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John H. Pollard
Macquarie University, jpollard@efs.mq.edu.au

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Improving Mortality: A Rule of Thumb and Regulatory Tool

John H. Pollard*

Abstract†

We develop a simple exact formula for determining cohort life expectancies under constant continuous uniform improvement in mortality using only a cross-sectional (period) Gompertz life table for the lives concerned and a simple approximation applicable to all life tables. The present values of annuities for such lives can be determined simply and accurately across the whole age span.

Key words and phrases: *Gompertz, life annuities, cohort mortality, cross-sectional mortality, period life table*

1 Introduction

The latter part of the twentieth century has seen mortality rates in many developed countries improving steadily over prolonged periods.

*John H. Pollard completed his B.Sc. degree at the University of Sydney with first class honors in 1963 and his Ph.D. at the University of Cambridge in 1967. After a post-doctoral year at the University of Chicago, he returned to Australia, becoming Professor and Chair of Actuarial Studies at Macquarie University in 1977.

Professor Pollard is the author of seven books on mortality, statistics, life insurance, and non-life insurance. His research publications range from mortality and population mathematics to stochastic interest models and reserving in non-life insurance. A past president of the Institute of Actuaries of Australia and of the Statistical Society of Australia, he has advised a number of major insurance companies, the World Health Organization, and the Australian Government and was a director of Swiss Re Australia from 1983 to 2001. He chaired the Mortality Committee of the International Union for the Scientific Study of Population from 1979 to 1983.

Dr. Pollard's address is: Division of Economic and Financial Studies, Macquarie University, NSW 2109, AUSTRALIA. Internet address: *jpollard@efs.mq.edu.au*

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Similar improvements have been observed with life insurance data. It is interesting to inquire about the effects such steady improvements have on expectation of life and make comparisons with the life expectancies reported in commonly prepared cross-sectional life tables. For insurance companies offering life annuities, allowance must be made for future mortality improvements to avoid the adverse financial consequences such improvements would otherwise have on company funds.

In a changing mortality environment, the value of an immediate annuity, for example, must depend inter alia on the age of the proposer and the year the annuity commences. Different values are therefore required according to each of these two variables, and a separate underlying life table must be computed for each combination. A simple change in the annual improvement rate or use of an updated base cross-sectional table necessitates a complete recalculation of all the underlying life tables and values. For a regulator, checking values used can be tedious.

Under the Gompertz law of mortality, a simple exact formula can be obtained for a cohort expectation of life at any age under constant uniform mortality improvement. An approximate rule derived from this formula, which is surprisingly accurate across a wide range of life tables, may be expressed as follows.

$$\text{cohort } \mathring{e}_x = \frac{9}{9 - 100r} \times \text{base } \mathring{e}_{x+150r} \tag{1}$$

where the annual rate of mortality improvement is $100r\%$ and $0 \leq 100r \leq 3$. A precise definition of "constant uniform mortality improvement" is given later.

Accurate approximations for the associated continuous life annuities can be obtained using

$$\text{cohort } \bar{a}_x = \text{base} \quad \bar{a}_x \times \left(\frac{\bar{a}_{\overline{n}|}}{\bar{a}_{\overline{v}|}} \right) \tag{2}$$

where $n = \text{cohort } \mathring{e}_x$ and $v = \text{base } \mathring{e}_x$.

The accuracy of the above formulae is such that errors introduced in using them are relatively minor compared with the uncertainty in selecting an assumed annual rate of mortality improvement. The formulae have the important advantage that a special generation life table does not need to be prepared for each and every age in a particular base year.

2 Expectation of Life Under Gompertz

The force of mortality under the well-known Gompertz law of mortality may be expressed in the following form:

$$\mu_x = k \exp[k(x - m)] \tag{3}$$

where m is the mode of the curve of deaths and k reflects the rate at which mortality increases with age. (See, for example, Benjamin and Pollard, 1993, page 298; Pollard, 1991.) The complete expectation of life at age x is

$$\mathring{e}_x = \frac{1}{k} \exp(\frac{\mu_x}{k}) E_1(\mu_x/k) \tag{4}$$

where $E_1(\cdot)$, an exponential integral, is defined by

$$E_1(t) = \int_1^\infty \frac{e^{-tu}}{u} du.$$

A proof of this result is given in the appendix, where a power series expansion for $E_1(\cdot)$ is provided.

For the purposes of this paper, the important thing to note in respect of equation (4) is that the complete expectation of life is simply equal to a function of the single variable μ_x/k divided by k and may be written

$$\mathring{e}_x = \frac{f(\mu_x/k)}{k}.$$

3 Improving Mortality

If mortality is improving at a constant instantaneous rate of r per annum at all ages and at all future times, a life age x in the base year and subject to a force of mortality μ_x at that time, will experience a force of mortality t years later of $\mu_{x+t} \exp(-rt)$. If the base cross-sectional table follows the Gompertz law of equation (3), $\mu_{x+t} = \mu_x \exp(kt)$, then the force of mortality t years after the base year for the life under consideration is

$$\mu_{x+t} e^{-rt} = \mu_x e^{(k-r)t}.$$

In other words, the cohort of lives age x in the base year will experience Gompertz mortality over their subsequent lifetimes with mortality increasing with age at a rate of $k-r$ compared with k in the base cross-sectional table. The cohort expectation of life is therefore

$$ {}_r\mathring{e}_x = \frac{f(\frac{\mu_x}{k-r})}{k-r}. $$

If we select an age $x+n$ in the cross-sectional life table such that

$$ \frac{\mu_{x+n}}{k} = \frac{\mu_x}{k-r}, \tag{5} $$

then

$$ \begin{aligned} (k-r)\,{}_r\mathring{e}_x &= f\left(\frac{\mu_x}{(k-r)}\right) \\ &= f(\frac{\mu_{x+n}}{k}) \\ &= k\mathring{e}_{x+n}. \end{aligned} $$

To obtain the improving mortality expectation of life, ${}_r\mathring{e}_x$, all we have to do is multiply the cross-sectional expectation of life at age $x+n$ by $k/(k-r)$. Because in the base Gompertz life table $\mu_{x+n} = \mu_x \exp(kn)$, the n required to satisfy equation (5) is

$$ n = \frac{1}{k}\ln\left[\frac{k}{(k-r)}\right]. \tag{6} $$

All the above formulae are exact. We now note that for most cross-sectional life tables, a Gompertz approximation to the adult age mortality will indicate a value of k in the neighborhood of 0.09, in which case

$$ n = \frac{1}{0.09}\ln\left(\frac{0.09}{0.09-r}\right) \approx 130r \tag{7} $$

which is exact when $r=0$. For $r = 0.01$, 0.02, and 0.03 and $k = 0.09$, equation (7) provides values of n of 1.31, 2.79, and 4.51, respectively. As a rule of thumb, therefore, we simply add 1.5 to the age for each percentage point in the annual mortality improvement rate (and pro rata between). The cross-sectional expectation of life at the resultant age is then multiplied by $9/(9-100r)$.

4 The Accuracy of the Rule of Thumb

The expectations of life at ages 30–35 in English Life Table 15 (Males) are shown in Table 1. As an example, let us imagine that this is the base cross-sectional life table and we require the expectation of life for a male age 30 at the time of the base table under a regime of continually improving mortality at a rate of 2% per annum.

Table 1
Expectation of Life
According to English Life Table 15 (Males)

Age x	30	31	32	33	34	35
$\mathring{e}_x$	44.88	43.92	42.96	42.01	41.05	40.09

According to the rule of thumb of the previous section, we simply take the expectation of life at age 33 according to the base table and multiply by 9/7. The approximate expectation of life is $42.01 \times 9/7 = 54.01$. Exact calculation using a specially prepared cohort life table produces a value of 54.10. Given the large difference between the cross-sectional $\mathring{e}_x(44.88)$ and the cohort $\mathring{e}_x(54.10)$, the rule of thumb produces a remarkably accurate approximation (only -0.2% error).

Further comparisons are presented in Tables 2, 3, and 4 using English Life Table 15 (Male), English Life Table 15 (Female), and the base cross-sectional table underlying the a(90) (Male) Life Table, none of which has a strict Gompertz shape. Interestingly, the rule of thumb works well, even at the juvenile ages, except when the annual mortality improvement rate is high. Although juvenile mortality does not follow the Gompertz pattern, it is now so low in developed-country populations that further mortality improvements affecting the complete expectation of life are largely concentrated in the later adult ages.

Continuous long-term mortality improvements at rates approaching 2% would be exceptional; the fact that the rule of thumb becomes progressively less accurate for values of r above 0.02 is therefore of little concern. Actuaries will usually be interested in monetary functions associated with the improving mortality, particularly annuities. The most accurate simple way of evaluating the latter approximately is to apply equation (2); that is, to multiply the life annuity in the base cross-sectional table by the ratio of the continuous annuity certain with a term equal to the approximated expectation of life under the improving mortality regime to the continuous annuity certain with its term equal to the base table expectation of life. The examples in Tables 2, 3, and 4 reveal that the approximations are remarkably accurate across the entire age range.

5 Concluding Remarks

The rule of thumb we have described for evaluating expectation of life and annuity values under continuously improving mortality regimes provides surprisingly accurate approximations for these life table and monetary functions. Attempts at gaining greater accuracy by using values of k derived from the actual base life table itself (e.g., setting k equal to the force of mortality at the mode of the curve of deaths and using a value for the age adjustment calculated using the exact equation (5)) fail to produce worthwhile improvements in the accuracy of calculated life expectancies and, given the uncertainty in the long-term mortality improvement rate, the additional work involved in making the more refined calculations is not justified.

Where the base life table relates to a mortality experience s years earlier rather than the date when the life of interest was age x, and where there has been continuous mortality improvements at rate p over the intervening period, the Gompertz law with $k = 0.09$ indicates that the life should be treated as being of age $x - ps/0.09$. This well-known adjustment should work well at all except the juvenile ages.

Regulators concerned with the solvency of life insurance companies need simple effective rules to apply to companies and their products. The rule we propose has potential applications for these purposes.

References

Abramowitz, M. and Stegun, I.A. *Handbook of Mathematical Functions.* New York: Dover Publications, 1964.

Benjamin, B., and Pollard, J.H. *The Analysis of Mortality and Other Actuarial Statistics.* London, England: Faculty of Actuaries and Institute of Actuaries, 1993.

Chemical Rubber Publishing Company. *Standard Mathematical Tables,* Twelfth edition. Cleveland, Ohio: Chemical Rubber Publishing Company, 1963.

Pollard, J.H. "Fun with Gompertz." *Genus* 47 (1991): 1-20.

Appendix

As is well known, the probability of survival from age x to age $x + t$ is

$$_tp_x = \exp\left(-\int_0^t \mu_{x+s}\mathrm{d}s\right),$$

and the complete expectation of life at age x is

$$\mathring{e}_x = \int_0^\infty {}_tp_x\mathrm{d}t.$$

Using the Gompertz force of mortality given in equation (3) yields

$$_tp_x = \exp\left[-\left(\frac{\mu_x}{k}\right)\left(e^{kt}-1\right)\right]. \tag{8}$$

Substituting $z = e^{kt}$ and integrating with respect to z yields

$$\mathring{e}_x = \frac{e^{\mu_x/k}}{k}\int_1^\infty \frac{1}{z}e^{-z\mu_x/k}dz = \frac{e^{\mu_x/k}}{k}E_1(\mu_x/k) \tag{9}$$

where $E_1(t)$ is the exponential integral defined by

$$E_1(t) = \int_1^\infty \frac{e^{-tu}}{u}du = -\gamma - \ln t - \sum_{n=1}^{\infty}\frac{(-t)^n}{n\,n!}$$

and $\gamma = 0.5772157...$ is Euler's constant. See, for example, CRC Standard Mathematical Tables page 315 or Abramowitz and Stegun (1965, Chapter 5) for more on this integral. Abramowitz and Stegun also provide accurate approximations to $E_1(t)$.

Table 2
Effect of Continually Improving Mortality
On English Life Table Number 15 (Male)
Life Expectancies and Life Annuities

	Age						
	0	5	10	20	30	60	90
Expectation of Life							
ELT15 Table	73.41	69.13	64.20	54.45	44.88	17.85	3.51
Improvement 1% pa							
Approximation	81.69	76.11	70.55	59.65	48.87	18.82	3.58
Exact Value	81.04	76.11	70.57	59.64	48.95	19.03	3.59
Error (%)	0.8	0.0	-0.0	0.0	-0.2	-1.1	-0.4
Improvement 2% pa							
Approximation	91.40	85.08	78.72	66.33	54.01	20.11	3.69
Exact Value	90.24	84.78	78.55	66.21	54.10	20.46	3.69
Error (%)	1.2	0.4	0.2	0.2	-0.2	-1.7	0.0
Improvement 3% pa							
Approximation	104.43	97.03	89.63	75.24	60.85	21.91	3.92
Exact Value	98.58	92.96	86.47	73.30	60.03	22.19	3.80
Error (%)	5.9	4.4	3.7	2.6	1.4	-1.3	3.1
Continuous Life Annuity at 5% pa Interest							
ELT15 Table	19.59	19.54	19.30	18.65	17.69	11.08	3.06
Improvement 1% pa							
Approximation	19.78	19.74	19.53	18.97	18.08	11.45	3.12
Exact Value	19.77	19.75	19.54	18.95	18.07	11.46	3.12
Error (%)	0.0	-0.0	-0.0	0.1	0.1	-0.1	0.0
Improvement 2% pa							
Approximation	19.92	19.92	19.75	19.27	18.49	11.91	3.20
Exact Value	19.92	19.94	19.76	19.25	18.44	11.87	3.19
Error (%)	0.0	-0.1	-0.0	0.1	0.3	0.3	0.3
Improvement 3% pa							
Approximation	20.03	20.06	19.93	19.55	18.90	12.51	3.38
Exact Value	20.04	20.08	19.94	19.51	18.80	12.31	3.26
Error (%)	-0.0	-0.1	-0.0	0.2	0.5	1.6	3.7

Table 3
Effect of Continually Improving Mortality
On English Life Table Number 15 (Female)
Life Expectancies and Life Annuities

	Age						
	0	5	10	20	30	60	90
Expectation of Life							
ELT15 Table	78.96	74.56	69.61	59.75	49.94	22.08	4.35
Improvement 1% pa							
Approximation	87.73	82.21	76.64	65.56	54.53	23.46	4.42
Exact Value	86.82	81.81	76.29	65.30	54.38	23.58	4.48
Error (%)	1.0	0.5	0.5	0.4	0.3	-0.5	-1.3
Improvement 2% pa							
Approximation	98.39	92.05	85.68	73.04	60.44	25.27	4.57
Exact Value	96.07	90.54	84.42	72.15	59.91	25.40	4.62
Error (%)	2.4	1.7	1.5	1.2	0.9	-0.5	-1.1
Improvement 3% pa							
Approximation	112.57	105.15	97.73	83.00	68.32	27.74	4.83
Exact Value	103.82	98.28	92.02	79.20	66.03	27.59	4.78
Error (%)	8.4	7.0	6.2	4.8	3.5	0.5	1.0
Continuous Life Annuity at 5% pa Interest							
ELT15 Table	19.81	19.78	19.60	19.08	18.28	12.69	3.70
Improvement 1% pa							
Approximation	19.96	19.95	19.80	19.35	18.63	13.12	3.75
Exact Value	19.95	19.95	19.80	19.35	18.61	13.09	3.79
Error (%)	0.1	0.0	0.0	0.0	0.1	0.2	-1.1
Improvement 2% pa							
Approximation	20.07	20.09	19.97	19.60	18.98	13.63	3.87
Exact Value	20.08	20.10	19.97	19.59	18.94	13.53	3.88
Error (%)	-0.0	-0.0	0.0	0.1	0.2	0.7	-0.3
Improvement 3% pa							
Approximation	20.16	20.19	20.11	19.82	19.32	14.27	4.06
Exact Value	20.16	20.21	20.11	19.80	19.24	13.99	3.98
Error (%)	0.0	-0.1	0.0	0.1	0.4	2.0	2.0

Table 4
Effect of Continually Improving Mortality
On a(90) Base Life Expectancies and Life Annuities

	Age			
	20	30	60	90
Expectation of Life				
Base Table	56.03	46.34	19.11	4.04
Improvement 1% pa				
Approximation	61.35	50.50	20.26	4.14
Exact Value	61.59	50.75	20.46	4.16
Error (%)	-0.4	-0.5	-1.0	-0.5
Improvement 2% pa				
Approximation	68.26	55.84	21.77	4.30
Exact Value	68.41	56.18	22.08	4.28
Error (%)	-0.2	-0.6	-1.4	0.4
Improvement 3% pa				
Approximation	77.48	62.96	23.85	4.56
Exact Value	75.31	62.11	24.01	4.42
Error (%)	2.9	1.4	-0.6	3.2
Continuous Life Annuity at 5% pa Interest				
Base Table	18.80	17.87	11.53	3.47
Improvement 1% pa				
Approximation	19.10	18.25	11.94	3.55
Exact Value	19.10	18.25	11.93	3.55
Error (%)	0.0	0.0	0.1	0.0
Improvement 2% pa				
Approximation	19.39	18.64	12.44	3.67
Exact Value	19.38	18.62	12.36	3.63
Error (%)	0.1	0.1	0.6	1.1
Improvement 3% pa				
Approximation	19.63	19.03	13.08	3.87
Exact Value	19.63	18.96	12.82	3.72
Error (%)	0.0	0.4	2.0	4.0