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Dynamic Funding and Investment Strategy for Defined Benefit Pension Schemes: A Model Incorporating Asset-Liability Matching Criteria

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Abstract

This paper studies the dynamic funding policy and investment strategy for defined benefit pension plans using one of the most comprehensive dynamic

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pension models to date. The model includes three investable assets: one risk-free and two risky. The optimal plan decisions are formulated as a stochastic control problem that is solved using dynamic programming. The objective function uses performance measures to take into account the stability and solvency of the plan. The model is then applied to a Taiwanese pension.

Key words and phrases: *optimal contribution, asset allocation, dynamic programming, performance measure*

1 Introduction

Although most pension liabilities are long-term in nature, traditional defined benefit pension plan management is based on one-period assumptions.[1] The pension plan manager seeks an optimal investment decision for the next period, based on the plan's current experience, current market conditions, and expectations about future contributions, returns, and risks. Such a short-sighted mechanism has two drawbacks: (i) the accumulation of a sequence of single-period optimal decisions across each of n periods may not be optimal for the n periods taken as a whole; and (ii) single-period decisions have difficulties in dealing with the investment and funding sides of a pension plan because the interaction between investments and funding appears only in the multi-period setting.

An important tool that can be used to assist plan managers in developing optimal funding policies over many periods is stochastic optimal control theory. This theory can be used to solve long-term financial planning problems through global optimization across periods instead of local optimization within a period.

Control theory has been developed by engineers since the 1930s. Its applications to economics emerged in the 1950s. [See Petit (1990) for more on this.] Several authors, including Samuelson (1969), Merton (1971, 1990), Brennan and Schwartz (1982), Karatzas et al., (1986), Brennan, Schwartz, and Lagnado (1997), Boyle and Yang (1997), Brennan and Schwartz (1998), and Sorensen (1999), have studied optimal consumption and investment problems using control theory. Although the popularity of stochastic control theory was hindered by its inher-

[1] Traditional pension management usually employs a mean-variance approach. Sharpe (1991) describes the mean-variance approach as a highly parsimonious characterization of investors' goals, employing a myopic view (i.e., one period at a time) and focusing on only two aspects of the probability distribution of possible returns over that period.

ent complexity, it is becoming more popular today due to the ready availability of high-speed computers.

The application of control theory to pension plan management started with O'Brien (1986, 1987) who constructed a stochastic model for the pension plan and studied the optimal funding policies for target funding ratios. Cairns (1995, 1996, 2000) introduced asset allocation into the control process to study the optimal funding and investment strategies needed to minimize certain quadratic loss functions. Chang (1999, 2000) applied their methods to a real pension plan in Taiwan using service tables and stochastic asset returns to numerically solve for optimal funding policies over various time horizons. Applications of control theory to other actuarial problems can be seen in Runggaldier (1998) and Schäl (1998). Runggaldier reviews the concepts and solution methods, while Schäl focuses on the dynamic programming for piecewise deterministic Markov processes.

In this paper we construct one of the most comprehensive dynamic models of a pension plan to date, numerically solve the stochastic control problem, and provide illustrations of the optimal investment and funding strategies. Compared to Cairns (1995, 1996, 2000), we have a richer set of liability dynamics, and we have included risk-free as well as risky assets. Compared with Chang (1999, 2000), we consider not only funding policies, but also asset allocations. In addition we consider more risk factors for invested assets. Furthermore, our performance measures (also called loss functions) take into account the stability of contributions and the security (the funding ratio) of the pension plan.

The features of our models are summarized as follows:

1. The dynamics of the plan's demography can be explicitly incorporated into investment decisions under different evaluation time horizons.

2. The optimal funding and investment strategies of the plan can be formalized with specific risk performance measures through a computerized system.

3. The contribution risk and solvency risk associated with any funding policy and investment strategy can be assessed given any evaluation horizon.

The paper is organized as follows. Section 2 describes the model (i.e., the basic framework, the dynamics of invested assets, and performance measures used) of the proposed dynamic optimization scheme.

Section 3 develops the solution to the optimal equation. Section 4 provides a practical example to illustrate the usefulness of the theory presented. A summary and closing comments are given in Section 5.

2 The Model

In this section, we formulate the funding and investment decisions of pension funds as an stochastic optimal control problem. These decisions are modeled through a continuous-time stochastic process over a specific evaluation time horizon.

2.1 The Basic Framework

The following notation is used for the various stochastic processes[2] used in the paper:

T = Management's planning horizon;

$\mathcal{F}_t$ = Plan's history up to time t;

$F(t)$ = Total assets of the pension plan at time t excluding any contributions made at time t;

$\mathrm{d}\delta(t,F)$ = Rate of investment return in $(t, t+\mathrm{d}t)$;

$C(t)$ = Contributions at time t;

$B(t)$ = Retirement benefit payment rate at time t;

σ_B = Volatility of $B(t)$;

$Z_i(t)$ = Wiener processes ($i \in$ {NC, B, W) at time t;

$\mathrm{AL}(t)$ = Total plan accrued liabilities at time t;

r' = Valuation rate for accrued liabilities;

$\mathrm{NC}(t)$ = Normal cost rate at time t;

$W(t)$ = Reduction rate in retirement liability at time t due to withdrawal payment;[3]

[2]Throughout this paper we assume that all stochastic processes are defined on appropriate probability spaces.

[3]Because death benefits are not currently included in the Taiwanese Labor Standard Law and withdrawal benefits are not paid from the accumulated pension fund, we consider only the retirement benefits payments and the reduced withdrawal liability in this paper.

σ_{NC} = Volatility of $\mathrm{NC}(t)$; and

σ_W = Volatility of $W(t)$.

The σs are assumed to be constants. The term rate, as used with respect to $C(t)$, $\mathrm{NC}(t)$, $B(t)$, and $W(t)$, refers to the amount paid in an infinitesimal time interval. For example, $C(t)\mathrm{d}t$ is the amount contributed in $(t, t+\mathrm{d}t)$.

The funding level $F(t)$ and accrued liabilities $\mathrm{AL}(t)$ are described by the following stochastic differential equations:

$$\mathrm{d}F(t) = F(t)\mathrm{d}\delta(t,F) + C(t)\mathrm{d}t - B(t)\mathrm{d}t + \sigma_B\,\mathrm{d}Z_B(t), \tag{1}$$

$$\begin{aligned}\mathrm{dAL}(t) = {} & (\mathrm{AL}(t)r' + \mathrm{NC}(t) - B(t) - W(t))\mathrm{d}t + \sigma_{\mathrm{NC}}\,\mathrm{d}Z_{\mathrm{NC}}(t) \\ & + \sigma_B\,\mathrm{d}Z_B(t) + \sigma_W\,\mathrm{d}Z_W(t).\end{aligned} \tag{2}$$

2.2 The Dynamics of Invested Assets

We assume three types of assets are available to the pension plan: cash, stocks, and consol bonds.[4] The proportion of the pension funds invested in stocks, consol bonds, and cash at time t is denoted by $P(t) = (a(t), b(t), c(t))$, where

$$a(t) + b(t) + c(t) = 1.$$

Following Brennan, Schwartz, and Lagnado (1997), we model the instantaneous rate of return on the stock portfolio $\mathrm{d}S(t)/S(t)$, the short rate $r(t)$, and the long rate $l(t)$, as the following joint stochastic process:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu_S\mathrm{d}t + \sigma_S\mathrm{d}Z_S(t), \tag{3}$$

$$\mathrm{d}r(t) = \mu_r\mathrm{d}t + \sigma_r\mathrm{d}Z_r(t), \tag{4}$$

$$\mathrm{d}l(t) = \mu_l\mathrm{d}t + \sigma_l\mathrm{d}Z_l(t), \tag{5}$$

where the subscripts S, r, l refer to stocks, short rate, and long rate, respectively, μ_i and σ_i ($i \in \{S, r, l\}$) are constant parameters and $\mathrm{d}Z_i$ ($i \in \{S, r, l\}$) are increments to Wiener processes.

We note that the price of a consol bond, $B_c(t)$, is inversely proportional to its yield and the total return of a consol bond is the sum of

[4]Consol bonds are bonds with infinite time to maturity, i.e., they never mature.

the yield and the price change. Also, from a simple application of Ito's lemma, the instantaneous total return on the consol bond can be proved to be:

$$\frac{dB_c(t)}{B_c(t)} + l(t)dt = \left(l(t) - \frac{\mu_l}{l(t)} + \frac{\sigma_l^2}{l(t)^2}\right)dt - \frac{\sigma_l}{l(t)}dZ_l(t). \tag{6}$$

The investment return of the pension plan between time t and $t + dt$, $d\delta(t, F)$, can be formulated as:

$$d\delta(t,F) = a(t)\frac{dS(t)}{S(t)} + b(t)\left(\frac{dB_c(t)}{B_c(t)} + l(t)dt\right) + (1 - a(t) - b(t))r(t)dt. \tag{7}$$

Hence, the instantaneous changes in the pension's assets in equation (1) can be rewritten as

$$\begin{aligned} dF(t) = F(t)\left[a(t)\mu_S + b(t)\left(l - \frac{\mu_l}{l(t)} + \frac{\sigma_l^2}{l^2(t)}\right) + (1 - a(t) - b(t))r(t)\right]dt \\ + (C(t) - B(t))dt + F(t)a(t)\sigma_S dZ_S(t) - F(t)b(t)\frac{\sigma_l}{l(t)}dZ_l(t) \\ + \sigma_B dZ_B(t). \end{aligned} \tag{8}$$

The dynamics of the pension plan for the fund and the accrued liabilities can then be jointly written as:

$$\begin{pmatrix} dF(t) \\ dAL(t) \end{pmatrix} \equiv dX(t) = \mu_X(t)dt + \sigma_X(t)d\xi(t), \tag{9}$$

where

$$\mu_X(t) = \begin{pmatrix} [a\mu_S + b(l - \mu_l/l + \sigma_l^2/l^2) + (1 - a - b)r]F + C - B \\ AL \cdot r' + NC - B - W \end{pmatrix},$$

$$\sigma_X(t) = \begin{pmatrix} Fa\sigma_S & -Fb\sigma_l/l & \sigma_B & 0 & 0 \\ 0 & 0 & \sigma_B & \sigma_{NC} & \sigma_W \end{pmatrix},$$

and $d\xi = (dZ_S, dZ_l, dZ_B, dZ_{NC}, dZ_W)^T$ is a five dimensional standard Wiener process with covariance matrix $[\sigma_{ij}]$. We adopt the notation σ_{Sl} to denote the covariance between Wiener processes Z_S and Z_l; other covariances are represented similarly.

Note: In the definition of $\mu_X(t)$ and $\sigma_X(t)$ above the function definitions are abbreviated by dropping (t) so that, for example, $F(t)$ is written as F and $a(t)$ is written as a. When no confusion arises, (t) and subscripts t or T will be omitted. This convention is used throughout the rest of this paper.

2.3 Performance Measures

A good performance measure should consider the two most important factors in pension plan valuations: (i) the contribution rate risk (i.e., level of funding deficiency), which is the difference between normal costs and contributions; and (ii) the level of unfunded liabilities, which is the difference between accrued liabilities and assets. The level of funding deficiency affects the stability of the plan, while the level of unfunded liabilities affects the solvency of the plan. Following Haberman and Sung (1994), we design our performance measure (also called a loss function) to take into account the contribution rate and solvency risks to give:

$$L(t, X, \{C, P\}) = (\mathrm{NC}(t) - C(t))^2 + k(\eta \mathrm{AL}(t) - F(t))^2, \qquad (10)$$

where k is a constant chosen subjectively by the pension fund manager to adjust for the difference in size between the contribution rate risk and the solvency risk, and η is the target funding ratio. The parameter k reflects the relative importance of matching contributions with normal costs and matching plan assets with accrued liabilities.

3 The Optimal Equation

Assume first that the performance measure $L(t, X, \{C, P\})$ is discounted continuously by a constant rate ρ. If we let $B[X_T, T]$ be a function measuring the loss associated with the state of the pension plan at the end of period, then the problem of choosing the optimal asset allocation and funding policy for a fixed evaluation time horizon T can be formulated as:

$$\inf\{\int_0^T e^{-\rho t} L(t, X, \{C, P\}) \mathrm{d}t + B[X_T, T]\},$$

subject to the asset and liability dynamics specified in equations (1) and (2), respectively. Furthermore, we have $C \geq 0$, a, b, $c \in R$, and $F(t) \geq 0$. Define

$$J(s,X,\{C,P\}) = \mathbb{E}_{s,X}[\int_s^T e^{-\rho t}L(t,X,\{C,P\})\mathrm{d}t + B[X_T,T]|\mathcal{F}_s] \quad (11)$$

where $\mathbb{E}_{s,X}$ represents the expectation conditioned on being in the state X at time s, given information $\mathcal{F}_s$. As X is assumed to follow a time-homogeneous Markov process, only the information at s is needed (information before time s can be ignored).

Let us assume that there exit optimal strategies $\{C^*,P^*\}$ that form a set of admissible control functions that minimize equation (11), i.e.,

$$I(s,X) = \inf_{(C,P)} J(s,X,\{C,P\}) = J(s,X,\{C^*,P^*\}).$$

Then by the principle of optimality [Bellman (1957)], we can express the Bellman-Dreyfus fundamental equation of optimality as

$$\begin{aligned}
\inf_{(C,P)} \{&[l(l(-B+C-(-1+a+b)Fr+bFl+aF\mu_S)-bF\mu_l)+bF\sigma_l^2]\frac{1}{l^2}I_F \\
&+(-B+\mathrm{NC}-W+r'\mathrm{AL})I_{\mathrm{AL}} \\
&+(\sigma_B^2+\sigma_{\mathrm{NC}}^2+\sigma_W^2+2\,\sigma_{\mathrm{NC}}\sigma_W\sigma_{\mathrm{NC},W}+2\,\sigma_{\mathrm{NC}}\sigma_B\sigma_{\mathrm{NC},B}+2\,\sigma_W\sigma_B\sigma_{WB})\frac{1}{2}I_{\mathrm{AL,AL}} \\
&+I_{F,F}(l\sigma_B^2+(\sigma_{\mathrm{NC}}(al\sigma_S\sigma_{\mathrm{NC},S}-b\sigma_l\sigma_{\mathrm{NC},l})F+\sigma_W(al\sigma_S\sigma_{SW}-b\sigma_l\sigma_{Wl})) \\
&+\sigma_B(l\sigma_{\mathrm{NC}}\sigma_{\mathrm{NC},B}+aFl\sigma_S\sigma_{SB}+l\sigma_W\sigma_{WB}-bF\sigma_l\sigma_{lB})) \\
&+e^{-\rho s}((\mathrm{NC}-C)^2+k(F-\eta\mathrm{AL})^2) \\
&+[l^2\sigma_B^2+(b^2\sigma_l^2+al\sigma_S(al\sigma_S-2b\sigma_l\sigma_{Sl}))F^2 \\
&+2Fl\sigma_B(al\sigma_S\sigma_{SB}-b\sigma_l\sigma_{lB})]\frac{1}{2l^2}I_{F,F}\}+I_s=0 \quad (12)
\end{aligned}$$

where the subscripts of the I denote partial derivatives, i.e., $I_s = \partial I/\partial s$, $I_{F,F} = \partial^2 I/\partial F^2$, $I_{F,\mathrm{AL}} = \partial^2 I/\partial F\partial\mathrm{AL}$, etc. Taking the partial derivatives with respect to C and P and using the boundary condition $I(T,X) = B[X,T]$, we obtain from the first order condition that:

$$C^* = \mathrm{NC} - e^{\rho s}I_F/2 \quad (13)$$

$$\begin{aligned}
a^* = &\frac{1}{Fl\,I_{F,F}\sigma_S^2\sigma_l(-1+\sigma_{Sl}^2)}[lI_F(l(-r+l)-\mu_l)\sigma_S\sigma_{Sl}+I_F\sigma_S\sigma_l^2\sigma_{Sl} \\
&+l\sigma_l(I_F(-r+\mu_S)+\sigma_S(I_{F,F}\sigma_B(\sigma_{SB}-\sigma_{Sl}\sigma_{lB}) \\
&+I_{FL}(\sigma_{\mathrm{NC}}\sigma_{\mathrm{NC},S}+\sigma_W\sigma_{SW}-\sigma_{\mathrm{NC}}\sigma_{\mathrm{NC},l}\sigma_{Sl}+\sigma_B\sigma_{SB} \\
&-\sigma_W\sigma_{Sl}\sigma_{Wl}-\sigma_B\sigma_{Sl}\sigma_{lB}))] \quad (14)
\end{aligned}$$

$$b^* = \frac{1}{FI_{F,F}\sigma_S\sigma_l^2(-1+\sigma_{Sl}^2)}[lI_F(-r+\mu_S)\sigma_l\sigma_{Sl}$$
$$+\sigma_S(I_F(-rl^2+l^3-l\mu_l+\sigma_l^2)+l\sigma_l(I_{F,F}\sigma_B(\sigma_{Sl}\sigma_{SB}-\sigma_{lB})$$
$$-I_{F,\mathrm{AL}}(\sigma_N\sigma_{\mathrm{NC},l}-\sigma_{\mathrm{NC}}\sigma_{\mathrm{NC},S}\sigma_{Sl}-\sigma_W\sigma_{SW}\sigma_{Sl}$$
$$-\sigma_B\sigma_{Sl}\sigma_{SB}+\sigma_W\sigma_{Wl}+\sigma_B\sigma_{lB}))] \tag{15}$$

The Hamilton-Jacobi-Bellman equation (Fleming and Rishel, 1975) is:

$$0 = I_s + I_{\mathrm{AL}}(-B+\mathrm{NC}-W+\mathrm{AL}r')$$
$$+\frac{1}{2}I_{\mathrm{AL,AL}}(\sigma_B^2+\sigma_{\mathrm{NC}}^2+\sigma_W^2+2\ \sigma_{\mathrm{NC}}\sigma_W\sigma_{\mathrm{NC},W}$$
$$+2\ \sigma_W\sigma_B\sigma_{WB}+2\ \sigma_{\mathrm{NC}}\sigma_B\sigma_{\mathrm{NC},B})$$
$$+\frac{I_{F,\mathrm{AL}}^2}{2I_{F,F}(-1+\sigma_{Sl}^2)}(\sigma_{\mathrm{NC}}(\sigma_{\mathrm{NC},l}-\sigma_{\mathrm{NC},S}\sigma_{Sl})$$
$$+\sigma_W(-\sigma_{SW}\sigma_{Sl}+\sigma_{Wl})+\sigma_B(-\sigma_{Sl}\sigma_{SB}+\sigma_{lB}))^2$$
$$-\frac{I_{F,\mathrm{AL}}^2(\sigma_{\mathrm{NC}}\sigma_{\mathrm{NC},S}+\sigma_W\sigma_{SW}+\sigma_B\sigma_{SB})^2}{2I_{F,F}}$$
$$+\frac{I_FI_{F,\mathrm{AL}}(r-\mu_S)(\sigma_{\mathrm{NC}}\sigma_{\mathrm{NC},S}+\sigma_W\sigma_{SW}+\sigma_B\sigma_{SB})}{I_{F,F}\sigma_S}$$
$$-\frac{1}{2}I_{F,F}\sigma_B^2(-1+\sigma_{SB}^2)+\frac{I_FI_{F,\mathrm{AL}}(\sigma_S(l^2(-r+l)-l\mu_l+\sigma_l^2)}{lI_{F,F}\sigma_S\sigma_l(-1+\sigma_{Sl}^2)}$$
$$+l(-r+\mu_S)\sigma_l\sigma_{Sl})(\sigma_{\mathrm{NC}}(-\sigma_{\mathrm{NC},l}+\sigma_{\mathrm{NC},S}\sigma_{Sl})$$
$$+\sigma_W(\sigma_{SW}\sigma_{Sl}-\sigma_{Wl})+\sigma_B(\sigma_{Sl}\sigma_{SB}-\sigma_{lB}))$$
$$+\frac{I_{F,\mathrm{AL}}\sigma_B}{(-1+\sigma_{Sl}^2)}(\sigma_{\mathrm{NC}}(-\sigma_{\mathrm{NC},l}+\sigma_{\mathrm{NC},S}\sigma_{Sl})+\sigma_W(\sigma_{SW}\sigma_{Sl}-\sigma_{Wl})$$
$$+\sigma_B(\sigma_{Sl}\sigma_{SB}-\sigma_{lB}))(\sigma_{Sl}\sigma_{SB}-\sigma_{WB})+I_{F,\mathrm{AL}}\sigma_B\sigma_{\mathrm{NC}}(\sigma_{\mathrm{NC},B}-\sigma_{\mathrm{NC},S}\sigma_{SB})$$
$$+I_{F,\mathrm{AL}}\sigma_B(-\sigma_B(-1+\sigma_{SB}^2)+\sigma_W(-\sigma_{SW}\sigma_{SB}+\sigma_{WB}))$$
$$+\frac{I_F^2(\sigma_S(l((r-l)l+\mu_l)-\sigma_l^2)+l(r-\mu_S)\sigma_l\sigma_{Sl})^2}{2l^2I_{F,F}\sigma_S^2\sigma_l^2(-1+\sigma_{Sl}^2)}$$
$$-\frac{1}{4}e^(\rho s)I_F^2-\frac{I_F^2(r-\mu_S)^2}{2I_{F,F}\sigma_S^2}+\frac{I_{F,F}\sigma_B^2(-\sigma_{Sl}\sigma_{SB}+\sigma_{lB})^2}{2(-1+\sigma_{Sl}^2)}$$
$$+ke^(-\rho s)(F-\eta\mathrm{AL})^2+\frac{I_F\sigma_B(\sigma_S(l^2(-r+l)-l\mu_l+\sigma_l^2)}{l\sigma_S\sigma_l(-1+\sigma_{Sl}^2)}$$
$$+l(-r+\mu_S)\sigma_l\sigma_{Sl})(\sigma_{Sl}\sigma_{SB}-\sigma_{lB})$$
$$-\frac{1}{\sigma_S}I_F((B-\mathrm{NC}-Fr)\sigma_S+(-r+\mu_S)\sigma_B\sigma_{SB}). \tag{16}$$

The function (C^*, P^*) is our candidate for the optimal strategy. Because $I(s, X)$ is unknown, the description is incomplete. We therefore have to guess a solution for $I(s, X)$ that has finite number of parameters, and then use the partial differential equation (16) to identify its parameters. Because our loss function is quadratic, our guess is:

$$I[s, X] = X^t \Phi X + \Psi^t X + d(s), \; 0 \le s \le T.$$

where $d(s)$ depends only on s, and

$$\Phi = \begin{pmatrix} a_1(s) & a_2(s) \\ a_2(s) & a_3(s) \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}.$$

Substituting Φ, Ψ, and $d(s)$ into the ordinary system of differential equations and noting that the optimal strategy must hold for all (s, X), we can solve for the coefficient matrices Φ, Ψ, and $d(s)$ in $I[s, X]$ by checking the coefficients in equations for F^2, FAL, AL^2, F, AL, and the constant term. The boundary condition is $I(T, X) = B[X, T] = 0$, i.e., $a_1(T) = a_2(T) = a_3(T) = b_1(T) = b_2(T) = d(T) = 0$, because the plan manager who adopts this optimal strategy must be able to match contributions with the normal costs and match assets with accrued liabilities at the end of the evaluation period.

After inspecting the formulas for a, b, and C, we find that we need only to compute I_F, $I_{F,F}$, and $I_{F,\text{AL}}$ via the solution of $a_1(s)$, $a_2(s)$, and $b_1(s)$. The system of differential equations involving $a_1(s)$, $a_2(s)$, and $b_1(s)$ (with (s) removed for convenience) is as follows:

$$\begin{aligned} 0 = {} & l^2(r - \mu_S)^2\sigma_l^2 a_1 - 2l(r - \mu_S)\sigma_S\sigma_l(l^2(-r + l) - l\mu_l + \sigma_l^2)\sigma_{Sl}a_1 \\ & + \sigma_S^2(l^2((r - l)l + \mu_l)^2 a_1 + \sigma_l^4 a_1 + l\sigma_l^2(-kle^{-\rho s} \\ & - 2((2r - l)l + \mu_l)a_1 + le^{\rho s}a_1^2 - la_1' + l\sigma_{Sl}^2(ke^{-\rho s} \\ & + 2ra_1 - e^{\rho s}a_1^2 + a_1'))) \end{aligned} \tag{17}$$

$$\begin{aligned} 0 = {} & 2(l^2(r - \mu_S)^2\sigma_l^2 a_2 - 2l(r - \mu_S)\sigma_S\sigma_l(l^2(-r + l) - l\mu_l + \sigma_l^2)\sigma_{Sl}a_2 \\ & + \sigma_S^2(l^2((r - l)l + \mu_l)^2 a_2 + \sigma_l^4 a_2 + l\sigma_l^2(kl\eta e^{-\rho s} \\ & + (-2\mu_l + l(-3r + 2l - r' + e^{\rho s}a_1))a_2 \\ & - la_2' - l\sigma_{Sl}^2(k\eta e^{-\rho s}) + (r - r' - e^{\rho s}a_1)a_2 + a_2')))) \end{aligned} \tag{18}$$

$$
\begin{aligned}
0 = {} & l^2(r-\mu_S)^2\sigma_l^2 b_1 + 2l(r-\mu_S)\sigma_S\sigma_l(l\sigma_l((\sigma_{\text{NC}}(-\sigma_{\text{NC},S} + \sigma_{\text{NC},l}\sigma_{Sl}) \\
& + \sigma_W(-\sigma_{SW} + \sigma_{Sl}\sigma_{Wl}))a_2 - \sigma_B(\sigma_{SB} - \sigma_{Sl}\sigma_{lB})(a_1 + a_2) \\
& + (l((r-l)l + \mu_l) - \sigma_l^2)\sigma_{Sl}b_1) \\
& + \sigma_S^2(2l^2(l(-r+l) - \mu_l)\sigma_l((\sigma_{\text{NC}}(-\sigma_{\text{NC},l} + \sigma_{\text{NC},S}\sigma_{Sl}) \\
& + \sigma_W(\sigma_{SW}\sigma_{Sl} - \sigma_{Wl}))a_2 \\
& + \sigma_B(\sigma_{Sl}\sigma_{SB} - \sigma_{lB})(a_1 + a_2)) + 2l\sigma_l^3((\sigma_{\text{NC}}(-\sigma_{\text{NC},l} + \sigma_{\text{NC},S}\sigma_{Sl}) \\
& + \sigma_W(\sigma_{SW}\sigma_{Sl} - \sigma_{Wl}))a_2 + \sigma_B(\sigma_{Sl}\sigma_{SB} - \sigma_{lB})(a_1 + a_2)) \\
& + l^2((r-l)l + \mu_l)^2 b_1 + \sigma_l^4 b_1 \\
& + l\sigma_l^2(-2(B - \text{NC} + W)l(-1 + \sigma_{Sl}^2)a_2 \\
& + (-2\mu_l + l(-3r + 2l + r\sigma_{Sl}^2))b_1 - l(-1 + \sigma_{Sl}^2)a_1(2B - 2\text{NC} \\
& + e^{\rho s}b_1) + l(-1 + \sigma_{Sl}^2)b_1')). \qquad (19)
\end{aligned}
$$

Numerical methods can now be used to solve the above system of differential equations for Φ, Ψ and d in order to obtain the optimal strategy C^*, P^* for $t \in [0, T]$). A Mathematica® subroutine that implements the sophisticated implicit Adams and Gear formulae with higher orders is used to solve the system of differential equations numerically.

4 An Illustration

We apply the results in Sections 2 and 3 to a defined benefit pension plan sponsored be a Taiwanese semi-conductor and electronic company. According to the Labor Standard Law enacted by the Taiwanese government in 1984, each employer is required to contribute from 2% to 15% of its employees' pensionable payroll to a government-managed trust fund. This trust fund is guaranteed a minimum return by the Taiwanese government. This mandatory pension plan is a defined benefit scheme in which a participant's retirement benefit is based on the participant's length of service and final salary. Although the pension plan has minimum guaranteed returns from the government, the plan is still subject to an insolvency risk because the contributions coupled with the investment returns may not be able to match the benefit payments.

The Taiwanese company's pension plan covers 2,768 members with initial assets of 254 million NT dollars and an accrued liability of 380 million NT dollars. We use the open group with size-constrained assumption to project the evolution of the plan's workforce (Winklevoss, 1993). The total number of employees is assumed to remain unchanged, i.e., each departing employee is immediately replaced by a new employee. The plan's service table is given in Tables A1 and A2 in the appendix.[5] Table A3 shows the assumptions made for new entrants. The entry age normal cost method (Anderson, 1992) is used to determine accrued liabilities, normal costs, and benefit retirement benefits.

The retirement benefit, $B(t)$, is formulated according to the current Labor Standard Law for the private pension plans in Taiwan. This law stipulates that the retirement benefit can only be paid as lump-sum payment to the retiree. The retirement benefit for a qualified plan member, $B(t)$, can be written as the accumulated service credits multiple by the average final six-month salary. Each plan member receives two service credits each year of service for the first fifteen years of service and one service credit each year after fifteen years of service, up to a maximum of forty-five service credits.

Suppose an active plan member j entered the plan at age e_j and is currently age x_j at with annual salary of $S^{(j)}$. If the annual salary growth rate is set at 3%, then the projected retirement benefit, $\text{RBEN}^{(j)}$, can be written as

$$\text{RBEN}^{(j)} = \begin{cases} 2x_e S^{(j)}(1.03)^{60-x_j} & \text{if } 60 - e_j \leq 15 \\ S^{(j)}(1.03)^{60-x_j} \times \min(x_e + 15, 45) & \text{if } 60 - e_j \geq 15 \end{cases}$$

Let $\mathcal{R}_t$ denote the set of active plan members who retire at time t and $\mathcal{W}_t$ denote the set of active plan members who withdraw at time t. The aggregated retirement benefit payments and reduced retirement liability payments at t, respectively, for the plan are given by:

$$B(t) = \sum_{j \in \mathcal{R}_t} \text{RBEN}^{(j)} \quad \text{and}$$
$$W(t) = \sum_{j \in \mathcal{W}_t} \text{RBEN}^{(j)}.$$

For practical reasons, continuous time processes are simulated at weekly intervals. Salaries are assumed to increase 3% per annum and the valuation interest rate is $ln(1.03)/52$ per week. Because we do not have the data to estimate the volatilities of normal costs and the retirement and withdrawal benefits, we first simulate 100 sets of NC, B,

[5] These are simplified service tables that give only the multiple decrement survival probability $p_x^{(\tau)}$ (i.e., the probability that a person age x remains in the plan at age $x+1$) for males and females. The illustrated pension plan used by Chang and Cheng (2002) is different from the plan used in this study. They focus on a public pension plan (Tai-PERS), while we discuss private pension plans.

and W, then use the simulated standards as the volatilities in equation (2). With regard to the parameters associated with the loss function, the rate ρ is assumed to be $ln(1.06)/52$ per week. Two target funding ratios, 0.75 and 1, are used for comparisons, and we subjectively choose $k = 0.0001$.

Table 1

Simulated Paths NC(t), AL(t), $l(t)$ and $r(t)$ With NC(t) and AL(t) Measured in 1,000,000

t	NC(t)	AL(t)	$l(t)$	$r(t)$
0	0.947	381.0	0.0700	0.0350
25	0.958	388.0	0.0710	0.0370
50	0.973	393.0	0.0715	0.0360
75	0.986	402.0	0.0730	0.0340
100	0.995	409.0	0.0735	0.0320
125	1.010	417.0	0.0730	0.0330
150	1.020	421.0	0.0760	0.0310
175	1.030	431.0	0.0740	0.0300
200	1.040	439.0	0.0720	0.0290
225	1.048	442.0	0.0715	0.0310
250	1.058	449.0	0.0750	0.0300
275	1.060	453.0	0.0730	0.0300
300	1.072	462.0	0.0725	0.0315
325	1.073	470.0	0.0740	0.0305
350	1.080	475.0	0.0730	0.0330
375	1.083	480.0	0.0725	0.0320
400	1.092	490.0	0.0715	0.0300
425	1.099	500.0	0.0700	0.0320
450	1.110	508.0	0.0690	0.0340
475	1.120	518.0	0.0710	0.0330
500	1.160	526.0	0.0720	0.0310
520	1.200	540.0	0.0740	0.0320

Notes: t is in weeks; NC(t) and AL(t) in 1,000,000 NT dollars.

The parameters for the dynamics of the assets are taken from Brennan, Schwartz, and Lagnado (1998):

$$\frac{dS}{S} = 0.009992\ dt + 0.041\ dW_S,$$
$$dr = 0.000158\ dt + 0.005472\ dW_r,$$
$$dl = -0.0002236\ dt + 0.00304\ dW_l.$$

The initial values for r and l are 3.5% and 7%, respectively.

The simulated paths of normal costs, accrued liabilities, short rate, and long rate are shown in Table 1.

We assume the following covariance (or correlation) matrix:[6]

$$\begin{bmatrix} 1 & \sigma_{S,\mathrm{NC}} & \sigma_{W,\mathrm{NC}} & \sigma_{B,\mathrm{NC}} & \sigma_{l,\mathrm{NC}} \\ \sigma_{\mathrm{NC},S} & 1 & \sigma_{WS} & \sigma_{BS} & \sigma_{lS} \\ \sigma_{\mathrm{NC},W} & \sigma_{SW} & 1 & \sigma_{BW} & \sigma_{lW} \\ \sigma_{\mathrm{NC},B} & \sigma_{SB} & \sigma_{WB} & 1 & \sigma_{lB} \\ \sigma_{\mathrm{NC},l} & \sigma_{Sl} & \sigma_{Wl} & \sigma_{Bl} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.2 & -0.3 & 0.1 & 0.4 \\ 0.2 & 1 & -0.25 & -0.5 & 0.3 \\ -0.3 & -0.25 & 1 & 0.1 & -0.2 \\ 0.1 & -0.5 & 0.1 & 1 & 0.1 \\ 0.4 & 0.3 & -0.2 & 0.1 & 1 \end{bmatrix}$$

To illustrate the solved optimal control law, we first simulate a set of paths that includes a short rate path and a long rate path. These paths are then plugged into the system of differential equations given in equations (17), (18), and (19) to solve for the optimal strategies corresponding to the (t, X). The solved optimal strategies are the optimal contributions per week and the corresponding optimal contribution rates[7] under 5-year and 10-year evaluation periods and target funding ratios of $\eta = 1$ and $\eta = 0.75$. Tables 2, 3, and 4 show results for $\eta = 1$.

[6] As we are using standard Wiener processes, the covariances and the correlations are the same.

[7] Weekly contributions are measured in dollars, while weekly contribution rates are measured by the ratio of weekly contributions to salary.

Table 2
Optimal Contribution Rates and Contributions with $\eta = 1$ Under 10-Year and 5-Year Time Horizons

	Contribution Rates		Contributions	
Week	10-Years	5-Years	10-Years	5-Years
0	0.0427	0.0426	0.960	2.510
25	0.0430	0.0429	0.968	3.030
50	0.0435	0.0435	0.977	3.400
75	0.0440	0.0440	0.989	3.580
100	0.0445	0.0445	1.001	3.750
125	0.0450	0.0450	1.016	3.900
150	0.0454	0.0454	1.021	4.080
175	0.0458	0.0458	1.031	4.270
200	0.0463	0.0463	1.041	4.300
225	0.0466	0.0467	1.050	4.350
250	0.0469	0.0469	1.058	4.420
260		0.0470		4.430
275	0.0473		1.064	
300	0.0477		1.074	
325	0.0478		1.078	
350	0.0481		1.082	
375	0.0482		1.084	
400	0.0486		1.092	
425	0.0487		1.099	
450	0.0491		1.106	
475	0.0493		1.112	
500	0.0495		1.116	
520	0.0497		1.120	

Notes: Contributions are measured in 1,000,000 NT dollars.

The optimal weekly contributions and contribution rates are shown in Table 2 for $\eta = 1$. The optimal weekly contributions and contribution rates increase steadily with time. Such increases are reasonable because normal costs increase with the aging of the employees in the plan. Table 3 show the resulting fund when the pension plan adopts the optimal investment strategies under different evaluation periods (5 and 10 years) and a target funding ratio of 1. Table 4 shows the evolution of the optimal mix of stocks, bonds and cash under different evaluation periods and a target funding ratio of 1. A summary of the results is given in Table 5.

Table 3
Evolution of the Optimal Fund and Fund Ratios with $\eta = 1$ Under 10-Year and 5-Year Time Horizons

	Fund ($F(t)$)		Fund Ratio	
Week	10-Years	5-Years	10-Years	5-Years
0	2.510	2.510	0.485	0.580
25	2.930	3.030	0.550	0.670
50	3.400	3.400	0.620	0.740
75	3.600	3.580	0.685	0.800
100	3.800	3.750	0.720	0.830
125	3.950	3.900	0.750	0.875
150	4.100	4.080	0.760	0.900
175	4.280	4.270	0.780	0.905
200	4.300	4.300	0.800	0.915
225	4.350	4.350	0.810	0.950
250	4.420	4.420	0.820	0.975
260		4.430		1.000
275	4.480		0.830	
300	4.510		0.850	
325	4.600		0.860	
350	4.680		0.880	
375	4.750		0.895	
400	4.800		0.900	
425	4.850		0.905	
450	4.920		0.920	
475	5.000		0.935	
500	5.080		0.950	
520	5.140		1.000	

Notes: The fund is measured in 100,000,000 NT dollars.

To quickly achieve the target funding ratio, the plan manager has to take some unusual positions (such as large amounts of short or long positions) at the beginning. These extreme positions are mainly driven by the parameters of financial market processes and the choice of k. As the optimal control law is sensitive to the estimated or chosen parameters, the plan manager should pay close attention to the choice of parameters.

Table 4
Proportions of Stocks, Bonds, and Cash
Under 10-Year and 5-Year Time Horizons with $\eta = 1$

	10-Year Horizon			5-Year Horizon		
Weeks	Stocks	Bonds	Cash	Stocks	Bonds	Cash
0	-4.100	8.800	-3.700	-3.800	8.800	-4.000
25	-2.750	4.800	-1.050	-2.250	4.200	-0.950
50	-1.400	2.600	-0.200	-1.350	2.500	-0.150
75	-0.650	1.600	0.050	-0.750	1.900	-0.150
100	-0.250	1.000	0.250	-0.400	1.400	0.000
125	-0.200	0.500	0.700	-0.250	0.700	0.550
150	-0.050	0.300	0.750	-0.060	0.300	0.760
175	-0.010	0.150	0.860	-0.010	0.150	0.860
200	0.000	0.050	0.950	0.001	0.050	0.949
225	0.000	0.001	0.999	-0.001	0.002	0.999
250	0.010	0.002	0.988	0.010	0.005	0.985
260				0.012	0.200	0.788
275	0.015	0.004	0.981			
300	0.010	0.006	0.984			
325	0.000	0.002	0.998			
350	0.005	0.003	0.992			
375	0.020	-0.050	1.030			
400	0.001	0.000	0.999			
425	0.005	-0.100	1.095			
450	0.080	-0.200	1.120			
475	0.004	-0.100	1.096			
500	0.001	0.005	0.994			
520	-0.100	0.500	0.600			

Table 5
Statistics on Asset Weights

Period	Asset Class	Minimum	Median	Mean	Maximum	Standard Deviation
Given $\eta = 0.75$ and $k = 0.0001$						
5 Years	stock	-0.7057	-0.0421	-0.1203	0.0061	0.1672
10 Years	stock	-0.7108	-0.0034	-0.0625	0.1083	0.1636
5 Years	long term bond	-0.0728	0.1792	0.2853	1.5888	0.3434
10 Years	long term bond	-0.2357	-0.0154	0.1272	1.6067	0.3655
5 Years	cash	0.1132	0.8697	0.8350	1.0668	0.1809
10 Years	cash	0.1031	1.0175	0.9354	1.1399	0.2076
Given $\eta = 1.00$ and $k = 0.0001$						
5 Years	stock	-3.9395	-0.1611	-0.6277	0.0024	0.9193
10 Years	stock	-4.0801	-0.0076	-0.3197	0.0950	0.7961
5 Years	long term bond	-0.0489	0.7039	1.4995	8.8692	1.8794
10 Years	long term bond	-0.2082	0.0147	0.7282	8.8981	1.6783
5 Years	cash	-3.9653	0.4364	0.1281	1.0465	0.9769
10 Years	cash	-3.9161	0.9929	0.5915	1.1188	0.8932

The funding ratio volatility and the trading activity volatility increase with the difference between the current funding ratio and the target funding ratio. Notice that the volatility of trading volatilities, fund levels, and funding ratios when $\eta = 1$ are greater than those $\eta = 0.75$. This is reasonable because the larger the difference is, the more the assets have to be increased and thus the more aggressive the trading must be. Furthermore, we observe that shorter evaluation periods result in higher volatilities. A possible explanation is that shorter evaluation periods make the optimal trading strategies more sensitive to financial markets because the plan manager has a shorter time to achieve the goal.

5 Summary and Closing Comments

Stochastic control is potentially a helpful tool for managing pension plans. It represents a significant improvement over the one-period approach traditionally used by plan managers because it can explicitly consider the inter-period dynamics and aim at long-term rather than short-term optimality. Furthermore, dynamic control models can simultaneously handle plan funding policies and investment decisions.

Our model is the most comprehensive one so far as it combines the merits from different models. The Haberman and Sung (1994) approach is used to develop our objective function, i.e., we consider the contribution risk (the stability of contributions) and the solvency risk (the security of funds). The Brennan, Schwartz, and Lagnado (1997) model is used to enlarge the set of investable assets so that it contains both risk-free and risky assets. For liabilities, we employ the stochastic simulations in Chang (1999, 2002) that explicitly characterize the plan members. These allow us to derive a system of differential equations, for which the solution represents optimal funding policies and asset allocations. We then apply the theoretical model to an actual Taiwanese pension plan and numerically obtain optimal solutions.

There are three areas for further research:

- One may add short sale constraints into our model because our optimal strategies usually involve certain amount of short sales. Most pension funds, however, are not allowed to engage in short sales because of the associated downside risk;
- One may want to include transaction costs. Note that high transaction costs may reduce the relative advantage of active trading

to passive trading, which might result in different optimal trading strategies; and finally,

- Include optimal hedging policies.

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Appendix

Table A1
Simplified Male Service Table
Survival Probabilities

x	$p_x^{(\tau)}$	x	$p_x^{(\tau)}$
15	0.860048	38	0.937401
16	0.859854	39	0.937279
17	0.859326	40	0.983539
18	0.859181	41	0.983369
19	0.859278	42	0.983176
20	0.801437	43	0.982957
21	0.801529	44	0.982763
22	0.801620	45	0.982559
23	0.801712	46	0.982352
24	0.801708	47	0.982149
25	0.881447	48	0.981952
26	0.881441	49	0.981673
27	0.881436	50	0.990293
28	0.881438	51	0.989965
29	0.881445	52	0.989596
30	0.937983	53	0.989173
31	0.937946	54	0.988972
32	0.937887	55	0.988738
33	0.937804	56	0.988465
34	0.937741	57	0.988150
35	0.937668	58	0.987794
36	0.937585	59	0.987288
37	0.937496	60	0

Note: $p_x^{(\tau)}$ = Probability a male plan member age x remains in the plan at age $x+1$.

Table A2
Simplified Female Service Table
Survival Probabilities

x	$p_x^{(\tau)}$	x	$p_x^{(\tau)}$
15	0.860180	38	0.938062
16	0.860134	39	0.938021
17	0.860081	40	0.984412
18	0.860059	41	0.984340
19	0.860054	42	0.984258
20	0.802068	43	0.984166
21	0.802065	44	0.984053
22	0.802064	45	0.983935
23	0.802063	46	0.983814
24	0.802061	47	0.983691
25	0.881830	48	0.983571
26	0.881818	49	0.983423
27	0.881800	50	0.992203
28	0.881781	51	0.992035
29	0.881759	52	0.991851
30	0.938305	53	0.991652
31	0.938279	54	0.991392
32	0.938258	55	0.991124

Note: $p_x^{(\tau)}$ = Probability a female plan member age x remains in the plan at age $x + 1$.

Table A3
Basic Statistics on New Entrants

Age Interval	Number of New Entrants	Average Annual Salary
15 19	82	23,356
20 24	163	27,660
25 29	273	38,404
30 34	88	38,718
35 39	17	46,297
40 44	7	43,305
45 49	4	36,053