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Triangulation

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Triangulation
Expository Paper

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Part IB.
Triangulation

Map making has been a scientific endeavor for mankind since the beginning of recorded human history (ca. 5000 years ago) and it is today more sophisticated than ever before. In terms of trigonometric functions there is evidence that dates back to Babylonian times that angles and distances from points on a triangle were utilized in measurement with significant amounts of work in this field done by ancient Greeks, Indians, as well as Arabic mathematicians. For example the ancient Egyptians utilized the trigonometric functions for surveying properties in order to determine how much of their land had washed away when the Nile River would flood, additionally, the near perfect squareness and the north-south orientation of the Great Pyramid of Giza built c. 2700 BC, affirm the ancient Egyptians skills in surveying.

Historically, angles and distances were measured through the utilization of various devices. Distances were measured in ancient times with lengths based on the human body, and there were remarkably many different systems of measurement as a result. Eventually, one standard of measure that gained a universal nature was the Egyptian cubit. Distance measures became somewhat standardized and the surveying tapes were made out of more reliable materials. In order to measure the angles relatively crude compasses were utilized. Later in time, the compass with it’s variations due to location on the earth and magnetic declination was refined with more carefully scribed discs that that were capable of providing better angular resolution which allowed for more accurate mapping. Today mapping is a much more sophisticated process with the use of total stations. These measuring devices have made the shift from being an opticalmechanical device to being fully electronic with computer and software support. Thus, there has been a tremendous advancement in the mapping capabilities since the earliest attempts.

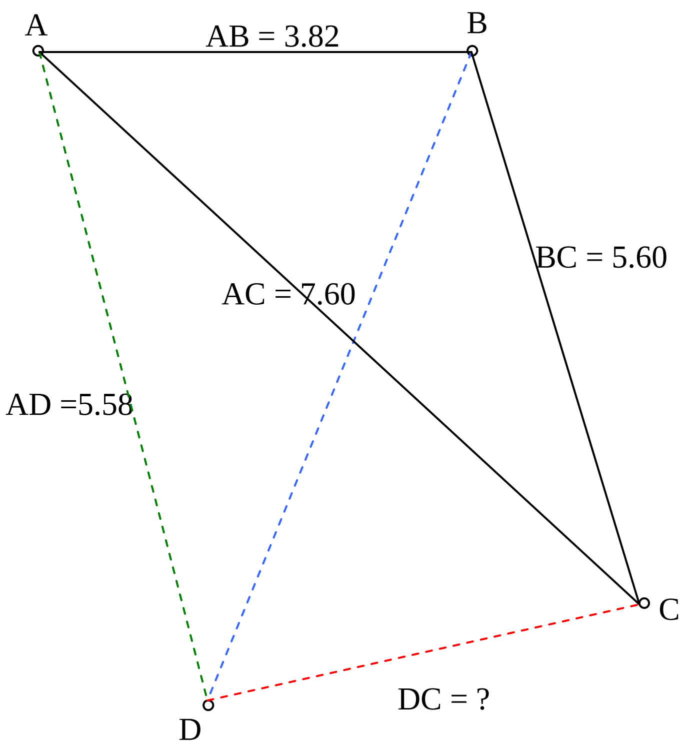
The Cassini family was an integral part of the advancement of mapping over the course of about 200 years (1650-1850) and they were experts at the triangulation method of mapping. Giovanni Cassini was born in Italy. He was the first of the famous Cassini family of astronomers and as such, is often known as Cassini I. He showed great intellectual curiosity and was especially interested in poetry, mathematics and astronomy during his schooling by the Jesuits in Italy. In 1650, Cassini I became professor of mathematics and astronomy at the University of Bologna. From his works, it can be seen that at this time Cassini believed in an Earth centered solar system, with comets coming from beyond Saturn but originating from the Earth. Observations lead him in 1659 to present an Earth centered system, with the moon and sun orbiting the Earth and the other planets orbiting the sun. Later he came to accept a version of the Copernican model. Beginning in 1664 Cassini I was able to observe with new powerful telescopes more of the solar system and with these instruments Cassini I made several new discoveries. One of his discoveries dealt with the speed of light. Had Cassini I not doubted his data, we would today know that the speed of light was related to the work of Cassini I. However, Cassini I was unable to accept his own ideas, and he soon rejected his ideas and looked for other explanations for the discrepancy. It is rather ironic that it was Cassini's data that was used by Römer in calculating the speed of light. Cassini's brilliant discoveries gave him an international reputation and led to him being invited to Paris by Louis XIV in

1668. Cassini I became head of the Paris Observatory in 1671 and he changed his views about returning to Italy and he became a French citizen.

One of his sons, Jacques was educated at the Paris Observatory and at the age of fourteen defended a thesis on optics. In 1694 Jacques began to work on projects with his father and like his father he was interested in astronomy. Additionally, Jacques was also interested in accurate map making. With his father he proposed a method to measure longitude by means of the eclipses of the stars and the planets by the moon. Unfortunately, Cassini I and II both believed that the Earth was elongated at the poles rather that flattened as suggested by Newton. Cassini II had several sons, one to be known later as Cassini III who continued the work of both his father and grandfather. Cassini III gained a tremendous amount of experience working with his father, although to his family's dismay, Cassini III found the data to oppose his family's view on the elongation of the earth at the poles. Cassini III also conducted at about this time the most accurate survey to have been carried out in France by setting up a large number of triangulation points based on the meridian through Paris. The basic principle that Cassini III utilized in his triangulation was the one describe by Gemma Frisius in the 16th century. This triangulation problem relies on the fact that if one triangle's angles and sides are known then successive triangles can be built. Each one of these triangles standing on the side of the previous triangle having a known side and if the angles of the new triangle can be accurately measured then one can successively build new triangles and as a result create an accurate map. Cassini III's idea to map France accurately was a tremendous undertaking that was estimated by Cassini III to take twenty years to complete, but because of numerous interruptions took over thirty years to complete. As a result, Cassini III who died of small pox did not see the mapping of France to completion. The project was completed by his son Dominique Cassini who had been working with his father for nearly ten years. After the death of his father, Cassini III, Dominique succeeded his father at the Paris Observatory and completed the mapping of France in 1790.

Today we have in memory of the Cassini family a space probe mapping the Saturn and the moons of Saturn. Notably, just today, new radar images from NASA's Cassini spacecraft revealed geological features similar to those found on Earth on Xanadu, an Australia-sized, region on the moon Titan. It is only fitting that today some 200 years later, Cassini is still mapping and involved in new discoveries in astronomy.

a) Given the quadrilateral ABCD below, we can use the given information, a ruler and a protractor and basic facts from plane geometry to double and half ∆ABC.



Triangle ABC can be doubled by folding A over B to create a new segment AB' that is twice the length of AB, by repeating this process for segments AC folding over C to create AC' doubles the length of segment AC creates a similar triangle that is twice as big. In order to do so the segments need to be extended through the use of a rule so that the doubled segments lie on the same line. One can halve the triangle by doing a similar process, but instead of folding to create a twice as big triangle, one will fold A to B to create a line segment half as long. At this point one could then fold point A to point C to create segment AC' that is half as long, this will create a triangle that is half the size of the original triangle ABC.

b) State the law of sines in the context of our triangle ABC and show how to compute the lengths of AC and CB.

The sides of the triangle are to one another in the same ratio as the sines of their opposite angles:

$$\frac{SineA}{BC} = \frac{SineB}{AC} = \frac{SineC}{AB}$$

If angle A is equal to 45.16 degrees and angle B is 105.96, then we know because of 180 degrees in a triangle that 180 – 45.16 -105.96 which is 28.92 degrees and is equal to angle C. Given segment AB is equal to 3.82 cm then we can solve for the ratios of angles to side to find the lengths of AC and BC as follows.

$$\frac{Sin45.16}{BC} = \frac{Sin28.92}{3.82}$$ Cross multiplying and solving for a gives the following

sin 45.16*3.82 =sin 28.92a, dividing both sides by the sine of 28.92 gives
BC = 5.60cm

To find the other segment we can repeat the process, and have the following:

$$\frac{Sin105.92}{AC} = \frac{Sin28.92}{3.82}$$

Cross multiplying and solving for b gives the following

sin 105.92*3.82 =sin 28.92*AC, dividing both sides by the sin of 28.92 gives

AC = 7.60cm

c) Explain how by taking only line of sight angle measurements at A and at B and by using the length of segment AB and the law of sines, you can determine the length of the dashed segment DC. Can you determine the length of DC no matter where DC is placed?

$\angle BAD = 79.93$ is a line of sight measurement
$\angle$ DBA = 70.05 is a line of sight measurement
$\angle$ ADB = 180 -79.93 – 70.05 = 40.03

Thus, by using the law of sines we get the following:

$$\frac{Sin79.93}{BD} = \frac{Sin40.02}{3.82}$$ Cross multiplying and solving for AD gives the following

sin 79.93*3.82 =sin 40.02*BD, dividing both sides by the sin 40.02 gives
BD = 5.85cm

Then continuing with this method to find further information about the polyhedron, we find side AD by also using the law of sines.

$\frac{Sine 79.93}{5.85} = \frac{Sine 70.04}{AD}$ Cross multiple and solve again gives the following:

AD = 5.58

Now using the law of cosines which is as follows:

$$DC^2 = AC^2 + AD^2 - 2(AC)(AD)CosA$$

DC ^{2} = $7.60^2 + 5.85^2$ -2*7.60*5.85*cos(79.93 -45.16)

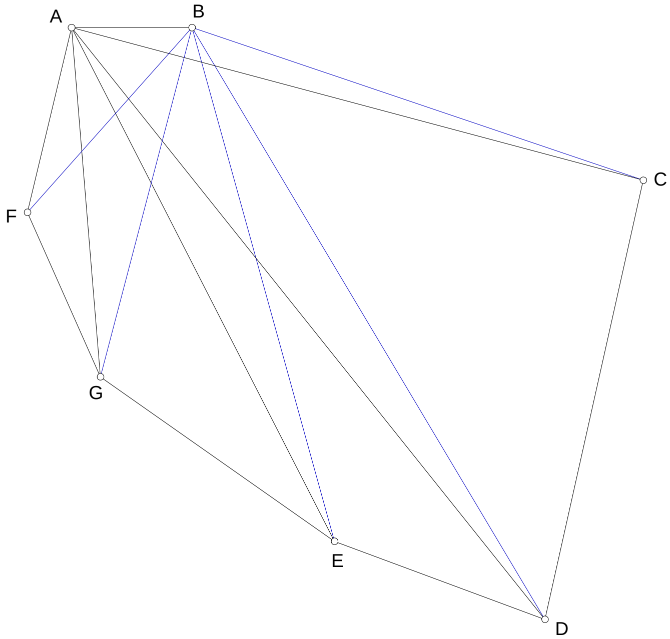
DC $^2 = 18.93935$

DC = $\sqrt{18.93935}$

DC = 4.35 cm

d) Assume the points below represent the sight points A, B and the grain elevator near Bakers chocolate factory is point C, the water tower at 84th and Yankee Hill is point D, Madonna ProActive at 56th and Pine Lake is point E, the transmission towers at roughly 14th and Old Cheney is point G, and finally, the State Capitol in downtown Lincoln is point F. Using only information from sightings from points A and B explain how using the practice problem distances and compass headings may be obtained from B to C to D to E to F to G to A. Assume North is straight up and that the actual distance from A to B is 1 mile.

m $\angle ABC = 159.94$
m $\angle BAC = 18.00$
AB = 0.93cm



Using the information from the previous page concerning the measures of angle ABC and angle BAC, the measure of angle BCA can be found as follows:

$$\angle BCA = 180 - \angle ABC - \angle BAC$$
$$\angle BCA = 2.06$$

$$\frac{Sin 18.00}{BC} = \frac{Sin 2.06}{.93cm}$$

Solving as we did in the practice problem using the law of sines allows us to find the length of BC

BC = 7.995cm

To convert 7.995cm to miles we use the unit conversion ratio from the initial information given in the problem as follows:

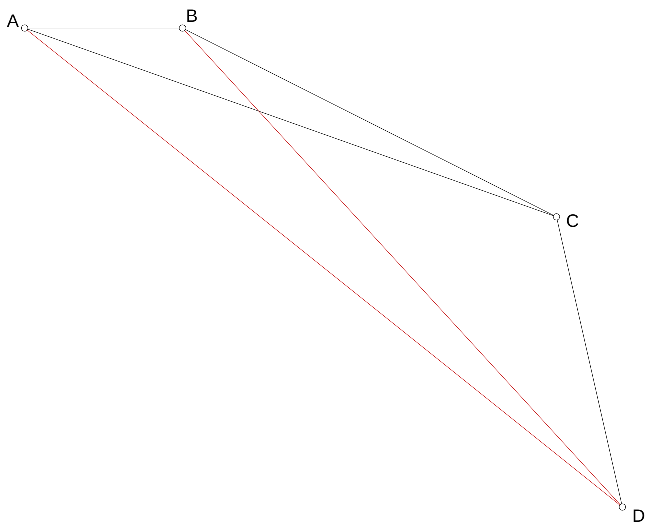
7.995cm * $\frac{1 mile}{0.93 cm}$ = 8.60 miles from point B to point C at a compass heading of 108.00 degrees.

To find segment AC we do the same operation as above using the law of sines.

AC = $\frac{Sin 159.94}{AC} = \frac{\sin 2.06}{0.93cm}$

Segment AC has a measure as found by the use of law of sines equal to 8.87cm

Now we will construct an additional triangle which will enable the calculation of the distance from C to D.



Given the following information, the law of sines, and the law of cosines the distance from C to D can be found.

$m\angle ABD = 121.75$

$m\angle BAD = 54.50$

$m\angle BDA = 180 - 121.75 - 54.50 = 3.75°$

Finding segment AD by the law of sines gives the following: $\frac{\sin 121.75}{AD} = \frac{\sin 3.75}{0.93cm}$ thus, segment AD = 12.09cm.

Now given side length AD and side length AC and an angle measurement for $\angle CAD = \angle$ BAD- $\angle BAC = 36.50°$, we can use the law of cosines to solve for segment CD. Below is the law of cosines.

$a^2 = b^2 + c^2 - 2 * b * c * \cos A$

DC ^{2} = 8.87 ^{2} +12.09 ^{2} $+ 8.87 * 12.09$*cos 36.50

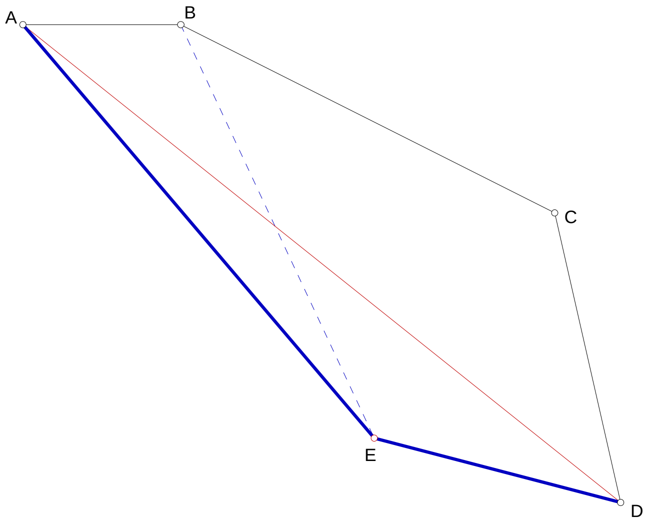
DC $= \sqrt{52.4365}$

DC = 7.24cm

At this point, we need to convert from cm to miles given that 0.93 cm is equal to 1 mile by the following:

7.24cm * $\frac{1mile}{0.93cm}$ = 7.78 miles at a compass heading of 144.50 degree from sight point A to the water tower at 84th and Yankee Hill.

Continuing this process of using the previous point to determine future point will allow us to move from point D, the water tower at 84th and Yankee Hill to Madonna Proactive at 56th and Pine Lake. Here we will also have line of sight measurements as well as previous triangle legs.



We know the following information for the point E in the above diagram:

$m\angle BAE = 68.50$

$m\angle BAD = 54.5$

$m\angle ABE = 106.25$

From this information, we can find the measure of angle AEB by subtracting from 180 the measures of the angle BAE and ABE. This gives a measure for AEB of 6.25 degrees.

The length of segment AE is equal to the following ratio: $\frac{\sin 6.25}{0.93} = \frac{\sin 106.25}{AE}$

AE = 8.20cm and from the previous triangle solution, we know that segment AD = 12.09 cm so with an angle measurement, once again the law of cosines may be utilized to find the distance between two points.

Angle DAE is needed in order to use the law of cosines and we find the angle by the following triangle relationship: $m\angle DAE = m\angle BAE - m\angle BAD$

= 68.50 – 54.50

= 14.00 degrees is the measure of angle DAE

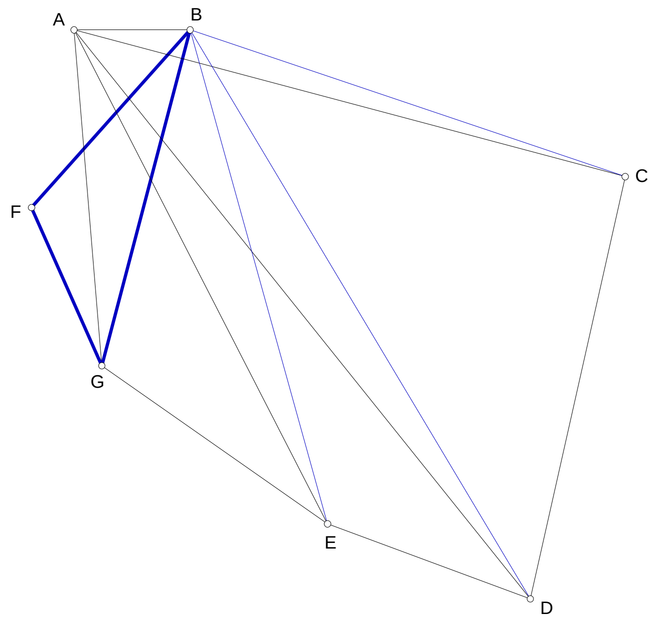
Given two legs of the triangle and an angle the law of cosines can be utilized again to find the distance between D and E.

$$ED^2 = AE^2 + AD^2 - 2(AD)(AE)\cos\angle DAE$$
$$ED^2 = 8.20^2 + 12.09^2 - 2*8.20*12.09*\cos 14.00$$
$$ED^2 = 21.02$$

ED = 4.58cm

Again converting this cm measurement into miles with the unit conversion of 0.93 cm being equal to 1 mile gives the distance from D to E as 4.92 miles at a compass heading of 158.00 degrees from sight point A to Madonna Proactive.

This process of utilization of the previous triangle leg and new line of sight measurements continues as one advances from point E to point G.



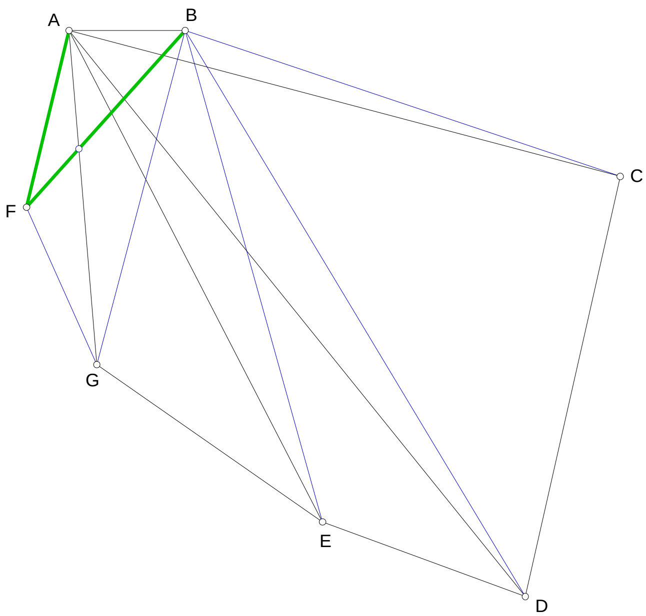
Again by taking line of sight measurements, it is found that angle BAG is 102.50 degrees and angle ABG is 68.50 degrees. Therefore, the measure of angle AGB is 180 degrees minus the measures of AGB and BAG or in terms of degrees 9.00 degrees.

Use the law of sines to find the next leg needed, so that the law of cosines may be used to calculate a measure for segment AG of 5.53 cm.

Using the law of cosines again results in a value for segment EG of 4.76 cm. Converting this measurement using the unit ratio gives a distance from EG of 5.12 miles and a compass heading of 192.50 degrees from sight point A to the transmission tower at 14th and Old Cheney.

This process of utilization of the previous triangle leg and new line of sight measurements continues as one moves from point F to the starting point A. At this point the process changes somewhat and we only need to find the last leg of the triangle, and can do so with just the law of sines. At this point we have arrived back at the starting point of the diagram, point A.
With the diagram below we can utilize just the law of sines and solve for a leg of the triangle and have both the distance between the points A and F as well as the compass heading.

Using the information from above, when the distance from G to F was found we use the value we found for the length AF, which was 3.57cm. Converting this into miles with the unit conversion of 0.93cm being equal to 1 mile we find that AF is 3.84 miles or in terms of the problem it is 3.84 miles from the State Capitol to the overpass at 27th St. and I-80 with a compass heading of 207.25 degrees from the sight point A to the State Capitol.



e) What happens if you are a bit off in your measurements? For example what error is introduced if you are off by one degree in measuring angles ABC and BAC? How does this change if B is moved further east so that AB is 2 miles? Check some other errors (for example if you are off by only 0.1 degree or if you are off in the original AB by 20 feet.

If in the original measurement of $\angle ABC$ the measurement is off by only one degree, this incorrect measurement causes $\angle BAC$ to either be one degree larger at 3.06 degrees or one degree smaller at 1.06 degrees. If we utilize these numbers in the law of sines as we did in part d, the calculations yield a tremendous error in the distance measured. For example, if the measurement of $\angle ABC$ resulted in $\angle BAC$ being 1.06 degrees, then the resulting calculations give a measurement in centimeters for segment BC of 15.53 cm. This is significant as the previous calculated distance for segment BC was 8.60 cm. This yields an error of 80%. Unfortunately, as all of the succeeding measurements are dependent upon the initial measurements, this original error compounds itself as on calculates the distances of the succeeding points. Lengthening the original leg to two miles enables the measurements to be made somewhat easier due to the sight lines being longer. Smaller errors as suggested in the question such as 0.1 degrees will create smaller errors in the calculations, but even with a 0.1 degree error in the angle measurement, there will be a 7.1% error in the length of segment BC. Again this is compounded throughout the additional sightings and measurements of the triangulation and results in significant errors towards the end of the series of triangles. A miscalculation of 20 feet in the measurement again does not seem significant, 20 feet out of a mile is only 0.38 % of the mile and seems as if it could be discounted, but again this small error compounds itself throughout the triangulation problem as a distance error has an impact on all of the future distance measurements. If 20 feet seems insignificant, drive on the county roads in Saline County after dark and find that the surveying between Saline and Seward counties is off by about 20 feet, and as a result, many of the gravel roads do not match up, and have interesting little bends in them at the county line!

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