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Geometric Study of the Category of Matrix Factorizations

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GEOMETRIC STUDY OF THE CATEGORY OF MATRIX FACTORIZATIONS

by

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We study the geometry of matrix factorizations in this dissertation. It contains two parts. The first one is a Chern-Weil style construction for the Chern character of matrix factorizations; this allows us to reproduce the Chern character in an explicit, understandable way. Some basic properties of the Chern character are also proved (via this construction) such as functoriality and that it determines a ring homomorphism from the Grothendieck group of matrix factorizations to its Hochschild homology. The second part is a reconstruction theorem of hypersurface singularities. This is given by applying a slightly modified version of Balmer's tensor triangular geometry to the homotopy category of matrix factorizations.

DEDICATION

To my parents, for their love and support.

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Chapter 1

Introduction

Matrix factorizations were introduced by Eisenbud in his 1980 paper [10]. He showed that the free resolution of every finitely generated module over $R = S/f$ (where S is a regular local ring) is given by a matrix factorization. In particular every such resolution is eventually 2-periodic. This in turn allowed him to show that matrix factorizations describe all maximal Cohen-Macaulay modules without free summands. Since this groundbreaking work matrix factorizations have been a common tool in commutative algebra. Buchweitz introduced the notion of the stable derived category of a ring in 1986. In his famous unpublished manuscript [4], he showed that the homotopy category of matrix factorizations gives one of four equivalent descriptions of the stable derived category of a hypersurface ring. This was rediscovered by Orlov for schemes in his series of papers [19], [20], [21].

Matrix factorizations play an important role in many areas of pure mathematics and mathematical physics. As in Eisenbud's original work, it is a classical tool in the study of hypersurface singularity algebras. In the geometric setting, the category of matrix factorizations measures the failure of coherent sheaves on the hypersurface $f = 0$ to have a finite locally free resolution [19]. It also appears prominently in

the work of Khovanov and Rozansky [16] on link homology. Recently, Carqueville and Murfet study matrix factorizations in the context of topological field theory [5]. Following the suggestion of Kontsevich, matrix factorizations were used by physicists to describe D-branes of type B in Landau-Ginzburg models [13], [14]. They found applications in various approaches to mirror symmetry and the study of the sigma model/Landau-Ginzburg correspondence [3],[9],[11],[15],[25].

We study matrix factorizations in the following two ways. First, we give a Chern-Weil style construction of the Chern character of matrix factorizations; this allows us to produce the Chern character in an explicit, understandable way. We also prove some basic properties of the Chern character via this construction such as functoriality and that the Chern character induces a map on the Grothendieck group of the homotopy category of matrix factorizations to its Hochschild homology. Second, we apply Balmer's theory of tensor triangular geometry to the homotopy category of matrix factorizations.

We discuss these two questions in more detail in the following subsections.

1.1 Chern character

Classically, Chern classes are topological invariants of vector bundles on a smooth manifold. It is in general quite hard to know whether two vector bundles are the same. The Chern classes provide one way of addressing the question: if the Chern classes of a pair of vector bundles are not the same, then the vector bundles are different. However, the converse is not true.

Chern classes can be used to construct a homomorphism of rings, called the Chern character, from the K-theory of a smooth manifold to (the completion of) its rational cohomology. The Chern character is of great importance for many reasons. For

example, it appears in the Grothendieck-Riemann-Roch Theorem [2].

The dg-category of matrix factorizations can be thought of as the derived category of sheaves on a noncommutative space [6]. Therefore it is reasonable to expect a theory of Chern characters for matrix factorizations.

In [26], Shklyarov gives a nice interpretation of the Chern character and the Riemann-Roch theorem in the context of dg-categories. In practice, it's hard to calculate what exactly the Hochschild homology and the Chern character map are. Polishchuk and Vaintrob [23] solved this problem for the case when $Q = k[[x_1, \cdots, x_n]]$ (k is a field of characteristic 0) and w is an isolated singularity by studying the dg-structure of matrix factorizations. Dyckerhoff-Murfet [7] produces the same Chern character by an explicit description of a local duality isomorphism. Recently, Carqueville-Murfet [5] studies the bicategory of Landau-Ginzburg models. Their main result is the existence of adjoints in this bicategory and a description of evaluation and coevaluation maps in terms of Atiyah classes and homological perturbation. They are able to recover the Chern character as an application of their theory. In fact, their Chern character now works for any noetherian $\mathbb{Q}$-algebra k. Most recently, Platt [22] gives an explicit formula for the boundary bulk map, and in the case when the matrix factorization admits a connection, an explicit formula for the Chern character.

Following the idea of Dyckerhoff-Murfet [8], I use the Atiyah class At of a matrix factorization to give an algebraic Chern-Weil type construction. This construction allows me to extend the Chern character to work in the more general situation where Q is any finitely generated smooth k-algebra (k a commutative ring containing $\mathbb{Q}$) and f is any element of Q. The Chern character map I construct (Definition 3.3.3) turns out to agree with the recent one of Platt [22], who defines a Chern character map using the sophisticated machinery of homotopy theory of dg categories. One advantage of my construction of the Chern character map is that it does not rely on

such a complicated theory, and I am able to establish its basic properties using only elementary methods. For example, I prove the Chern character map is independent of all of the choices made in its definition by using only the relatively elementary notions of homotopy of matrix factorizations. In addition, I am able to establish basic properties of the Chern character map, such as functoriality and the fact that it determines a ring homomorphism from the Grothendieck group of matrix factorizations to its Hochschild homology (Corollary 3.3.15).

1.2 Reconstruction of hypersurface singularities

The homotopy category of matrix factorizations for a given ring Q and an element $f \in Q$, denoted by $[MF(Q, f)]$, has a natural structure of triangulated category. It is well known that this category is equivalent (as a triangulated category) to the singularity category defined by Buchweitz [4] (for modules) and later rediscovered by Orlov [19] (for schemes), i.e., $[MF(Q, f)] \cong \mathfrak{D}^b_{Sing}(Q/f)$.

Balmer's tensor triangular geometry associates a locally ringed space to a given tensor triangular category. We would like to apply Balmer's theory to the category of matrix factorizations and see if the space given by Balmer's theory gives a reconstruction of the hypersurface singularity.

Balmer's theory requires a tensor product. Luckily there is a natural tensor product of matrix factorizations (denoted by $\otimes_{mf}$), but it does not behave exactly the way we want. In fact, it has two problems. First, given any two matrix factorizations $\mathcal{M} \in [MF(Q, f)]$ and $\mathcal{N} \in [MF(Q, g)]$, we have $\mathcal{M} \otimes_{mf} \mathcal{N} \in [MF(Q, f + g)]$. Second, the tensor identity is a matrix factorization of $0 \in Q$. Therefore, we need to modify Balmer's theory a little bit. We can solve the first problem in two ways. The first is to look at the graded tensor triangulated category $(\coprod_{i \geqslant 1}[MF(Q, if)], \otimes_{mf})$ and the

second one is to look at the triangulated category $([MF(Q,f)], \otimes^{\frac{1}{2}})$ with a modified tensor product $\otimes^{\frac{1}{2}} = \frac{1}{2} \circ \otimes_{mf}$ (See Section 4.1.3 for details of the functor $\frac{1}{2}$). Note that this will also solve the second problem. The solution to the tensor identity problem is to look at a pseudo tensor triangulated category introduced in Section 4.1.1. We can show that no matter which pseudo tensor triangulated category you want to use, i.e., either $(\coprod_{i>0}[MF(Q,if)], \otimes_{mf})$ or $([MF(Q,f)], \otimes^{\frac{1}{2}})$, you will get a reconstruction theorem.

There is already a good support theory for matrix factorizations due to many people. We prove the reconstruction theorems by applying this slightly modified pseudo version theory of Balmer to the support of matrix factorizations. To be more specific, we prove $Sing(Q/f) \cong Spc'(K)$ (Chapter 4) by showing that the support is in fact a classifying support data in the pseudo sense.

Chapter 2

Matrix factorizations

We recall the theory of matrix factorizations in this chapter. Everything is Noetherian in this thesis. In this chapter, Q denotes a commutative ring, f is an element of Q, all modules over Q to which we refer will be assumed to be finitely generated.

Definition 2.0.1. A *matrix factorization* of $f \in Q$ is a $\mathbb{Z}/2$-graded Q-module $M = M_0 \oplus M_1$, where M is a finitely generated projective Q-module, together with a degree 1 endomorphism

$$d = \begin{bmatrix} 0 & d_1 \\ d_0 & 0 \end{bmatrix}$$

such that $d \circ d = f \cdot 1_M$.

Equivalently, a matrix factorization for (Q, f) consists of a pair of finitely generated projective Q-modules M_0 and M_1 and Q-linear maps $d_0 : M_0 \to M_1$ and $d_1 : M_1 \to M_0$ such that each composition is multiplication by f:

$$d_0 \circ d_1 = f \cdot 1_{M_1} \text{ and } d_1 \circ d_0 = f \cdot 1_{M_0}.$$

We visualize a matrix factorization as

$$\mathcal{M} = (M_1 \underset{d_0}{\overset{d_1}{\leftrightarrows}} M_0) \text{ or } (M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_1)$$

depending on the context that we are working with. Note, we write the degree 0 part on the right and the degree 1 part on the left for the first version. For the second version, we have the degree 0 piece in the middle and degree 1 pieces elsewhere.

Example 2.0.2. Let $Q = \mathbb{C}[[x]]$ and $f = x^n$, then we have factorizations $(Q \underset{x^{n-i}}{\overset{x^i}{\leftrightarrows}} Q)$, for all i.

A more interesting example is the following:

Example 2.0.3. Given $Q = \mathbb{C}[[x, y, z]], f = xy + yz + zx$, then $(Q^2 \underset{d_0}{\overset{d_1}{\leftrightarrows}} Q^2)$, with $d_1 = \begin{bmatrix} z & y \\ x & -x-y \end{bmatrix}$ and $d_0 = \begin{bmatrix} x+y & y \\ x & -z \end{bmatrix}$ is a matrix factorization of f.

Definition 2.0.4. A *strict morphism* of matrix factorizations from $\mathcal{M}$ to $\mathcal{N}$ is a $\mathbb{Z}/2$-graded Q-linear map of degree zero $\alpha : \mathcal{M} \to \mathcal{N}$ such that $d^{\mathcal{N}} \circ \alpha = \alpha \circ d^{\mathcal{M}}$.

Equivalently, a strict morphism is a pair of Q-linear maps $\alpha_0 : M_0 \to N_0$ and $\alpha_1 : M_1 \to N_1$ such that the two evident squares commute.

We write $MF(Q, f)$ for the category of all matrix factorizations of (Q, f) with morphisms given by the set of strict morphisms. It is an exact category in the sense of Quillen: a sequence of strict morphisms is exact if and only if it is so in both degrees.

Definition 2.0.5. Two strict morphisms $\alpha, \beta : \mathcal{M} \to \mathcal{N}$ are *homotopic* if there exist morphisms $h_1 \in \mathrm{Hom}(M_1, N_0)$ and $h_0 \in \mathrm{Hom}(M_0, N_1)$ such that

$$d^N \circ h_0 + h_1 \circ d^M = \alpha - \beta.$$

We visualize a homotopy as a diagram,

$$\begin{tikzcd} M_1 \arrow[r, "d^M"] \arrow[d, "\alpha-\beta"'] & M_0 \arrow[r, "d^M"] \arrow[d, "\alpha-\beta"] \arrow[dl, dotted, "h_0"] & M_1 \arrow[d, "\alpha-\beta"] \arrow[dl, dotted, "h_1"] \\ N_1 \arrow[r, "d^N"'] & N_0 \arrow[r, "d^N"'] & N_1 \end{tikzcd}$$

Being homotopic is an equivalence relation and is preserved by composition of strict morphisms. The category $[MF(Q,f)]$ is obtained from $MF(Q,f)$ by leaving the objects unchanged but modding out the hom sets by this equivalence relation.

A strict morphism $\alpha : \mathcal{M} \to \mathcal{N}$ that becomes an isomorphism in $[MF(Q,f)]$ is called a *homotopy equivalence*, i.e., α is a homotopy equivalence if and only if there exists a strict morphism $\beta : \mathcal{N} \to \mathcal{M}$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are each homotopic to the appropriate identity map.

Definition 2.0.6. For $\mathcal{M} \in MF(Q,f)$, define the *shift* of $\mathcal{M}$, written $\mathcal{M}[1] \in MF(Q,f)$, to be:

$$\left(M_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} M_0 \right)[1] = \left(M_0 \underset{-d_1}{\overset{-d_0}{\rightleftarrows}} M_1 \right).$$

We define $\mathcal{M}[n]$ to be the iteration of n applications of $[1]$, if n is positive.

Notice that $\mathcal{M}[2] = (\mathcal{M}[1])[1] = \mathcal{M}$. Thus we define $\mathcal{M}[-1] = \mathcal{M}[1]$ and more generally $\mathcal{M}[-n] = \mathcal{M}[n]$ for $n > 0$.

Definition 2.0.7. We define the *cone* of a strict morphism $\alpha : \mathcal{M} \to \mathcal{N}$ to be the following matrix factorization:

$$\operatorname{cone}(\alpha) = \left(N_1 \oplus M_0 \underset{\begin{bmatrix} d_0^N & \alpha_1 \\ 0 & -d_1^M \end{bmatrix}}{\overset{\begin{bmatrix} d_1^N & \alpha_0 \\ 0 & -d_0^M \end{bmatrix}}{\rightleftarrows}} N_0 \oplus M_1 \right).$$

There are canonical maps

$$\mathcal{N} \to \text{cone}(\alpha) \;\text{ and }\; \text{cone}(\alpha) \to \mathcal{M}[1],$$

just as for the category of chain complexes. These will give the "distinguished triangles" in the triangulated structure discussed in the next proposition.

Proposition 2.0.8. *(Proposition 3.3 of [19]) For any Q and $f \in Q$, the category $[MF(Q,f)]$ is a triangulated category. The shift functor is $\mathcal{M} \mapsto \mathcal{M}[1]$ and the distinguished triangles are those isomorphic (in $[MF(Q,f)]$) to triangles of the form*

$$\mathcal{M} \xrightarrow{\alpha} \mathcal{N} \xrightarrow{canonical} \text{cone}(\alpha) \xrightarrow{canonical} \mathcal{M}[1]$$

for any strict morphism α.

Definition 2.0.9. Given two matrix factorizations $\mathcal{M} \in MF(Q,f)$ and $\mathcal{N} \in MF(Q',f')$, where Q, Q' are commutative rings and elements $f \in Q$, $f' \in Q'$. We define the *tensor product* of $\mathcal{M}$ and $\mathcal{N}$ to be

$$\mathcal{M} \otimes_{mf} \mathcal{N} := ((M_1 \otimes N_0) \oplus (M_0 \otimes N_1) \underset{d_{M\otimes N}}{\overset{d_{M\otimes N}}{\rightleftarrows}} (M_0 \otimes N_0) \oplus (M_1 \otimes N_1)),$$

where $d_{\mathcal{M}\otimes_{mf}\mathcal{N}} = d_M \otimes 1 + (-1)^{|\cdot|} \otimes d_N$ making it into a matrix factorization of $f \otimes 1 + 1 \otimes f'$. To be more precise, we have

$$d_{\mathcal{M}\otimes_{mf}\mathcal{N}}(m \otimes n) = d_M(m) \otimes n + (-1)^{|m|} m \otimes d_N(n)$$

for simple, homogeneous tensors $m \otimes n$. For further details, see [8] [32].

As in the Definition 2.4 of [8], the tensor product $-\otimes_{mf}-$ of matrix factorizations is

well-defined on the homotopy category of matrix factorizations, i.e., $-\otimes_{mf}-$ preserves matrix factorization homotopy.

Proposition 2.0.10. *(Lemma 2.2 of [32])*

Given any three matrix factorizations $\mathcal{M}$,$\mathcal{N}$ and $\mathcal{L}$, we have

$$(\mathcal{M} \oplus \mathcal{N}) \otimes_{mf} \mathcal{L} \cong (\mathcal{M} \otimes_{mf} \mathcal{L}) \oplus (\mathcal{N} \otimes_{mf} \mathcal{L}).$$

For a complex of Q-modules, we have the following definition.

Definition 2.0.11. Given any complex $C^\bullet$ of Q-modules we denote by $C^\bullet_{\mathbb{Z}/2}$ the $\mathbb{Z}/2$-*folding*, which has $\bigoplus\limits_{i\in 2\mathbb{Z}} C^i$ in degree zero and $\bigoplus\limits_{i\in 2\mathbb{Z}+1} C^i$ in degree one, together with the obvious differential. Note $C^\bullet_{\mathbb{Z}/2}$ is a matrix factorization of 0.

Remark 2.0.12. *Therefore, we can talk about tensor products (in the sense of Definition 2.0.9) between complexes of projective Q-modules and matrix factorizations. If one of the factors in the tensor product is simply a projective Q-module, or more generally a bounded complex of projective Q-modules, we first view it as a factorization of zero using the $\mathbb{Z}/2$-folding (for the case of a single module, we follow the usual convention by placing it in the degree* 0 *piece of a complex), then tensor it with the other matrix factorization; i.e., $P^\bullet \otimes \mathcal{M} := P^\bullet_{\mathbb{Z}/2} \otimes_{mf} \mathcal{M}$ for a complex $P^\bullet$ of projective modules. We have the following proposition addressing the problem of compatibility.*

Proposition 2.0.13. *Given two complexes $X^\bullet$ and $Y^\bullet$ of Q-modules, we have $(X^\bullet \otimes_{cx} Y^\bullet)_{\mathbb{Z}/2} = X^\bullet_{\mathbb{Z}/2} \otimes_{mf} Y^\bullet_{\mathbb{Z}/2}$, where $\otimes_{cx}$ stands for the usual tensor product of complexes.*

Proof. First note that the underlying modules for $(X^\bullet \otimes_{cx} Y^\bullet)_{\mathbb{Z}/2}$ and $X^\bullet_{\mathbb{Z}/2} \otimes_{mf} Y^\bullet_{\mathbb{Z}/2}$ are identical.

Indeed, We have

$$((X^\bullet \otimes_{cx} Y^\bullet)_{\mathbb{Z}/2})_1 = \bigoplus_{k \text{ is odd}} (\bigoplus_{i+j=k} (X^i \otimes Y^j))$$

and

$$(X^\bullet_{\mathbb{Z}/2} \otimes_{mf} Y^\bullet_{\mathbb{Z}/2})_1 = [(\bigoplus_{i \text{ is odd}} X^i) \otimes (\bigoplus_{j \text{ is even}} Y^j)] \bigoplus [(\bigoplus_{i \text{ is even}} X^i) \otimes (\bigoplus_{j \text{ is odd}} Y^j)] = \bigoplus_{k \text{ is odd}} (\bigoplus_{i+j=k} (X^i \otimes Y^j)).$$

Similarly $((X^\bullet \otimes_{cx} Y^\bullet)_{\mathbb{Z}/2})_0 = (X^\bullet_{\mathbb{Z}/2} \otimes_{mf} Y^\bullet_{\mathbb{Z}/2})_0$.

The fact that the differentials are the same can be seen by carefully keeping track of where elements go. $\square$

Proposition 2.0.14. *For any matrix factorization $\mathcal{X} \in [MF(Q, f)]$,*

$$\mathcal{X} \otimes_{mf} -, \quad - \otimes_{mf} \mathcal{X} : [MF(Q, g)] \to [MF(Q, f + g)]$$

are triangulated functors.

Proof. We prove this for the functor $- \otimes_{mf} \mathcal{X}$, the other one follows since $\mathcal{X} \otimes_{mf} -$ is naturally isomorphic to $- \otimes_{mf} \mathcal{X}$.

Given any distinguished triangle $\mathcal{M} \xrightarrow{\alpha} \mathcal{N} \to \text{cone}(\alpha) \to \mathcal{M}[1]$ of $[MF(Q, g)]$, we need to check:

1. $(\mathcal{M} \otimes_{mf} \mathcal{X})[1] \simeq \mathcal{M}[1] \otimes_{mf} \mathcal{X}$, and
2. $\mathcal{M} \otimes_{mf} \mathcal{X} \xrightarrow{\alpha \otimes 1} \mathcal{N} \otimes_{mf} \mathcal{X} \to \text{cone}(\alpha) \otimes_{mf} \mathcal{X} \to (\mathcal{M}[1]) \otimes_{mf} \mathcal{X} (\simeq (\mathcal{M} \otimes_{mf} \mathcal{X})[1])$ is also a distinguished triangle.

Say $\mathcal{M} = (M_1 \underset{d_0^M}{\overset{d_1^M}{\rightleftarrows}} M_0), \mathcal{N} = (N_1 \underset{d_0^N}{\overset{d_1^N}{\rightleftarrows}} N_0)$ and $\mathcal{X} = (X_1 \underset{d_0^X}{\overset{d_1^X}{\rightleftarrows}} X_0)$,

For 1, by definition, $(\mathcal{M} \otimes_{mf} \mathcal{X})[1]$ is the matrix factorization

$$(M_0 \otimes X_0) \oplus (M_1 \otimes X_1) \underset{-\begin{bmatrix} d_1^M \otimes 1 & 1 \otimes d_1^X \\ -1 \otimes d_0^X & d_0^M \otimes 1 \end{bmatrix}}{\overset{-\begin{bmatrix} d_0^M \otimes 1 & -1 \otimes d_1^X \\ 1 \otimes d_0^X & d_1^M \otimes 1 \end{bmatrix}}{\rightleftarrows}} (M_1 \otimes X_0) \oplus (M_0 \otimes X_1)$$

and $\mathcal{M}[1] \otimes_{mf} \mathcal{X}$ is the following

$$(M_0 \otimes X_0) \oplus (M_1 \otimes X_1) \underset{\begin{bmatrix} -d_1^M \otimes 1 & -1 \otimes d_1^X \\ 1 \otimes d_0^X & -d_0^M \otimes 1 \end{bmatrix}}{\overset{\begin{bmatrix} -d_0^M \otimes 1 & 1 \otimes d_1^X \\ -1 \otimes d_0^X & -d_1^M \otimes 1 \end{bmatrix}}{\rightleftarrows}} (M_1 \otimes X_0) \oplus (M_0 \otimes X_1)$$

so they are in fact equal to each other.

For 2, first notice that the morphism $\mathcal{M} \otimes_{mf} \mathcal{X} \xrightarrow{\alpha \otimes 1} \mathcal{N} \otimes_{mf} \mathcal{X}$ is

$$\begin{tikzcd}
(M_1 \otimes X_0) \oplus (M_0 \otimes X_1) \arrow[r] \arrow[d, "(\alpha_1 \otimes 1, \alpha_0 \otimes 1)"'] & (M_0 \otimes X_0) \oplus (M_1 \otimes X_1) \arrow[r] \arrow[d, "(\alpha_0 \otimes 1, \alpha_1 \otimes 1)"] & (M_1 \otimes X_0) \oplus (M_0 \otimes X_1) \arrow[d, "(\alpha_1 \otimes 1, \alpha_0 \otimes 1)"] \\
(N_1 \otimes X_0) \oplus (N_0 \otimes X_1) \arrow[r] & (N_0 \otimes X_0) \oplus (N_1 \otimes X_1) \arrow[r] & (N_1 \otimes X_0) \oplus (N_0 \otimes X_1)
\end{tikzcd}$$

Therefore by definition $cone(\alpha \otimes 1)$ is the matrix factorization

$$cone(\alpha\otimes 1)_1 \underset{\begin{bmatrix} d_0^N \otimes 1 & -1 \otimes d_1^X & \alpha_1 \otimes 1 & \\ 1 \otimes d_0^X & d_1^N \otimes 1 & & \alpha_0 \otimes 1 \\ & & -d_1^M \otimes 1 & -1 \otimes d_1^X \\ & & 1 \otimes d_0^X & -d_0^M \otimes 1 \end{bmatrix}}{\overset{\begin{bmatrix} d_1^N \otimes 1 & 1 \otimes d_1^X & \alpha_0 \otimes 1 & \\ -1 \otimes d_0^X & d_0^N \otimes 1 & & \alpha_1 \otimes 1 \\ & & -d_0^M \otimes 1 & 1 \otimes d_1^X \\ & & -1 \otimes d_0^X & -d_1^M \otimes 1 \end{bmatrix}}{\rightleftarrows}} cone(\alpha\otimes 1)_0$$

where $cone(\alpha\otimes 1)_1 = (N_1 \otimes X_0) \oplus (N_0 \otimes X_1) \oplus (M_0 \otimes X_0) \oplus (M_1 \otimes X_1)$ and $cone(\alpha\otimes 1)_0 = (N_0 \otimes X_0) \oplus (N_1 \otimes X_1) \oplus (M_1 \otimes X_0) \oplus (M_0 \otimes X_1)$.

Also, by definition $cone(\alpha) \otimes_{mf} \mathcal{X}$ is equal to

$$(cone(\alpha)\otimes_{mf} \mathcal{X})_1 \underset{\begin{bmatrix} d_0^N \otimes 1 & \alpha_1 \otimes 1 & -1 \otimes d_1^X & \\ & -d_1^M \otimes 1 & & -1 \otimes d_1^X \\ 1 \otimes d_0^X & & d_1^N \otimes 1 & \alpha_0 \otimes 1 \\ & 1 \otimes d_0^X & & -d_0^M \otimes 1 \end{bmatrix}}{\overset{\begin{bmatrix} d_1^N \otimes 1 & \alpha_0 \otimes 1 & 1 \otimes d_1^X & \\ & -d_0^M \otimes 1 & & 1 \otimes d_1^X \\ -1 \otimes d_0^X & & d_0^N \otimes 1 & \alpha_1 \otimes 1 \\ & -1 \otimes d_0^X & & -d_1^M \otimes 1 \end{bmatrix}}{\rightleftarrows}} (cone(\alpha)\otimes_{mf} \mathcal{X})_0$$

where $(cone(\alpha) \otimes_{mf} \mathcal{X})_1 = (N_1 \otimes X_0) \oplus (M_0 \otimes X_0) \oplus (N_0 \otimes X_1) \oplus (M_1 \otimes X_1)$ and $(cone(\alpha) \otimes_{mf} \mathcal{X})_0 = (N_0 \otimes X_0) \oplus (M_1 \otimes X_0) \oplus (N_1 \otimes X_1) \oplus (M_0 \otimes X_1)$.

Now it's not hard to see that there is an isomorphism $\eta = (\eta_0, \eta_1)$ between $cone(\alpha \otimes 1)$ and $cone(\alpha) \otimes_{mf} \mathcal{X}$, where $\eta_1 : cone(\alpha \otimes 1)_1 \to (cone(\alpha) \otimes_{mf} \mathcal{X})_1$ sends

$$\begin{bmatrix} n_1 \otimes x_0 \\ n_0 \otimes x_1 \\ m_0 \otimes x_0' \\ m_1 \otimes x_1' \end{bmatrix} \text{ to } \begin{bmatrix} n_1 \otimes x_0 \\ m_0 \otimes x_0' \\ n_0 \otimes x_1 \\ m_1 \otimes x_1' \end{bmatrix}.$$

Similarly for $\eta_0 : cone(\alpha \otimes 1)_0 \to (cone(\alpha) \otimes_{mf} \mathcal{X})_0$, where it sends

$$\begin{bmatrix} n_0 \otimes x_0 \\ n_1 \otimes x_1 \\ m_1 \otimes x_0' \\ m_0 \otimes x_1' \end{bmatrix} \text{ to } \begin{bmatrix} n_0 \otimes x_0 \\ m_1 \otimes x_0' \\ n_1 \otimes x_1 \\ m_0 \otimes x_1' \end{bmatrix}.$$

$\square$

Chapter 3

Chern character

3.1 Algebraic Chern-Weil Theory

Here we review the basic Chern-Weil theory from the algebraic point of view, which will be used later in our construction. From now on, Q is a finitely generated commutative k-algebra, where k is a commutative ring. All modules are finitely generated. Also, let $\Omega^1_{Q/k}$ be the Q-module of differential 1-forms and $\Omega^n_{Q/k} := \bigwedge^n_Q \Omega^1_{Q/k}$ the Q-module of differential n-forms. For details, see [18].

Definition 3.1.1. Let Q be a commutative k-algebra and E a Q-module. A *connection* on the Q-module E is a k-linear map $\nabla : E \to \Omega^1 \otimes_Q E$ such that for any $e \in E$ and $q \in Q$ the following Leibniz rule holds:

$$\nabla(qe) = (dq) \otimes e + q\nabla(e).$$

Just like the exterior differential operator d, a connection ∇ can be extended

canonically to a map, which we still denote by ∇,

$$\nabla : \Omega^\bullet \otimes_Q E \to \Omega^{\bullet+1} \otimes_Q E$$

such that for any homogeneous element $u \in \Omega^\bullet$ and $e \in E$

$$\nabla(u \otimes e) = (du) \otimes e + (-1)^{|u|} u \wedge \nabla(e).$$

Example 3.1.2. For $E = Q$, the exterior differential operator d is a connection. More generally, if $E = Q^r$,

$$\Omega^\bullet \otimes_Q E \cong (\Omega^\bullet)^r \text{ and } d \cdot I_r : (\Omega^\bullet)^r \to (\Omega^{\bullet+1})^r$$

is a connection for Q^r.

Every finitely generated projective Q-module E possesses a connection. Given such an E, choose an idempotent e in $M_r(Q)$ for some r such that $E = Im(e)$. Then, from the connection on Q^r that we defined in the previous example, we can extract a connection on E through the following composition:

$$\Omega^\bullet \otimes_Q E \hookrightarrow \Omega^\bullet \otimes_Q Q^r \xrightarrow{d \cdot I_r} \Omega^{\bullet+1} \otimes_Q Q^r \xrightarrow{1 \otimes e} \Omega^{\bullet+1} \otimes_Q E$$

Definition 3.1.3. This connection on $E = Im(e)$ is called the *Levi-Civita* connection by analogy with the classical situation in differential geometry.

Definition 3.1.4. The *curvature* R of a connection ∇ on a finitely generated Q-module E is defined to be

$$R = \nabla \circ \nabla : E \to \Omega^2 \otimes_Q E$$

It can be shown that R is Q-linear.

Proposition 3.1.5. *(Example 4.2.6 of [12])*

1. *Let E_1 and E_2 be projective Q-modules with connections ∇_1 and ∇_2, respectively. Then for $e_1 \in E_1$ and $e_2 \in E_2$, we set*

$$\nabla(e_1 \oplus e_2) = \nabla_1(e_1) \oplus \nabla_2(e_2).$$

   This defines a natural connection on the direct sum $E_1 \oplus E_2$.

2. *In order to define a connection on the tensor product $E_1 \otimes_Q E_2$ one defines*

$$\nabla(e_1 \otimes e_2) = \nabla_1(e_1) \otimes e_2 + e_1 \otimes \nabla_2(e_2).$$

   Note that the second component naturally lands in $E_1 \otimes_Q (\Omega^1 \otimes_Q E_2)$, so we need to apply an isomorphism $\sigma : E_1 \otimes_Q \Omega^1 \to \Omega^1 \otimes_Q E_1$ $(e_1 \otimes w \longmapsto w \otimes e_1)$ to make it into an element of the target module $\Omega^1 \otimes_Q E_1 \otimes_Q E_2$.

Before getting into the next proposition, we want to inform the reader that by $exp(R)$ we mean the series $1 + R + \frac{R^2}{2!} + \frac{R^3}{3!} + \cdots + \frac{R^n}{n!} + \cdots \in \prod_n End_Q(E) \otimes_Q \Omega^{2n}$. In order to do this, we need to make the extra assumption that $k \supset \mathbb{Q}$. The exterior operator d can be extended to maps $\Omega^n_{Q/k} \to \Omega^{n+1}_{Q/k}$ (for any $n \in \mathbb{N}$) by

$$d(a_0 da_1 \cdots da_n) = da_0 da_1 \cdots da_n.$$

Since $d(1) = 0$ it is immediate that $d^2 = 0$, and the following sequence

$$Q = \Omega^0_{Q/k} \xrightarrow{d} \Omega^1_{Q/k} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{Q/k} \xrightarrow{d} \cdots$$

is a complex called the *de Rham complex* of Q over k. The homology groups of the de Rham complex are denoted $H^n_{DR}(Q)$ and are called the *de Rham cohomology groups* of Q over k.

Proposition 3.1.6. *(Proposition 8.1.6 of [18]) The homogeneous component of degree 2n of $ch(E,\nabla) \coloneqq tr(exp(R))$ is a cycle in $\Omega^{2n}_{Q|k}$ (of the de Rham complex), where tr stands for the trace map for projective modules (details at Section 3.3.1).*

This proposition implies that $ch(E,\nabla)$ defines a cohomology class in the de Rham cohomology of Q.

Theorem 3.1.7. *(Theorem-Definition 8.1.7 of [18]) The cohomology class of $ch(E,\nabla) \coloneqq tr(exp(R))$ is independent of the connection ∇ and defines an element*

$$ch(E) \in \prod_{n\geqslant 0} H^{2n}_{DR}(Q)$$

*which is called the "*Chern character*" of the finitely generated projective Q-module E.*

Theorem 3.1.8. *(Theorem 8.2.4 of [18])*

The Chern character induces a ring homomorphism $ch : K_0(Q) \to H^{even}_{DR}(Q)$.

3.2 Main constructions

Given a k-algebra Q, k a Noetherian commutative ring, for a matrix factorization $\mathcal{E} = (E_1 \xrightarrow{A} E_0 \xrightarrow{B} E_1)$ of $f \in Q$ (so the odd endomorphism of this matrix factorization is $d = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$), choose connections $\nabla_i : E_i \to \Omega^1_{Q/k} \otimes_Q E_i$ for $i = 0,1$. By Proposition 3.1.5, ∇_0 and ∇_1 induce a natural connection for the underlying module $E = E_0 \oplus E_1$ of the form

$$\nabla = \begin{bmatrix} \nabla_0 & 0 \\ 0 & \nabla_1 \end{bmatrix}.$$

Definition 3.2.1 (Dyckerhoff-Murfet [8])**.** The *Atiyah class* of $\mathcal{E}$, written $At_{\mathcal{E},\nabla}$ or simply just At if there is no confusion, is the map

$$\nabla \circ d - (1 \otimes d) \circ \nabla =^{def} At_{\mathcal{E},\nabla} : \mathcal{E} \to \Omega^1[1] \otimes_{mf} \mathcal{E}.$$

See Remark 2.0.12 for details of the tensor product of a module and a matrix factorization.

It's easy to check that the Atiyah class is a Q-module homomorphism from the $\mathbb{Z}/2$-graded Q-module E to the $\mathbb{Z}/2$-graded Q-module $\Omega^1 \otimes E$.

Compositions of Atiyah classes are defined in the following way. For example, by definition, we have $(1 \otimes At_{\mathcal{E},\nabla}) \circ At_{\mathcal{E},\nabla} : \mathcal{E} \to \Omega^1[1] \otimes_{mf} \mathcal{E} \to \Omega^1[1] \otimes_{mf} \Omega^1[1] \otimes_{mf} \mathcal{E}$, which is defined as

$$((1 \otimes \nabla) \circ (1 \otimes d) - (1 \otimes 1 \otimes d) \circ (1 \otimes \nabla)) \circ (\nabla \circ d - (1 \otimes d) \circ \nabla).$$

For simplicity, we denote this composition by $\widetilde{At^2_{\mathcal{E},\nabla}}$. Similarly, we can define $\widetilde{At^i_{\mathcal{E}}}$ (for natural numbers $i \geqslant 2$) recursively by

$$\widetilde{At^i} := (1_{\underbrace{\Omega^1 \otimes \cdots \otimes \Omega^1}_{i-1}} \otimes At) \circ \widetilde{At^{i-1}}.$$

Hence the map

$$\widetilde{At^i} : \mathcal{E} \to \overbrace{\Omega^1[1] \otimes_{mf} \cdots \otimes_{mf} \Omega^1[1]}^{i} \otimes_{mf} \mathcal{E}$$

has i copies of Ω^1 in the target.

Definition 3.2.2. Define At^i to be the composition:

$$\mathcal{E} \xrightarrow{\widetilde{At^i}} \overbrace{\Omega^1[1] \otimes_{mf} \cdots \otimes_{mf} \Omega^1[1]}^{i} \otimes_{mf} \mathcal{E} \xrightarrow{\wedge} \Omega^i[i] \otimes_{mf} \mathcal{E}.$$

Note that we have $At = At^1 = \widetilde{At^1}$.

3.2.1 Basic construction: the strict morphism φ

Definition 3.2.3. Define $\mathcal{E}^{(1)} = (\, Q \xrightarrow{df\wedge} \Omega^1 \,) \otimes_{mf} \mathcal{E}$, with Q in degree 0 and Ω^1 in degree 1.

Explicitly, $\mathcal{E}^{(1)}$ is by definition the following :

$$\mathcal{E}^{(1)} = \left(\begin{matrix} E_1 \\ \oplus \\ \Omega^1 \otimes E_0 \end{matrix} \xrightarrow{\overline{A}} \begin{matrix} E_0 \\ \oplus \\ \Omega^1 \otimes E_1 \end{matrix} \xrightarrow{\overline{B}} \begin{matrix} E_1 \\ \oplus \\ \Omega^1 \otimes E_0 \end{matrix} \right)$$

with $\overline{A} = \begin{bmatrix} A & 0 \\ df\wedge & -B \end{bmatrix}$ and $\overline{B} = \begin{bmatrix} B & 0 \\ df\wedge & -A \end{bmatrix}$. For details, see Proposition 2.0.13.

Note that we have the following diagram (commutativity will be checked below in Proposition 3.2.4)

$$\begin{tikzcd}
E_1 \arrow[r, "A"] \arrow[d, "\varphi_1"'] & E_0 \arrow[r, "B"] \arrow[d, "\varphi_0"] & E_1 \arrow[d, "\varphi_1"] \\
\begin{matrix} E_1 \\ \oplus \\ \Omega^1 \otimes E_0 \end{matrix} \arrow[r, "\overline{A}"] & \begin{matrix} E_0 \\ \oplus \\ \Omega^1 \otimes E_1 \end{matrix} \arrow[r, "\overline{B}"] & \begin{matrix} E_1 \\ \oplus \\ \Omega^1 \otimes E_0 \end{matrix}
\end{tikzcd} \tag{1}$$

$$\text{where} \qquad \varphi_1 = \begin{bmatrix} 1 \\ \nabla_0 A - (1 \otimes A)\nabla_1 \end{bmatrix}, \qquad \varphi_0 = \begin{bmatrix} 1 \\ \nabla_1 B - (1 \otimes B)\nabla_0 \end{bmatrix}.$$

Therefore we make the following definition:

Definition 3.2.4. Define $\varphi_{\mathcal{E},\nabla} : \mathcal{E} \to \mathcal{E}^{(1)}$ to be the morphism $\begin{bmatrix} 1 \\ At_{\mathcal{E}} \end{bmatrix}$.

Proposition 3.2.5. *$\varphi_{\mathcal{E},\nabla}$ is a strict morphism of matrix factorizations.*

Proof. Here we check the commutativity for the square on the left of (1); that is $\overline{A} \circ \varphi_1(x) = \varphi_0 \circ A(x)$ for any $x \in E_1$. It's enough to check the commutativity for the second component. Letting π_2 be the projection to the second component, we have

$$\begin{aligned} \pi_2 \circ \overline{A} \circ \varphi_1(x) &= df \wedge x - B(\nabla_0 A - A\nabla_1)(x) \\ &= df \wedge x - B\nabla_0 A(x) + f\nabla_1(x) \\ &= df \wedge x + f\nabla_1(x) - B\nabla_0 A(x) \\ &= \nabla_1(f \cdot x) - B\nabla_0 A(x) \\ &= (\nabla_1 B - B\nabla_0)(A(x)) \\ &= \pi_2 \circ \varphi_0 \circ A(x). \end{aligned}$$

The commutativity of the right square can be proved in a similar way. Therefore $\varphi_{\mathcal{E},\nabla}$ is a strict morphism of matrix factorizations.

□

Proposition 3.2.6. *$\varphi_{\mathcal{E},\nabla}$ is independent of the choice of connections up to homotopy.*

Proof. Suppose we choose other connections for the E_i's, say $\nabla_i' : E_i \to \Omega^1 \otimes_Q E_i$. We show that $\varphi = \varphi_{\mathcal{E},\nabla}$ is homotopic to $\varphi' = \varphi_{\mathcal{E},\nabla'}$.

First, $\nabla - \nabla'$ is a morphisms of Q-modules: for any $q \in Q$, $x \in E_i$,

$$(\nabla - \nabla')(q\cdot x) = \nabla(q\cdot x) - \nabla'(q\cdot x) = (dq\wedge x + q\cdot\nabla(x)) - (dq\wedge x + q\cdot\nabla'(x)) = q\cdot(\nabla - \nabla')(x).$$

Therefore, we can define $\alpha_0 = \begin{bmatrix} 0 \\ (\nabla_0 - \nabla_0') \end{bmatrix}, \alpha_1 = \begin{bmatrix} 0 \\ (\nabla_1 - \nabla_1') \end{bmatrix}$, which live in the following diagram

$$\begin{tikzcd}
E_1 \arrow[rr, "A"] \arrow[d] & & E_0 \arrow[rr, "B"] \arrow[d, "\varphi-\varphi'"] \arrow[dll, dotted, "\alpha_0"] & & E_1 \arrow[d] \arrow[dll, dotted, "\alpha_1"] \\
\begin{smallmatrix} E_1 \\ \oplus \\ \Omega^1\otimes E_0 \end{smallmatrix} \arrow[rr, "\overline{A}"'] & & \begin{smallmatrix} E_0 \\ \oplus \\ \Omega^1\otimes E_1 \end{smallmatrix} \arrow[rr, "\overline{B}"'] & & \begin{smallmatrix} E_1 \\ \oplus \\ \Omega^1\otimes E_0 \end{smallmatrix}
\end{tikzcd}$$

It's easy to check that $\overline{A}\circ\alpha_0 + \alpha_1\circ B = \varphi - \varphi'$ and similarly for the other square.

$\square$

Therefore, we usually drop the ∇ from the notation $\varphi_{\mathcal{E},\nabla}$ to simply write it as $\varphi_{\mathcal{E}}$. When Q is local or if we take E_i to be free Q-modules, the Atiyah clase is typically like that of the following example.

Example 3.2.7. For $\mathcal{E} = (Q^n \xrightarrow{A} Q^n \xrightarrow{B} Q^n)$, $df = AdB + (dA)B$, where $dA = (d_{Q/k}(a_{ij}))$ for a matrix $A = (a_{ij})$. Since φ is independent of choice of connection, we can choose the exterior differential d for the Atiyah class, i.e, $\nabla_i = d$ for $i = 0, 1$. First note that we have $d\circ A - A\circ d = dA\cdot$, because $(d\circ A - A\circ d)(x) = d(A\cdot x) - A\cdot(dx) = dA\cdot x + A\cdot dx - A\cdot dx = dA\cdot x$. Therefore

$$At_{\mathcal{E}} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} - \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & d\circ A - A\circ d \\ d\circ B - B\circ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & dA \\ dB & 0 \end{bmatrix},$$

hence $At^i_{\mathcal{E}} =$

$$\begin{bmatrix} \overbrace{dAdB\cdots dAdB}^{i} & 0 \\ 0 & \underbrace{dBdA\cdots dBdA}_{i} \end{bmatrix} \text{(i even)}$$

or

$$\begin{bmatrix} 0 & \overbrace{dAdB\cdots dA}^{i} \\ \underbrace{dBdA\cdots dB}_{i} & 0 \end{bmatrix} \text{(i odd).}$$

Definition 3.2.8. Define $\mathcal{E}^{(i)} := (\ Q \xrightarrow{df\wedge} \Omega^1\ )^{\otimes i} \otimes_{mf} \mathcal{E}$. Also define morphisms

$$\varphi_{\mathcal{E}}^{(i)} := 1^{\otimes i-1} \otimes \varphi_{\mathcal{E}} : \quad \mathcal{E}^{(i-1)} \to \mathcal{E}^{(i)}.$$

By our definition, $\varphi_{\mathcal{E}} = \varphi_{\mathcal{E}}^{(1)}$. Note that $\varphi_{\mathcal{E}}^{(i)}$ can be written in the form: $\begin{bmatrix} I_{2^i} \\ At_{\mathcal{E}} \cdot I_{2^i} \end{bmatrix}$, where I_{2^i} means the $2^i \times 2^i$ identity matrix.

The following illustrates what we mean by $\mathcal{E}^{(i)}$ and $\varphi_{\mathcal{E}}^{(i)}$.

$$\begin{tikzcd}
\mathcal{E} = & E_1 \arrow[r, "A"] \arrow[d] & E_0 \arrow[r, "B"] \arrow[d, "\varphi"] & E_1 \arrow[d] \\
\mathcal{E}^{(1)} = & \begin{array}{c} E_1 \\ \oplus \\ \Omega^1 \otimes E_0 \end{array} \arrow[r, "\overline{A}"] \arrow[d] & \begin{array}{c} E_0 \\ \oplus \\ \Omega^1 \otimes E_1 \end{array} \arrow[r, "\overline{B}"] \arrow[d] & \begin{array}{c} E_1 \\ \oplus \\ \Omega^1 \otimes E_0 \end{array} \arrow[d] \\
\mathcal{E}^{(2)} = & \begin{array}{c} E_1 \\ \oplus \\ \Omega^1 \otimes E_0 \\ \bigoplus \\ \Omega^1 \otimes E_0 \\ \oplus \\ \Omega^1 \otimes \Omega^1 \otimes E_1 \end{array} \arrow[r, "\overline{\overline{A}}"] \arrow[d] & \begin{array}{c} E_0 \\ \oplus \\ \Omega^1 \otimes E_1 \\ \bigoplus \\ \Omega^1 \otimes E_1 \\ \oplus \\ \Omega^1 \otimes \Omega^1 \otimes E_0 \end{array} \arrow[r, "\overline{\overline{B}}"] \arrow[d] & \begin{array}{c} E_1 \\ \oplus \\ \Omega^1 \otimes E_0 \\ \bigoplus \\ \Omega^1 \otimes E_0 \\ \oplus \\ \Omega^1 \otimes \Omega^1 \otimes E_1 \end{array} \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

and for $e \in \mathcal{E}$,

$$\varphi_{\mathcal{E}}(e) = \begin{bmatrix} 1 \\ At \end{bmatrix}(e) = \begin{bmatrix} e \\ At(e) \end{bmatrix},$$

$$\varphi_{\mathcal{E}}^{(1)}\left(\begin{bmatrix} e \\ At(e) \end{bmatrix}\right) = \begin{bmatrix} 1 & \\ & 1 \\ At & \\ & At \end{bmatrix} \cdot \begin{bmatrix} e \\ At(e) \end{bmatrix} = \begin{bmatrix} e \\ At(e) \\ At(e) \\ \widetilde{At^2}(e) \end{bmatrix} = \begin{bmatrix} e \\ 2At(e) \\ \widetilde{At^2}(e) \end{bmatrix}$$

and

$$\varphi_{\mathcal{E}}^{(2)}\left(\begin{bmatrix} e \\ 2At(e) \\ \widetilde{At^2}(e) \end{bmatrix}\right) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ At & & \\ & At & \\ & & At \end{bmatrix} \cdot \begin{bmatrix} e \\ 2At(e) \\ \widetilde{At^2}(e) \end{bmatrix} = \begin{bmatrix} e \\ 2At(e) \\ \widetilde{At^2}(e) \\ At(e) \\ 2\widetilde{At^2}(e) \\ \widetilde{At^3}(e) \end{bmatrix} = \begin{bmatrix} e \\ 3At(e) \\ 3\widetilde{At^2}(e) \\ \widetilde{At^3}(e) \end{bmatrix}, \cdots$$

Corollary 3.2.9. *$\varphi_{\mathcal{E}}^{(i)}$ is a strict morphism of matrix factorizations and it is independent of the choice of connections (up to homotopy) for any i.*

3.2.2 The map φ^n

For any natural number n, denote the complex

$$Q \xrightarrow{ndf\wedge} \Omega^1 \xrightarrow{(n-1)df\wedge} \Omega^2 \xrightarrow{(n-2)df\wedge} \cdots \xrightarrow{df\wedge} \Omega^n \qquad (*)$$

by $\Omega^{(n)}_{Q,df}$, where $idf\wedge$ denotes left multiplication by idf (i.e., $w_1\wedge\cdots\wedge w_n \longmapsto idf\wedge w_1\wedge\cdots\wedge w_n$), for any $0\leqslant i\leqslant n$.

There is a natural map of chain complexes $(Q\to\Omega^1)^{\otimes n}\to\Omega^{(n)}_{Q,df}$, induced from natural Q-module homomorphisms of the following diagram

$$\begin{tikzcd} (Q\oplus\Omega^1)^{\otimes n} \arrow[r, hook] \arrow[dr, dashed] & (\bigoplus_{i\geqslant 0}\Omega^i)^{\otimes n} \arrow[d, "\wedge"] \\ & \bigoplus_{i\geqslant 0}\Omega^i \end{tikzcd}$$

We denote this map again by $\wedge$.

Proposition 3.2.10. *The map $\wedge : (Q\to\Omega^1)^{\otimes n}\to\Omega^{(n)}_{Q,df}$ is a map of complexes.*

Proof. The map $\wedge$ is obviously a Q-module homomorphism, so we just need to show $\wedge$ commutes with the differentials of complexes.

Let us denote the differential in $(Q\to\Omega^1)^{\otimes n}$ by ∂ and the differential in $\Omega^{(n)}_{Q,df}$ by ∂'. Note that for an element $u\in\Omega^m$, $\partial' u=(n-m)df\wedge u$ and for $v\in Q\oplus\Omega^1$,

$$\partial(v)=\begin{cases} df\wedge v, & \text{if } |v|=0\\ 0, & \text{else.}\end{cases}$$

Therefore, for $a_1\otimes\cdots\otimes a_m\in\Omega^m$, where $a_i\in\Omega^1$,

$$\wedge\partial(a_1\otimes\cdots\otimes a_m)$$

$$= \wedge(\sum_{i=1}^{m}(-1)^{|a_1|+\cdots+|a_{i-1}|}a_1 \otimes \cdots \otimes \partial(a_i) \otimes \cdots \otimes a_m)$$

$$= \sum_{i=1}^{m}(-1)^{|a_1|+\cdots+|a_{i-1}|}a_1 \wedge \cdots \wedge \partial(a_i) \wedge \cdots \wedge a_m$$

$$= \sum_{i=1}^{m}(-1)^{|a_1|+\cdots+|a_{i-1}|}a_1 \wedge \cdots \wedge (df \wedge a_i) \wedge \cdots \wedge a_m$$

$$= \sum_{i=1 \text{ and } |a_i|=0}^{m}(-1)^{2(|a_1|+\cdots+|a_{i-1}|)}df \wedge a_1 \wedge \cdots \wedge a_m$$

$$= (n-m)df \wedge a_1 \wedge \cdots \wedge a_m$$

Also,

$$\partial' \wedge (a_1 \otimes \cdots \otimes a_m)$$

$$= \partial'(a_1 \wedge \cdots \wedge a_m)$$

$$= (n-m)df \wedge a_1 \wedge \cdots \wedge a_m$$

This completes the proof.

□

We obviously have :

Corollary 3.2.11. $\wedge \otimes 1_{\mathcal{E}} : (Q \to \Omega^1)^{\otimes n} \otimes_{mf} \mathcal{E} \to \Omega^{(n)}_{Q,df} \otimes_{mf} \mathcal{E}$ *is a strict morphism of matrix factorizations, for any matrix factorization* $\mathcal{E}$.

Definition 3.2.12. Define $\varphi^n : \mathcal{E} \to \Omega^{(n)}_{Q,df} \otimes_{mf} \mathcal{E}$ to be the composition $(\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(n)} \circ \varphi^{(n-1)} \circ \cdots \circ \varphi^{(1)}$; i.e., φ^n is the composition of the following chain of strict morphisms

$$\mathcal{E} \xrightarrow{\varphi^{(1)}} \mathcal{E}^{(1)} \xrightarrow{\varphi^{(2)}} \mathcal{E}^{(2)} \xrightarrow{\varphi^{(3)}} \cdots \xrightarrow{\varphi^{(n)}} \mathcal{E}^{(n)} = (Q \to \Omega^1)^{\otimes n} \otimes_{mf} \mathcal{E} \xrightarrow{\wedge \otimes 1_{\mathcal{E}}} \Omega^{(n)}_{Q,df} \otimes_{mf} \mathcal{E}$$

Corollary 3.2.13. φ^n *is a strict morphism of matrix factorizations and is independent of choice of connections up to homotopy.*

Proof. We know that each of the $\varphi^{(i)}$'s is independent of choice of connections, and thus φ^n is too.

□

Proposition 3.2.14. *We have* $(\wedge \otimes 1_{\mathcal{E}}) \circ (1_{\Omega^1[1]^{\otimes(i-1)}} \otimes At) = (\wedge \otimes 1_{\mathcal{E}}) \circ (1_{\Omega^{i-1}} \otimes At) \circ (\wedge \otimes 1_{\mathcal{E}})$*, for all* i*; that is, the following diagram commutes:*

$$\begin{tikzcd}
\overbrace{\Omega^1[1] \otimes_{mf} \cdots \otimes_{mf} \Omega^1[1]}^{i-1} \otimes_{mf} \mathcal{E} \arrow[r, "\wedge\otimes 1_{\mathcal{E}}"] \arrow[d, "1_{\Omega^1[1]^{\otimes(i-1)}}\otimes At"'] & \Omega^{i-1}[i-1] \otimes_{mf} \mathcal{E} \arrow[d, "1_{\Omega^{i-1}}\otimes At"] \\
\underbrace{\Omega^1[1] \otimes_{mf} \Omega^1[1] \otimes_{mf} \cdots \otimes_{mf} \Omega^1[1]}_{i} \otimes_{mf} \mathcal{E} \arrow[dr, "\wedge\otimes 1_{\mathcal{E}}"'] & \Omega^{i-1}[i-1] \otimes_{mf} \Omega^1[1] \otimes_{mf} \mathcal{E} \arrow[d, "\wedge\otimes 1_{\mathcal{E}}"] \\
 & \Omega^i[i] \otimes_{mf} \mathcal{E}
\end{tikzcd}$$

Proof. We can check this directly. For example, it is obvious for $i = 1$. □

For the sake of simplicity, we will drop all the $1 \otimes \cdots \otimes 1$ if there is no confusion from now on. For example, for the above proposition, we will in fact write it as $(\wedge \otimes 1_{\mathcal{E}}) \circ At = (\wedge \otimes 1_{\mathcal{E}}) \circ At \circ (\wedge \otimes 1_{\mathcal{E}})$.

Similarly, we have the following corollary.

Corollary 3.2.15. $(\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(i)} = (\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(i)} \circ (\wedge \otimes 1_{\mathcal{E}})$*, for any* i*. That is, we have the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{E}^{(i-1)} \arrow[r, "\varphi^{(i)}"] \arrow[d, "\wedge\otimes 1_{\mathcal{E}}"'] & \mathcal{E}^{(i)} \arrow[r, "\wedge\otimes 1_{\mathcal{E}}"] & \Omega^{(i)}_{Q,df} \otimes_{mf} \mathcal{E} \\
\Omega^{(i-1)}_{Q,df} \otimes_{mf} \mathcal{E} \arrow[r, "\varphi^{(i)}"'] & \Omega^{(i-1)}_{Q,df} \otimes_{mf} (Q \to \Omega^1) \otimes_{mf} \mathcal{E} \arrow[ur, "\wedge\otimes 1_{\mathcal{E}}"'] &
\end{tikzcd}$$

Proof. Note that $(\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(i)} = (\wedge \otimes 1_{\mathcal{E}}) \circ \begin{bmatrix} I_{2^i} \\ At \cdot I_{2^i} \end{bmatrix}$

$$= \begin{bmatrix} I_{2^i} \\ (\wedge \otimes 1_{\mathcal{E}}) \circ At \cdot I_{2^i} \end{bmatrix}$$

$$= \begin{bmatrix} I_{2^i} \\ (\wedge \otimes 1_{\mathcal{E}}) \circ At \circ (\wedge \otimes 1_{\mathcal{E}}) \cdot I_{2^i} \end{bmatrix}$$

$$= (\wedge \otimes 1_{\mathcal{E}}) \circ \begin{bmatrix} I_{2^i} \\ At \cdot I_{2^i} \end{bmatrix} \circ (\wedge \otimes 1_{\mathcal{E}})$$

$$= (\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(i)} \circ (\wedge \otimes 1_{\mathcal{E}}).$$

$\square$

Corollary 3.2.16. *We have* $\varphi^n = \sum_{i=0}^{n} \binom{n}{i} At^i \in \Omega^{(n)}_{Q,df} \otimes_{mf} \mathcal{E}$.

Proof. The base case is clear (look at the statement right before Corollary 3.2.8).

By induction, say $\varphi^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} At^i$, then

$$\varphi^n = (\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(n)} \circ \varphi^{(n-1)} \circ \cdots \circ \varphi$$

$$= (\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(n)} \circ (\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(n-1)} \circ \cdots \circ \varphi$$

$$= (\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(n)} \circ \varphi^{n-1}$$

$$= (\wedge \otimes 1_{\mathcal{E}}) \circ \varphi^{(n)} \circ \sum_{i=0}^{n-1} \binom{n-1}{i} At^i$$

$$= \begin{bmatrix} I_{2^n} \\ (\wedge \otimes 1_{\mathcal{E}}) \circ At \cdot I_{2^n} \end{bmatrix} \circ \sum_{i=0}^{n-1} \binom{n-1}{i} At^i$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} At^i + At(\sum_{i=0}^{n-1} \binom{n-1}{i} At^i)$$

$$= \sum_{i=0}^{n} \binom{n}{i} At^i. \qquad \square$$

3.2.3 The map $\widetilde{\varphi^n}$

For any integer n, define $(\Omega^\bullet, df, n)$ to be the complex :

$$Q \xrightarrow{df\wedge} \Omega^1 \xrightarrow{df\wedge} \Omega^2 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega^n \ .$$

From now on, we assume in addition that $k \supset \mathbb{Q}$. Under this assumption, there is an isomorphism of complexes $\Omega^{(n)}_{Q,df} \to (\Omega^\bullet, df, n)$ defined by

$$\begin{tikzcd}
Q \arrow[r, "ndf\wedge"] \arrow[d, "="] & \Omega^1 \arrow[r, "(n-1)df\wedge"] \arrow[d, "\frac{1}{n}"] & \Omega^2 \arrow[r, "(n-2)df\wedge"] \arrow[d, "\frac{1}{n(n-1)}"] & \cdots \arrow[r, "df\wedge"] & \Omega^n \arrow[d, "\frac{1}{n!}"] \\
Q \arrow[r, "df\wedge"] & \Omega^1 \arrow[r, "df\wedge"] & \Omega^2 \arrow[r, "df\wedge"] & \cdots \arrow[r, "df\wedge"] & \Omega^n
\end{tikzcd} .$$

We compose φ^n with the above isomorphism to get a morphism which is now called $\widetilde{\varphi^n}$; i.e.,

$$\widetilde{\varphi^n} : \ \mathcal{E} \longrightarrow (\Omega^\bullet, df, n) \otimes_{mf} \mathcal{E} \ .$$

By Proposition 3.2.15, we have an expression for $\widetilde{\varphi^n}$:

$$\widetilde{\varphi^n} = \sum_{i=0}^{n} \frac{1}{i!} At^i.$$

3.3 The Chern character for matrix factorizations and its basic properties

For a k-algebra Q, with k a commutative unital ring that contains $\mathbb{Q}$, we constructed (for a given $n \in \mathbb{N}$) a strict morphism of matrix factorizations $\widetilde{\varphi^n} : \mathcal{E} \to (\Omega^\bullet, df, n) \otimes_{mf} \mathcal{E}$ in last section. Now, we can define a Chern character for matrix factorizations.

3.3.1 Supertrace

Let Q be any commutative ring and M a finitely generated projective Q-module. Let $M^* = Hom_Q(M, Q)$ be the dual of M. Consider the two maps

$$End_Q(M) \xleftarrow{\ \xi\ } M^* \otimes_Q M \xrightarrow{\ \varepsilon\ } Q$$

given by $\xi(\alpha \otimes n)(m) = \alpha(m)n$, with $m, n \in M, \alpha \in M^*$ and by $\varepsilon(\alpha \otimes n) = \alpha(n)$ respectively. If M is a finitely generated projective Q-module, then ξ is an isomorphism and the composite $\varepsilon \circ \xi^{-1}$ is the standard trace map: $\varepsilon \circ \xi^{-1} = tr$. Suppose that M is free of finite rank over Q, with Q a k-algebra. Then a Q-linear map $M \to M \otimes_Q \Omega^*_{Q/k}$, upon choice of basis, is a matrix with coefficients in $\Omega^*_{Q/k}$, that is, an element of $End_Q(M) \otimes_Q \Omega^*_{Q/k}$. Since a projective module is a direct summand of a free module, the same is true when M is projective, i.e.,

$$Hom_Q(M, M \otimes_Q \Omega^\bullet) \cong End_Q(M) \otimes_Q \Omega^\bullet.$$

Therefore, when $\mathcal{M}$ a matrix factorization, the underlying module M is projective so

At^i can be viewed as an element of $\Omega^i_{Q/k} \otimes_Q End_Q(M)$.

Definition 3.3.1. Given a $\mathbb{Z}/2$-graded finitely generated projective Q-module M and an endomorphism T of M of degree 0, using that $End_Q(M)_0 = End_Q(M_0 \oplus M_1)_0 = End_Q(M_0) \oplus End_Q(M_1)$, define

$$str(T) := tr(T_0) - tr(T_1) \in Q$$

where $T = T_0 \oplus T_1$ with $T_i \in End_Q(M_i)$, $i = 0, 1$.

Proposition 3.3.2.

1. *If $\alpha, \beta : \mathcal{E} \to \mathcal{E}$ are strict morphisms of matrix factorizations and α is homotopic to β, then $str(\alpha) = str(\beta)$.*

2. *Supertrace str is an invariant under cyclic permutations, i.e.,*

$$str(\alpha_1 \circ \cdots \circ \alpha_n) = str(\alpha_{\sigma(1)} \circ \cdots \circ \alpha_{\sigma(n)})$$

for σ a cyclic permutation of n elements.

Proof. Say $\mathcal{E} = (E_1 \underset{B}{\overset{A}{\rightleftarrows}} E_0)$

1. There are Q-module homomorphisms $x : E_0 \to E_1$ and $y : E_1 \to E_0$ such that $Ax + yB = \alpha_0 - \beta_0$ and $By + xA = \alpha_1 - \beta_1$. So

$$(Ax + yB) - (By + xA) = (\alpha_0 - \beta_0) - (\alpha_1 - \beta_1),$$

$$(Ax - xA) - (yB - By) = (\alpha_0 - \alpha_1) - (\beta_0 - \beta_1),$$

$$tr(Ax - xA) - tr(yB - By) = tr(\alpha_0 - \alpha_1) - tr(\beta_0 - \beta_1),$$

$$(tr(Ax) - tr(xA)) - (tr(yB) - tr(By)) = 0 = str(\alpha) - str(\beta),$$

$$str(\alpha) = str(\beta).$$

2. This is obvious since tr is an invariant under cyclic permutations.

$\square$

3.3.2 Chern character

Before making the definition of the Chern character of a matrix factorization, let's first recall the definition of a smooth algebra and prove a few propositions necessary for the definition.

Definition 3.3.3. 1. If k is an algebraically closed field, then Q is *smooth of relative dimension* d if it is of finite type, its dimension is d, and the module $\Omega^1_{Q/k}$ of differentials is a finitely generated locally free Q-module of rank d.

2. Let k be an arbitrary field, $\overline{k}$ its algebraic closure. Then Q is *smooth of relative dimension* d if $Q \otimes_k \overline{k}$ is smooth of relative dimension d over $\overline{k}$.

3. Let $\theta : Q \to Q'$ be a ring map, then θ is *smooth of relative dimension* d if it is flat, finitely presented, and for all primes $\mathfrak{p}$ of Q, the fibre ring $k(\mathfrak{p}) \otimes_Q Q'$ is smooth of relative dimension d over $k(\mathfrak{p})$, where $k(\mathfrak{p})$ is the residue field at $\mathfrak{p}$.

Proposition 3.3.4. *Suppose Q is a smooth k-algebra of relative dimension d with k a commutative unital ring that contains $\mathbb{Q}$. Then* $str(\widetilde{\varphi^d}) = str(\widetilde{\varphi^{d+1}}) = \cdots$.

Proof. We have $str(At^{d+1}) \in \Omega^{d+1} \otimes_{mf} \mathcal{E} = 0$. Therefore

$$str(\widetilde{\varphi^{d+1}}) = \sum_{i=0}^{d+1} \frac{1}{i!} str(At^i) = 1 + str(At) + \cdots + \frac{1}{d!} str(At^d) + \frac{1}{(d+1)!} str(At^{d+1})$$

$$= 1 + str(At) + \cdots + \frac{1}{d!} str(At^d) + 0$$

$$= str(\widetilde{\varphi^d})$$

i.e., $str(\widetilde{\varphi^d}) = str(\widetilde{\varphi^{d+1}})$ and hence $str(\widetilde{\varphi^d}) = str(\widetilde{\varphi^{d+i}})$ for any $i \geqslant 2$. □

Proposition 3.3.5. *Given any matrix factorization $\mathcal{E} = (E_1 \xrightarrow{A} E_0 \xrightarrow{B} E_1) \in [MF(Q,f)]$, $df \wedge str(At^i_{\mathcal{E}}) = 0$ in Ω^{i+1} for any i. If i is an odd integer, $str(At^i) = 0$.*

Proof. For the underlying finitely generated projective Q-module $E = E_0 \oplus E_1$ of $\mathcal{E}$, the trace homomorphism tr is $End(E) \otimes \Omega^\bullet \to \Omega^\bullet$, so $str(At^i_{\mathcal{E}}) \in \Omega^i$ since it's the difference of two elements in Ω^i.

It's enough to check this locally, so we adopt the notations used in Example 3.2.6. In particular, since the map $\Omega^{i+1}_{Q/k} \hookrightarrow \Omega^{i+1}_{Q[\frac{1}{f}]/k}$ is injective, it suffices to check $df \wedge str(At^i_{\mathcal{E}}) = 0$ after inverting f. Therefore we may assume A and B are invertible matrices with entries in $Q[\frac{1}{f}]$. Notice that $str(At^i) = 0$ when i is odd so we just need to show this when i is an even integer.

Since A is invertible, $B = f \cdot A^{-1}$, therefore

$$dB = df \cdot A^{-1} + f \cdot dA^{-1},$$

since $A^{-1} \cdot A = I \Longrightarrow dA^{-1} = -A^{-1} dA \cdot A^{-1}$, we get

$$dB = df \cdot A^{-1} - f A^{-1} dA \cdot A^{-1}$$

Also, since $df = dA \cdot B + A \cdot dB$ and say $i = 2l$,

$$df \wedge str(At^i_{\mathcal{E}}) = 2tr(df\, \overbrace{dAdBdAdB \cdots dAdB}^{l})$$

$$= 2tr(df dA(df \cdot A^{-1} - fA^{-1}dA \cdot A^{-1}) \cdots dA(df \cdot A^{-1} - fA^{-1}dA \cdot A^{-1}))$$

$$= 2tr(df dA(-fA^{-1}dA \cdot A^{-1}) \cdots dA(-fA^{-1}dA \cdot A^{-1}))$$

$$= (-1)^l 2tr(\underbrace{dA \cdot A^{-1} \cdots dA \cdot A^{-1}}_{\text{even numbers}}) = 0.$$

We have the last equality because switching matrices of odd forms is only going to introduces a sign. Also, nothing is the product $(dA \cdot A^{-1})^{\text{even}} = (dA \cdot A^{-1})^{\text{odd}} \cdots (dA \cdot A^{-1})^{\text{odd}}$ stays the same no matter how you switch. Therefore it has to be 0.

□

By the above proposition, we know that $str(At^i)$ vanishes when i is odd and is a cycle (of the complex $(\Omega^\bullet, df, n)$) when i is even, so it defines an element of the homology.

Now we are ready to give our definition of the Chern character. Assume that Q is now a smooth k-algebra of relative dimension n with k a commutative unital ring that contains $\mathbb{Q}$.

Definition 3.3.6. We define the *Chern character* of $\mathcal{E}$ to be

$$ch(\mathcal{E}) := str(\widetilde{\varphi^n}) = \sum_{i=0}^{n} \frac{1}{i!} str(At^i) \in H_0((\Omega^\cdot, df, n)_{\mathbb{Z}/2}) = \bigoplus_{i=0}^{n} \frac{ker(\Omega^{2i} \xrightarrow{df} \Omega^{2i+1})}{im(\Omega^{2i-1} \xrightarrow{df} \Omega^{2i})}.$$

Recall $(\Omega^\cdot, df, n)$ is the complex

$$Q \xrightarrow{df\wedge} \Omega^1 \xrightarrow{df\wedge} \Omega^2 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega^n$$

and $(\Omega^{\cdot}, df, n)_{\mathbb{Z}/2}$ the $\mathbb{Z}/2$-folding of this complex. Also, notice that by Proposition 3.3.5, we have $str(At^i) = 0$ when i is an odd integer. Therefore the Chern character is in fact $ch(\mathcal{E}) = \sum_{i\geqslant 0} \frac{1}{(2i)!} str(At^{2i})$.

Our definition of the Chern character is the same as the one in Platt [22]. For the special case when $Q = k[[x_1, \cdots, x_n]]$ and $f \in Q$ an isolated singularity, see the following example.

Example 3.3.7. Let $f \in Q = k[x_1, \cdots, x_n]$ be an isolated singularity at the origin (that is, the localizations at every prime except the maximal ideal $m = (x_1, \cdots, x_n)$ is regular), $\mathcal{E} = (Q^r \underset{B}{\overset{A}{\rightleftarrows}} Q^r)$ a matrix factorization with $d = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. E is a free Q-module, so as before, we can choose d, the exterior differential operator to be the connection and we get that $At = \begin{bmatrix} 0 & dA \\ dB & 0 \end{bmatrix}$. Also, notice that in this situation $(\Omega^{\bullet}, df, n)$ is exact except in position n (the dimension of Q) [31]. Therefore we have that $ch(\mathcal{E}) = \sum_{i=0}^{n} \frac{1}{i!} str(At^i) = \frac{1}{n!} str(dAdB \cdots dAdB)$, which agrees with the Chern character obtained by [14], [22] and [24]. It differs by a sign with the ones in [5], [7] and [23].

Proposition 3.3.8. *Given matrix factorizations* $\mathcal{E} = (E_1 \xrightarrow{A} E_0 \xrightarrow{B} E_1)$ *and* $\mathcal{E}' = (E_1' \xrightarrow{C} E_0' \xrightarrow{D} E_1')$ *in* $[MF(Q, f)]$, *a strict morphism* $\beta : \mathcal{E}' \to \mathcal{E}$, *the following diagram commutes up to homotopy*

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "\widetilde{\varphi_{\mathcal{E}}}"] \arrow[d, "\beta"] & (Q \xrightarrow{df \wedge} \Omega^1) \otimes_{mf} \mathcal{E}' \arrow[d, "1\otimes\beta"] \\ \mathcal{E} \arrow[r, "\widetilde{\varphi_{\mathcal{E}'}}"] & (Q \xrightarrow{df \wedge} \Omega^1) \otimes_{mf} \mathcal{E} \end{tikzcd} \cdot$$

Proof. Recall that $\widetilde{\varphi_{\mathcal{E}}} = \begin{bmatrix} 1 \\ At_{\mathcal{E}} \end{bmatrix} = 1 + At_{\mathcal{E}}$.

Choose connections ∇_i and ∇_i', then we can construct module homomorphisms $\psi_0 = \begin{bmatrix} 0 \\ (\nabla_0\beta - (1\otimes\beta)\nabla_0') \end{bmatrix}, \psi_1 = \begin{bmatrix} 0 \\ (\nabla_1\beta - (1\otimes\beta)\nabla_1') \end{bmatrix}$, which lives in the diagram

$$
\begin{tikzcd}
E_1' \arrow[rr, "C"] \arrow[d] & & E_0' \arrow[rr, "D"] \arrow[d, "\widetilde{\varphi_E}\circ\beta-(1\otimes\beta)\circ\widetilde{\varphi_{E'}}"] \arrow[dll, dotted, "\psi_0"'] & & E_1' \arrow[d] \arrow[dll, dotted, "\psi_1"] \\
\begin{smallmatrix} E_1 \\ \oplus \\ \Omega^1\otimes E_0 \end{smallmatrix} \arrow[rr, "\overline{A}"'] & & \begin{smallmatrix} E_0 \\ \oplus \\ \Omega^1\otimes E_1 \end{smallmatrix} \arrow[rr, "\overline{B}"'] & & \begin{smallmatrix} E_1 \\ \oplus \\ \Omega^1\otimes E_0 \end{smallmatrix}
\end{tikzcd}
$$

where $\widetilde{\varphi_{\mathcal{E}}}\circ\beta - (1\otimes\beta)\circ\widetilde{\varphi_{\mathcal{E}'}}$ is the matrix $\begin{bmatrix} 0 \\ At_{\mathcal{E}}\circ\beta - (1\otimes\beta)\circ At_{\mathcal{E}'} \end{bmatrix}$.

First, ψ_0 and ψ_1 are indeed module homomorphisms:

$$
\begin{aligned}
\psi_0(q\cdot x) &= \nabla_0\beta(q\cdot x) - (1\otimes\beta)\nabla_0'(q\cdot x) \\
&= \nabla_0(q\cdot\beta(x)) - (1\otimes\beta)(dq\wedge x + q\cdot\nabla_0'(x)) \\
&= dq\wedge\beta(x) + q\cdot\nabla_0\beta(x) - dq\wedge\beta(x) - (1\otimes\beta)(q\cdot\nabla_0'(x)) \\
&= q\cdot\nabla_0\beta(x) - q\cdot(1\otimes\beta)(\nabla_0'(x)) \\
&= q(\nabla_0\beta - (1\otimes\beta)\nabla_0')(x) \\
&= q\cdot\psi_0(x)
\end{aligned}
$$

The same argument shows that ψ_1 is also a module homomorphism.

We want to show that ψ_0 and ψ_1 give us a homotopy. For the degree 0 part, we need to show $(\widetilde{\varphi_{\mathcal{E}}}\circ\beta - (1\otimes\beta)\circ\widetilde{\varphi_{\mathcal{E}'}})_0 = \overline{A}\circ\psi_0 + \psi_1\circ D$.

Recall that $\overline{A} = \begin{bmatrix} A & 0 \\ df\wedge & -B \end{bmatrix}$. Elements are 2×1 column vectors, the equality in the first row is easy to see so we just check the equality for the second row. Hence,

$$\begin{aligned}
&(At_{\mathcal{E}} \circ \beta - (1 \otimes \beta) \circ At_{\mathcal{E}'})_0 \\
&= (\nabla_1 B - B\nabla_0)\beta - (1 \otimes \beta)(\nabla_1' D - D\nabla_0') \\
&= \nabla_1 B\beta - B\nabla_0\beta - (1 \otimes \beta)\nabla_1' D + (1 \otimes \beta) D\nabla_0' \\
&= \nabla_1 \beta D - B\nabla_0\beta - (1 \otimes \beta)\nabla_1' D + B(1 \otimes \beta)\nabla_0' \\
&= (\nabla_1\beta - (1 \otimes \beta)\nabla_1')D - B(\nabla_0\beta - (1 \otimes \beta)\nabla_0') \\
&= \psi_1 \circ D + \overline{A} \circ \psi_0
\end{aligned}$$

In the above calculation, we use $B\beta = \beta D$ and $(1 \otimes \beta)D = B(1 \otimes \beta)$ by the fact that β is a strict morphism of matrix factorizations, i.e., the following commutative diagram:

$$\begin{tikzcd}
E_1' \arrow[r, "C"] \arrow[d, "\beta"'] & E_0' \arrow[r, "D"] \arrow[d, "\beta"'] & E_1' \arrow[d, "\beta"'] \\
E_1 \arrow[r, "A"] & E_0 \arrow[r, "B"] & E_1
\end{tikzcd}$$

□

Corollary 3.3.9. *Under the same hypothesis as in Proposition 3.3.8, we have* $\widetilde{\varphi^n_{\mathcal{E}}} \circ \beta \sim (1 \otimes 1 \otimes \cdots \otimes 1 \otimes \beta) \circ \widetilde{\varphi^n_{\mathcal{E}'}}$

Proof. Indeed,

$$\widetilde{\varphi^n_{\mathcal{E}}} \circ \beta = \widetilde{\varphi^{n-1}_{\mathcal{E}}} \circ \widetilde{\varphi_{\mathcal{E}}} \circ \beta \sim \widetilde{\varphi^{n-1}_{\mathcal{E}}} \circ (1 \otimes \beta) \circ \widetilde{\varphi_{\mathcal{E}'}}$$

then by induction

$$\widetilde{\varphi^n_{\mathcal{E}}} \circ \beta \sim (1 \otimes 1 \otimes \cdots \otimes 1 \otimes \beta) \circ \widetilde{\varphi^n_{\mathcal{E}'}}.$$

□

Corollary 3.3.10. *We have $ch(\mathcal{E}) = ch(\mathcal{E}')$ for homotopy equivalent matrix factorizations $\mathcal{E}$ and $\mathcal{E}'$.*

Proof. Say we have $\mathcal{E} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathcal{E}'$, such that $\alpha \circ \beta \sim 1_{\mathcal{E}'}$ and $\beta \circ \alpha \sim 1_{\mathcal{E}}$, by Corollary 3.3.6

$$\widetilde{\varphi^n_{\mathcal{E}}} \circ \beta \circ \alpha \sim (1 \otimes 1 \otimes \cdots \otimes 1 \otimes \beta) \circ \widetilde{\varphi^n_{\mathcal{E}'}} \circ \alpha$$

Therefore, by Proposition 3.3.2

$$\begin{aligned} &str(\widetilde{\varphi^n_{\mathcal{E}}}) \\ =&str(\widetilde{\varphi^n_{\mathcal{E}}} \circ \beta \circ \alpha) \\ =&str((1^{\otimes n} \otimes \beta) \circ \widetilde{\varphi^n_{\mathcal{E}'}} \circ \alpha) \\ =&str(\alpha \circ (1^{\otimes n} \otimes \beta) \circ \widetilde{\varphi^n_{\mathcal{E}'}}) \\ =&str(\widetilde{\varphi^n_{\mathcal{E}'}}) \end{aligned}$$

This gives

$$ch(\mathcal{E}) = ch(\mathcal{E}').$$

□

Theorem 3.3.11. *Given any distinguished triangle*

$$\mathcal{P} \xrightarrow{\theta} \mathcal{Q} \longrightarrow cone(\theta) \longrightarrow \mathcal{P}[1]$$

in $[MF(Q,f)]$, we have

$$ch(\mathcal{Q}) = ch(\mathcal{P}) + ch(cone(\theta)). \qquad (*)$$

Proof. We will prove this theorem by explicit calculation of the Chern character.

First, it's enough to check equality for the even components, as discussed in the proof of Proposition 3.3.5. By definition $ch(\mathcal{E}) = str(\widetilde{\varphi_{\mathcal{E}}^{n}}) = str(\sum_{i=0}^{n} \frac{1}{i!} At_{\mathcal{E}}^{i}) = \sum_{i=0}^{n} \frac{1}{i!} str(At_{\mathcal{E}}^{i})$ for any matrix factorization $\mathcal{E}$, so it's enough to prove $str(At_{\mathcal{Q}}^{2i}) = str(At_{\mathcal{P}}^{2i}) + str(At_{cone(\theta)}^{2i})$, for all even integers $2i$ between 2 and n.

Say $\mathcal{P} = (P_1 \xrightarrow{A} P_0 \xrightarrow{B} P_1)$ and $\mathcal{Q} = (Q_1 \xrightarrow{C} Q_0 \xrightarrow{D} Q_1)$, the mapping cone is

$$cone(\theta) = (Q_1 \oplus P_0 \underset{\begin{bmatrix} D & f_1 \\ 0 & -A \end{bmatrix}}{\overset{\begin{bmatrix} C & f_0 \\ 0 & -B \end{bmatrix}}{\rightleftarrows}} Q_0 \oplus P_1).$$

Choose any connections ∇_0^P and ∇_1^P for $\mathcal{P}$, similarly ∇_0^Q and ∇_1^Q for $\mathcal{Q}$. We have induced connections for $cone(\theta)$:

$$\nabla_1^{cone(\theta)} = \begin{bmatrix} \nabla_0^{\mathcal{Q}} & \\ & \nabla_1^{\mathcal{P}} \end{bmatrix} \qquad \nabla_0^{cone(\theta)} = \begin{bmatrix} \nabla_1^{\mathcal{Q}} & \\ & \nabla_0^{\mathcal{P}} \end{bmatrix}$$

Since the Chern character is independent of choice of connections, we use these to compute the Atiyah class $At_{cone(\theta)}$ for $cone(\theta)$, which is just

$$At_{cone(\theta)} = \begin{bmatrix} \nabla_1^{\mathcal{Q}} & & & \\ & \nabla_0^{\mathcal{P}} & & \\ & & \nabla_0^{\mathcal{Q}} & \\ & & & \nabla_1^{\mathcal{P}} \end{bmatrix} \cdot \begin{bmatrix} & & C & f_0 \\ & & & -B \\ D & f_1 & & \\ & -A & & \end{bmatrix} - \begin{bmatrix} & & C & f_0 \\ & & & -B \\ D & f_1 & & \\ & -A & & \end{bmatrix} \cdot \begin{bmatrix} \nabla_1^{\mathcal{Q}} & & & \\ & \nabla_0^{\mathcal{P}} & & \\ & & \nabla_0^{\mathcal{Q}} & \\ & & & \nabla_1^{\mathcal{P}} \end{bmatrix}$$

$$= \begin{bmatrix} & & X & * \\ & & & Z \\ Y & * & & \\ & W & & \end{bmatrix}$$

where

$$X = (\nabla_1^{\mathcal{Q}} C - C \nabla_0^{\mathcal{Q}}),$$

$$Y = (\nabla_0^{\mathcal{Q}} D - D \nabla_1^{\mathcal{Q}}),$$

$$Z = (B \nabla_1^{\mathcal{P}} - \nabla_0^{\mathcal{P}} B), \text{ and}$$

$$W = (A \nabla_0^{\mathcal{P}} - \nabla_1^{\mathcal{P}} A).$$

Hence

$$At^2_{cone(\theta)} = \begin{bmatrix} XY & * & & \\ & ZW & & \\ & & YX & * \\ & & & WZ \end{bmatrix}$$

Therefore,

$$At^{2i}_{cone(\theta)} = \begin{bmatrix} (XY)^i & * & & \\ & (ZW)^i & & \\ & & (YX)^i & * \\ & & & (WZ)^i \end{bmatrix}$$

for any even integer $2i$ between 1 and n.

This gives that

$$\begin{aligned}
& str(At^{2i}_{cone(\theta)}) \\
&= tr\begin{bmatrix}(XY)^i & * \\ & (ZW)^i\end{bmatrix} - tr\begin{bmatrix}(YX)^i & * \\ & (WZ)^i\end{bmatrix} \\
&= 2tr((XY)^i) - 2tr((WZ)^i).
\end{aligned}$$

hence

$$\begin{aligned}
str(At^{2i}_{\mathcal{P}}) + str(At^{2i}_{cone(\theta)}) &= 2tr((WZ)^i) + 2tr((XY)^i) - 2tr((WZ)^i) \\
&= 2tr((XY)^i) = str(At^{2i}_{\mathcal{Q}}).
\end{aligned}$$

□

3.3.3 Grothendieck group

Recall that the Grothendieck group $K_0(T)$ of a triangulated category T is the free abelian group generated by isomorphism classes of objects of T, modulo the relations $[X]+[Z]=[Y]$ for distinguished triangles $X \to Y \to Z \to X[1]$.

Corollary 3.3.12. *The Chern character induces a map from $K_0[MF(Q,f)]$ to $H_0((\Omega^{\cdot}, df)_{\mathbb{Z}/2})$.*

Proof. Any distinguished triangle is isomorphic (in the homotopy category) to a triangle of the form of Theorem 3.3.8. Now apply Corollary 3.3.7 and Theorem 3.3.8. □

Now we will prove that the Chern character is a ring homomorphism.

Lemma 3.3.13. $\bigoplus_{f \in Q} K_0([MF(Q,f)])$ *is a ring via* $[\mathcal{E}]_f \cdot [\mathcal{F}]_g := [\mathcal{E} \otimes_{mf} \mathcal{F}]_{f+g}$.

Proof. First we have to show that the above multiplication is well-defined.

Since we know from the definition that $-\otimes_{mf}-$ preserves homotopy equivalences of matrix factorizations. For any given $\mathcal{E}\simeq\mathcal{E}'$ and $\mathcal{F}\simeq\mathcal{F}'$, we have $\mathcal{E}\otimes_{mf}\mathcal{F}\simeq\mathcal{E}\otimes_{mf}\mathcal{F}'\simeq\mathcal{E}'\otimes_{mf}\mathcal{F}'$, so the tensor product is well-defined on the free abelian group generated by isomorphism classes of matrix factorizations; we denote this group by $\bigoplus\limits_{f\in Q}\mathbb{Z}([MF(Q,f)])$.

Now, let's show that $\bigoplus\limits_{f\in Q}\mathbb{Z}([MF(Q,f)])$ is a commutative ring under the above multiplication.

1. $(\mathcal{E})_f\cdot(\mathcal{F})_g\in\bigoplus\limits_{f\in Q}\mathbb{Z}([MF(Q,f)])$.

2. $\mathcal{E}\otimes_{mf}\mathcal{F}\cong\mathcal{F}\otimes_{mf}\mathcal{E}$ so $(\mathcal{E})_f\cdot(\mathcal{F})_g=(\mathcal{F})_g\cdot(\mathcal{E})_f$.

3. $((\mathcal{E})_f\cdot(\mathcal{F})_g)\cdot(\mathcal{G})_h=(\mathcal{E}\otimes_{mf}\mathcal{F})_{f+g}\cdot(\mathcal{G})_h=((\mathcal{E}\otimes_{mf}\mathcal{F})\otimes_{mf}\mathcal{G})_{f+g+h}$

   Also, $(\mathcal{E})_f\cdot((\mathcal{F})_g\cdot(\mathcal{G})_h)=(\mathcal{E})_f\cdot((\mathcal{F}\otimes_{mf}\mathcal{G})_{g+h})=(\mathcal{E}\otimes_{mf}(\mathcal{F}\otimes_{mf}\mathcal{G}))_{f+g+h}$,

   where $(\mathcal{E})$ means the isomorphism class of $\mathcal{E}$. Hence the above shows that the multiplication is associative.

4. There is an identity $\mathbf{1}=(0\underset{0}{\overset{0}{\rightleftarrows}}Q)\in MF(Q,0)$ such that $(1)_0\cdot(\mathcal{E})_f=(\mathcal{E})_f$.

   Indeed, $\mathbf{1}\otimes_{mf}\mathcal{E}$ equals to

$$0\oplus E_1 \underset{\begin{bmatrix} & -1\otimes e_1 \\ 1\otimes e_0 & \end{bmatrix}}{\overset{\begin{bmatrix} & 1\otimes e_1 \\ -1\otimes e_0 & \end{bmatrix}}{\rightleftarrows}} 0\oplus E_0.$$

   Therefore we have an isomorphism of $\mathbf{1}\otimes_{mf}\mathcal{E}$ and $\mathcal{E}$, i.e., $(1)_0\cdot(\mathcal{E})_f=(\mathcal{E})_f$.

   Similarly, we also have $(\mathcal{E})_f\cdot(1)_0=(\mathcal{E})_f$.

The above shows that the isomorphism classes of all matrix factorizations is a monoid under $-\otimes_{mf}-$, so $\bigoplus_{f\in Q}\mathbb{Z}([MF(Q,f)])$ is in fact a commutative ring.

Finally, to show that this multiplication is well-defined on the quotient group, it's enough to prove that the subgroup

$$\{[\mathcal{Q}]-[\mathcal{P}]-[\mathcal{W}] : \mathcal{P}\to\mathcal{Q}\to\mathcal{W}\to\mathcal{P}[1] \text{ a distinguished triangle}\}$$

is an ideal inside $\bigoplus_{f\in Q}\mathbb{Z}([MF(Q,f)])$. This amounts to the following fact: tensor product is a triangulated functor (which is Proposition 2.0.14).

□

Proposition 3.3.14. *Define $K_f(Q) := \bigoplus_{i\in\mathbb{Z}_{\geqslant 0}} K_0([MF(Q,if)])$; this is in fact a subring of $\bigoplus_{f\in Q} K_0([MF(Q,f)])$.*

Proof. Given any $[a],[b]\in K_f(Q)$, $[a]+[b]\in K_f(Q)$; $[a]\cdot[b]\in K_f(Q)$; $[\mathbf{1}]\in K_f(Q)$; $[-a]\in K_f(Q)$ (since $[-a]=[a[1]]\in K_f(Q)$).

□

Lemma 3.3.15. $\bigoplus_{f\in Q} H_0((\Omega^{\cdot},df,n)_{\mathbb{Z}/2})$ *is a commutative ring via* $\wedge$.

Proof. First, assume n is an odd integer. A similar proof works when n is even. Recall that $(\Omega^{\bullet},df,n)$ is the following complex:

$$Q \xrightarrow{df\wedge} \Omega^1 \xrightarrow{df\wedge} \Omega^2 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega^n .$$

Therefore the $\mathbb{Z}/2$-folding $(\Omega^{\bullet},df,n)_{\mathbb{Z}/2}$ is the matrix factorization $\mathcal{E}=(E_1 \xrightarrow{D_1} E_0 \xrightarrow{D_0} E_1)$ where

$$E_1=\Omega^1\oplus\Omega^3\oplus\cdots\oplus\Omega^n, \qquad E_0=Q\oplus\Omega^2\oplus\Omega^4\oplus\cdots\oplus\Omega^{n-1}$$

and $$D_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ df\wedge & 0 & \cdots & \\ 0 & df\wedge & 0 & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & df\wedge \end{bmatrix}_{\frac{n+1}{2}\times\frac{n+1}{2}}, \quad D_0 = \begin{bmatrix} df\wedge & 0 & \cdots & \\ 0 & df\wedge & 0 & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & df\wedge \end{bmatrix}_{\frac{n+1}{2}\times\frac{n+1}{2}}.$$

Hence $H_0((\Omega^{\cdot}, df, n)_{\mathbb{Z}/2}) = \dfrac{ker D_0}{im D_1}$ is a ring by properties of the wedge product. For example, for any $a, b, c \in \bigoplus_{i \text{ even}} \Omega^i$, we have

$$(a \wedge b) \wedge c = a \wedge (b \wedge c).$$

It is not hard to see that elements on the two sides of the above equation determine the same element in homology. The same holds for other conditions to make a set into a ring. It is commutative since we are dealing only with even forms (in general $u \wedge v = (-1)^{ij} v \wedge u$ for $a \in \Omega^i$ and $b \in \Omega^j$; therefore, i, j even means $u \wedge v = v \wedge u$). From this we see that $\bigoplus_{i \in \mathbb{Z}_{\geqslant 0}} H_0((\Omega^{\cdot}, df, n)_{\mathbb{Z}/2})$ is a commutative ring.

It is clear that $\bigoplus_{i \in \mathbb{Z}_{\geqslant 0}} H_0((\Omega^{\cdot}, idf, n)_{\mathbb{Z}/2})$ is a subring of $\bigoplus_{f \in Q} H_0((\Omega^{\cdot}, df, n)_{\mathbb{Z}/2})$.

□

Theorem 3.3.16. *Given two matrix factorizations $\mathcal{E} \in [MF(Q, f)]$ and $\mathcal{F} \in [MF(Q, g)]$, we have the following commutative diagram*

$$\begin{tikzcd} \mathcal{E} \otimes_{mf} \mathcal{F} \arrow[r, "\widetilde{\varphi^n_{\mathcal{E}}} \otimes \widetilde{\varphi^n_{\mathcal{F}}}"] \arrow[dr, "\widetilde{\varphi^n_{\mathcal{E}\otimes_{mf}\mathcal{F}}}"'] & ((\Omega^\bullet, df, n) \otimes_{mf} \mathcal{E}) \otimes_{mf} ((\Omega^\bullet, dg, n) \otimes_{mf} \mathcal{F}) \arrow[d, "\wedge\circ(1\otimes\tau\otimes 1)"] \\ & (\Omega^\bullet, df + dg, n) \otimes_{mf} (\mathcal{E} \otimes_{mf} \mathcal{F}) \end{tikzcd}$$

where $\tau : \mathcal{E} \otimes_{mf} (\Omega^\bullet, df, n) \to (\Omega^\bullet, df, n) \otimes_{mf} \mathcal{E}$ is the isomorphism $\tau(a \otimes b) =$

$(-1)^{|a||b|} b \otimes a$.

Remark 3.3.17. *The above diagram makes sense, since the relative dimension of Q over k is n. After changing the position, wedging things together, terms with degree higher than n vanish.*

Proof. (of Theorem 3.3.16) First, by Proposition 3.1.5, for underlying modules E and F, if we choose a connection ∇_E for E and ∇_F for F, then there is a natural connection for the tensor product: $\nabla_E \otimes 1 + 1 \otimes \nabla_F$. Also, the differential for the tensor product of the two matrix factorizations is given by $d_{\mathcal{E}\otimes_{mf}\mathcal{F}}(e \otimes f) = d_{\mathcal{E}}(e) \otimes f + (-1)^{|e|} e \otimes d_{\mathcal{F}}(f)$, where $e \in E$ and $f \in F$. After a careful calculation, we have that

$$At_{\mathcal{E}\otimes_{mf}\mathcal{F}}(e \otimes f) = At_{\mathcal{E}}(e) \otimes f + (-1)^{|e|} e \otimes At_{\mathcal{F}}(f), \qquad (*)$$

i.e., $At_{\mathcal{E}\otimes_{mf}\mathcal{F}} = At_{\mathcal{E}} \otimes 1 + \tau(1 \otimes At_{\mathcal{F}})$, where τ is the map in the statement of the theorem.

Another observation we want to make before looking into $\widetilde{\varphi^n_{E\otimes F}}$ is that

$$\wedge \circ (At_{\mathcal{E}} \otimes 1) \circ \tau(1 \otimes At_{\mathcal{F}}) = \wedge \circ \tau(1 \otimes At_{\mathcal{F}}) \circ (At_{\mathcal{E}} \otimes 1).$$

In fact, we have

$$\wedge \circ (At_{\mathcal{E}} \otimes 1) \circ (\tau(1 \otimes At_{\mathcal{F}}))(e \otimes f) = \wedge \circ (At_{\mathcal{E}} \otimes 1)((-1)^{|e|} \sigma(e \otimes At_{\mathcal{F}}(f))),$$

where σ is the same as τ but doesn't introduce a sign. Say for simplicity that $At_{\mathcal{F}}(f) = u \otimes f'$ and $At_{\mathcal{E}}(e) = w \otimes e'$ (these should really be sums of simple tensors, nonetheless,

the idea is the same and the case for simple tensors is more clear), then the above is

$$\wedge \circ (At_{\mathcal{E}} \otimes 1) \circ ((-1)^{|e|} u \otimes e \otimes f')$$

$$= (-1)^{|e|} \wedge (u \otimes At_{\mathcal{E}}(e) \otimes f')$$

$$= (-1)^{|e|} \wedge (u \otimes w \otimes e' \otimes f')$$

$$= (-1)^{|e|} \cdot (u \wedge w \otimes e' \otimes f')$$

$$= -(-1)^{|e|} \cdot (w \wedge u \otimes e' \otimes f')$$

$$= -(-1)^{|e|} \wedge \sigma(At_{\mathcal{E}}(e) \otimes At_{\mathcal{F}}(f)).$$

For $\wedge \circ \tau(1 \otimes At_{\mathcal{F}}) \circ (At_{\mathcal{E}} \otimes 1)(e \otimes f)$, we have

$$\wedge \circ \tau(1 \otimes At_{\mathcal{F}}) \circ (At_{\mathcal{E}} \otimes 1)(e \otimes f)$$

$$= \wedge \circ \tau(1 \otimes At_{\mathcal{F}})(At_{\mathcal{E}}(e) \otimes f)$$

$$= \wedge \circ \tau(1 \otimes At_{\mathcal{F}})(w \otimes e' \otimes f)$$

$$= \wedge \circ \tau(w \otimes e' \otimes At_{\mathcal{F}}(f))$$

$$= \wedge \circ \tau(w \otimes e' \otimes u \otimes f')$$

$$= (-1)^{|e'|} \cdot \wedge(w \otimes u \otimes e' \otimes f')$$

$$= (-1)^{|e|+1} \cdot (w \wedge u \otimes e' \otimes f')$$

$$= -(-1)^{|e|} \wedge \sigma(At_{\mathcal{E}}(e) \otimes At_{\mathcal{F}}(f)).$$

Therefore, the two compositions of the operators $At_{\mathcal{E}} \otimes 1$ and $\tau(1 \otimes At_{\mathcal{F}})$ are the

same after $\wedge$ and more importantly we get $-(-1)^{|e|} \wedge \sigma(At_{\mathcal{E}}(e) \otimes At_{\mathcal{F}}(f))$ applying to the element $e \otimes f$. Then it is not hard to see that

$$\wedge(At_{\mathcal{E}} \otimes 1)^k \circ (\tau(1 \otimes At_{\mathcal{F}}))^s = \begin{cases} \wedge\sigma(At^k_{\mathcal{E}}(e) \otimes At^s_{\mathcal{F}}(f)), & \text{if one of } k, s \text{ is an even integer} \\ -(-1)^{|e|} \wedge \sigma(At^k_{\mathcal{E}}(e) \otimes At^s_{\mathcal{F}}(f)), & \text{if both } k \text{ and } s \text{ are odd integers.} \end{cases}$$

We can compute $\widetilde{\varphi^n_{E\otimes F}}$ by formula $(*)$, the degree i^{th} piece is (remember the notation ~ indicates we have already applied $\wedge$ to the Atiyah class)

$$\frac{1}{i!} At^i_{\mathcal{E}\otimes_{mf}\mathcal{F}} = \frac{1}{i!}(At_{\mathcal{E}} \otimes 1 + \tau(1 \otimes At_{\mathcal{F}}))^i$$

$$= \begin{cases} \frac{1}{i!} \sum_{k+s=i} \binom{i}{k} \wedge \sigma(At^k_{\mathcal{E}} \otimes At^s_{\mathcal{F}}), & \text{if one of } k, s \text{ is even} \\ -(-1)^{|e|} \frac{1}{i!} \sum_{k+s=i} \binom{i}{k} \wedge \sigma(At^k_{\mathcal{E}} \otimes At^s_{\mathcal{F}}), & \text{if both } k \text{ and } s \text{ are odd} \end{cases}$$

$$= \begin{cases} \sum_{k+s=i} \frac{1}{k!s!} \wedge \sigma(At^k_{\mathcal{E}} \otimes At^s_{\mathcal{F}}), & \text{if one of } k, s \text{ is even} \\ -(-1)^{|e|} \sum_{k+s=i} \frac{1}{k!s!} \wedge \sigma(At^k_{\mathcal{E}} \otimes At^s_{\mathcal{F}}), & \text{if both } k \text{ and } s \text{ are odd} \end{cases}$$

since $\frac{1}{i!} \cdot \binom{i}{k} = \frac{1}{i!} \cdot \frac{i!}{k!(i-k)!} = \frac{1}{i!} \cdot \frac{i!}{k!s!} = \frac{1}{k!s!}$.

Meanwhile, the i^{th} component for $\widetilde{\varphi^n_E} \otimes \widetilde{\varphi^n_F}$ is $\sum_{k+s=i} \frac{1}{k!s!} At^k_{\mathcal{E}} \otimes At^s_{\mathcal{F}}$. Therefore, say $At^k_{\mathcal{E}}(e) = w' \otimes \overline{e}$ and $At^s_{\mathcal{F}}(f) = u' \otimes \overline{f}$ with $w' \in \Omega^k$ and $u' \in \Omega^s$ (hence $|\overline{e}| = |e| + 1$ if k is odd and $|\overline{e}| = |e|$ if k is even), we have

$$(\wedge \circ (1 \otimes \tau \otimes 1))(\sum_{k+s=i} \frac{1}{k!s!} At^k_{\mathcal{E}}(e) \otimes At^s_{\mathcal{F}}(f))$$

$$= (-1)^{|\overline{e}|\cdot s} \cdot \sum_{k+s=i} \frac{1}{k!s!} \cdot \wedge\sigma(At^k_{\mathcal{E}}(e) \otimes At^s_{\mathcal{F}}(f))$$

$$= \begin{cases} -(-1)^{|e|} \cdot \sum\limits_{k+s=i} \dfrac{1}{k!s!} \cdot \wedge\sigma(At_{\mathcal{E}}^k(e) \otimes At_{\mathcal{F}}^s(f)), & \text{if both } k \text{ and } s \text{ are odd} \\ \sum\limits_{k+s=i} \dfrac{1}{k!s!} \cdot \wedge\sigma(At_{\mathcal{E}}^k(e) \otimes At_{\mathcal{F}}^s(f)), & \text{otherwise.} \end{cases}$$

This completes the proof of the theorem. □

Corollary 3.3.18. *The Chern character* $ch : K_f(Q) \to \bigoplus_{i \in \mathbb{Z}_{\geqslant 0}} H_0((\Omega^{\cdot}, idf, n)_{\mathbb{Z}/2})$ *is a ring homomorphism, i.e.,*

$$ch([\mathcal{E}] \cdot [\mathcal{F}]) = ch([\mathcal{E}])ch([\mathcal{F}]).$$

Proof. Theorem 3.3.13 tells us that $\widetilde{\varphi^n_{\mathcal{E}\otimes_{mf}\mathcal{F}}} = (\wedge \circ (1 \otimes \tau \otimes 1)) \circ (\widetilde{\varphi^n_{\mathcal{E}}} \otimes \widetilde{\varphi^n_{\mathcal{F}}})$. The Corollary follows by applying str to this equation. □

3.3.4 Functoriality

Consider a k-algebra homomorphism $\varphi : R \to S$ that sends $f \in R$ to $g \in S$. For any matrix factorization $\mathcal{E} = (E_1 \underset{B}{\overset{A}{\rightleftarrows}} E_0) \in MF(R, f)$, there is then a naturally induced matrix factorization

$$\mathcal{E} \otimes_{mf} S = (E_1 \otimes_R S \underset{B\otimes 1}{\overset{A\otimes 1}{\rightleftarrows}} E_0 \otimes_R S) \in MF(S, g).$$

It is obvious that $E_1 \otimes_R S$ and $E_0 \otimes_R S$ are finitely generated projective S-modules. Also, we do have $(1 \otimes B) \circ (1 \otimes A) = (1 \otimes A) \circ (1 \otimes B) = g \cdot id$: in fact, $(1 \otimes B) \circ (1 \otimes A)(s \otimes e_1) = s \otimes f \cdot e_1$, but since we are talking about S-modules, $s \otimes f \cdot e_1 = \varphi(f) \cdot s \otimes e_1 = g \cdot s \otimes e_1 = g \cdot (s \otimes e_1)$.

Definition 3.3.19. For a k-algebra homomorphism φ as above, define a functor $\varphi_* : [MF(R, f)] \to [MF(S, g)]$ that sends $\mathcal{E}$ to $\varphi_*(\mathcal{E}) := \mathcal{E} \otimes_{mf} S$ and a strict morphism $\alpha = (\alpha_0, \alpha_1) : \mathcal{E} \to \mathcal{F}$ to a strict morphism $\varphi_*(\alpha) := (\alpha_0 \otimes 1, \alpha_1 \otimes 1) : \varphi_*(\mathcal{E}) \to \varphi_*(\mathcal{F})$.

It is obvious that the functor φ_* is well-defined on the homotopy category of matrix factorizations. Also we can talk about $\varphi_*(E) = E \otimes_R S$ for a finitely generated projective R-module E by regarding E as a matrix factorization of 0 and applying the above definition. In particular, there is a natural map $\mu : \Omega^1_{R/k} \otimes_R S = \varphi_*(\Omega^1_{R/k}) \to \Omega^1_{S/k}$ which sends $d_{R/k}(r) \otimes s$ to $s \cdot d_{S/k}(\varphi(r))$.

Let us prove a lemma before getting into the statement about funtoriality.

Lemma 3.3.20. *For φ as above and any finitely generated projective R-module E, there is a naturally induced connection $\nabla_{E \otimes_R S} \coloneqq \mu(\nabla_E \otimes 1) + \sigma(1 \otimes d_{S/k})$ for the S-module $\varphi_*(E)$, i.e.,*

$$\nabla_{E \otimes_R S} : \varphi_*(E) = E \otimes_R S \to \Omega^1_{S/k} \otimes_S (E \otimes_R S) \cong \Omega^1_{S/k} \otimes_R E.$$

Proof. First, notice that we have the following two compositions:

$$E \otimes_k S \xrightarrow{\nabla_E \otimes 1} (\Omega^1_{R/k} \otimes_R E) \otimes_k S \cong \Omega^1_{R/k} \otimes_R (E \otimes_k S) \cong (\Omega^1_{R/k} \otimes_R S) \otimes_k E \xrightarrow{\mu} \Omega^1_{S/k} \otimes_k E$$

and

$$E \otimes_k S \xrightarrow{1 \otimes d_{S/k}} E \otimes_R \Omega^1_{S/k} \xrightarrow{\sigma} \Omega^1_{S/k} \otimes_k E$$

Let's denote the sum of the above two compositions by $\nabla_{E \otimes_k S}$. It is obvious that they are both k-linear, one can also show that $\nabla_{E \otimes_k S}$ is in fact R-linear by checking directly. Hence we get an induced map:

$$\begin{tikzcd} E \otimes_k S \arrow[r, "\nabla_{E \otimes_k S}"] \arrow[d] & \Omega^1_{S/k} \otimes_k E \arrow[r, two heads] & \Omega^1_{S/k} \otimes_R E \\ E \otimes_R S \arrow[ur, dashed] & & \end{tikzcd}$$

which we denote by $\nabla_{E \otimes_R S}$.

Now the only thing left to check is that $\nabla_{E\otimes_R S}$ satisfies the Leibniz rule, i.e.,

$$\nabla_{E\otimes_k S}(s\cdot(e\otimes s')) = d_{S/k}(s)\otimes(e\otimes s') + s\cdot\nabla_{E\otimes_k S}(e\otimes s'),$$

for any $e\in E$, $s,s'\in S$.

Let's prove it using the same technique as in the proof of Theorem 3.3.16. Say $\nabla_E(e) = d_{R/k}(r)\otimes e' \in \Omega^1_{R/k}\otimes_R E$,

$$\nabla_{E\otimes_R S}(s\cdot(e\otimes s')) = \nabla_{E\otimes_R S}(e\otimes ss')$$

$$= \mu(\nabla_E(e)\otimes ss') + \sigma(e\otimes d_{S/k}(ss'))$$

$$= \mu(d_{R/k}(r)\otimes e'\otimes ss') + d_{S/k}(s)\cdot s' + s\cdot d_{S/k}(s')\otimes e$$

$$= ss'\cdot d_{S/k}(\varphi(r))\otimes e' + d_{S/k}(s)\cdot s' + s\cdot d_{S/k}(s')\otimes e.$$

Meanwhile,

$$d_{S/k}(s)\otimes(e\otimes s')+s\cdot\nabla_{E\otimes_R S}(e\otimes s') = s'\cdot d_{S/k}(s)\otimes e+s\cdot(\mu(\nabla_E\otimes 1)(e\otimes s')+\sigma(1\otimes d_{S/k})(e\otimes s'))$$

$$= s'\cdot d_{S/k}(s)\otimes e + s\cdot(\mu(\nabla_E(e)\otimes s') + d_{S/k}(s')\otimes e)$$

$$= s'\cdot d_{S/k}(s)\otimes e + s\cdot d_{S/k}(s')\otimes e + s\cdot\mu(d_{R/k}(r)\otimes e'\otimes s')$$

$$= s'\cdot d_{S/k}(s)\otimes e + s\cdot d_{S/k}(s')\otimes e + s\cdot(s' d_{S/k}(\varphi(r))\otimes e').$$

$\square$

For simplicity (and to make future calculations easier), we follow the usual convention of denoting $\nabla_{E\otimes_R S}$ by $\nabla_E\otimes 1 + 1\otimes d_{S/k}$. Now we can state and prove

Proposition 3.3.21. *(Functoriality)*

Under the above hypotheses and the extra condition that both R and S are smooth k-algebras with the same relative dimension n, we have $\varphi_* \circ ch = ch \circ \varphi_*$.

Proof. By our formula for the Chern character, it's enough to show $\varphi_*(At(\mathcal{E})) = At(\varphi_*(\mathcal{E}))$ for a matrix factorization $\mathcal{E}$.

By Lemma 3.3.20, choose the natural connection $\nabla_E \otimes 1 + 1 \otimes d_{S/k}$ for $E \otimes_R S$, so the Atiyah class of $\varphi_*(\mathcal{E}) = \mathcal{E} \otimes_{mf} S$ is

$$\begin{bmatrix} \nabla_E \otimes 1 + 1 \otimes d_{S/k} & \\ & \nabla_E \otimes 1 + 1 \otimes d_{S/k} \end{bmatrix} \begin{bmatrix} & A \otimes 1 \\ B \otimes 1 & \end{bmatrix} -$$

$$\begin{bmatrix} & A \otimes 1 \\ B \otimes 1 & \end{bmatrix} \begin{bmatrix} \nabla_E \otimes 1 + 1 \otimes d_{S/k} & \\ & \nabla_E \otimes 1 + 1 \otimes d_{S/k} \end{bmatrix}$$

$$= \begin{bmatrix} & (\nabla_E A - A \nabla_E) \otimes 1 \\ (\nabla_E B - B \nabla_E) \otimes 1 & \end{bmatrix}$$

The Atiyah class of $\mathcal{E}$ is

$$\begin{bmatrix} & \nabla_E A - A \nabla_E \\ \nabla_E B - B \nabla_E & \end{bmatrix}$$

Now is obvious from the definition of φ_* on strict morphisms that $\varphi_*(At(\mathcal{E})) = At(\varphi_*(\mathcal{E}))$.

$\square$

Chapter 4

Reconstruction of hypersurface singularities

In this chapter, we will give details about the reconstruction theorem mentioned in the introduction. First, let us introduce the notion of a pseudo tensor triangulated category.

4.1 Pseudo tensor triangulated category

4.1.1 Pseudo tensor triangulated category

Here we give the definition of a pseudo tensor triangulated category, which is in fact just a tensor triangulated category (in the sense of Balmer [1]) with no tensor identity.

Definition 4.1.1. A *pseudo tensor triangulated category* is a triangulated category K equipped with a tensor product $\otimes : K \times K \to K$ such that, for any $a, b, c \in K$, there

is a natural isomorphism, called the *associator*

$$\alpha_{a,b,c} : \quad (a \otimes b) \otimes c \simeq a \otimes (b \otimes c)$$

and a natural isomorphism, called the *braiding*

$$B_{a,b} : a \otimes b \simeq b \otimes a.$$

We require that the associator satisfies the *pentagon identity*, which says this diagram commutes:

$$\begin{tikzcd}
 & (a \otimes b) \otimes (c \otimes d) \arrow[dr, "\alpha_{a,b,c\otimes d}"] & \\
((a \otimes b) \otimes c) \otimes d \arrow[ur, "\alpha_{a\otimes b,c,d}"] \arrow[d, "\alpha_{a,b,c}\otimes 1_d"'] & & a \otimes (b \otimes (c \otimes d)) \\
(a \otimes (b \otimes c)) \otimes d \arrow[r, "\alpha_{a,b\otimes c,d}"] & a \otimes ((b \otimes c) \otimes d) \arrow[ur, "1_a\otimes\alpha_{b,c,d}"'] &
\end{tikzcd}$$

We also require the associator and the braiding to satisfy the *hexagon identity*:

$$\begin{tikzcd}
(a \otimes b) \otimes c \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "B_{a,b}\otimes 1_c"'] & a \otimes (b \otimes c) \arrow[r, "B_{a,b\otimes c}"] & (b \otimes c) \otimes a \arrow[d, "\alpha_{b,c,a}"] \\
(b \otimes a) \otimes c \arrow[r, "\alpha_{b,a,c}"'] & b \otimes (a \otimes c) \arrow[r, "1_y\otimes B_{a,c}"'] & b \otimes (c \otimes a)
\end{tikzcd}$$

Last, we require that $- \otimes a$ and $a \otimes -$ are triangulated functors.

Note that the tensor product commutes with finite coproducts:

Proposition 4.1.2. *Let K be a pseudo tensor triangulated category. Then for $a, b, c \in K$, we have*

$$(a \oplus b) \otimes c \simeq (a \otimes c) \oplus (b \otimes c).$$

Proof. This is Proposition 2.14 of [27]. □

Removing the tensor identity is the only modification we made compared to Balmer's original idea. Other notions like thick ⊗-ideal, radical and so on are exactly the ones defined by Balmer [1]. These will be recalled at the beginning of the following subsection.

4.1.2 Pseudo spectrum

Everything in this section is due to Balmer [1]. Balmer does not concern the pseudo case but most of his theory still works without the tensor identity. Here in this section I just list the ones I need later (i.e., the results that still work without the existence of the tensor identity). All of Balmer's original proofs in [1] are still valid.

Let us recall Balmer's theory of tensor triangular geometry (with slight modification to the pseudo case).

Definition 4.1.3. Consider a pseudo tensor triangulated category K. A thick *tensor-ideal* A of K is a full subcategory containing 0 and such that the following conditions are satisfied:

1. A is triangulated: for any distinguished triangle $a \to b \to c \to a[1]$ in K if two out of a, b and c belong to A, then so does the third;

2. A is thick: if an object $a \in A$ splits in K as $a \simeq b \oplus c$ then both summands b and c belong to A;

3. A is a tensor-ideal: if $a \in A$ and $b \in K$ then $a \otimes b$ also belongs to A.

A *prime* of K is a proper thick tensor ideal $P \subsetneq K$ such that

$$a \otimes b \in P \Longrightarrow a \in P \text{ or } b \in P.$$

Let the set $Spc'(K)$ be the collection of all primes of K. (We use $'$ to indicate the pseudo category but $Spc'(K) = Spc(K)$ if K contains a tensor identity. Similarly Z', U', $supp'$ denote the analogues of $Z, U, supp$). For any family of objects $S \subset K$ we denote by $Z'(S)$ the following subset of $Spc'(K)$:

$$Z'(S) = \{P \in Spc'(K) : S \cap P = \varnothing\}.$$

It is clear that $\cap Z'(S_i) = Z'(\cup S_i)$ and $Z'(S_1) \cup Z'(S_2) = Z'(S_1 \oplus S_2)$ where $S_1 \oplus S_2 := \{a_1 \oplus a_2 : a_i \in S_i \text{ for } i = 1, 2\}$ by checking directly. We also have $Z'(K) = \varnothing$ and $Z'(\varnothing) = Spc'(K)$, hence the collection $\{Z'(S) \subset Spc'(K) : S \subset K\}$ defines the closed subsets of a topology on $Spc'(K)$. We call this the *Zariski topology* on $Spc'(K)$. The open complement of $Z'(S)$ is written

$$U'(S) := Spc'(K) \backslash Z'(S) = \{P \in Spc'(K) : S \cap P \neq \varnothing\}.$$

For any object $a \in K$, denote by $supp'(a)$ the following closed subset of $Spc'(K)$:

$$supp'(a) := Z'(\{a\}) = \{P \in Spc'(K) : a \notin P\}$$

which we call the *support* of the object $a \in K$.

A collection of objects $S \subset K$ is called *(tensor) multiplicative* if $a_1, a_2 \in S \Rightarrow$

$a_1 \otimes a_2 \in S$. Note that in the original definition (Definition 2.1 of [1]) of a tensor multiplicative collection of objects, it contains the tensor identity, hence is always not empty, but for us it can be an empty collection.

Now we start to list the modified statements of Balmer and check that his original proofs still work. Basically what we do here is simply to remove all conditions related to the tensor identity from the original statements and add necessary extra conditions to make the modified version work. The category K in the following statements is a non-zero **pseudo** tensor triangulated category.

Lemma 4.1.4. ***[Lemma 2.2 of [1]]***

*Let K be a non-zero **pseudo** tensor triangulated category. Let $J \subset K$ be a thick $\otimes$-ideal and $S \subset K$ a **non-empty** $\otimes$-multiplicative family of objects such that $S \cap J = \varnothing$. Then there exists a prime ideal $P \in Spc'(K)$ such that $J \subset P$ and $P \cap S = \varnothing$.*

Lemma 4.1.5. ***[Lemma 2.6 (b) of [1]]***

For any two objects $a, b \in K$, we have $U'(a \oplus b) = U'(a) \cap U'(b)$.

Remark 4.1.6. ***[Remark 2.7 of [1]]***

Since for any $S \subset K$, we have $U'(S) = \bigcup_{a \in S} U'(a)$, it follows from Lemma 4.1.5 that $\{U'(a) | a \in K\}$ is a basis of the topology on $Spc'(K)$. Equivalently, their complements $\{supp'(a) | a \in K\}$ form a basis of closed subsets.

Proposition 4.1.7. ***[Proposition 2.8 of [1]]***

Let $W \subset Spc'(K)$ be a subset of the pseudo spectrum. Its closure is

$$\overline{W} = \bigcap_{a \in K \, such \ that \ W \subset supp'(a)} supp'(a).$$

Definition 4.1.8. [Definition 3.1 of [1]]

A *support data* on a pseudo tensor triangulated category $(K, \otimes)$ is a pair (X, σ) where σ is an assignment which associates to any object $a \in K$ a closed subset $\sigma(a) \subset X$ such that

0. $X = \bigcup_{a \in K} \sigma(a)$ (This replaces Balmer's original condition: $\sigma(\mathbf{1}) = X$)
1. $\sigma(0) = \varnothing$
2. $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$
3. $\sigma(a[1]) = \sigma(a)$
4. $\sigma(a) \subset \sigma(b) \cup \sigma(c)$ for any distinguished triangle $a \to b \to c \to a[1]$.
5. $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$.

A *morphism* $\varphi : (X, \sigma) \to (Y, \tau)$ of support data on the same category K is a continuous map $\varphi : X \to Y$ such that $\sigma(a) = \varphi^{-1}(\tau(a))$ for all objects $a \in K$. Such a morphism is an isomorphism if and only if φ is a homeomorphism.

Proposition 4.1.9. ***[Proposition 2.9 of [1]]***

For any point $P \in Spc'(K)$ its closure in $Spc'(K)$ is

$$\overline{\{P\}} = \{\mathcal{Q} \in Spc'(K) | \mathcal{Q} \subset P\}.$$

In particular, if $\overline{\{P_1\}} = \overline{\{P_2\}}$, then $P_1 = P_2$. (i.e., $Spc'(K)$ is T_0.)

Lemma 4.1.10. ***[Lemma 3.3 of [1]]***

Let X be a set and $\varphi_1, \varphi_2 : X \to Spc'(K)$ be two maps such that $\varphi_1^{-1}(supp'(a)) = \varphi_2^{-1}(supp'(a))$ for all $a \in K$, then $\varphi_1 = \varphi_2$.

Lemma 4.1.11. ***[Lemma 3.4 of [1]]***

Let (X,σ) be a support data on K and $Y \subset X$ any subset. Then the full subcategory of K with objects $\{a \in K | \sigma(a) \subset Y\} =: K_Y$ is a thick $\otimes$-ideal.

Theorem 4.1.12. ***[Theorem 3.2 of [1]]***

Let $(K,\otimes)$ be a pseudo tensor triangulated category. The pseudo spectrum $(Spc'(K), supp')$ defined above is the final support data on K in the sense of Definition 4.1.8. In other words, $(Spc'(K), supp')$ is a support data and for any support data (X,σ) on K there exists a unique continuous map $\varphi : X \to Spc'(K)$ such that $\sigma(a) = \varphi^{-1}(supp'(a))$ for any object $a \in K$. Explicitly, the map φ is defined, for all $x \in X$, by

$$\varphi(x) = \{a \in K | x \notin \sigma(a)\}.$$

Proof. We only need to check the modified condition :$X = \bigcup_{a\in K} \sigma(a)$, i.e., $Spc'(K) = \bigcup_{a\in K} supp'(a)$. This is obvious. In fact, the direction $\supseteq$ is trivial and if P is a prime (so proper), we can find $a \in K \backslash P$ and then by definition $P \in supp'(a)$. Also, $supp'(a)$ is defined to be closed.

The rest is exactly Balmer's original proof in [1].

□

Definition 4.1.13. [Definition 4.1 of [1]]

The *radical* $\sqrt{J}$ of a thick $\otimes$-ideal $J \subset K$ is defined to be

$$\sqrt{J} := \{a \in K | \exists n \geqslant 1 \text{ such that } a^{\otimes n} \in J\}.$$

A thick subcategory J is called *radical* if $\sqrt{J} = J$.

Lemma 4.1.14. ***[Lemma 4.2 of [1]]***

$\sqrt{J}$ is a thick $\otimes$-ideal equal to the intersection $\bigcap_{P \supset J} P$ of all the primes $P \in Spc'(K)$ containing J.

Definition 4.1.15. [Definition 5.1 of [1]]

A support data (X, σ) on a pseudo tensor triangulated category K is a *classifying support data* if the following two conditions hold:

1. The topological space X is Noetherian and any non-empty irreducible closed subset $Z \subset X$ has a unique generic point: $\exists! x \in Z$ with $\overline{\{x\}} = Z$.

2. We have a bijection $\theta : \{Y \subset X | Y \text{specialization closed}\} \longleftrightarrow \{J \subset K | J \text{ radical thick } \otimes\text{-ideal}\}$ defined by $Y \longmapsto \{a \in K | \sigma(a) \subset Y\}$, with inverse $J \longmapsto \sigma(J) := \bigcup_{a \in J} \sigma(a)$.

Theorem 4.1.16. ***[Theorem 5.2 of [1]]***

Suppose that (X, σ) is a classifying support data on K. Then the canonical map $\varphi : X \to Spc'(K)$ of theorem 4.1.12 is a homeomorphism.

4.1.3 The functor λ

As mentioned in the introduction, tensor products of two matrix factorizations do not behave the way we want so we need to modify them a little bit. To do this, we first introduce the following functors.

Definition 4.1.17. For any $\lambda \in Q^\times$, define a functor $\lambda : MF(Q, f) \to MF(Q, \lambda f)$. λ sends an object $\mathcal{M} = (M_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} M_0) \in MF(Q, f)$ to the object $(M_1 \underset{\lambda d_0}{\overset{d_1}{\rightleftarrows}} M_0) \in MF(Q, \lambda f)$; it sends a strict morphism $\alpha = (\alpha_0, \alpha_1) : \mathcal{M} \to \mathcal{N}$ to the morphism $\lambda(\alpha) = (\alpha_0, \alpha_1) : \lambda(\mathcal{M}) \to \lambda(\mathcal{N})$, i.e.,

$$\begin{tikzcd}
M_1 \arrow[r, "d^M"] \arrow[d, "\alpha_1"'] & M_0 \arrow[r, "d^M"] \arrow[d, "\alpha_0"] & M_1 \arrow[d, "\alpha_1"] \\
N_1 \arrow[r, "d^N"'] & N_0 \arrow[r, "d^N"'] & N_1
\end{tikzcd}$$

gets sent to

$$\begin{tikzcd}
M_1 \arrow[r, "d^M"] \arrow[d, "\alpha_1"'] & M_0 \arrow[r, "\lambda d^M"] \arrow[d, "\alpha_0"] & M_1 \arrow[d, "\alpha_1"] \\
N_1 \arrow[r, "d^N"'] & N_0 \arrow[r, "\lambda d^N"'] & N_1
\end{tikzcd}$$

where (α_1, α_0) remains the same.

Thus λ induces a functor, which we still call λ, from the category $[MF(Q,f)]$ to $[MF(Q,\lambda f)]$, via the following lemma

Lemma 4.1.18. *The functor λ maps a homotopy (h_0,h_1) to the homotopy $(h_0,\lambda^{-1}h_1)$.*

Proof. Indeed, we have $d_1^{\mathcal{N}}h_0 + h_1 d_0^{\mathcal{M}} = \varphi_0 - \psi_0$ and $d_0^{\mathcal{N}}h_1 + h_0 d_1^{\mathcal{M}} = \varphi_1 - \psi_1$. These equations can be rewritten as $d_1^{\mathcal{N}}h_0 + (\lambda^{-1}h_1)(\lambda d_0^{\mathcal{M}}) = \varphi_0 - \psi_0$ and $(\lambda d_0^{\mathcal{N}})(\lambda^{-1}h_1) + h_0 d_1^{\mathcal{M}} = \varphi_1 - \psi_1$ which means $(h_0,\lambda^{-1}h_1)$ is a homotopy hence $\lambda(\varphi) \sim \lambda(\psi)$.

□

Now, for any $\lambda \in Q^\times$, we have functors

$$\lambda : [MF(Q,f)] \rightleftarrows [MF(Q,\lambda f)] : \lambda^{-1};$$

they are obviously inverses to each other. Hence the two categories $[MF(Q,f)]$ and $[MF(Q,\lambda f)]$ are equivalent categories. We next show they are equivalent as triangulated categories.

Proposition 4.1.19. *For any $\lambda \in Q^\times$, $\lambda : [MF(Q,f)] \to [MF(Q,\lambda f)]$ is a triangulated functor.*

Proof. First, λ is clearly additive and there is a natural isomorphism $\phi_{\mathcal{M}} = (1_{\mathcal{M}}, \lambda) : \lambda(\mathcal{M}[1]) \to (\lambda(\mathcal{M}))[1]$ for any $\mathcal{M} = (M_1 \overset{d_1}{\underset{d_0}{\rightleftarrows}} M_0) \in [MF(Q,f)]$.

In fact, we have

$$\lambda(\mathcal{M}[1]) = \lambda((M_0 \overset{-d_0}{\underset{-d_1}{\rightleftarrows}} M_1)) = (M_0 \overset{-d_0}{\underset{-\lambda d_1}{\rightleftarrows}} M_1),$$

and

$$(\lambda(\mathcal{M}))[1] = (M_1 \overset{d_1}{\underset{\lambda d_0}{\rightleftarrows}} M_0)[1] = (M_0 \overset{-\lambda d_0}{\underset{-d_1}{\rightleftarrows}} M_1).$$

$\phi_{\mathcal{M}} = (1_{\mathcal{M}}, \lambda)$ is well-defined since we have the following commutative diagram:

$$\begin{tikzcd} M_0 \arrow[r, "-d_0"] \arrow[d, "1"'] & M_1 \arrow[r, "-\lambda d_1"] \arrow[d, "\lambda"] & M_0 \arrow[d, "1"] \\ M_0 \arrow[r, "-\lambda d_0"'] & M_1 \arrow[r, "-d_1"'] & M_0 \end{tikzcd}$$

where $\phi_{\mathcal{M}} = (1_{\mathcal{M}}, \lambda)$ is an isomorphism since there is an obvious inverse $\phi_{\mathcal{M}}^{-1} = (1_{\mathcal{M}}, \lambda^{-1})$ so $\lambda(\mathcal{M}[1])$ and $(\lambda(\mathcal{M}))[1]$ are isomorphic in $[MF(Q,\lambda f)]$.

Now, let's show that the functor λ maps distinguished triangles to distinguished triangles. Note that distinguished triangles in $[MF(Q,\lambda f)]$ have the form:

$$\mathcal{M} \xrightarrow{p} \mathcal{N} \to cone(p) \to \mathcal{M}[1]$$

where

$$\text{cone}(p) = \left(N_1 \oplus M_0 \underset{\begin{bmatrix} d_0^{\mathcal{N}} & p_1 \\ 0 & -d_1^{\mathcal{M}} \end{bmatrix}}{\overset{\begin{bmatrix} d_1^{\mathcal{N}} & p_0 \\ 0 & -d_0^{\mathcal{M}} \end{bmatrix}}{\rightleftarrows}} N_0 \oplus M_1 \right).$$

So

$$\lambda(cone(p)) = \left(N_1 \oplus M_0 \underset{\lambda\begin{bmatrix} d_0^{\mathcal{N}} & p_1 \\ 0 & -d_1^{\mathcal{M}} \end{bmatrix}}{\overset{\begin{bmatrix} d_1^{\mathcal{N}} & p_0 \\ 0 & -d_0^{\mathcal{M}} \end{bmatrix}}{\rightleftarrows}} N_0 \oplus M_1 \right)$$

and

$$cone(\lambda(p)) = \left(N_1 \oplus M_0 \underset{\begin{bmatrix} \lambda d_0^{\mathcal{N}} & p_1 \\ 0 & -d_1^{\mathcal{M}} \end{bmatrix}}{\overset{\begin{bmatrix} d_1^{\mathcal{N}} & p_0 \\ 0 & -\lambda d_0^{\mathcal{M}} \end{bmatrix}}{\rightleftarrows}} N_0 \oplus M_1 \right).$$

We have an isomorphism $(1, \begin{bmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{bmatrix}) : cone(\lambda(p)) \to \lambda(cone(p))$. Therefore the resulting triangle

$$\lambda(\mathcal{M}) \xrightarrow{\lambda(p)} \lambda(\mathcal{N}) \to \lambda(cone(p)) \to \lambda(\mathcal{M}[1])$$

is isomorphic (in $[MF(Q, \lambda f)]$) to the triangle

$$\lambda(\mathcal{M}) \xrightarrow{\lambda(p)} \lambda(\mathcal{N}) \to \text{cone}(\lambda(p)) \to (\lambda(\mathcal{M}))[1]$$

hence also a distinguished triangle. This completes the proof.

$\square$

Corollary 4.1.20. *$[MF(Q,f)]$ and $[MF(Q,\lambda f)]$ are equivalent triangulated categories, for any $\lambda \in Q^\times$.*

Now we will define a modified tensor product for matrix factorizations which gives the triangulated category $[MF(Q,f)]$ a pseudo structure.

Definition 4.1.21. Assume $2 \in Q^\times$, for any $\mathcal{M},\mathcal{N} \in [MF(Q,f)]$, define their new tensor product as $\frac{1}{2}(\mathcal{M} \otimes_{mf} \mathcal{N})$, the resulting matrix factorization then belongs to $[MF(Q,f)]$.

Hence we have a tensor operation for the triangulated category $[MF(Q,f)]$:

$$[MF(Q,f)] \times [MF(Q,f)] \xrightarrow{\otimes_{mf}} [MF(Q,2f)] \xrightarrow{\frac{1}{2}} [MF(Q,f)]$$

From now on, we will simply denote this modified tensor product by $\otimes^{\frac{1}{2}}$.

Proposition 4.1.22. *$([MF(Q,f)], \otimes^{\frac{1}{2}})$ is a pseudo tensor triangulated category.*

Proof. Given any three matrix factorizations $\mathcal{M} = (M_1 \underset{B}{\overset{A}{\rightleftarrows}} M_0), \mathcal{N} = (N_1 \underset{D}{\overset{C}{\rightleftarrows}} N_0), \mathcal{L} = (L_1 \underset{F}{\overset{E}{\rightleftarrows}} L_0) \in [MF(Q,f)]$, we need to check all conditions in Definition 4.1.1 :

1. $\mathcal{M} \otimes^{\frac{1}{2}} \mathcal{N} \simeq \mathcal{N} \otimes^{\frac{1}{2}} \mathcal{M}$
2. $(\mathcal{M} \otimes^{\frac{1}{2}} \mathcal{N}) \otimes^{\frac{1}{2}} \mathcal{L} \simeq \mathcal{M} \otimes^{\frac{1}{2}} (\mathcal{N} \otimes^{\frac{1}{2}} \mathcal{L})$
3. $\mathcal{M} \otimes^{\frac{1}{2}} -$ and $- \otimes^{\frac{1}{2}} \mathcal{M}$ are triangulated functors
4. The pentagon identity

5. The hexagon identity

The first two are straightforward since we already know that $- \otimes_{mf} -$ is commutative and associative [32]. To prove these for $\otimes^{\frac{1}{2}} = \frac{1}{2} \circ \otimes_{mf}$, we just need to insert $\frac{1}{2}$ in appropriate places, so essentially the same proofs work.

The third condition is obvious since we already proved in Proposition 2.0.14 and Proposition 4.1.19 that $\mathcal{M} \otimes_{mf} -$ and $\frac{1}{2}$ are triangulated functors, hence so is their composition $\otimes^{\frac{1}{2}}$.

The remaining two conditions can be shown by explicitly writing down everything and checking directly. Doing so is tedious but straightforward, so we omit the proofs here.

$\square$

4.2 Support theory for matrix factorizations

There is a well developed support theory for matrix factorizations due to many people in different contexts. Our main reference is Walker's forthcoming paper [30], which develops support theory in the language of matrix factorizations. I am grateful to Professor Walker for sharing his work with me.

Let us recall everything that we need about the support theory in this section.

Definition 4.2.1. [Walker [30]]

A matrix factorization $\mathcal{M}$ of $MF(Q, f)$ is *contractible* if the identity map on $\mathcal{M}$ is homotopic to the zero map, i.e., there is a degree 1 map from the $\mathbb{Z}/2$-graded module M to itself ($\Longleftrightarrow$ Q-linear maps $h_0 : M_0 \to M_1$ and $h_1 : M_1 \to M_0$) such that $h_1 \circ d_0 + d_1 \circ h_0 = id_{M_0}$ and $h_0 \circ d_1 + d_0 \circ h_1 = id_{M_1}$.

When the ring Q is regular of finite Krull dimension, the *support* of $\mathcal{M}$ is defined to be:

$$supp_{mf}(\mathcal{M}) = \{\mathfrak{p} \in Spec(Q) | \mathcal{M}_{\mathfrak{p}} \text{ is not contractible}\}.$$

Equivalently, the above definition is the same as

$$supp_{mf}(\mathcal{M}) = \{\mathfrak{p} \in Spec(Q) | \mathcal{M}_{\mathfrak{p}} = \mathcal{M} \otimes_{mf} Q_{\mathfrak{p}} \neq 0 \text{ in } [MF(Q_{\mathfrak{p}}, f)]\}.$$

From now on, we will assume that the ring Q is regular of finite Krull dimension and f is a non zero divisor.

Proposition 4.2.2. ***[Walker [30]]***

For $\mathcal{M}, \mathcal{N} \in MF(Q, f)$, if $\alpha : \mathcal{M} \to \mathcal{N}$ is a homotopy equivalence, then $supp_{mf}(\mathcal{M}) = supp_{mf}(\mathcal{N})$.

This is obvious from the definition of support.

Proposition 4.2.3. ***[Walker [30]]***

For any $\mathcal{M} \in MF(Q,f)$*,* $supp_{mf}(\mathcal{M})$ *is contained in* $Spec(Q/f)$ *and is specialization closed. In particular, when* $f \in Q$ *is a non-zero divisor,* $supp_{mf}(\mathcal{M})$ *is contained in*

$$\mathrm{Sing}(Q/f) := \{\mathfrak{p} \in \mathrm{Spec}(Q/f) \subseteq \mathrm{Spec}(Q) \mid \ (Q/f)_{\mathfrak{p}} \textit{ is not regular local ring}\}.$$

Since we are looking at matrix factorizations associated to a non zero divisor f, we can really think of the definition of support as

$$supp_{mf}(\mathcal{M}) = \{\mathfrak{p} \in Sing(Q/f) | \mathcal{M}_{\mathfrak{p}} \neq 0 \text{ in } [MF(Q_{\mathfrak{p}}, f)]\}.$$

We will use this as the definition of support from now on.

Proposition 4.2.4. ***[Walker [30]]***

For every closed subset Z *of* $Spec(Q)$ *that is contained in* $Sing(Q/f)$*, there is an object* $\mathcal{M} \in MF(Q,f)$ *such that* $supp_{mf}(\mathcal{M}) = Z$*.*

Before getting into the next few propositions, we need to introduce the notion of $\mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M},\mathcal{N})$ for any two given matrix factorizations $\mathcal{M} \in MF(Q,f)$ and $\mathcal{N} \in MF(Q,g)$. This is the hom object for the dg category of matrix factorizations. The reason why we metion it is that it makes the proof of Lemma 4.3.1 (5) nicer. Since it takes a while to write down the definition of $\mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M},\mathcal{N})$ and we don't need it anywhere else, we will omit it here and refer the reader to [23] or [30]. Note that one can still prove Lemma 4.3.1 (5) without using $\mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M},\mathcal{N})$.

Proposition 4.2.5. ***[Walker [30]]***

For any $\mathcal{M},\mathcal{N} \in MF(Q,f)$, we have

$$supp_{mf}(\mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M},\mathcal{N})) = supp_{mf}(\mathcal{M}) \cap supp_{mf}(\mathcal{N})$$

and

$$supp_{mf}(\mathcal{M}^*) = supp_{mf}(\mathcal{M})$$

where $\mathcal{M}^* := \mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M},(0 \rightleftarrows Q)) \in MF(Q,-f)$.

Corollary 4.2.6. ***[Walker [30]]***

For any $\mathcal{M} \in MF(Q,f)$, $supp_{mf}(\mathcal{M})$ is a closed subset of $Spec(Q/f)$. In particular, when f is a non zero divisor, it's a closed subset of $Sing(Q/f)$.

Proposition 4.2.7. ***[Walker [30]]***

For any $\mathcal{M} \in MF(Q,f)$, $\mathcal{N} \in MF(Q,g)$, $\mathcal{T} \in MF(Q,h)$ we have an isomorphism

$$\mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M} \otimes_{mf} \mathcal{N},\mathcal{T}) \cong \mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M},\mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{N},\mathcal{T}))$$

in $MF(Q,h-f-g)$ that is natural in $\mathcal{M},\mathcal{N}$, and $\mathcal{T}$.

Theorem 4.2.8. ***[Walker [30]]***

There exists a bijective correspondence

$$\{specialization\ closed\ subsets\ of\ Sing(Q/f)\} \longleftrightarrow \{thick\ subcategories\ of\ [MF(Q,f)]\}$$

given by

$$Z \longmapsto \{\mathcal{M} \in [MF(Q,f)] | supp_{mf}(\mathcal{M}) \subset Z\}$$

and

$$\bigcup_{\mathcal{M} \in T} supp_{mf}(\mathcal{M}) \leftarrow\!\shortmid T.$$

Remark 4.2.9. *This theorem is also proved by Stevenson [28] and Takahashi [29] in a different context.*

4.3 Proofs

We will show that the support $supp_{mf}$ mentioned in the last section is a classifying support data for the pseudo tensor triangulated category $([MF(Q,f)], \otimes^{\frac{1}{2}})$. Therefore we get a reconstruction of $Sing(Q/f)$ by Theorem 4.1.16.

Lemma 4.3.1. *For any $\mathcal{M}, \mathcal{N} \in MF(Q,f)$, their supports have the following properties:*

1. $supp_{mf}(0) = \varnothing$
2. $supp_{mf}(\mathcal{M} \oplus \mathcal{N}) = supp_{mf}(\mathcal{M}) \cup supp_{mf}(\mathcal{N})$
3. $supp_{mf}(\mathcal{M}[1]) = supp_{mf}(\mathcal{M})$
4. $supp_{mf}(\mathcal{M}) \subset supp_{mf}(\mathcal{N}) \cup supp_{mf}(\mathcal{L})$ *for any distinguished triangle* $\mathcal{M} \to \mathcal{N} \to \mathcal{L} \to \mathcal{M}[1]$
5. $supp_{mf}(\mathcal{M} \otimes^{\frac{1}{2}} \mathcal{N}) = supp_{mf}(\mathcal{M}) \cap supp_{mf}(\mathcal{N})$

Note that in (5)*, the tensor product* $\otimes^{\frac{1}{2}}$ *is the modified one defined before, that is* $\otimes^{\frac{1}{2}} = \dfrac{1}{2} \circ \otimes_{mf}$.

Proof. 1. $supp_{mf}(0) = \{\mathfrak{p} \in Spec(Q/f) | 0_{\mathfrak{p}} = 0 \text{ is not contractible}\} = \varnothing$

2. $supp_{mf}(\mathcal{M} \oplus \mathcal{N}) = supp_{mf}(\mathcal{M}) \cup supp_{mf}(\mathcal{N})$

   ($\subseteq$) Say $\mathfrak{p} \in supp_{mf}(\mathcal{M} \oplus \mathcal{N})$, so $(\mathcal{M} \oplus \mathcal{N})_{\mathfrak{p}} \cong \mathcal{M}_{\mathfrak{p}} \oplus \mathcal{N}_{\mathfrak{p}} \neq 0$, then $\mathfrak{p} \in supp_{mf}(\mathcal{M}) \cup supp_{mf}(\mathcal{N})$. Otherwise $\mathcal{M}_{\mathfrak{p}} = 0$ and $\mathcal{N}_{\mathfrak{p}} = 0$ will force $\mathcal{M}_{\mathfrak{p}} \oplus \mathcal{N}_{\mathfrak{p}} = 0$, contraction.

   ($\supseteq$) Let $\mathfrak{p} \in supp_{mf}(M) \cup supp_{mf}(N)$, without loss of generality, we may assume that $\mathfrak{p} \in supp_{mf}(M)$, i.e., $M_{\mathfrak{p}} \neq 0$, so $(\mathcal{M} \oplus \mathcal{N})_{\mathfrak{p}} \cong \mathcal{M}_{\mathfrak{p}} \oplus \mathcal{N}_{\mathfrak{p}} \neq 0$. Therefore $\mathfrak{p} \in supp_{mf}(\mathcal{M} \oplus \mathcal{N})$.

3. $supp_{mf}(\mathcal{M}[1]) = supp_{mf}(\mathcal{M})$

   For $\mathcal{M} = (M_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} M_0)$, we know $\mathcal{M}[1] = (M_0 \underset{-d_1}{\overset{-d_0}{\rightleftarrows}} M_1)$, so it is obvious.

4. $supp_{mf}(\mathcal{M}) \subset supp_{mf}(\mathcal{N}) \cup supp_{mf}(\mathcal{L})$ for any distinguished triangle $\mathcal{M} \to \mathcal{N} \to \mathcal{L} \to \mathcal{M}[1]$

   Proposition 2.0.14 showed that $\mathcal{E} \otimes_{mf} -$ maps distinguished triangles to distinguished triangles, therefore by taking $\mathcal{E} = Q_{\mathfrak{p}} = (0 \rightleftarrows Q_{\mathfrak{p}})$, we have a distinguished triangle in $MF(Q_{\mathfrak{p}}, f)$

$$\mathcal{M}_{\mathfrak{p}} \to \mathcal{N}_{\mathfrak{p}} \to \mathcal{L}_{\mathfrak{p}} \to \mathcal{M}_{\mathfrak{p}}[1].$$

   Let $\mathfrak{p} \in supp_{mf}(\mathcal{M}) (\Longleftrightarrow \mathcal{M}_{\mathfrak{p}} \neq 0)$, then $\mathfrak{p} \in supp_{mf}(\mathcal{N}) \cup supp_{mf}(\mathcal{L})$. Otherwise $\mathcal{N}_{\mathfrak{p}}$ and $\mathcal{L}_{\mathfrak{p}}$ are both 0, this forces $\mathcal{M}_{\mathfrak{p}} = 0$, contradiction.

5. $supp_{mf}(\mathcal{M} \otimes^{\frac{1}{2}} \mathcal{N}) = supp_{mf}(\mathcal{M}) \cap supp_{mf}(\mathcal{N})$

   For Proposition 4.2.7, take $\mathcal{T} = (0 \rightleftarrows Q)$, we have the following

$$\mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M} \otimes_{mf} \mathcal{N}, (0 \rightleftarrows Q)) \cong \mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M}, \mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{N}, (0 \rightleftarrows Q))),$$

i.e.,

$$(\mathcal{M} \otimes_{mf} \mathcal{N})^* \cong \mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M}, \mathcal{N}^*)$$

$$\begin{aligned}
\text{Therefore, } supp_{mf}(\mathcal{M} \otimes_{mf} \mathcal{N}) &= supp_{mf}((\mathcal{M} \otimes_{mf} \mathcal{N})^*) && \text{(Proposition 4.2.5)}\\
&= supp_{mf}(\mathcal{H}\mathrm{om}_{\mathrm{mf}}(\mathcal{M}, \mathcal{N}^*)) && \text{(the above isomorphism)}\\
&= supp_{mf}(\mathcal{M}) \cap supp_{mf}(\mathcal{N}^*) && \text{(Proposition 4.2.5)}\\
&= supp_{mf}(\mathcal{M}) \cap supp_{mf}(\mathcal{N}) && \text{(Proposition 4.2.5)}
\end{aligned}$$

Also, notice that we have $supp_{mf}(\mathcal{M}) = supp_{mf}(\lambda(\mathcal{M}))$ for any $\lambda \in Q^\times$. Indeed, by Remark 4.1.18, the functor λ preserves homotopy and the definition of support is the collection of primes where the localized matrix factorization is not contractible. Hence $supp_{mf}(\mathcal{M} \otimes^{\frac{1}{2}} \mathcal{N}) = supp_{mf}(\frac{1}{2}(\mathcal{M} \otimes_{mf} \mathcal{N})) = supp_{mf}(\mathcal{M} \otimes_{mf} \mathcal{N}) = supp_{mf}(\mathcal{M}) \cap supp_{mf}(\mathcal{N})$.

□

Corollary 4.3.2. *$(Sing(Q/f), supp_{mf})$ is a support data on $([MF(Q,f)], \otimes^{\frac{1}{2}})$.*

Proof. First, $supp_{mf}$ is well-defined, i.e.,

1. $Sing(Q/f)$ is a topological space.

2. Given $\alpha : \mathcal{M} \to \mathcal{N}$ a homotopy equivalence, $supp_{mf}(\mathcal{M}) = supp_{mf}(\mathcal{N})$.

3. For any $\mathcal{M} \in [MF(Q,f)]$, $supp_{mf}(\mathcal{M})$ is a closed subset of $Sing(Q/f)$.

(1) is trivial. (2) is true by Proposition 4.2.2 . (3) is Corollary 4.2.6.

Also, $Sing(Q/f) = \bigcup_{\mathcal{M} \in [MF(Q,f)]} supp_{mf}(\mathcal{M})$. The containment $\supseteq$ is obvious ((3) above). The other containment is Proposition 4.2.4. Indeed, for any point $x \in Sing(Q/f)$, its closure $\overline{\{x\}} \subseteq Sing(Q/f)$. Then by Proposition 4.2.4, there is an object $\mathcal{M} \in [MF(Q,f)]$ such that $supp_{mf}(\mathcal{M}) = \overline{\{x\}}$, so $x \in \overline{\{x\}} = supp_{mf}(\mathcal{M})$.

The remaining conditions are proved by the previous lemma. Therefore, $(Sing(Q/f), supp_{mf})$ is a support data.

$\square$

Proposition 4.3.3. *We have that $(Sing(Q/f), supp_{mf})$ is a classifying support data on $[MF(Q,f)]$.*

Proof. We will denote the tensor product $\otimes^{\frac{1}{2}}$ simply by $\otimes$ in the proof to avoid unclear notations like $x^{\otimes^{\frac{1}{2}} n}$, $x \in [MF(Q,f)]$.

1. The fact that $Sing(Q/f)$ is noetherian and any non-empty irreducible closed subset $Z \subset Sing(Q/f)$ has a unique generic point comes from algebraic geometry.

2. We need to show that there is a bijection

   $$\theta : \{Y \subset X | \; Y \text{ specialization closed}\} \longleftrightarrow \{J \subset [MF(Q,f)] | \; J \text{ radical thick } \otimes\text{-ideal}\}$$

   given by

   $$Y \longmapsto \{\mathcal{E} \in [MF(Q,f)] | supp_{mf}(\mathcal{E}) \subset Y\}$$

   and

   $$\bigcup_{\mathcal{E} \in J} supp_{mf}(\mathcal{E}) \leftarrow\!\shortmid J.$$

   From the Theorem 4.2.8 above, there exists a bijective correspondence

   $$\theta_w : \{\text{specialization closed subsets of } Sing(Q/f)\} \longleftrightarrow \{\text{thick subcategories of } [MF(Q,f)]\}$$

   given by

   $$Z \longmapsto \{\mathcal{E} \in [MF(Q,f)] | supp_{mf}(\mathcal{E}) \subset Z\}$$

and

$$\bigcup_{\mathcal{E}\in T} supp_{mf}(\mathcal{E}) \leftarrow\!\shortmid T.$$

Take θ to be θ_w. From Theorem 4.2.8, we know that $\theta_w(Y)$ is thick. $\theta_w(Y)$ is also a radical $\otimes$-ideal. Indeed, we always have $\theta_w(Y) \subset \sqrt{\theta_w(Y)}$; for the other direction, notice that if $x \in \sqrt{\theta_w(Y)}$, i.e., $x^{\otimes n} \in \theta_w(Y)$, then $supp_w(x^{\otimes n}) = supp_{mf}(x) \subset \theta_w(Y)$ (Lemma 4.3.1 part (5)), then $x \in \theta_w(Y)$, done. The fact that $\theta_w(Y)$ is a $\otimes$-ideal is proven as follows: say $x \in \theta_w(Y)$, i.e., $supp_{mf}(x) \subset Y$, for any $a \in [MF(Q,f)]$, we have $supp_{mf}(a \otimes x) = supp_{mf}(a) \cap supp_{mf}(x) \subset Y$, done.

The above tells us that θ_w is well-defined so the only thing left is to show that it's a bijection. This is Theorem 4.2.8.

□

Corollary 4.3.4. *We have an isomorphism* $Sing(Q/f) \cong Spc'([MF(Q,f)])$.

Proof. The previous proposition tells us that $(Sing(Q/f), supp_{mf})$ is a classifying support data on $[MF(Q,f)]$. Now apply Theorem 4.1.16. □

4.4 Reconstruction via graded tensor triangulated category

We mentioned in the introduction that we can also get a reconstruction theorem using the usual tensor product $-\otimes_{mf}-$ of matrix factorizations by looking at a larger category: $K = \bigsqcup_{i>0} [MF(Q, if)]$.

The definition of a graded tensor triangulated category that I use here is the one developed by Yu-Han Liu [17]. Liu's original definition concerns about categories graded by a monoid but we don't have that. We only consider categories graded by $\mathbb{Z}_{>0}$ but this is not a big problem. Exactly the same construction still works.

We won't recall the definition of a graded tensor triangulated category but rather refer the reader to Liu's paper. However, we do want to remind the reader that by definition an object in K is a tuple $(\mathcal{E}_i)_{i>0}$ (all but finitely many $\mathcal{E}_i$'s are nonzero), where $\mathcal{E}_i \in [MF(Q, if)]$ for any $i > 0$.

The reconstruction theorem is essentially the same as the one we gave before. However, we do need to make a change to the definition of support for objects in the graded category K.

Definition 4.4.1. We use the support theory in Section 4.2 to define a support on the category K, which we denote by $supp_{gr}$, as the following,

$$supp_{gr}((\mathcal{E}_i)) = \bigcup_{i>0} supp_{mf}(\mathcal{E}_i) \in X := Sing(Q/f).$$

Here we need to assume $char(k) = 0$ to make the support an element of $Sing(Q/f)$ (if $char(k) = 0$, we have $Sing(Q/f) = Sing(Q/nf), \forall n$).

We have the following easy consequence of Lemma 4.3.1.

Corollary 4.4.2. $(Sing(Q/f), supp_{gr})$ *is a support data on* K.

Now we are ready to prove

Proposition 4.4.3. $(Sing(Q/f), supp_{gr})$ *is a classifying support data on* K.

Remark 4.4.4. *The following proof is essentially the same as the one for Proposition 4.3.3. However, we do need to use the universal property of a support data to modify the proof a little bit. Instead of just saying what the modification is, it's better to write down a complete proof here.*

Proof. Again, we adopt the notation $\otimes$ for $\otimes^{\frac{1}{2}}$ for the same reason as in the proof of Proposition 4.3.3.

1. The fact that $Sing(Q/f)$ is noetherian and any non-empty irreducible closed subset $Z \subset Sing(Q/f)$ has a unique generic point again comes from algebraic geometry.

2. We need to show that there is a bijection

   $$\theta : \{Y \subset Sing(Q/f) | \ Y \text{ specialization closed}\} \longleftrightarrow \{J \subset K_+ | \ J \text{ radical thick } \otimes\text{-ideal}\}$$

   given by
   $$Y \longmapsto \{E \in K_+ | supp_{gr}(E) \subset Y\}$$

   and
   $$\bigcup_{E \in J} supp_{gr}(E) \leftarrow\!\shortmid J.$$

   From the Theorem 4.2.8 above, we know there exists a bijective correspondence

   $$\theta_w : \{\text{specialization closed subsets of } Sing(Q/f)\} \longleftrightarrow \{\text{thick subcategories of } [MF(Q,f)]\}$$

given by

$$Z \longmapsto \{\mathcal{E} \in [MF(Q,f)] | supp_{mf}(\mathcal{E}) \subset Z\}$$

and

$$\bigcup_{\mathcal{E} \in T} supp_{mf}(\mathcal{E}) \leftarrow\!\shortmid T.$$

From Theorem 4.2.8, we know that $\theta_w(Y)$ is thick, therefore $\theta(Y)$ is thick. $\theta(Y)$ is also a radical $\otimes$-ideal. Indeed, we always have $\theta(Y) \subset \sqrt{\theta(Y)}$; for the other direction, notice that if $x \in \sqrt{\theta(Y)}$, i.e., $x^{\otimes n} \in \theta(Y)$, then $supp_{gr}(x^{\otimes n}) = supp_{gr}(x) \subset \theta(Y)$ (Corollary 4.4.2), then $x \in \theta(Y)$, done. The fact that $\theta(Y)$ is a $\otimes$-ideal is the following: say $x \in \theta(Y)$, i.e., $supp_{gr}(x) \subset Y$, for any $a \in K_+$, we have $supp_{gr}(a \otimes x) = supp_{gr}(a) \cap supp_{gr}(x) \subset Y$ (Corollary 4.4.2), done.

The fact that $\theta^{-1}(J)$ is specialization closed can be checked directly: we get $J \to \bigcup_{E \in J} supp_{gr}(E) = \bigcup_{E \in J} \bigcup_{i>0} supp_{mf}(\mathcal{E}_i)$ where all the $supp_{mf}(\mathcal{E}_i)$s are closed, i.e., a union of closed subsets, therefore specialization closed.

The above two paragraphs tell us that θ is well-defined so the only thing left is to show that it's a bijection. The idea of the proof is essentially Theorem 4.2.8 (but as mentioned at the beginning of the proof, we do need to change Walker's original argument a little bit).

It is clear that for any Y in the left-hand side we have

$$\bigcup_{E \in K, supp_{gr}(E) \subset Y} supp_{gr}(E) \subset Y.$$

The opposite containment holds since we may write Y as a union of closed subsets of $Sing(Q/f)$, and for any such closed subset W specialization closed. There is an $\mathcal{E}_i \in [MF(Q,if)] (\subset K_+)$ with $supp_{mf}(\mathcal{E}_i) = W$ by Proposition 4.2.4.

Likewise, it is clear that for any J in the right-hand side, we have a containment

$$J \subset \{E \in K_+ | supp_{gr}(E) \subset \bigcup_{F\in J} supp_{gr}(F)\}$$

of radical $\otimes$-ideals. (That the right hand side is a radical $\otimes$-ideal is easy to check, as we did above.) For the opposite containment, given E in the right-hand side, since $supp_{gr}(E)$ is closed and X is a Noetherian space, $supp_{gr}(E)$ is contained in a finite union of supports of objects of J and hence, by taking direct sums, $supp_{gr}(E) \subset supp_{gr}(F)$ for some $F \in J$. Since $supp_{gr}$ is a support data, there is a unique continuous map φ such that $\varphi(supp_{gr}(M)) = supp_+(M)$, therefore $supp_+(E) \subset supp_+(F)$, i.e., $E \notin P \Longrightarrow F \notin P$, for any prime P.

To show that $E \in J$. We have $J = \sqrt{J} = \bigcap_{P\supset J} P$ by Lemma 4.4.14 for any radical $\otimes$-ideal J. $F \in J = \sqrt{J} = \bigcap_{P\supset J} P$, so $F \in P$ for any $P \supset J$. So $E \in P$ for any $P \supset J$, if not, $E \notin P \Longrightarrow F \notin P$, contradiction. Therefore $E \in J$.

□

Corollary 4.4.5. *We have an isomorphism* $Sing(Q/f) \cong Spc'(K)$.

Remark 4.4.6.

1. *As in many reconstruction type theorems, we really should show that the reconstruction theorem we proved is not only a reconstruction of the underlying topological spaces but rather a reconstruction of schemes. In Balmer's theory of tensor triangulated geometry, one way to construct a structure sheaf is to look at the endomorphism ring of the tensor identity. We don't have the tensor identity in our categories (neither* $[MF(Q,f)]$ *nor* $\bigsqcup_{i>0}[MF(Q,if)]$*), however, we do know that there is a tensor identity for* $-\otimes_{mf}-$: $Q \cong (0 \rightleftarrows Q) \in [MF(Q,0)]$.

We might still be able to use it to give a sheaf for our topological space but it's not clear what exactly the construction should be at this stage.

There is another way to construct a sheaf for our space, also due to Balmer; unfortunately, again, more work needs to be done.

2. *In fact, it is very likely that we should really consider a 'Proj-construction' for the graded category* $K = \bigsqcup_{i \geqslant 0} [MF(Q, if)]$ *to see what we can get. Also, this larger category contains the tensor identity so it might automatically solve the problem in the above statement. However, the author is not able to settle this now. It will be considered in the future.*

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