2002

Some Comments on the Pricing of an Exotic Excess of Loss Treaty

Jean-Francois Walhin
Secura Belgian Re, jfw@secura-re.com

Journal of Actuarial Practice Vol. 10, 2002

Some Comments on the Pricing of an Exotic Excess of Loss Treaty

Jean-François Walhin*

Abstract

This paper uses a multivariate analog of Panjer's algorithm to develop a method for pricing a complex excess of loss treaty. The treaty is such that some layers inure to the benefit of other layers. The structure of this treaty is discussed. Numerical examples are provided.

Key words and phrases: *multivariate Panjer's algorithm, paid reinstatements, inuring layers, order of claims*

1 Excess of Loss Treaties

1.1 The Basics

The classic collective risk model assumes that an insurer has a portfolio of similar policies that experiences N claims in a year—any other period, such as a quarter or month, will do. The sizes of the claims are $X_1, X_2, \ldots, X_N$ and are independent and identically distributed with common distribution function $F(x)$. The aggregate losses faced by the insurer for a year, S, is then given by

$$S = X_1 + \cdots + X_N.$$

One way to manage these losses is through reinsurance. Excess of loss reinsurance is a means to share risks between the insurer and the reinsurer. The insurer always remains liable for the part of the claim below

*Jean-François Walhin, Ph.D., is a civil engineer, an actuary (fellow of ARAB-KVBA) and holds a Ph.D. in sciences from the Université Catholique de Louvain (UCL). He is R&D Manager at Secura Belgian Re and invited professor at UCL.

Dr. Walhin's address is: Université Catholique de Louvain, Institut des Sciences Actuarielles, Grand-Rue, 54 B-1348 Louvain-la-Neuve, BELGIUM. Internet address: *jfw@secura-re.com*

a given attachment point (deductible) P. The reinsurer, on the other hand, pays the excess of each loss above P and up to a limit $P + L$, thus offering some capacity between P and the limit $P + L$. The quantity L is often called the amount of capacity offered. So, for each claim X_i, the liability of the excess of loss reinsurer is R_i, where

$$R_i = \min(L, \max(0, X_i - P)). \tag{1}$$

In practice, reinsurers use the term $L\,\mathbf{xs}\,P$ to refer to the contract defined in equation (1). The aggregate claims of the reinsurer is

$$S_R = R_1 + \cdots + R_N. \tag{2}$$

Sometimes a line of business is protected by several reinsurance treaties such that for each X_i, the j^{th} reinsurance treaty pays

$$R_i^{(j)} = \min(L_j, \max(0, X_i - P_j)), \tag{3}$$

where $P_{j+1} = P_j + L_j$ for $j = 1, 2, 3, \ldots$. In this case the j^{th} reinsurance treaty is called the j^{th} layer.

When the reinsurer offers the capacity L $k + 1$ times per year, we say there are k reinstatements, $k = 0, 1, \ldots$. The annual liability of the reinsurer is $\min\{(k + 1)L, S_R\}$. Reinstatements may be paid or free. If they are free, then the insurer can use the k reinstatements without payment of a reinstatement premium. If they are paid, however, the insurer has to pay the reinsurer a reinstatement premium each time the insurer uses the whole layer or part of the layer. Reinstatement premiums are usually defined as a percentage of the initial reinsurance premium. Thus, for example, the reinsurer would state: $L\,\mathbf{xs}\,P$ with three reinstatements payable at 100%. This means that the offered capacity may be used up to four times, and each time it is used the insurer has to pay a reinstatement premium, which is 100% of the original premium prorated for the used capacity.

In order to reduce the aggregate claims paid by the reinsurer (and hence the reinsurance premium charged) the reinsurance treaty may include an annual aggregate deductible, AAD. In the general case where there are k reinstatements and an annual aggregate deductible of AAD, the annual liability of the reinsurer is:

$$S^{(\mathrm{AL})} = \min\{(k + 1)L, \max(0, S_R - \mathrm{AAD})\}. \tag{4}$$

The following practical example is used to illustrate the ideas and terminology mentioned above. Consider the following treaty:

Treaty 1. *An excess of loss treaty with two layers,*

- *Layer 1: 100 xs 100 with two reinstatements payable at* 150% *after an annual aggregate deductible of* 50. *The reinsurance premium is* 25.
- *Layer 2: 300 xs 200 with one reinstatement payable at* 100%. *The reinsurance premium is* 10.

Suppose further that the insurer experiences the following claims under the treaty: 120, 250, 150, 130. The claims are paid as follows:

- Layer 1: Within layer 1, the claims for the reinsurer are: 20, 100, 50, 30. The aggregate claims is 200. As there is an annual aggregate deductible, the individual claims within the layer are aggregated (to give 200), and the insurer pays 50 of the 200. The remaining 150 is to be paid by the reinsurer. As 150 = 100 + 50, layer 1 is used completely once with 50 remaining. Fortunately there are two reinstatements, so the remaining 50 is paid by the reinsurer. As the reinstatements are not free, the reinsurer will ask for two reinstatement premiums: the first for the full layer used, i.e., for the full reinstatement premium, which is $150\% \times 25 = 37.5$; and the second for the partial (50/100) layer used, i.e., for $150\% \times \frac{50}{100} \times 25 = 18.75$.
- Layer 2: Within layer 2 the attachment point is 200 per claim, so the claims faced by the reinsurer are 0, 50, 0, 0. The aggregate claims is 50, which will be paid by the reinsurer. As a reinstatement is payable, however, there is a compulsory reinstatement premium: $100\% \times \frac{50}{300} \times 10 = 1.666$.

These results are summarized below.

Reinsurance Premium	
Layer 1:	25 + 37.5 + 18.75 = 81.25
Layer 2:	10 + 1.666 = 11.666
Losses Paid	
By insurer:	100 + 100 + 100 + 100 + 50 = 450.
By reinsurer for layer 1:	200 − 50 = 150.
By reinsurer for layer 2:	50.

1.2 The Notion of Inuring

Recently, I have been given the opportunity to price the following excess of loss treaty:

Treaty 2.

- *Layer 1: 7.5* **xs** *2.5 with three reinstatements payable at* 100% *after an annual aggregate deductible of* 10.
- *Layer 2: 15* **xs** *2.5 with three reinstatements payable at* 100% *after an annual aggregate deductible of* 5.
- *Layer 3: 22.5* **xs** *2.5 with two reinstatements payable at* 100%.

Notice that in Treaty 1, layer 2 is such that the upper limit of layer 1 is the attachment point of layer 2. Thus, we must apply layer 1 first, i.e., layer 1 has priority over layer 2. In Treaty 2, however, each layer has the same priority, 2.5, which is why I call this an exotic excess of loss treaty. A rule has to be given to assign a priority to each layer, i.e., to know which layer has to pay first, second, and third. The rule is

- Layer 1 inures to the benefit of layer 2; and
- Layers 1 and 2 inure to the benefit of layer 3.

This means that an excess claim must be paid by layer 3 unless layer 1 or layer 2 is able to pay for it, and layer 1 must pay before layer 2.

There are two ways to interpret the term inure:

(i) Each claim, from ground up (i.e., from the first dollar) is reduced by application of the lower layer; and

(ii) The part of each claim within the layer is reduced by application of the lower layer.

Let us analyze these interpretations with a numerical example. Assume the following claims hit Treaty 2: 20, 5, and 25; and that these claim amounts are from ground up. Under interpretation (i), the results are given in Table 1. Now let us change the order of claims hitting Treaty 2 to 5, 25, and 20. The results are given in Table 2. We observe that, though the sum of total payments within the layers has not changed (42.5), the distribution of these payments has changed, making it questionable whether interpretation (i) makes for good actuarial practice.

It is instructive to analyze what happens in case of a loss larger than the 25, i.e., larger than the largest limit (the one of layer 3). The results are given in Table 3. Observe that due to the reductions of the losses, a loss larger than 25 is still paid entirely (except the deductible) by the reinsurance. The total payments now becomes 52.5.

Next we analyze interpretation (ii) in Table 4. Now let us change the order of claims; the results are shown in Table 5. Notice that when the order of claims changes, the total payments within the layers remain the same. It is easier to analyze the treaty on an aggregate basis as is shown in Table 6. Finally, we analyze the case of a loss larger than the 25, i.e. larger than the largest limit (the one of layer 3), in Table 7. Here, even with larger claims, the sum of total payments remains the same.

Given the results and our observations, we use interpretation (ii) as our definition of inuring.

Table 1
Interpretation (i)

	20	5	25	Total
In layer 1	7.5	2.5	7.5	
Paid by layer 1	0*	0*	7.5	7.5
Reduced claims	20	5	17.5	
In layer 2	15	2.5	15	
Paid by layer 2	10**	2.5	15	27.5
Reduced claim	10	2.5	2.5	
In layer 3	7.5	0	0	
Paid by layer 3	7.5	0	0	7.5
Total payment				42.5

Notes: *Due to the annual aggregate deductible of layer 1; and **Due to the annual aggregate deductible of layer 2.

2 The General Mathematical Model

2.1 The Aggregate Loss Model

Our analysis will be conducted within the collective risk model. This model essentially states:

- the number of claims, N, is a random variable N;
- the claim amounts $X_1, X_2, \ldots, X_N$ are independent realizations of a random variable X; and
- X and N are independent.

Table 2
Interpretation (i)
With Changed Order of Claims

	5	25	20	Total
In layer 1	2.5	7.5	7.5	
Paid by layer 1	0	0	7.5	7.5
Reduced claims	5	25	12.5	
In layer 2	2.5	15	10	
Paid by layer 2	0	12.5	10	22.5
Reduced claim	5	12.5	2.5	
In layer 3	2.5	10	0	
Paid by layer 3	2.5	10	0	12.5
Total payment				42.5

Table 3
Interpretation (i)
With One Claim Larger than 25

	5	35	20	Total
In layer 1	2.5	7.5	7.5	
Paid by layer 1	0	0	7.5	7.5
Reduced claims	5	35	12.5	
In layer 2	2.5	15	10	
Paid by layer 2	0	12.5	10	22.5
Reduced claim	5	22.5	2.5	
In layer 3	2.5	20	0	
Paid by layer 3	2.5	20	0	22.5
Total payment				52.5

Table 4
Interpretation (ii)

	20	5	25	Total
In layer 1	7.5	2.5	7.5	17.5
Paid by layer 1	0	0	7.5	
In layer 2	15	2.5	15	32.5
Paid by layer 2	10	2.5	15	
In layer 3	17.5	2.5	22.5	42.5
Paid by layer 3	17.5	2.5	22.5	
Real payment by layer 1	0	0	7.5	7.5
Real payment by layer 2	10	2.5	7.5	20
Real payment by layer 3	7.5	0	7.5	15
Total payment				42.5

Notes: Real payment by layer 2=Paid by layer 2−Real payment by layer 1; and Real payment by layer 3=Paid by layer 3−Real payment by layer 2−Real payment by layer 1.

Table 5
Interpretation (ii)
With Changed Order of Claims

	5	25	20	Total
In layer 1	2.5	7.5	7.5	17.5
Paid by layer 1	0	0	7.5	
In layer 2	2.5	15	15	32.5
Paid by layer 2	0	12.5	15	
In layer 3	2.5	22.5	17.5	42.5
Paid by layer 3	2.5	22.5	17.5	
Real payment by layer 1	0	0	7.5	7.5
Real payment by layer 2	0	12.5	7.5	20
Real payment by layer 3	2.5	10	2.5	15
Total payment	2.5	22.5	17.5	42.5

Table 6
Interpretation (ii)
With Changed Order of Claims
Analyzed on an Aggregate Basis

	5	25	20	Total
In layer 1	2.5	7.5	7.5	17.5
In layer 2	2.5	15	15	32.5
In layer 3	2.5	22.5	17.5	42.5

Payments in layer 1: $17.5 - 10 = 7.5.$
Payments in layer 2: $32.5 - 7.5 - 5 = 20.$
Payments in layer 3: $42.5 - 20 - 7.5 = 15.$

Table 7
Interpretation (ii)
With One Claim Larger than 25

	20	35	5	Total
In layer 1	7.5	7.5	2.5	17.5
In layer 2	15	15	2.5	32.5
In layer 3	17.5	22.5	2.5	42.5

For more on collective risk models, see, for example, Klugman et al., (1998, Chapter 4).

In our model, we assume the number of claims, N, is a Poisson random variable with mean λ and that the claims follow a limited Pareto distribution with distribution function $F_X(x)$ given by

$$F_X(x) = \begin{cases} 0 & x \leq A, \\ (A^{-\alpha} - x^{-\alpha})/(A^{-\alpha} - B^{-\alpha}) & A < x \leq B, \\ 1 & x > B \end{cases}$$

where A, B, and α are non-negative constants. A limited Pareto distribution is used because it is known from the treaty that there are no losses above a certain threshold, and we are interested only in losses above a certain attachment point. Nevertheless, our approach can be used with any distribution.

A useful tool for determining probabilities in the collective risk model is Panjer's algorithm (Panjer, 1981). This algorithm requires that the distribution of X be of lattice type; therefore, the limited Pareto distribution is made discrete using, for example, the local moment matching method with one moment. [See, for example, Gerber (1982).] This method ensures that the sum of the masses is 1 and that the first moment of the distribution is conserved.

For a given span $h = (B - A)/m$, it is not difficult to show that the probabilities of the lattice version of X are given by $\tilde{f}_X$:

$$\begin{aligned} \tilde{f}_X(A) &= 1 - \frac{(A+h)^{1-\alpha} - A^{1-\alpha} - (1-\alpha)hB^{-\alpha}}{h(1-\alpha)(A^{-\alpha} - B^{-\alpha})}, \\ \tilde{f}_X(A+jh) &= \frac{2(A+jh)^{1-\alpha} - (A+(j-1)h)^{1-\alpha} - (A+(j+1)h)^{1-\alpha}}{h(1-\alpha)(A^{-\alpha} - B^{-\alpha})} \\ & \qquad j = 1, \ldots, m-1, \\ \tilde{f}_X(B) &= 1 - \tilde{f}_X(A) - \tilde{f}_X(A+h) - \cdots - \tilde{f}_X(B-h). \end{aligned}$$

2.2 An Exotic Excess of Loss Model

Let D be the common priority of all layers, L_j be the limit of layer j, AAD_j be the annual aggregate deductible of layer j, and AAL_j be the annual aggregate limit of layer j. In a classical excess of loss treaty, one would naturally define the part of the loss X_i hitting the various layers as: $R_i^{(1)}$ for the first layer, $R_i^{(2)}$ for the second layer, and $R_i^{(3)}$ for the third layer. We have

$$
\begin{aligned}
R_i^{(1)} &= \min(L_1, \max(0, X_i - D)),\\
R_i^{(2)} &= \min(L_2, \max(0, X_i - D)),\\
R_i^{(3)} &= \min(L_3, \max(0, X_i - D)).
\end{aligned}
$$

In our exotic excess of loss treaty the aggregate claims for each layer is given by

$$S_1 = \min(\mathrm{AAL}_1, \max(0, \sum_{i=1}^{N} R_i^{(1)} - \mathrm{AAD}_1)), \tag{5}$$

$$S_2 = \min(\mathrm{AAL}_2, \max(0, \sum_{i=1}^{N} R_i^{(2)} - S_1 - \mathrm{AAD}_2)), \tag{6}$$

$$S_3 = \min(\mathrm{AAL}_3, \max(0, \sum_{i=1}^{N} R_i^{(3)} - S_2 - S_1 - \mathrm{AAD}_3)). \tag{7}$$

The term $-S_1$ in equation (6) indicates that layer 1 inures to the benefit of layer 2. Similarly, the term $-S_2 - S_1$ in equation (7) indicates that layers 1 and 2 inure to the benefit of layer 3.

In order to price Treaty 2, we now need the distributions of S_1, S_2, and S_3. Now the distribution of S_1 is easy to obtain by applying Panjer's algorithm to the case where N is Poisson. The problem is more complicated, however, for S_2 and S_3. Indeed, S_1 and $R_1^{(2)} + \cdots + R_N^{(2)}$ are correlated, and S_1, S_2, and $R_1^{(3)} + \cdots + R_N^{(3)}$ are also correlated. Thus, the joint distribution of $\sum R_i^{(1)}$, $\sum R_i^{(2)}$, and $\sum R_i^{(3)}$ is needed. Fortunately, a multivariate analog of Panjer's algorithm exists; see Walhin and Paris (2000) or Sundt (1999).

Let

$$f^{(S)}(s_1, s_2, s_3) = \mathbb{P}[S_1 = s_1, S_2 = s_2, S_3 = s_3] \tag{8}$$

$$f^{(R)}(r_1, r_2, r_3) = \mathbb{P}[R^{(1)} = r_1, R^{(2)} = r_2, R^{(3)} = r_3] \tag{9}$$

$$f_0^{(R)} = \mathbb{P}[R^{(1)} = 0, R^{(2)} = 0, R^{(3)} = 0]$$

The multivariate analog of the Panjer's algorithm is as follows:

$$f^{(S)}(0,0,0) = \Psi_N(f_0^{(R)}) \tag{10}$$

$$f^{(S)}(s_1,s_2,s_3) = \frac{1}{(1-af_0^{(R)})} \sum_{x_1,x_2,x_3}^{s_1,s_2,s_3} \Big(f^{(S)}(s_1-x_1,s_2-x_2,s_3-x_3) \times f^{(R)}(x_1,x_2,x_3) \times [a+b\frac{x_1}{s_1}] \Big), \quad s_1 \geq 1 \tag{11}$$

$$f^{(S)}(s_1,s_2,s_3) = \frac{1}{(1-af_0^{(R)})} \sum_{x_1,x_2,x_3}^{s_1,s_2,s_3} \Big(f^{(S)}(s_1-x_1,s_2-x_2,s_n-x_n) \times f^{(R)}(x_1,x_2,x_3) \times [a+b\frac{x_2}{s_2}] \Big), \quad s_2 \geq 1 \tag{12}$$

$$f^{(S)}(s_1,s_2,s_3) = \frac{1}{(1-af_0^{(R)})} \sum_{x_1,x_2,x_3}^{s_1,s_2,s_3} \Big(f^{(S)}(s_1-x_1,s_2-x_2,s_n-x_n) \times f^{(R)}(x_1,x_2,x_3) \times [a+b\frac{x_3}{s_3}] \Big), \quad s_3 \geq 1 \tag{13}$$

where $\Psi_N(u) = \mathbb{E}[u^N]$ denotes the probability generating function of N, the probabilities $p_n = \mathbb{P}[N = n]$ satisfy

$$p_n = (a + \frac{b}{n})p_{n-1}$$

for $n = 1, 2, \ldots$, and

$$\sum_{x_1,x_2,x_3}^{s_1,s_2,s_3} g(x_1,x_2,x_3) = \sum_{x_1=0}^{\min(s_1,m_1)} \sum_{x_2=0}^{\min(s_2,m_2)} \sum_{x_3=0}^{\min(s_3,m_3)} g(x_1,x_2,x_3) - g(0,0,0),$$

where

$$m_1 = \max(x|f^{(R)}(x,x_2,x_3) > 0),$$
$$m_2 = \max(x|f^{(R)}(x_1,x,x_3) > 0),$$

and

$$m_3 = \max(x|f^{(R)}(x_1,x_2,x) > 0).$$

Though the execution of this algorithm can be time-consuming, we can take advantage of a particular feature of the vector $(R^{(1)}, R^{(2)}, R^{(3)})$:

$$R_i^{(1)} \leq R_i^{(2)} \leq R_i^{(3)} \quad \text{for } i = 1, 2, \ldots.$$

This implies that the sums in the algorithm may be rewritten as:

$$\sum_{x_1,x_2,x_3}^{s_1,s_2,s_3} g(x_1,x_2,x_3) = \sum_{x_3=0}^{\min(s_3,m_3)} \sum_{x_2=0}^{\min(s_2,m_2,x_3)} \sum_{x_1=0}^{\min(s_1,m_1,x_2)} g(x_1,x_2,x_3) - g_0$$

where $g_0 = g(0,0,0)$. As a corollary, we have $S_1 \le S_2 \le S_3$. Therefore, the algorithm needs only to be evaluated for values of $(S_1 = s_1, S_2 = s_2, S_3 = s_3)$ such that $s_1 \le s_2 \le s_3$. Moreover, only a few of the values of the random vector $(R^{(1)}, R^{(2)}, R^{(3)})$ have a positive probability. It may be more efficient to rewrite the algorithm as:

$$f^{(S)}(0,0,0) = \Psi_N(f^{(R)}(0,0,0))$$

$$f^{(S)}(s_1,s_2,s_3) = \frac{1}{(1 - af^{(R)}(0,0,0))} \sum_{j=0}^{t} [a + b\frac{\mathbf{Z}(j,1)}{s_1}] \times f_{S_1,S_2,S_3}(s_1 - \mathbf{Z}(j,1), s_2 - \mathbf{Z}(j,2), s_3 - \mathbf{Z}(j,3))\mathbf{Z}(j,4), \quad s_1 \ge 1$$

$$f^{(S)}(s_1,s_2,s_3) = \frac{1}{(1 - af^{(R)}(0,0,0))} \sum_{j=0}^{t} [a + b\frac{\mathbf{Z}(j,2)}{s_2}] \times f_{S_1,S_2,S_3}(s_1 - \mathbf{Z}(j,1), s_2 - \mathbf{Z}(j,2), s_3 - \mathbf{Z}(j,3))\mathbf{Z}(j,4), \quad s_2 \ge 1$$

$$f^{(S)}(s_1,s_2,s_3) = \frac{1}{(1 - af^{(R)}(0,0,0))} \sum_{j=0}^{t} [a + b\frac{\mathbf{Z}(j,3)}{s_3}] \times f_{S_1,S_2,S_3}(s_1 - \mathbf{Z}(j,1), s_2 - \mathbf{Z}(j,2), s_3 - \mathbf{Z}(j,3))\mathbf{Z}(j,4), \quad s_3 \ge 1$$

where $\mathbf{Z}$ denotes a matrix with t rows and four columns. The number of rows in $\mathbf{Z}$ represents the number of points of $(R^{(1)}, R^{(2)}, R^{(3)})$ with positive mass [(excluding the possible point $(0,0,0)$]. Column j represents the value of the R^j for $j = 1,2,3$. The fourth column gives the probability associated with the realization $(R^{(1)} = \mathbf{Z}(j,1), R^{(2)} = \mathbf{Z}(j,2), R^{(3)} = \mathbf{Z}(j,3))$.

3 Numerical Results for Treaty 2

Let us develop actual prices for the exotic treaty given in Treaty 2. Two cases are considered: free reinstatements and paid reinstatements. In both cases we use a span $h = 2.5$ to discretize the distribution.

The following parameters values are used in this section.

$$
\begin{array}{llll}
A = 2.5, & B = 25, & \alpha = 0.85, & \lambda = 10.61 \\
D = 2.5, & L_1 = 7.5, & L_2 = 15, & L_3 = 22.5 \\
\mathrm{AAD}_1 = 10, & \mathrm{AAD}_2 = 5, & \mathrm{AAD}_3 = 0, &
\end{array}
$$

$\mathrm{AAL}_1 = 4 \times 7.5 = 30$, $\mathrm{AAL}_2 = 4 \times 15 = 60$, and $\mathrm{AAL}_3 = 3 \times 22.5 = 67.5$.

3.1 Free Reinstatements

First we find the pure premiums for the three layers using equations (5), (6), and (7):

$$\mathbb{E}[S_1] = 21.20, \quad \mathbb{E}[S_2] = 17.31, \quad \mathbb{E}[S_3] = 7.18.$$

Summing these premiums we obtain 45.68, which is the premium for an unlimited cover:

$$\mathbb{E}[N] \times \mathbb{E}[\max(0, X - D)] = 10.61 \times 4.3056 = 45.68.$$

This shows that, using the data given, this arrangement is about the same as an unlimited cover. The total liability offered by the reinsurance is $30 + 60 + 67.5 = 157.5$.

We can simplify the structure (for pricing purposes) and assume, for example, the following cover (which offers essentially the same capacity):

- 7.5 **xs** 2.5 with 9 reinstatements
- 7.5 **xs** 10 with 7 reinstatements
- 7.5 **xs** 17.5 with 5 reinstatements.

This cover offers the same aggregate liability: $67.5 + 52.5 + 37.5 = 157.5$, but the prices obtained by the classical univariate Panjer algorithm are

$$\mathbb{E}[S_1] = 34.47, \quad \mathbb{E}[S_2] = 9.06, \quad \mathbb{E}[S_3] = 2.06,$$

for a total of 45.59, which is a little smaller than the premium for an unlimited cover (i.e., for cover without an annual aggregate limit). This suggests that this alternative has a cover that is a little smaller than the one given by the exotic excess of loss treaty.

If we change the covers to

- 7.5 **xs** 2.5 with 12 reinstatements;
- 7.5 **xs** 10 with 6 reinstatements; and
- 7.5 **xs** 17.5 with 3 reinstatements,

we obtain the following prices:

$$\mathbb{E}[S_1] = 34.50, \quad \mathbb{E}[S_2] = 9.11, \quad \mathbb{E}[S_3] = 2.06.$$

The sum of these prices is 45.67, which is almost an unlimited cover. Thus, it is not difficult to offer an almost unlimited cover to the insurer without the exotic cover.

If these premiums are then applied to the exotic excess of loss treaty, we obtain the following expected gains for the reinsurer:

$$\mathbb{E}[G_1] = 13.3, \quad \mathbb{E}[G_2] = -8.2, \quad \mathbb{E}[G_3] = -5.12,$$

i.e., a total loss of 0.02. This shows that for free reinstatements, the reinsurer who participates on all the layers (with the same share) can almost replicate any treaty. We will see in the next section that this situation dramatically changes when there are paid reinstatements.

3.2 Paid Reinstatements

Recall that the original exotic excess of loss treaty (Treaty 2) is with paid reinstatements. Using the joint distribution of S_1, S_2, and S_3 given in equations (5), (6) and (7), then from Sundt (1991) it is easy to calculate the pure premium, P_i:

$$P_i = \frac{\mathbb{E}[S_i]}{1 + \sum_{j=1}^{k} \frac{c_j}{L_i} \mathbb{E}[\max(0, S_i - (j-1)L_i)]} \tag{14}$$

where k is the number of reinstatemnts and c_j is the price of the j^{th} reinstatement. With all reinstatements at 100% we obtain the following prices:

$$P_1 = 6.22, \quad P_2 = 8.15, \quad P_3 = 5.44.$$

The premiums obtained within the classical excess of loss treaty

- 7.5 **xs** 2.5 with 12 reinstatements;
- 7.5 **xs** 10 with 6 reinstatements; and
- 7.5 **xs** 17.5 with 3 reinstatements

are

$$P_1 = 6.16, \quad P_2 = 4.11, \quad P_3 = 1.61.$$

If these premiums are then applied to the exotic excess of loss treaty, we obtain the following expected gains for the reinsurer:

$$\mathbb{E}[G_1] = -0.22, \quad \mathbb{E}[G_2] = -8.61, \quad \mathbb{E}[G_3] = -5.06,$$

i.e., a total loss of 13.89. This loss shows the importance of using the correct model to price each layer.

3.3 Annual Aggregate Deductibles

Finally we analyze the effect of the annual aggregate deductibles within the exotic excess of loss treaty. First, let us assume that there are no annual aggregate deductibles. Table 8 shows the premiums already derived for various levels of annual aggregate deductibles (AAD).

Table 8
Premiums for Various Layers
With and Without Reinstatements

	Free Reinstatements	Reinstatements @ 100%
$AAD_1 = AAD_2 = AAD_3 = 0$		
L1	26.49	6.91
L2	16.92	8.11
L3	2.27	2.06
$AAD_1 = 10$, $AAD_2 = 5$, $AAD_3 = 0$		
L1	21.13	6.22
L2	17.37	8.15
L3	7.18	5.44

Table 9 shows the premiums with $AAD_1 = 20$ and $AAD_2 = 10$. The effect of the annual aggregate deductibles on the first two layers is nil.

Indeed, the sum of the premiums with free reinstatements is constant (45.68). The only effect of these annual aggregate deductibles is to distribute the claims differently between the layers. With very large annual aggregate deductibles, $AAD_1 = 60$, $AAD_2 = 90$, and $AAD_3 = 0$, there is an effect with respect to the total liability of the reinsurer. So far the effect of the annual aggregate deductibles is due to the annual aggregate limit of the third layer. If this layer has unlimited reinstatements, then we would not observe the effect of large deductibles on layers 1 and 2. The final part of Table 9 shows the case with an annual aggregate deductible on layer 3. In this case the sum of the premiums with free reinstatements is 38.67, showing the effect of the annual aggregate deductible on the third layer. Clearly only an annual aggregate deductible on the third layer has the effect of a classical annual aggregate deductible.

Table 9
Premiums for Various Layers
With and Without Reinstatements

	Reinstatements	
	Free	@ 100%
$AAD_1 = 20$, $AAD_2 = 10$, $AAD_3 = 0$		
L1	14.12	5.26
L2	19.50	8.54
L3	12.07	7.86
$AAD_1 = 60$, $AAD_2 = 90$, $AAD_3 = 0$		
L1	0.35	0.33
L2	0.04	0.04
L3	43.41	16.47
$AAD_1 = 10$, $AAD_2 = 5$, $AAD_3 = 15$		
L1	21.13	6.22
L2	17.37	8.15
L3	0.17	0.17

4 Closing Comments

We have shown how to price an exotic excess of loss treaty using a multivariate analog of Panjer's algorithm. The exotic structure presents mathematical difficulties that can be avoided by using a certain definition of inuring that ensures the order of the claims has no effect on the pricing. Numerical examples show that it is important to correctly identify the actuarial model in order to obtain accurate pricing. We also show that other classical reinsurance structures may also provide a similar level of cover.

Our calculations were based on a step size of $h = 2.5$, which may appear too large. A short sensitivity analysis, however, shows that halving the step size to $h = 1.25$ does not significantly affect the premiums for the original reinsurance program.

References

Gerber, H.U., "On the Numerical Evaluation of the Distribution of Aggregate Claims and its Stop-Loss Premiums." *Insurance: Mathematics and Economics* 1 (1982): 13–18.

Klugman, S.A., Panjer, H.H., and Willmot, G.E. *Loss Models. From Data to Decisions.* New York, New York: Wiley, 1998.

Panjer, H.H. "Recursive Evaluation of a Family of Compound Distributions." *Astin Bulletin* 12 (1981): 22–26.

Sundt, B. "On Excess of Loss Reinsurance with Reinstatement." *Bulletin of Swiss Actuaries* (1991): 1–15.

Sundt, B. "On Multivariate Panjer's Recursions." *Astin Bulletin* 29 (1999): 29–46.

Walhin, J.F. and Paris, J. "Excess-of-Loss Reinsurance with Reinstatements and Ruin of the Cedent." *Blatter Deutsche Gesellschaft für Versicherungsmathematik* 24 (2000): 615–627.