7-2007

The Volume of a Platonic Solid

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Master of Arts in Teaching (MAT)
Masters Exam

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In partial fulfillment of the requirements for the Master of Arts in Teaching with a Specialization in the Teaching of Middle Level Mathematics in the Department of Mathematics.
Jim Lewis, Advisor

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Expository Paper

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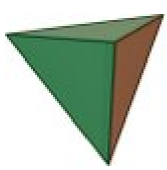
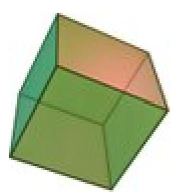
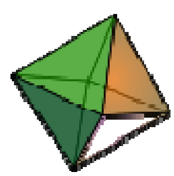
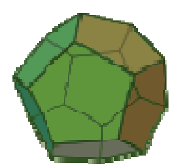
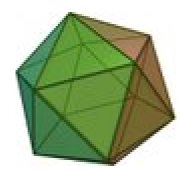
A regular tetrahedron and a regular octahedron are two of the five known Platonic Solids. These five “special” polyhedra look the same from any vertex, their faces are made of the same regular shape, and every edge is identical. The earliest known description of them as a group is found in Plato’s *Timaeus*, thus the name Platonic Solids. Plato theorized the classical elements were constructed from the regular solids. The tetrahedron was considered representative of fire, the hexahedron or cube represented earth, the octahedron stood for gas or air, the dodecahedron represented vacuum or ether (which is made up of pure electromagnetic energy) , and the icosahedron was water.

The Five Convex Regular Polyhedra (Platonic solids)				
Tetrahedron	Hexahedron or Cube	Octahedron	Dodecahedron	Icosahedron
				

There are special relationships which exist between these three dimensional shapes. Duality is one of the distinct patterns among the platonic solids. It associates the regular polyhedra into pairs called duals where the vertices of one correspond to the faces of the other. These dual pairs are the octahedron and the cube, and the dodecahedron and the icosahedron. The tetrahedron is a self dual. These duals have the unique property of each having the same number of vertices as the other has faces. For example, the cube has 8 vertices and 6 faces, and the octahedron has 8 faces and 6 vertices.

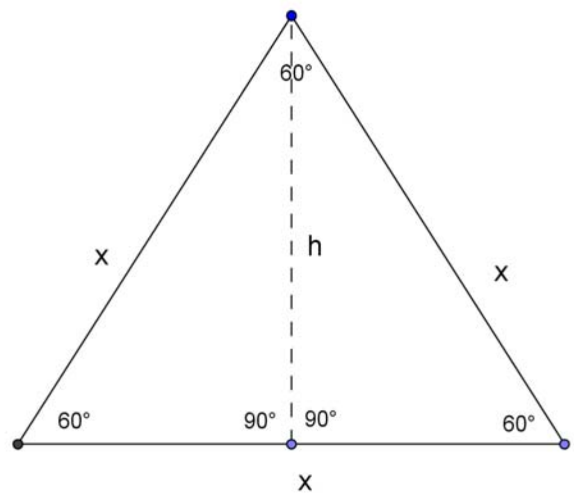
Another notable association among the platonic solids is the way in which one solid inscribes in another. Using basic geometric principles relative volume may be calculated. This relationship offers up a way to find the volume of the platonic solids which fit inside the cube.

The volume of the hexahedron or cube is easily calculated by cubing the measure of one of the sides of the cube. However, the lack of obvious right angles in the 4 remaining platonic solids makes finding their volumes more difficult. This expository paper will consider finding the volume of a regular tetrahedron and a regular octahedron by using two methods. One method will be utilizing trigonometry and the Pythagorean Theorem, and the second method will be inscribing the regular tetrahedron and the regular octahedron in a cube.

Volume of a Regular Tetrahedron Using Trigonometry and the Pythagorean Theorem

A tetrahedron is a triangular pyramid. As with any pyramid, the volume of a tetrahedron is $V = 1/3\, AH$ where A is the area of the base and H the height from the base to the apex or highest point. This applies for each of the four choices of the base, so the distances from the apexes to the opposite faces are inversely proportional to the areas of these faces.

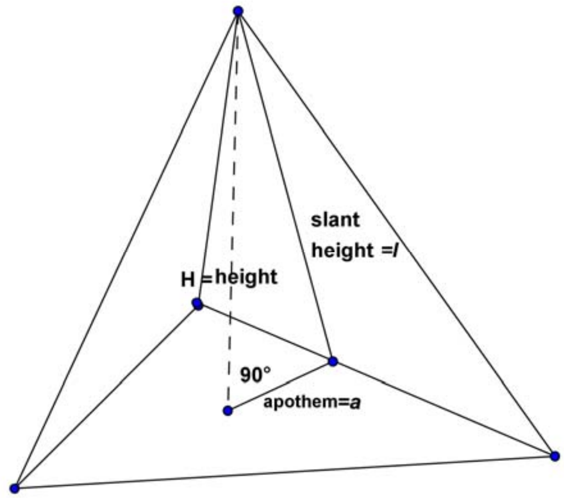
In order to find the volume of a regular tetrahedron the area of its base must be calculated. Let the length of each side of the triangular base of the tetrahedron be x. The height of the triangular base is designated by h, and it is known that the angles in an equilateral triangle are all 60^{o}.



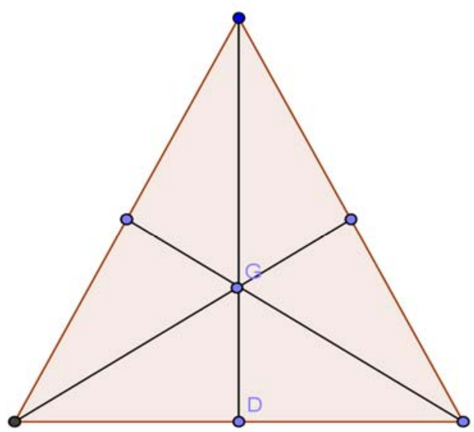
Trigonometry may be used to calculate the area of the base. To find the height of the shape, I know $\frac{h}{x} = \sin 60^{\circ}$, and thus, $\frac{h}{x} = \frac{\sqrt{3}}{2}$. Multiplying both sides of this equation by *x*, gives the height of the base of the regular tetrahedron $h = \frac{x\sqrt{3}}{2}$. The formula for the area of a triangle is $\frac{1}{2}bh$. Plugging in the height I now have the equation for the area of the base: $A = \frac{1}{2}x\left(\frac{x\sqrt{3}}{2}\right)$. Simplifying this gives the area of the base to be $\frac{x^2\sqrt{3}}{4}$.

Putting this into my original volume formula $V = \frac{1}{3}AH$ gives: $V = \frac{1}{3}(\frac{x^2\sqrt{3}}{4})H$.

Finding the height of the regular tetrahedron will take several steps.



We now find the apothem which is the line segment from the center of the base to the edge of the base. I know the slant height of the tetrahedron is just the height of the equilateral triangle, which was just shown to be $\frac{x\sqrt{3}}{2}$.



V

Segment DG is the apothem

To find the length of the apothem (i.e. the line segment GD) we note that the angle at G (<VGD) is 60^{o} and $\tan(60) = \frac{x/2}{a}$. Solving for a we obtain

$$a = \frac{x}{2\tan(60)} = \frac{x}{2\sqrt{3}}$$

Knowing the apothem, I use the Pythagorean Theorem to find the height, *H*.

The slant height = *l*, and the apothem = *a*.

$$H^2 + a^2 = l^2$$

$$H^2 + \left(\frac{x}{2\sqrt{3}}\right)^2 = \left(\frac{x\sqrt{3}}{2}\right)^2$$

$$H^2 + \frac{x^2}{12} = \frac{3x^2}{4}$$

$$H = \sqrt{\frac{2}{3}}x$$

Plugging this expressions for height into the volume formula for the regular tetrahedron gives:

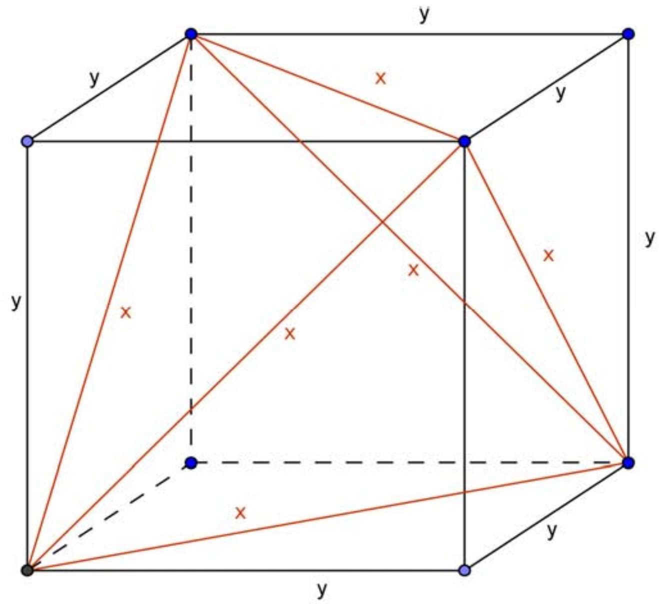
$$V = \frac{1}{3}\left(\frac{x^2\sqrt{3}}{4}\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\right)x$$

$$V = \frac{x^3\sqrt{2}}{12}.$$

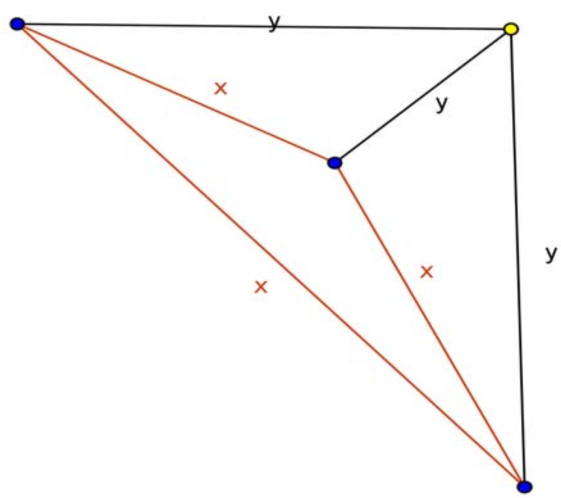
<u>Inscribing a Regular Tetrahedron in a Cube to Find Its Volume</u>

Inscribing a regular tetrahedron in a cube may be done by letting each edge of the tetrahedron be a diagonal of a face of the cube. Inside the cube and outside the tetrahedron, are several tetrahedra whose volumes are easy to compute since there are some right angles involved.

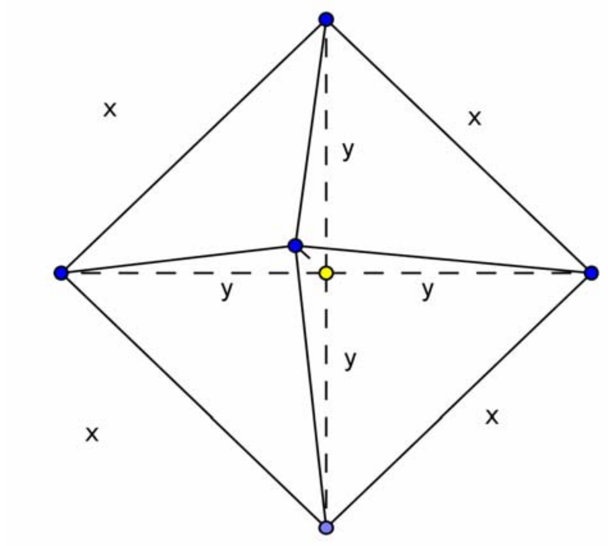
The volume of the inscribed tetrahedron may be found by subtracting the combined volume of the four tetrahedra outside of the regular tetrahedron from the volume of the cube. Let *y* be the length of the sides of the cube, and let *x* be the lengths of the sides of the regular tetrahedron. Note that $x = y\sqrt{2}$ since the edges of the tetrahedron are the diagonals of the square faces of the cube.



One of the cut-off tetrahedra (the pieces of the cube which are outside of the inscribed tetrahedron) would look like the following picture:



All four of these tetrahedral may be combined to form a new pyramid. Piecing the cut-off tetrahedra together would result in a pyramid with a square base, with a top-view as shown:



The base of this pyramid is a square formed with four isosceles triangles. The ratio between the sides of a 45-45-90 isosceles triangle is 1: 1: $\sqrt{2}$. Thus, the triangles in the base of this pyramid have sides of lengths y, y, and $y\sqrt{2}$ (i.e. $x = y\sqrt{2}$). I will use the formula for the volume of a pyramid to find the volume cut-off from the cube.

$$V = \frac{1}{3}\ A\ h$$

$$V = \frac{1}{3}(y\sqrt{2})^2\ (y)$$

$$V = \frac{1}{3}(2y^2)\ (y)$$

$$V = \frac{1}{3}(2y^3)$$

$$V = \frac{2}{3}y^3$$

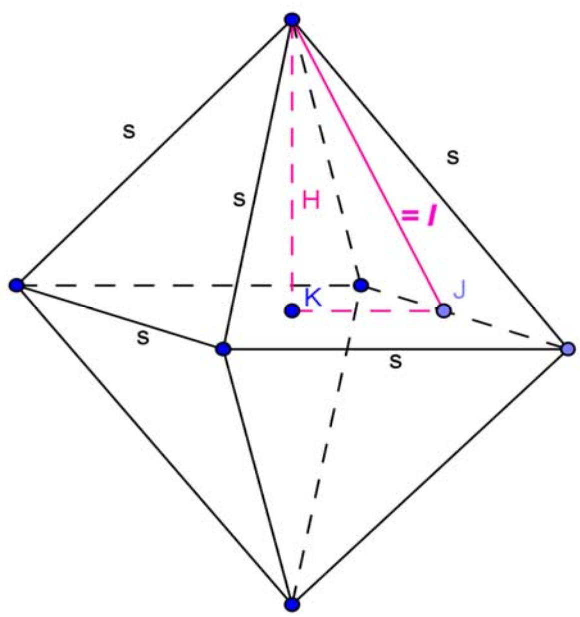
Subtracting the volume of the four tetrahedra cut off of the cube yields the volume of the inscribed regular tetrahedron. The volume of the cube less the volume of the cut-off pieces, equals the volume of the inscribed regular tetrahedron.

$$y^3 - \frac{2}{3}\ y^3 = \frac{1}{3}\ y^3$$

Thus, the volume of a regular tetrahedron is $\frac{1}{3}$ the volume of the cube in which it is inscribed.

Volume of a Regular Octahedron Using Trigonometry and Pythagorean Theorem

A regular octahedron is a platonic solid with 8 equal triangular faces. It is formed by 2 pyramids with square bases. The volume of a regular octahedron may be found by finding the volume of one of the pyramids and multiplying that volume by 2. In the case of the regular octahedron I know the area of the square base to be the square of the length of a side. I will use *s* to designate the length of the sides of the regular octahedron. Then I will use my earlier result for the height of an equilateral triangle $h = \frac{\sqrt{3}}{2}s$, which will be the slant height, *l*, of my tetrahedron.



The apothem, or *a*, of the pyramid is labeled segment KJ. The length of a is half of the octahedron's side so $a = \frac{1}{2}s$. The height of the pyramid may now be found using Pythagorean Theorem.

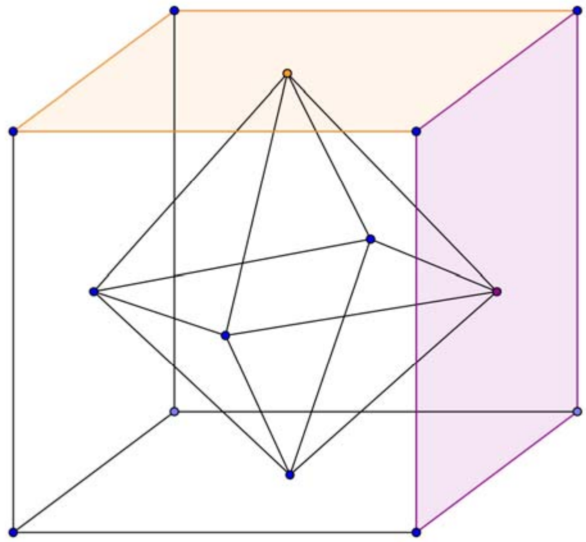
$$H^2 + a^2 = l^2$$
$$H^2 + \left(1\frac{s}{2}\right)^2 = \left(\frac{\sqrt{3}}{2}s\right)^2$$
$$H^2 + \frac{s^2}{4} = \frac{3s^2}{4}$$
$$H^2 = \frac{2s^2}{4}$$
$$H^2 = \frac{s^2}{2}$$
$$H = \sqrt{\frac{s^2}{2}}$$
$$H = \frac{s}{\sqrt{2}},$$

The height may now be plugged into the volume formula for a pyramid and doubled to find the volume of the regular octahedron.

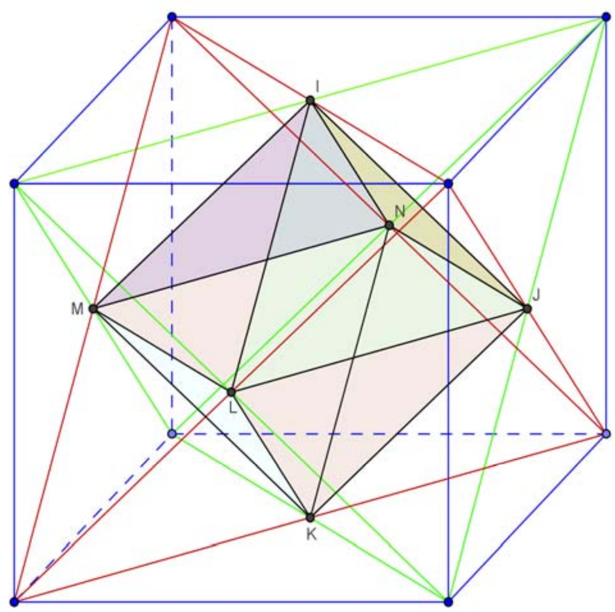
$$V = 2\left(\frac{1}{3}\right)AH$$
$$V = 2\left(\frac{1}{3}\right)(s^2)\left(\frac{s}{\sqrt{2}}\right)$$
$$V = \frac{2s^3}{3\sqrt{2}}$$
$$V = \frac{\sqrt{2}s^3}{3}.$$

Inscribing a Regular Octahedron in a Cube

A regular octahedron is a dual of a cube. Each vertex of the regular octahedron touches the center of one of the faces of the cube.

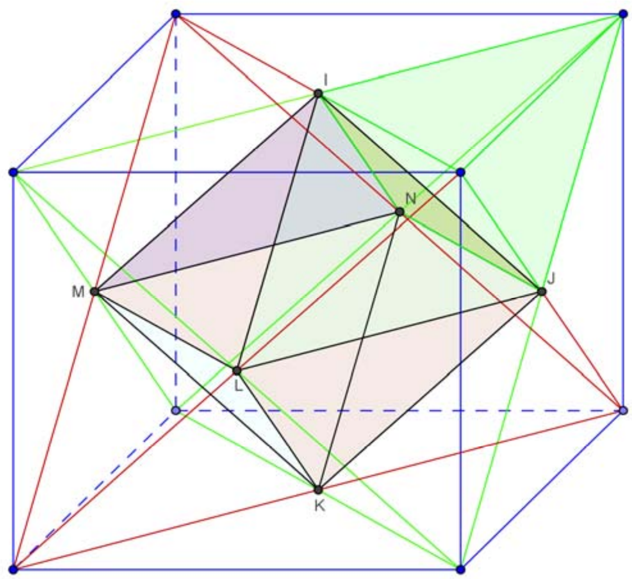


It is an interesting fact that this octahedron may be formed by the intersection of two tetrahedra inscribed in a cube.



The regular octahedron INJKLM, formed with black line segments, has vertices at the center of each face of the cube. The vertices are also formed by the intersection of the edges of the green regular tetrahedron and the red regular tetrahedron.

Since the edges of the tetrahedra are diagonals of the cube faces, and it is known that the diagonals of a square bisect each other, the intersection points for the two tetrahedra occur at the center of the faces of the cube. There are two tetrahedra inscribed in the cube. Focus on the green tetrahedron and one of its smaller pyramids truncated by its intersection with the red tetrahedron.



The shaded green truncated pyramid is similar to the original green tetrahedron since the edges of the truncated pyramid are each $\frac{1}{2}$ as long as the original tetrahedron. Thus, the truncated pyramid is similar to the original tetrahedron at a $\frac{1}{2}$ scale factor. Other dimensions are also proportional. Since volume is proportional to the scale factor cubed, the volume of each of the truncated pyramids is $(\frac{1}{2})^3$ or $\frac{1}{8}$ of the volume of the original, green tetrahedron. Having cut off 4 pieces each with $\frac{1}{8}$ of the volume of the original, their total volume is $\frac{1}{2}$ of the volume of the original, green tetrahedron. There remains $\frac{1}{2}$ of the original green tetrahedron volume to form the inscribed octahedron.

Since the green tetrahedron has been found earlier in this paper to be $\frac{1}{3}$ the volume of the cube in which it is inscribed, the octahedron is $\frac{1}{2}$ of $\frac{1}{3}$ of the volume of the cube. Thus, the inscribed octahedron is $\frac{1}{6}$ the volume of the cube.

In Conclusion

The following table shows the relationships between the cube, regular tetrahedron, and the regular octahedron.

SHAPE	**SIDE LENGTH**	**RELATIONSHIPS**	**VOLUME**
Cube	c		c^3
Tetrahedron	S	$\sqrt{2}c = S$	$\frac{S^3}{6\sqrt{2}} = \frac{\sqrt{2}S^3}{12}$
Octahedron	s	$S = 2s$ or $s = \frac{c}{\sqrt{2}}$	$\frac{\sqrt{2}s^3}{3}$

Letting the sides of the cube be of unit length yields the following:

when c = 1, then the Volume of the regular tetrahedron = $\frac{\sqrt{2}^3}{6\sqrt{2}} = \frac{1}{3}$;

when c = 1, then the Volume of the regular octahedron = $\frac{\sqrt{2}(\frac{1}{\sqrt{2}})^3}{3} = \frac{1}{6}$.

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