2002

# A Note on the Parallelogram Method for Computing the On-Level Premium

David P.M. Scollnik
*University of Calgary*, scollnik@math.ucalgary.ca

Wai Man Sara Lau
*University of Calgary*, wmlau@math.ucalgary.ca

Journal of Actuarial Practice Vol. 10, 2002

# A Note on the Parallelogram Method for Computing the On-Level Premium

David P.M. Scollnik* and Wäi Man Sara Lau†

## Abstract‡

This paper discusses the differences appearing in the descriptions of the parallelogram method for the determination of earned premium at current rate levels given by McClenahan (1996) and Brown and Gottlieb (2001). It observes that the former is consistent with the method of extending exposures while the latter is not. An illustration is provided. This paper also discusses two other approaches to the determination of the earned premium.

Key words and phrases: *earned premium, extending exposures, ratemaking*

## 1 Introduction

For the purpose of ratemaking, it is often necessary to determine the dollars of earned premium at current rates. The method of extending exposures (also known as the extension of exposures technique) simply re-rates each policy using the current rate manual and the existing distribution of earned exposures. This is the best method when detailed

*David P.M. Scollnik, A.S.A., Ph.D., is an associate professor of actuarial science and statistics at the University of Calgary. He holds a Ph.D. in statistics from the University of Toronto.

Dr. Scollnik's address is: Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., Calgary, Alberta T2N 1N4, CANADA. Internet address: *scollnik@math.ucalgary.ca* and <http://www.math.ucalgary.ca/~scollnik>

†Wai Man Sara Lau is a graduate student in the Department of Mathematics and Statistics.

Ms. Lau's address is: Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., Calgary, Alberta T2N 1N4, CANADA. Internet address: *wmlau@math.ucalgary.ca*

‡This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada (NSERC). The authors would like to thank the editor for helpful comments that have improved the presentation of this paper.

data and appropriate rating software are both available. When this is not the case, the so-called parallelogram method can be used instead. The parallelogram method adjusts calendar year earned premiums to reflect the effect of all rate changes made since these earned premiums were written. McClenahan (1996, pages 42–44) and Brown and Gottlieb (2001, pages 73–76) describe these two methods in some detail.

Careful readers of McClenahan (1996) and Brown and Gottlieb (2001) will note that, whereas the geometrical interpretations given to the parallelogram method initially appear to be the same in both sources, the implementation details regarding the definition and calculation of the on-level factors differ. Practitioners and students taking the SOA Course 5 and/or CAS Exam 5 will benefit from an explicit mention of the discrepancy along with an illustration and discussion clarifying the discrepancy. This paper also discusses two other approaches to the determination of the earned premium.

## 2 Illustration

Suppose that the experience period consists of three calendar years, $Z$, $Z+1$, and $Z+2$, during which the earned premiums were 1250, 1575, and 1620, respectively. Suppose $P$ is the rate level effective $1/1/Z-1$ and rate increases (applied to newly issued policies or renewals) were introduced as follows:

- $+25\%$ effective $7/1/Z$, and
- $+28\%$ effective $4/1/Z+1$.

Figure 1 shows the rates in effect during the experience period, under the standard assumptions that all policies have a one-year term and policy issue dates are uniformly distributed over time. Under these assumptions, the parallelogram method can be used to determine the proportion of policies in each year that were written at the various premium rate levels.[1] Using the methodologies in McClenahan (1996) or Brown and Gottlieb (2001), the reader can verify that these proportions are as given in Table 1.

[1] These standard assumptions can be modified using techniques available in the casualty actuarial literature as in Miller and Davis, 1976.

**Figure 1**
**Experience Period and Rate Changes**

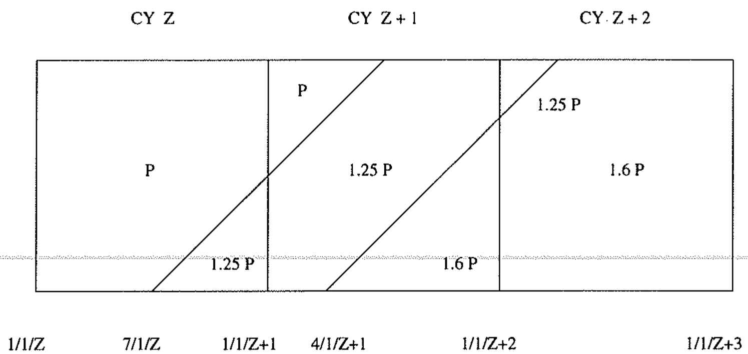



**Table 1**
**Proportion of Policies Within Each Calendar Year Written at the Premium Rates Effective On $1/1/Z-1$, $7/1/Z$, and $4/1/Z+1$**

| Calendar Year | $1/1/Z-1$ | $7/1/Z$ | $4/1/Z+1$ |
|---|---|---|---|
| $Z$ | 0.875 | 0.12500 | 0.00000 |
| $Z+1$ | 0.125 | 0.59375 | 0.28125 |
| $Z+2$ | 0.000 | 0.03125 | 0.96875 |

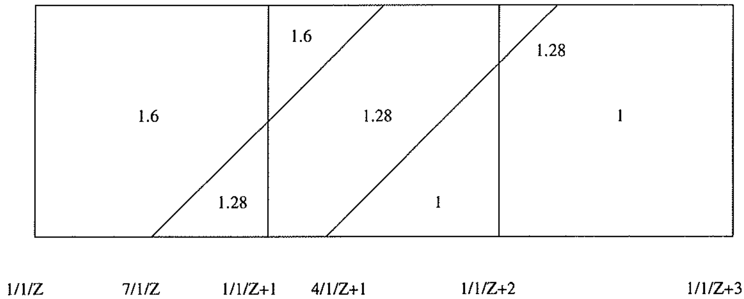
## 2.1 Using the Brown and Gottlieb (2001) Method

At this stage the parallelogram method as described in Brown and Gottlieb (2001) proceeds by determining the multiplicative factors that should be applied to each premium band within each calendar year in order to calculate the earned premium at the current rate level. These rate promotion factors are given in Table 2 and are illustrated in Figure 2.

**Table 2**
**Rate Promotion Factors Applied to the Policies Within Each Calendar Year Written at the Premium Rates Effective On $1/1/Z-1$, $7/1/Z$, and $4/1/Z+1$**

| Calendar Year | $1/1/Z-1$ | $7/1/Z$ | $4/1/Z+1$ |
|---|---|---|---|
| $Z$ | $1.25\times 1.28 = 1.6$ | 1.28 | - |
| $Z+1$ | $1.25\times 1.28 = 1.6$ | 1.28 | 1 |
| $Z+2$ | - | 1.28 | 1 |

**Figure 2**
**Rate Promotion Factors Applied to the Policies Within Each Calendar Year Written at the Premium Rates Effective On $1/1/Z-1$, $7/1/Z$, and $4/1/Z+1$**




The next step is to determine the weighted average on-level factor for each calendar year as follows:

| Year | On-Level Factor |
|---|---|
| $Z$ | $0.875 \times 1.6 + 0.125 \times 1.28 = 1.56$ |
| $Z+1$ | $0.125 \times 1.6 + 0.59375 \times 1.28 + 0.28125 = 1.24125$ |
| $Z+2$ | $0.03125 \times 1.28 + 0.96875 = 1.00875$ |

The proportions in Table 1 are used as the weights in the calculations above. The on-level factor corresponding to a particular calendar year can be interpreted as the weighted average of the required multiplicative factors needed to bring that year's earned premium to the current rate.

**Table 3**
**Development of Earned Premium at Current Rates Using the Methodology in Brown and Gottlieb (2001)**

| Calendar Year | Earned Premium | On-Level Factor | Earned Premium at Current Rates |
|---|---|---|---|
| $Z$ | 1250 | 1.56000 | 1950.000 |
| $Z+1$ | 1575 | 1.24125 | 1954.969 |
| $Z+2$ | 1620 | 1.00875 | 1634.175 |
| Total: | | | 5539.144 |

The earned premiums for the different calendar years under consideration are reported in the second column of Table 3. The estimated earned premiums at the current rate level are developed in the fourth column of Table 3 using calculated on-level factors.

## 2.2 Using the Method Described in McClenahan (1996)

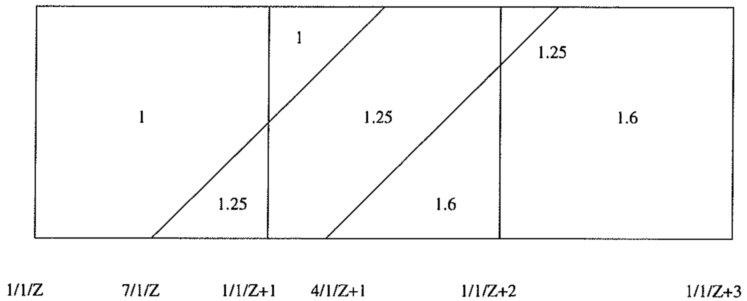
The traditional methodology described in McClenahan (1996) also begins with the determination of the proportions in Table 1. Instead of developing the rate promotion factors in Table 2, however, one determines the relationship each premium rate class within each calendar year bears to the earliest rate in effect at the beginning of the period under examination.

For instance, the earned premium in calendar year Z +1 that was written between $7/1/Z$ and $4/1/Z+1$ was written at a level that is equal to 1.25 times the rate that was in effect on $1/1/Z-1$. These relations or factors are reported in Table 4 and are illustrated in Figure 3.

**Table 4**
**Relation of the Written Premium Rates**
**Within Each Calendar Year to the Earliest Premium Rate**

| Year | $1/1/Z-1$ | $7/1/Z$ | $4/1/Z+1$ |
|---|---|---|---|
| $Z$ | | 1.25 | - |
| $Z+1$ | | 1.25 | $1.25 \times 1.28 = 1.6$ |
| $Z+2$ | - | 1.25 | $1.25 \times 1.28 = 1.6$ |

**Figure 3**
**Relation of the Written Premium Rates**
**Within Each Calendar Year to the Earliest Premium Rate**



The next step is to determine the weighted average of these factors for each calendar year as follows:

| Year | Weighted Average Factor |
|---|---|
| $Z$ | $0.875 + 0.125 \times 1.25 = 1.03125$ |
| $Z+1$ | $0.125 + 0.59375 \times 1.25 + 0.28125 \times 1.6 = 1.3171875$ |
| $Z+2$ | $0.03125 \times 1.25 + 0.96875 \times 1.6 = 1.5890625$ |

As before, the proportions in Table 1 are used as the weights in the calculation of the weighted average factors above. These weighted average factors are not yet the on-level factors as they are traditionally defined. [For example, as in McClenahan (1996).]

Rather, the on-level factor for a given calendar year is determined by dividing the current rate level (i.e., $1.6P$ in this example) by $P$ and by the weighted average factor for the year under consideration. So, the on-level factors for this example are given as follows:

| Year | On-Level Factor |
|---|---|
| $Z$ | $1.6/1.03125 = 1.551515$ |
| $Z+1$ | $1.6/1.3171875 = 1.214709$ |
| $Z+2$ | $1.6/1.5890625 = 1.006883$ |

It is evident that these on-level factors differ from the ones constructed using the methodology given in Brown and Gottlieb (2001), as do the resulting estimates of the earned premium at current rates given in Table 5.

**Table 5**
**Development of Earned Premium at Current Rates Using the Traditional Methodology as in McClenahan (1996)**

| Calendar Year | Earned Premium | On-Level Factor | Earned Premium at Current Rates |
|---|---|---|---|
| $Z$ | 1250 | 1.551515 | 1939.394 |
| $Z+1$ | 1575 | 1.214709 | 1913.167 |
| $Z+2$ | 1620 | 1.006883 | 1631.150 |
| Total: | | | 5483.711 |

# 3 Interpretation of the Results

## 3.1 Practical Interpretation of the Results

In Section 2, we stated that the parallelogram method can be used to determine what proportion of policies in each year were written at the various premium rate levels. The differences we have observed in the two methodologies arise from the fact that McClenahan (1996) interprets these proportions as proportions of earned exposures in a year, whereas Brown and Gottlieb (2001, page 75) implicitly develop them as proportions of earned exposures but then use them as proportions of earned premium dollars. Clearly, both cannot be correct under the assumption that policy issues are uniformly distributed within a calendar year during which a rate change occurs.

Suppose that an additional 1000 earned units of exposure were discovered in the books for each of calendar years $Z$, $Z+1$, and $Z+2$, in the context of the previous illustration, and we wanted to determine the addition to our estimate of the current earned premium. Using the method of extending exposures, this value is simply $3 \times 1000 \times 1.6 = 4800$.

If we assume that policy issues were uniformly distributed within each year, then the additional earned premiums in calendar years $Z$, $Z+1$, and $Z+2$ are given by 1031.25, 1317.1875, and 1589.0625, respectively. Using the on-level factors developed in Section 2.2, we find that McClenahan's method estimates the addition to current earned premium as follows:

$$
\begin{aligned}
& 1031.25 \times 1.551515 + 1317.1875 \times 1.214709 + 1589.0625 \times 1.006883 \\
& = 1600 + 1600 + 1600 \\
& = 4800.
\end{aligned}
$$

This result is consistent with that given by the method of extending exposures.

On the other hand, using the on-level factors developed in Section 2.1 we find that Brown and Gottlieb's method estimates the addition to current earned premium as follows:

$$
\begin{aligned}
& 1031.25 \times 1.56 + 1317.1875 \times 1.24125 + 1589.0625 \times 1.00875 \\
& = 1608.75 + 1634.96 + 1602.97 \\
& = 4846.68.
\end{aligned}
$$

This result is not consistent with that given by the method of extending exposures. It also demonstrates that 1000 units of earned exposure in a calendar year will yield different additions to current earned premium, depending on the calendar year to which it is assigned. This is not the answer that most practitioners are looking for.

This illustration clarifies the fact that the parallelogram methodology described in McClenahan (1996) is the one that should be used under the assumption that policy issue dates are uniformly distributed over time. When the policy issue dates are uniformly distributed, McClenahan's method yields results that are consistent with the method of extending exposures, and with traditional methods in the casualty actuarial literature, such as Kallop (1975) and Miller and Davis (1976).

The on-level factors defined in Brown and Gottlieb (2001) are appropriate if the *dollars of earned premium income* are uniformly distributed over any calendar year so that the proportions given by the parallelogram method were proportions of earned premium. This is not the assumption made in Brown and Gottlieb (2001, pages 73 and 76) (cf. Brown, 1993, page 74), however, nor can it ever be consistent with the assumption of uniform policy issues in a period containing a rate change.

## 3.2 Mathematical Interpretation of the Results

To explain the difference in the two methods in a more formal fashion, consider the following: Suppose that there are $n(k)$ premium bands in the calendar year $Z+k$, and the base premium at the start of the experience period (i.e., on $1/1/Z$) is $P$. Then, for $i = 1, 2, \ldots n(k)$, define the following:

$$\begin{aligned}
P_i^{(k)} &= \text{Premium for band } i \text{ in calendar year } Z+k;\\
f_i^{(k)} &= \text{Factor applied to } P \text{ to give } P_i^{(k)}, \text{ i.e.,}\\
P_i^{(k)} &= f_i^{(k)} \times P;\\
P^{(\text{cur})} &= \text{Current premium rate at the end of the experience period;}\\
g_i^{(k)} &= \text{Factor applied to } P_i^{(k)} \text{ to give } P^{(\text{cur})}, \text{ i.e.,}\\
P^{(\text{cur})} &= f_i^{(k)} \times g_i^{(k)} \times P; \quad \text{and}\\
A_i^{(Z)} &= \text{Proportion of policies written in band } i, \text{ with}
\end{aligned}$$

$$A_1^{(k)} + \cdots + A_{n(k)}^{(k)} = 1.$$

The on-level factor for calendar year $Z + k$ can now be given as follows: The McClenahan (1996) on-level factor, $\text{OLF}_M^{(k)}$, is given by

$$\text{OLF}_M^{(k)} = \frac{p^{(\text{cur})}}{P} \frac{1}{\sum_{i=1}^{n(k)} A_i^{(k)} P_i^{(k)}/P} = \frac{p^{(\text{cur})}/P}{\sum_{i=1}^{n(k)} A_i^{(k)} f_i^{(k)}} = \left[ \sum_{i=1}^{n(k)} \frac{A_i^{(k)}}{g_i^{(k)}} \right]^{-1}.$$

On the other hand, the Brown and Gottlieb (2001) on-level factor, $\text{OLF}_{BG}^{(k)}$, is given by

$$\text{OLF}_{BG}^{(k)} = \sum_{i=1}^{n(k)} A_i^{(k)} \frac{p^{(\text{cur})}}{P_i^{(k)}} = \frac{p^{(\text{cur})}}{P} \sum_{i=1}^{n(k)} \frac{A_i^{(k)}}{f_i^{(k)}} = \sum_{i=1}^{n(k)} A_i^{(k)} g_i^{(k)}.$$

It is evident that $\text{OLF}_{BG}^{(k)}$ is not equal to $\text{OLF}_M^{(k)}$, in general.

In Section 3.1, we observed that $\text{OLF}_{BG}^{(k)}$ would be appropriate if the dollars of earned premium income were uniformly distributed over any calendar year so that the proportions given by the parallelogram method were proportions of earned premium. In this case,

$$A_i^{(k)} = \frac{A_i^{(k)} P_i^{(k)}}{\sum_{i=1}^{n(k)} A_i^{(k)} P_i^{(k)}}, \quad \text{which implies} \quad P_i^{(k)} = \sum_{i=1}^{n(k)} A_i^{(k)} P_i^{(k)},$$

for $i = 1, 2, \ldots, n(k)$. Under this assumption, we have

$$\text{OLF}_{BG}^{(k)} = \sum_{i=1}^{n(k)} A_i^{(k)} \frac{p^{(\text{cur})}}{P_i^{(k)}} = \frac{p^{(\text{cur})} \sum_{i=1}^{n(k)} A_i^{(k)}}{\sum_{i=1}^{n(k)} A_i^{(k)} P_i^{(k)}} = \frac{p^{(\text{cur})}/P}{\sum_{i=1}^{n(k)} A_i^{(k)} f_i^{(k)}} = \text{OLF}_M^{(k)}.$$

This demonstrates that $\text{OLF}_{BG}^{(k)}$ is equal to $\text{OLF}_M^{(k)}$, and hence its usage will reproduce the method of extending exposures, only in a very special (and unrealistic) case.

## 4 Further Discussion and New Approaches

Of course, the assumption (as in McClenahan, 1996) that policy issue dates are uniformly distributed over time is unlikely to be consistent with actual experience. When the actual past levels of exposures are known and available, the procedure described in Miller and Davis (1976) can be used to determine the premium adjustment factors and on-level premiums. When this is not the case, the following variation on the parallelogram method might be used instead.

The basic idea behind this variation is to use the observed earned premium in each calendar year in order to better approximate the true underlying levels of exposure and then reprice these exposure levels at the current premium rate. The standard assumption that policy issue dates are uniformly distributed over time is replaced with the assumption that they are uniformly distributed at a constant rate between any two adjacent premium rate change dates. This twist on the parallelogram method will be illustrated in the context of the continuing example from Section 2.

Let $P_1$ be the initial premium rate and let $E_1$ denote the constant policy issue rate in effect prior to $7/1/Z$. Then, $P_1 \times E_1$ is the annual rate at which earned premium was being generated prior to $7/1/Z$. Let $E_2$ denote the constant policy issue rate in effect between $7/1/Z$ and $4/1/Z+1$ and $E_3$ the rate thereafter. Then the earned premium in each of the three calendar years under consideration should satisfy these relations:

| Year | Earned Premium |
|---|---|
| $Z$ | $1250 = P_1 \times (E_1 \times 0.875 + E_2 \times 1.25 \times 0.125)$ |
| $Z+1$ | $1575 = P_1 \times (E_1 \times 0.125 + E_2 \times 1.25 \times 0.59375$ $+E_3 \times 1.6 \times 0.28125)$ |
| $Z+2$ | $1620 = P_1 \times (E_2 \times 1.25 \times 0.03125$ $+E_3 \times 1.6 \times 0.96875)$. |

Solving this system of linear equations yields the values:

$$\begin{aligned} P_1 \times E_1 &= 1195.162602, \\ P_1 \times E_2 &= 1307.089431, \\ P_1 \times E_3 &= 1012.220528. \end{aligned}$$

Hence, the earned premium at the current rate (i.e., $1.6 \times P_1$) in each calendar year is as follows:

| Year | Earned Premium at Current Rates |
|---|---|
| $Z$ | $1934.645529 = P_1 \times 1.6 \times (E_1 \times 0.875 + E_2 \times 0.125)$ |
| $Z+1$ | $1936.266717 = P_1 \times 1.6 \times (E_1 \times 0.125 + E_2 \times 0.59375$ |
| $Z+2$ | $1634.296290 = P_1 \times 1.6 \times (E_2 \times 0.03125 + E_3 \times 0.96875)$ |
| Total: | 5505.208536. |