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Further Remarks on Risk Sources Measuring: The Case of a Life Annuity Portfolio

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Abstract§

The paper considers a model that allows the actuary to measure the riskiness connected to the randomness of projected mortality tables in evaluating a portfolio of life annuities, obtaining a measure to reflect the risk associated with the randomness of the projection. The coherence of the risk parameters with the specific nature of the considered risk sources is also discussed.

Numerical examples illustrate the results, showing the importance of the risk components in terms of the number of policies and comparing measure tools obtained by means of two procedures.

Key words and phrases: *investment risk, insurance risk, longevity risk, random projected mortality tables*

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1 Introduction

As a life insurance business can be viewed as a dynamic risk process, actuaries must have at their disposal tools to control the various factors affecting this risk process. The two most important risks life insurers face are investment risk and demographic risk. The investment risk is the risk to the market value of the insurer's assets due to the random movements of the financial market. [See, for example, Beekman and Fuelling (1990) and (1991), Frees (1998), Parker (1994a) and (1994b), and Zaks (2001).] The demographic risk is the risk of premature death (in the case of life insurance) or excessive longevity (in the case of annuities).[1]

Focusing on the annuity business, demographic risk consists of two components: (i) the insurance risk, which is due to the random deviations of the number of deaths from their expected values; and (ii) the longevity risk, which is due to improvements in mortality rates.[2] The longevity risk combines the effects of two phenomena: (i) "rectangularization," which refers to the higher concentration of deaths around the mode of the curve of deaths; and (ii) "expansion," which refers to the increase in the mode of the curve of deaths over time. Insurance risk can be viewed as a pooling risk because it decreases as the number of policies in-force increases, while longevity risk is a non-pooling risk because it is not affected by the number of policies in-force.

Given the potentially adverse impact of the longevity risk on the stability of a portfolio of annuities, mortality tables used to value annuity contracts must take into account the anticipated improvements in future mortality, i.e., a mortality projection. In other words, tables should be constructed based on anticipated decreases in future mortality rates. Failure to include future improvements in mortality could cause a significant underestimation of future obligations.

Though several authors [e.g., Pitacco (1997), Marocco et al., (1998), Olivieri (1998), Olivieri et al., (1999) and Coppola et al., (2000))] have studied the longevity risk, the Coppola et al., (2000) paper is of particu-

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[1] In the case of insurance with a death benefit, the effect of improving mortality is to delay the time of death thus postponing the payment of the death benefit. In the case of annuities, however, the effect of improving mortality is to increase the duration of the insurer's payments to the annuitant.

[2] Due to the constant advances in public health and safety and better nutrition, people in most societies around the world are living longer and healthier lives. In the developed economies, there is a trend of increasing sales of annuity contracts to pay for retirement. This combination of increasing longevity and increasing sales of annuity contracts requires actuaries to have a deeper understanding of mortality trends, and hence the longevity risk. [See, also, UP-Task Force (1996).]

lar interest to us. Coppola et al., identify and characterize the two risk factors for a life annuity portfolio and analyze the demographic risk by taking into account only the contribution of the longevity risk. They present a model of the global riskiness of the portfolio and provide an expression for the contribution of each risk component to the value of the entire portfolio.

Our current paper is a follow-up to the Coppola et al., (2000) paper. It uses a stochastic framework for the interest rates and a deterministic framework for the longevity risk. The basic underlying distribution of an individual's future lifetime is Weibull. The longevity risk is modeled by constructing three different projected mortality tables, each representing a particular scenario for improving mortality. This leads to a more general model of the present value of the portfolio. We propose measurement tools for determining the riskiness of the average cost per policy due to the randomness in choosing projected mortality tables, while still taking into account the effect of interest rates and random mortality deviations.

This paper is organized as follows: In Section 2 we review some valuation results already presented by Coppola et al., (2000) and introduce the stochastic model for interest rates. Section 3 presents equations to decompose the total riskiness taking into consideration the randomness of the projection together with the interest randomness and the random mortality deviations. In particular we obtain different equations following two procedures, both based on a risk decomposition using variance, but conditioning on different risk sources. The asymptotic behavior of the obtained risk parameters, when the portfolio's size tends to infinity, is also discussed. In Section 4 the main results of the paper are complemented by some numerical examples.

2 Portfolio Valuations

Let us consider a portfolio consisting of c individuals age exactly x each of whom has a whole life annuity-immediate policy (or contract) paying \$1 per year for life. Let us introduce the following notation:

$K_i(x)$ = Curtate future lifetime of the i^{th} policy;

Z_i = Present value of the annuity contract for the i^{th} policy;

$Z(c)$ = Present value of the entire annuity portfolio of c contracts; and

$\delta(s)$ = Stochastic force of interest at time $s \geq 0$.

It follows that

$$Z_i = \sum_{h=1}^{K_i(x)} \exp(-y(h)) \tag{1}$$

where

$$y(h) = \int_0^h \delta(s)ds; \tag{2}$$

in addition,

$$Z(c) = \sum_{i=1}^{c} Z_i. \tag{3}$$

As in Coppola et al., (2000), the following assumptions are made:

(i) The times of death $K_1(x), K_2(x), \ldots, K_c(x)$ are mutually independent and identically distributed random variables;

(ii) Given the sequence $y(1), y(2), \ldots$, the Z_1, Z_2,..., Z_c are independent and identically distributed; and

(iii) The times of death $K_1(x), K_2(x), \ldots, K_c(x)$ and the interest rate process $\delta(s)$ are mutually independent.

The first two moments are given in Coppola et al., (2000) and are as follows:

$$\begin{aligned}
\mathbb{E}[Z_i | \{y(h)\}_{h=1}^{\infty}] &= \sum_{h=1}^{\infty} {}_h p_x e^{-y(h)} \\
\mathbb{E}[Z_i] &= \sum_{h=1}^{\infty} {}_h p_x \mathbb{E}[e^{-y(h)}] \\
\mathbb{E}[Z_i^2 | \{y(h)\}_{h=1}^{\infty}] &= \sum_{h=1}^{\infty} {}_h p_x e^{-2y(h)} + 2 \sum_{h=2}^{\infty} {}_h p_x \sum_{r=1}^{h-1} e^{-y(r)-y(h)} \\
\mathbb{E}[Z_i^2] &= \sum_{h=1}^{\infty} {}_h p_x \mathbb{E}[e^{-2y(h)}] + 2 \sum_{h=2}^{\infty} {}_h p_x \sum_{r=1}^{h-1} \mathbb{E}[e^{-y(r)-y(h)}] \\
\mathbb{E}[Z(c)] &= c \sum_{h=1}^{\infty} {}_h p_x \mathbb{E}[e^{-y(h)}], \quad \text{and} \\
\mathbb{E}[Z(c)^2] &= \sum_{i=1}^{c} \mathbb{E}[Z_i^2] + \sum_{\substack{i,j=1 \\ i \neq j}}^{c} \mathbb{E}[Z_i Z_j].
\end{aligned}$$

By assumptions (i), (ii), and (iii), we can write [see, Coppola et al., (2000)]

$$\mathbb{E}[Z_i Z_j] = \mathbb{E}[\mathbb{E}[Z_i Z_j | \{y(h)\}_{h=1}^{\infty}]] \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_h p_x \, {}_k p_x \mathbb{E}[e^{-y(h)-y(k)}].$$

It follows that

$$\begin{aligned}\mathbb{E}[Z(c)^2] &= c\mathbb{E}[Z_i^2] + \sum_{\substack{i,j=1 \\ i \neq j}}^{c} \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_h p_x \, {}_k p_x \mathbb{E}[e^{-y(h)-y(k)}] \\ &= c\mathbb{E}[Z_i^2] + c(c-1) \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_h p_x {}_k p_x \mathbb{E}[e^{-y(h)-y(k)}]. \qquad (4)\end{aligned}$$

The first two moments of the average cost per policy of the portfolio under consideration, $Z(c)/c$, are

$$\mathbb{E}[\frac{Z(c)}{c}] = \sum_{h=1}^{\infty} {}_h p_x \mathbb{E}[e^{-y(h)}] \qquad (5)$$

$$\mathbb{E}[\left(\frac{Z(c)}{c}\right)^2] = \frac{1}{c}\mathbb{E}[Z_i^2] + \frac{c-1}{c} \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_h p_x \, {}_k p_x \mathbb{E}[e^{-y(h)-y(k)}]. \qquad (6)$$

2.1 The Stochastic Interest Rate Environment

In order to get a realistic description of the insurance environment, we consider the risk arising from fluctuations of the rate of interest process $\delta(t)$. The interest rate process is viewed as a sum of two components: a deterministic component, $i(t)$, which can be estimated on the basis of the company's investment policy, and a stochastic component, $X(t)$, which describes the deviations of the interest rate process from its expected values. Thus

$$\delta(t) = i(t) + X(t).$$

The $X(t)$ process is assumed to be an Ornstein-Uhlenbeck process, with parameters $\beta > 0$ and $\sigma > 0$, and initial position $X(0) = 0$. Ornstein-Uhlenbeck processes are characterized by the following stochastic differential equation

$$dX(t) = -\beta X(t)dt + \sigma dW(t)$$

where $W(t)$ is a standard Wiener process. See, for example, Arnold (1974) or Gard (1998) for more on stochastic differential equations.

The present value at time 0 of a payment of one monetary unit at time t is given by

$$v(t)F(t) = e^{-y(t)} = e^{-\int_0^t (i(s)+X(s))ds}$$

where

$$v(t) = e^{-\int_0^t i(s)ds} \quad \text{and } F(t) = e^{-\int_0^t X(s)ds}$$

are the deterministic and stochastic discounting factors, respectively. Using the fact that $F(t)$ is log-normal and $\mathbb{E}[X(t)] = 0$, Coppola et al., (2000) demonstrate that:

$$\mathbb{E}[F(t)] = e^{\frac{1}{2}\phi(t)} \tag{7}$$

$$\mathbb{V}\text{ar}[F(t)] = e^{\phi(t)}[e^{\phi(t)} - 1] \tag{8}$$

$$\mathbb{C}\text{ov}[F(h), F(k)] = e^{\frac{1}{2}[\phi(h)+\phi(k)]}[e^{\Phi(h,k)} - 1] \tag{9}$$

where

$$\phi(t) = \mathbb{V}\text{ar}[\int_0^t X(s)ds], \quad \text{and} \tag{10}$$

$$\Phi(h,k) = \mathbb{C}\text{ov}[\int_0^h X(s)ds, \int_0^k X(s)ds] \tag{11}$$

is the autocovariance function of $F(t)$.

3 A Measure of Projection Randomness

To take into account the influence of the randomness in projections, we use a well known variance decomposition equation for estimating the importance of different risk sources in portfolio valuations.[3] Coppola et al., (2000) obtain measurement tools for estimating the impact of some risk components, i.e., the insurance and the investment risks, when different projected mortality tables are used.

[3]See, for example, Coppola et al., (2000), Coppola et al., (1999), Frees (1998), Parker (1997).

3.1 Conditioning on the Random Survival Function

Let P denote the survival function used to construct the survival probabilities in the projected table. The variance of the average cost per policy can then be split in two components:

$$\mathbb{V}\mathrm{ar}[\frac{Z(c)}{c}] = \mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|P]] + \mathbb{E}[\mathbb{V}\mathrm{ar}[\frac{Z(c)}{c}|P]]. \tag{12}$$

The first term on the right side of equation (12) is a measure of the variability of $Z(c)/c$ due to the effect of the randomness of the projection. The second term measures the effect of the other risk components (random interest rates and random mortality deviations), the effects of the projection randomness having been averaged out. As

$$\mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|P] = \mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{1}{c}\sum_{i=1}^{c} Z_i|P]] = \mathbb{V}\mathrm{ar}[\mathbb{E}[\sum_{h=1}^{K_i(x)} e^{-y(h)}|P]],$$

it is clear that $\mathbb{V}\mathrm{ar}[\mathbb{E}[Z(c)/c|P]]$ is a measure of a systematic risk, which is independent of the size of the portfolio c. This agrees with the nature of the risk due to the randomness of projection. Thus we have the following definition

Definition 1A: $\mathbb{V}\mathrm{ar}[\mathbb{E}[Z(c)/c|P]]$ is a measure of the projection risk.

The second term on the right side of equation (12) can again split as follows

$$\begin{aligned}\mathbb{E}[\mathbb{V}\mathrm{ar}[\frac{Z(c)}{c}|P]] &= \mathbb{E}[\mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]] \\ &+ \mathbb{E}[\mathbb{E}[\mathbb{V}\mathrm{ar}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]].\end{aligned} \tag{13}$$

Now, it is easy to see that

$$\begin{aligned}\mathbb{E}[\mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]] &= \frac{1}{c^2}\mathbb{E}[\mathbb{V}\mathrm{ar}[\mathbb{E}[Z(c)|\{y(h)\}_{h=1}^{\infty}]|P]] \\ &= \frac{1}{c^2}\mathbb{E}[\mathbb{V}\mathrm{ar}[c\sum_{h=1}^{\infty} {}_hp_x e^{-y(h)}|P]] \\ &= \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{E}[{}_hp_x\, {}_kp_x]\mathbb{C}\mathrm{ov}[e^{-y(h)}, e^{-y(k)}].\end{aligned}$$

Also, observing that $\mathbb{V}\mathrm{ar}[\mathbb{E}[Z(c)/c|\{y(h)\}_{h=1}^{\infty}]]$ is a measure of the variability of $Z(c)/c$ due to the effect of the stochastic discounting factors, the effect due to the randomness of mortality having been averaged out [see, also Coppola et al., (2000)] and is independent of c. This leads to the following definition

Definition 2A. $\mathbb{E}[\mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]]$ *is a measure of the portfolio's investment risk.*

From equation (3) we have

$$\mathbb{E}[\mathbb{E}[\mathbb{V}\mathrm{ar}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]] = \frac{1}{c^2}\mathbb{E}[\mathbb{E}[\mathbb{V}\mathrm{ar}[Z(c)|\{y(h)\}_{h=1}^{\infty}]P]].$$

We observe that $\mathbb{E}[\mathbb{E}[\mathbb{V}\mathrm{ar}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]]$ is a measure of the variability of $Z(c)/c$ due to random deviations in the number of deaths for a given mortality table. Notice the effect of pooling risks: as c tends to infinity, this measure tends to zero.

Definition 3A. $\mathbb{E}[\mathbb{E}[\mathbb{V}\mathrm{ar}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]]$ *is a measure of the insurance risk.*

3.2 Conditioning on the Interest Rate Process

The variance of $Z(c)/c$ can be decomposed in another way:

$$\mathbb{V}\mathrm{ar}[\frac{Z(c)}{c}] = \mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]] + \mathbb{E}[\mathbb{V}\mathrm{ar}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]]. \tag{14}$$

The first term on the right side of equation (14) provides a measure of the variability of $Z(c)/c$ due to stochastic interest rates (discount factors), while the effect of the demographic components (projection and deviations) has been averaged out. As

$$\mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]] = \sum_{h=1}^{\infty}\sum_{k=1}^{\infty}\mathbb{E}[{}_hp_x]\mathbb{E}[{}_kp_x]\mathbb{C}\mathrm{ov}[e^{-y(h)}, e^{-y(k)}], \tag{15}$$

we note that $\mathbb{V}\mathrm{ar}[\mathbb{E}[Z(c)/c|\{y(h)\}_{h=1}^{\infty}]]$ does not depend on c, the portfolio's size, as we can expect in the case of a systematic risk. This suggests the following alternative definition of investment risk:

Definition 2B. $\mathbb{V}\text{ar}[\mathbb{E}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]]$ *is a measure of the investment risk.*

The second term on the right side of equation (14) can be split in turn as follows

$$\mathbb{E}[\mathbb{V}\text{ar}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]] = \mathbb{E}[\mathbb{V}\text{ar}[\mathbb{E}[\frac{Z(c)}{c}|P]|\{y(h)\}_{h=1}^{\infty}]] + \mathbb{E}[\mathbb{E}[\mathbb{V}\text{ar}[\frac{Z(c)}{c}|P]|\{y(h)\}_{h=1}^{\infty}]]. \quad (16)$$

Now we observe that $\mathbb{E}[\mathbb{V}\text{ar}[\mathbb{E}[Z(c)/c|P]|\{y(h)\}_{h=1}^{\infty}]]$, which is a measure of the variability of $Z(c)/c$ due to the randomness of the projection, does not depend on c, as we expect by virtue of the systematic nature of this kind of risk. In fact

$$\begin{aligned}\mathbb{E}[\mathbb{V}\text{ar}[\mathbb{E}[\frac{Z(c)}{c}|P]|\{y(h)\}_{h=1}^{\infty}]] &= \mathbb{E}[\mathbb{V}\text{ar}[\sum_{h=1}^{\infty} {}_{h}p_{x}e^{-y(h)}|\{y(h)\}_{h=1}^{\infty}]] \\ &= \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{C}\text{ov}[{}_{h}p_{x}, {}_{k}p_{x}]\mathbb{E}[e^{-y(h)-y(k)}].\end{aligned} \quad (17)$$

This suggests the following alternative definition:[4]

Definition 1B. $\mathbb{E}[\mathbb{V}\text{ar}[\mathbb{E}[\frac{Z(c)}{c}|P]|\{y(h)\}_{h=1}^{\infty}]]$ *is a measure of the projection risk.*

Because

$$\mathbb{E}[\mathbb{E}[\mathbb{V}\text{ar}[\frac{Z(c)}{c}|P]|\{y(h)\}_{h=1}^{\infty}]] = \mathbb{E}[\mathbb{E}[\mathbb{V}\text{ar}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]] \quad (18)$$

where the variance is calculated with respect to the random deviations of mortality, $\mathbb{E}[\mathbb{E}[\mathbb{V}\text{ar}[Z(c)/c|P]|\{y(h)\}_{h=1}^{\infty}]]$ is just equal to the measure provided by Definition 3A. So we have an alternative definition of insurance risk:

Definition 3B. $\mathbb{E}[\mathbb{E}[\mathbb{V}\text{ar}[\frac{Z(c)}{c}|P]|\{y(h)\}_{h=1}^{\infty}]]$ *is a measure of the insurance risk.*

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[4]Note that Definitions x.A and x.B are alternative definitions of the same concept, where x can be 1, 2 or 3.

At this point let us consider the difference between the two investment risk measures given in Definitions 2A and 2B respectively. Note that

$$\mathbb{E}[\mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]] - \mathbb{V}\mathrm{ar}\mathbb{E}[\mathbb{E}[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]]$$
$$= \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{C}\mathrm{ov}[{}_hp_x, {}_kp_x]\mathbb{C}\mathrm{ov}[e^{-y(h)}, e^{-y(h)}]. \quad (19)$$

Now, let us consider the difference between the two projection risk measures given in Definitions 1A and 1B

$$\mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|P]] - \mathbb{E}[\mathbb{V}\mathrm{ar}[\mathbb{E}[\frac{Z(c)}{c}|P]|\{y(h)\}_{h=1}^{\infty}]]$$
$$= -\sum_{h=1}^{\omega-1-x}\sum_{k=1}^{\omega-1-x} \mathbb{C}\mathrm{ov}[{}_hp_x, {}_kp_x]\mathbb{C}\mathrm{ov}[e^{-y(h)}, e^{-y(h)}]. \quad (20)$$

4 A Numerical Example

Consider a portfolio of life annuities, each policy being issued to each of $c = 1000$ lives age $x = 65$. We assume that the underlying survival function for the group of insured lives follows a Weibull distribution, i.e.,

$$s(x) = \exp(-(\frac{x}{\alpha})^{\gamma}), \quad x > 0$$

where α and γ are positive constant parameters. The Weibull model is often used because it is simple and fits well the statistical observations and it easily represents mortality related to adult ages; see, for example, Kefitz and Beekman (1984) for specific details.

Specifically we assume that the current mortality is such that $\alpha = 82.7$ and $\gamma = 7.00$. The projected mortality are obtained by choosing α and γ to reflect a survival function with mortality rates that are pessimistic (i.e., higher than expected), realistic (i.e., as expected), and optimistic (i.e., lower than expected). Table 1 displays the values of α and γ for the various projections.

Moreover, we consider projected survival tables by choosing the parameters α and γ corresponding to a survival function for contemporaries and to three projected tables with increasing survival probabilities. [See Olivieri (1998)]

Table 1
Parameters Used for the Projections

Contemporary Mortality Table	82.7	7.00
Pessimistic Mortality Projection	83.5	8.00
Realistic Mortality Projection	85.2	9.15
Optimistic Mortality Projection	87	10.45

With regards to the force of interest rate process, we calculate the parameters β and σ as described in Coppola et al., (2000). In particular, recall that the stochastic process $X(t)$ defined in Section 2.1 represents the deviations of the force of interest from its expected value. Thus, the differences between actual observed rates and their forecasted values are used to estimate β and σ by means of the covariance equivalence principle. [See, for example, Pandit and Wu (1983), or Parker (1994a or b)).]

For this example, however, our illustrations, the Italian short term (three months) bond series for the period 1993-1996 is used. It turns out that the constant deterministic component is $i = 0.09$, and the parameters of the deviation process, $X(t)$, are $\beta = 0.11$ and $\sigma = 0.005$.

Table 2 displays certain values for the average cost per policy for each type of mortality table. The insurance system scenario is such that the probability of choosing a type of projected mortality table is 0.2, 0.6, 0.2 for the pessimistic, the realistic, and the optimistic projections, respectively. Table 3 shows the variance decomposition for each definition.

In particular, according to the first variance decomposition presented in Section 3.1, we obtain the values in the first column on the basis of Definitions 1A, 2A, 3A. In the second column, according to the second variance decomposition presented in Section 3.2, the values are obtained by means of Definitions 1B, 2B, 3B.

Concerning the mean value, investment risk, and variance, we can note that the values in Table 3 are greater than the corresponding values in Table 2 for the realistic projection and smaller than those for the optimistic projection. On the contrary, the insurance risk in Table 3 is smaller than the insurance risk in Table 2 for the realistic projection, and it is greater than insurance risk in Table 2 for the optimistic projection. Moreover, in Table 3 a new set of risk parameter appears, which is the projection risk. Note the contribution of the projection risk is higher than the insurance risk.

Table 2
Properties of $Z(c)/c$ for $c = 1000$ and $x = 65$

	Contemporary	Projected Mortality Table Type		
	Mortality	Pessimistic	Realistic	Optimistic
Mean Value	7.11024	7.33410	7.64704	8.01712
Variance	0.42786	0.46240	0.51639	0.58740
Investment Risk	0.42088	0.45613	0.51099	0.58299
Insurance Risk	0.00698	0.00627	0.0054	0.00441

Table 3
Variance Decompositions

	First Decomp.	Second Decomp.
Mean Value	7.65833	7.65833
Projection Risk	0.04693	0.04601
Investment Risk	0.51442	0.51534
Insurance Risk	0.00538	0.00538
Variance	0.56672	0.56672

Finally, we note also that the differences between the projection and the investment risk measures are very small in the two decomposition procedures.

5 Closing Comments

We have considered a model for a portfolio of identical life annuities under the assumption that both mortality and interest rates are random, and the rate of return consists of two components: a deterministic one, which takes into consideration the existing investments of the company, and a stochastic one, which is modeled by an Ornstein-Uhlenbeck process.

The main part of the paper analyzes the effect of the randomness of the projected mortality rates in the valuation of an annuity portfolio. This study points out the importance of the systematic risk component due to the randomness of the survival functions used in constructing the mortality tables. Further information about the analysis of the projection risk could be obtained by scenario testing on different mortality tables.

In the context of a life annuity portfolio, in which the three risks under consideration are the mortality risk, the projection risk and the financial risk, the equations we have derived are easy to implement by practitioners. Moreover in this way the overall variance is obtained simply adding the three contributions.

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