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## Decompositions of Betti Diagrams

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DECOMPOSITIONS OF BETTI DIAGRAMS

by

Courtney Gibbons

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

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For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Luchezar Avramov and Roger Wiegand

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# DECOMPOSITIONS OF BETTI DIAGRAMS

Courtney Gibbons, Ph.D.

University of Nebraska, 2013

Advisors: Professors Luchezar Avramov and Roger Wiegand

In this dissertation, we are concerned with decompositions of Betti diagrams over standard graded rings and the information about that ring and its modules that can be recovered from these decompositions. In Chapter 2, we study the structure of modules over short Gorenstein graded rings and determine a necessary condition for a matrix of nonnegative integers to be the Betti diagram of such a module. We also describe the cone of Betti diagrams over the ring  $\mathbb{k}[x, y]/(x^2, y^2)$ , and we provide an algorithm for decomposing Betti diagrams, even for modules of infinite projective dimension. Chapter 3 represents work done jointly with Christine Berkesch, Jesse Burke, and Daniel Erman. There we give a complete description of the cone of Betti diagrams over a standard graded hypersurface ring of the form  $\mathbb{k}[x, y]/(q)$ , where  $q$  is a homogeneous quadric. In this setting we also provide an algorithm for decomposing Betti diagrams. In both Chapters 2 and 3, the coefficients of the decompositions paint a picture of some aspect of the module theory over the ring.

## DEDICATION

I dedicate this dissertation to Marilyn Johnson.

*Botte buona fa buon vino.*

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*When you hear music, after it's over, it's gone, in the air. You can never capture it again.*

–Eric Dolphy

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# Chapter 1

## Introduction

This dissertation is concerned with Betti diagrams, which encode the numerical information in free resolutions of graded modules over graded rings. In general, it is not possible to determine exactly when a matrix of integers is the Betti diagram of a module. But what if we're allowed to scale Betti diagrams? In 2006, M. Boij and J. Söderberg had the novel idea to consider Betti diagrams “up to rational multiple” [BS08]. This insight developed into a program for embedding Betti diagrams into a rational vector space and analyzing the convex cone that they span. In this setting, properties of the cone can be translated to properties of the Betti diagrams. For a ring  $R$ , if we find a set of rays that minimally generate the cone, then we are able to determine when a ray in the ambient rational vector space contains a multiple of a Betti diagram.

Over a polynomial ring, Boij and Söderberg conjectured that special diagrams, those with the property of being pure, generated the cone of Betti diagrams, and they proved this conjecture for Cohen–Macaulay modules over polynomial rings in two variables [BS08]. For polynomial rings in more variables, one of the crucial breakthroughs involved constructing modules with specific Betti numbers. With some hypotheses

on the field, D. Eisenbud, G. Fløystad, and J. Weyman constructed the modules for an arbitrary number of variables in [EFW11]. The other crucial breakthrough was showing that the pure diagrams generate the cone; Eisenbud and F. Schreyer proved the conjectures in general in [ES09].

Since then, more work has been done to study Boij–Söderberg theory over polynomial rings (see, for example, [BS12, Erm09, EES11, EE12, ES10, GJM<sup>+</sup>13, McC11]; [Flø12] offers a concise expository survey). Simultaneously, many of the same players were working to extend Boij–Söderberg theory to other graded rings, including quotients of polynomial rings (see [BEKS10, BEKS11, BBCI<sup>+</sup>10, KS13]). The results in this dissertation continue this trend; the result in Chapter 3 was the first extension of Boij–Söderberg theory to a standard graded ring that is not a polynomial ring. To aid the reader, we have collected the necessary results from convex geometry in Appendix A; this appendix was originally written in slightly less generality by C. Berkesch, J. Burke, D. Erman, and C. Gibbons for our paper [BBEG12].

From now on, we focus on modules over quotients of polynomial rings. These rings and modules are intuitively simple to understand as they can be viewed as a direct sum of finite dimensional vector spaces over the coefficient field, with multiplication following the usual conventions for polynomial multiplication. Not only do such graded rings and modules arise in many settings, but they themselves generate interesting questions to study. In fact, Boij–Söderberg theory was born in order to answer one of these interesting questions about bounds on the multiplicity of Cohen–Macaulay modules over a polynomial ring [HS98].

Next, we provide some framework for the results in this thesis. Let  $\mathbb{k}$  be any field. A ring  $R$  is called a *standard graded  $\mathbb{k}$ -algebra* provided there are  $\mathbb{k}$ -vector spaces

$R_j$  such that  $R = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} R_j$ , and

$$R_0 = \mathbb{k} \quad \text{and} \quad R_j = R_1^j \text{ for } j \geq 1.$$

We refer to  $R_j$  as the **degree  $j$  component of  $R$** . Similarly, an  $R$ -module  $M$  is said to be **graded** provided, for each  $j \in \mathbb{Z}$ , there exist  $\mathbb{k}$ -vector spaces  $M_j$  such that  $M = \bigoplus_{j \in \mathbb{Z}} M_j$ , and, for each  $k \in \mathbb{Z}$ , the  $R$ -action on  $M$  satisfies  $R_k M_j \subseteq M_{j+k}$ . We call  $M_j$  the **degree  $j$  component of  $M$** .

In this dissertation, our standard graded  $\mathbb{k}$ -algebras arise as quotients of  $S = \mathbb{k}[x_1, \dots, x_n]$ , a standard graded  $\mathbb{k}$ -algebra whose degree  $j$  component, denoted  $S_j$ , is the  $\mathbb{k}$ -span of forms of degree  $j$ ; monomials of degree  $j$  form a basis for  $S_j$ . For an ideal  $I$  generated by homogeneous polynomials of degree at least 2, we give the ring  $R = S/I$  the grading inherited from  $S$ . We use the notation  $\mathfrak{m} = (x_1, \dots, x_n)$  to denote the homogeneous maximal ideals of  $S$  and  $R$  simultaneously.

We will consider only finitely generated graded  $R$ -modules. At times, it will be useful to adjust the grading of a module, a process referred to as twisting the module. By  $R(-d)$ , we denote the cyclic, free  $R$ -module with the grading  $R(-d)_j = R_{j-d}$ , and we call this module the **twist of  $R$  by  $-d$** . For example,  $R(-1)$  denotes the cyclic free module with its generator in degree 1. For an  $R$ -module  $M$ , the **twist of  $M$  by  $d$**  is the module with grading  $M(-d)_j = M_{j-d}$ .

Set  $h_j(M) = \dim_{\mathbb{k}}(M_j)$ . Note that  $M_j = 0$  for  $j \ll 0$  because  $M$  is finitely generated and  $R_j = 0$  when  $j < 0$ . The **Hilbert series of  $M$**  is the formal Laurent series

$$\mathcal{H}_M(s) = \sum_j h_j(M) s^j.$$

At times it is useful to consider the coefficients of this series, which we write as a

**Hilbert sequence,**

$$\text{Hilb}_R(M) = (\cdots, h_{-1}(M), h_0(M), h_1(M), \cdots).$$

A homomorphism of graded  $R$ -modules,  $\varphi : M \rightarrow N$ , is said to be **graded** provided  $\varphi(M_j) \subseteq N_j$  for all  $j \in \mathbb{Z}$ . A complex  $F_\bullet$  of graded free modules and graded  $R$ -module homomorphisms,

$$F_\bullet : \quad 0 \longleftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\partial_3} \cdots$$

is called a **graded free resolution of  $M$**  provided

$$\text{coker } \partial_1 = M \quad \text{and} \quad \text{im } \partial_{i+1} = \ker \partial_i.$$

If  $\text{im } \partial_{i+1} \subseteq \mathfrak{m}F_i$ , then the resolution is called **minimal**. Fix a minimal free resolution  $F_\bullet$  of  $M$  with differential  $\partial$ , and let  $\epsilon : F_0 \rightarrow M$  be the augmentation map (i.e., the map for which  $\text{im } \epsilon \cong \text{coker } \partial_1$ ). For  $i = 1$ , the **first syzygy of  $M$**  is the module  $\text{Syz}_1(M) = \ker \epsilon$ . For each integer  $i \geq 2$ , the  **$i$ -th syzygy of  $M$**  is the module  $\text{Syz}_i(M) = \ker \partial_{i-1}$ . For each  $i \geq 1$ , there exist nonnegative integers  $\beta_{i,j}(M)$  such that

$$F_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}(M)}.$$

The numbers  $\beta_{i,j}(M)$  are independent of the choice of  $F_\bullet$ , and  $\beta_{i,j}(M)$  is called the  **$i, j$ -th graded Betti number of  $M$** . The number  $\beta_{0,j}$  measures the minimal number of degree  $j$  generators of  $M$ . When  $i \geq 1$ , the number  $\beta_{i,j}(M)$  measures the minimal number of degree  $j$  generators of the  $i$ -th syzygy module. The **Poincaré series of**

$M$  is the formal power series

$$\mathcal{P}_M(t) = \sum_{i \geq 0} \sum_{j \in \mathbb{Z}} \beta_{i,j} s^j t^i$$

with coefficients in  $\mathbb{Z}[s, s^{-1}]$ . A module's Betti diagram and Poincaré series encode precisely the same data.

In Chapter 2, we will discuss negative syzygy modules using minimal injective resolutions. A graded  $R$ -module  $T$  is called a **graded essential extension** of  $M$  provided there is a graded inclusion  $M \hookrightarrow T$  such that  $U \cap M \neq 0$  for every nonzero submodule  $U \subseteq T$ . If  $M \neq T$ , the extension is called **proper**. An  $R$ -module is said to be injective provided it has no proper graded essential extension. The **graded injective hull of  $M$**  is a module that is both injective and a graded essential extension of  $M$ . Every  $R$ -module has a graded injective hull that is unique up to isomorphism [BH93, Chapter 3, Section 6]. A complex  $E^\bullet$  of graded injective modules and graded  $R$ -module homomorphisms,

$$E^\bullet : \quad \cdots \longleftarrow E^2 \xleftarrow{\partial^2} E^1 \xleftarrow{\partial^1} E^0 \longleftarrow 0,$$

is called a **minimal graded injective resolution of  $M$**  provided  $\ker \partial^{i+1} = \operatorname{im} \partial^i$ ,  $E^0$  is the injective hull of  $M$ , and each subsequent  $E^i$  is the injective hull of  $\operatorname{im} \partial^i$ . For each positive integer  $i$ , the **negative  $i$ -th syzygy of  $M$**  (also called the  **$i$ -th cosyzygy of  $M$** ) is the module  $\operatorname{Syz}_{-i}(M) = \operatorname{im} \partial^i$ .

Let  $\mathbb{V}$  denote the space of column-finite  $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ -indexed matrices with entries in  $\mathbb{Q}$ ; each element of  $\mathbb{V}$  is a map  $(m_{j,i}) : \mathbb{Z} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Q}$  where, for fixed  $i$ , there are only finitely many  $j$  for which  $m_{j,i} \neq 0$ . Define the **Betti diagram** of  $M$ , denoted  $\beta(M)$ , to be the matrix for which  $\beta_{i,j+i}(M)$  occurs in column  $i$  and row  $j$ . This indexing

convention stems from the commutative algebra software Macaulay2 [M2] . When displaying a Betti diagram, we use the symbol  $*$  to identify the  $(0, 0)$ -th entry:

$$\beta(M) = * \begin{pmatrix} \vdots & \vdots & \vdots & \cdots \\ \beta_{0,0}(M) & \beta_{1,1}(M) & \beta_{2,2}(M) & \cdots \\ \beta_{0,1}(M) & \beta_{1,2}(M) & \beta_{2,3}(M) & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

We say that  $\beta(M)$  is **pure** provided that for each  $i$ , there exists at most one  $j$  for which  $\beta_{i,j}(M) \neq 0$ ; that is, each column has at most one nonzero entry. Pure diagrams play an important role in Boij-Söderberg theory for the polynomial ring and continue to do so in this dissertation.

At times we will wish to consider a particular type of subspace that restricts the rows for which a Betti diagram may be nonzero. Namely, we define the subspace  $\mathbb{V}_r$  of  $\mathbb{V}$  to be

$$\mathbb{V}_r = \{v \in \mathbb{V} \mid v_{i,j} = 0 \text{ unless } |j - i| \leq r\}.$$

**Definition 1.0.1.** The **cone of Betti diagrams over  $R$**  is defined to be

$$B_{\mathbb{Q}}(R) := \left\{ \sum_{R\text{-modules } M} a_M \beta(M) \mid a_M \in \mathbb{Q}_{\geq 0} \text{ and almost all } a_M \text{ are zero} \right\} \subseteq \mathbb{V};$$

i.e., it is the positive hull of the set of Betti diagrams of finitely generated  $R$ -modules. In the cone  $B_{\mathbb{Q}}(R)$ , a ray  $v$  is called **extremal** if it is not a subset of the positive hull of  $B_{\mathbb{Q}}(R) \setminus \{v\}$ .

We say  $M$  is an **indecomposable  $R$ -module** if it is nonzero and cannot be written as a direct sum of two nonzero  $R$ -modules. When  $M = \bigoplus_{\ell} N_{(\ell)}^{a_{\ell}}$  is a decomposition of  $M$  into indecomposable modules  $N_{(\ell)}$ , we refer to  $a_{\ell}$  as the **multiplicity of  $N_{(\ell)}$  in the decomposition of  $M$**  (or simply as the **multiplicity of  $N_{(\ell)}$  in  $M$** ; when the

Krull–Remak–Schmidt theorem applies, this number is independent of the decomposition of  $M$ ). Observe that  $\beta(M \oplus N) = \beta(M) + \beta(N)$ ; therefore, if  $\beta(M)$  cannot be expressed as a nonnegative integer combination of Betti diagrams of two or more nonzero modules, then the module  $M$  itself must be indecomposable. This interplay gives some idea of the interplay between representation theory for modules and an analogous theory for Betti diagrams; however, there is one big difference. Representation theory for modules is very hard, while understanding the decomposition of Betti diagrams has moved rather quickly. For example, there is no hope of classifying  $\mathbb{k}[x, y]$ -modules up to isomorphism, but  $B_{\mathbb{Q}}(S)$  has a simple and elegant description: the extremal rays of the cone are exactly the rays spanned by pure Betti diagrams of Cohen–Macaulay  $S$ -modules, as shown in [BS12].

In Chapter 2, we study Betti diagrams over short Gorenstein standard graded  $\mathbb{k}$ -algebras. A standard graded  $\mathbb{k}$ -algebra is called *short* provided  $R_3 = 0$ . These short Gorenstein rings are Koszul, meaning that the minimal free resolution of  $\mathbb{k}$  is linear (that is, the Betti diagram of  $\mathbb{k}$  occupies a single row). The aim is to determine information about indecomposable direct summands of modules with a given Betti table. Some of these summands are directly recoverable from the Betti diagram (see, for instance, Theorem 2.1.10), while some can at least be said to appear at most a specified number of times (see, for instance, Proposition 2.2.1). In the same chapter, we also study the cone of Betti diagrams over short Gorenstein rings of embedding dimension 2 from the point of view of Boij–Söderberg theory. That is, we find the extremal Betti diagrams that span the cone and describe a decomposition algorithm for writing any Betti diagram as a nonnegative combination of the extremal Betti diagrams.

Chapter 3 represents work done jointly with Christine Berkesch, Jesse Burke, and Daniel Erman. In that chapter, we study a class of quadric hypersurfaces. A *degree*



$d$  **hypersurface** is a ring of the form  $R = S/(f)$  where  $f$  is a single homogeneous polynomial of degree  $d$ . When  $d = 2$ , the hypersurface is called **quadric**. Like free resolutions of modules over polynomial rings, free resolutions of modules over hypersurfaces exhibit certain behaviors that are well understood. Consider a degree  $d$  hypersurface  $R$  and an  $R$ -module  $M$ . The main result of [Eis80] is that there exists a free resolution  $F_\bullet$  of  $M$  where  $F_i = F_{i+2}(d)$  and  $\partial_i = \partial_{i+2}$  for all  $i \geq \text{depth}(R) + 1$ . We show this explicitly in Chapter 3 using a construction due to [Sha69]. Compare this result to Hilbert's syzygy theorem, which states that over a polynomial ring  $S$ , every module has a finite free resolution of length at most  $\text{depth}(S)$ . The interplay between resolutions of an  $R$ -module over  $S$  and  $R$  plays a key role in Chapter 3.

In each cone we study throughout this dissertation, the extremal rays form a subset of the set of rays spanned by pure diagrams. This is not the case for every cone, as evidenced by the following proposition. In particular, the following proposition indicates that considering hypersurfaces of degree greater than 2 will introduce a new set of obstacles to overcome.

**Proposition 1.0.2.** *Let  $R = \mathbb{k}[x_1, \dots, x_n]/(f)$  be any hypersurface ring with  $n > 1$  and  $\deg(f) > 2$ . Then  $\beta(\mathbb{k})$  is not pure, yet it lies on an extremal ray in  $B_{\mathbb{Q}}(R)$ .*

**Proof.** In [Tat57], J. Tate provides a construction for the minimal free resolution of  $\mathbb{k}$  over  $R$ . In this construction, since  $\deg(f) > 2$  and  $n > 1$ , the second syzygy module  $\text{Syz}_2(\mathbb{k})$  has minimal generators in degrees 2 and  $\deg(f) - 1$ , so  $\beta(\mathbb{k})$  is not pure.

We next claim that, if  $M$  is any module generated in degree 0, then  $\beta_{1,1}(M) \leq n \cdot \beta_{0,0}(M)$ , with equality if and only if  $M$  is a direct sum of copies of  $\mathbb{k}$ . To see this, we first set  $a = \beta_{0,0}(M)$ . Since the degree 1 component of  $\text{Syz}_1(M)$  is a  $\beta_{1,1}(M)$ -dimensional  $\mathbb{k}$ -vector subspace of  $R_1^a$ , the inequality follows. Now, if  $M \cong \mathbb{k}^a$ , then

equality holds since  $\beta_{1,1}(\mathbb{k}) = n$ . Conversely, if  $\beta_{1,1}(M) = n \cdot a$ , then each generator of  $M$  is annihilated by  $\mathfrak{m}$ , and so  $M \cong \mathbb{k}^a$ .

Finally, to see that  $\beta(\mathbb{k})$  is extremal, suppose that  $\beta(\mathbb{k}) = \sum_i a_i \beta(M_{(i)})$  for finitely many  $R$ -modules  $M_{(i)}$  and positive rational numbers  $a_i$ . This implies that

$$\sum_i a_i \beta_{0,0}(M_{(i)}) = \beta_{0,0}(\mathbb{k}) = 1.$$

Using this, and the claim above, we have

$$n = \beta_{1,1}(\mathbb{k}) = \sum_i a_i \beta_{1,1}(M_{(i)}) \leq \sum_i n a_i \beta_{0,0}(M_{(i)}) = n$$

and so  $\sum_i a_i \beta_{1,1}(M_{(i)}) = \sum_i n a_i \beta_{0,0}(M_{(i)})$ . Since  $\beta_{1,1}(M_{(i)}) \leq n \beta_{0,0}(M_{(i)})$  for all  $i$ , we must have equality, and so each  $M_{(i)}$  is a direct sum of copies of  $\mathbb{k}$ .  $\square$

## Chapter 2

### Short Gorenstein rings

In this chapter, we focus our attention on short Gorenstein rings, which were first studied by G. Sjödin in [Sjö79]. More recently, L. Avramov, S. Iyengar, and L. Şega studied these rings in the papers [AIŞ08] and [AIŞ10]. Recall that a standard graded  $\mathbb{k}$ -algebra  $R$  is called short provided  $R_3 = 0$ . When  $R$  is short and Gorenstein with embedding dimension  $e \geq 2$ , it follows that  $R$  is completely determined by the multiplication  $R_1 \times R_1 \rightarrow R_2$ . Classifying short Gorenstein rings, then, is equivalent to the classification of quadratic forms [Jac85, Section 6.3]. Thus, when  $\mathbb{k}$  is algebraically closed and  $\text{char } \mathbb{k} \neq 2$ , every short Gorenstein ring with embedding dimension  $e$  is isomorphic to the ring

$$\mathbb{k}[x_1, \dots, x_e]/(x_i^2 - x_j^2, x_i x_j \mid 1 \leq i < j \leq e).$$

For any ring  $R$ , a nonzero  $R$ -module  $M$  is said to be *linear* provided there exists an integer  $d$  for which  $\beta_{i,j}(M) = 0$  whenever  $j - i \neq d$ ; that is, the nonzero entries of  $\beta(M)$  are concentrated in a single row. A linear module for which  $d = 0$  is said to be *Koszul*. If a module isn't linear, it is said to be *nonlinear*. Over short Gorenstein

rings, the nonlinear indecomposable modules are exactly those modules that have a twist of  $\mathbb{k}$  as a syzygy (see [Sjö79] and [AIŞ08]).

This chapter proceeds as follows. In Section 2.1, we record results about indecomposable modules from [Sjö79] and [AIŞ08]. Then we obtain information about the indecomposable summands of a module from its Betti diagram. Sjödin's shows that every  $R$ -module has a rational Poincaré series, and in fact, that each Poincaré series can be written as a rational function with a fixed denominator determined by  $e$  alone. This observation underpins our work. The behavior of the Betti numbers of nonlinear indecomposable modules is also crucial to this endeavor.

For example, over  $R = \mathbb{k}[x, y, z]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$ , we will be able to compute that the module  $(\text{Syz}_{-2}(\mathbb{k}))(-3)$  is generated in degree 0 and has the Betti diagram

$$\beta((\text{Syz}_{-2}(\mathbb{k}))(-3)) = \begin{matrix} * \\ \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & & \vdots & \\ - & \cdots & - & - & \cdots & - & \cdots \\ 3 & 1 & - & - & \cdots & - & \cdots \\ - & - & 1 & 3 & \cdots & \beta_{n,n}(\mathbb{k}) & \cdots \\ - & - & - & - & \cdots & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix} \end{matrix}.$$

Observe that the nonzero Betti numbers move from row 0 to row 1. In fact, for each  $n \geq 1$ ,  $\text{Syz}_{-n}(\mathbb{k})(-n-1)$ , has nonzero Betti numbers exactly in row 0 for columns 0 through  $n-1$  and row 1 for columns  $n$  and higher. We exploit this behavior later.

In Section 2.1 we also consider indecomposable linear modules. In particular, given the Hilbert series of a linear module, we can determine if it is negative syzygy of some ideal generated by linear forms. For example, over the ring above, we'll see that a module generated in a single degree with Hilbert series  $2 + 5s$  is indecomposable, is a negative syzygy of an ideal generated by a single form, and is linear. Unfortunately, in general we can't determine when a linear module is indecomposable by its Hilbert

series alone. For example, over the same ring the modules  $M = \text{Syz}_1(\mathbb{k})$ ,  $N = \mathbb{k}(-1)$ , and  $L = (\text{Syz}_1(xR))(1)$  have Hilbert series  $\mathcal{H}_M(s) = 3s + s^2$ ,  $\mathcal{H}_N(s) = s$ , and  $\mathcal{H}_L(s) = 2s + s^2$ . These modules are linear (see Remark 2.1.1(iv)) and indecomposable (because they are cyclic or syzygies of cyclic modules), but  $\mathcal{H}_M(s) = \mathcal{H}_{N \oplus L}(s)$ .

In Section 2.2, we focus on short Gorenstein rings with embedding dimension  $e = 2$  and the cone of Betti diagrams over such a ring. When  $\mathbb{k}$  is algebraically closed and of characteristic other than 2,  $R \cong \mathbb{k}[x, y]/(x^2, y^2)$ , and this ring and its indecomposable modules have been thoroughly studied. Both L. Kronecker and K. Weierstrass are credited with classifying the indecomposable modules over this ring by classifying pairs of commuting matrices of square zero [Kro74, Wei68], and the reader may turn to J. Dieudonné [Die46] for a more modern treatment. As in the main results of Boij–Söderberg theory for the standard graded polynomial ring [ES09], we drop the assumptions on  $\mathbb{k}$  and prove that  $B_{\mathbb{Q}}(R)$  has a simplicial fan structure (see Theorem 2.2.9).

Something interesting to note about this cone is that, for a nonzero linear form  $f$ , the modules  $(R/(f))$  and  $\mathbb{k}$  are both pure and have linear free resolutions with the same twists in each homological degree; in the parlance of Boij–Söderberg theory, we say that they have the same degree sequences (see Definition 2.2.6). However, their Betti diagrams are not scalar multiples of one another. In the case of a polynomial ring or a hypersurface, each Betti diagram that spans an extremal ray is uniquely determined by its degree sequence and the assumption that some module with the given degree sequence is Cohen–Macaulay. Meanwhile, for short Gorenstein rings with  $e = 2$ , the modules above show that degree sequences are too coarse to identify Betti diagrams spanning extremal rays. Indeed, we need Betti diagrams that represent the two possible kinds of Betti number growth over this ring: Betti numbers that are constant and Betti numbers that grow linearly.

## 2.1 Short Gorenstein rings of embedding dimension at least 2

Let  $\mathbb{k}$  be a field and let  $R$  be a short Gorenstein graded  $\mathbb{k}$ -algebra of embedding dimension  $e \geq 2$ . The Hilbert series of  $R$  is  $\mathcal{H}_R(s) = 1 + es + s^2$ . This ring is Koszul; that is,  $\mathbb{k}$  is a Koszul  $R$ -module. We begin by recording some of the structure of modules over short Gorenstein rings.

**Remark 2.1.1** ([Sjö79, Lemma 3],[AIS08, Theorem 4.6]). Set the notation

$$C^{(n)} = (\text{Syz}_{-n}(\mathbb{k}))(-n-1), \text{ and let}$$

$$K^{(p,q)} \text{ denote a Koszul module with Hilbert series } p + qs.$$

For an indecomposable  $R$ -module  $M$ , the following statements hold:

- (i) If  $M$  is free, then  $M \cong R(-j)$  for some  $j$ .
- (ii) If  $M$  is linear and nonfree, then there exist nonnegative integers  $p$  and  $q$  such that  $M = K^{(p,q)}(-j)$  for some  $j \in \mathbb{Z}$ .
- (iii) The module  $M$  is nonlinear if and only if  $M$  is isomorphic to  $C^{(n)}(-j)$  for some  $n \geq 1$  and  $j \in \mathbb{Z}$ .
- (iv) If  $n \geq 1$ , the module  $M$  is isomorphic to  $C^{(n)}(-j)$  if and only if  $M$  has Hilbert series  $\mathcal{H}_M(s) = b_{n-1}s^j + b_n s^{j+1}$ , where the sequence  $(b_n)_{n \in \mathbb{Z}}$  is defined via

$$b_n = \begin{cases} 0, & n < 0; \\ \beta_{n,n}(\mathbb{k}), & n \geq 0. \end{cases} \quad (2.1.A)$$

**Example 2.1.2.** Many non-isomorphic modules may be labeled  $K^{(p,q)}$ . Consider the ring  $R = \mathbb{k}[x, y, z]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$ . The modules  $Rx(1)$  and  $Ry(1)$  are both indecomposable and nonfree, and both have Hilbert series  $1 + s$ . By Remark 2.1.1, they are both Koszul, and both are labeled  $K^{(1,1)}$ . These modules aren't isomorphic, however, because  $y$  annihilates  $Rx(1)$  but not  $Ry(1)$ .

**Lemma 2.1.3.** *Over a short Gorenstein ring  $R$ , every module  $M$  has a direct sum decomposition*

$$M = \bigoplus_j \left( R^{r_j} \oplus K^{(p_j, q_j)} \oplus \bigoplus_{n \geq 1} (C^{(n)})^{c_{n,j}} \right) (-j). \quad (2.1.B)$$

Moreover, the numbers  $r_j$ ,  $p_j$ ,  $q_j$ , and  $c_{n,j}$  are uniquely determined.

**Proof.** The Krull–Remak–Schmidt theorem applies to short Gorenstein rings [Ati56, Theorem 1] (see also [Yos90]), so each  $R$ -module  $M$  is uniquely a direct sum of indecomposable  $R$ -modules. We see from Remark 2.1.1(i)-(iii) that this decomposition uniquely determines the numbers  $r_j$  and  $c_{n,j}$  for all  $n \geq 1$  and  $j \in \mathbb{Z}$ . Taking the direct sum of all nonfree, linear indecomposable modules generated in degree  $j$ , we obtain a module  $K^{(p_j, q_j)}(-j)$  where  $p_j$  and  $q_j$  are uniquely determined.  $\square$

In the next few pages, we define the building blocks for Poincaré series of  $R$ -modules and provide some structural results about the Poincaré series of the modules appearing in the right hand side of (2.1.B).

**Definition 2.1.4.** Define the rational functions  $f$  and  $g$  in  $\mathbb{Q}(s, t)$  by

$$f = \frac{1}{1 - est + s^2t^2}; \quad (2.1.C)$$

$$g = \begin{cases} \frac{1}{(1 - st)}, & e = 2, \\ \frac{t}{1 - est + s^2t^2}, & e \geq 3; \end{cases} \quad (2.1.D)$$

For  $n \geq 1$ , let  $\theta_n(t)$  and  $\rho_n(t)$  denote the unique polynomials in  $\mathbb{Q}(s)[t]$  such that  $s^{n+1}t^n = \theta_n(t)(1 - est + s^2t^2) + \rho_n(t)$  with  $\deg_t \rho_n \leq 1$  (these polynomials exist and are unique by the Euclidean division theorem). Recall the sequence  $(b_n)_{n \in \mathbb{Z}}$  defined in Equation (2.1.A), and define

$$h_n(t) = b_0 s^{n-1} t^{n-1} + b_1 s^{n-2} t^{n-2} + \cdots + b_{n-1} + \theta_n(t). \quad (2.1.E)$$

Each polynomial  $h_n(t)$  has  $t$ -degree  $n - 1$ .

**Remark 2.1.5.** From [Sjö79, Lemma 5], we have  $\mathcal{P}_k(t) = f$ , and the sequence  $(b_n)$  satisfies the recursion  $b_{n+1} = eb_n - b_{n-1}$  for all  $n \geq 0$ .

Using homological arguments, Sjödin shows the recursion directly to conclude that  $\mathcal{P}_k(t) = f$ . Indeed, these conditions are equivalent. We have  $\sum_{n \geq 0} b_n s^n t^n = f$  if and only if

$$\sum_{n \geq 0} b_n s^n t^n (1 - est + s^2t^2) = 1.$$

Compare  $s^n t^n$  coefficients on each side. The above equation holds exactly when  $b_n - eb_{n-1} + b_{n-2} = 0$  for  $n \geq 2$  and  $b_1 - eb_0 = b_{-1}$  (since  $b_{-1} = 0$ ).

**Lemma 2.1.6.** *The rational functions  $f$ ,  $g$ , and  $h_n$ ,  $n \geq 1$ , are linearly independent*



over  $\mathbb{Q}(s)$ . For each  $n \geq 1$ ,  $h_n(t)$  is in  $\mathbb{Q}[s, t]$ . There is an equality

$$\frac{\rho_n(t)}{1 - est + s^2t^2} = \begin{cases} s \cdot f - ns \cdot g, & e = 2, \\ -b_{n-2}s \cdot f + b_{n-1}s^2 \cdot g, & e \geq 3. \end{cases} \quad (2.1.F)$$

**Proof.** Define  $G_n(s, t) = s^{n+1}t^n - b_{n-1}s^2t + b_{n-2}s$ . We claim  $\frac{G_n(s, t)}{1 - est + s^2t^2}$  is in  $\mathbb{Q}[s, t]$ . Observe that  $G_1(s, t) = 0$  and  $G_2(s, t) = s(1 - est + s^2t^2)$ , so the claim holds when  $n = 1$  and when  $n = 2$ . Fix  $n \geq 3$ , and assume the claim for all  $k < n$ . In the induction step, we'll use that  $b_{n-1} = eb_{n-2} - b_{n-3}$  and  $b_{n-2} = eb_{n-3} - b_{n-4}$  by Remark 2.1.5. Now we have

$$\begin{aligned} G_n(s, t) &= s^{n+1}t^n - b_{n-1}s^2t + b_{n-2}s \\ &= (s^{n+1}t^n - es^n t^{n-1} + s^{n-1}t^{n-2}) + es^n t^{n-1} - s^{n-1}t^{n-2} - b_{n-1}s^2t + b_{n-2}s \\ &= s^{n-1}t^{n-2}(1 - est + s^2t^2) \\ &\quad + es^n t^{n-1} - s^{n-1}t^{n-2} - (eb_{n-2} - b_{n-3})s^2t + (eb_{n-3} - b_{n-4})s \\ &= s^{n-1}t^{n-2}(1 - est + s^2t^2) \\ &\quad + e(s^n t^{n-1} - b_{n-2}s^2t + b_{n-3}s) - (s^{n-1}t^{n-2} - b_{n-3}s^2t + b_{n-4}s) \\ &= s^{n-1}t^{n-2}(1 - est + s^2t^2) + eG_{n-1}(s, t) - G_{n-2}(s, t), \end{aligned}$$

and each term is divisible by  $1 - est + s^2t^2$  (in  $\mathbb{Q}[s, t]$ ). This proves the claim. Now,

$$s^{n+1}t^n = \frac{G_n(s, t)}{1 - est + s^2t^2}(1 - est + s^2t^2) + b_{n-1}s^2t - b_{n-2}s,$$

so by the uniqueness of  $\theta_n(t)$  and  $\rho_n(t)$ ,

$$\theta_n(t) = \frac{G_n(s, t)}{1 - est + s^2t^2} \in \mathbb{Q}[s, t] \quad \text{and} \quad \rho_n(t) = b_{n-1}s^2t - b_{n-2}s.$$

Therefore  $h_n(t) \in \mathbb{Q}[s, t]$  and, moreover, when  $e \geq 3$ , (2.1.F) follows immediately.

When  $e = 2$ , a trivial induction using Remark 2.1.5 yields  $b_{n-1} = n$  and  $b_{n-2} = n - 1$ .

Then  $\rho_n(t) = ns^2t - (n - 1)s = s - ns(1 - st)$ , so (2.1.F) holds for  $e = 2$ .

Finally, we address the linear independence of  $f$ ,  $g$ , and  $h_n$ ,  $n \geq 1$ . When  $e = 2$ , the polynomials  $(1 - st)^2f$ ,  $(1 - st)^2g$ , and  $(1 - st)^2h_n(t)$  have  $t$ -degrees 0, 1, and  $n + 1$  respectively, so they are linearly independent over  $\mathbb{Q}(s)$ . This implies  $f$ ,  $g$ , and  $h_n(t)$ ,  $n \geq 1$ , are linearly independent over  $\mathbb{Q}(s)$ . A similar argument holds for  $e \geq 3$ .  $\square$

**Lemma 2.1.7** (Poincaré series and Betti numbers of special modules). *We have the following equalities*

$$\mathcal{P}_R(t) = h_1(t) = 1; \tag{2.1.G}$$

$$\mathcal{P}_{K^{(p,q)}}(t) = \begin{cases} (p - q) \cdot f + q \cdot g, & e = 2, \\ p \cdot f - qs \cdot g, & e \geq 3; \end{cases} \tag{2.1.H}$$

and, for each  $n \geq 1$ ,

$$\mathcal{P}_{C^{(n)}}(t) = h_n(t) + \begin{cases} s \cdot f - ns \cdot g, & e = 2; \\ -b_{n-2}s \cdot f + b_{n-1}s^2 \cdot g, & e \geq 3. \end{cases} \tag{2.1.I}$$

Furthermore, for a module  $K^{(p,q)}$ ,

$$\beta_{n,n}(K^{(p,q)}) = \begin{cases} pb_n - qb_{n-1}, & \text{for all } n \geq 0; \\ e\beta_{n-1,n-1}(K^{(p,q)}) - \beta_{n-2,n-2}(K^{(p,q)}) & \text{for all } n \geq 2. \end{cases} \quad (2.1.J)$$

Finally,

$$\beta_{i,j}(C^{(n)}) = \begin{cases} b_{n-i-1}, & i < n \text{ and } j = i; \\ b_{i-n}, & i \geq n \text{ and } j = i + 1; \\ 0, & \text{otherwise.} \end{cases} \quad (2.1.K)$$

**Remark 2.1.8.** If  $M$  and  $N$  are  $R$ -modules and  $N = \text{Syz}_k(M)$ ,  $k \geq 0$ , then

$$\begin{aligned} \beta_{i,j}(N) &= \beta_{i+k,j}(M), \quad \text{and} \\ t^k \mathcal{P}_N(t) &= \sum_{i \geq k} \sum_j \beta_{i,j}(M) s^j t^i. \end{aligned}$$

**Proof of Lemma 2.1.7.** Formula (2.1.G) is clearly true, and (2.1.H) is known (see [AIS08, Proposition 3.1]). Next, we prove (2.1.J). We claim  $\mathcal{P}_{K^{(p,q)}}(t) = (p - qst) \cdot f$ . This is clear from (2.1.H) when  $e \geq 3$ , and when  $e = 2$ ,

$$\mathcal{P}_{K^{(p,q)}}(t) = (p - q) \cdot f + q \cdot g = (p - q + (1 - st)q) \cdot f = (p - qst) \cdot f.$$

Set  $\beta_n = \beta_{n,n}(K^{(p,q)})$ ; since  $K^{(p,q)}$  has a linear free resolution,  $\mathcal{P}_{K^{(p,q)}}(t) = \sum_{n \geq 0} \beta_n s^n t^n$ .

Moreover, since  $f = \mathcal{P}_k(t) = \sum_n b_n s^n t^n$ ,

$$\sum_{n \geq 0} \beta_n s^n t^n = (p - qst) \sum_{n \geq 0} b_n s^n t^n.$$

Comparing coefficients of  $s^n t^n$ , we obtain  $\beta_{n,n} = pb_n - qb_{n-1}$ . For  $n \geq 0$ ,  $\beta_n = pb_n - qb_{n-1}$ , and our desired recursion follows from Remark 2.1.5. Therefore (2.1.J) holds. In particular,

$$p = \beta_{0,0}(K^{(p,q)}) \quad \text{and} \quad q = e\beta_{0,0}(K^{(p,q)}) - \beta_{1,1}(K^{(p,q)}). \quad (2.1.L)$$

For (2.1.I), we use that the minimal free resolution of  $C^{(n)}$  is obtained by splicing together a minimal injective resolution of  $\mathbb{k}(-n-1)$  and a minimal free resolution of  $\mathbb{k}(-n-1)$ :

$$\begin{array}{ccccccc} \cdots \leftarrow R(1)^{b_n} & \longleftarrow & R(0)^{b_{n-1}} & \leftarrow \cdots \leftarrow & R(-n+1)^{b_0} & \longleftarrow & R(-n-1)^{b_0} \leftarrow \cdots \\ & & \swarrow & & \swarrow & & \swarrow \\ & & C^{(n)} & & \mathbb{k}(-n-1) & & \\ & \swarrow & & \searrow & \swarrow & & \searrow \\ 0 & & & & 0 & & 0 \end{array}$$

From this resolution, (2.1.K) is clear, and

$$\begin{aligned} \mathcal{P}_{C^{(n)}}(t) &= b_{n-1} + b_{n-2}st + \cdots + b_0 s^{n-1} t^{n-1} + t^n \mathcal{P}_{\mathbb{k}(-n-1)}(t) \\ &= b_{n-1} + b_{n-2}st + \cdots + b_0 s^{n-1} t^{n-1} + \frac{s^{n+1} t^n}{1 - est + s^2 t^2}, \end{aligned}$$

where the second equality follows from Remark 2.1.8. From here, (2.1.I) follows from the fact that  $\mathcal{P}_{\mathbb{k}}(t) = f$ .  $\square$

**Proposition 2.1.9.** *Fix a Betti diagram  $\beta$  and suppose  $M$  is any  $R$ -module with  $\beta(M) = \beta$ . Write  $M$  as in Lemma 2.1.3:*

$$M = \bigoplus_j \left( R^{r_j} \oplus K^{(p_j, q_j)} \oplus \bigoplus_{n \geq 1} (C^{(n)})^{c_{n,j}} \right) (-j).$$

For all  $n \geq 2$  and  $j \in \mathbb{Z}$ , the numbers  $r_j + c_{1,j}$ ,  $c_{n,j}$ ,  $p_j$ , and  $q_j - c_{1,j-1}$  are uniquely

determined by  $\beta$ , and we have the following equations:

When  $e = 2$ ,

$$\begin{aligned} \mathcal{P}_M(t) &= \sum_j (r_j + c_{1,j})s^j + \sum_j \sum_{n \geq 2} c_{n,j} s^j \cdot h_n(t) \\ &\quad + \sum_j \left( p_j - q_j + \sum_{n \geq 1} c_{n,j-1} \right) s^j \cdot f + \sum_j \left( q_j - \sum_{n \geq 1} n c_{n,j-1} \right) s^j \cdot g, \end{aligned} \quad (2.1.M)$$

and when  $e \geq 3$ ,

$$\begin{aligned} \mathcal{P}_M(t) &= \sum_j (r_j + c_{1,j})s^j + \sum_j \sum_{n \geq 2} c_{n,j} s^j \cdot h_n(t) \\ &\quad + \sum_j \left( p_j - \sum_{n \geq 1} c_{n,j-1} b_{n-2} \right) s^j \cdot f + \sum_j \left( \sum_{n \geq 1} c_{n,j-1} b_{n-1} - q_j \right) s^{j+1} \cdot g. \end{aligned} \quad (2.1.N)$$

Define  $\ell = \max_{j \in \mathbb{Z}} \ell_j$  where

$$\ell_j = \begin{cases} 0, & c_{n,j} = 0 \text{ for all } n \geq 1, \\ \max\{n \mid c_{n,j} \neq 0\}, & \text{otherwise,} \end{cases}$$

The  $\ell$ -th syzygy of  $M$  decomposes as a direct sum of linear  $R$ -modules.

**Proof.** Equations (2.1.M) and (2.1.N) follow from taking the Poincaré series of the right hand side of Equation (2.1.B). By Lemma 2.1.6, the coefficients  $\alpha, \gamma, \delta_n$  in  $\mathbb{Q}(s)$  of  $f, g$ , and  $h_n(t)$  respectively are uniquely determined. Without loss of generality, we may assume that  $M$  is generated in nonnegative degrees. Then  $\alpha, \gamma$ , and  $\delta_n$  are each in  $\mathbb{Q}[s]$ . Subsequently, the coefficient of  $s^j$  in each of  $\alpha, \gamma$ , and  $\delta_n$  is uniquely determined. This proves that, for all  $n \geq 2$  and  $j \in \mathbb{Z}$ ,  $\beta$  determines the numbers  $r_j + c_{1,j}, c_{n,j}, p_j$ , and  $q_j - c_{1,j-1}$  (note that in the coefficient of  $f$  in (2.1.N), the term  $c_{n,j-1} b_{n-2}$  is zero when  $n = 1$ ).

For all  $n \leq \ell$ , the  $\ell$ -th syzygy of  $C^{(n)}(-j)$  is a linear module, so the  $\ell$ -th syzygy of every indecomposable summand of  $M$  is linear.  $\square$

**Theorem 2.1.10.** *Let  $R$  be a short Gorenstein graded  $\mathbb{k}$ -algebra of embedding dimension  $e \geq 2$ . Let  $M$  be a finitely generated  $R$ -module with no non-zero free summand, and set*

$$\begin{aligned} u &= \min\{j \mid M_j \neq 0\}, \\ v &= \max\{j \mid M_j \neq 0\}, \quad \text{and} \\ w &= \max\left\{n \mid b_{n-1} \leq \max_j \{\beta_{0,j}(M)\}\right\} \quad \text{for the sequence } (b_n) \text{ defined in (2.1.A)}. \end{aligned}$$

*Each one of the following sets of data determines the others:*

- (i) *The Betti diagram  $\beta(M)$ .*
- (ii) *The Betti numbers*

$$\left\{ \beta_{i,j}(M) \mid 0 \leq i \leq w+1, u \leq j-i \leq v+1 \right\}.$$

- (iii) *The numbers  $c_{n,j}$ ,  $p_j$ , and  $q_j$  in Equation (2.1.B) for all integers  $j$  and positive integers  $n$ .*

*The module  $\text{Syz}_w(M)$  is the direct sum of linear modules:*

$$\text{Syz}_w(M) = \bigoplus_{j=u+w}^{v+1+w} K^{(p'_j, q'_j)}(-j), \quad \text{where}$$

$$p'_j = \beta_{w,j}(M) \quad \text{and} \tag{2.1.O}$$

$$q'_j = e\beta_{w,j}(M) - \beta_{w+1,j+1}(M).$$

Furthermore,

$$\mathcal{P}_M(t) = \sum_{i=0}^{w-1} \sum_{j=u+i}^{v+1+i} \beta_{i,j}(M) s^j t^i + t^w \mathcal{P}_{\text{Syz}_w(M)}(t). \quad (2.1.P)$$

**Example 2.1.11.** The hypothesis that  $M$  has no nonzero free summand is necessary. For an  $R$ -module  $M$ , let  $c_{1,0}(M)$  denote the multiplicity of  $C^{(1)}$  in the representation of  $M$  as a direct sum of indecomposable modules (the number  $c_{1,j}(M)$  is independent of the choice of decomposition by the Krull–Remak–Schmidt theorem). Suppose  $l \in R_1$  is a nonzero linear form. Then  $Rl = K^{(1,1)}(-1)$ ; indeed,  $Rl$  has the Hilbert series  $s + s^2$ , is indecomposable, and therefore by Remark 2.1.1(iv) is linear. Consider  $M = C^{(1)} \oplus Rl$  and  $N = R \oplus \mathbb{k}(-1)$ . By Lemma 2.1.7,  $\mathcal{P}_M(t) = \mathcal{P}_N(t) = 1 + s \cdot f$ . Thus  $M$  and  $N$  have the same Betti diagram  $\beta$ , but  $c_{1,0}(M) = 1$  and  $c_{1,0}(N) = 0$ .

We proceed now with the proof of the theorem.

**Proof of Theorem 2.1.10.** Because  $R$  and its modules have finite length,  $u$ ,  $v$ , and  $w$  are integers, and  $w \geq 0$ .

It is clear that (i) determines (ii).

Next, we prove (ii) determines (i). First, we show that rows  $u \leq j - i \leq v + 1$  determine  $\beta(M)$ . By assumption,  $M_k = 0$  when  $k < u$  or  $k > v$ , and therefore the degree of each generator of  $M$  is bounded between  $u$  and  $v$ . We claim that the Betti diagram of each indecomposable  $R$ -module generated in degree  $k$  has nonzero entries in (at most) rows  $k \leq j - i \leq k + 1$ . Indeed,  $\beta_{i,j}(R(-k)) = 0$  and  $\beta_{i,j}(K^{(p,q)}(-k)) = 0$  if  $j - i \neq k$ , and by (2.1.K),  $\beta_{i,j}(C^{(n)}(-k)) = 0$  if  $j - i < k$  or  $k + 1 < j - i$ . Therefore  $\beta_{i,j}(M) = 0$  when  $j - i < u$  and  $j - i > v + 1$ .

Next, we show that columns  $0 \leq i \leq w + 1$  determine  $\beta$ . Recall  $\ell$  as defined in Proposition 2.1.9. We claim  $w \geq \ell$ . Fix a  $j \in \mathbb{Z}$ . If  $\ell_j = 0$ , then since  $w \geq 0$ ,  $w \geq \ell_j$ . If  $\ell_j \neq 0$ , then  $C^{(\ell_j)}(-j)$  is a summand of  $M$ , and hence  $\beta_{0,j}(M) \geq \beta_{0,j}(C^{(\ell_j)}(-j))$ .

By (2.1.K),  $\beta_{0,j}(C^{(\ell_j)}(-j)) = b_{\ell_j-1}$ . Therefore  $b_{\ell_j-1} \leq \max_j \{\beta_{0,j}(M)\}$ , and therefore  $\ell_j \leq w$ . It follows that  $w \geq \max_{j \in \mathbb{Z}} \ell_j = \ell$ .

Let  $N = \text{Syz}_w(M)$ . By Proposition 2.1.9,  $N$  is the direct sum of linear modules; i.e., there are integers  $p'_j, q'_j$  such that  $N = \bigoplus_j K^{(p'_j, q'_j)}(-j)$ . Using (2.1.L) and Remark 2.1.8 we have

$$\begin{aligned} \mathcal{P}_N(t) &= \sum_j \mathcal{P}_{K^{(p'_j, q'_j)}}(t) s^j \quad \text{where} \\ p'_j &= \beta_{0,j}(N) = \beta_{w,j}(M) \quad \text{and} \\ q'_j &= e\beta_{0,j}(N) - \beta_{1,j+1}(N) = e\beta_{w,j}(M) - \beta_{w+1,j+1}(M). \end{aligned}$$

Therefore,  $\mathcal{P}_N(t)$  is determined by  $\{\beta_{w,j}(M), \beta_{w+1,j}(M) \mid j \in \mathbb{Z}\}$ . By Remark 2.1.8,

$$\sum_{i \geq w} \sum_j \beta_{i,j}(M) s^j t^i = t^w \mathcal{P}_N(t), \quad (2.1.Q)$$

and thus (ii) determines (i).

These arguments also show that  $N$  and  $\mathcal{P}_M(t)$  are as described in the statement of the theorem.

Finally, with the hypothesis that  $r_j = 0$  for all  $j \in \mathbb{Z}$ , the equivalence of (i) and (iii) follows from Proposition 2.1.9.  $\square$

**Remark 2.1.12.** Suppose  $M$  is an  $R$ -module having no nonzero free summand, and write  $M$  as in Equation (2.1.B):

$$M = \bigoplus_j K^{(p_j, q_j)}(-j) \oplus \bigoplus_j \bigoplus_{n \geq 1} C^{(n)}(-j)^{c_{n,j}}.$$

In practice, we can use Theorem 2.1.10 to write  $\mathcal{P}_M(t)$  in terms of the rational functions  $f, g$ , and  $h_n(t)$ ,  $n \geq 1$ , as defined in Definition 2.1.4. The benefit of doing so is



that, applying Equation (2.1.M) or (2.1.N) (depending on the value of  $e$ ), we recover the various  $p_j$ ,  $q_j$ , and  $c_{n,j}$  parameters for  $M$ . We outline this process below.

Let  $N = \text{Syz}_w(M)$ , and observe that  $\mathcal{P}_N(t)$  can be written as rational function with the denominator  $1 - est + s^2t^2$ . Equation (2.1.O) shows how the coefficients in the numerator of this rational function can be deduced from columns  $w$  and  $w + 1$  of  $\beta(M)$ . In this way, using the nonzero entries in columns 0 through  $w + 1$  of  $\beta(M)$ , we recover Equation (2.1.P). Using Euclidean division over the polynomial ring  $\mathbb{Q}(s)[t]$ , we can transform  $\mathcal{P}_N(t)$  (and thus  $\mathcal{P}_M(t)$ ) to a  $\mathbb{Q}(s)$ -linear combination of  $f$ ,  $g$ , and  $h_n$ ,  $n \geq 1$ .

Now, depending on  $e$ , choose the appropriate equation in Proposition 2.1.9 and match coefficients to determine  $p_j$ ,  $q_j$ , and  $c_{n,j}$ .

**Example 2.1.13.** For  $R$  with  $e = 3$ , there does not exist an  $R$ -module  $M$ , having no nonzero free summand, for which the nonzero entries in columns  $0 \leq i \leq 3$  of  $\beta(M)$  are given by

$$*\begin{pmatrix} 5 & 1 & 1 & 2 \\ 2 & 1 & 3 & 8 \end{pmatrix}.$$

Seeking a contradiction, assume such a module  $M$  exists. Reading the first column of the alleged Betti diagram, we see  $\max_j\{\beta_{0,j}(M)\} = 5$ . Then, for  $w$  as in the statement of the theorem,  $w = 2$  since, for the ring  $R$ ,  $b_0 = 1$ ,  $b_1 = 3$ , and  $b_2 = 8$ . Theorem 2.1.10 requires columns 0 through 3, and we have this data available.

First, we work with  $\text{Syz}_2(M)$ . From the given data, we have

$$\text{Syz}_2(M) = K^{(p',q')}(-2) \oplus K^{(p'',q'')}(-3), \text{ where}$$

$$p' = \beta_{2,2}(M) = 1,$$

$$q' = e\beta_{2,2}(M) - \beta_{3,3}(M) = 1,$$

$$p'' = \beta_{2,3}(M) = 3, \text{ and}$$

$$q'' = e\beta_{2,3}(M) - \beta_{3,4}(M) = 1.$$

Using the technique outlined in the previous remark, we obtain

$$\begin{aligned} \mathcal{P}_M(t) &= 5 + 2s + st + s^2t + s^2t^2\mathcal{P}_{K(1,1)}(t) + s^3t^2\mathcal{P}_{K(3,1)}(t) \\ &= 5 + 2s + st + s^2t + s^2t^2 \cdot f - s^3t^2 \cdot g + 3s^3t^2 \cdot f - s^4t^2 \cdot g \\ &= 5 + 2s + st + s^2t + \frac{(s^2 + 3s^3)t^2 - (s^3 + s^4)t^3}{1 - 3st + s^2t^2} \\ &= 3 + 2s + 2 \cdot f + (-5s + s^2) \cdot g. \end{aligned}$$

Write  $M$  as in Equation (2.1.B):

$$M = \bigoplus_{j=0}^1 K^{(p_j, q_j)}(-j) \oplus \bigoplus_{j=0}^1 \bigoplus_{n \geq 1} C^{(n)}(-j)^{c_{n,j}}.$$

From Equation (2.1.N), we see that the  $2s$  term in  $\mathcal{P}_M(t)$  implies that  $c_{1,1} = 2$ , but this contradicts the given data (since row 3 is uniformly 0). The coefficient of  $g$  in  $\mathcal{P}_M(t)$  also implies that  $c_{1,0} - q_1 = 1$ . We see  $c_{1,0} = 3$ , so we obtain  $q_1 = 2$ . But  $0s + 2s^2$  is an inadmissible Hilbert series for a linear module generated in degree 1.

To see another way we may apply this theorem, we invoke results due to Avramov, Iyengar, and Şega.

**Remark 2.1.14** (Hilbert series of Koszul modules).

- (i) If  $M$  is a Koszul  $R$ -module with Hilbert series  $p + qs$ , by [AIŠ10, Corollary 1.7] the following inequalities hold:

$$1 \leq p \quad \text{and} \quad 0 \leq \frac{q}{p} \leq \left( \frac{e + \sqrt{e^2 - 4}}{2} \right) < e.$$

- (ii) For a pair of integers  $p$  and  $q$  satisfying

$$1 \leq p \quad \text{and} \quad 0 \leq \frac{q}{p} \leq (e - 1),$$

there exists a Koszul  $R$ -module with Hilbert series  $p + qs$  [AIŠ10, Theorem 2].

**Example 2.1.15.** In this example, we will use Theorem 2.1.10 and Remark 2.1.14 to identify the Hilbert series of each indecomposable linear summand of an  $R$ -module, based on an initial window if its Betti table.

Suppose  $R$  has  $e = 3$  and  $M$  is the  $R$ -module generated in degrees 0 and 1 with the presentation  $R(0)^3 \oplus R(-1) \xleftarrow{A} R(-1) \oplus R(-2)^3$ , where the free modules have the standard bases and

$$A = \left[ \begin{array}{c|ccc} y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & z^2 & 0 & 0 \\ \hline 0 & 0 & y & z \end{array} \right].$$

We see from its presentation that  $M$  has no free summands. We leave it to the reader to verify, using pencil and paper or [M2], that the nonzero entries in columns  $0 \leq i \leq 3$  of  $\beta(M)$  are given by

$$* \begin{pmatrix} 3 & 1 & 1 & 2 \\ 1 & 3 & 8 & 21 \end{pmatrix}.$$

By Theorem 2.1.10, the given Betti numbers uniquely determine  $\beta(M)$  and  $\mathcal{P}_M(t)$ . Using the earlier remark, we find  $c_{1,0} = 1$ ,  $p_0 = 2$ ,  $q_0 = 5$ ,  $p_1 = 1$ , and  $q_1 = 1$ ; every other parameter is 0. Therefore  $M = K^{(1,1)}(-1) \oplus K^{(2,5)} \oplus C^{(1)}$ . We claim that each of the summands on the right hand side is indecomposable. Indeed,  $C^{(1)}$  is indecomposable (it's the negative syzygy of an indecomposable module), and  $K^{(1,1)}(-1)$  is indecomposable (it's cyclic). Finally, if  $K^{(2,5)} = K^{(1,q)} \oplus K^{(1,q')}$ , then without loss of generality  $\frac{q}{1} \geq 3$ , but this violates Remark 2.1.14(i). Therefore  $K^{(2,5)}$  is indecomposable.

As an aside, the existence of  $K^{(2,5)}$  shows that the converse of Remark 2.1.14(ii) does not hold; indeed,

$$(e - 1) = 2 < \frac{5}{2} = \frac{q_0}{p_0}.$$

**Proposition 2.1.16.** *Let  $R$  be a short Gorenstein standard graded  $\mathbb{k}$ -algebra with  $e \geq 2$  and let  $M$  be an  $R$ -module. Fix a positive integer  $a \leq e - 1$  and a positive integer  $n$ . The following statements are equivalent:*

(i) *There exists an ideal  $I$  minimally generated by  $l_1, \dots, l_a \in R_1$  such that*

$$M \cong (\text{Syz}_{-n}(I)(-n + 1)).$$

(ii) *The module  $M$  is Koszul and has Hilbert series*

$$H_M(s) = (b_{n-1} - ab_{n-2}) + (b_n - ab_{n-1})s.$$

*Further, when the above hold,  $M$  is indecomposable.*

**Example 2.1.17.** Proposition 2.1.16 provides an infinite number of modules outside the range in Remark 2.1.14(ii). When  $e \geq 3$  and  $n \geq 2$ , consider  $K^{(p,q)}$  where

$p = (b_n - b_{n-1})$  and  $q = (b_{n+1} - b_n)s$ . Then, using Remark 2.1.5, we see that  $q > (e - 1)p$ :

$$b_{n+1} - b_n = b_{n+1} - eb_n + (e - 1)b_n = (e - 1)b_n - b_{n-1} > (e - 1)(b_n - b_{n-1}),$$

where the last inequality is strict because we have assumed  $n \geq 2$  (therefore  $b_{n-1} > 0$ ).

One can use the following remark to check if the coefficients of a Hilbert series are of the form in Proposition 2.1.16(ii).

**Remark 2.1.18.** For each  $e \geq 2$  and  $n \geq 0$ ,

$$b_n = \begin{cases} n + 1, & e = 2, \\ \frac{1}{\sqrt{e^2 - 4}} (\phi^{n+1} - \bar{\phi}^{n+1}), & e \geq 3, \end{cases} \quad (2.1.R)$$

$$\text{where } \phi = \frac{e + \sqrt{e^2 - 4}}{2} \quad \text{and} \quad \bar{\phi} = \frac{e - \sqrt{e^2 - 4}}{2}. \quad (2.1.S)$$

**Proof.** When  $e = 2$ , a trivial induction argument by Remark 2.1.5 shows  $b_n = n + 1$ .

When  $e \geq 3$ , we have  $\phi - \bar{\phi} = \frac{2\sqrt{e^2 - 4}}{2} = b_0\sqrt{e^2 - 4}$  and  $\phi^2 - \bar{\phi}^2 = \frac{4e\sqrt{e^2 - 4}}{4} = b_1\sqrt{e^2 - 4}$ . Fix  $n \geq 2$  and suppose  $\phi^{k+1} - \bar{\phi}^{k+1} = b_k\sqrt{e^2 - 4}$  when  $k < n$ . Observe that  $\phi + \bar{\phi} = e$  and  $\phi\bar{\phi} = 1$ . Then, using Remark 2.1.5,

$$\begin{aligned} b_n\sqrt{e^2 - 4} &= \sqrt{e^2 - 4}(eb_{n-1} - b_{n-2}) \\ &= (\phi + \bar{\phi})(\phi^n - \bar{\phi}^n) - \phi^{n-1} + \bar{\phi}^{n-1} \\ &= \phi^{n+1} - \phi\bar{\phi}^n + \bar{\phi}\phi^n - \bar{\phi}^{n+1} - \phi^{n-1} + \bar{\phi}^{n-1} \\ &= \phi^{n+1} - \bar{\phi}^{n+1}. \end{aligned}$$

□

**Remark 2.1.19.** Let  $L = K^{(p,q)}(-k)$  with  $q \neq 0$ , and suppose  $L$  is indecomposable. Let  $E(L)$  denote the injective hull of  $L$ . Then  $E(L) = R(-k+1)^q$ .

To see this, let  $\mathfrak{m}$  denote the homogeneous maximal ideal of  $R$ , and recall that  $L_j$  is the degree  $j$  component of  $L$ . On the one hand,  $\mathfrak{m}^2 L = L_{k+2} = 0$  by hypothesis. Because  $\mathfrak{m} L_{k+1} \subseteq L_{k+2} = 0$ , it follows that  $L_{k+1} \subseteq \text{soc}_R(L)$ . On the other hand, since  $L$  is indecomposable,  $\text{soc}_R(L) \subseteq \mathfrak{m} L \subseteq L_{k+1}$  (see, for example, [Sjö79, Lemma 3]). Therefore  $\text{soc}_R(L) = L_{k+1} \cong \mathbb{k}(-k-1)^q$ .

Note that  $\mathbb{k}(-2) \cong \text{soc}_R(R)$ , and therefore,  $\mathbb{k}(-k-1)^q \cong \text{soc}_R(R(-k+1)^q)$ . This induces a (graded) essential extension  $L \rightarrow R(-k+1)^q$ .

**Proof of Proposition 2.1.16.** Since the socle of  $R$  is one-dimensional, every ideal of  $R$  is indecomposable. Thus, when statement (i) holds,  $M$  is indecomposable since it is the negative syzygy of an indecomposable  $R$ -module (namely,  $I$ ).

We first show (i) implies (ii). Let  $I$  be minimally generated by  $a$  linear forms. Define  $N^{(k)} = (\text{Syz}_{-k}(I))(-k+1)$ . Observe that  $N^{(1)} = \text{Syz}_{-1}(I) = R/I$ ; this module has the Hilbert series  $1 + (e-a)s = b_0 - ab_{-1} + (b_1 - ab_0)s$ , and there is an exact sequence

$$0 \leftarrow R/I \leftarrow R(0) \leftarrow I \leftarrow 0.$$

For  $m \geq 1$ , set  $p_m = b_{m-1} - ab_{m-2}$  and  $q_m = b_m - ab_{m-1}$ , and assume that  $N = N^{(m)}$  has the Hilbert series  $p_m + q_m s$ . For  $m \geq 1$ , we have

$$q_m = b_m - ab_{m-1} \geq b_m - (e-1)b_{m-1} = b_m - eb_{m-1} + b_{m-1} = -b_{m-2} + b_{m-1} > 0.$$

Then  $N$  injects into  $R(1)^{q_m}$  by Remark 2.1.19.

Since Hilbert series are additive on short exact sequences, the sequence

$$0 \leftarrow \text{Syz}_{-1}(N) \leftarrow R(1)^{q_m} \leftarrow N \leftarrow 0$$

implies  $\text{Syz}_{-1}(N)$  has the Hilbert series

$$\begin{aligned} (s^{-1} + e + s)q_m - (p_m + q_ms) &= s^{-1}q_m + eq_m - p_m \\ &= (b_m - ab_{m-1})s^{-1} + e(b_m - ab_{m-1}) - (b_{m-1} - ab_{m-2}) \\ &= (b_m - ab_{m-1})s^{-1} + (b_{m+1} - ab_m) \\ &= p_{m+1}s^{-1} + q_{m+1}, \end{aligned}$$

where the third equality follows from Remark 2.1.5. Now,

$$(\text{Syz}_{-1}(N))(-1) = (\text{Syz}_{-m-1}(I))(-m+1-1) = N^{(m+1)},$$

and therefore  $N^{(m+1)}$  has the Hilbert series  $p_{m+1} + q_{m+1}s$ . Furthermore, since  $R(-1)^{p_m}$  surjects onto  $N^{(m)}(-1)$ , we have an exact sequence

$$0 \leftarrow N^{(m+1)} \leftarrow R(0)^{p_{m+1}} \leftarrow R(-1)^{p_m}.$$

Thus, by induction, there exists a linear exact sequence

$$0 \leftarrow N^{(k)} \leftarrow R(0)^{p_k} \leftarrow R(-1)^{p_{k-1}} \dots \leftarrow R(-k+1) \leftarrow I(-k+1) \leftarrow 0.$$

Now  $I(-k+1)$  is generated in degree  $k$ , so  $R(-k)^a$  surjects onto  $I(-k+1)$ , and the free resolution of  $I(-k+1)$  is linear by Remark 2.1.1(iv). Therefore,  $N^{(k)}$  is Koszul.

For (ii) implies (i), consider  $M$  as in the statement of (ii). Let  $I$  be an ideal

minimally generated by  $a$  linear forms, and keep the notation  $N^{(k)} = (\text{Syz}_{-k}(I))(-k+1)$ . The modules  $M$  and  $N = N^{(n)}$  are Koszul and have the same Hilbert series, and therefore they have the same Poincaré series by Lemma 2.1.7. This implies that the modules  $M' = (\text{Syz}_{n-1}(M))$  and  $(\text{Syz}_{n-1}(N))$  have the same Hilbert series; indeed, they are both linear, so (2.1.L) applies.

Observe that  $N' = (\text{Syz}_{-1}(I))(-n+1) = (R/I)(-n+1)$ , and therefore  $\mathcal{H}_{N'} = s^{n-1}(1 - (e-a)s)$ . A linear module  $K$  generated in a single degree with Hilbert series  $1 - (e-a)s$  is cyclic, and hence there is an ideal  $J$  minimally generated by  $a$  linear forms such that  $K \cong (R/J)$ . Therefore, for some such  $J$ ,  $\text{Syz}_n(M) = \text{Syz}_1(M') = J(-n+1)$ . Taking  $(-n)$ -th syzygies, we have  $M \cong (\text{Syz}_{-n}(J))(-n+1)$ .  $\square$

## 2.2 Short Gorenstein rings of embedding dimension 2

In this section, we restrict our focus to short Gorenstein rings of embedding dimension  $e = 2$ .

Given an  $R$ -module  $M$ , let  $c_{1,j}(M)$  denote the multiplicity of  $C^{(n)}(-j)$  in the representation of  $M$  as a direct sum of indecomposable  $R$ -modules (the number  $c_{1,j}(M)$  is independent of the choice of decomposition by the Krull–Remak–Schmidt theorem). For a given Betti diagram  $\beta$ , define

$$c_j(\beta) = \sup\{c_{1,j}(M) \mid M \text{ is an } R\text{-module and } \beta(M) = \beta\}.$$

Because  $\beta(M \oplus N) = \beta(M) + \beta(N)$  for all  $R$ -modules  $M$  and  $N$ , we see that  $c_j(\beta)$  is the largest possible coefficient of  $\beta(C^{(1)}(-j))$  in any decomposition of  $\beta$ . The following theorem shows that there exists a decomposition of  $\beta$  in which this maximum



value is achieved as the coefficient of  $\beta(C^{(1)}(-j))$  simultaneously for each  $j$ , and, moreover, that the remaining coefficients are uniquely determined once we choose this maximum value for each  $j$ .

**Proposition 2.2.1.** *Let  $R$  be a short Gorenstein standard graded  $\mathbb{k}$ -algebra with  $e = 2$ . Every Betti diagram  $\beta$  over  $R$  can be written uniquely as a nonnegative integral combination of the Betti diagrams of*

$$(i) \quad R(-j),$$

$$(ii) \quad K^{(1,1)}(-j),$$

$$(iii) \quad K^{(1,0)}(-j), \quad \text{and}$$

$$(iv) \quad C^{(n)}(-j)$$

in which the coefficient of  $\beta(C^{(1)}(-j))$  is  $c_j(\beta)$  for each  $j \in \mathbb{Z}$ .

**Definition 2.2.2.** We refer to the modules of types (i)-(iv) in the statement of the proposition as *realization modules*.

**Remark 2.2.3.** The modules  $K^{(1,0)}$  and  $K^{(1,1)}$  exist. In particular,  $\mathbb{k} = K^{(1,0)}$ , and for any nonzero linear form  $l$ ,  $Rl(-1) = K^{(1,1)}$  (see Example 2.1.11).

**Example 2.2.4.** The best we can hope for is uniqueness up to some condition on coefficients in the decomposition. As in Example 2.1.11, there is an equality

$$\beta(R) + \beta(K^{(1,0)}(-1)) = \beta(C^{(1)}) + \beta(K^{(1,1)}(-1)).$$

**Proof of Proposition 2.2.1.** Suppose  $\beta \in B_{\mathbb{Q}}(R)$  is a Betti diagram. Let  $M$  be an arbitrary  $R$ -module with  $\beta(M) = \beta$ , and write

$$M = \bigoplus_j \left( R^{r_j} \oplus K^{(p_j, q_j)} \oplus \bigoplus_{n \geq 1} (C^{(n)})^{c_{n,j}} \right) (-j)$$

as in Lemma 2.1.3. By Proposition 2.1.9,  $\beta$  uniquely determines  $c_{n,j}$  for all  $n \geq 2$  and  $j \in \mathbb{Z}$ , so we may assume  $c_{n,j} = 0$  for all  $n \geq 2$  and  $j \in \mathbb{Z}$ . Another application of Proposition 2.1.9 implies that  $\beta$  uniquely determines  $u_j$ ,  $v_j$ , and  $w_j$  in  $\mathbb{Z}$  such that

$$\mathcal{P}_M(t) = \left( \sum_j u_j s^j \right) h_1(t) + \left( \sum_j v_j s^j \right) \cdot f + \left( \sum_j w_j s^j \right) \cdot g, \quad (2.2.A)$$

and, in particular,  $u_j = r_j + c_{1,j}$ ,  $v_j = p_j - q_j + c_{1,j-1}$ , and  $w_j = q_j - c_{1,j-1}$ .

Define  $d_j = \min\{u_j, v_{j+1}\}$ .

First, we exhibit a decomposition of  $\beta$  into the Betti diagrams of realization modules where  $d_j$  is the coefficient of  $\beta(C^{(1)}(-j))$ . Indeed, the module

$$N = \bigoplus_j \left( R(-j)^{u_j - d_j} \oplus K^{(1,0)}(-j)^{v_j - d_{j-1}} \oplus K^{(1,1)}(-j)^{w_j + d_{j-1}} \oplus C^{(1)}(-j)^{d_j} \right) \quad (2.2.B)$$

induces such a decomposition, where  $u_j - d_j$  is the coefficient of  $\beta(R(-j))$ ,  $v_j - d_{j-1}$  is the coefficient of  $\beta(K^{(1,0)}(-j))$ , and so on. The fact that this module has the same Poincaré series as  $M$  follows from applying Lemma 2.1.7 to each summand. For all  $j \in \mathbb{Z}$ , the coefficients  $u_j - d_j$ ,  $v_j - d_{j-1}$ ,  $w_j + d_{j-1}$  are nonnegative integers that are uniquely determined by  $\beta$  and  $d_j$ .

Next, we show  $d_j = c_j(\beta)$ . The module  $N$  in (2.2.B) satisfies  $\beta(N) = \beta$ , and thus  $d_j = c_{1,j}(N) \leq c_j(\beta)$ . On the other hand, for any  $M$  with  $\beta(M) = \beta$ , Remark 2.1.14(i)

implies that  $p_j \geq q_j$  for all  $j \in \mathbb{Z}$  since  $e = 2$ . It follows that

$$d_j = \min\{r_j + c_{1,j}(M), p_{j+1} - q_{j+1} + c_{1,j}(M)\} \geq c_{1,j}(M).$$

Since  $M$  was arbitrarily chosen, this implies that  $d_j \geq c_j(\beta)$ . Therefore  $d_j = c_j(\beta)$ . □

Next, we describe the structure of the cone. For the relevant notions from convex geometry, we direct the reader to Appendix A. First, we describe a partial order on the realization modules by refining the partial order on degree sequences given in [SE10].

**Remark 2.2.5** (Partial order on degree sequences [SE10]). Given two strictly increasing sequences of integers  $(d_0, d_1, \dots, d_r)$  and  $(d'_0, d'_1, \dots, d'_{r'})$  (where  $r, r' \leq \infty$ ), declare

$$(d_0, d_1, \dots, d_r) \geq (d'_0, d'_1, \dots, d'_{r'})$$

provided  $r \leq r'$  and  $d_i \geq d'_i$  for  $i = 1, 2, \dots, r$ .

**Definition 2.2.6.** An  $R$ -module  $M$  with a pure minimal free resolution is said to have the *degree sequence*  $d = (d_0, d_1, \dots)$ , where  $d_i \in \mathbb{Z}$ , provided  $\beta_{i,j}(M) \neq 0$  if and only if  $j = d_i$ . We provide the degree sequences for the realization modules in the table below.

Module	Degree Sequence
$R(-j)$	$(j)$
$K^{(1,0)}(-j)$	$(j, j+1, j+2, \dots)$
$K^{(1,1)}(-j)$	$(j, j+1, j+2, \dots)$
$C^{(n)}(-j), n \geq 1$	$d_i = j + i$ when $i < n$ , $d_i = j + i + 1$ when $i \geq n$ ; $(j, j+1, \dots, \underbrace{j+n-1}_{i=n-1}, \underbrace{j+n+1}_{i=n}, j+n+2, \dots)$

We will use the partial order in Remark 2.2.5 as a starting point for building a partial order on realization modules. First, we will examine how the degree sequences for the realization modules are ordered.

**Example 2.2.7** (Ordering degree sequences of realization modules). Over  $R$ , the partial order in Remark 2.2.5 totally orders the degree sequences of nonfree realization modules. For a realization module  $M$ , let  $\delta(M)$  denote its degree sequence. It is tedious but easy to verify that, for all  $j \in \mathbb{Z}$ , we have

$$\begin{aligned} \dots > \delta(K^{(1,0)}(-j-1)) > \delta(C^{(1)}(-j)) > \delta(C^{(2)}(-j)) > \dots \\ \dots > \delta(C^{(n)}(-j)) > \delta(C^{(n+1)}(-j)) > \dots > \delta(K^{(1,0)}(-j)) > \dots \end{aligned}$$

Next, consider the free realization modules. When  $k \leq j$ ,  $\delta(R(-j)) > \delta(C^{(n)}(-k))$  for all  $n \geq 1$ . However, when  $k > j$ , the only comparison involving  $\delta(R(-j))$  is  $\delta(R(-k)) > \delta(R(-j))$ . Indeed, for any nonfree realization module  $M$  generated in degree  $k$ ,  $\delta(M)$  is “longer” than  $\delta(R(-j))$  but  $k \geq j$ , so these degree sequences are incomparable by Remark 2.2.5.

The partial order in Remark 2.2.5 induces a relation  $\geq$  on realization modules, but it is not a partial order: for example, the modules  $K^{(1,0)}$  and  $K^{(1,1)}$  have the same degree sequence  $(0, 1, 2, 3, 4, \dots)$ . We will fix this by declaring that  $K^{(1,1)}(-j) >$

$K^{(1,0)}(-j)$  for each  $j$ . Also, we will introduce some strict inequalities involving the free realization modules. The resulting partially ordered set will have the following property consistent with the theme of Boij-Söderberg theory: every Betti diagram decomposes as a sum of Betti diagrams of realization modules forming a chain.

**Proposition 2.2.8.** *The set of realization modules,*

$$P = \left\{ R(-j), K^{(1,1)}(-j), K^{(1,0)}(-j), C^{(n)}(-j) \mid j \in \mathbb{Z}, n \geq 1 \right\},$$

*is a poset with respect to the relation  $>$  induced by the partial order on their degree sequences and the additional relations*

$$(1) \ K^{(1,1)}(-j) > K^{(1,0)}(-j) \text{ for each } j \in \mathbb{Z},$$

$$(2) \ M(-k-1) > R(-j) \text{ for each } k \geq j \in \mathbb{Z} \text{ and } M \in \{K^{(1,1)}, C^{(n)}\}, \text{ and}$$

$$(3) \ K^{(1,0)}(-k-2) > R(-j) \text{ for each } k \geq j \in \mathbb{Z}.$$

*In particular,  $K^{(1,0)}(-j-1)$  and  $R(-j)$  are incomparable for all  $j \in \mathbb{Z}$ .*

**Proof.** From Example 2.2.7, we see that the set

$$P' = \left\{ C^{(n)}(-j), K^{(1,1)}(-j), K^{(1,0)}(-j) \mid j \in \mathbb{Z}, n \geq 1 \right\}$$

is totally ordered:

$$\dots > K^{(1,1)}(-1) > K^{(1,0)}(-1) > C^{(1)}(0) > \dots > C^{(n)}(0) > \dots > K^{(1,1)}(0) > K^{(1,0)}(0) > C^{(1)}(1) > \dots$$

We also see that the set

$$P'' = \left\{ C^{(n)}(-j), K^{(1,1)}(-j), R(-j) \mid j \in \mathbb{Z}, n \geq 1 \right\}$$

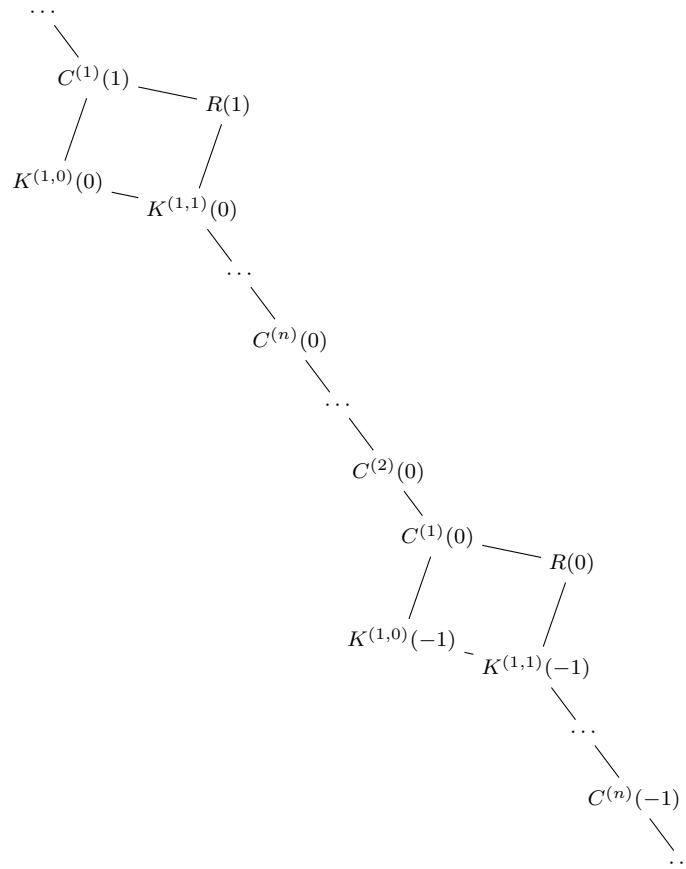


Figure 2.1: The poset of realization modules, where  $M > N$  if there is an downward path from  $N$  to  $M$ . Thinking of this diagram as a directed graph with each edge oriented downward, it is (directed) acyclic. This gives another proof that  $\geq$  is a partial order on  $P$ .

is totally ordered:

$$\dots > K^{(1,1)}(-1) > R(0) > C^{(1)}(0) > \dots > C^{(n)}(0) > \dots > K^{(1,1)}(0) > R(1) > C^{(1)}(1) > \dots .$$

Temporarily declare the additional relations

$$(4) \quad K^{(1,0)}(-j-1) \geq R(-j) \text{ for each } j \in \mathbb{Z}.$$

Observe that with these relations, the set  $P$  is totally ordered. Therefore, without relations (4),  $\geq$  is antisymmetric.

It remains to show that without relations (4),  $\geq$  is still transitive. We claim that there is no element  $M \in P$  such that

$$K^{(1,0)}(-j-1) > M > R(-j) \quad \text{or} \quad R(-j) > M > K^{(1,0)}(-j-1).$$

We know that  $>$  totally orders each of the sets  $P'$  and  $P''$ , and in each set we see that  $K^{(1,0)}(-j-1)$  and  $R(-j)$  have the same immediate predecessor,  $C^{(1)}(-j)$ , and the same immediate successor,  $K^{(1,1)}(-j-1)$ . Since  $>$  is antisymmetric, this proves the claim, and therefore  $>$  is transitive.  $\square$

**Theorem 2.2.9.** *Let  $\Phi : P \rightarrow B_{\mathbb{Q}}(R)$  denote the set map  $M \mapsto \beta(M)$ . Then*

$$\Sigma(P, \Phi) = \bigcup_{\substack{\text{finite chains} \\ p_1 < \dots < p_s}} \text{pos}(\Phi(p_1), \dots, \Phi(p_s))$$

*is a simplicial fan.*

*Moreover,  $B_{\mathbb{Q}}(R) = \text{supp}(\Sigma(P, \Phi))$ ; that is, every Betti diagram over  $R$  decomposes uniquely as a sum of Betti diagrams of realization modules forming a chain in  $P$ .*

**Proof.** We refer the reader to Appendix A for the relevant background on convex geometry.

To show that  $\Sigma(P, \Phi)$  is a simplicial fan, we must show the following:

- (i) Every finite chain of realization modules in  $P$  maps to a linearly independent set in  $B_{\mathbb{Q}}(R)$ , and
- (ii) Given two chains  $P_1$  and  $P_2$  in  $P$ , the intersection

$$\text{pos}(\Phi(P_1)) \cap \text{pos}(\Phi(P_2)) = \text{pos}(\Phi(P_1 \cap P_2))$$

(that is, that the intersection is a proper face of both cones).

For (i), suppose for  $j \in \mathbb{Z}$  and  $n \geq 1$  that there exist rational numbers  $r_j$ ,  $a_j$ ,  $a'_j$ , and  $c_{n,j}$  such that only finitely many are nonzero and

$$0 = \sum_j r_j \beta(R(-j)) + \sum_j a_j \beta(K^{(1,0)}) + \sum_j a'_j \beta(K^{(1,1)}(-j)) + \sum_j \sum_{n \geq 1} c_{n,j} \beta(C^{(n)}(-j)).$$

Passing to Poincaré series, we obtain

$$\begin{aligned} 0 = \sum_j (r_j + c_{1,j}) s^j + \sum_j \sum_{n \geq 2} c_{n,j} s^j \cdot h_n(t) \\ + \sum_j \left( a_j + \sum_{n \geq 1} c_{n,j-1} \right) s^j \cdot f + \sum_j \left( a'_j - \sum_{n \geq 1} n c_{n,j-1} \right) s^j \cdot g. \end{aligned}$$

Now, Lemma 2.1.6 implies that, for each  $j \in \mathbb{Z}$ , the following equalities hold:

$$c_{n,j} = 0 \text{ for all } n \geq 2, \quad r_j + c_{1,j} = 0, \quad a_{j+1} + c_{1,j} = 0, \quad \text{and} \quad a'_{j+1} - c_{1,j} = 0.$$

For all  $n \geq 1$  and  $j \in \mathbb{Z}$ , the coefficients  $r_j$ ,  $a_{j+1}$ ,  $a'_{j+1}$ , and  $c_{n,j}$  are all 0 unless there exists at least one integer  $j$  for which  $r_j \neq 0$ . When this happens,  $a_{j+1}$  is also nonzero, but  $R(-j)$  and  $K^{(1,0)}(-j-1)$  do not lie in a chain. Therefore the image of a chain in  $P$  is linearly independent in  $\mathbb{V}$ .

For (ii), the inclusion  $\text{pos}(\Phi(P_1 \cap P_2)) \subseteq \text{pos}(\Phi(P_1)) \cap \text{pos}(\Phi(P_2))$  is evident.

For the remaining inclusion, suppose  $\beta \in \text{pos}(\Phi(P_1))$ , and write

$$\beta = \sum_j r_j \beta(R(-j)) + \sum_j a_j \beta(K^{(1,0)}) + \sum_j a'_j \beta(K^{(1,1)}(-j)) + \sum_j \sum_{n \geq 1} c_{n,j} \beta(C^{(n)}(-j)) \tag{2.2.C}$$

where  $r_j$ ,  $a_j$ ,  $a'_j$ , and  $c_{n,j}$  are nonnegative rational numbers, and, for a realization



module  $N$ , the coefficient of  $\beta(N) \neq 0$  only if  $N \in P_1$ .

Passing to Poincaré series, we see that every module with Betti diagram  $\beta$  has the Poincaré series

$$\begin{aligned} \mathcal{P}(t) = & \sum_j (r_j + c_{1,j}) s^j + \sum_j \sum_{n \geq 2} c_{n,j} s^j \cdot h_n(t) \\ & + \sum_j \left( a_j + \sum_{n \geq 1} c_{n,j-1} \right) s^j \cdot f + \sum_j \left( a'_j - \sum_{n \geq 1} n c_{n,j-1} \right) s^j \cdot g. \end{aligned}$$

Let  $M$  be an  $R$ -module satisfying  $\beta(M) = \beta$ , and write

$$M = \bigoplus_j \left( R^{r_j(M)} \oplus K^{(p_j(M), q_j(M))} \oplus \bigoplus_{n \geq 1} (C^{(n)})^{c_{n,j}(M)} \right) (-j).$$

Recall  $c_j(\beta) = \sup\{c_{1,j}(M) \mid M \text{ is an } R\text{-module and } \beta(M) = \beta\}$ .

We claim  $c_{1,j} = c_j(\beta)$  and that the remaining coefficients in decomposition 2.2.C are integers. By Proposition 2.1.9 and Lemma 2.1.6, for all  $j \in \mathbb{Z}$ , we have the equalities

$$\begin{aligned} c_{n,j} &= c_{n,j}(M) \quad \text{for all } n \geq 2, \\ r_j + c_{1,j} &= r_j(M) + c_{1,j}(M), \\ a_{j+1} + c_{1,j} &= p_{j+1}(M) - q_{j+1}(M) + c_{1,j}(M), \quad \text{and} \\ a'_{j+1} - c_{1,j} &= q_{j+1}(M) - c_{1,j}(M). \end{aligned}$$

If  $r_j = 0$ , then the equalities above imply that  $r_j$ ,  $a_j$ ,  $a'_j$ , and  $c_{n,j}$  are integers for all  $n \geq 1$  and  $j \in \mathbb{Z}$ . Moreover, in this case,  $c_{1,j} = r_j(M) + c_{1,j}(M) \geq c_{1,j}(M)$  with equality throughout when  $M$  is the module induced from the decomposition (2.2.C).

If  $r_j \neq 0$ , then  $a_{j+1} = 0$  because  $R(-j)$  and  $K^{(1,0)}(-j-1)$  cannot both belong to

$P_1$ . In this case, the equalities above imply that  $r_j$ ,  $a_j$ ,  $a'_j$ , and  $c_{n,j}$  are integers for all  $n \geq 1$  and  $j \in \mathbb{Z}$ . Furthermore,  $c_{1,j} = p_{j+1}(M) - q_{j+1}(M) + c_{1,j}(M)$ . We have already observed that  $p_{j+1}(M) - q_{j+1}(M) \geq 0$ . Hence  $c_{1,j} \geq c_{1,j}(M)$  with equality when  $p_{j+1} = q_{j+1} = a'_{j+1}$ , and this occurs when  $M$  is the module induced from the decomposition (2.2.C).

Therefore, for all  $j \in \mathbb{Z}$ ,  $c_{1,j} = c_j(M)$  in (2.2.C). If  $P_2$  is another chain for which  $\beta \in \text{pos}(\Phi(P_2))$ , the same argument shows that  $c_j(\beta)$  is the coefficient of  $\beta(C^{(1)}(-j))$  in a decomposition of  $\beta$  into the Betti diagrams of modules in  $P_2$ ; it also shows that all the coefficients in this decomposition are integers. Proposition 2.2.1 implies that a decomposition of  $\beta$  with these properties is unique. Thus, for any realization module  $N$  for which  $\beta(N)$  has a nonzero coefficient in the decomposition (2.2.C), we have  $N \in P_1 \cap P_2$ . Therefore  $\beta \in \text{pos}(\Phi(P_1 \cap P_2))$ .

For the equality  $B_{\mathbb{Q}}(R) = \text{supp}(\Sigma(P, \Phi))$ , it is immediate that  $\text{supp}(\Sigma(P, \Phi)) \subseteq B_{\mathbb{Q}}(R)$ , and Proposition 2.2.1 implies Betti diagrams of realization modules span  $B_{\mathbb{Q}}(R)$ . Therefore  $B_{\mathbb{Q}}(R) \subseteq \text{supp}(\Sigma(P, \Phi))$ .  $\square$

**Remark 2.2.10.** The main obstruction to generalizing the work done for this ring to short Gorenstein rings in general is that we do not know precisely which pairs  $p$  and  $q$  satisfying the conditions in Remark 2.1.14(i) are realized as the coefficients of the Hilbert series of a Koszul module. Because of this blind spot, although we know we will need the cyclic free modules and the twisted cosyzygies of  $\mathbb{k}$  as part of a set of modules whose Betti diagrams span extremal rays in the cone, we do not know exactly which Koszul modules we will need to add to this set.

## Chapter 3

# Hypersurfaces of low embedding dimension

This chapter represents work done jointly with Christine Berkesch, Jesse Burke, and Daniel Erman.

In this chapter, we investigate Boij–Söderberg theory for two classes of graded hypersurface rings, where the existence of infinite free resolutions is the primary complicating factor. Our main result is a complete description of the cone of Betti diagrams over a standard graded quadric hypersurface ring of the form  $\mathbb{k}[x, y]/(q)$ . As in the case of a standard graded polynomial ring, there is a partial order on the extremal rays of the cone which gives it the structure of a simplicial fan. We obtain a similar result for standard graded rings of the form  $\mathbb{k}[x]/(x^n)$  for any  $n \geq 2$ . Although there has been recent work on extending Boij–Söderberg theoretic results to rings other than the polynomial ring [BF11, Flø10, BBCI<sup>+</sup>10, BEKS11], the main result of this paper provides the first example of another graded ring for which the cone of Betti diagrams is entirely understood.

As in the case of a polynomial ring, our description of the cone of Betti diagrams

for the hypersurfaces above depends on the notion of a pure resolution; recall that this is a minimal resolution of the form

$$R(-d_0)^{\beta_0} \longleftarrow R(-d_1)^{\beta_1} \longleftarrow R(-d_2)^{\beta_2} \longleftarrow \cdots .$$

We refer to  $(d_0, d_1, d_2, \dots)$  as the **degree sequence** of the pure resolution. If  $\beta_n \neq 0$  but  $\beta_i = 0$  for  $i > n$ , (that is, the corresponding module has projective dimension  $n$ ), we write the degree sequence  $(d_0, \dots, d_n, \infty, \infty, \dots)$ . Thus every degree sequence is either a strictly increasing sequence of integers or of the form  $(d_0, \dots, d_n, \infty, \infty)$  where  $d_0 < \dots < d_n$  for some  $n \geq 0$ .

The simplest hypersurface ring  $R$  is one of embedding dimension 1. The extremal ray description of this cone, provided in Proposition 3.0.11, follows from the structure theorem of finitely generated modules over a principal ideal domain. We give an equivalent description of this cone in terms of facets in Theorem 3.2.4.

**Proposition 3.0.11.** *Let  $R = \mathbb{k}[x]/(x^n)$ . The extremal rays of  $B_{\mathbb{Q}}(R)$  are the rays in  $\mathbb{V}$  spanned by:*

- i. the Betti diagrams of those modules of finite projective dimension having a pure resolution of the form  $(d_0, \infty, \infty, \dots)$ ;*
- ii. the Betti diagrams of those modules of infinite projective dimension having a pure resolution of type  $(d_0, d_1, d_0 + n, d_1 + n, d_0 + 2n, d_1 + 2n, \dots)$ .*

Our main result is a complete description of the cone  $B_{\mathbb{Q}}(R)$  when  $R$  is a quadric hypersurface ring of embedding dimension 2. We state here its description in terms of extremal rays; see Theorem 3.1.4 for a description in terms of facets.

**Theorem 3.0.12.** *Let  $q$  be any quadric in  $\mathbb{k}[x, y]$ , and let  $R = \mathbb{k}[x, y]/(q)$ . The extremal rays of  $B_{\mathbb{Q}}(R)$  are the rays in  $\mathbb{V}$  spanned by:*

- i. the Betti diagrams of those Cohen–Macaulay modules of finite projective dimension having a pure resolution of the form  $(d_0, d_1, \infty, \dots)$ ;*
- ii. the Betti diagrams of those finite length modules of infinite projective dimension having a pure resolution of type  $(d_0, d_1, d_1 + 1, d_1 + 2, d_1 + 3, d_1 + 4, \dots)$ .*

As in the main results of Boij–Söderberg theory for the standard graded polynomial ring [ES09], for both types of hypersurfaces  $R$  above, our results provide a simplicial fan structure on  $B_{\mathbb{Q}}(R)$ .

**Theorem 3.0.13.** *Let  $R$  be a standard graded hypersurface ring of the form  $\mathbb{k}[x]/(x^n)$  for any  $n \geq 2$  or  $\mathbb{k}[x, y]/(q)$ , where  $q$  is any homogeneous quadric. Then the cone of Betti diagrams  $B_{\mathbb{Q}}(R)$  has the structure of a simplicial fan induced by a partial order on its extremal rays.*

From the simplicial fan structure, we obtain decomposition algorithms for  $R$ -Betti diagrams as in [ES09, BS12], as well as  $R$ -analogues of the Multiplicity Conjectures (see Section 3.3).

New phenomena arise in the hypersurface case that are not seen in the case of a standard graded polynomial ring. To begin with, some of the functionals used to provide a halfspace description of  $B_{\mathbb{Q}}(R)$  have no analogue in the polynomial ring case. One set of these functionals directly measures the nonminimality of the standard resolution. This resolution, introduced in [Sha69, Section 3] (see also [Eis80, Section 7]), builds a free  $R$ -resolution from a minimal free  $S$ -resolution. The resulting functionals thus directly reflect the passage from the polynomial to hypersurface case.

Another interesting difference comes from the simplicial structure on  $B_{\mathbb{Q}}(R)$ . Unlike the polynomial ring, we cannot simply use the termwise partial order on  $R$ -degree sequences. Instead, we introduce partial orders that take into account the infinite resolutions that occur over a hypersurface ring, see Definitions 3.1.1 and 3.2.1.

Finally, we observe that for hypersurface rings, it is no longer the case that every Cohen–Macaulay module with a pure resolution lies on an extremal ray. This already happens in the context of Theorem 3.0.12. For instance, let  $M$  be the maximal Cohen–Macaulay module  $(x) \subseteq R = \mathbb{k}[x, y]/(x^2)$ . The Betti diagram of  $M$  is not extremal, since it decomposes as

$$\beta(M) = \begin{matrix} * \\ \begin{pmatrix} - & - & - & \cdots \\ 1 & 1 & 1 & \cdots \\ - & - & - & \cdots \end{pmatrix} \end{matrix} = \frac{1}{2} \begin{matrix} * \\ \begin{pmatrix} - & - & - & \cdots \\ 1 & 2 & 2 & \cdots \\ - & - & - & \cdots \end{pmatrix} \end{matrix} + \frac{1}{2} \begin{matrix} * \\ \begin{pmatrix} - & - & - & \cdots \\ 1 & - & - & \cdots \\ - & - & - & \cdots \end{pmatrix} \end{matrix}.$$

### 3.1 Quadric hypersurfaces of embedding dimension 2

Set  $S = \mathbb{k}[x, y]$  and  $R = S/(q)$  for a quadric  $q$  in  $S$ . In this section, we give a full description of the cone of Betti diagrams of  $R$ -modules and include a proof of Theorem 3.0.12.

**Definition 3.1.1.** We say that

$$d = (d_0, d_1, d_2, \dots) \in \prod_{i \in \mathbb{N}} (\mathbb{Z} \cup \{\infty\})$$

is an  ***$R$ -degree sequence*** if it has the form

- i.  $d = (d_0, \infty, \infty, \infty, \dots)$  with  $d_0 < \infty$ ,
- ii.  $d = (d_0, d_1, \infty, \infty, \dots)$  with  $d_0 < d_1 < \infty$ , or
- iii.  $d = (d_0, d_1, d_1 + 1, d_1 + 2, d_1 + 3, d_1 + 4, \dots)$  with  $d_0 < d_1 < \infty$ .

We define a partial order  $\leq$  on  $R$ -degree sequences as follows. We do a termwise comparison on the first two entries; in the case of a tie, we then do a termwise

comparison on the remaining entries. In other words, for two  $R$ -degree sequences  $d, d'$  we say that  $d \leq d'$  if and only if either

- $d_0 \leq d'_0$  and  $d_1 \leq d'_1$ , with one of these inequalities being strict, or
- $d_0 = d'_0$ ,  $d_1 = d'_1$ , and  $d_n \leq d'_n$  for all  $n \geq 2$ .

Definition 3.1.1 leads to a decomposition algorithm (see Section 3.3) and fits into the framework of [BEKS10].

Recall that the  $\mathbb{Q}$ -vector space  $\mathbb{V}$  is the set of column-finite matrices with columns indexed by  $i \in \mathbb{Z}_{\geq 0}$  and rows indexed by  $j \in \mathbb{Z}$ . For each  $R$ -degree sequence  $d$ , we define a matrix  $\pi_d \in \mathbb{V}$  as follows. Define

$$(\pi_d)_{j,i} = \begin{cases} 1 & \text{if } j = d_i - i \neq \infty \text{ and } 0 \leq i \leq 1, \\ 2 & \text{if } j = d_i - i \neq \infty \text{ and } i \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 3.1.2.** Three degree sequences and their corresponding Betti diagrams appear below.

$$\pi_{(0,\infty,\dots)} = \begin{matrix} * \\ \begin{pmatrix} \vdots & \vdots \\ - & - & \dots \\ 1 & - & \dots \\ - & - & \dots \\ - & - & \dots \\ \vdots & \vdots \end{pmatrix} \end{matrix} \quad \pi_{(1,2,\infty,\dots)} = \begin{matrix} * \\ \begin{pmatrix} \vdots & \vdots & \vdots \\ - & - & - & \dots \\ - & - & - & \dots \\ 1 & 1 & - & \dots \\ - & - & - & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} \end{matrix} \quad \pi_{(0,3,4,5,\dots)} = \begin{matrix} * \\ \begin{pmatrix} \vdots & \vdots & \vdots \\ - & - & - & \dots \\ 1 & - & - & \dots \\ - & - & - & \dots \\ - & 2 & 2 & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} \end{matrix}$$

We define functionals on  $v \in \mathbb{V}$  as follows:

$$\begin{aligned}\epsilon_{j,i}(v) &= v_{j,i}, \\ \alpha_k(v) &= \epsilon_{k,1}(v) - \epsilon_{k+1,2}(v), \quad \text{and} \\ \gamma_k(v) &= \sum_{j \leq k} (2\epsilon_{j,0}(v) - 2\epsilon_{j+1,1}(v) + \epsilon_{j+2,2}(v)).\end{aligned}$$

Observe that the functional  $\gamma_k$  is well-defined for any  $v \in \mathbb{V}$  because  $v$  is column-finite.

**Example 3.1.3.** The functional  $\gamma_2$  applied to a Betti diagram  $\beta(M)$  is given by taking the dot product of  $\beta(M)$  with the following diagram:

$$* \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \\ 2 & -2 & 1 & 0 & \cdots \\ 2 & -2 & 1 & 0 & \cdots \\ 2 & -2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

**Theorem 3.1.4.** *The following cones in  $\mathbb{V}$  are equal:*

- i. *The cone  $B_{\mathbb{Q}}(R)$  spanned by the Betti diagrams of all finitely generated  $R$ -modules.*
- ii. *The cone  $D$  spanned by  $\{\pi_d \mid d \text{ is an } R\text{-degree sequence}\}$ .*
- iii. *The cone  $F$  defined to be the intersection of the halfspaces*
  - a)  $\{\epsilon_{j,i} \geq 0\}$  *for all  $i \geq 0$  and  $j = 0$  or  $2$ ;*
  - b)  $\{\alpha_k \geq 0\}$  *for all  $k \in \mathbb{Z}$ ;*
  - c)  $\{\gamma_k \geq 0\}$  *for all  $k \in \mathbb{Z}$ ;*



d)  $\{\pm(\epsilon_{j,i} - \epsilon_{j+1,i+1}) \geq 0\}$  for  $i \geq 2, j \in \mathbb{Z}$ .

To prove Theorem 3.1.4, we show the inclusions  $D \subseteq B_{\mathbb{Q}}(R) \subseteq F \subseteq D$  which are contained in Lemmas 3.1.5, 3.1.6, and 3.1.8, respectively. The proof of Lemma 3.1.5 is straightforward, and the proof of Lemma 3.1.8 largely follows the techniques involving convex polyhedral geometry from [BS08]. By contrast, the proof of Lemma 3.1.6 requires new ideas. In particular, we use a construction due to J. Shamash in [Sha69] that constructs a (not necessarily minimal)  $R$ -free resolution from an  $S$ -free resolution; see also [Eis80, Section 7]. We briefly recall this construction now.

Let  $G_{\bullet}$  be a graded free  $S$ -resolution of an  $R$ -module  $M$  (recall that  $S = \mathbb{k}[x, y]$ ) Since multiplication by  $q$  is nullhomotopic on  $G_{\bullet}$ , there are homotopy maps  $s_1, s_2$ :

$$\begin{array}{ccccccc} 0 & \longleftarrow & G_0(-2) & \longleftarrow & G_1(-2) & \longleftarrow & G_2(-2) \longleftarrow 0 \\ & & \downarrow q & & \searrow^{s_1} & \downarrow q & \searrow^{s_2} & \downarrow q \\ 0 & \longleftarrow & \bar{G}_0 & \longleftarrow & \bar{G}_1 & \longleftarrow & \bar{G}_2 \longleftarrow 0. \end{array}$$

Now set  $\bar{G}_i = G_i \otimes R$ ,  $\bar{\partial}_i = \partial_i \otimes R$ , and  $\bar{s}_i = s_i \otimes R$  for  $i = 1, 2$ . The resulting complex

$$\begin{array}{ccccccc} 0 & \longleftarrow & \bar{G}_0 & \xleftarrow{\bar{\partial}_1} & \bar{G}_1 & \xleftarrow{\begin{pmatrix} \bar{\partial}_2 & \bar{s}_1 \end{pmatrix}} & \bar{G}_2 & \xleftarrow{\begin{pmatrix} \bar{s}_2 \\ \bar{\partial}_1 \end{pmatrix}} & \bar{G}_1(-2) & \xleftarrow{\begin{pmatrix} \bar{\partial}_2 & \bar{s}_1 \end{pmatrix}} & \bar{G}_2(-2) & \xleftarrow{\dots} & \dots \\ & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\ & & & & \bar{G}_0(-2) & & \bar{G}_0(-4) & & & & & & \end{array}$$

is an  $R$ -free resolution of  $M$ . Note that there are additional maps  $G_i \rightarrow G_{i+2d-1}$  in the construction given in [Sha69, Section 3]. These maps are 0 in our context because  $G_i = 0$  when  $i \geq 3$ .

**Lemma 3.1.5.** *There is an inclusion  $D \subseteq B_{\mathbb{Q}}(R)$ .*

**Proof.** It suffices to show that, for each  $R$ -degree sequence  $d$ , there exists an  $R$ -

module  $M$  with  $\beta(M) = \pi_d$ . If  $d = (d_0, \infty, \dots)$ , we simply choose  $M = R(-d_0)$ . For the other cases, fix  $\ell$ , a linear form not a scalar multiple of  $x$ , that is a nonzero divisor on  $R$  (i.e.,  $\ell$  does not divide  $q$ ). Such an  $\ell$  exists in any characteristic. If  $d = (d_0, d_1, \infty, \dots)$ , we set  $M = R(-d_0)/(\ell^{d_1-d_0})$ .

Finally, if  $d = (d_0, d_1, d_1+1, d_1+2, \dots)$ , we set  $M$  to be  $R(-d_0)/(\ell^{d_1-d_0}, x\ell^{d_1-d_0-1})$ . To see that  $M$  has the desired Betti diagram, we first consider the minimal  $S$ -free resolution. By hypothesis  $q, \ell^{d_1-d_0}, x\ell^{d_1-d_0-1}$  are a minimal set of generators in  $S$ . Applying the Hilbert-Burch theorem, see e.g. [Eis95, 20.15], the  $S$ -free resolution of  $M$  has the form:

$$\begin{array}{ccccccc}
& & & & S(-d_0 - 2) & & \\
& & & & \oplus & & \\
0 & \longleftarrow & S(-d_0) & \xleftarrow{\partial_1} & & \longleftarrow & S(-d_1 - 1)^2 \longleftarrow 0. \\
& & & & \oplus & & \\
& & & & S(-d_1)^2 & & 
\end{array}$$

where  $\partial_1 = \begin{bmatrix} q, & \ell^{d_1-d_0}, & x\ell^{d_1-d_0-1} \end{bmatrix}$ . We fix homotopies  $s_1, s_2$  for multiplication by  $q$  on this resolution:

$$\begin{array}{ccccccc}
& & & & S(-d_0 - 4) & & \\
& & & & \oplus & & \\
0 & \longleftarrow & S(-d_0 - 2) & \longleftarrow & & \longleftarrow & S(-d_1 - 3)^2 \longleftarrow 0 \\
& & \downarrow q & \searrow s_1 & \downarrow q & \searrow s_2 & \downarrow q \\
& & & & S(-d_1 - 2)^2 & & \\
& & & & \oplus & & \\
& & & & S(-d_0 - 2) & & \\
0 & \longleftarrow & S(-d_0) & \xleftarrow{\partial_1} & & \longleftarrow & S(-d_1 - 1)^2 \longleftarrow 0. \\
& & & & \oplus & & \\
& & & & S(-d_1)^2 & & 
\end{array}$$

Since  $\ell$  does not divide  $q$  we see that the component of  $s_1$  that maps  $S(-d_0 - 2)$  to  $S(-d_0 - 2)$  must be 1. By degree considerations, the maps  $s_1$  and  $s_2$  cannot contain

any other unit entries.

The standard resolution of  $M$  is now given by

$$\begin{array}{ccccccc}
& & R(-d_0 - 2) & & R(-d_1 - 1)^2 & & R(-d_0 - 4) \\
0 \longleftarrow & R(-d_0) \longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow \cdots . \\
& & R(-d_1)^2 & & R(-d_0 - 2) & & R(-d_1 - 2)^2 & 
\end{array}$$

The maps  $R(-d_0 - 2n) \leftarrow R(-d_0 - 2n)$  are the only nonminimal part of this resolution.

It follows that  $M$  has a minimal free  $R$ -resolution of the form

$$0 \longleftarrow R(-d_0) \longleftarrow R(-d_1)^2 \longleftarrow R(-d_1 - 1)^2 \longleftarrow R(-d_1 - 2)^2 \longleftarrow \cdots ,$$

which yields the desired Betti diagram. □

**Lemma 3.1.6.** *There is an inclusion  $B_{\mathbb{Q}}(R) \subseteq F$ .*

**Proof.** Fix a finitely generated graded  $R$ -module  $M$ . We must show that the inequalities defining  $F$  are nonnegative on  $\beta(M)$ . Certainly  $\epsilon_{j,i}(\beta(M)) = \beta_{i,j}(M) \geq 0$  for all  $i$  and  $j$ , completing case (iiia). For case (iiid), the minimal resolution of  $M$  is given by a matrix factorization after at most two steps by [Eis80, Theorem 4.1]. By [Eis95, Lemma 20.11],  $\text{Syz}_2(M)$  has depth 2 and is thus maximal Cohen–Macaulay. After extending the base field to its algebraic closure (which does not affect Betti diagrams), the homogeneous quadric  $q$  is, without loss of generality, either  $x^2$  or  $xy$ . The matrix factorizations of these quadrics over an algebraically closed field are classified (see [Yos90, Example 6.5 and p. 76]). Thus the resolution of  $M$  after at most 2 steps is given by one of the matrix factorizations above; one easily checks for these that (iiid) hold.

For case (iiib), we show that  $\alpha_k(\beta(M)) \geq 0$  by showing that it measures the rank

of a map. Fix a minimal free  $S$ -resolution  $G_\bullet$  of  $M$  as above, and let  $s_1$  and  $s_2$  denote the homotopies occurring in the standard resolution of  $M$  over  $R$ . Let  $\beta_{i,j}^S(M)$  denote the  $i, j$ -th graded Betti numbers of  $M$  over  $S$ . Let  $\sigma_{i,j}$  be the composition of the maps

$$\sigma_{i,j}: S(-j)^{\beta_{i-1,j-2}^S(M)} \hookrightarrow G_{i-1}(-2) \xrightarrow{s_i} G_i \rightarrow S(-j)^{\beta_{i,j}^S(M)}.$$

With the chosen basis, the entries of  $\sigma_{i,j}$  have degree 0, so  $\sigma_{i,j}$  is a matrix of elements of  $\mathbb{k}$ . We claim that

$$\alpha_k(\beta(M)) = \beta_{1,k}(M) - \beta_{2,k+1}(M) = \text{rank } \sigma_{2,k} \geq 0.$$

It follows from this construction that

$$\begin{aligned} \beta_{1,k}(M) &= \beta_{1,k}^S(M) - \text{rank } \sigma_{1,k} \\ \text{and } \beta_{3,k+2}(M) &= \beta_{1,k}^S(M) - \text{rank } \sigma_{1,k} - \text{rank } \sigma_{2,k}. \end{aligned}$$

Thus  $\beta_{1,k}(M) - \beta_{3,k+2}(M) = \text{rank } \sigma_{2,k}$ . As noted above in the proof of (iiid),  $\beta_{2,k+1}(M) = \beta_{3,k+2}(M)$ , which yields the claim.

Finally, for case (iiic), or  $\gamma_k$ , let  $\phi_1: F_1 \rightarrow F_0$  be a minimal presentation of  $M$  over  $R$  and set

$$F'_0 = \bigoplus_{j \leq k} R(-j)^{\beta_{0,j}(M)} \quad \text{and} \quad F'_1 = \bigoplus_{j \leq k+1} R(-j)^{\beta_{1,j}(M)}.$$

There are natural split inclusions  $F'_0 \subseteq F_0$  and  $F'_1 \subseteq F_1$ . In particular,  $\phi_1$  induces a map  $\phi'_1: F'_1 \rightarrow F'_0$ . We set  $N = \text{coker}(\phi'_1)$ , and note that  $\phi'_1$  is a minimal presentation of  $N$ . As such,  $\beta_{0,j}(M) = \beta_{0,j}(N)$  for all  $j \leq k$ , and  $\beta_{1,j}(M) = \beta_{1,j}(N)$  for all  $j \leq k+1$ . In addition, we claim that  $\beta_{2,j}(M) = \beta_{2,j}(N)$  for all  $j \leq k+2$ . To see this,

consider the diagram

$$\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Syz}_2(N) & \longrightarrow & F'_1 & \longrightarrow & F'_0 & \longrightarrow & N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Syz}_2(M) & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M \longrightarrow 0,
\end{array}$$

where we view  $\text{Syz}_2(N)$ ,  $\text{Syz}_2(M)$  as submodules of  $F'_1, F_1$  respectively. By the Snake Lemma,  $\text{Syz}_2(N)$  is a submodule of  $\text{Syz}_2(M)$ . For a fixed basis of  $F_1$ , any element of  $\text{Syz}_2(M)$  may be written as a linear combination of the basis elements with coefficients in  $R_{\geq 1}$ . Thus for an element  $x \in \text{Syz}_2(M)$  of degree  $j$  with  $j \leq k+2$ , we see that the basis elements whose corresponding coefficients are nonzero in a decomposition of  $x$  have degree at most  $j-1 \leq k+1$ . In particular, these basis elements are in  $F'_1$ , and hence  $\text{Syz}_2(N)_j = \text{Syz}_2(M)_j$  for all  $j \leq k+2$ , which implies the claim.

By the definition of  $\gamma_k$ , we have now shown that  $\gamma_k(\beta(M)) = \gamma_k(\beta(N))$ . It thus suffices to show that  $\gamma_k(\beta(N)) \geq 0$ . We achieve this by showing that  $\gamma_k(\beta(N)) = h_{k+2}(N)$ , where  $h_{k+2}(N)$  denotes the Hilbert function of  $N$  in degree  $k+2$ .

The Hilbert function of  $N$  can be computed entirely in terms of  $\beta(N)$ :

$$\begin{aligned}
h_{k+2}(N) &= \sum_{j \in \mathbb{Z}} \sum_{i=0}^{\infty} (-1)^i \beta_{i,j}(N) h_{k+2}(R(-j)) \\
&= \sum_{j \in \mathbb{Z}} \sum_{i=0}^{\infty} (-1)^i \beta_{i,j}(N) h_{k+2-j}(R) \\
&= \sum_{\ell \in \mathbb{Z}} \sum_{i=0}^{\infty} (-1)^i \beta_{i,i+\ell}(N) h_{k+2-i-\ell}(R).
\end{aligned}$$

Since  $\beta_{0,j}(N) = 0$  for  $j > k$ ,  $\beta_{1,j}(N) = 0$  for  $j > k + 1$ , and  $h_i(R) = 2$  for all  $i > 0$ , we have that

$$\begin{aligned} h_{k+2}(N) &= \sum_{\ell \leq k} \sum_{i=0}^{\infty} (-1)^i \beta_{i,i+\ell}(N) h_{k+2-i-\ell}(R) \\ &= \sum_{\ell \leq k} \left( \sum_{i=0}^{k+1-\ell} (-1)^i \beta_{i,i+\ell}(N) \cdot 2 \right) + (-1)^{k+2-\ell} \beta_{k+2-\ell,k+2}(N) \cdot 1. \end{aligned}$$

By applying (iiid) twice, we see that  $\beta_{i,j}(N) = \beta_{i+2,j+2}(N)$  for all  $i \geq 2$ . Using this to cancel, we obtain

$$h_{k+2}(N) = \sum_{\ell \leq k} (2\beta_{0,\ell}(N) - 2\beta_{1,\ell+1}(N) + \beta_{2,\ell+2}(N)) = \gamma_k(\beta(N)).$$

□

For the final inclusion in the proof of Theorem 3.1.4, we compare the cone  $D$  (which is defined in terms of extremal rays) and the cone  $F$  (which is defined in terms of halfspaces). As we see in Lemma A.0.3, it is easier to move between these two descriptions in the case of a simplicial fan, so we first construct a simplicial fan  $\Sigma$  whose support is contained in  $D$ .

**Lemma 3.1.7.** *For every finite chain  $C$  of  $R$ -degree sequences, the cone*

$$\text{pos}(C) = \mathbb{Q}_{\geq 0}\{\pi_d \mid d \in C\}$$

*is simplicial. The collection of these simplicial cones forms a simplicial fan.*

**Proof.** The diagrams  $\pi_d$  from any finite chain  $C$  are linearly independent. This follows from the fact that for any degree sequence  $d$ ,  $\pi_d$  has a nonzero entry in a position such that, for every degree sequence  $d'$  in the chain  $C$  with  $d < d'$ ,  $\pi_{d'}$  has a

zero in the corresponding position.

For the second statement, we need to show that these cones meet along faces. Using the observation above, the proof of [BS08, 2.9] applies directly to our situation.

□

**Lemma 3.1.8.** *There is an inclusion  $F \subseteq D$ .*

**Proof.** Let  $\Sigma$  be the simplicial fan constructed in Lemma 3.1.7, and let  $\text{supp}(\Sigma)$  denote its support, as defined in Appendix A. By construction,  $\text{supp}(\Sigma) \subseteq D$ , so it suffices to prove that  $F \subseteq \text{supp}(\Sigma)$ .<sup>1</sup> Now, we have a simplicial fan  $\Sigma$  defined in terms of extremal rays, and we seek to determine its boundary halfspaces, as defined in Appendix A. Then to prove the Lemma it will be enough to show that each of the boundary halfspaces of  $\Sigma$  is contained in the list of halfspaces defining  $F$ .

In order to apply Lemma A.0.3, we first reduce to the case of a full-dimensional, equidimensional simplicial fan in a finite dimensional vector space. For each  $m \in \mathbb{Z}_{\geq 0}$ , define the subspace  $\mathbb{V}_m$  of  $\mathbb{V}$  to be

$$\mathbb{V}_m = \{v \in \mathbb{V} \mid v_{j,i} = 0 \text{ unless } -m + i \leq j \leq m + i\}.$$

Note that  $\mathbb{V}_m$  contains the Betti diagram of any module with generators in degrees at least  $-m$  and with regularity at most  $m$ .

Set  $\Sigma_m = \Sigma \cap \mathbb{V}_m$  and  $F_m = F \cap \mathbb{V}_m$ . Observe that

$$\Sigma_m = \{\text{pos}(C) \mid C \text{ is a chain in } \mathbf{P}_m\},$$

where  $\mathbf{P}_m = \{\text{degree sequences } d \mid \pi_d \in \mathbb{V}_m\}$ , so by Lemma 3.1.7,  $\Sigma_m$  is a simplicial

---

<sup>1</sup>A priori,  $\text{supp}(\Sigma)$  is a (not necessarily convex) subcone of  $D$ ; the proof of Theorem 3.2.4 implies that  $\text{supp}(\Sigma) = D$ .

fan. Since  $\mathbb{V} = \bigcup_{m \geq 0} \mathbb{V}_m$ , it is enough to show that  $F_m \subseteq \text{supp}(\Sigma_m)$  for all  $m \geq 0$ .

Next, we define the finite dimensional vector space

$$\overline{\mathbb{V}}_m = \{v \in \mathbb{V}_m \mid v_{j,i} = 0 \text{ unless } i \leq 2\},$$

and consider the projection  $\Phi_m: \mathbb{V}_m \rightarrow \overline{\mathbb{V}}_m$ . Since every pure diagram  $\pi_d$  satisfies the functional of type (iiid) in the definition of  $F$ , it follows that  $\Phi_m$  induces an isomorphism of  $\Sigma_m$  onto its image, which we denote by  $\overline{\Sigma}_m$ . There is also an isomorphism of  $F_m$  onto its image  $\overline{F}_m$ , since the defining halfspaces of  $F_m$  contain (iiid). It thus suffices to show that  $\overline{F}_m \subseteq \text{supp}(\overline{\Sigma}_m)$  for all  $m$ .

We claim that  $\overline{\Sigma}_m$  is  $(\dim \overline{\mathbb{V}}_m)$ -equidimensional. Every maximal chain of degree sequences in  $\mathbf{P}_m$  begins with  $(-m, -m+1, m+n, \dots)$  and ends with  $(m, \infty, \infty, \dots)$ . For a fixed maximal chain  $C$ , there is a unique  $m' \leq m$  such that  $C$  is

$$\begin{aligned} (-m, -m+1, -m+n, \dots) &< \dots < (m', m+1, m'+n, \dots) \\ &< (m', \infty, \infty, \dots) < \dots < (m, \infty, \infty, \dots). \end{aligned} \quad (3.1.A)$$

From this observation, it follows that

$$|C| = ((m+m)') + 2(2m+1) + (m-m'+1) = 6m+3.$$

Since the set  $\{\pi_d\}$  is linearly independent for  $d \in C$  by Lemma 3.1.7, these diagrams form a basis of  $\overline{\mathbb{V}}_m$ . It follows that  $\overline{\Sigma}_m$  is a  $(\dim \overline{\mathbb{V}}_m)$ -equidimensional simplicial fan.

We now record a collection of supporting halfspaces which define  $\overline{F}_m$ :

- i.  $\{\epsilon_{j,i} \geq 0\}$  for all  $i \geq 0, j \in \mathbb{Z} \cap [-m+i, m+i]$ ;
- ii.  $\{\alpha_k \geq 0\}$  for all  $k \in \mathbb{Z} \cap [-m, m]$ ;



iii.  $\{\gamma_{m,k} \geq 0\}$  for all  $k \in \mathbb{Z} \cap [-m, m]$ , where for  $k \in \mathbb{Z} \cap [-m, m]$  we set

$$\gamma_{m,k} = \sum_{j=-m}^k (2\epsilon_{j,0} - 2\epsilon_{j+1,1} + \epsilon_{j+2,2}).$$

To complete the proof, we show that each boundary halfspace of  $\bar{\Sigma}_m$  corresponds to a supporting halfspace of  $\bar{F}_m$ . By Lemma A.0.3, each boundary halfspace of  $\bar{\Sigma}_m$  is determined by (at least one) boundary facet, and hence is determined by some submaximal chain in the poset  $\mathbf{P}_m$  that is uniquely extended to a maximal chain. The proof of [BS08, Proposition 2.12] applies in our context, showing that each boundary halfspace of  $\bar{\Sigma}_m$  depends on only a small part of any submaximal chain to which it corresponds. Namely, such a halfspace is determined by the unique  $R$ -degree sequence  $d$  that extends a corresponding chain to a maximal one, along with its two neighbors  $d' < d''$  in this extended chain, if they exist. We write this data as  $\cdots < d' < d^\wedge < d'' < \cdots$ . By direct inspection of  $\mathbf{P}_m$  (see Figure 3.1 for the case  $m = 1$ ), the submaximal chains that can be uniquely extended are of the following forms:

(a)  $\cdots < d' < d^\wedge < d'' < \cdots$ , where  $d'$  and  $d''$  have projective dimension 1 and either  $d''_0 - d'_0 = 1$  or  $d''_1 - d'_1 = 1$  (but not both). For instance,

$$\cdots < (0, 1, \infty, \infty, \dots) < (0, 2, 3, 4, \dots)^\wedge < (0, 2, \infty, \infty, \dots) < \cdots$$

(b)  $\cdots < d' < d^\wedge < d'' < \cdots$ , where  $d'$  and  $d''$  have infinite projective dimension and either  $d''_0 - d'_0 = 1$  or  $d''_1 - d'_1 = 1$  (but not both). For instance,

$$\cdots < (-1, 1, 2, 3, \dots) < (-1, 1, \infty, \infty, \dots)^\wedge < (-1, 2, 3, 4, \dots) < \cdots$$

(c)  $\cdots < (d'_0, d'_0 + 1, \infty, \infty, \dots) < d^\wedge < (d'_0 + 1, d'_0 + 2, d'_0 + 3, d'_0 + 4, \dots) < \cdots$ . For

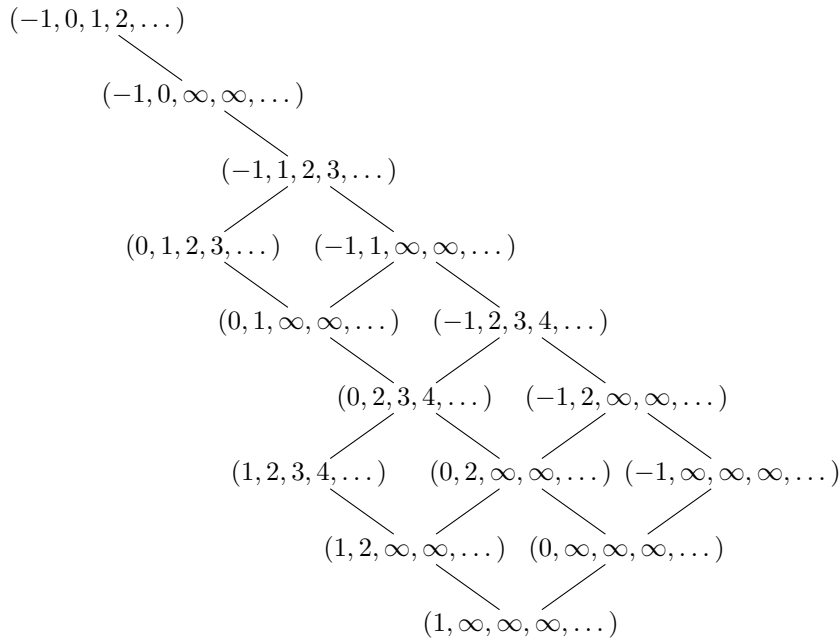


Figure 3.1: The poset of degree sequences whose Betti diagrams lie in  $\mathbb{V}_1$ , where  $d > d'$  if there is an downward path from  $d'$  to  $d$ .

instance,

$$\cdots < (0, 1, \infty, \infty, \dots) < (0, 2, 3, 4, \dots)^\wedge < (1, 2, 3, 4, \dots) < \cdots .$$

(d)  $\cdots < d' < d^\wedge < d'' < \cdots$ , where  $d'$  and  $d''$  differ by two in the first entry. For

instance,

$$\cdots < (-1, 2, 3, 4, \dots) < (0, 2, 3, 4, \dots)^\wedge < (1, 2, 3, 4) < \cdots \text{ or}$$

$$\cdots < (1, \infty, \infty, \infty, \dots) < (2, \infty, \infty, \infty, \dots)^\wedge < (3, \infty, \infty, \infty, \dots) < \cdots .$$

(e)  $\cdots < (d_0, m+1, m+2, m+3, \dots) < (d_0, m+1, \infty, \infty, \dots)^\wedge < (d_0, \infty, \infty, \infty) <$

$\dots$ , for instance

$$\dots < (0, 2, 3, 4) < (0, 2, \infty, \infty)^\wedge < (0, \infty, \infty, \infty) < \dots$$

- (f)  $\dots < d' < d^\wedge$ , where  $d' = (m, m+1, \infty, \infty, \dots)$  and  $d = (m, \infty, \infty, \infty, \dots)$  is the maximal element of its chain.
- (g)  $\dots < d' < d^\wedge$ , where  $d' = (m-1, \infty, \infty, \infty, \dots)$  and  $d = (m, \infty, \infty, \infty, \dots)$  is the maximal element of its chain.
- (h)  $d^\wedge < d'' < \dots$ , where  $d$  is the minimal element of its chain.

We can show on a case by case basis that each boundary halfspace of  $\bar{\Sigma}_m$  (as determined by a submaximal chain from the list above) corresponds to one of the halfspaces defining  $\bar{F}_m$ ; we provide details for a portion of case (a). Consider a submaximal chain  $C$  of the form

$$\dots < (d_0, d_1 - 1, \infty, \dots) < (d_0, d_1, d_2, \dots)^\wedge < (d_0, d_1, \infty, \dots) < \dots,$$

where  $d_2 = d_1 + 1$ . Note that  $\epsilon_{d_2,2}^*(\pi_c) = 0$  for all  $c = (c_0, c_1, \dots) \in C$  because either  $c_2 < d_2$  or  $c_2 > d_2$ . This shows that  $\pi_c$  lies in the hyperplane  $\{\epsilon_{d_2,2}^* = 0\}$  for all  $c \in C$ . Since, in addition,  $\epsilon_{d_2,2}^*(\pi_{d_0,d_1,d_2,\dots}) = 1$ , it follows that  $C$  corresponds to the halfspace  $\{\epsilon_{0,d_0+1}^* \geq 0\}$ .

Using similar arguments, we see that a submaximal chain of type (a) or (h) corresponds to  $\{\epsilon_{d_2,2}^* \geq 0\}$ ; type (b) corresponds to  $\{\alpha_{d_1} \geq 0\}$  and type (e) corresponds to  $\{\alpha_{m+1} \geq 0\}$ ; type (c) or (f) to  $\{\gamma_{m,d'_0} \geq 0\}$ ; and finally, chains of types (d) and (g) correspond to  $\{\epsilon_{m,0}^* \geq 0\}$ .  $\square$

**Proof of Theorem 3.0.12.** Let  $E$  be the cone spanned by Betti diagrams of extremal modules of finite projective dimension and extremal modules of infinite projective dimension with the stated degree sequences. We see that  $D \subseteq E$  by Lemma 3.1.5, noting that the modules there are extremal of finite projective dimension or of infinite projective dimension with the correct degree sequence. Thus by Theorem 3.1.4, we have  $B_{\mathbb{Q}}(R) = D$ , as desired.  $\square$

## 3.2 Hypersurfaces of embedding dimension 1 and degree at least 2

Set  $S = \mathbb{k}[x]$  and  $R = S/(x^n)$  for some  $n \geq 2$ . In this section, we give a full description of the cone of Betti diagrams of  $R$ -modules, as well as its implications for the cone of Betti diagrams of maximal Cohen–Macaulay modules over any standard graded hypersurface ring.

**Definition 3.2.1.** We say that

$$d = (d_0, d_1, \dots) \in \prod_{i \in \mathbb{N}} (\mathbb{Z} \cup \{\infty\})$$

is an  *$R$ -degree sequence* if it has the form

- i.  $d = (d_0, \infty, \infty, \infty, \dots)$  or
- ii.  $d = (d_0, d_1, d_2, \dots)$ , where  $d_0 < d_1$  and  $d_{i+2} - d_i = n$  for all  $i \geq 0$ .

We define a partial order on  $R$ -degree sequences as follows: if  $d$  has finite projective dimension and  $d'$  has infinite dimension, then we say that  $d < d'$ ; otherwise, we use the termwise partial order.

Given an  $R$ -degree sequence  $d = (d_0, d_1, \dots)$ , we define a diagram  $\pi_d \in \mathbb{V}$  by

$$(\pi_d)_{j,i} = \begin{cases} 1 & \text{if } j = d_i - i \neq \infty, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 3.2.2.** If  $n = 3$ , then

$$\pi_{(0,\infty,\infty,\infty,\dots)} = * \begin{pmatrix} \vdots & \vdots & & & & & \\ - & - & \dots & & & & \\ * & 1 & - & \dots & & & \\ - & - & \dots & & & & \\ - & - & \dots & & & & \\ \vdots & \vdots & & & & & \end{pmatrix} \quad \text{and} \quad \pi_{(0,1,3,4,\dots)} = * \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & & \\ - & - & - & - & - & \dots & \\ * & 1 & 1 & - & - & - & \dots \\ - & - & 1 & 1 & - & \dots & \\ - & - & - & - & 1 & \dots & \\ - & - & - & - & - & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}.$$

To describe the cone  $B_{\mathbb{Q}}(R)$ , we define the following functionals on  $v \in \mathbb{V}$ :

$$\epsilon_{j,i}(v) = v_{j,i}, \quad \alpha_{k,i}(v) = \epsilon_{k,i}(v) - \epsilon_{k+n,i+2}(v), \quad \theta_k(v) = \sum_{j \leq k} \epsilon_{j,2}(v) - \sum_{j \leq k-n+1} \epsilon_{j,1}(v),$$

$$\text{and} \quad \eta_k(v) = \sum_{j \leq k} (\epsilon_{j,1}(v) - \epsilon_{j+1,2}(v)).$$

**Example 3.2.3.** The functional  $\eta_3$  applied to a Betti diagram  $\beta(M)$  is given by

taking the dot product of  $\beta(M)$  with the following diagram:

$$* \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & & & \\ 0 & 1 & -1 & 0 & 0 & \cdots & \\ 0 & 1 & -1 & 0 & 0 & \cdots & \\ 0 & 1 & -1 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & & & \end{pmatrix}.$$

**Theorem 3.2.4.** *The following cones in  $\mathbb{V}$  are equal:*

- i. The cone  $B_{\mathbb{Q}}(R)$  spanned by the Betti diagrams of all finitely generated  $R$ -modules.*
- ii. The cone  $D$  spanned by  $\pi_d$  for all  $R$ -degree sequences  $d$ .*
- iii. The cone  $F$  defined as the intersection of the halfspaces*
  - a)  $\{\epsilon_{j,i} \geq 0\}$  for  $i = 0, 1, 2$  and  $j \in \mathbb{Z}$ ;*
  - b)  $\{\alpha_{k,0} \geq 0\}$  for all  $k \in \mathbb{Z}$ ;*
  - c)  $\{\theta_k \geq 0\}$  for all  $k \in \mathbb{Z}$ ;*
  - d)  $\{\eta_k \geq 0\}$  for all  $k \in \mathbb{Z}$ ;*
  - e)  $\{\pm\alpha_{k,i} \geq 0\}$  for all  $i \geq 1, k \in \mathbb{Z}$ ;*
  - f)  $\{\pm\eta_{\infty} \geq 0\}$ .*

**Proof.** The equality  $B_{\mathbb{Q}}(R) = D$  follows from the structure theorem of finitely generated modules over principal ideal domains. Using extremal rays, it is also straightforward to check that  $B_{\mathbb{Q}}(R) \subseteq F$ . We complete the proof by showing that  $F \subseteq D$ .

For this final inclusion, note that the proof of Lemma 3.1.7 also holds in this context, so that  $\Sigma = \{\text{pos}(C) \mid C \text{ is a finite chain of } R\text{-degree sequences}\}$  is a sim-

plial fan; it suffices to prove that  $F \subseteq \text{supp}(\Sigma)$ . Let  $\bar{\mathbb{V}}$  denote the natural projection of  $\mathbb{V}$  that sends  $v \in \mathbb{V}$  to its first three columns, denoted  $v \mapsto \bar{v}$ . Denote the respective images of  $F$  and  $\Sigma$  under this map by  $\bar{F}$  and  $\bar{\Sigma}$ . For  $m \geq 0$ , let  $P_m = \{\text{degree sequences } d \mid \bar{\pi}_d \in \bar{\mathbb{V}}_m\}$ , and define  $\bar{\mathbb{V}}_m$ ,  $\bar{\Sigma}_m$ , and  $\bar{F}_m$  in a manner analogous to the proof of Lemma 3.1.8. Since every pure diagram  $\pi_d$  satisfies the functionals of types (iiie) and (iiif) in the definition of  $F$ , it now suffices to show that  $\bar{F}_m \subseteq \text{supp}(\bar{\Sigma}_m)$  for all  $m \geq 0$ . Note that  $\bar{F}_m$  and  $\text{supp}(\bar{\Sigma}_m)$  are both contained in the subspace  $\bar{\mathbb{W}}_m$  of  $\bar{\mathbb{V}}_m$  given by

$$\bar{\mathbb{W}}_m = \{\bar{v} \in \bar{\mathbb{V}}_m \mid \eta_\infty(\bar{v}) = 0 \text{ and } \bar{v}_{j,2} = 0 \text{ for } -m+2 \leq j < n-m.\}.$$

We thus view them as objects in there.

Direct computation shows that  $\bar{\Sigma}_m \subseteq \bar{\mathbb{W}}_m$  is a full-dimensional, equidimensional simplicial fan. To work towards  $\bar{F}_m \subseteq \text{supp}(\bar{\Sigma}_m)$ , note that defining halfspaces for  $\bar{F}_m \subseteq \bar{\mathbb{W}}_m$  are:

$$\begin{aligned} & \{\epsilon_{j,i} \geq 0\} \quad \text{for } i \in \{0, 1, 2\} \text{ and } j \in \mathbb{Z} \cap [-m+i, m+i], \\ & \{\alpha_{j,0} \geq 0\} \quad \text{for } j \in \mathbb{Z} \cap [-m, m+2-n], \\ & \left\{ \theta_{m,k} = \sum_{j=-m+2}^k \epsilon_{j,2} - \sum_{j=-m+1}^{k-n+1} \epsilon_{j,1} \geq 0 \right\} \quad \text{for } k \in \mathbb{Z} \cap [n-m-1, m+2], \quad \text{and} \\ & \left\{ \eta_{km} = \sum_{j=-m+1}^k (\epsilon_{j,1} - \epsilon_{j+1,2}) \geq 0 \right\} \quad \text{for } k \in \mathbb{Z} \cap [-m+1, m+1]. \end{aligned}$$

Each boundary halfspace of  $\bar{\Sigma}_m$  depends on certain submaximal chains given by data of the form  $\cdots < d' < \widehat{d''} < d'' < \cdots$ . Such submaximal chains take the following forms:

$$(a) \quad \cdots < (d_0, d_1, \dots) < (d_0 + 1, d_1, \dots)^\wedge < (d_0 + 2, d_1, \dots) < \cdots, \text{ where } d_1 < \infty;$$

- (b)  $\cdots < (d_0, d_1, \dots) < (d_0, d_1 + 1, \dots)^\wedge < (d_0, d_1 + 2, \dots) < \cdots$ , where  $d_1 < \infty$ ;
- (c)  $\cdots < (d_0, d_0 + 1, \dots) < (d_0, d_0 + 2, \dots)^\wedge < (d_0 + 1, d_0 + 2, \dots) < \cdots$ ;
- (d)  $\cdots < (d_0, d_0 + n - 1, \dots) < (d_0 + 1, d_0 + n - 1, \dots)^\wedge < (d_0 + 1, d_0 + n, \dots) < \cdots$ ;
- (e)  $\cdots < (m - n + 2, m, \dots) < (m - n + 2, m + 1, \dots)^\wedge < (-m, \infty, \dots) < \cdots$ ;
- (f)  $\cdots < (m - n + 2, m + 1, \dots) < (-m, \infty, \dots)^\wedge < (-m + 1, \infty, \dots) < \cdots$ ;
- (g)  $\cdots < (d_0, \infty, \dots) < (d_0 + 1, \infty, \dots)^\wedge < (d_0 + 2, \infty, \dots) < \cdots$ ;
- (h)  $(-m, -m + 1, \dots)^\wedge < (-m, -m + 2, \dots) < \cdots$ ;
- (i)  $\cdots < (m - 1, \infty, \dots) < (m, \infty, \dots)^\wedge$ .

One may now verify that the boundary halfspaces corresponding to these submaximal chains are, respectively:

- (a)  $\{\epsilon_{d_0+1+n,2} \geq 0\}$ ;
- (b)  $\{\epsilon_{d_1+1,1} \geq 0\}$ ;
- (c)  $\{\theta_{m,d_0+n} \geq 0\}$ ;
- (d)  $\{\eta_{d_0+n-1,m} \geq 0\}$ ;
- (e)  $\{\epsilon_{m+1,1} \geq 0\}$ ;
- (f)  $\{\alpha_{-m,0} \geq 0\}$ ;
- (g)  $\{\alpha_{d_0+1,0} \geq 0\}$ ;
- (h)  $\{\epsilon_{-m+1,1} \geq 0\}$ ;
- (i)  $\{\epsilon_{m,0} \geq 0\}$  if  $n > 2$  or  $\{\alpha_{m,0} \geq 0\}$  if  $n = 2$ .



As each of these halfspaces appears in our definition of  $\overline{F}_m$  above, we obtain  $F \subseteq D$ , as desired.  $\square$

As illustrated by the following corollary, Theorem 3.2.4 has implications for the study of Betti diagrams of maximal Cohen–Macaulay modules over any standard graded hypersurface ring.

**Corollary 3.2.5.** *Let  $T = \mathbb{k}[x_1, \dots, x_r]/(f)$  for any homogeneous  $f$  of degree at least 2, and let  $B_{\mathbb{Q}}^{MCM}(T)$  denote the cone of Betti diagrams of maximal Cohen–Macaulay  $T$ -modules. Then there is an inclusion*

$$B_{\mathbb{Q}}^{MCM}(T) \subseteq B_{\mathbb{Q}}(R),$$

where  $R = \mathbb{k}[x]/(x^{\deg(f)})$ . These cones are equal if  $\text{char}(\mathbb{k})$  does not divide  $\deg(f)$ .

**Proof.** Let  $n$  be the degree of the homogeneous polynomial  $f$ , so that  $R = \mathbb{k}[x]/(x^n)$ . Recall that  $T = \mathbb{k}[x_1, \dots, x_r]/(f)$ , and let  $M$  be a maximal Cohen–Macaulay  $T$ -module. To show that  $B_{\mathbb{Q}}^{MCM}(T) \subseteq B_{\mathbb{Q}}(R)$ , we find an  $R$ -module  $M'$  with the same Betti diagram.

We may assume  $\mathbb{k}$  is infinite by taking a flat extension. Then we find a sequence of  $M$ - and  $R$ -regular linear forms  $(\ell_1, \dots, \ell_{r-1})$ . Note that  $T/(\ell_1, \dots, \ell_{r-1}) \cong R$ . Applying [Avr98, Corollary 1.2.4], we see that  $\beta^T(M) = \beta^{T/(\ell_1, \dots, \ell_{r-1})}(M/(\ell_1, \dots, \ell_{r-1}))$ , as desired.

For the second statement, assume that  $(\deg f, \text{char}(\mathbb{k})) = 1$ . Since  $B_{\mathbb{Q}}(R) = D$ , it is enough to show that for each  $\pi_d \in D$ , there exists a maximal Cohen–Macaulay  $T$ -module  $M_d$  such that  $\beta^T(M_d) = \pi_d$ . If  $d = (d_0, \infty, \dots)$ , then  $\beta^T(T(-d_0)) = \pi_d$ .

Now consider the case that  $d = (d_0, d_1, d_0 + n, \dots)$ , where without loss of generality  $d_0 = 0$  and hence  $d_1 < n$ . In [BHS88], it is shown that there exists a matrix

factorization of  $f$  that can be decomposed into a product of  $n$  matrices of linear forms. Suppose  $A_1 A_2 \cdots A_n$  is such a decomposition. If  $M = \text{coker}(A_1 \cdots A_n)$  is presented by this matrix of  $(d_i)$ -forms, then it follows that  $\beta^T(M) = \pi_d$ . Hence  $B_{\mathbb{Q}}^{\text{MCM}}(T) = B_{\mathbb{Q}}(R)$ , as desired.  $\square$

### 3.3 Multiplicity conjectures and decomposition algorithms

In this section,  $R$  denotes a standard graded hypersurface rings of the form  $\mathbb{k}[x]/(x^n)$  for any  $n$  or  $\mathbb{k}[x, y]/(q)$ , where  $q$  is any homogeneous quadric. We first note that Theorem 3.0.13 follows from the proofs of Lemma 3.1.8 and Theorem 3.2.4, as they provide the desired simplicial structure.

This simplicial structure gives rise to a greedy decomposition algorithm of Betti diagrams into pure diagrams, as in [ES09, §1]. The key fact is that, since the cone  $B_{\mathbb{Q}}(R)$  is simplicial, for any module  $M$ , there is a finite chain of degree sequences  $d_1 < \dots < d_n$  such that  $\beta(M)$  is a positive linear combination of the  $\pi_{d_i}$ . And as noted in Lemma 3.1.7, the diagram  $\pi_{d_i}$  has a nonzero entry in a position in which, for all  $j > i$ ,  $\pi_{d_j}$  has a zero entry. We now present a detailed example to illustrate the algorithm.

**Example 3.3.1.** Let  $R = S/(x^2)$  and  $M = \text{coker} \begin{pmatrix} x & xy^2 & y^4 \\ 0 & y^3 & xy^3 \end{pmatrix}$ . Then we have

$$\beta(M) = \begin{pmatrix} * & 2 & 1 & 1 & 1 & \cdots \\ - & - & - & - & \cdots \\ - & 1 & - & - & \cdots \\ - & 1 & 1 & 1 & \cdots \end{pmatrix}.$$

We decompose  $\beta(M)$  by first considering the minimal  $R$ -degree sequence that could possibly contribute to  $\beta(M)$ , which is  $(0, 1, 2, 3, \dots)$ . We then subtract  $\frac{1}{2}\pi_{(0,1,2,3,\dots)}$ , as this is the largest multiple that can be removed while remaining inside  $B_{\mathbb{Q}}(R)$ .

This yields

$$\beta(M) - \frac{1}{2}\pi_{(0,1,2,3,\dots)} = \begin{matrix} * \\ \left( \begin{array}{cccc} \frac{3}{2} & - & - & - & \dots \\ - & - & - & - & \dots \\ - & 1 & - & - & \dots \\ - & 1 & 1 & 1 & \dots \end{array} \right) \end{matrix}.$$

We next subtract one copy of  $\pi_{(0,3,\infty,\infty,\dots)}$ , to obtain

$$\beta(M) - \frac{1}{2}\pi_{(0,1,2,3,\dots)} - \pi_{(0,3,\infty,\infty,\dots)} = \begin{matrix} * \\ \left( \begin{array}{cccc} \frac{1}{2} & - & - & - & \dots \\ - & - & - & - & \dots \\ - & - & - & - & \dots \\ - & 1 & 1 & 1 & \dots \end{array} \right) \end{matrix}.$$

Note that the remaining Betti diagram equals  $\frac{1}{2}\pi_{(0,4,5,6,\dots)}$ . In particular,  $\beta(M)$  lies in the face corresponding to the chain

$$(0, 1, 2, 3, \dots) < (0, 3, \infty, \infty, \dots) < (0, 4, 5, 6, \dots).$$

The existence of these simplicial structures also gives rise to  $R$ -analogues of the Herzog–Huneke–Srinivasan Multiplicity Conjectures. We say that an  $R$ -degree sequence  $d$  is **compatible** with a Betti diagram  $\beta(M)$  if  $\beta_{i,d_i}(M) \neq 0$  when  $d_i < \infty$ .

**Corollary 3.3.2.** *Let  $M$  be an  $R$ -module generated in a single degree. Let  $\underline{d} = (d_0, d_1, \dots)$  be the minimal  $R$ -degree sequence compatible with  $\beta(M)$ , and let  $\bar{d} = (\bar{d}_0, \bar{d}_1, \dots)$  be the maximal  $R$ -degree sequence compatible with  $\beta(M)$ .*

i. We have

$$e(M) \leq \beta_0(M) \cdot e(\pi_{\underline{d}}).$$

ii. If  $\overline{d}_1 < \infty$ , then

$$\beta_0(M) \cdot e(\pi_{\underline{d}}) \leq e(M) \leq \beta_0(M) \cdot e(\pi_{\overline{d}}),$$

with equality on either side if and only if  $\underline{d} = \overline{d}$ .

**Proof.** Since  $M$  is generated in a single degree, we may assume that  $\underline{d}_0 = 0$ . By Theorem 3.0.13, there is a unique chain  $\underline{d} = d^0 < d^1 < \dots < d^s = \overline{d}$  for which

$$\beta(M) = \sum_{i=0}^s a_i \pi_{d^i}. \quad (3.3.A)$$

If  $\overline{d} = (0, \infty, \infty, \dots)$ , then  $M$  has dimension 1 and  $e(M) = a_s e(\pi_{\overline{d}})$ . Since  $a_s \leq \beta_{0,0}(M)$ , this proves (i) in the case that  $\overline{d}_1 = \infty$ .

We now assume that  $\overline{d}_1 = \infty$ , and prove (ii), which implies (i) for this remaining case. We first compute the multiplicity of  $\pi_d$  for any  $R$ -degree sequence  $d$  of the form  $d = (0, d_1, d_2, d_3, \dots)$  with  $d_1 < \infty$ . We consider separately the cases  $\infty = d_2 = d_3 = \dots$  and  $d_i = d_1 + i - 1$  for all  $i \geq 2$ .

We may assume that  $\mathbb{k}$  is infinite by taking a flat extension. For the first case, we may assume after a possible change of coordinates that  $y$  is a nonzero divisor on  $R$ . Then the Betti diagram of  $R/(y^{d_1})$  equals  $\pi_{(0, d_1, \infty, \infty, \dots)}$ , and hence

$$e(\pi_{(0, d_1, \infty, \infty, \dots)}) = e((R/(y^{d_1}))) = 2d_1.$$

For the remaining case, the Betti diagram of  $R/(y^{d_1}, xy^{d_1-1})$  equals  $\pi_{(0, d_1, d_1+1, d_1+2, \dots)}$ ,

and hence

$$e(\pi_{(0,d_1,d_1+1,d_1+2,\dots)}) = e(R/(y^{d_1}, xy^{d_1-1})) = 2d_1 - 1.$$

Note that, since  $\overline{d_1} < \infty$ , every pure diagram  $\pi_{d^i}$  arising in the decomposition (3.3.A) satisfies  $d_0^i = 0$  and  $d_1^i < \infty$ . Therefore

$$e(\pi_{d^0}) < e(\pi_{d^1}) < \dots < e(\pi_{d^s}).$$

By convexity, this implies (ii). □

# Appendix A

## Convex geometry

The main results in this dissertation include descriptions of convex cones in a rational vector space. Here we provide some background on the relevant convex geometry. For a full and thorough course, we recommend [Zie95, Chapters 1,2,7].

Fix a  $\mathbb{Q}$ -vector space  $V$ . A subset  $C \subseteq V$  is a **convex cone** if it is closed under addition and multiplication by elements of  $\mathbb{Q}_{\geq 0}$ . For a subset  $B \subseteq V$ , the notation  $\text{pos}(B)$  refers to the **positive hull of  $B$** , defined as  $\text{pos}(B) := \{\sum_{b \in B} a_b b \mid a_b \in \mathbb{Q}_{\geq 0}\}$ , which is clearly a cone. A **ray** is the  $\mathbb{Q}_{\geq 0}$ -span of an element of  $V$ . A ray in a positive hull  $\text{pos}(B)$  is an **extremal ray** of  $\text{pos}(B)$  if it does not lie in  $\text{pos}(B \setminus \{b\})$ .

We say  $C$  is a  **$n$ -dimensional simplicial cone** if  $C = \text{pos}(B)$  for a set of  $n$

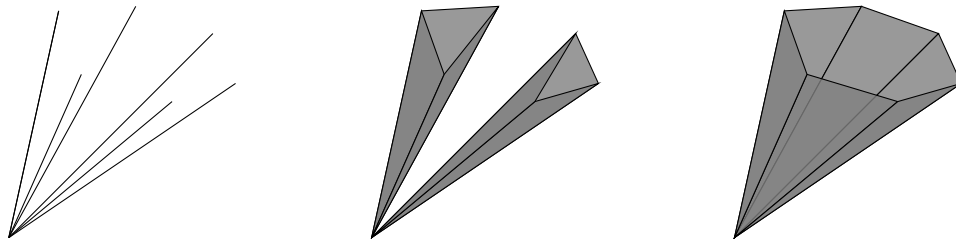


Figure A.1: Visual convex geometry. From left to right: a collection of rays, a simplicial fan containing those rays, the positive hull of the rays that is also the cone over the simplicial fan.

linearly independent vectors  $B$ . An  $m$ -**dimensional face** of  $C$  is a subset of the form  $\text{pos}(B')$ , for  $B'$  a subset of  $m$  vectors of  $B$ . A **facet** of  $C$  is an  $(n-1)$ -dimensional face.

A **simplicial fan**  $\Sigma$  is a collection of simplicial cones  $\{C_i\}$  such that  $C_i \cap C_j$  is a face of both  $C_i$  and  $C_j$  for all  $i, j$ . We refer to  $\bigcup_i C_i \subseteq V$  as the **support** of  $\Sigma$ , denoted  $\text{supp}(\Sigma)$ . We say that a subset  $\Sigma$  of  $V$  has the **structure of a simplicial fan** if  $\Sigma$  is the support of some simplicial fan.

A simplicial fan  $\Sigma$  that is a finite union of cones is  $m$ -**equidimensional** if each maximal cone has dimension  $m$ . A **facet** of an equidimensional fan is a facet of any maximal cone, and it is a **boundary facet** if it is contained in exactly one maximal cone.

If  $\dim V$  is finite and  $\Sigma$  is  $(\dim V)$ -equidimensional, then each boundary facet  $F$  determines a unique, up to scalar, functional  $L: V \rightarrow \mathbb{Q}$  such that  $L$  vanishes along  $F$  and is nonnegative on the (unique) maximal cone containing  $F$ ; we refer to the halfspace  $\{L \geq 0\}$  as a **boundary halfspace** of the fan.

Simplicial fan structures that come from posets arise several times in this dissertation. Let  $P$  be a poset and assume that there is a map  $\Phi: P \rightarrow V$  such that  $\Phi(p_1), \dots, \Phi(p_s)$  is linearly independent in  $V$  for all chains  $p_1 < \dots < p_s$  in  $P$  and such that the union of simplicial cones

$$\Sigma(P, \Phi) := \{\text{pos}(\{\Phi(p_1), \dots, \Phi(p_s)\}) \mid s \in \mathbb{Z}_{\geq 0} \text{ and } p_1 < \dots < p_s \text{ is a chain in } P\}$$

is a simplicial fan. When  $P$  is finite, this fan is referred to as a **geometric realization** of  $P$ . More generally, when a cone is the support of a simplicial fan arising from a poset, we say that the cone is **simplicial with respect to a poset**. In Chapter 2,  $P$  is the poset of realization modules and  $\Phi$  is the map  $M \mapsto \beta(M)$ . In Chapter 3,

$P$  is the poset of  $R$ -degree sequences, and  $\Phi$  is the map  $d \mapsto \pi_d \in \mathbb{V}$ . If  $\dim V$  is finite, then maximal cones of  $\Sigma(P, \Phi)$  are in bijection with maximal chains in  $P$ , and submaximal chains in  $P$  are in bijection with facets of  $\Sigma(P, \Phi)$ .

**Lemma A.0.3.** *Let  $V$  be an  $m$ -dimensional  $\mathbb{Q}$ -vector space,  $P$  be a finite poset,  $\Phi: P \rightarrow V$  as above, and  $\Sigma(P, \Phi)$  be an  $m$ -equidimensional simplicial fan. Then there is a bijective map:*

$$\left\{ \begin{array}{l} \text{submaximal chains of } P \text{ that} \\ \text{lie in a unique maximal chain of} \\ P \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{boundary} \\ \text{facets of} \\ \Sigma(P, \Phi) \end{array} \right\}$$

that is given by

$$p_1 < \cdots < p_{m-1} \mapsto \text{pos}(\{\Phi(p_1), \dots, \Phi(p_{m-1})\}).$$

In addition, since  $p_1 < \cdots < p_{m-1}$  lies in a unique maximal cone, there is a unique  $q \in P$  which extends this to a maximal chain. The boundary halfspace determined by this submaximal chain is the halfspace  $\{L \geq 0\}$ , where  $L(\Phi(p_i)) = 0$  and  $L(\Phi(q)) > 0$ . Though more than one submaximal chain may determine the same boundary halfspace, each boundary halfspace corresponds to at least one such chain.  $\square$

**Example A.0.4.** Let  $P$  be the poset from Figure 3.1. We continue with the notation of the proof of Lemma 3.1.8, letting  $D'_1$  be the simplicial fan on  $P$ . Since  $P$  has 12 maximal chains, it follows that  $D'_1$  is the union of 12 simplicial cones (of dimension 9). Consider the maximal chain corresponding to the lower left boundary of Figure 3.1: there are 7 submaximal chains that uniquely extend to this maximal chain. More precisely, there are respectively 0, 2, 2, 0, 1, 1, 0, 1 such submaximal chains of type (a)–(h).



Although simplicial fans are not necessarily convex, we can always construct a convex cone from a simplicial fan.

**Lemma A.0.5.** *Let  $V$  be an  $m$ -dimensional  $\mathbb{Q}$ -vector space, and let  $\Sigma$  be an  $m$ -equidimensional simplicial fan. Let  $\{\{L_k \geq 0\}\}$  be the set of boundary halfspaces of  $\Sigma$ . The convex cone  $\bigcap_k \{L_k \geq 0\}$  is a subset of the support of  $\Sigma$ .*

**Proof.** The arguments in the proof of Theorem 2.15 of [Zie95] show that  $\bigcap_k \{L_k \geq 0\}$  is the largest convex cone contained in the support of  $\Sigma$ .  $\square$

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