case we assume the response is a function of this alternate decomposition. A sufficient, but not too restrictive, condition to satisfy the Clausius-Duhem inequality is then given by assuming each element is dissipative and that its free energy does not depend on the temperature gradient. This sufficient condition can be written as

$$\begin{aligned}
&\overline{\mathbf{T}} = \int_{-\infty}^{\infty} \overline{\mathbf{T}}_s^e ds,\\
&\overline{\eta} = -\int_{-\infty}^{\infty} [\partial_\theta(\psi_s^\dagger) - \frac{1}{\rho}\Delta\overline{\mathbf{T}}_s^{\theta T} : \overline{\boldsymbol{\alpha}}_s] ds,\\
&-\Delta\overline{\mathbf{T}}_s^{bT} : \overline{\mathbf{L}}_s^{ve} \leq 0,\\
&\frac{1}{\theta}\overline{\mathbf{q}}_s \circ \mathbf{g} \leq 0.
\end{aligned} \tag{8.14}$$

A fairly general viscoelastic model can be developed by taking for each element a nonlinear evolution equation (flow law) of the form

$$\overline{\mathbf{L}}_s^{ve} \equiv \overline{\mathbf{C}}_s : \Delta\overline{\mathbf{T}}_s^{bT}, \tag{8.15}$$

where $\overline{\mathbf{C}}_s$ is a fourth order coefficient function depending on the state $\overline{\mathcal{S}}_s$ at the current time but not the rates, excluding dependence on the temperature gradient. This is the form we consider for our study, which, from the Clausius-Duhem inequality, must always satisfy

$$\Delta\overline{\mathbf{T}}_s^{bT} : \overline{\mathbf{C}}_s : \Delta\overline{\mathbf{T}}_s^{bT} \geq 0, \tag{8.16}$$

for every admissible over stress $\Delta\overline{\mathbf{T}}_s^b$.

There are at least two ways to evaluate the incremental perturbation of the internal parameters in this case. The first is to directly follow a similar process as in Section 4.4.. We will first do this, but there is an alternate solution that is based on relating the perturbations from the two solutions which we will present at the end of the section.

The evolution equations given for the thermal deformation gradients for each element during the base history and the total history are directly given by

$$\begin{aligned}
&\dot{\overline{\mathbf{F}}}_s^\theta \equiv \overline{\boldsymbol{\alpha}}_s \overline{\mathbf{F}}_s^\theta \dot{\theta},\\
&\dot{\overline{\mathbf{F}}}_s^{\theta *} \equiv \overline{\boldsymbol{\alpha}}_s^* \overline{\mathbf{F}}_s^{\theta *} \dot{\theta}^*,
\end{aligned} \tag{8.17}$$

and the flow rules for the viscoelastic deformation gradients are directly given by

$$\begin{aligned}
\dot{\overline{\mathbf{F}}}_s^{ve} &\equiv (\overline{\mathbf{C}}_s : \Delta\overline{\mathbf{T}}_s^{bT})\overline{\mathbf{F}}_s^{ve}, \\
\dot{\overline{\mathbf{F}}}_s^{ve*} &\equiv (\overline{\mathbf{C}}_s^* : \Delta\overline{\mathbf{T}}_s^{b*T})\overline{\mathbf{F}}_s^{ve*}.
\end{aligned} \tag{8.18}$$

Next, we will use the following relations

$$\begin{aligned}
\overline{\mathbf{F}}_s^{ve*} &= \tilde{\overline{\mathbf{F}}}_s^{ve}\overline{\mathbf{F}}_s^{ve}, \\
\dot{\overline{\mathbf{F}}}_s^{ve*} &= \dot{\tilde{\overline{\mathbf{F}}}}_s^{ve}\overline{\mathbf{F}}_s^{ve} + \tilde{\overline{\mathbf{F}}}_s^{ve}\dot{\overline{\mathbf{F}}}_s^{ve}, \\
\overline{\mathbf{F}}_s^{\theta*} &= \tilde{\overline{\mathbf{F}}}_s^{\theta}\overline{\mathbf{F}}_s^{\theta}, \\
\dot{\overline{\mathbf{F}}}_s^{\theta} &= \mathbf{F}_s^{ve}\dot{\mathbf{F}}_s^{\theta}\mathbf{F}_s^{\theta-1}\mathbf{F}_s^{ve-1}\overline{\mathbf{F}}_s^{\theta}.
\end{aligned} \tag{8.19}$$

in manipulate the two sets of evaluation equations (8.17) and (8.18) for the two histories while eliminating the higher order infinitesimal terms. From this we get the effects of the perturbation on the evolution rules through the following two coupled first order differential equations with the unknown incremental internal variables $\tilde{\overline{\mathbf{H}}}_s^{\theta}$ and $\tilde{\overline{\mathbf{H}}}_s^{ve}$ given by

$$\begin{aligned}
\dot{\tilde{\overline{\mathbf{H}}}}_s^{\theta} =& \overline{\boldsymbol{\alpha}}_s\tilde{\overline{\mathbf{H}}}_s^{\theta}(\dot{\theta} + \delta\dot{\theta}) + \overline{\boldsymbol{\alpha}}_s\delta\dot{\theta} + (\delta\overline{\boldsymbol{\alpha}}_s)(\dot{\theta} + \delta\dot{\theta}) - \tilde{\overline{\mathbf{H}}}_s^{\theta}\overline{\boldsymbol{\alpha}}_s\dot{\theta}, \\
\dot{\tilde{\overline{\mathbf{H}}}}_s^{ve} =& \overline{\mathbf{C}}_s : [\overline{\mathbf{F}}_s^{\theta T}\mathbf{F}_s^{eT}\tilde{\mathbf{H}}_s^{eT}\overline{\mathbf{T}}_s^{eT}\mathbf{F}_s^{e-T}\overline{\mathbf{F}}_s^{\theta-T} - \overline{\mathbf{F}}_s^{\theta T}\mathbf{F}_s^{eT}\overline{\mathbf{T}}_s^{eT}\tilde{\mathbf{H}}_s^{eT}\mathbf{F}_s^{e-T}\overline{\mathbf{F}}_s^{\theta-T} \qquad (8.20) \\
& - \overline{\mathbf{F}}_s^{\theta T}\mathbf{F}_s^{eT}\overline{\mathbf{T}}_s^{eT}\mathbf{F}_s^{e-T}\tilde{\overline{\mathbf{H}}}_s^{\theta T}\overline{\mathbf{F}}_s^{\theta-T} \\
& + \overline{\mathbf{F}}_s^{\theta T}\tilde{\overline{\mathbf{H}}}_s^{\theta T}\mathbf{F}_s^{eT}\overline{\mathbf{T}}_s^{eT}\mathbf{F}_s^{e-T}\overline{\mathbf{F}}_s^{\theta-T} + \overline{\mathbf{F}}_s^{\theta T}\mathbf{F}_s^{eT}(\delta\overline{\mathbf{T}}_s^{eT})\mathbf{F}_s^{e-T}\overline{\mathbf{F}}_s^{\theta-T} - \delta\overline{\mathbf{T}}_s^{bT}] \\
& + (\delta\overline{\mathbf{C}}_s) : \Delta\overline{\mathbf{T}}_s^{bT} + (\overline{\mathbf{C}}_s : \Delta\overline{\mathbf{T}}_s^{bT})\tilde{\overline{\mathbf{H}}}_s^{ve} - \tilde{\overline{\mathbf{H}}}_s^{ve}(\overline{\mathbf{C}}_s : \Delta\overline{\mathbf{T}}_s^{bT}), \qquad (8.21)
\end{aligned}$$

where, $\delta\overline{\boldsymbol{\alpha}}_s = \overline{\boldsymbol{\alpha}}_s^* - \overline{\boldsymbol{\alpha}}_s$ and $\delta\overline{\mathbf{C}}_s = \overline{\mathbf{C}}_s^* - \overline{\mathbf{C}}_s$ are, respectively, the incremental thermal expansion parameter and the incremental viscoelastic parameter give by the perturbations

$$\delta\overline{\boldsymbol{\alpha}}_s = \overline{\mathbf{a}}_s^e : (\tilde{\mathbf{H}}_s^e\mathbf{F}_s^e) + \overline{\mathbf{a}}_s^{ve} : (\tilde{\overline{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{ve}) + \overline{\mathbf{a}}_s^{\theta} : (\tilde{\overline{\mathbf{H}}}_s^{\theta}\overline{\mathbf{F}}_s^{\theta}) + \overline{\boldsymbol{\alpha}}_s^{\theta}\delta\theta, \tag{8.22}$$

$$\delta\overline{\mathbf{C}}_s = \overline{\mathbf{D}}_s^e : (\tilde{\mathbf{H}}_s^e\mathbf{F}_s^e) + \overline{\mathbf{D}}_s^{ve} : (\tilde{\overline{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{ve}) + \overline{\mathbf{D}}_s^{\theta} : (\tilde{\overline{\mathbf{H}}}_s^{\theta}\overline{\mathbf{F}}_s^{\theta}) + \overline{\mathbf{C}}_s^{\theta}\delta\theta. \tag{8.23}$$

Since $\overline{\boldsymbol{\alpha}}_s(t)$ and $\overline{\mathbf{C}}_s(t)$ are assumed to be functions of the state of the element, we define the following associated coefficients which are the derivatives of $\overline{\boldsymbol{\alpha}}_s$ and $\overline{\mathbf{C}}_s$ with respect to their associate variables,

$$
\begin{aligned}
&\overline{\mathbf{a}}_s^{e} \equiv \partial_{\mathbf{F}_s^e}(\overline{\boldsymbol{\alpha}}_s^{\dagger}),\\
&\overline{\mathbf{a}}_s^{ve} \equiv \partial_{\overline{\mathbf{F}}_s^{ve}}(\overline{\boldsymbol{\alpha}}_s^{\dagger}),\\
&\overline{\mathbf{a}}_s^{\theta} \equiv \partial_{\overline{\mathbf{F}}_s^{\theta}}(\overline{\boldsymbol{\alpha}}_s^{\dagger}),\\
&\overline{\boldsymbol{\alpha}}_s^{\theta} \equiv \partial_{\theta}(\overline{\boldsymbol{\alpha}}_s^{\dagger}),
\end{aligned} \tag{8.24}
$$

and

$$
\begin{aligned}
&\overline{\mathbf{D}}_s^{e} \equiv \partial_{\mathbf{F}_s^e}(\overline{\mathbf{C}}_s^{\dagger}),\\
&\overline{\mathbf{D}}_s^{ve} \equiv \partial_{\overline{\mathbf{F}}_s^{ve}}(\overline{\mathbf{C}}_s^{\dagger}),\\
&\overline{\mathbf{D}}_s^{\theta} \equiv \partial_{\overline{\mathbf{F}}_s^{\theta}}(\overline{\mathbf{C}}_s^{\dagger}),\\
&\overline{\mathbf{C}}_s^{\theta} \equiv \partial_{\theta}(\overline{\mathbf{C}}_s^{\dagger}),
\end{aligned} \tag{8.25}
$$

where $\overline{\mathbf{a}}_s^{e}$, $\overline{\mathbf{a}}_s^{ve}$, $\overline{\mathbf{a}}_s^{\theta}$ and $\overline{\mathbf{C}}_s^{\theta}$ are fourth order tensor functions, $\overline{\boldsymbol{\alpha}}_s^{\theta}$ is a second order tensor, $\overline{\mathbf{D}}_s^{e}$, $\overline{\mathbf{D}}_s^{ve}$ and $\overline{\mathbf{D}}_s^{\theta}$ are sixth order tensor functions.

The two systems of equations given in (8.20) are the counterparts of equations in (4.70) and can be solved by using the same approach. The equations in (8.20) can be organized into the forms

$$
\dot{\widetilde{\overline{\mathbf{H}}}}{}_s^{\theta} = \overline{\mathbf{B}}_s^{\theta} : \widetilde{\overline{\mathbf{H}}}{}_s^{\theta} + \overline{\mathbf{B}}_s^{ve} : \widetilde{\overline{\mathbf{H}}}{}_s^{ve} + \overline{\mathbf{B}}_s : \check{\mathbf{H}} + \overline{\boldsymbol{\theta}}_s, \tag{8.26}
$$

$$
\dot{\widetilde{\overline{\mathbf{H}}}}{}_s^{ve} = \overline{\mathbf{Y}}_s^{ve} : \widetilde{\overline{\mathbf{H}}}{}_s^{ve} + \overline{\mathbf{Y}}_s^{\theta} : \widetilde{\overline{\mathbf{H}}}{}_s^{\theta} + \overline{\mathbf{Y}}_s : \check{\mathbf{H}} + \overline{\boldsymbol{\eta}}_s. \tag{8.27}
$$

In these two simplified forms $\overline{\mathbf{B}}_s^{\theta}$, $\overline{\mathbf{B}}_s^{ve}$, $\overline{\mathbf{B}}_s$, $\overline{\mathbf{Y}}_s^{ve}$, $\overline{\mathbf{Y}}_s^{\theta}$ and $\overline{\mathbf{Y}}_s$ are fourth order tensors, $\overline{\boldsymbol{\theta}}_s$ and $\overline{\boldsymbol{\eta}}_s$ are second order tensors. In order to solve the tensor differential equations (8.26), we should represent the tensors in a curvilinear coordinate system and then solve the differential equations in the component form. For simplicity, here we choose an orthonormal base and write the corresponding component form in this base as

$$\dot{\widetilde{\overline{H}}}{}^{\theta}_{s\_ij} = \overline{\mathsf{B}}^{\theta}_{s\_ijkl}\widetilde{\overline{H}}{}^{\theta}_{s\_kl} + \overline{\mathsf{B}}^{ve}_{s\_ijkl}\widetilde{\overline{H}}{}^{ve}_{s\_kl} + \overline{\mathsf{B}}_{s\_ijkl}\check{H}_{kl} + \overline{\theta}_{s\_ij},$$
$$\dot{\widetilde{\overline{H}}}{}^{ve}_{s\_ij} = \overline{\mathsf{Y}}^{ve}_{s\_ijkl}\widetilde{\overline{H}}{}^{ve}_{s\_kl} + \overline{\mathsf{Y}}^{\theta}_{s\_ijkl}\widetilde{\overline{H}}{}^{\theta}_{s\_kl} + \overline{\mathsf{Y}}_{s\_ijkl}\check{H}_{kl} + \overline{\eta}_{s\_ij}. \quad (8.28)$$

To solve this system of first order differential equations, we organize the unknows into a one-dimensional array denoted by $\overline{\boldsymbol{\chi}}_s$ that takes the form

$$\overline{\boldsymbol{\chi}}_s \equiv (\widetilde{\overline{H}}{}^{ve}_{s\_11}, \widetilde{\overline{H}}{}^{ve}_{s\_12}, \widetilde{\overline{H}}{}^{ve}_{s\_13}, \widetilde{\overline{H}}{}^{ve}_{s\_21}, ..., \widetilde{\overline{H}}{}^{\theta}_{s\_11}, \widetilde{\overline{H}}{}^{\theta}_{s\_12}, \widetilde{\overline{H}}{}^{\theta}_{s\_13}, \widetilde{\overline{H}}{}^{\theta}_{s\_21}, ...). \quad (8.29)$$

To do this, we introduce two transformations T_{ijk} and K_{ijk} which can transform the components of $\widetilde{\overline{\mathbf{H}}}{}^{ve}_s$ and $\widetilde{\overline{\mathbf{H}}}{}^{\theta}_s$ into the one-dimensional array $\overline{\boldsymbol{\chi}}_s$ through the relations

$$\overline{\chi}_{s\_i} = T_{ijk}\widetilde{\overline{H}}{}^{ve}_{s\_jk} + K_{ijk}\widetilde{\overline{H}}{}^{\theta}_{s\_jk}, \quad (8.30)$$

with the inverse transformations given as

$$\widetilde{\overline{H}}{}^{ve}_{s\_ij} = T^{-1}_{ijk}\overline{\chi}_{s\_k},$$
$$\widetilde{\overline{H}}{}^{\theta}_{s\_ij} = K^{-1}_{ijk}\overline{\chi}_{s\_k}. \quad (8.31)$$

After substituting the transformations (8.31) into (8.28), and combining the equations we get the first order system of differential equations

$$\dot{\overline{\boldsymbol{\chi}}}_s(t) = \overline{\mathbf{A}}_s(t)\overline{\boldsymbol{\chi}}_s(t) + \overline{\mathbf{f}}_s(t), \quad (8.32)$$

where the components of the coefficient matrix $\overline{A}_{s\_mn}(t)$ and the components of the inhomogeneous array $\overline{f}_{s\_m}(t)$ are given as

$$\overline{A}_{s\_mn}(t) = T_{mij}\overline{\mathsf{Y}}^{ve}_{s\_ijkl}(t)T^{-1}_{kln} + K_{mij}\overline{\mathsf{B}}^{ve}_{s\_ijkl}(t)T^{-1}_{kln} + K_{mij}\overline{\mathsf{B}}^{\theta}_{s\_ijkl}(t)K^{-1}_{kln}$$
$$+ T_{mij}\overline{\mathsf{Y}}^{\theta}_{s\_ijkl}(t)K^{-1}_{kln}, \quad (8.33)$$
$$\overline{f}_{s\_m}(t) = T_{mij}\overline{\mathsf{Y}}_{s\_ijkl}(t)\check{H}_{kl}(t) + T_{mij}\overline{\eta}_{s\_ij}(t) + K_{mij}\overline{\mathsf{B}}_{s\_ijkl}(t)\check{H}_{kl}(t) + K_{mij}\overline{\theta}_{s\_ij}(t). \quad (8.34)$$

The general solution to the system (8.32) is provided by Myskis [Myskis, 1975] as

$$\overline{\boldsymbol{\chi}}_s(t) = \int_{t_i}^{t} \overline{\mathbf{Y}}_s(t,\tau)\overline{\mathbf{f}}_s(\tau)d\tau + \overline{\mathbf{Y}}_s(t,t_i)\overline{\boldsymbol{\chi}}_{si}, \tag{8.35}$$

under the initial condition $\overline{\boldsymbol{\chi}}_{si} = \overline{\boldsymbol{\chi}}_s(t_i)$, and where,

$$\begin{aligned}\overline{\mathbf{Y}}_s(t,t_i) =& \mathbf{I} + \int_{t_i}^{t} \overline{\mathbf{A}}_s(\tau_1)d\tau_1 + \int_{t_i}^{t} \overline{\mathbf{A}}_s(\tau_1) \int_{t_i}^{\tau_1} \overline{\mathbf{A}}_s(\tau_2)d\tau_2 d\tau_1 \\ &+ \int_{t_i}^{t} \overline{\mathbf{A}}_s(\tau_1) \int_{t_i}^{\tau_1} \overline{\mathbf{A}}_s(\tau_2) \int_{t_i}^{\tau_2} \overline{\mathbf{A}}_s(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \dots\ .\end{aligned} \tag{8.36}$$

From the general solution for $\overline{\boldsymbol{\chi}}_s(t)$ given in (8.35) and the inverse transformations from $\overline{\boldsymbol{\chi}}_s(t)$ to $\widetilde{\overline{\mathbf{H}}}{}_s^{ve}(t)$ and $\widetilde{\overline{\mathbf{H}}}{}_s^{\theta}(t)$, we then can calculate the increments of the internal variables $\check{\overline{\mathbf{H}}}{}_s^{ve}(t)$ and $\check{\overline{\mathbf{H}}}{}_s^{\theta}(t)$.

The following is an alternate method to achieve a similar result.

Alternate Method :

For the base history the deformation gradient is decomposed by using the alternate method as

$$\mathbf{F}(t) = \mathbf{F}_s^{e}(t)\overline{\mathbf{F}}{}_s^{\theta}(t)\overline{\mathbf{F}}{}_s^{ve}(t), \tag{8.37}$$

where the relations between the derivatives of the components for the two decompositions are given by

$$\begin{aligned}\dot{\overline{\mathbf{F}}}{}_s^{\theta} &= \mathbf{F}_s^{ve}\dot{\mathbf{F}}_s^{\theta}\mathbf{F}_s^{\theta-1}\mathbf{F}_s^{ve-1}\overline{\mathbf{F}}{}_s^{\theta}, \\ \dot{\overline{\mathbf{F}}}{}_s^{ve} &= \overline{\mathbf{F}}{}_s^{ve}\mathbf{F}_s^{\theta-1}\mathbf{F}_s^{ve-1}\dot{\mathbf{F}}_s^{ve}\mathbf{F}_s^{\theta}.\end{aligned} \tag{8.38}$$

In a similar manner, the deformation gradient of the total history is decomposed for the alternate method as

$$\mathbf{F}^{*}(t) = \mathbf{F}_s^{e*}(t)\overline{\mathbf{F}}{}_s^{\theta*}(t)\overline{\mathbf{F}}{}_s^{ve*}(t), \tag{8.39}$$

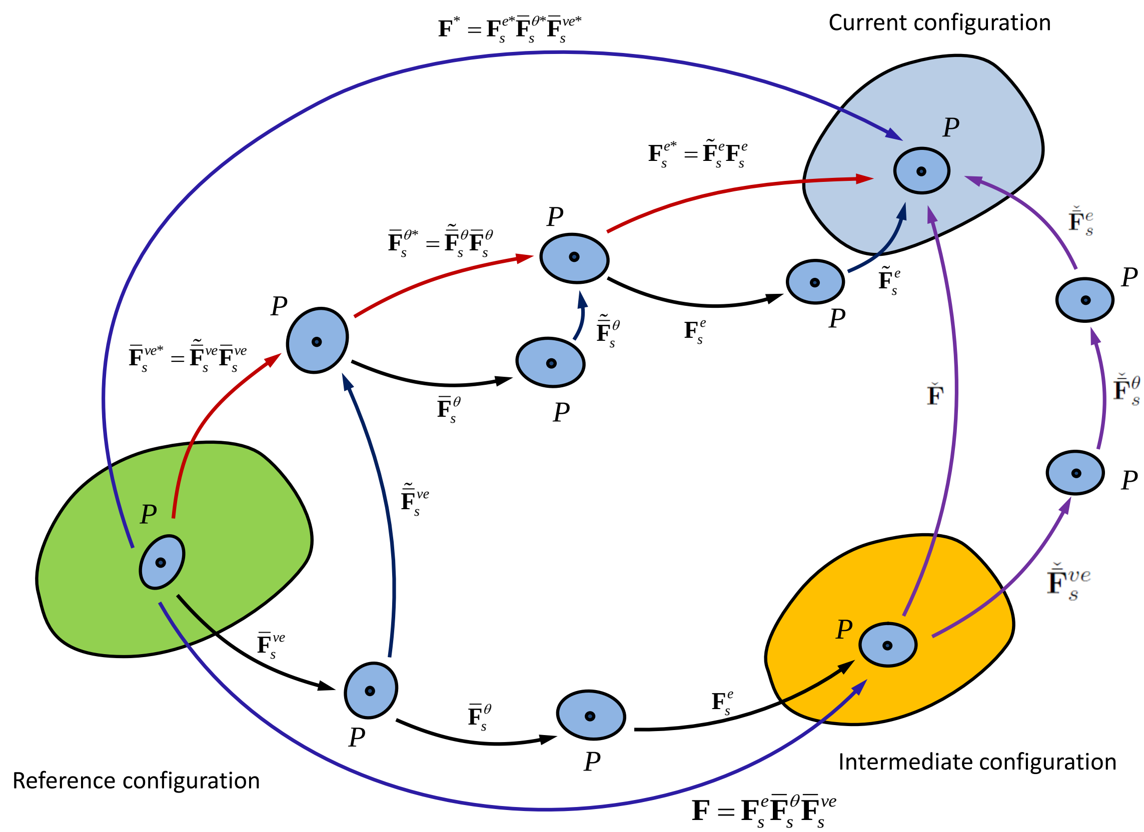


Figure 8.1: Elastic-viscoelastic and thermal parts.

where

$$\overline{\boldsymbol{\alpha}}_s^* = \mathbf{F}_s^{ve*}\boldsymbol{\alpha}_s^*\mathbf{F}_s^{ve*-1}, \tag{8.40}$$

$$\mathbf{L}_s^{ve*} = \overline{\mathbf{F}}_s^{\theta*}\overline{\mathbf{L}}_s^{ve*}\overline{\mathbf{F}}_s^{\theta*-1}. \tag{8.41}$$

The relations of the elastic, viscoelastic and thermal parts in the base history and in the total history are described in Figure 8.1, and the associated perturbations are

$$\begin{aligned}
\delta\mathbf{F}_s^e(t) &= \mathbf{F}_s^{e*}(t) - \mathbf{F}_s^e(t) = \tilde{\mathbf{F}}_s^e(t)\mathbf{F}_s^e(t) - \mathbf{F}_s^e(t) = \tilde{\mathbf{H}}_s^e(t)\mathbf{F}_s^e(t),\\
\delta\overline{\mathbf{F}}_s^\theta(t) &= \overline{\mathbf{F}}_s^{\theta*}(t) - \overline{\mathbf{F}}_s^\theta(t) = \widetilde{\overline{\mathbf{F}}}_s^\theta(t)\overline{\mathbf{F}}_s^\theta(t) - \overline{\mathbf{F}}_s^\theta(t) = \widetilde{\overline{\mathbf{H}}}_s^\theta(t)\overline{\mathbf{F}}_s^\theta(t),\\
\delta\overline{\mathbf{F}}_s^{ve}(t) &= \overline{\mathbf{F}}_s^{ve*}(t) - \overline{\mathbf{F}}_s^{ve}(t) = \widetilde{\overline{\mathbf{F}}}_s^{ve}(t)\overline{\mathbf{F}}_s^{ve}(t) - \overline{\mathbf{F}}_s^{ve}(t) = \widetilde{\overline{\mathbf{H}}}_s^{ve}(t)\overline{\mathbf{F}}_s^{ve}(t).
\end{aligned} \tag{8.42}$$

The perturbations of the thermodynamic stresses in each element in this case are given by

$$\delta\overline{\mathbf{T}}_s^{eT} = \overline{\mathbf{E}}_s^{e} : (\tilde{\mathbf{H}}_s^{e}\mathbf{F}_s^{e}) + \overline{\mathbf{E}}_s^{ve} : (\widetilde{\overline{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{ve}) + \overline{\mathbf{E}}_s^{\theta} : (\widetilde{\overline{\mathbf{H}}}_s^{\theta}\overline{\mathbf{F}}_s^{\theta}) + \overline{\mathbf{E}}_s^{\theta}\delta\theta, \tag{8.43}$$

$$\delta\overline{\mathbf{T}}_s^{bT} = \overline{\mathbf{E}}_s^{be} : (\tilde{\mathbf{H}}_s^{e}\mathbf{F}_s^{e}) + \overline{\mathbf{E}}_s^{bve} : (\widetilde{\overline{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{ve}) + \overline{\mathbf{E}}_s^{b\theta} : (\widetilde{\overline{\mathbf{H}}}_s^{\theta}\overline{\mathbf{F}}_s^{\theta}) + \overline{\mathbf{E}}_s^{b\theta}\delta\theta, \tag{8.44}$$

and the perturbation of the Cauchy stress can be obtained from

$$\delta\overline{\mathbf{T}} = \int_{-\infty}^{\infty} \delta\overline{\mathbf{T}}_s^{e} ds. \tag{8.45}$$

The superimposed deformation gradient $\check{\mathbf{F}}(t)$ can be directly separated by using an alternate form as

$$\check{\mathbf{F}}(t) = \check{\overline{\mathbf{F}}}_s^{e}(t)\check{\overline{\mathbf{F}}}_s^{\theta}(t)\check{\overline{\mathbf{F}}}_s^{ve}(t), \tag{8.46}$$

therefore,

$$\begin{aligned}\mathbf{F}^*(t) =& [\tilde{\mathbf{F}}_s^{e}(t)\mathbf{F}_s^{e}(t)][\widetilde{\overline{\mathbf{F}}}_s^{\theta}(t)\overline{\mathbf{F}}_s^{\theta}(t)][\widetilde{\overline{\mathbf{F}}}_s^{ve}(t)\overline{\mathbf{F}}_s^{ve}(t)] \\ =& \check{\mathbf{F}}(t)\mathbf{F}(t) = \check{\overline{\mathbf{F}}}_s^{e}(t)\check{\overline{\mathbf{F}}}_s^{\theta}(t)\check{\overline{\mathbf{F}}}_s^{ve}(t)\mathbf{F}_s^{e}(t)\overline{\mathbf{F}}_s^{\theta}(t)\overline{\mathbf{F}}_s^{ve}(t),\end{aligned} \tag{8.47}$$

where the relations with the "∼" variables are defined by

$$\check{\overline{\mathbf{F}}}_s^{e}(t) = \tilde{\mathbf{F}}_s^{e}(t), \tag{8.48}$$

$$\check{\overline{\mathbf{F}}}_s^{\theta}(t) = \mathbf{F}_s^{e}(t)\widetilde{\overline{\mathbf{F}}}_s^{\theta}(t)\mathbf{F}_s^{e-1}(t), \tag{8.49}$$

$$\check{\overline{\mathbf{F}}}_s^{ve}(t) = \mathbf{F}_s^{e}(t)\overline{\mathbf{F}}_s^{\theta}(t)\widetilde{\overline{\mathbf{F}}}_s^{ve}(t)\overline{\mathbf{F}}_s^{\theta-1}(t)\mathbf{F}_s^{e-1}(t), \tag{8.50}$$

$$\check{\mathbf{H}} \approx \tilde{\mathbf{H}}_s^{e} + \mathbf{F}_s^{e}\overline{\mathbf{F}}_s^{\theta}\widetilde{\overline{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{\theta-1}\mathbf{F}_s^{e-1} + \tilde{\mathbf{F}}_s^{e}\mathbf{F}_s^{e}\widetilde{\overline{\mathbf{H}}}_s^{\theta}\mathbf{F}_s^{e-1}, \tag{8.51}$$

where $\tilde{\mathbf{F}}_s^{e}(t)$, $\widetilde{\overline{\mathbf{F}}}_s^{ve}(t)$, $\widetilde{\overline{\mathbf{F}}}_s^{\theta}(t)$, $\check{\overline{\mathbf{F}}}_s^{e}(t)$, $\check{\overline{\mathbf{F}}}_s^{\theta}(t)$, and $\check{\overline{\mathbf{F}}}_s^{ve}(t)$ are assumed close to $\mathbf{I}$.

The two decompositions for the total history should be equal throughout the loading process, which can be described as

$$\mathbf{F}^*(t) = \mathbf{F}_s^{e*}(t)\mathbf{F}_s^{ve*}(t)\mathbf{F}_s^{\theta*}(t) = [\tilde{\mathbf{F}}_s^{e}(t)\mathbf{F}_s^{e}(t)][\tilde{\mathbf{F}}_s^{ve}(t)\mathbf{F}_s^{ve}(t)][\tilde{\mathbf{F}}_s^{\theta}(t)\mathbf{F}_s^{\theta}(t)], \tag{8.52}$$

$$\mathbf{F}^*(t) = \mathbf{F}_s^{e*}(t)\overline{\mathbf{F}}_s^{\theta*}(t)\overline{\mathbf{F}}_s^{ve*}(t) = [\tilde{\mathbf{F}}_s^{e}(t)\mathbf{F}_s^{e}(t)][\widetilde{\overline{\mathbf{F}}}_s^{\theta}(t)\overline{\mathbf{F}}_s^{\theta}(t)][\widetilde{\overline{\mathbf{F}}}_s^{ve}(t)\overline{\mathbf{F}}_s^{ve}(t)]. \tag{8.53}$$

After equating them and organization we obtain

$$\widetilde{\overline{\mathbf{F}}}_s^{\theta}\overline{\mathbf{F}}_s^{\theta}\widetilde{\overline{\mathbf{F}}}_s^{ve}\overline{\mathbf{F}}_s^{ve} = \tilde{\mathbf{F}}_s^{ve}\mathbf{F}_s^{ve}\tilde{\mathbf{F}}_s^{\theta}\mathbf{F}_s^{\theta}. \tag{8.54}$$

Since the above equation should be satisfied at all times, we take the time derivative of both sides and manipulate the equation to get

$$\dot{\tilde{\mathbf{F}}}_s^{ve}\mathbf{F}_s^{ve}\tilde{\mathbf{F}}_s^{\theta}\mathbf{F}_s^{\theta} + \tilde{\mathbf{F}}_s^{ve}\dot{\mathbf{F}}_s^{ve}\tilde{\mathbf{F}}_s^{\theta}\mathbf{F}_s^{\theta} - \tilde{\mathbf{F}}_s^{ve}\mathbf{F}_s^{ve}\tilde{\mathbf{F}}_s^{\theta}\mathbf{F}_s^{ve-1}\dot{\mathbf{F}}_s^{ve}\mathbf{F}_s^{\theta} = \widetilde{\overline{\mathbf{F}}}_s^{\theta}\overline{\mathbf{F}}_s^{\theta}\dot{\widetilde{\overline{\mathbf{F}}}}_s^{ve}\overline{\mathbf{F}}_s^{ve}. \tag{8.55}$$

Equations (8.54) and (8.55) can be further simplified by organization and by using the smallness of the incremental displacement gradients of the internal parameters $\tilde{\mathbf{H}}_s^{ve} = \tilde{\mathbf{F}}_s^{ve} - \mathbf{I}$, $\tilde{\mathbf{H}}_s^{\theta} = \tilde{\mathbf{F}}_s^{\theta} - \mathbf{I}$, $\widetilde{\overline{\mathbf{H}}}_s^{ve} = \widetilde{\overline{\mathbf{F}}}_s^{ve} - \mathbf{I}$, and $\widetilde{\overline{\mathbf{H}}}_s^{\theta} = \widetilde{\overline{\mathbf{F}}}_s^{\theta} - \mathbf{I}$ to get

$$\begin{aligned}
\widetilde{\overline{\mathbf{H}}}_s^{\theta} =& \tilde{\mathbf{H}}_s^{ve} + \mathbf{F}_s^{ve}\tilde{\mathbf{H}}_s^{\theta}\mathbf{F}_s^{ve-1} - \overline{\mathbf{F}}_s^{\theta}\widetilde{\overline{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{\theta-1}, \\
\dot{\widetilde{\overline{\mathbf{H}}}}_s^{ve} =& \overline{\mathbf{F}}_s^{\theta-1}(\dot{\tilde{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{\theta} + \dot{\tilde{\mathbf{H}}}_s^{ve}\mathbf{F}_s^{ve}\tilde{\mathbf{H}}_s^{\theta}\mathbf{F}_s^{\theta}\overline{\mathbf{F}}_s^{ve-1} + \dot{\mathbf{F}}_s^{ve}\tilde{\mathbf{H}}_s^{\theta}\mathbf{F}_s^{\theta}\overline{\mathbf{F}}_s^{ve-1} - \mathbf{F}_s^{ve}\tilde{\mathbf{H}}_s^{\theta}\mathbf{F}_s^{ve-1}\dot{\mathbf{F}}_s^{ve}\mathbf{F}_s^{\theta}\overline{\mathbf{F}}_s^{ve-1} \\
& + \overline{\mathbf{F}}_s^{\theta}\widetilde{\overline{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{\theta-1}\dot{\tilde{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{\theta} - \mathbf{F}_s^{ve}\tilde{\mathbf{H}}_s^{\theta}\mathbf{F}_s^{ve-1}\dot{\tilde{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{\theta} - \tilde{\mathbf{H}}_s^{ve}\dot{\tilde{\mathbf{H}}}_s^{ve}\overline{\mathbf{F}}_s^{\theta}).
\end{aligned} \tag{8.56}$$

From equations (8.56) combined with equations in (8.38) the values for $\widetilde{\overline{\mathbf{H}}}_s^{ve}$ and $\widetilde{\overline{\mathbf{H}}}_s^{\theta}$ in the alternate decomposition can be evaluated from $\tilde{\mathbf{H}}_s^{ve}$ and $\tilde{\mathbf{H}}_s^{\theta}$ in the original decomposition we use in the development by solving the first order differential system.

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