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Results on edge-colored graphs and pancyclicity

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RESULTS ON EDGE-COLORED GRAPHS AND PANCYCLICITY

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RESULTS ON EDGE-COLORED GRAPHS AND PANCYCLICITY

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This thesis focuses on determining when a graph with additional structure contains certain subgraphs, particularly circuits, cycles, or trees. The specific problems and presented results include a blend of many fundamental graph theory concepts such as edge-coloring, routing problems, decomposition problems, and containing cycles of various lengths. The three primary chapters in this thesis address the problems of finding eulerian circuits with additional restrictions, decomposing the edge-colored complete graph K_n into rainbow spanning trees, and showing a 4-connected claw-free and $N(3,2,1)$-free graph is pancyclic.

Let G be an eulerian digraph with a fixed edge coloring (incident edges may have the same color). A *compatible circuit* of G is an eulerian circuit such that every two consecutive edges in the circuit have different colors. We characterize the existence of a compatible circuit for digraphs avoiding certain vertices of outdegree three. For certain families of digraphs where all the vertices are of outdegree three we also have a characterization for when there is a compatible circuit. From our characterizations we develop a polynomial time algorithm that determines the existence of a compatible circuit in an edge-colored eulerian digraph and produces a compatible circuit if one exists.

A *rainbow spanning tree* T is a spanning tree of an edge-colored graph where all the edges of T have different colors. Brualdi and Hollingsworth conjectured that every properly edge-colored K_n ($n \geq 6$ and even) using exactly $n-1$ colors has $n/2$ edge-

disjoint rainbow spanning trees, and they proved there are at least two edge-disjoint rainbow spanning trees. Kaneko, Kano, and Suzuki strengthened the conjecture to include any proper edge coloring of K_n, and they proved there are at least three edge-disjoint rainbow spanning trees.

We prove that if $n \geq 1{,}000{,}000$ then an edge-colored K_n, where each color appears on at most $n/2$ edges, contains at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees.

The final result focuses on showing a 4-connected, claw-free, and $N(3, 2, 1)$-free graph is pancyclic. A graph G is *pancyclic* if it contains cycles of all lengths from 3 to $|V(G)|$. There has been interest in determining which pairs of forbidden subgraphs imply a 4-connected graph is pancyclic. In the last chapter we present a result that helps complete the classification of which 4-connected, claw-free, and N-free graphs are pancyclic.

DEDICATION

To my parents.

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Chapter 1

Introduction

This thesis focuses on determining when a graph with additional structure contains certain subgraphs, particularly circuits, cycles, or trees. The specific problems and results presented include a blend of many fundamental graph theory concepts such as edge-coloring, routing problems, decomposition problems, and containing cycles of various lengths. The three primary chapters in this thesis address the problems of finding eulerian circuits with additional restrictions, decomposing the edge-colored complete graph K_n into rainbow spanning trees, and showing a 4-connected claw-free and $N(3, 2, 1)$-free graph is pancyclic.

In Chapter 3 we investigate certain routings in eulerian digraphs. An *eulerian digraph* is a directed graph (edges have directions like one-way streets) that has a walk that visits each edge exactly once and starts and ends at the same vertex. Such a walk is called an *eulerian circuit*. Eulerian circuits can be used to solve routing problems for mail delivery, garbage collection, and other routes where each edge needs to be visited exactly once.

Often times these routes may have undesirable turns that should be avoided. For example, U-turns can be difficult for delivery trucks and may not be desirable for a mail carrier [46]. UPS uses routes that avoid U-turns and left turns to reduce the time of deliveries and number of accidents, saving millions of dollars [48]. While eulerian circuits are often used in routing problems, eulerian circuits do not restrict U-turns from occurring and in fact they frequently occur. This naturally leads us to the question of when an eulerian digraph has an eulerian circuit that avoids certain types of turns.

To answer this question we introduce several new definitions. A *colored eulerian digraph* is an eulerian digraph where each edge is assigned a color (incident edges may have the same color). A *compatible circuit* is an eulerian circuit in a colored eulerian digraph such that no two consecutive edges in the circuit have the same color. We can use colored eulerian digraphs to find eulerian circuits that avoid certain types of turns.

A compatible circuit for an edge-colored eulerian *undirected* graph is defined similarly. Kotzig [39] gave simple necessary and sufficient conditions for when an edge-colored undirected graph has a compatible circuit by considering the size of the largest color class incident to a vertex. Finding a compatible circuit in a colored eulerian undirected graph has received considerable attention, since Pevzner [51] proved that this problem can be used to solve the small-scale DNA physical mapping problem known as the Double Digest Problem.

Benkouar et al. [6] provided a polynomial time algorithm for finding a compatible circuit in colored eulerian undirected graphs. They claimed that a similar algorithm also works for eulerian digraphs. However, their sufficient condition is false. Our work shows that determining if a colored eulerian digraph has a compatible circuit is more complicated and subtle than the undirected case.

Isaak [34] used the digraph version to find universal cycles of permutations, which are generalizations of De Bruijn sequences. Isaak [34] and Fleischner and Fulmek [26] gave sufficient conditions for when a colored eulerian digraph contains a compatible circuit.

We give a characterization of when a colored eulerian digraph G has a compatible circuit when G has no vertices of outdegree three. The proof ideas come from constructing an auxiliary graph H from G, which is an edge-colored undirected graph that models certain barriers in G. Finding a rainbow spanning tree in H turns out to be equivalent to finding a compatible circuit in G. A *rainbow spanning tree* is a spanning tree in an edge-colored graph where each edge has a different color.

This reduces the problem of finding compatible circuits to finding a rainbow spanning tree in an edge-colored graph. Broersma and Li [10] provided a characterization of when an edge-colored graph contains a rainbow spanning tree using the Matroid Intersection Theorem. In Chapter 2.1.1 we provide a graph-theoretical proof of the same result. From these results, we show that if G is a colored eulerian digraph with no vertices of outdegree three, then there is a polynomial time algorithm to determine if G has a compatible circuit and gives a compatible circuit if one exists.

We also investigate digraphs where all the vertices have outdegree three, and in certain cases we can characterize when a compatible circuit exists. Finally, we consider the more general setting where instead of colors each edge of G has a list of acceptable following turns. A compatible circuit is an eulerian circuit of G such that all the turns are acceptable. We show that determining if G has a compatible circuit in this more general setting is NP-complete.

In Chapter 4 we investigate finding edge-disjoint rainbow spanning trees. Brualdi and Hollingsworth [11] studied the number of edge-disjoint rainbow spanning trees in a properly edge-colored complete graph K_n with exactly $n-1$ colors. Brualdi and Hollingsworth [11] and Kaneko, Kano, and Suzuki [37] conjecture that every properly edge-colored K_n ($n \geq 5$) should have $\lfloor n/2 \rfloor$ edge-disjoint rainbow spanning trees.

Notice that this conjecture states that when n is even the edge set of K_n decomposes into $n/2$ edge-disjoint rainbow spanning trees. The best previous result towards the conjecture was by Kaneko, Kano and Suzuki [37] who showed that every properly edge-colored K_n ($n \geq 6$) contains at least three edge-disjoint rainbow spanning trees.

In joint work with Stephen Hartke and Paul Horn we show that the number of rainbow spanning trees in K_n is close to the conjectured linear bound.

Theorem 1.1 (Carraher, Hartke, Horn)**.** *Let K_n be a properly edge-colored complete graph, with $n \geq 1{,}000{,}000$. There are at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees in K_n.*

Our proof technique is to randomly construct $t := \lfloor n/(1000 \log n) \rfloor$ edge-disjoint subgraphs of G and show that all the subgraphs have a rainbow spanning tree with positive probability. Each random subgraph we consider is distributed as an Erdős-Rényi random graph $G(n, 1/t)$. To prove each subgraph has a rainbow spanning tree, we use the result by Broersma and Li [10], carefully analyze the structure of the random graphs, and apply Bernstein's inequality and convexity arguments. Then we show using the union sum bound that the probability of all t subgraphs simultaneously having rainbow spanning trees has nonzero probability.

In Chapter 5 we study pancyclicity, which is a generalization of hamiltonicity. A *hamiltonian cycle* in a graph G is a cycle through all the vertices of G. Determining if a graph G has a hamiltonian cycle is a classic topic studied in graph theory and is a NP-complete problem. Hence there is much interest in finding sufficient conditions for when a graph has a hamiltonian cycle, such as the well known degree conditions by Dirac [15] and Ore [50]. Another approach is to forbid certain induced subgraphs to force a hamiltonian cycle. A graph G is $\mathcal{F}$-free if G contains no member of the family of graphs $\mathcal{F}$ as an induced subgraph. A claw-free graph is a graph that is $K_{1,3}$-free.

Another graph that appears in this chapter is the *generalized net* $N(i, j, k)$, which is obtained by identifying an endpoint of each of the paths P_{i+1}, P_{j+1} and P_{k+1} with distinct vertices of a triangle. A well known conjecture by Matthews and Sumner [45] has motivated much research into hamiltonicity of claw-free graphs.

Conjecture 1.2 (Matthews and Sumner [45]). *If G is a 4-connected claw-free graph, then G is hamiltonian.*

The Matthews-Sumner Conjecture has inspired research into properties that generalize hamiltonicity in 4-connected claw-free graphs. We consider whether such a graph G is *pancyclic*, meaning that G contains cycles of length s, for $3 \leq s \leq |V(G)|$.

Gould, Łuczak and Pfender [30] provided a characterization of which pair of subgraphs in $\mathcal{F}$ must be forbidden in a 3-connected graph to force pancyclicity. Gould [29] posed the problem of characterizing which pairs of forbidden subgraphs imply that a 4-connected graph is pancyclic. Initial results include Ferrara, Morris and Wenger [23] and Ferrara et al. [22], who showed that if G is 4-connected, claw-free, and avoids either P_9 or a family of graphs known as bulls, then G is pancyclic.

In joint work with Michael Ferrara, Tim Morris, and Michael Santana we provide a partial answer to the problem of characterizing which pairs of forbidden subgraphs imply a 4-connected graph is pancyclic. Although all authors significantly contributed to our main result Theorem 1.3, I was the primary author for the $N(3, 2, 1)$ case, which is the content in Chapter 5. We show in the pair $\mathcal{F} = \{X, Y\}$ of forbidden subgraphs, that X is either the claw or $K_{1,4}$. In the case when X is the claw, our full results provide a complete characterization.

Theorem 1.3 (Carraher, Ferrara, Morris, Santana). *Let Y be a connected graph with at least three edges. Every 4-connected $\{K_{1,3}, Y\}$-free graph is pancyclic if and only* *if Y is an induced subgraph of P_9, the Łuczak graph, or the generalized net $N(i,j,k)$ with $i+j+k=6$.*

The papers [22, 23, 30] handle all the cases except when Y is $N(2,2,2)$, $N(3,2,1)$, or $N(4,1,1)$. In Chapter 5 we present the case $Y = N(3,2,1)$. The proofs for the other two nets are similar. We directly show that every 4-connected $\{K_{1,3}, N(3,2,1)\}$-free graph has 3-, 4-, and 5-cycles, and inductively show that if G has an s-cycle $(5 \leq s < n)$ then G has an $(s+1)$-cycle. Thus a 4-connected $\{K_{1,3}, N(3,2,1)\}$-free graph must be pancyclic.

Chapter 2

Background

2.1 Graph theory

A *graph* G is composed of a vertex set $V(G)$, and an edge set $E(G)$ where an edge is an unordered pair of vertices. If we allow $E(G)$ to be a multiset, then we call G a *multigraph*. If we allow vv to be an edge, then G has a *loop*. A *simple graph* is a graph with no multiple edges and no loops. Two vertices u and v are *adjacent* if there is an edge uv, and the vertices u and v are *incident* to the edge uv. A vertex u is a *neighbor* of v if uv is an edge. The *neighborhood of* v $N(v)$ is the set of neighbors of v. The *degree* of a vertex v, denoted $\deg(v)$, is the number of incident edges to v, where loops are counted twice. A vertex of degree 0 is called an *isolated vertex*.

Let $n(G) = |V(G)|$, which is called the *order* of G. Often when G is understood, we refer to n as the number of vertices of G. The complete graph K_n is a simple graph on n vertices with every possible edge.

A *subgraph* H of G is a graph where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* is a subgraph H of G, where if $u, v \in V(H)$ and $uv \in E(G)$, then $uv \in E(H)$. A graph G is called H-free if G does not have H as an induced subgraph.

A *walk* in a graph is a sequence of vertices and incident edges. A *path* is a walk that does not repeat any vertex. A *cycle* is a closed walk that does not repeat any vertex and we do not specify the first vertex but keep the cyclic order of the vertices. A *circuit* is a closed walk that does not repeat any edges and again we do not specify the first edge but keep the cyclic order of the edges. A graph is *connected* if there is a walk between every pair of vertices. The *components* of a graph are the maximal connected subgraphs. A graph is *k-connected* if between every pair of vertices u and v there are k vertex-disjoint paths between u and v. A subgraph T of G is called a *spanning tree* when $V(T) = V(G)$, $|E(T)| = n - 1$, and T is connected.

An *eulerian circuit* is a walk in a graph G that starts and ends at the same vertex and crosses each edge exactly once. We refer to graphs with an eulerian circuit as an eulerian graph or just as eulerian. The theorem below provides a characterization for when a graph has an eulerian circuit.

Theorem 2.1. *A graph G with no isolated vertices is eulerian if and only if G is connected and the degree of every vertex is even.*

A *digraph* (also called a *directed graph*) is composed of a vertex set $V(G)$, and an edge set $E(G)$ where an edge is an ordered pair of vertices, i.e. now the edges have directions like one-way streets. The *tail* of an edge is the starting vertex, and the *head* is the ending vertex. The indegree of a vertex $\deg^-(v)$ is the number of edges whose head is v, and the outdegree of a vertex $\deg^+(v)$ is the number of edges whose tail is v. A *walk* in a digraph G is a sequence of vertices and incident edges, where each vertex is the tail of the following edge. A *circuit* in a digraph G is a closed walk that never visits any edge more than once. A digraph G is *strongly connected* if there is a directed walk between every pair of vertices in $V(G)$. The theorem below provides a characterization for when a digraph has an eulerian circuit.

Theorem 2.2. *A digraph G with no isolated vertices is eulerian if and only if G is strongly connected and* $\deg^-(v) = \deg^+(v)$ *for every vertex* $v \in V(G)$.

An *edge coloring* of a graph (or digraph) is a function $f : E(G) \to \{1, 2, \ldots, k\}$ that assigns each edge one color from some set of colors $\{1, 2, \ldots, k\}$. A *color class* is the set of edges with the same color.

For more information about basic graph theory concepts and terminology see West [65]. In Chapter 4 we use probabilistic arguments. For an introduction to the application of probability to combinatorics and graph theory see Alon and Spencer [5].

2.1.1 Rainbow spanning trees

Several of the problems we consider have connections to rainbow spanning trees. In this section we give the definition of a rainbow spanning tree and provide a characterization of when an edge-colored graph has a rainbow spanning tree.

Definition 2.3. Let H be an undirected multigraph with a fixed edge coloring (incident edges may have the same color). A *rainbow spanning tree* is a spanning tree of H that has at most one edge from each color class.

Broersma and Li [10] showed that determining the largest rainbow spanning forest of H can be solved by applying the Matroid Intersection Theorem [18] (see Schrijver [56, p. 700]) to the graphic matroid and the partition matroid on the edge set of H defined by the color classes. Schrijver [56] translated the conditions of the Matroid Intersection Theorem into necessary and sufficient conditions for the existence of a rainbow spanning tree, stated below in Theorem 2.4. Suzuki [59] gave a graph-theoretical proof of the same theorem.

Theorem 2.4. *A graph H has a rainbow spanning tree if and only if, for every partition π of $V(H)$ into s parts, there are at least $s-1$ different colors represented between the parts of π.*

Theorem 2.4 is similar to a result by Tutte [61] and Nash-Williams [49] on finding k edge-disjoint spanning trees. For completeness we provide a graph-theoretical proof of Theorem 2.4 that is different from Suzuki's proof.

Proof. For any partition π of $V(H)$ into s parts, a spanning tree must have at least $s-1$ edges between the partitions, since a tree is connected. Thus, there must be at least $s-1$ distinct colored edges between the parts of π in a rainbow spanning tree, and hence in H as well.

To prove sufficiency, assume that for every partition π of $V(H)$ into s parts the number of color classes between parts is at least $s-1$. Let $1,\ldots,k$ be the k color classes. For any subset S of the edges of H, let $\chi(S,i)$ denote the number of edges in S with the color i, and let $\sigma(S)=\sum_{i:\chi(S,i)>0}(\chi(S,i)-1)$. When viewing a spanning tree T as a set of edges, then $\sigma(T)\geq 0$ for all spanning trees, with equality if and only if T is a rainbow spanning tree.

Let T be a spanning tree of H that minimizes the value $\sigma(T)$ over all spanning trees. We want to show that $\sigma(T)=0$. Assume that $\sigma(T)>0$. Then there exist at least two edges in the tree T with the same color c. Label all the edges of T with color c with a_1. We inductively extend the edge labeling of T in the following way: suppose the labels $a_1,\ldots,a_{i-1}$ have been assigned. An unlabeled edge e of T is labeled a_i if there exists an edge e' in T labeled a_{i-1} and an edge f of H that has the same color as e such that $T+f-e'$ is a spanning tree of H. The process terminates when no such unlabeled edge exists. Note that the process may terminate leaving some edges of T unlabeled.

If there is an edge e_i with label a_i in T and an edge f in H where $T+f-e_i$ is a tree, where there are no edges of T with the same color as f, then we can create another spanning tree with smaller σ value than T. Add f to the tree T and delete the edge e_i to form the tree $T_i = T+f-e_i$. Since e_i was labeled with a_i there exists an edge e_{i-1} of T with label a_{i-1} and an edge f_{i-1} that has the same color as e_i. Add f_{i-1} and delete e_{i-1} to form the tree T_{i-1}. Continuing this process, we obtain the tree T_1, where $T_1 = T+f-e_i+f_{i-1}-e_{i-1}+\cdots-e_1 = T+f+(-e_i+f_{i-1})+(-e_{i-1}+f_{i-2})+\cdots+(-e_2-f_1)-e_1$. Note that e_i and f_{i-1} have the same color for $i \geq 2$ and adding f and removing e_1 gives us that $\sigma(T_1) = \sigma(T) - 1 < \sigma(T)$. By the extremal choice of T such an edge f can not exist.

Therefore, for every edge e_i with label a_i all the edges f in H which lie between the components of $T - e_i$ have the same color as some edge of T. Let R be the set of edges in T that receive a label. The components of $T - R$ create a partition π of the vertices. We claim this partition contradicts the hypothesis. There are $n - 1 - |R|$ unlabeled edges in T, so $s - 1 = n - 1 - (n - 1 - |R|) = |R|$. Each edge f that lies between two parts of π also lies between the components of $T - e$ for some labeled edge e, hence by the minimality of T edge f has the same color as an edge in R. The number of different colors in R is strictly less than $|R|$, since at least two of the edges are colored c (i.e. labeled a_1). Thus, for this partition the number of colors between the parts is smaller than $|R| = s - 1$, contradicting the hypothesis. $\square$

There are many known polynomial algorithms for finding maximum weight common independent sets (see Schrijver [56, p. 705-707]). Most notable are results by Edmonds [18] and Lawler [40]. Therefore there is a polynomial time algorithm for finding a rainbow spanning tree in an edge-colored graph H by finding a common independent set in the graphic and partition matroid of size $n - 1$. The proof of

Theorem 2.4 also gives a polynomial time algorithm for finding a rainbow spanning tree of H or finding a partition π that demonstrates no rainbow spanning tree exists.

Chapter 3

Compatible circuits

3.1 Introduction

A *colored eulerian digraph* is an eulerian digraph G with a fixed edge coloring ϕ (incident edges may have the same color). A *compatible circuit* of G is an eulerian circuit such that every two consecutive edges in the circuit have different colors. We prove necessary and sufficient conditions for the existence of a compatible circuit in colored eulerian digraphs that do not have certain vertices of outdegree three. The methods that we use give a polynomial time algorithm determining the existence of a compatible circuit and producing one if it exists. We investigate graphs where all the vertices are of outdegree three, and in certain cases we can characterize when there is a compatible circuit. Finally we show the problem of determining if an eulerian digraph where each edge is given some fixed list of acceptable transitions has an eulerian circuit that has all acceptable transitions is NP-complete.

Fleischner and Fulmek [26] provided sufficient conditions for the existence of a compatible circuit when the number of colors at each vertex is large. Isaak [34] gave stronger conditions for the existence of compatible circuits in digraphs and used these results to show the existence of certain universal cycles of permutations. In Section 3.3 we expand upon Isaak's methods to determine when we can make local changes at a vertex to construct a compatible circuit. See Fleischner [25] for an overview of topics on eulerian digraphs, including compatible circuits in colored eulerian digraphs.

Kotzig [39] gave necessary and sufficient conditions for the existence of a compatible circuit in colored eulerian undirected graphs[1]. Our result for digraphs is analogous to Kotzig's result. In an important application, Pevzner [51] used compatible circuits in undirected eulerian graphs to reconstruct DNA from its segments. Benkouar et al. [6] gave a polynomial time algorithm for finding a compatible circuit in a colored eulerian undirected graph, providing an alternate proof of Kotzig's Theorem. They claimed that a similar algorithm holds for digraphs and gave a statement characterizing the existence of a compatible circuit in a colored eulerian digraph. However, their sufficient condition is false, as shown by the graph on the right in Figure 3.1 below.

There are many other results on finding subgraphs avoiding monochromatic transitions. Bollobás and Erdős [8] initiated the study of properly edge-colored hamiltonian cycles (which they called alternating hamiltonian cycles) in edge-colored complete graphs, and this study was continued in papers such as [3, 4]. The problem of finding properly edge-colored paths and circuits in edge-colored digraphs has been studied in several articles such as Gourvès et al. [31]; see the survey paper by Gutin and Kim [32] for an overview. Finding subgraphs with all edges having different colors (called rainbow or heterochromatic) has also been well studied. Kano and Li [38] gave a survey paper on recent results about monochromatic and rainbow subgraphs in edge-colored graphs. In our methods for eulerian digraphs we use results on rainbow spanning trees in edge-colored undirected multigraphs.

Fleischner and Jackson [24] studied a related, but different, notion of compatible

[1] This answered a question Kotzig claimed was posed by Nash-Williams (see [26, p. VI.1]).

circuits in eulerian digraphs and graphs. They showed that an eulerian digraph of minimum degree $2k$ has a set S of $k/2 - 1$ eulerian tours such that each pair of adjacent edges of D is consecutive in at most one tour of S. Jackson [35] studied a variation of the problem for graphs by considering restricting certain transitions and found sufficient conditions for the existence of a compatible circuit.

This chapter is organized as follows. In Section 3.2 we introduce definitions and elementary necessary conditions for the existence of a compatible circuit. Section 3.3 investigates when local changes at each vertex can be used to construct a compatible circuit. Section 3.4 gives a characterization of when a colored eulerian digraph avoiding certain vertices of outdegree three has a compatible circuit. This characterization leads to a polynomial time algorithm for finding a compatible circuit. Also in this section, we determine the fewest number of monochromatic transitions required in any eulerian circuit of a colored eulerian digraph that avoids certain vertices of outdegree three. Section 3.5 investigates graphs with only vertices of outdegree three and characterizes when certain colorings of these digraphs have a compatible circuit. In Section 3.6 we investigate eulerian digraphs where each edge has some list of allowed transitions and when there exists an eulerian circuit that has all allowed transitions. Determining if there is an eulerian circuit that has all allowed transitions is a generalization of finding a compatible circuit in a colored eulerian digraph. We show the problem of determining if an eulerian digraph has a compatible circuit when each edge is given a list of allowed transitions is NP-complete. Finally we conclude the chapter with some open questions in Section 3.7.

3.2 Preliminaries

In this section we introduce the basic definitions and describe elementary necessary conditions for the existence of a compatible circuit.

Throughout the chapter we let G be an eulerian digraph with a fixed edge coloring ϕ (incident edges may have the same color). We refer to G as a *colored eulerian digraph*. A *monochromatic transition* is two consecutive edges in a walk of G that have the same color. A *compatible circuit* T is a an eulerian circuit with no monochromatic transitions. Our goal is to determine whether a colored eulerian digraph has a compatible circuit. Though an eulerian circuit is defined to be a cyclic sequence of consecutive edges, it is determined by the transitions at each vertex of the digraph G. We will focus on finding transitions at each vertex such that the resulting eulerian circuit is a compatible circuit.

Fleischner and Fulmek [26] considered a more general setting where the head and tail of each directed edge can receive different colors. Such graphs can be handled in our setting by subdividing edges with different colors on the head and tail. Similarly we can assume that a colored eulerian digraph is loopless, since subdividing the loop twice and coloring the new middle edge a new color results in a colored eulerian digraph with no loops. Henceforth we consider only loopless colored eulerian digraphs.

Now we establish some notation that will be used throughout the rest of the chapter. Let G be a colored eulerian digraph, and v a vertex of G. Define $E^+(v)$ to be the set of outgoing edges incident to v, and $E^-(v)$ to be the set of incoming edges incident to v. For each vertex v define $C_i(v)$ to be the set of incident edges to v that are colored with the color i. We assume there are a total of k colors in the edge coloring. We refer to the sets $C_1(v), \ldots, C_k(v)$ as the *color classes* of v. Let $\gamma(v)$ denote the size of the largest color class at v. For each color i, define $C_i^+(v)$ to be the set of outgoing edges incident to v that are colored i, and similarly $C_i^-(v)$ for incoming edges.

An eulerian circuit T determines a matching between $E^-(v)$ and $E^+(v)$ corresponding to the transitions incident to a vertex v. Hence, if $\gamma(v) > \deg^+(v)$ for some vertex v, then G does not have a compatible circuit; by the Pigeonhole Principle there will be a transition from an edge in $E^-(v)$ to $E^+(v)$ with the same color. Thus if G has a compatible circuit, then $\gamma(v) \leq \deg^+(v)$ for all vertices v.

Definition 3.1. Let G be a colored eulerian digraph. Assume v is a vertex of G, where $\gamma(v) = \deg^+(v)$, and $C_i(v)$ is a largest color class. If $C_i(v)$ lies completely inside $E^+(v)$ or $E^-(v)$, then there are no restricted transitions at v. Otherwise, if a compatible circuit exists it must match $C_i^+(v)$ to $E^-(v) - C_i^-(v)$, and $C_i^-(v)$ to $E^+(v) - C_i^+(v)$.

Construct a new colored eulerian digraph G' from G, where for each vertex v with $\gamma(v) = \deg^+(v)$ and where the largest color class is not contained in $E^+(v)$ or $E^-(v)$, the vertex v is split into two new vertices v_1 and v_2. In G' the vertex v_1 has incoming edges $E^-(v) - C_i^-(v)$ and outgoing edges $C_i^+(v)$, and v_2 has incoming edges $C_i^-(v)$ and outgoing edges $E^+(v) - C_i^+(v)$ as depicted in Figure 3.1. The resulting digraph G' is reduced: a *reduced colored eulerian digraph* G is a loopless colored eulerian digraph with $\gamma(v) \leq \deg^+(v)$ for all vertices $v \in V(G)$, and if $\gamma(v) = \deg^+(v)$, then the largest color class is either all the incoming or all the outgoing edges.

Lemma 3.2. *Let G be a colored eulerian digraph and G' be the reduced eulerian digraph created from G. Then G has a compatible circuit if and only if G' has a compatible circuit.*

Proof. A compatible circuit T of G' viewed as a series of edges is also a compatible circuit in G. If G has a compatible circuit T, then for each vertex v with $\gamma(v) =$

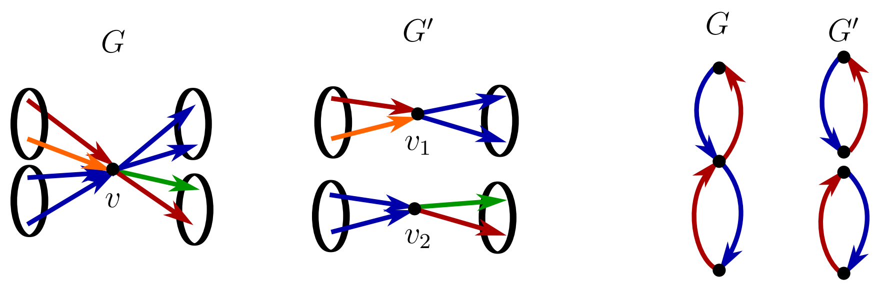

Figure 3.1: The vertex v is replaced with the vertices v_1 and v_2 when $\gamma(v) = \deg^+(v)$ and the largest color class has both incoming and outgoing edges. The example on the the right is a digraph with no compatible circuit.

$\deg^+(v)$ the edges in $C_i^+(v)$ must be matched to $E^-(v) - C_i^-(v)$ and the edges in $C_i^-(v)$ must be matched to $E^+(v) - C_i^+(v)$. So T is also a compatible circuit in G'. □

Lemma 3.2 shows the equivalence between the colored eulerian digraph G and the reduced eulerian digraph G'. Hence, throughout the rest of the chapter we consider only reduced eulerian digraphs.

3.3 Fixable vertices

In the case when $\gamma(v) = \deg^+(v)$, if the largest color class is contained in $E^+(v)$ or $E^-(v)$, then there are no restricted transitions. Thus, every eulerian circuit has no monochromatic transitions between $E^-(v)$ and $E^+(v)$. In this section we investigate when we can change the transitions of an eulerian circuit at a single vertex v to create a new eulerian circuit with no monochromatic transitions at v.

Definition 3.3. An eulerian circuit T determines a matching between $E^+(v)$ and $E^-(v)$ by considering the segments $S_1, \ldots, S_d$ between successive appearances of v in T. We refer to these segments $S_1, \ldots, S_d$ as *excursions* of T. There is a natural matching between $E^+(v)$ and $E^-(v)$ where the first edge of S_i is matched to the last edge of S_i. We wish to find nonmonochromatic transitions at v such that the excursions are combined into one circuit.

Since it is not immediate which matchings of $E^+(v)$ and $E^-(v)$ arise from the excursions of some eulerian circuit, we consider *any* matching M between $E^+(v)$ and $E^-(v)$. Label the edges incident to v as $e_1^-, e_1^+, \ldots, e_d^-, e_d^+$, where $d = \deg^+(v)$ and $e_i^- \in E^-(v)$ is matched in M to $e_i^+ \in E^+(v)$ for $i = 1, \ldots, d$. The *excursion graph* $L_M(v)$ is the colored digraph with vertex set consisting of v and the disjoint union of $N^-(v)$ and $N^+(v)$. The edge set of $L_M(v)$ consists of all edges in G incident to v, along with edges from $e_i^+(v)$ to $e_i^-(v)$ for all i. The edges incident to v retain their color from G and the new edges receive a new color $k+1$ not in G. Note that the excursion graph $L_M(v)$ is a colored eulerian digraph consisting of cycles containing v of length three; see the right graph in Figure 3.2 for an example. When the matching M arises from an eulerian circuit T, we write $L_T(v)$ to denote the excursion graph $L_M(v)$.

Definition 3.4. Let G be a colored eulerian digraph. A vertex v is *fixable* if $L_M(v)$ has a compatible circuit for every matching M between $E^+(v)$ and $E^-(v)$.

The usefulness of fixable vertices is clear from the following proposition.

Proposition 3.5. *Let G be a colored eulerian digraph. If all the vertices of G are fixable, then G has a compatible circuit.*

Proof. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G. Let T_0 be an arbitrary eulerian circuit of G. Since v_1 is fixable, the excursion graph $L_{T_0}(v_1)$ has a compatible circuit W_0. The circuit W_0 determines a set of transitions between $E^-(v_1)$ and $E^+(v_1)$. Let $d_1 = \deg^+(v_1)$. We use these transitions to alter the circuit T_0 by rearranging the order in which the excursions $S_1, \ldots, S_{d_1}$ occur according to the transitions found in the compatible circuit. Since $L_{T_0}(v_1)$ is compatible the resulting trail is an eulerian trail of G which we call T_1. The eulerian trail T_1 has no monochromatic transitions at v_1 and introduces no new monochromatic transitions at other vertices. For each $i = 2, \ldots, n$ we repeat the previous process with the excursion graph $L_{T_{i-1}}(v_i)$. We obtain a sequence of eulerian circuits $T_0, T_1, \ldots, T_n$, where T_i has no monochromatic transitions at v_j for $1 \leq j \leq i$. Thus, T_n has no monochromatic transitions and hence is a compatible circuit of G. □

From this proposition we see that reduced colored eulerian digraphs that do not have compatible circuits must have nonfixable vertices. In the rest of this section we characterize fixable vertices. We use the same approach taken by Isaak [34] to find compatible circuits. Our proof uses Meyniel's Theorem [47] rather than Ghouila-Houri's Theorem [28], which gives a slightly stronger result.

Theorem 3.6 (Meyniel [47])**.** *Let G be a digraph on n vertices with no loops. If*

$$\deg^+(x) + \deg^-(x) + \deg^+(y) + \deg^-(y) \geq 2n - 1$$

for all pairs of nonadjacent vertices x and y in G, then G is hamiltonian.

As a direct consequence of Meyniel's Theorem we have the following proposition.

Proposition 3.7. *Let G be a colored eulerian digraph with a vertex v. If $\gamma(v) \leq \deg^+(v) - 2$ or $\gamma(v) = \deg^+(v) - 1$ and the second largest color class has size strictly smaller than $\deg^+(v) - 1$, then v is fixable.*

Proof. Consider the excursion graph $L_M(v)$ for an arbitrary matching M between $E^+(v)$ and $E^-(v)$. Let $S_1, \ldots, S_d$ denote the directed 3-cycles of $L_M(v)$, where $d = \deg^+(v)$.

Create a new digraph D with vertex set $S_1, \ldots, S_d$, where there is a directed edge from S_i to S_j if S_j can follow S_i in a compatible circuit, i.e. $i \neq j$ and $e_i^-(v)$ is not the same color as $e_j^+(v)$. By construction, a hamiltonian cycle of D corresponds to a compatible circuit of $L_M(v)$. We apply Meyniel's Theorem to show that D is hamiltonian.

Assume that S_i and S_j are distinct nonadjacent vertices in D. Then $e_i^-(v)$ and $e_j^+(v)$ have the same color, and $e_i^+(v)$ and $e_j^-(v)$ have the same color. Without loss of generality assume that $e_i^-(v)$ has color 1 and $e_i^+(v)$ has color 2. The outdegree of S_i in D is at least $d - |C_1^+(v)| - 1$, since $e_i^-(v)$ has color 1, there are $|C_1^+(v)|$ edges with color 1 on the other 3-cycles, and there is no loop at vertex S_i in D. Similarly the indegree of S_i is at least $d - |C_2^-(v)| - 1$, the outdegree of S_j is at least $d - |C_2^+(v)| - 1$, and the indegree of S_j is at least $d - |C_1^-(v)| - 1$. Therefore, the sum of the indegree and outdegree of the vertices S_i and S_j in D is at least

$$4d - (|C_1^+(v)| + |C_1^-(v)| + |C_2^+(v)| + |C_2^-(v)|) - 4 = 4d - |C_1(v)| - |C_2(v)| - 4.$$

Without loss of generality assume that the size of $C_1(v)$ is at least as large as the size of $C_2(v)$. By hypothesis, $|C_1(v)| \leq \deg^+(v) - 1$ and $|C_2(v)| \leq \deg^+(v) - 2$. Therefore the sum of the degrees of S_i and S_j in D is

$$\deg^+(S_i) + \deg^-(S_i) + \deg^+(S_j) + \deg^-(S_j) \geq 4d - (2d-1) - (2d-2) - 4 = 2d - 1.$$

By Meyniel's Theorem, D has a hamiltonian cycle, so $L_M(v)$ contains a compatible circuit. Since the choice of M was arbitrary, v is a fixable vertex. $\square$

As a consequence of Proposition 3.7, if each vertex of a colored eulerian digraph has enough color classes, then the graph has a compatible circuit.

Corollary 3.8. *Let G be a reduced colored eulerian digraph. If each vertex has at least five different color classes then G has a compatible circuit.*

Proof. If each vertex v has at least five different color classes and $\gamma(v) \leq \deg^+(v) - 1$, then the second largest color class has size at most $\deg^+(v) - 2$. By Proposition 3.7, all the vertices where $\gamma(v) < \deg^+(v)$ are fixable, and since G is reduced all the vertices with $\gamma(v) = \deg^+(v)$ are fixable. By Proposition 3.5, G has a compatible circuit. □

Notice that Meyniel's Theorem does not apply when $C_x(v)$ and $C_y(v)$ are both of size $\gamma(v) = \deg^+(v) - 1$. The next proposition provides a characterization of fixable vertices.

Proposition 3.9. *Let G be a colored eulerian digraph, and v a vertex of G, where $\gamma(v) = \deg^+(v) - 1$ and $\deg^+(v) \geq 2$. Then the graph $L_M(v)$ has a compatible circuit unless the following properties hold:*

1. *The two largest color classes, $C_x(v)$ and $C_y(v)$, are of size $\deg^+(v) - 1$.*
2. *The color classes $C_x(v)$ and $C_y(v)$ have both incoming and outgoing edges (i.e. the sets $C_x^-(v)$, $C_x^+(v)$, $C_y^-(v)$, and $C_y^+(v)$ are all nonempty).*
3. *The matching M matches $C_x^-(v)$ to $C_y^+(v)$; $C_x^+(v)$ to $C_y^-(v)$; and the two edges not in $C_x(v)$ and $C_y(v)$ are matched together (hence one is an incoming edge and the other is an outgoing edge).*

If the above properties hold, then $L_M(v)$ does not have a compatible circuit.

Figure 3.2 illustrates the properties of the proposition.

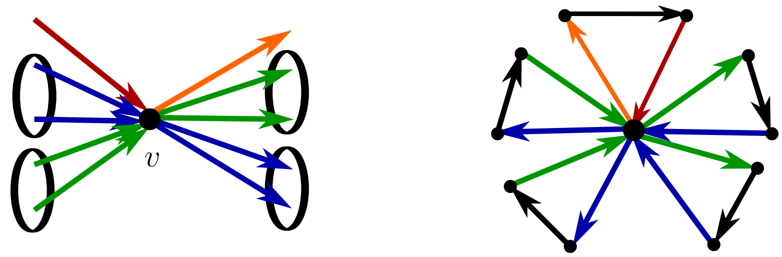

Figure 3.2: All nonfixable vertices have the form of the vertex on the left. The graph to the right is an excursion graph for a nonfixable vertex of outdegree 5 with no compatible circuit.

Proof. First we show that the colored eulerian digraph $L_M(v)$ satisfying the above properties does not have a compatible circuit. Notice that the 3-cycles where $C_x^-(v)$ is matched to $C_y^+(v)$ can not transition to the 3-cycles where $C_y^-(v)$ is matched to $C_x^+(v)$. The only other 3-cycle in the excursion graph is created by the two edges not in the largest color class. This 3-cycle can transition from the 3-cycles where $C_x^-(v)$ is matched to $C_y^+(v)$ (or vice versa), but it can not be used to transition back. So in every eulerian circuit in $L_M(v)$ there is a monochromatic transition at v.

We prove the vertices not satisfying the above properties are fixable by induction on $d = \deg^+(v)$. Proposition 3.7 handles the case when the second largest color class is of size strictly less than $\deg^+(v) - 1$.

The base cases are when $d = 2, 3, 4$. The case for $d = 2$ is an outdegree two vertex where each edge has a distinct color. This vertex has no restrictions, so it is fixable. For $d = 3, 4$ we used Sage [58] to examine all possible color combinations in the case where we have two color classes of size $\deg^+(v) - 1$. We colored the six (or eight) edges and checked all possible hamiltonian cycles on three (or four) vertices to find a compatible circuit. We found a compatible circuit except in the cases where the properties in the proposition hold. This required checking 180 configurations for $d = 3$ and 1120 configurations for $d = 4$.

Assume that $d > 4$. Pick an edge e_1 in $C_x(v)$ to match with an edge e_2 in $C_y(v)$. Then fixing this transition and splitting the transition off the digraph of $L_M(v)$ creates a new digraph $L'_M(v)$, which has only $d-1$ directed 3-cycles. We will pick this transition so we can apply induction to the new colored digraph $L'_M(v)$.

Let $C_*(v)$ denote the two edges incident to v not colored x or y. Suppose that the two edges in $C_*(v)$ are matched together; i.e. $C_*(v) = \{e_i^-(v), e_i^+(v)\}$ for some i. If the properties above do not hold, then either $C_x^+(v) = \emptyset$, $C_x^-(v) = \emptyset$, or there exists j such that $e_j^-(v)$ and $e_j^+(v)$ both belong to $C_x(v)$.

First consider the case when $C_x^+(v) = \emptyset$; the case when $C_x^-(v) = \emptyset$ uses a symmetric argument. Pick an edge $e_j^-(v)$ in $C_x(v) = C_x^-(v)$ and match it with an edge $e_k^+(v)$ in $C_y(v) = C_y^+(v)$, where $j \neq k$. Fix this transition and split those two edges off the vertex v. This operation combines the 3-cycles j and k to form one cycle with outgoing edge $e_k^+(v)$ and incoming edge $e_j^-(v)$. Contract and recolor the appropriate edges to form a new excursion graph $L'_M(v)$. The new digraph still avoids the matching described above, and hence we can apply induction to $L'_M(v)$.

Now consider the case when there exists j such that $e_j^-(v)$ and $e_j^+(v)$ both belong to $C_x(v)$. If $e_k^-(v)$ and $e_k^+(v)$ have the same color for $k = 1, \ldots i-1, i+1, \ldots, d$, then fix a transition $e_k^-(v)$ and $e_\ell^+(v)$, where $e_\ell^+(v)$ is colored y. Splitting this transition off results in a new excursion graph avoiding the matching described above, and hence we can apply induction. Otherwise there exists $k \neq i, j$ with either $e_k^+(v) \in C_x(v)$ and $e_k^-(v) \in C_y(v)$, or $e_k^-(v) \in C_x(v)$ and $e_k^+(v) \in C_y(v)$. If $e_k^-(v) \in C_y(v)$, match $e_k^-(v)$ to $e_j^+(v)$, and if $e_k^+(v) \in C_y(v)$, match $e_j^-(v)$ to $e_k^+(v)$. After splitting off this transition we can apply induction.

We now consider the case when the edges of $C_*(v)$ are not matched together. Without loss of generality we assume that $C_*(v) \cap E^+(v) \neq \emptyset$; otherwise we can use the symmetric argument by switching $+$ and $-$. Let $e_i^+(v) \in C_*(v)$, and assume that the other edge of $C_*(v)$ is either $e_j^+(v)$ or $e_j^-(v)$. Without loss of generality assume that $e_i^-(v) \in C_x(v)$ (the same argument will work if $e_i^-(v) \in C_x(v)$ by switching x and y). If $C_y^+(v) - \{e_j^+(v)\} \neq \emptyset$, then we can match $e_i^-(v)$ to an edge from this set, and we will still have the two edges in $C_*(v)$ not matched. Therefore after splitting off this transition we can apply induction.

Assume that $C_y^+(v) - \{e_j^+(v)\} = \emptyset$. Then $C_x(v) = C_x^+(v) \cup e_i^-(v)$, where $C_x^+(v) = E^+(v) - \{e_i^+(v), e_j^+(v)\}$. Since $d > 4$ we know that $|C_x^+(v)| > 2$ and $|C_y^-(v) - \{e_j^-(v)\}| > 2$; therefore we can match an edge from $C_x^+(v)$ to an edge in $C_y^-(v) - \{e_j^-(v)\}$. After splitting off this transition, we can apply induction. $\square$

Corollary 3.10. *Let G be a colored eulerian digraph. A vertex v is fixable if and only if v satisfies one of the following:*

1. $\gamma(v) \leq \deg^+(v) - 2$.
2. $\gamma(v) = \deg^+(v) - 1$ *and the second largest color class has size strictly smaller than* $\deg^+(v) - 1$.
3. $\gamma(v) = \deg^+(v) - 1$ *and there are two color classes of size* $\gamma(v) = \deg^+(v) - 1$, *where the two edges not in the largest two color classes are both incoming or both outgoing edges.*
4. $\gamma(v) = \deg^+(v) - 1$ *and there are two color classes of size* $\gamma(v) = \deg^+(v) - 1$, *where one of the largest color classes has only incoming edges or only outgoing edges.*
5. $\gamma(v) = \deg^+(v)$, *where a largest color class* $C_i(v)$ *has only incoming edges or only outgoing edges.*

Proof. By Proposition 3.7, Proposition 3.9, and the discussion of vertices v with $\gamma(v) = \deg^+(v)$, we know that the above vertices are fixable.

In the case when $\gamma(v) > \deg^+(v)$ the excursion graph $L_M(v)$ never has a compatible circuit. So all vertices v with $\gamma(v) > \deg^+(v)$ are not fixable. If $\gamma(v) = \deg^+(v)$ and if a largest color class $C_i(v)$ has both incoming and outgoing edges then we can create a matching M where $C_i^+(v)$ is matched to $E^-(v) - C_i^-(v)$. When this happens the excursion graph $L_M(v)$ does not have compatible circuit since the 3-cycles with an edge from $C_i^+(v)$ can never be matched with the 3-cycles from $C_i^-(v)$.

Proposition 3.9 shows that the only nonfixable vertices with $\gamma(v) = \deg^+(v) - 1$ are those with two color classes of size $\deg^+(v) - 1$, and the two other edges have an incoming and outgoing edge. □

Proposition 3.11. *Let G be a colored eulerian digraph, v a fixable vertex of G, and T a (not necessarily compatible) eulerian circuit of G. Then there exists a polynomial time algorithm to find a compatible circuit in $L_T(v)$.*

Proof. Berman and Liu [7] gave a polynomial time algorithm for finding a hamiltonian cycle in a digraph that satisfies the hypothesis of Meyniel's Theorem. Hence for a fixable vertex v that satisfies the hypotheses in Proposition 3.7 we can find in polynomial time a hamiltonian cycle in D, and thus a compatible circuit in $L_T(v)$.

In the case when $\gamma(v) = \deg^+(v)$ and all the incoming or outgoing edges have the same color then any eulerian circuit of $L_T(v)$ is compatible.

Finally, when the two largest color classes have size $\gamma(v) = \deg^+(v) - 1$, the proof of Theorem 3.9 gives a polynomial time algorithm: we iteratively match edges from the largest color classes together until we have an excursion graph with only four excursions. □

3.4 Graphs with no S_3 vertices

In this section we examine reduced colored eulerian digraphs with nonfixable vertices. Let G be a reduced eulerian digraph, and let S be the set of nonfixable vertices in G described in Lemma 3.9: these vertices have two color classes of size $\gamma(v) = \deg^+(v) - 1$, where both largest color classes have incoming and outgoing edges and there is one incoming and one outgoing edge not in the largest two color classes. Let S_3 be the nonfixable vertices v with $\deg^+(v) = 3$ with exactly three color classes.

For the rest of this section we will assume that $S_3 = \emptyset$. First, we will describe several related graphs that model the important properties of colored eulerian digraphs when $S_3 = \emptyset$.

Definition 3.12. Let G be a reduced colored eulerian digraph, and S be the set of nonfixable vertices. For each vertex $v \in S$ let $C_x(v)$ and $C_y(v)$ denote the two largest color classes, and define $C_*(v) = \{e^+(v), e^-(v)\}$ to be the two edges incident to v not in $C_x(v) \cup C_y(v)$. Note that since $S_3 = \emptyset$, the set $C_*(v)$ is well defined.

We construct a new colored digraph G_S by splitting all vertices in S as follows: for $v \in S$ replace v with three new vertices v_1, v_2, and v_3, where

- v_1 is incident to the edges in $C_*(v)$,
- v_2 is incident to the edges in $C_x^-(v) \cup C_y^+(v)$, and
- v_3 is incident to the edges in $C_y^-(v) \cup C_x^+(v)$.

Notice that when the two edges in $C_*(v)$ are deleted, the resulting vertex v' has $\gamma(v') = \deg^+(v')$ and using the splitting from Definition 3.1 on v' results in the creation of the vertices v_2 and v_3.

Definition 3.13. Define the *component graph* H_G of G_S as follows: the vertices of H_G are the strong components of G_S. (Note that since $\deg^+(v) = \deg^-(v)$ for all vertices in G_S, the strong components are also the weak components.) For each vertex $v \in S$ there is an edge in H_G labeled with v between D_1 and D_2 where the component D_1 contains v_1 and D_2 contains v_2, and another edge in H_G labeled v between D_1 and D_3 where the component D_1 contains v_1 and D_3 contains v_3.

Note that the component graph H_G is an undirected edge labeled multigraph. See Figures 3.3 and 3.6 for pictures of the graphs G, G_S, and H_G.

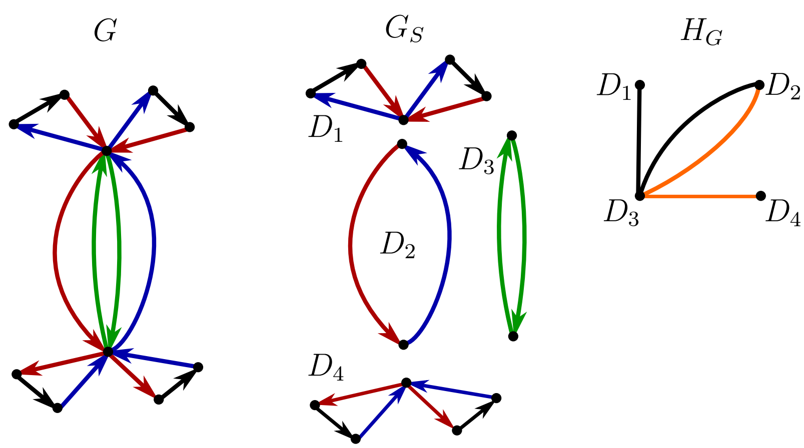

Figure 3.3: An example of a colored eulerian digraph with two nonfixable vertices and the auxiliary graphs G_S and H_G.

Definition 3.14. A *2-trail* is a set of two incident edges in H_G with the same label v. Either of the edges may be a loop, or the edges may be multiple edges.

Notice that in H_G the edges with label $v \in S$ form a 2-trail, and so the edge set of H_G can be thought of as a union of 2-trails. As we will see in Theorem 3.16, there is a compatible circuit in G if and only if there exists an appropriate connected subgraph of H_G. We will need the following definition to make this precise.

Definition 3.15. A *2-trail traversal* is a set E' of edges in H_G with exactly one edge from each 2-trail such that the spanning subgraph of H_G with edge set E' is connected.

Theorem 3.16. *Let G be a reduced colored eulerian digraph with no nonfixable vertices of outdegree three with exactly three color classes (i.e. $S_3 = \emptyset$). Then the graph G has a compatible circuit if and only if the component graph H_G has a 2-trail traversal.*

Proof. ($\Rightarrow$) Let T be a compatible circuit of G. From T we will construct a 2-trail traversal E' of H_G.

For each vertex $v \in S$, let $C_x(v)$ and $C_y(v)$ be the two largest color classes at v, and let $C_*(v) = \{e^-(v), e^+(v)\}$ denote the two edges not in $C_x(v) \cup C_y(v)$. In T the two edges in $C_*(v)$ satisfy exactly one of the three following conditions:

1. the two edges in $C_*(v)$ are matched together, or
2. $e^-(v)$ is matched to an edge in $C_y^+(v)$ and $e^+(v)$ is matched to an edge in $C_x^-(v)$, or
3. $e^-(v)$ is matched to an edge in $C_x^+(v)$ and $e^+(v)$ is matched to an edge in $C_y^-(v)$.

T is a cyclic sequence of edges in G, so it is a sequence of edges in G_S. Since T is an eulerian circuit, the sequence visits each edge exactly once in G and in G_S. Let v be a nonfixable vertex of G, and $e^-(v)$ be the incoming edge incident to v_1 in G_S. Let e be the edge following $e^-(v)$ in T.

If the edge e is in the same component in G_S as $e^-(v)$, then e is $e^+(v)$. Let f_v be an arbitrary edge in H_G labeled v. If the edge e is in a different component in G_S than $e^-(v)$, then e has as its tail the vertex v_2 or v_3. There is an edge f_v labeled v in H_G that corresponds to the vertices v_1 and the tail of the edge e. Notice that the two edges in $C_*(v)$ satisfy condition 2 or 3 above. The edge e' preceding $e^+(v)$ in T

is such that the head of e' is the tail of e in G_S; hence f_v is consistent for the edges $e^-(v)$ and e, and e' and $e^+(v)$.

Let $E' = \{f_v : v \in S\}$ denote the set of edges f_v corresponding to the nonfixable vertices as described above. By construction there is exactly one edge from each 2-trail in E'. We now show that the spanning subgraph of H_G with edge set E' is connected. Let D_1 and D_2 be two vertices of H_G. Since G_S has no isolated vertices, we can pick an edge e_1 in the component D_1 in G_S and an edge e_2 in the component D_2 in G_S. Start at the edge e_1 in the circuit T and follow the circuit until we reach the edge e_2. As we move along the sequence T, whenever we move from an edge in one component in G_S to an edge in another component, there is a corresponding edge in E' between these components in H_G. Thus, following the circuit in G_S from e_1 to e_2 induces a walk in H_G from D_1 to D_2 using only edges of E'. Hence the spanning subgraph of H_G with edge set E' is connected, and so E' is a 2-trail traversal.

($\Leftarrow$) Now assume that H_G has a 2-trail traversal E'. We will form a new graph G' from G_S by identifying pairs of vertices according to the 2-trail traversal. The operation of identifying two vertices v_1 and v_i is to remove v_1 and v_i from the graph and create a new vertex v' that is incident to the disjoint union of the edges incident to v_1 and v_i.

First we assume that $\deg^+(v_2) \geq 2$ and $\deg^+(v_3) \geq 2$ in G_S for all nonfixable vertices $v \in S$. Each edge in the 2-trail traversal corresponds to two vertices v_1 and v_i for $i \in \{2, 3\}$ in G_S. For all edges e in E', identify the corresponding vertex v_1 with the vertex v_i. Call the resulting graph G'.

The graph G' is a colored digraph with $\deg^+(v) = \deg^-(v)$ for all vertices $v \in V(G')$. Since $\deg^+(v_i) \geq 2$ the vertex in G' created by identifying v_1 and v_i is a fixable vertex by Corollary 3.10, so all vertices of G' are fixable. We claim that G' is connected. Let r' and s' be two vertices in G'. If r' was created by identifying vertices v_1 and v_i, arbitrarily pick v_1 or v_i and call it r. If not, then r' is also a vertex in G_S, but we write r for the vertex r' in G_S. We choose s similarly. Let D_1 be the component in G_S containing r, and $\widehat{D}$ be the component containing s. Since E' is a 2-trail traversal there is a walk $W = D_1D_2 \dots D_k$, where $\widehat{D} = D_k$, in H_G using edges of E'. Since D_iD_{i+1} is an edge in E' there exist vertices $x_i \in D_i$ and $y_{i+1} \in D_{i+1}$ identified together in G'; call this vertex z_i. Since y_i and x_i are in the same component D_i in G_S there exists a walk from y_i to x_i, and hence there is a walk from z_{i-1} to z_i. There are also walks from r to z_1 and z_{k-1} to s in G'. Concatenating these walks forms a walk from r to s in G'. Thus G' is connected.

By Proposition 3.5, G' has a compatible circuit T. Since incident edges in G' are incident in G, the circuit T is a compatible circuit of G.

In the case when v_2 or v_3 has outdegree one we need to be more careful. In the case when the vertex v_1 is identified with the vertex v_i of outdegree one it does not form a fixable vertex in G' if the edges in $C_*(v)$ have the same color. Instead the created vertex requires an additional split as in Definition 3.1, which could disconnect G'.

We claim that if H_G has a 2-trail traversal, then it has a 2-trail traversal E' with the property that for any edge $e \in E'$ where

- e corresponds to vertices v_1 and v_i, where $\deg^+(v_i) = 1$ in G_S, and
- both edges of $C_*(v)$ have the same color

then e is a bridge in the spanning subgraph of H_G with edge set E'.

By hypothesis there exists a 2-trail traversal. Both of the edges labeled v satisfy the two properties above only when $v \in S_3$. Since S_3 is empty, if E' has an edge e labeled v with the properties above, then we may replace the edge e with the other edge f labeled v, thereby reducing the number of edges in E' with the above properties.

As before, we form the graph G' by identifying vertices in G_S according to the edges in E'. Identified vertices in G' are fixable unless v_1 and v_i both have outdegree one in G_S and the two edges in $C_*(v)$ have the same color. In this case, the edge labeled v in E' is a bridge in the spanning subgraph of H_G with edge set E', so the identified vertex v' in G' is a cut vertex in the graph G'.

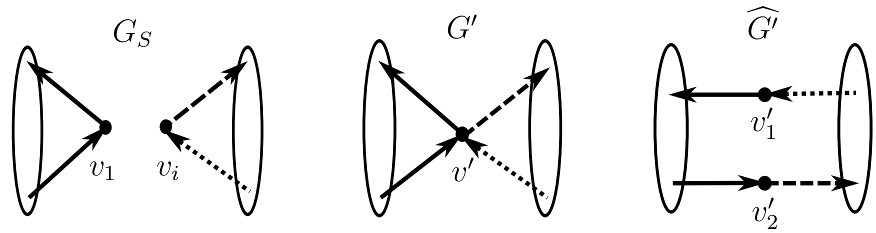

Figure 3.4: The identification of v_1 and v_i to form G' and splitting to create $\widehat{G'}$. The dotted and dashed edges denote different colors.

Let $e^-(v)$ and $e^+(v)$ be the incoming and outgoing edges of v_1, and e' and e be the incoming and outgoing edges of v_i. Then we split the vertex v' according to Definition 3.1. Replace the vertex v' with two new vertices v'_1 and v'_2, where v'_1 has incoming edge $e^-(v)$ and outgoing edge e and v'_2 has incoming edge e' and outgoing edge $e^+(v)$. Note that the split at v' preserves connectivity since v' is a cut vertex, as shown in Figure 3.4. Note the condition that v' being a cut vertex is sufficient, as shown in Figure 3.5.

Applying this splitting to all vertices v' in G' where v_1 and v_i both have degree one in G_S and the two edges in $C_*(v)$ have the same color results in a connected eulerian digraph $\widehat{G'}$, where $\deg^+(v) = \deg^-(v)$ for all vertices $v \in V(\widehat{G'})$ and where all the vertices are fixable. Again by Proposition 3.5, $\widehat{G'}$ has a compatible circuit T, and since incident edges in $\widehat{G'}$ are incident in G, the circuit T is a compatible circuit of G. □

Figure 3.3 shows a colored eulerian digraph with no compatible circuit since H_G does not have a 2-trail traversal. Figure 3.6 gives an example of a colored eulerian digraph with a compatible circuit and the associated graphs G_S, H_G, G', and $\widehat{G'}$.

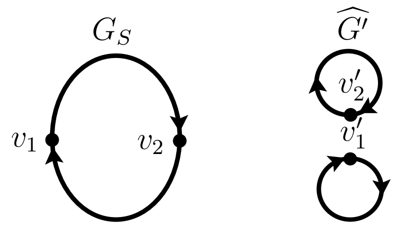

Figure 3.5: If v_1 and v_2 are in the same component and form a 2-cut in G_S, then identifying the vertices and splitting disconnects the components in $\widehat{G'}$.

Theorem 3.16 provides necessary and sufficient conditions for the existence of a compatible circuit in reduced colored eulerian graphs where $S_3 = \emptyset$ in terms of finding a 2-trail traversal in H_G. We can determine if H_G has a 2-trail traversal by finding a rainbow spanning tree, where the color classes are the 2-trails, and arbitrarily add an edge from every 2-trail that does not have an edge in the rainbow spanning tree. Therefore the problem of determining whether a 2-trail traversal exists in H_G can be solved by determining if H_G has a rainbow spanning tree. Section 2.1.1 discusses the existence of rainbow spanning trees in edge-colored graphs.

3.4.1 Polynomial time algorithm

Our results give a polynomial time algorithm that determines whether a colored eulerian digraph G has a compatible circuit. If G does, then the algorithm provides a compatible circuit, and if not then the algorithm provides a certificate that shows why such a compatible circuit does not exist.

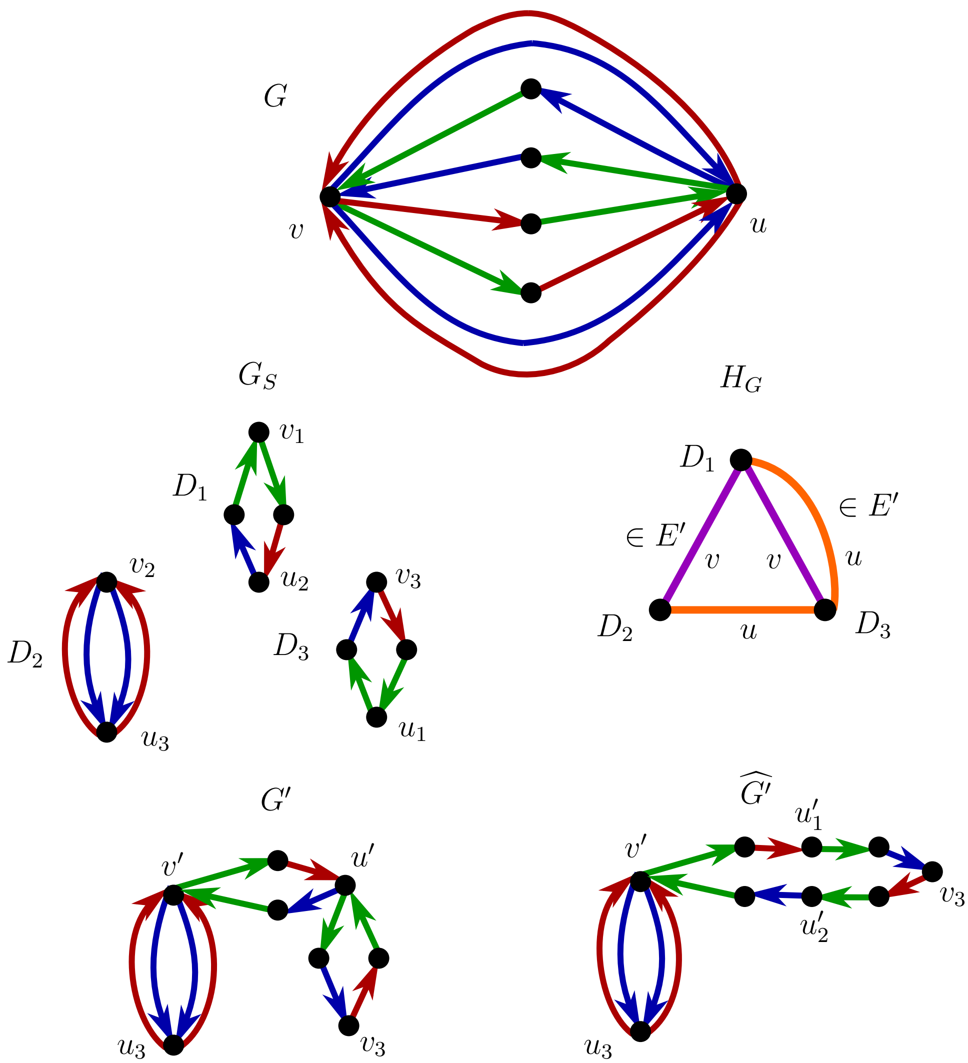

Figure 3.6: Above is an example of the auxiliary graphs G_S and H_G, where H_G has a 2-trail traversal E'. From this 2-trail traversal we can construct the graphs G' and $\widehat{G'}$.

Algorithm

Input: A colored eulerian digraph G with no nonfixable vertices of outdegree three with exactly three color classes.

Output: Compatible circuit of G or certificate which shows no such circuit exists.

Step 1: Check whether $\gamma(v) \leq \deg^+(v)$ for all vertices v. If not, then return as the certificate the vertex v such that $\gamma(v) > \deg^+(v)$.

Step 2: Create the reduced colored eulerian digraph: split all vertices with $\gamma(v) = \deg^+(v)$ according to Definition 3.1, and subdivide all loops to create graph G'. Check whether G' is connected. If not, then return a separation indicating that G' is disconnected.

Step 3: Find the nonfixable vertices of G and construct the auxiliary graphs G_S and H_G from Definitions 3.12 and 3.13.

Step 4: Determine whether H_G has a rainbow spanning tree. If not, return as the certificate the partition π of the vertices of H_G that demonstrates the obstruction for the rainbow spanning tree. If H_G has a rainbow spanning tree, find a rainbow spanning tree and call it R.

Step 5: From R construct a 2-trail traversal E' with the two properties described in the proof of Theorem 3.16. Form G' (or $\widehat{G'}$) by identifying vertices in G_S according to the 2-trail traversal of H_G, and split appropriate vertices according to the proof of Theorem 3.16.

Step 6: Find an eulerian circuit T_0 in G' (where T_0 is not necessarily compatible). For each vertex $v_i \in V(G')$, $i = 1, \ldots, n$, perform the step below:

Step 6a: Construct the excursion graph $L_{T_{i-1}}(v)$. Find a compatible circuit of $L_{T_{i-1}}(v)$, and use the transitions to rearrange the excursions of T_{i-1} at v to form the eulerian circuit T_i. Return the compatible circuit T_n of G.

Each of these steps can be completed in polynomial time. In particular, Step 4 follows from polynomial time algorithms for finding maximum common independent sets, as discussed in Section 2.1.1. Step 6a can be computed in polynomial time by Proposition 3.11. Thus the entire algorithm runs in polynomial time.

3.4.2 Number of monochromatic transitions

In this section we consider finding an eulerian circuit of a colored eulerian digraph with as few monochromatic transitions as possible. When a colored eulerian digraph has a compatible circuit the number of monochromatic transitions is zero. Our focus here is to colored eulerian digraphs that do not have a compatible circuit.

Recall that S is the set of nonfixable vertices described in Lemma 3.9: with two color classes of size $\gamma(v) = \deg^+(v) - 1$, where both largest color classes have incoming and outgoing edges and there is one incoming and one outgoing edge not in the largest two color classes. Again we assume there are no vertices in S_3. We restrict to this case so we can use the previous techniques from this section.

In a similar fashion to the proof of Theorem 3.16 we construct several auxiliary graphs that are used to count the number of monochromatic transitions.

Definition 3.17. Let G be a colored eulerian digraph with no vertices in S_3. Define a new colored digraph G_1, where each vertex v with exactly two color classes and $\gamma(v) = \deg^+(v)$ is split into two new vertices, v_1 and v_2, as in Definition 3.1.

Note the graph G_1 does not split all vertices with $\gamma(v) = \deg^+(v)$, but only the vertices with exactly two color classes.

Definition 3.18. Let G_S be the graph obtained from G by splitting all vertices v with $\gamma(v) = \deg^+(v)$ according to Definition 3.1 and splitting all nonfixable vertices in S according to Definition 3.12.

Define the *component graph* H_G of G_S as follows: the vertices of H_G are the strong components of G_S. For each vertex $v \in S$ there is an edge in H_G labeled v between D_1

and D_2, where the component D_1 contains v_1 and D_2 contains v_2, and another edge labeled v between D_1 and D_3 where the component D_1 contains v_1 and D_3 contains v_3. This definition is consistent with Definition 3.13.

Define H'_G as follows: the vertices of H'_G are the strong components of G_S. The graph H'_G contains H_G as a subgraph, and so in particular all the edges described above. In addition, for every vertex v with $\gamma(v) = \deg^+(v)$ there is an edge in H'_G between D_1 and D_2 labeled v where D_1 contains v_1 and D_2 contains v_2 in G_S.

Note that G_S can be obtained from G_1 by splitting the remaining vertices with $\gamma(v) = \deg^+(v)$ and the nonfixable vertices in S. In this way each component of G_1 corresponds to some collection of components of G_S. We refer to the collection of components in G_S that correspond to one component of G_1 as a *pseudocomponent*. The pseudocomponents do not intersect each other, since the components of G_1 are disjoint. Furthermore, the pseudocomponents correspond to a collection of vertices in H'_G, where the only edges between the pseudocomponents have a label v, where $\gamma(v) = \deg^+(v)$ and there are exactly exactly two color classes incident to v in G.

Figure 3.7 gives an example of a colored eulerian digraph G along with the graphs G_1, G_S, H_G, and H'_G.

The following technical lemma gives us a way to switch edges between two strong components to create one strong component.

Lemma 3.19. *Let G be a digraph with exactly two strong components D_1 and D_2, where $e_1 = uv$ is a directed edge in D_1 and $e_2 = wx$ is a directed edge in D_2. Let G' be the digraph formed from G by removing the edges e_1 and e_2 and adding the edges $e'_1 = ux$ and $e'_2 = wv$. Then G' is strongly connected.*

Proof. Let y and z be two vertices of G'. We want to show that there is a directed path from y to z in G'.

Since D_1 is a strong component there is a path P_1 from v to u in D_1 not using the edge e_1. Similarly, since D_2 is a strong component there is a path P_2 from x to w not using e_2.

If y and z are both in D_1, then consider a path Q in G from y to z in D_1. If Q uses the edge e_1, then replace e_1 with uxP_2wv, resulting in a path in G' from y to z. The case when y and z are both in D_2 is similar.

In the case when y is in D_1 and z is in D_2, there is a path Q_1 from y to u in D_1 not using e_1 and a path Q_2 from x to z in D_2 not using e_2. The path Q_1uxQ_2 is a path in G' from y to z. The case when y is in D_2 and z is in D_1 is similar. □

Next we give several definitions that are similar in nature to a 2-trail traversal in Definition 3.15.

Definition 3.20. A *traversal* is a subset E' of the edges of H'_G, such that the spanning subgraph with edge set E' is connected, and there is at least one edge from each 2-trail in E'.

The following definitions count the number of connected components in certain spanning subgraphs.

Definition 3.21. Let $\#(G)$ be the number of components in the graph G. Let $\mathrm{rsf}(H_G)$ be the minimum number of components among all rainbow spanning forests of H_G.

Theorem 3.22. *Let G be a loopless colored eulerian digraph with no vertices in S_3. Then every eulerian circuit of G has at least*

$$(\mathrm{rsf}(H_G) - 1) + (\#(G_1) - 1) + \sum_{v:\gamma(v)>\deg^+(v)} (\gamma(v) - \deg^+(v)). \qquad (\dagger)$$

monochromatic transitions. Furthermore, there exists an eulerian circuit with exactly (†) *monochromatic transitions.*

Proof. First we show (†) is a lower bound on the number of monochromatic transitions in any eulerian circuit T. The edges of T can be thought of as a sequence of edges in G, and as a sequence of edges in G_S. (The edges of G and G_S have a natural bijection.) Suppose $e_1, e_2, \ldots, e_m$ is the sequence of edges of T.

The circuit T naturally defines a spanning set of edges E' of H'_G defined in the following way: whenever e_i and e_{i+1} are in different components in G_S, the head of e_i is some vertex v_i, and the tail of e_{i+1} is some vertex v_j, where v is a nonfixable vertex in S or a vertex with $\gamma(v) = \deg^+(v)$ in G. Suppose e_i is in component D_k, and e_{i+1} is in component D_ℓ in G_S. If v has $\gamma(v) = \deg^+(v)$, then by definition H'_G has an edge labeled v between D_k and D_ℓ; add this edge to E'. If v is nonfixable, then there are two cases to consider. When v_i or v_j is the vertex v_1, then there is an edge labeled v between D_k and D_ℓ, add this edge to E'. When v_i and v_j are the vertices v_2 and v_3, there is a component $D_{\ell'}$ containing vertex v_1. Add both edges of the 2-trail to E'. Note that this 2-trail is a walk in H'_G from D_k to D_ℓ.

Every 2-trail that does not have an edge added in this way from T, arbitrarily add one edge from the 2-trail to E'.

We claim the set E' obtained in this fashion is a traversal. Note that whenever the sequence T moves from one component of G_S to another component, these two components are connected by a path in the subgraph of H'_G with edge set E'. Since T visits all the components, this implies that the subgraph with edge set E' is a spanning connected subgraph of H'_G.

Next we show that there are at least $(\mathrm{rsf}(H_G) - 1) + (\#(G_1) - 1)$ monochromatic transitions e_ie_{i+1} in T, where e_i is in component D_k in G_S and e_{i+1} is in D_ℓ in G_S, where $k \neq \ell$. To do this we consider the edges in the traversal and when they imply we have a monochromatic transition.

First we consider the edges in E' between the pseudocomponents of H'_G. Notice that the edges between the pseudocomponents are labeled v, where $\gamma(v) = \deg^+(v)$ with exactly two color classes in G. There must be at least $\#(G_1) - 1$ edges in E' of this type, since the subgraph of H'_G with edge set E' is connected. For each of these edge in E' labeled v, there are two distinct transitions e_ie_{i+1} and e_je_{j+1}, where

1. v in G is the head of e_i and e_j and the tail of e_{i+1} and e_{j+1}, and

2. e_i and e_{j+1} are in D_k and e_{i+1} and e_j are in D_ℓ, where D_k and D_ℓ are in different pseudocomponents of G_S (this implies they are in different components as well).

By construction the transitions e_ie_{i+1} and e_je_{j+1} are monochromatic. This implies there are at least $2(\#(G_1) - 1)$ monochromatic transitions between the pseudocomponents.

Next we consider the edges in E' that are not between the pseudocomponents. For every edge in E' labeled v, where $\gamma(v) = \deg^+(v)$ there is at least one monochromatic transition. For every 2-trail that has both edges in E' there is at least one monochromatic transition. The largest number of components in H'_G that can be connected without monochromatic transitions is $\mathrm{rsf}(H_G)$. We know the subgraph with edge set E' in H'_G is connected. Thus there are at least $\mathrm{rsf}(H_G) - \#(G_1)$ additional monochromatic transitions e_ie_{i+1}, where $e_i \in D_k$ and $e_{i+1} \in D_\ell$ and $\ell \neq k$.

Thus there are at least $(\mathrm{rsf}(H_G) - 1) + (\#(G_1) - 1)$ monochromatic transitions e_ie_{i+1} in T, where e_i is in component D_k in G_S and e_{i+1} is in D_ℓ in G_S, where $k \neq \ell$.

By the Pigeonhole Principle each vertex with $\gamma(v) > \deg^+(v)$ has at least $\gamma(v) - \deg^+(v)$ monochromatic transitions. Adding this value to $(\mathrm{rsf}(H_G) - 1) + (\#(G_1) - 1)$

gives the lower bound (†) on the number of monochromatic transitions in any eulerian circuit of G.

Now we demonstrate an eulerian circuit T of G with exactly (†) monochromatic transitions.

Step 1. Construct a traversal of H'_G, which is used to help find the desired eulerian circuit. Let F be a rainbow spanning forest of H_G of maximum size. By definition $\#(F)$ is the number of components of F. By construction the graph H'_G is connected, and H_G is a subgraph of H'_G. Thus, F is a subgraph of H'_G. We can pick $\#(H_G) - 1$ edges from H'_G such that these edges along with F form a spanning tree T of H'_G.

Recall the pseudocomponents of H'_G form a partition of the vertices of H'_G that corresponds to the components of G_1, where the only edges in H'_G between the partitioned vertices have $\gamma(v) = \deg^+(v)$ and exactly 2 color classes. Pick edges labeled v, where v is either nonfixable or $\gamma(v) = \deg^+(v)$ with at least 3 color classes to connect form exactly $\#(G_1)$ components. This can be done since the components of G_1 are connected, so the edges with these labels must be between the vertices inside the pseudocomponents. Next pick $\#(G_1)$ edges between the pseudocomponents to form a spanning tree T of H'_G.

There may be some 2-trails with no edge in T. For each 2-trail with no edge in T, pick the edge corresponding to vertices v_1 and v_2 and add it to T to form a spanning subgraph T'. The set T' is a spanning subgraph of H'_G. If there is a 2-trail edge e in T' labeled v between components containing vertices v_1 and v_i in G_S, where $i \in \{2, 3\}$ and $\deg^+(v_i) = 1$ and the edges in $C_*(v)$ have the same color, then the edge e in T' must be a bridge. If the edge is not a bridge, remove e from T' and add the other edge from the 2-trail to T'. If the other edge is also in T', this contradicts the maximality of the rainbow spanning tree, because removing e and adding the other edge of the

2-trail along with the other edges of F and an edge from a 2-trail that is in a cycle with e is a rainbow spanning subgraph in H_G with fewer than $\#(H_G)$ components.

The subgraph T' is the subgraph we use to help find the eulerian circuit with (†) monochromatic transitions.

Step 2. In this step we use T' to form an eulerian digraph G' obtained by identifying and augmenting certain vertices in G_S.

For each 2-trail edge with exactly one edge in T', glue the corresponding vertices together in G_S as in Theorem 3.16. If there is a 2-trail, where the edges in $C_*(v)$ are the same and $\deg^+(v_i) = 1$, then this edge is a bridge in T' and after identifying the vertices together preform the splitting to form vertices v_1' and v_i' as in Theorem 3.16. This result in a graph with exactly $\#(H_G)$ components where all the the edges in S except the vertices whose 2-trail edges both appear in T', are fixable.

Next, we use Lemma 3.19 to glue the remaining components together. For every 2-trail, where both edges are in T' both edges are bridges in T'. Otherwise, we can find a larger rainbow spanning forest. By Lemma 3.19, pick the incoming edge incident to v_1 and switch with an incoming edge incident to v_2, then pick the outgoing edge incident to v_1 and switch with an outgoing edge incident to v_3. If v_1, v_2 and v_3 are in different components in G_S, the result will combine all three components together. The new vertex v_1' is monochromatic and the vertices v_2' and v_3' are fixable. Since both edges of each 2-trail in T' are bridges in H_G' the vertices v_1, v_2 and v_3 are always in different components, since the changes at the other vertices can not connect the components containing v_1, v_2 and v_3 together.

For every vertex v in T' with $\gamma(v) = \deg^+(v)$ we use Lemma 3.19 to glue two different components together in G_S. Each edge labeled v is a bridge, and so v_1 and v_2 are in different components in G_S. Suppose v has at least three color classes. Suppose v_1 has incoming edges from the color class 1 of size $\deg^+(v)$ and v_2 has outgoing edges colored 1. If $N^+(v_1)$ is monochromatic, then pick an incoming edge e incident to v_2 not of that color. By Lemma 3.19 switch the edge e with the edge f, where f is any incoming edge incident to v_1. If $N^+(v_1)$ is not monochromatic, then let e be any incoming edge of v_2. In both cases, we have two new vertices v_1' and v_2', where v_1' is fixable and v_2' has exactly one monochromatic transition (all but one incoming edges are colored 1, all outgoing edges are colored 1).

In the case when v has exactly two color classes, pick any incoming edge incident to v_1 colored 1 and any incoming edge incident to v_2 colored 2. By switching these vertices, we end up with a new digraph with one fewer component and vertices v_1' and v_2', where v_1' and v_2' each have exactly one monochromatic transition.

After all the identifications, the resulting graph G' is a connected colored eulerian digraph, with the same edge set as G.

Step 3. For each vertex v with $\gamma(v) > \deg^+(v)$ we will replace v with three new vertices v_1, v_2 and v_3. Assume the largest color class of v is the color 1.

Arbitrarily pick $\gamma(v) - \deg^+(v)$ incoming and outgoing edges colored 1 to be incident to vertex v_1. Vertex v_2 is incident to all other incoming edges colored 1 and all outgoing edges not colored 1. Vertex v_3 is incident to all incoming edge not colored 1 and all remaining outgoing edges colored 1. If either v_2 or v_3 is not incident to any edges, remove the vertex from the graph.

If v_1 and v_2 are not in the same component, then select one incoming edge incident to v_1 and an incoming edge incident to v_2 and use Lemma 3.19 to form a connected digraph. Similarly, if v_1 and v_3 are not in the same component, then select one outgoing edge incident to v_1 and an outgoing edge incident to v_3 and use Lemma 3.19 to form a connected digraph. Therefore, there is a way to replace v with three new vertices v_1, v_2 and v_3 such that the new digraph is connected, v_1 has 1 color class and $\deg^+(v_1) = \gamma(v) - \deg^+(v)$, and v_2 and v_3 are fixable vertices.

Let G'' be the resulting digraph obtained from G' by replacing each vertex v with $\gamma(v) > \deg^+(v)$ in this way. Note by construction G'' is eulerian.

Step 4. The graph G'' is an eulerian digraph, where all the vertices are fixable, except the vertices where:

1. the vertex v_1' from the nonfixable vertices with both edges of the 2-trail in T' have exactly one monochromatic transition, and v_2' and v_3' are fixable.

2. the vertices v_2', from vertices v where $\gamma(v) = \deg^+(v)$ and v has at least three color classes, has exactly one monochromatic transition,

3. the vertices v_1' and v_2', from vertices v where $\gamma(v) = \deg^+(v)$ and v has exactly two colors classes, that have exactly one monochromatic transitions each,

4. the vertices v_1, from vertices v where $\gamma(v) > \deg^+(v)$, has exactly $\gamma(v) - \deg^+(v)$ monochromatic transitions.

Thus, any eulerian circuit T in G'' is an eulerian circuit with exactly (†) monochromatic transitions. By construction the transitions in G'' are transitions in the original graph G. Thus T is an eulerian circuit in G with exactly (†) monochromatic transitions. $\square$

As in Section 2.6 there is a polynomial time algorithm to obtain an eulerian circuit with exactly (†) monochromatic transitions. The digraph G'' can be built in polynomial time, and any eulerian circuit of G'' has exactly (†) monochromatic transitions.

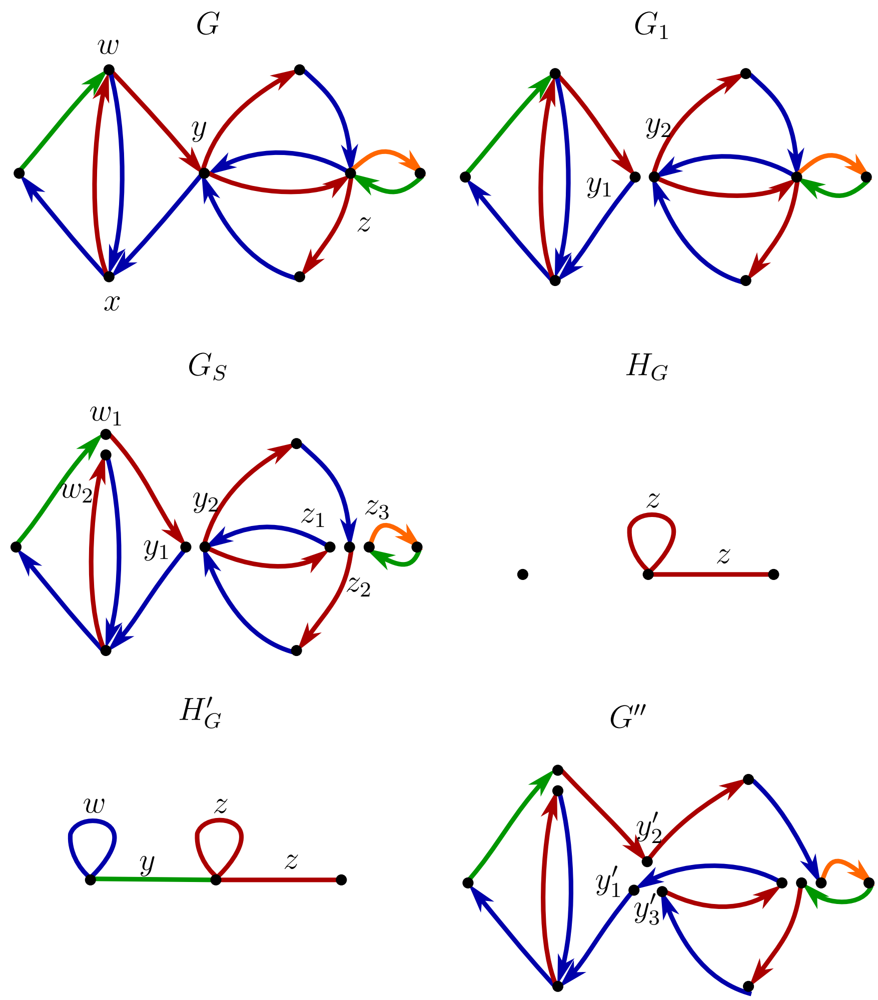

Figure 3.7: The graph G is a colored eulerian digraph with no compatible circuit. The auxiliary graphs G_1, G_S, H_G and H'_G are used to help construct the digraph G'', which is a colored eulerian digraph, where has an eulerian circuit with exactly (†) monochromatic transitions. In this example, the quantity from (†) is three.

3.5 Graphs with all vertices in S_3

In this section we investigate colored eulerian digraphs where all the vertices are in S_3. Recall that S_3 is the set of nonfixable vertices of outdegree three that have exactly three color classes, where each color class has one incoming edge and one outgoing edge. Throughout this section we consider edge-colorings of G, where the head and tail of each edge may have different colors. Notice that this is equivalent to the previous version by subdividing certain edges. Let ϕ denote the edge-coloring and for a directed edge uv, we write $\phi(uv) = (a, b)$ where a is the color of the tail and b is the color of the head. Up to renaming colors, we can assume that G is edge-colored using exactly the colors 1, 2 and 3. We are interested in which colorings of G have a compatible circuit and which coloring fail to have a compatible circuit when all the vertices of G are in S_3.

An eulerian circuit can be thought of as both a sequence of pair-wise incident edges in G, or as a collection of transitions at each vertex. First we introduce a formal definition of a transition.

Definition 3.23. A *transition* at v is a pair of incident edges at v consisting of one incoming edge and one outgoing edge. We write evh for a transition at v, where e is an incoming edge into v and h is an outgoing edge leaving v. A *transition system at* v is a collection of transitions at v, where each edge incident to v is in exactly one transition. A *transition system of* G is a collection of transition systems at v for all vertices v in $V(G)$. A transition system of G can be thought of as a function $f : E(G) \to E(G)$ that assigns an incoming edge e at v to an outgoing edge h at v, where evf is a transition in the transition system of G. We refer to such a function as a *transition function*.

A transition system determines a circuit decomposition of the graph G and vice versa. In particular every eulerian circuit T corresponds to a transition system of G, and hence we speak of the transitions of T.

Definition 3.24. A *pseudocompatible circuit* is an eulerian circuit T of a colored eulerian digraph G with all vertices in S_3, such that whenever there is a monochromatic transition e_1vh_1 centered at a vertex v, the other transitions e_2vh_2 and e_3vh_3 at v are also monochromatic transitions of T.

A pseudocompatible circuit is a weakening of the notion of a compatible circuit, but as we will see in the following lemma there is a strong connection between them.

Lemma 3.25. *Let G be a colored eulerian digraph with a pseudocompatible circuit T. Then G has a compatible circuit.*

Proof. The proof is by induction on the number of monochromatic transitions k in T. When $k = 0$, T is a compatible circuit.

Suppose $k > 0$. Let v be a vertex where all the transitions of T at v are monochromatic, and consider the excursion graph $L_T(v)$. Since there are three excursions, the excursion graph $L_T(v)$ has exactly two eulerian circuits. The excursions in the pseudocompatible circuit T correspond to an eulerian circuit of $L_T(v)$ with three monochromatic transitions. The other eulerian circuit of $L_T(v)$ has no monochromatic transitions. Replacing the transitions at v with these nonmonochromatic transitions in T gives a new eulerian circuit T' of G with fewer monochromatic transitions. □

Let G be an eulerian digraph with an edge coloring ϕ, and let v be a fixed vertex. We want to understand when changing only the colors of the tails of the outgoing edges of v maintains the property that G has a compatible circuit. Let ϕ' denote the new coloring obtained by changing the colors of the tails of the outgoing edges of v. For v to be in S_3 in ϕ', there is a permutation of the colors 1, 2 and 3 describing how to change the colors on the tails of the outgoing edges at v. The symmetric group on three elements describes the different ways to apply this type of permutation.

The symmetric group on three elements is isomorphic to the dihedral group of the triangle. We write $\mathcal{S}_3 = \{id, \rho_1, \rho_2, \tau_1, \tau_2, \tau_3\}$ for the symmetric group on $\{1, 2, 3\}$, where ρ_1 and ρ_2 are the elements of order 3 (rotations of the triangle) and τ_1, τ_2, and τ_3 are the elements of order 2 (reflections of the triangle). The following notation is inspired by the dihedral group of the triangle.

Definition 3.26. Let $\Gamma = \{id, \rho_1, \rho_2\}$ be the subgroup of size three in $\mathcal{S}_3$. A *rotation of ϕ at v* is a new edge-coloring ϕ' of G where all the edges have the same color except the tails of the edges outgoing from v are recolored by applying a fixed element of Γ.

Let $N = \{\tau_1, \tau_2, \tau_3\}$ be the set of permutations in the symmetric group with exactly one fixed point. A *reflection of ϕ at v* is a new edge-coloring ϕ' of G where all the edges have the same color except the tails of the edges outgoing from v are recolored by applying a fixed element of N.

Suppose we fix some permutation $\sigma \in \mathcal{S}_3$ and some vertex v in G, and recolor the heads of the incoming edges incident to v and the tails of the outgoing edges incident to v according to σ to obtain a new edge coloring ϕ'. Note that v has the same monochromatic transitions for both the edge colorings ϕ and ϕ', so the existence or nonexistence of a compatible circuit is preserved under permuting the colors incident to a vertex.

Observation 3.27. Up to recoloring edges incident to a vertex v, we can assume the head of the incoming edges for each coloring are the same for all edge colorings. Throughout the rest of the chapter we assume that the color of the head of each edge is fixed. Thus two colorings differ only on the color on the tails of the edges.

The next lemma shows that the existence of a compatible circuit is also preserved when rotating the edge coloring at a vertex.

Lemma 3.28. *Let G be a colored eulerian digraph where every vertex is a nonfixable vertex in S_3. Let ϕ be the starting edge coloring of G, v be an arbitrary vertex of G, and ϕ' be the edge coloring of G by applying a rotation of ϕ at v. Then G has a compatible circuit with edge coloring ϕ if and only if G has a compatible circuit with edge coloring ϕ'.*

Proof. Since rotations are invertible, we need only prove one direction of the statement. Let T be a compatible circuit of G with edge coloring ϕ. We claim that T is a pseudocompatible circuit of G with edge coloring ϕ'. Clearly T has no monochromatic transitions at vertices $w \neq v$.

Since T is a compatible circuit for ϕ, the edges incoming to v with heads colored 1, 2, and 3 in ϕ are matched to the outgoing edges with tail colors 2, 3, and 1 respectively; or 3, 2, 1 respectively. By applying a rotation at v to obtain ϕ', the incoming edges with heads colored 1, 2, and 3 are matched to the outgoing edges with tail colors, 1, 2, and 3 respectively; 2, 3, and 1 respectively; or 3, 2, 1 respectively. Thus for ϕ', the circuit T either has three monochromatic transitions at v or none. □

Given two edge colorings ϕ_1 and ϕ_2 of G (where ϕ_1 and ϕ_2 have the same color on the head of every edge), ϕ_1 can be transformed into ϕ_2 by applying a sequence of rotations and reflections. Lemma 3.28 shows that applying rotations preserves the existence (or nonexistence) of a compatible circuit. Therefore we can talk about classes of colorings, where the class of colorings either all have the property of having a compatible circuit or none of them have a compatible circuit.

Definition 3.29. An equivalence class is a set of colorings of G, where ϕ_1 and ϕ_2 are in the same equivalence class if the edge-coloring ϕ_1 can be obtained from ϕ_2 by a set of rotations at the vertices of G.

All the colorings in an equivalence class either all have a compatible circuit or none of them have a compatible circuit by Lemma 3.28. Observe that each equivalence class has the same number of colorings.

Lemma 3.30. *There is a bijection between the equivalence classes and the subsets of vertices of G. Hence the number of equivalence classes is 2^n.*

Proof. Let A_i and A_j be two different equivalence classes. Let $\phi_1 \in A_i$ and $\phi_2 \in A_j$ be two edge colorings. There is a series of reflections and rotations at vertices of G to transform ϕ_1 into ϕ_2. Let S_{ij} be the set of vertices that we introduced a reflection to ϕ_1 to obtain ϕ_2.

We argue that this set of vertices is the same for any representatives from A_i and A_j. Every coloring $\phi_3 \in A_j$ differs from ϕ_2 by rotations at vertices. Composing a rotation by a rotation is a rotation, and composing a rotation with a reflection is a reflection. Thus the set of vertices we need to introduce reflection to transform ϕ_1 into ϕ_3 is the set of vertices S_{12}. Similarly for any $\phi_4 \in A_i$ differs from ϕ_1 by rotations at vertices, so ϕ_4 differ by reflection at the same set of vertices. Thus the set S_{ij} is the same for any representatives from A_i and A_j.

Fix an equivalence class A_1 and a representative edge coloring $\phi \in A_1$. Let α_{A_1} be the function that maps the equivalence class A_1 to the empty set, and every other equivalence class A_j to the set of vertices S_{1j}. The map α_{A_1} is injective, since any two equivalence classes that differ from A_1 on the same set of vertices by reflections differ by rotations at all the vertices, thus they belong to the same equivalence class. The map α_{A_1} is surjective, since given any set S of vertices, introduce reflections at each vertex $v \in S$ gives a coloring in some equivalence class A_j, with $S_{1j} = S$. $\square$

This lemma shows that moving from one equivalence class to another occurs by adding reflections at a subset of the vertices.

Throughout the rest of this section we consider building auxiliary graphs, similar to flavor to Definitions 3.12 and 3.13. We use transitions functions from Definition 3.23 to help build auxiliary digraphs.

Definition 3.31. Let f be a transition function for a digraph G with all vertices in S_3. For each vertex v, there are three transitions, $e_i v h_i$ for $i \in \{1, 2, 3\}$. We form the graph G_f by replacing each vertex v with three new vertices v_i, where v_i is incident only to the edges e_i and h_i. Note that G_f is a collection of directed cycles that represents a circuit decomposition of G, and there is a natural bijection between the edges of G and G_f. Let $D_1, \ldots, D_r$ denote the connected components in G_f.

Define the component graph $H_{G,f,\phi}$ to be the graph obtained from G_f where the vertices of $H_{G,f,\phi}$ are the components of G_f, and for each vertex $v \in V(G)$, there is an edge labeled v between vertices D_k and D_ℓ in $H_{G,f,\phi}$ (k may equal ℓ), if v_i is in D_k and v_j is in D_ℓ for distinct $i, j \in \{1, 2, 3\}$. The *3-circuit with label* v is the set of three edges labeled v in $H_{G,f,\phi}$. We often refer to a 3-circuit without being specific about which label it receives. Note that a 3-circuit may include loops or double edges.

We call a 3-circuit with label v *dashed* if exactly one of the transitions at v_1, v_2 and v_3 in G_f is monochromatic and the other two transitions are not. Every 3-circuit that is not dashed is called *solid*. Note that a solid 3-circuit either has no monochromatic transitions or three monochromatic transitions.

Our goal is to use the auxiliary graphs to determine when a colored eulerian digraph with all vertices in S_3 has a compatible circuit, similar to Theorem 3.16. Notice that the edge coloring ϕ only determines if the 3-circuits are solid or dashed in $H_{G,f,\phi}$. In cases where we are working with a fixed edge coloring ϕ of G, we sometimes write $H_{G,f}$ instead of $H_{G,f,\phi}$. In fact any two colorings from the same equivalence class always give rise to the same component graph $H_{G,f}$.

Lemma 3.32. *Let ϕ_1 and ϕ_2 be two edge colorings in the same equivalence class and let f be any fixed transition function. Then H_{G,f,ϕ_1} and H_{G,f,ϕ_2} are the same graph.*

Proof. Recall that in the construction of $H_{G,f,\phi}$, that the coloring ϕ only affects whether the 3-circuit is dashed or solid. Since ϕ_1 and ϕ_2 are in the same equivalence class they differ by rotations at each vertex. Therefore by definition of solid and dashed 3-circuits, all the 3-circuits in H_{G,f,ϕ_1} and H_{G,f,ϕ_2} agree. $\square$

Next we define several definitions that discuss what happens when we change the transition function. Recall that the symmetric group of three elements was given by $\mathcal{S}_3 = \{id, \rho_1, \rho_2, \tau_1, \tau_2, \tau_3\}$, where id, ρ_1, and ρ_2 are rotations and τ_1, τ_2, and τ_3 are reflections..

Definition 3.33. A *rotation of f at v* is a new transition function f' where all the transitions remain the same except the transitions at v are reordered by some rotation.

A *reflection of f at v* is a new transition function f' where all the transitions are the same except the transitions at v are reordered by some reflection.

Note that if f' is obtained by a rotation of f at v, then the 3-circuit labeled v is either solid in both $H_{G,f}$ and $H_{G,f'}$, or dashed in both $H_{G,f}$ and $H_{G,f'}$. If f' is formed by a reflection at v of f, then the 3-circuit labeled v is dashed in one of $H_{G,f}$ and $H_{G,f'}$, and solid in the other.

Definition 3.34. A *3-circuit traversal* is a spanning connected subgraph E of $H_{G,f}$ such that there is exactly one edge from each dashed 3-circuit and for each solid 3-circuit either all or none of the edges in the 3-circuit are in E.

Suppose G has a compatible circuit T. We next define an important set of edges in $H_{G,f}$ that correspond to the compatible circuit T.

Definition 3.35. Let $T = e_1, e_2, \ldots, e_m$ be a pseudocompatible circuit in a colored eulerian digraph G with all vertices in S_3. Let f_T be the transition function given by $f(e_i) = e_{i+1}$ for $1 \leq i \leq m-1$ and $f(e_m) = e_1$. We refer to f_T as the transition function given by T.

The set $E_{T,f}$ represents the transitions of f_T that differ from f. Precisely, suppose $f_T(e_i) \neq f(e_i)$. Let v_i be the head of e_i and v_j be the tail of $f_T(e_i)$ in G_f, and note that v_i and v_j are distinct vertices. There is an edge x in $H_{G,f}$ corresponding to v_i and v_j in G_f. Whenever $f_T(e_i) \neq f(e_i)$, we add the edge x to $E_{T,f}$.

Lemma 3.37 below shows that given a pseudocompatible circuit T, the set $E_{T,f}$ is a 3-circuit traversal of $H_{G,f}$. To help prove Lemma 3.37 we prove the following result that tells us some of the structure of $E_{T,f}$.

Lemma 3.36. *Let T be a pseudocompatible circuit and f be a transition function of G. Then for each dashed 3-circuit in $H_{G,f}$ there is exactly one edge in $E_{T,f}$, and for each solid 3-circuit in $H_{G,f}$ either all or none of the edges in the 3-circuit are in $E_{T,f}$.*

Proof. A 3-circuit is dashed if there is exactly one monochromatic transition. Since there is no vertex in T with exactly one monochromatic transition, this implies there is at least one edge from each dashed 3-circuit in E_T. Checking the cases, we see that if there is more than one edge from a dashed 3-circuit, then T has exactly one monochromatic transition at the vertex v, but this contradicts that T is a pseudocompatible circuit.

In the case of a solid 3-circuit labeled v by checking cases we discover that having one or two edges of the solid 3-circuit in $E_{T,f}$ implies that T has exactly one monochromatic transition at v, which contradicts the fact that T is a pseudocompatible circuit. □

By Lemma 3.36 it is enough to show that $E_{T,f}$ is connected to prove that $E_{T,f}$ is a 3-circuit traversal.

Lemma 3.37. *Let T be a pseudocompatible circuit and f be a transition function of G. The set $E_{T,f}$ is a 3-circuit traversal in $H_{G,f}$.*

Proof. By Lemma 3.36 there is exactly one edge from each dashed 3-circuit in $E_{T,f}$ and either all of none of the edges from each solid 3-circuit. Therefore we want to show that the subgraph with edge set $E_{T,f}$ is connected in $H_{G,f}$. Since T visits all the edges of the graph G and G_f, it visits all the components of G_f. Each time the circuit T moves from one component D_j to another component D_k there is a corresponding edge in $E_{T,f}$ between the vertices D_j and D_k in $H_{G,f}$. Thus the subgraph with edge set $E_{T,f}$ is connected. □

The contrapositive of Lemma 3.37 tell us that if $H_{G,f}$ has no 3-circuit traversal, then G does not have a compatible circuit. The following lemma shows that we can determine in polynomial time whether $H_{G,f}$ has a 3-circuit traversal.

Lemma 3.38. *There exists a polynomial time algorithm that determines if $H_{G,f}$ contains a 3-circuit traversal, and produces a 3-circuit traversal if one exists.*

Proof. Let E denote all the edges in the solid 3-circuits in $H_{G,f}$. Since a 3-circuit traversal can contain all the edges form each solid 3-circuit, we may as well assume that E is part of the 3-circuit traversal if it exists. Consider the subgraph of $H_{G,f}$ with edge set E. We let E denote both the edge set and the subgraph.

If E is connected, then add an arbitrary edge from each dashed 3-circuit to obtain a 3-circuit traversal. If E is not connected, then form a new graph H_E by identifying

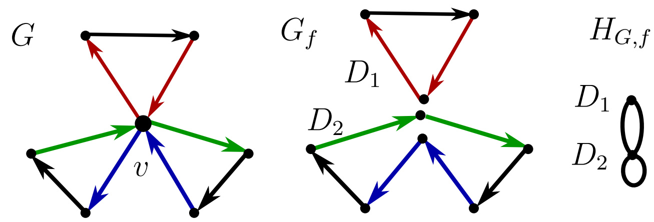

Figure 3.8: The graph G has no compatible circuit, but choosing f to match monochromatic edges together gives rise to $H_{G,f}$ that clearly has a 3-circuit traversal.

all the vertices in each component of E together into one vertex. The only edges between vertices in H_E are the dashed 3-circuit edges. Determining if there is a 3-circuit traversal is now a question of it we can pick one edge from each dashed 3-circuit such that the spanning subgraph is connected. This is precisely a problem of finding a rainbow spanning tree in H, which can be done in polynomial time as discussed in Section 2.1.1. $\square$

Lemma 3.37 shows there is a map M from pseudocompatible circuits to 3-circuit traversals in $H_{G,f}$ given by mapping T to $E_{T,f}$. We make two important observations about the map M. First, the map M is not surjective, as not every 3-circuit traversal necessarily has a corresponding pseudocompatible circuit. Figure 3.8 gives an example of a graph where $H_{G,f}$ has a 3-circuit traversal but G has no pseudocompatible circuit. Second, there can be several compatible circuits T and T' where $E_{T,f}$ and $E_{T',f}$ are the same 3-circuit traversal, so this mapping is not injective.

One important open question is whether given a graph G with no compatible circuit is there some choice for f such that $H_{G,f}$ does not have a 3-circuit traversal. If some transition function f always exists and can be found in polynomial time, then the problem of determining if a graph has a compatible circuit is in co-NP.

Next we define a more restrictive class of 3-circuit traversals that are contained in the image of the map M, and hence if they exist can be used to find a compatible circuit.

Definition 3.39. A *3-circuit tree traversal* is a spanning connected subgraph E of $H_{G,f}$ such that E is a 3-circuit traversal and the only cycles in the subgraph with edge set E are triangles where all the edges of each triangle have the same label. Note that such a triangle is a solid 3-circuit.

The existence of a 3-circuit tree traversal in $H_{G,f}$ is a sufficient condition for a colored eulerian digraph G to have a compatible circuit.

Lemma 3.40. *Let G be a colored eulerian digraph where all the vertices are in S_3, and let f be a transition function. If $H_{G,f}$ has a 3-circuit tree traversal, then G has a compatible circuit.*

Proof. By Lemma 3.25 it is enough to show that G has a pseudocompatible circuit.

Let E be a 3-circuit tree traversal of $H_{G,f}$. We proceed by induction on the number of vertices in $H_{G,f}$, which we denote by k. When $k = 1$, G_f has exactly one component. There are no dashed 3-circuits in $H_{G,f}$, otherwise E would contain a loop and not be a 3-circuit tree traversal. Since all the 3-circuits are solid, the transitions at each vertex v are all nonmonochromatic, or they all are monochromatic. Thus the one component of G_f is a directed cycle that corresponds to a pseudocompatible circuit in G.

Now suppose $k > 1$ and there is an edge with label v in E. We consider two cases based on if the 3-circuit with label v is dashed or solid.

In the case the 3-circuit labeled v is dashed there is exactly one edge e in E with label v. The edge e is a bridge in E, so in G_f there vertices v_i in component D_k and v_j in component D_k, where $k \neq \ell$. Suppose $e_1 v_i h_1$ and $e_2 v_j h_2$ are transitions in G_f. Let f' be a new transition function that has all the same transitions as f, except replaces

$e_1v_ih_1$ and $e_2v_jh_2$ with $e_1v_ih_2$ and $e_2v_jh_1$. By Lemma 3.19 the graph $G_{f'}$ has one less component than G_f. The graph $H_{G,f'}$ can be obtained from $H_{G,f}$ by identifying the vertices D_k and D_ℓ and changing the 3-circuit labeled v from dashed to solid. Therefore $E - \{g\}$, where g is the edge with label v, is a 3-circuit tree traversal of $H_{G,f'}$. By induction, since $H_{G,f'}$ has a 3-circuit tree traversal this implies that G has a pseudocompatible circuit.

In the case when the 3-circuit labeled v is solid there are three edges with label v in E that form a triangle. Suppose $e_1v_1h_1$, $e_2v_2h_2$, and $e_2v_3h_3$ are the transitions in f. Since the 3-circuit labeled v is a triangle, the vertices v_1, v_2, and v_3 are all in different components of G_f. Form a new transition function by replacing the above three transitions with $e_1v_1h_2$, $e_2v_2h_3$, and $e_3v_3h_1$. By Lemma 3.19 the graph $G_{f'}$ has fewer components than G_f. The graph $G_{f'}$ can be obtained from G_f by identifying the three components containing v_1, v_2, and v_3. Thus $H_{G,f'}$ has two fewer vertices, then $H_{G,f}$. The set $E-\{g_1, g_2, g_3\}$, where g_1, g_2, and g_3 are the edges of the 3-circuit with label v, is a 3-circuit tree traversal of $H_{G,f'}$. By induction, since $H_{G,f'}$ has a 3-circuit tree traversal this implies that G has a pseudocompatible circuit. $\square$

Lemma 3.40 shows that if we can find an appropriate function f such that $H_{G,f}$ has a 3-circuit tree traversal, then we know G has a compatible circuit. Not every choice of f gives rise to an appropriate transition function. Figure 3.9 gives an example of a graph G that has a compatible circuit, but $H_{G,f}$ does not contain a 3-circuit tree traversal. If G has a compatible circuit T, then there is an appropriate choice for a transition function, namely f_T. The component graph H_{G,f_T} has exactly one vertex and n solid 3-circuits, and so has a trivial 3-circuit tree traversal. The problem is finding a choice for f such that $H_{G,f}$ has a 3-circuit tree traversal. In general we do not know how to find such a function f, and this remains an open question. However,

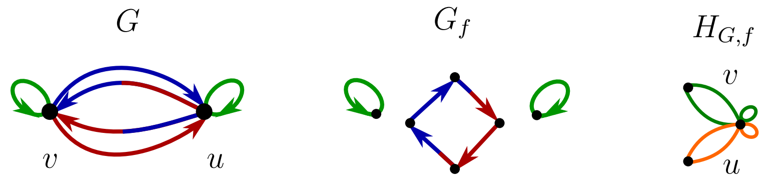

Figure 3.9: The graph G has a compatible circuit, but for the choice of f the graph $H_{G,f}$ does not have a 3-circuit tree traversal.

for some families of digraphs we can find an appropriate function f as discussed in the sections below.

In the next subsections we investigate specific functions f for certain colored eulerian digraphs G, and in many instances provide conditions for when a compatible exists or does not exist in G.

3.5.1 Half of the equivalence classes do not have a compatible circuit

In this section we consider a graph G with an edge-coloring where all the vertices are in S_3. We show that at least half the colorings fail to have compatible circuits.

Recall from the previous section that there are 2^n equivalence classes of edge-colorings. We will show that for at least 2^{n-1} of the equivalence classes, the edge-colorings in those equivalence classes do not have compatible circuits.

Let ϕ be an edge-coloring of G. Let f_ϕ be the transition function that maps each directed edge e_i to h_i, where $e_i v h_i$ is a monochromatic transition, i.e. the color on the head of e_i is the same as the tail color of h_i. Since each vertex is in S_3 this is a well defined map. The auxiliary graphs G_{f_ϕ} and H_{G,f_ϕ} can be used to show that certain edge-colorings of G do not have a compatible circuit.

Lemma 3.41. *Let G be an edge-colored eulerian digraph, where each vertex is in S_3. Let f be a transition function where every 3-circuit in $H_{G,f}$ is solid, and let v be a fixed vertex of G. Let f' and f'' be the two transition functions obtained from f by applying the two nontrivial rotations to f at v. The number of vertices in $H_{G,f}$, $H_{G,f'}$, and $H_{G,f''}$ all have the same parity.*

Proof. The proof is broken up into cases based on how many of the vertices v_1, v_2, and v_3 are in the same component. Suppose f has transitions $e_1v_1h_1$, $e_2v_2h_2$, and $e_3v_3h_3$; f' has transitions $e_1v_1h_2$, $e_2v_2h_3$, and $e_3v_3h_1$; and f'' has transitions $e_1v_1h_3$, $e_2v_2h_1$, and $e_3v_3h_2$.

Note that is is enough to show that the number of components in G_f, $G_{f'}$, and $G_{f''}$ have the same parity.

Case 1: The vertices v_1, v_2 and v_3 are all in the same component.

Up to renaming assume v_1, v_2 and v_3 appear in this order in the component D_k in G_f. We see that $G_{f'}$ reorders the directed cycle D_k and has the same number of components as G_f, and $G_{f''}$ splits D_k into three components. In either case the number of components in G_f, $G_{f'}$, and $G_{f''}$ have the same parity.

Case 2: The vertices v_1, v_2 and v_3 are all in different components.

Let P_i denote the path from h_i to e_i. The graph $G_{f'}$ combines the three components in G_f into one component given by $P_1P_2P_3$, and $G_{f''}$ combines the three components in G_f into one component given by $P_1P_3P_2$. Again the number of components in G_f, $G_{f'}$, and $G_{f''}$ have the same parity.

Case 3: Two of the vertices are in the same component and the other vertex is in a different component.

Without loss of generality assume v_1 and v_2 are in the component D_k and v_3 is in D_ℓ, where $k \neq \ell$ in G_f. Let P_1 denote the path from h_1 to e_2, P_2 be the path from

h_2 to e_1, and P_3 be the path from h_3 to e_3 in G_f.

The graph $G_{f'}$ replaces D_k and D_ℓ with the two components given by P_1P_3 and P_2, and the graph $G_{f''}$ replaces D_k and D_ℓ with the two components given by P_1 and P_2P_3. So the graphs G_f, $G_{f'}$, and $G_{f''}$ all have the same number of components. $\square$

Lemma 3.42. *Let G be an edge-colored eulerian digraph, where each vertex is in S_3. If the number of vertices in H_{G,f_ϕ} is even, then G does not have a compatible circuit.*

Recall that the number of vertices in H_{G,f_ϕ} is the number of components in G_{f_ϕ}.

Proof. Suppose that G has a compatible circuit T. We know that f_T is a transition function and H_{G,f_T} has one component with all solid 3-circuits.

The transitions in f_ϕ and f_T differ at each vertex by a rotation. By Lemma 3.41 the number of vertices in H_{G,f_ϕ} and H_{G,f_T} must have the same parity. However H_{G,f_ϕ} has an even number of vertices and H_{G,f_T} has one vertex, giving a contradiction. $\square$

Next we consider what happens to the parity of the number of components of G_{f_ϕ} when we apply a reflection to the edge coloring ϕ of G to obtain a new coloring ϕ'.

Lemma 3.43. *Let G have edge coloring ϕ and G' have edge coloring ϕ', where ϕ' is obtained from ϕ by adding a reflection at v for some vertex $v \in V(G)$. The number of components in G_{f_ϕ} and $G_{f_{\phi'}}$ have different parity.*

Note that G_{f_ϕ} is obtained from the graph G with edge coloring ϕ, and $G_{f_{\phi'}}$ is obtained from G with edge coloring ϕ'.

Proof. Suppose that $e_iv_ih_i$ are the transitions in f_ϕ given in G_{f_ϕ} at the vertex v. We can obtain the transition function $f_{\phi'}$ from f_ϕ by adding a reflection of f_ϕ at v. We consider the specific reflection that replaces the following transitions in G_{f_ϕ} $e_1v_1h_1$, $e_2v_2h_2$, and $e_3v_3h_3$ with the following transitions in $G'_{I'}$ $e_1v_1h_2$, $e_2v_2h_1$, and $e_3v_3h_3$.

Note that because of the reflection in of colors in ϕ', all the transitions are described above are monochromatic. The other two reflections have a similar proof.

The proof is broken up into cases based on whether the vertices v_1, v_2 and v_3 are in the same or different components. In each case we show that the number of components of G_{f_ϕ} and $G_{f_{\phi'}}$ always differ by one.

Case 1: The vertices v_1, v_2 and v_3 are all in the same component.

Without loss of generality the vertices v_1, v_2 and v_3 appear in this order in the components D_k in G_I. Let P_i denote the path from v_i to v_{i+1}, where addition is modulo three. We can obtain $G_{f_{\phi'}}$ from G_{f_ϕ} by replacing the component D_k with the components P_1 and P_2P_3.

Case 2: The vertices v_1, v_2 and v_3 are all in different components.

Let P_i denote the path from h_i to e_i. The graph $G_{f_{\phi'}}$ can be obtained from G_{f_ϕ} by replacing the three components with the two components given by P_1P_2 and P_3.

Case 3: Two of the vertices are in the same component and the other vertex is in a different component.

First consider the case when v_1 and v_2 are in the same component D_k in G_I. We can obtain $G_{f_{\phi'}}$ from G_{f_ϕ} by replacing D_k with the two components P_1 and P_2, where P_1 is the path from v_1 to v_2 and P_2 is the path from v_2 to v_1. Next we consider the case when v_1 and v_2 are in different components. Up to renaming we can assume v_1 and v_3 are in the same component D_k and v_2 is in D_ℓ in G_{f_ϕ}. Let P_1 be the path from h_1 to e_3, P_2 be the path from h_2 to e_2, and P_3 be the path from h_3 to e_1. The graph $G_{f_{\phi'}}$ can be obtained from G_{f_ϕ} by replacing D_k and D_ℓ with the single component $P_1P_3P_2$.

This proves that the parities of the number of components in G_{f_ϕ} and $G_{f'_\phi}$ is always different. $\square$

Combining Lemmas 3.42 and 3.43 we can prove that at least half the edge-colorings of G do not have a compatible circuit.

Corollary 3.44. *For every eulerian digraph G, where all vertices are in S_3, at least half the equivalence classes do not have the property of having a edge coloring with a compatible circuit.*

Proof. Fix an edge coloring ϕ in equivalence class A_1. We want to investigate how the parity of the number of components in G_{f_ϕ} changes by adding multiple reflections. By Lemma 3.30 we know that there is a bijection α_{A_1} between the number of reflections and each equivalence class. By Lemma 3.43 the edge colorings of the equivalence classes with an even number of reflections have the same parity of the number of components in G_{f_ϕ} as ϕ; and the equivalence classes with an odd number of reflection have a different parity of the number of components in G_{f_ϕ}.

Therefore half of the equivalence classes have an even number of components in G_{f_ϕ}, and by Lemma 3.42 none of the edge colorings in these equivalence classes have a compatible circuit. □

Recall that each equivalence class has the same number of edge colorings. Corollary 3.44 proves that at least half the edge colorings of G, where all vertices are in S_3, do not have a compatible circuit. There are graphs where exactly half the edge colorings have compatible circuits, but there are other examples where strictly fewer than half the edge colorings have a compatible circuit.

In the next two sections we investigate several specific families of colored eulerian digraphs that give rise to examples with strictly fewer than half the edge colorings have a compatible circuit. In these sections we use certain structural properties of G to pick a specific transition function f, and show that the auxiliary graph $H_{G,f}$ must have a 3-circuit tree traversal.

3.5.2 Digraphs obtained from cubic graphs

In this section we consider the family of eulerian digraphs obtained by taking a undirected loopless cubic graph F and creating a directed graph G by replacing each edge uv in F with directed edges uv and vu.

Consider the transition function γ that maps every edge uv to its antiparallel edge vu. The auxiliary graph G_γ is composed of 2-cycles corresponding to each of the edges of the cubic graph F. The 3-circuits in G_γ are in fact triangles, since vertices v_1, v_2, and v_3 can not be in the same components since F is loopless.

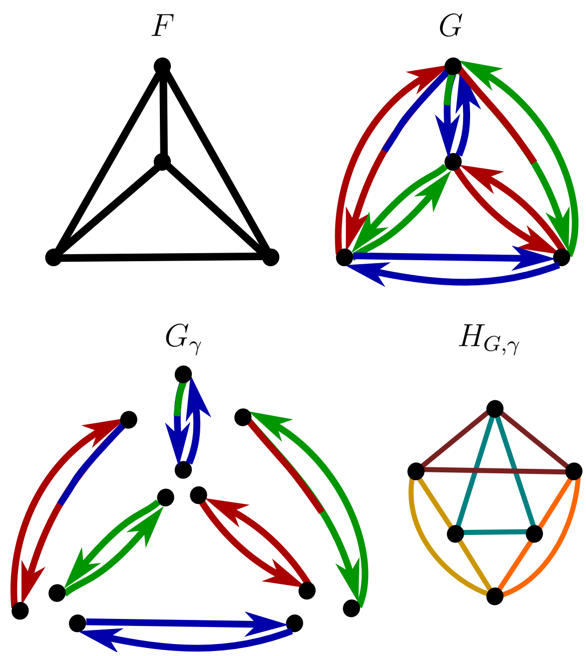

Figure 3.10: An example of a cubic graph F and the eulerian digraph G. Also shown are the auxiliary graphs G_γ and $H_{G,\gamma}$.

Theorem 3.45. *Let G be a colored eulerian digraph obtained from a cubic graph F where all vertices are in S_3 in G. The digraph G has a compatible circuit if and only if the component graph $H_{G,\gamma}$ has a 3-circuit tree traversal.*

Proof. (⇐) If $H_{G,\gamma}$ has a 3-circuit tree traversal, then G has a compatible circuit by Lemma 3.40.

(⇒) Suppose G has a compatible circuit. Our goal is to show that there is a compatible circuit T where $E_{T,\gamma}$ is in fact a 3-circuit tree traversal in $H_{G,\gamma}$.

Out of all pseudocompatible circuits T of G, select the one where $E_{T,\gamma}$ has as few edges as possible. In a slight abuse of notation we let $E_{T,\gamma}$ denote both the 3-circuit traversal and the subgraph of $H_{G,\gamma}$ with edge set $E_{T,\gamma}$.

Suppose there is a cycle D in $E_{T,\gamma}$ besides the solid 3-circuits. Let $D = D_1 \dots D_k$ denote a smallest cycle in $E_{T,\gamma}$ where the edges in D do not all have the same label, and the D_i are vertices in $H_{G,\gamma}$. First we make the observation that D contains at most one edge from each 3-circuit since D is a smallest cycle.

First we show that D must have at least one edge from a solid 3-circuit. Suppose that the edges of D come only from dashed 3-circuits. Each vertex D_i in D is adjacent to exactly two dashed 3-circuits, where both 3-circuits have an edge in D. Thus the cycle D is not connected in $E_{T,\gamma}$ to any vertices outside of $D_1, \dots, D_k$. Therefore D must be a hamiltonian cycle in $H_{G,\gamma}$ since $E_{T,\gamma}$ is connected. Since D is a hamiltonian cycle, fixing the transitions at all the components does not give rise to an eulerian circuit but two disconnected circuits, as shown in Figure 3.11. This contradicts that T is an eulerian circuit.

Thus D contains at least one edge from a solid 3-circuit. Up to renaming suppose that v is the label of a solid 3-circuit that appears between the components D_k, D_1 and B, where $B \notin \{D_1, D_2, \dots, D_k\}$.

Suppose that e_1vh_1, e_2vh_2, and e_3vh_3 are the transitions in γ at v. Note that up to renaming we can assume $e_1v_1h_1$ is in a component D_1 and $e_2v_2h_2$ is in a component D_k in G_γ. Let α be the transition function given by rotating the transitions of f_T at v to obtain e_ivh_i. Note that the permutation applied at v is a rotation since the

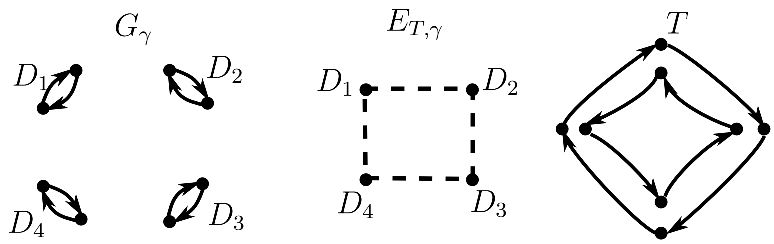

Figure 3.11: Above is a drawing of G_γ and $E_{T,\gamma}$ when D is a cycle from only dashed 3-circuit edges. and the transitions given for T. The dashed edges represent the cycle D in $H_{G,\gamma}$. As we can see when D is a hamiltonian cycle the result is not an eulerian circuit, but two disconnected directed cycles.

3-circuit labeled v is solid. The transition function can not give rise to an eulerian circuit, since this would be a new psuedocompatible circuit T' where $E_{T',\gamma}$ has fewer edges than $E_{T,\gamma}$. Thus the graph G_α has at least two components. Since α differs from f_T by a rotation, we know that G_α must have three components and the solid 3-circuit labeled v is a triangle between the components. Let P_1, P_2, and P_3 be the components of G_α. We will use the components P_1, P_2, and P_3 to find a new pseudocompatible circuit.

The transitions e_ivh_i are in both γ and α. We follow the circuit P_1 in the graph G_γ, and in particular we are going to consider what happens in the components $D_1, D_2, \ldots, D_k$. The transition $e_1v_1h_1$ is in the component D_1, so both of those edges are in P_1.

Assume the edges in D_i in G_γ are in P_1 for some fixed $i \geq 1$. If the edge from D_iD_{i+1} is dashed in $E_{T,\gamma}$, then the edges in the component D_{i+1} in G_γ are also edges in P_1, as shown in Figure 3.12. Next we consider the case when the edge D_iD_{i+1} is solid. Let the 3-circuit containing D_iD_{i+1} have label w. Since the 3-circuit labeled w is in $E_{T,\gamma}$ we know the transitions of γ at w are not the same as the transitions of f_T and α at w. Therefore at least four of the edges in D_i, D_{i+1}, and B' are in P_1, where

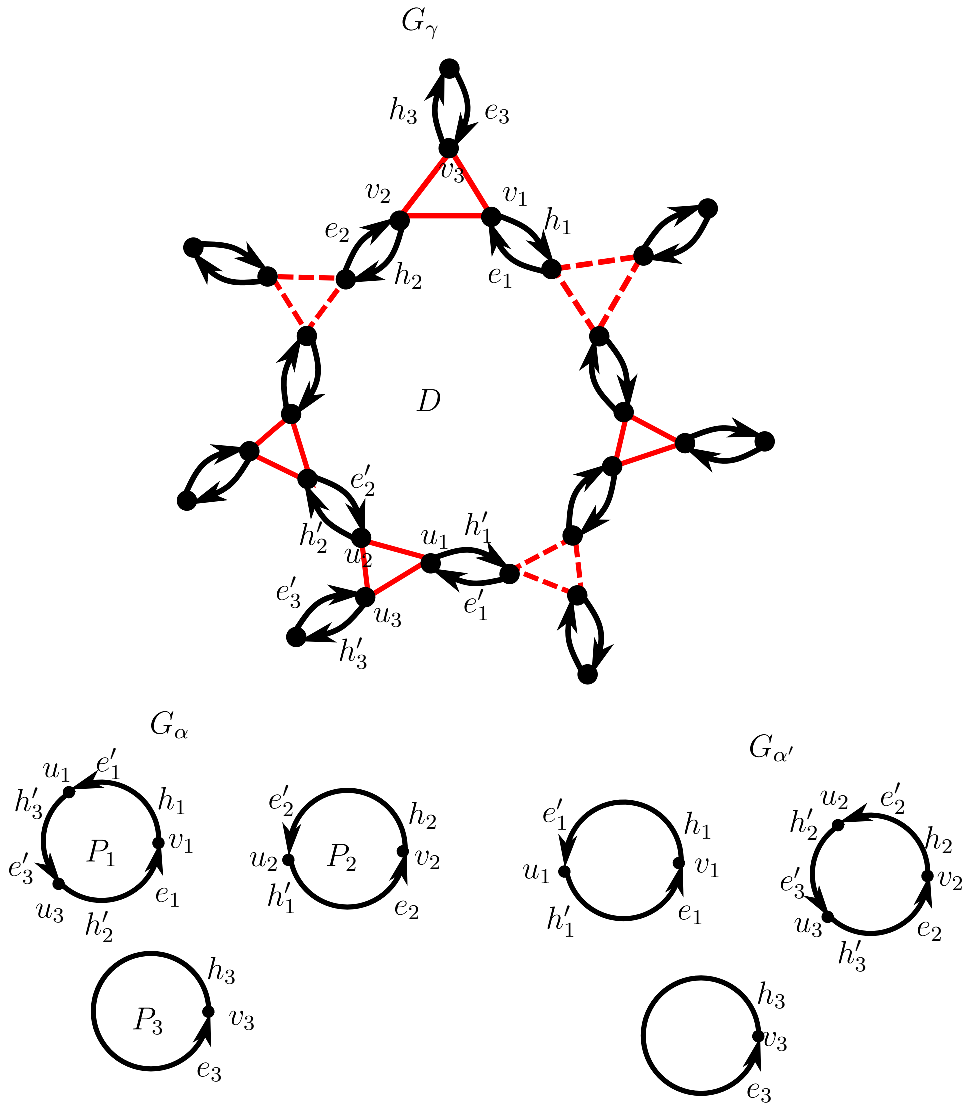

Figure 3.12: The figure above shows the graph G'_α, where the red triangles are the 3-circuits in $H_{G,\gamma}$. As we move along the 2-cycles in the components $D_1, \ldots, D_k$ there is a solid 3-circuit labeled u, where changing the transitions of u of α results in the transition function α'. Changing the transitions of $e_i v h_i$ in α' results in an eulerian circuit T'.

B' is a component containing a w_j. Let u be the first vertex label of a solid 3-circuit where D_i, D_{i+1}, and B' have exactly four edges in P_1 and the other two edges are in P_i for $i \in \{2, 3\}$. We know u must exist since D_k contains $e_2 v_2 h_2$ which is a transition in P_2.

Therefore two of the transitions of u are in P_1 and the other transition is in P_i for $i \in \{2, 3\}$. Let α' be the transition function obtained from α by rotating the transitions at u such that they agree with the transitions of γ. As we see in Figure 3.12 the graph $G_{\alpha'}$ has three components, and by changing the transitions at v to agree with the transitions of f_T we obtain an eulerian circuit T'.

The eulerian circuit T' has all the same transitions as T except for the transitions at u. The 3-circuit traversal $E_{T',\gamma}$ has fewer edges than $E_{T,\gamma}$, since $E_{T',\gamma}$ has all of the edges as $E_{T,\gamma}$ except the 3-circuit labeled u. This contradicts the minimality of T, which proves that $E_{T,\gamma}$ must have no nontrivial cycles. □

3.5.3 Planar digraphs

Next we show that certain edge-colored planar digraphs have a compatible circuit if and only if the associated component graph $H_{G,\psi}$ has a 3-circuit tree traversal. Suppose G is an edge-colored planar eulerian digraph where all the vertices are in S_3 and each face is bounded by a directed circuit. (Since G can have cut vertices the boundary of a face may be a circuit and not a cycle.) A *clockwise oriented face* is an internal face if its bounding circuit is oriented clockwise, or the outer face whose bounding circuit is oriented counterclockwise. Similarly a *counterclockwise oriented face* is an internal face if its bounding circuit is oriented counterclockwise, or the outer face whose bounding circuit is oriented clockwise. Note that if G is considered as an undirected planar graph, the dual of G with this embedding is bipartite, where the clockwise oriented faces and counterclockwise oriented faces form the two parts.

Every edge e in G borders two distinct faces, one of which is oriented clockwise and the other counterclockwise. Let ψ be the transition function that maps the edge

e to the edge f that follows e in the clockwise oriented face bordering e. Note that the components of the graph G_ψ are the clockwise oriented faces of G. See Figure 3.13 for an example.

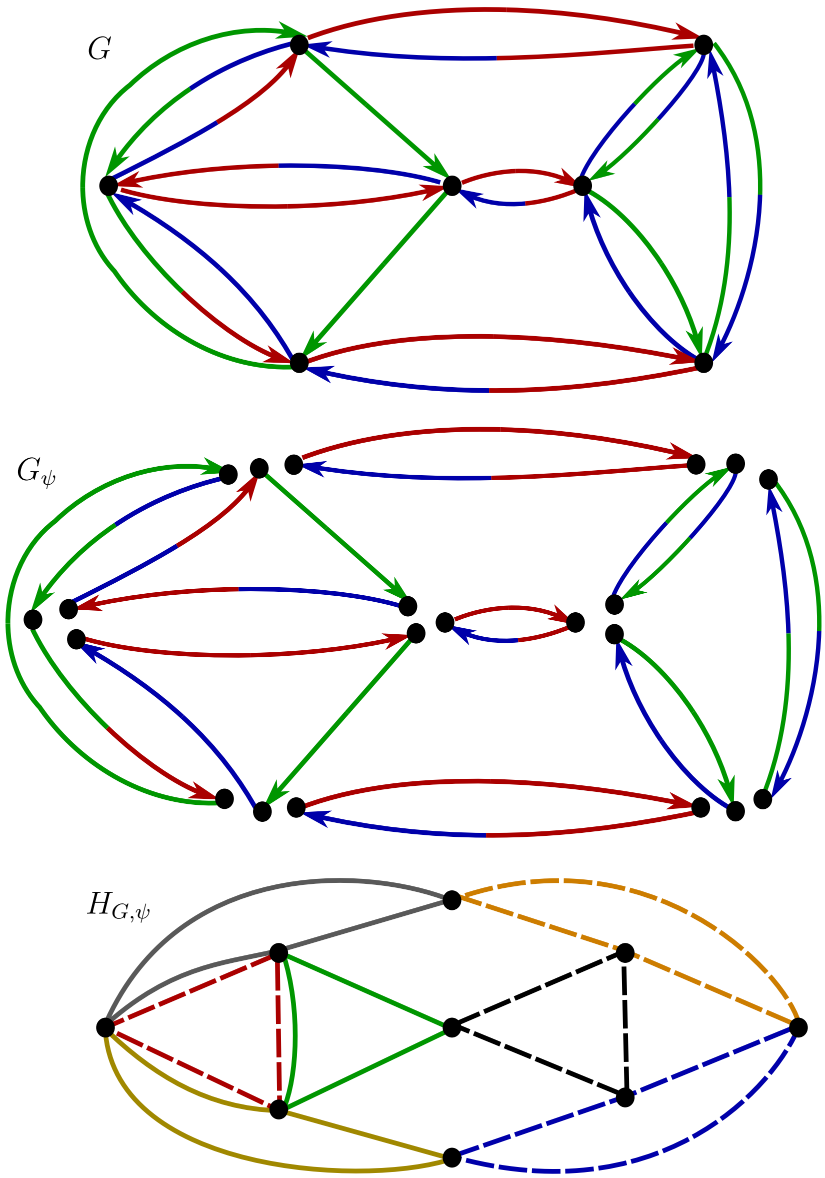

Figure 3.13: An example of a planar digraph where each face is a cycle. Less than half the edge colorings of this graph have a compatible circuit. The coloring above does not have a compatible circuit since $H_{G,\psi}$ does not have a 3-circuit tree traversal.

Theorem 3.46. *Let G be an edge-colored eulerian planar digraph where all the vertices are in S_3 and each face is bounded by a directed circuit. The digraph G has a compatible circuit if and only if the component graph $H_{G,\psi}$ has a 3-circuit tree traversal.*

Proof. ($\Leftarrow$) If $H_{G,\psi}$ has a 3-circuit tree traversal, then G has a compatible circuit by Lemma 3.40.

($\Rightarrow$) Suppose G has a compatible circuit. We create a new edge coloring ϕ' from the edge coloring ϕ of G as follows. For every solid 3-circuit labeled v in $H_{G,\psi}$, we rotate the colors on the tails of the outgoing edges of v such that the color of the head of each incoming edge is the same as the outgoing edge that does not share a face with the incoming edge, as in Figure 3.14. Note that with the edge coloring ϕ' the only nonmonochromatic transitions either follow the clockwise or counterclockwise oriented faces at the vertices whose corresponding 3-circuits are solid in $H_{G,\psi,\phi'}$.

Since ϕ' differs from ϕ by rotations, Lemma 3.28 implies that G with edge coloring ϕ' also has a compatible circuit. Lemma 3.32 shows that the graphs $H_{G,\psi,\phi}$ and $H_{G,\psi,\phi'}$ are the same graph. We will show that there is a 3-circuit tree traversal in $H_{G,\psi,\phi'}$, which implies $H_{G,\psi,\phi}$ also has a 3-circuit tree traversal.

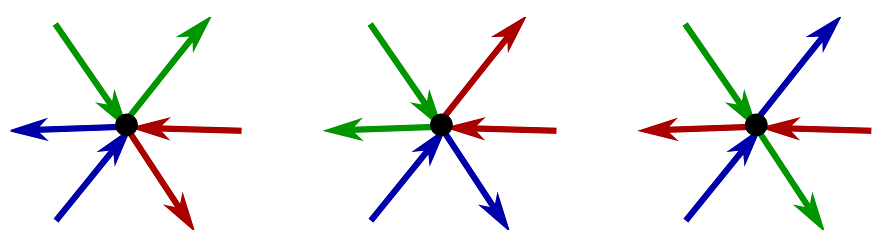

Figure 3.14: Rotate the outgoing colors of the tails such that they are colored as the figure on the left. The case when the incoming edge colors are red, green, and blue moving counterclockwise around the vertex v is similar.

Choose T to be a compatible circuit of G with the edge coloring ϕ' such that T

minimizes the number of edges in the 3-circuit traversal $E_{T,\psi}$ in $H_{G,\psi,\phi'}$. Notice that we are only considering compatible circuits and not pseudocompatible circuits. We claim that for this choice of T that $E_{T,\psi}$ is a 3-circuit tree traversal in $H_{G,\psi,\phi'}$.

Suppose $E_{T,\psi}$ is not a 3-circuit tree traversal. Let $D = D_1 D_2 \ldots D_k$ be the smallest cycle in $E_{T,\psi}$ that is not a solid 3-circuit. Let $V(D)$ denote the set of vertices of G that appear as labels on the edges of D.

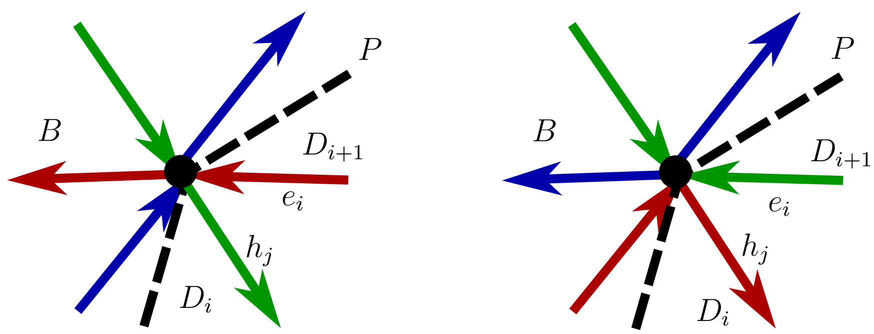

Figure 3.15: The vertex on the left corresponds to a solid 3-circuit, and the figure shows the curve P through the faces D_i and D_{i+1}. The vertex on the right corresponds to a dashed 3-circuit, and again the figure shows the curve P. Notice that in both cases there is no transition in T that crosses P.

There exists a closed planar curve P that does not cross any edge (except at the endpoints) of the planar embedding of G, moves through the faces D_1, D_2, $\ldots$, D_k, and passes through each of the vertices in $V(D)$. Note that there are edges both inside the region enclosed by P, and edges outside P, as shown in Figure 3.15.

We claim that the edges inside P and outside P are not connected in T, contradicting that T is an eulerian circuit. We say a transition evh *crosses* the curve P if one of the edges e or h is inside the region enclosed by P and the other edge is outside P.

Consider the edge $D_i D_{i+1}$ with label $v \in V(D)$ in the cycle D. Since each face is a directed circuit, the incoming and outgoing edges around each vertex alternate.

Thus there is a counterclockwise oriented face B incident to v and is adjacent to both of the faces D_i and D_{i+1}. We claim the two edges in G bordering the face B incident to v must form a transition in T. When the 3-circuit labeled v is dashed, the incoming edge into v adjacent to the face D_i goes to the outgoing edge of v adjacent to the face D_{i+1}, and the incoming edge into v adjacent to the face D_{i+1} goes to the outgoing edge of v adjacent to D_i. Therefore there is no transition across the curve P. When the 3-circuit labeled v is solid, the only nonmonochromatic transitions follow the clockwise oriented faces or follows the counterclockwise oriented faces, by how we choose the edge coloring ϕ'. Since the solid 3-circuit labeled v is in $E_{T,\psi}$ the transitions follow the counterclockwise oriented faces, which implies there is no transition that crosses the curve P. See Figure 3.15 for an illustration of these cases.

Since this argument holds for all $v \in V(D)$, there is no transition in T that crosses P, and hence T cannot reach both the edges of G inside P and the edges outside P. This contradicts that T is an eulerian circuit, and thus the subgraph $E_{T,\psi}$ must not have any cycles. □

3.5.4 Polynomial time algorithm to find a 3-circuit tree traversal

In this section we show that we can determine in polynomial time whether the auxiliary graph $H_{G,f}$ contains a 3-circuit tree traversal by considering spanning trees in a related 3-uniform hypergraph.

Definition 3.47. A *cycle* in a hypergraph H is a sequence of vertices and incident edges that start and end at the same vertex, i.e. $v_1, e_1, v_2, e_2, \ldots, v_k = v_1$, where v_i and v_{i+1} are vertices contained in the edge e_i. A *spanning tree* of H is a connected spanning subgraph having no cycles.

Lovász [41, 42] showed that there is a polynomial time algorithm to determine if a 3-uniform hypergraph has a spanning tree, and the algorithm provides a spanning tree if one exists.

There is a strong connection between spanning trees in 3-uniform hypergraphs and finding a 3-circuit tree traversal in $H_{G,f}$. The following definition constructs a hypergraph $H^*_{G,f}$ that is closely related to $H_{G,f}$.

Definition 3.48. Let $H_{G,f}$ be the auxiliary graph for an edge-colored eulerian digraph G where all vertices are in S_3, and let f be a transition function for G. Assume that $H_{G,f}$ has no dashed 3-circuit that consists of three loops. Let $H^*_{G,f}$ be the hypergraph with vertex set $V(H^*_{G,f}) = V(H_{G,f}) \cup \{u_v : \text{the 3-circuit labeled } v \text{ is dashed}\}$, i.e. we add a vertex for each dashed 3-circuit. The edges of H are given as follows: for each solid 3-circuit in $H_{G,f}$ that is a triangle in $H_{G,f}$, there is an edge in $H^*_{G,f}$; for each dashed 3-circuit with label v between only two vertices x and y in $H_{G,f}$, the hypergraph $H^*_{G,f}$ has the edge xyu_v; and for each dashed 3-circuit with label v between three vertices x, y and z in $H_{G,f}$, the hypergraph $H^*_{G,f}$ has the edges xyu_v, xzu_v, and yzu_v.

We make several observations about the hypergraph $H^*_{G,f}$. If $H_{G,f}$ has a dashed 3-circuit with three loops, then we immediately know that there is no 3-circuit tree traversal. The hypergraph $H^*_{G,f}$ is defined only if $H_{G,f}$ does not contain a dashed 3-circuit that consists of three loops. Every solid 3-circuit not in a triangle is never an edge in a 3-circuit tree traversal, so those edges are ignored when forming $H^*_{G,f}$.

Lemma 3.49. *Suppose the graph $H_{G,f}$ does not have a dashed 3-circuit with three loops. The graph $H_{G,f}$ has a 3-circuit tree traversal if and only if the 3-uniform hypergraph $H^*_{G,f}$ has a spanning tree.*

Proof. Suppose $H_{G,f}$ has a 3-circuit tree traversal E. We construct the following set T of edges of $H^*_{G,f}$. For each solid 3-circuit in E add the corresponding edge to T, and for each edge xy in E from a dashed 3-circuit with label v add the edge xyu_v to T. Since $H_{G,f}$ is connected, we know T spans at least the vertices in $H_{G,f}$. Further, since E contains one edge from each dashed 3-circuit we know the vertices in $\{u_v$: the 3-circuit labeled v is dashed$\}$ are also connected in T. Therefore the subgraph in $H^*_{G,f}$ with edge set T is connected.

Note the vertices u_v are in exactly one edge in T, so they do not belong to any cycle. Thus since E is a 3-circuit tree traversal there are no cycles in the subgraph with edge set T in $H^*_{G,f}$, hence it is a spanning tree.

If $H^*_{G,f}$ has a spanning tree T, then we have a corresponding set of edges in $H_{G,f}$ found in the following way: for each edge corresponding to a solid 3-circuit add that edge to E; for each edge xyu_v in T pick an edge in $H_{G,f}$ between the vertices x and y to be in E. Note when the dashed 3-circuit is between only two vertices there are two choices. The set E is a set of edges in $H_{G,f}$ that are connected and contain no cycles, except for the solid 3-circuits. Thus E is a 3-circuit tree traversal. $\square$

Corollary 3.50. *There is a polynomial time algorithm that determines if $H_{G,f}$ has a 3-circuit tree traversal, and provides a 3-circuit tree traversal if one exists.*

These results give rise to a polynomial time algorithm for determining if G has a compatible circuit when G for the eulerian digraphs discussed in Sections 3.5.2 and 3.5.3.

Algorithm

Input: A colored eulerian digraph G where all the vertices are in S_3 and G is either constructed from a cubic graph or a planar digraph where all the faces are directed cycles.

Output: Compatible circuit of G or a certificate which shows no such circuit exists.

Step 1: Construct the auxiliary graphs G_f and $H_{G,f}$, where $f = \gamma$ if G is constructed from a cubic graph and $f = \psi$ if G is a planar digraph where each face is a directed cycle.

Step 2: If there is a dashed 3-circuit with three loops, then $H_{G,f}$ can not have a 3-circuit tree traversal. Return $H_{G,f}$ and the dashed 3-circuit as a certificate. Otherwise, construct the 3-uniform hypergraph $H^*_{G,f}$ from $H_{G,f}$, and determine if $H^*_{G,f}$ has a spanning tree T. If $H^*_{G,f}$ does not have a spanning tree, then H is a certificate that G does not have a compatible circuit.

Step 3: If $H^*_{G,f}$ has a rainbow spanning tree, then construct the corresponding 3-circuit tree traversal E.

Step 4: Changing transition according to the 3-circuit tree traversal gives us a pseudocompatible circuit T'. For each vertex that has three monochromatic transitions, we can rearrange the excursions to finally obtain a compatible circuit T.

Each of these steps can be completed in polynomial time. This provides a polynomial time algorithm for certain families of graphs where all the vertices are in S_3. An open question is whether there is a polynomial time algorithm to determine if G has a compatible circuit when all the vertices are in S_3.

3.6 General compatible circuits

Let G be an eulerian digraph. An edge e is incident to an edge h if $e = uv$ and $h = vw$ for some vertices u, v, and w. The *acceptable transition set* for an edge e is a nonempty set of incident edges to e. We let $A(e)$ denote the acceptable transition set for the edge e. Throughout the rest of this section we assume that G is an eulerian digraph where all the edges have been assigned acceptable transition sets.

A *compatible circuit* is an eulerian circuit of G, call it $T = e_1, e_2, \ldots, e_m$, such that $e_{i+1} \in A(e_i)$ for $1 \leq i \leq m-1$ and $e_m \in A(e_1)$.

Definition 3.51. A vertex v is *fixable* if $L_M(v)$ has a compatible circuit for every matching M between $E^+(v)$ and $E^-(v)$.

Proposition 3.52. *Let G be a colored eulerian digraph. If all the vertices of G are fixable then G has a compatible circuit.*

Next we give a sufficient condition when a vertex is fixable. As in the case with edge-coloring we use Meyniel's Theorem.

Proposition 3.53. *Let v be a vertex in G. If for each incoming edge e_i^- we have $|A(e_i^-)| \leq \frac{\deg^+(v)}{2} + 1$, and for each outgoing edge e_i^+ we have e_i^+ appears in at least $d^+(v)/2 + 1$ acceptable transition sets, then the vertex v is fixable.*

Proof. Consider the excursion graph $L_M(v)$ for some vertex v and an arbitrary matching M. Let H be a directed graph whose vertex set is given by the excursions, and there is a directed edge S_iS_j between the vertices S_i and S_j in H if $i \neq j$ and $e_j^+ \in A(e_i^-)$. A hamiltonian cycle in H corresponds to a compatible circuit in $L_M(v)$. Note that H is a graph on $d^+(v)$ vertices.

We use Meyniel's Theorem to show that H has a Hamiltonian cycle. Let S_i and S_j be nonadjacent vertices in H. Then we know $e_j^+ \notin A(e_i^-)$ and $e_i^+ \notin A(e_j^-)$. We also know that $d_H^+(S_i)$ is at least $d^+(v)/2$, since $|A(e_i^-)| \geq d^+(v) + 1$ (note $e_i^+ \in A(e_i^-)$, but is not an edge in H). We also have $d_H^-(S_i)$ is at least $d^+(v)/2$, since e_i^+ is in at least $d^+(v)/2 + 1$ lists.

Thus,

$$d^+(S_i) + d^-(S_i) + d^+(S_j) + d^-(S_j) \geq 2d^+(v) \geq 2|V(H)| - 1$$

and so by Meyniel's Theorem the digraph H has a hamiltonian cycle. Since M was arbitrary, the graph $L_M(v)$ has a compatible circuit for all matchings M. This proves that v is fixable. □

Note that Proposition 3.53 gives sufficient conditions for when a vertex is fixable, but they are not necessary. One open question is providing necessary and sufficient conditions for when a vertex is fixable. Another related question is to determine if providing a lower bound on the size of the number allowed edges would imply a vertex is fixable.

3.6.1 Finding a general compatible circuit is NP-complete

In this section we show that determining if an eulerian digraph G with arbitrary allowed transitions has a compatible circuit is NP-complete. We reduce our problem to 3-SAT, which is a well known NP-complete problem. To show this reduction we first describe an eulerian digraph where each edge has a list of allowed transitions where compatible circuits will be related to satisfying assignments to a 3-SAT formula Φ.

Let the variables be $y_1, \ldots, y_r$ and their negations $\overline{y_1}, \ldots, \overline{y_r}$. We let x_i denote a *literal* which is an arbitrary element in $\{y_1, \ldots, y_r\} \cup \{\overline{y_1}, \ldots, \overline{y_r}\}$. Consider a 3-SAT formula Φ composed of k clauses, where each clause has the form $C_\ell = (x_{i_\ell} \vee x_{j_\ell} \vee x_{k_\ell})$. We may assume a clause C_ℓ does not contain both x_{i_ℓ} and $\overline{x_{i_\ell}}$ since otherwise such a clause is always true. We assume every clause has three distinct variables. Otherwise the clause $C_\ell = (x_{i_\ell} \vee x_{j_\ell} \vee x_{k_\ell})$ where $x_{i_\ell} = x_{j_\ell}$ is equivalent to the clauses $(x_{i\ell} \vee x_{k_\ell} \vee y) \wedge (x_{i_\ell} \vee x_{k_\ell} \vee \overline{y})$, where y is some new variable.

Definition 3.54. Let Φ denote a 3-SAT formula where there are k clauses and each clause contains three distinct variables. Let x_{i_ℓ}, x_{j_ℓ}, and x_{k_ℓ} be the literals in the clause C_ℓ, where x_{a_ℓ} corresponds to the variable y_{a_ℓ} or $\overline{y_{a_\ell}}$ for $a \in \{i, j, k\}$.

Let G_Φ denote the graph with vertex set $\{v_i : \text{for each variable } y_i\} \cup \{w\}$, and note that v_i corresponds to both the variable y_i and its negation $\overline{y_i}$. The graph G_Φ has for each clause $C_\ell = (x_{i_\ell} \vee x_{j_\ell} \vee x_{k_\ell})$ the following nine edges: $e_{i_\ell j_\ell} = v_{i_\ell} v_{j_\ell}$, $e_{j_\ell k_\ell} = v_{j_\ell} v_{k_\ell}$, $e_{k_\ell i_\ell} = v_{k_\ell} v_{i_\ell}$, $f_{wi_\ell} = wv_{i_\ell}$, $f_{i_\ell w} = v_{i_\ell} w$, $f_{wj_\ell} = wv_{j_\ell}$, $f_{j_\ell w} = v_{j_\ell} w$, $f_{wk_\ell} = wv_{k_\ell}$, and $f_{k_\ell w} = v_{k_\ell} w$. See Figure 3.16 for a picture of the nine edges in G_Φ corresponding to one clause. Here we assume that the literal x_{i_ℓ} corresponds to the variable y_{i_ℓ} or $\overline{y_{i_\ell}}$.

Next we define the acceptable transitions for each incoming edge into a vertex $v \in V(G_\Phi)$. The acceptable transitions for an incoming edge $f_{a_\ell w}$ into w are all outgoing edges, i.e. $A(f_{a_\ell w})$ is all outgoing edges of w. Suppose the variable y_i appears only in the clauses $C_{\ell_1}, \ldots, C_{\ell_t}$ and $\overline{y_i}$ appears only in the clauses $C_{\ell_{t+1}}, \ldots, C_{\ell_s}$. Up to renaming we assume that the literal x_{i_ℓ} always corresponds to y_i or $\overline{y_i}$. The acceptable transitions for each incoming edge into v_i in G_Φ is given by $A(f_{wi_\ell}) = \{f_{i_\ell w}, e_{i_\ell j_\ell}\}$ and $A(e_{k_\ell i_\ell}) = \{f_{i_{\ell+1} w}, e_{i_\ell j_\ell}\}$ for $1 \leq k \leq t$; and $A(f_{wi_\ell}) = \{f_{i_{\ell+1} w}, e_{i_\ell j_\ell}\}$ and $A(e_{k_\ell i_\ell}) = \{f_{i_\ell w}, e_{i_\ell j_\ell}\}$ for $t + 1 \leq k \leq s$.

Lemma 3.55. *There are exactly two allowed transition systems at each vertex v_i in G_Φ. The first has the transitions $e_{k_\ell i_\ell} e_{i_\ell j_\ell}$ and $f_{wi_\ell} f_{i_\ell w}$ for $1 \leq i \leq t$; and $e_{k_\ell i_\ell} f_{i_\ell w}$ and*

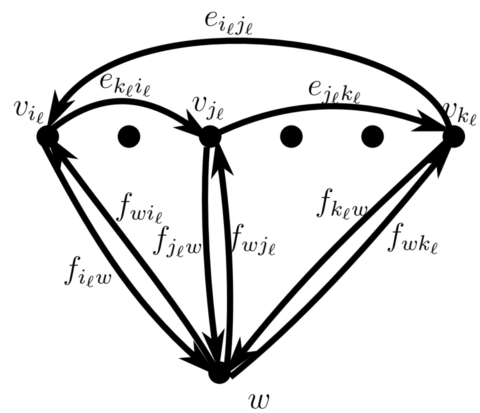

Figure 3.16: The figure shows the nine edges in G_Φ corresponding to the clause $C_\ell = (x_{i_\ell} \vee x_{j_\ell} \vee x_{k_\ell})$.

$f_{wi_\ell}e_{i_\ell j_\ell}$ for $t+1 \leq i \leq s$; and the second has $e_{k_\ell i_\ell}f_{i_{\ell+1}w}$ and $f_{wi_\ell}e_{i_\ell j_\ell}$ for $q \leq i \leq t$ and $e_{k_\ell i_\ell}e_{i_\ell j_\ell}$ and $f_{wi_\ell}f_{i_{\ell+1}w}$ for $t+1 \leq i \leq s$.

Proof. We prove that once a transition at v_i is selected, then all the rest are determined. Note the following implications:

$$\begin{aligned}
e_{wi_s}f_{i_1w} &\Rightarrow f_{wi_1}e_{i_1j_1} \Rightarrow e_{k_1i_1}f_{i_2w},\\
e_{k_\ell i_\ell}f_{i_{\ell+1}w} &\Rightarrow f_{wi_{\ell+1}}e_{i_{\ell+1}j_{\ell+1}} \Rightarrow e_{ki_{\ell+1}}f_{i_{\ell+2}w} \text{ for } 1 \leq \ell \leq t-1,\\
t_{k_ti_t}f_{i_{t+1}w} &\Rightarrow e_{k_{t+1}i_{t+1}}e_{i_{t+1}j_{t+1}} \Rightarrow f_{wi_{t+1}}f_{i_{t+2}w}, \text{ and}\\
f_{wi_\ell}f_{i_{\ell+1}w} &\Rightarrow e_{k_{\ell+1}i_{\ell+1}}e_{i_{\ell+1}j_{\ell+1}} \Rightarrow f_{wi_{\ell+1}}f_{i_{\ell+2}w} \text{ for } t+1 \leq \ell \leq s-1.
\end{aligned}$$

Therefore if we ever have one of the transitions $e_{k_\ell i_\ell}f_{i_{\ell+1}w}$ for $1 \leq \ell \leq t$ or the transition $f_{wi_\ell}f_{i_{\ell+1}w}$ for $t+1 \leq \ell \leq s$, then the rest of the transitions are forced and we have the second transition system from the lemma.

If none of these transition appear, then we must have the transitions from the first transition system. □

Our goal is to show each clause is true by showing the 3-cycle $e_{i_\ell j_\ell} e_{j_\ell k_\ell} e_{k_\ell i_\ell}$ is not disconnected from the graph. This corresponds to all the literals $x_{i_\ell j_\ell}, x_{j_\ell k_\ell}, x_{k_\ell i_\ell}$ being false.

Definition 3.56. Given a formula Φ and the graph G_Φ, we say the variable y_i is *false* if the transitions at v_i are the first set of transitions, and y_i is *true* if v_i has the second set of transitions.

Theorem 3.57. *Let Φ be a 3-SAT formula. The formula Φ has a satisfying assignment if and only if G_Φ has a compatible circuit.*

Proof. Let C_ℓ be a clause with literals x_{i_ℓ}, x_{j_ℓ}, and x_{k_ℓ}. If the transition systems at v_{i_ℓ}, v_{j_ℓ}, and v_{k_ℓ} have truth values such that the clause C_ℓ is false, then the 3-cycle $e_{i_\ell j_\ell} e_{j_\ell k_\ell} e_{k_\ell i_\ell}$ is disconnected from the graph. Otherwise, there is a walk starting and ending at w containing the edges $e_{i_\ell j_\ell}$, $e_{j_\ell k_\ell}$, and $e_{k_\ell i_\ell}$.

If Φ has a satisfying assignment, then choosing the corresponding transitions systems at each v_i gives rise to a compatible circuit of G_Φ (note that this does not determine the transitions at w, but we can arrange the excursions at w arbitrarily since there are no restrictions on the transitions).

If G_Φ has a compatible circuit, then the truth value for the variable y_i is given by the transition system at v_i. Since the edges $e_{i_\ell j_\ell}$, $e_{j_\ell k_\ell}$, and $e_{k_\ell i_\ell}$ for clause C_ℓ are in the compatible circuit the clause C_ℓ must be true. Since this holds for every clause, this is a satisfying assignment for Φ. $\square$

Corollary 3.58. *Determining if a digraph G has a compatible circuit where each edge has a list of acceptable transitions is NP-complete.*

Proof. This follows from Theorem 3.57 and the fact that forming G_Φ can be done in polynomial time. $\square$

3.7 Future work

Finally, we provide some open questions about compatible circuits in eulerian digraphs.

Question 1: Theorem 3.16 provides necessary and sufficient conditions for the existence of compatible circuits when there are no nonfixable vertices of outdegree three. In Section 3.5 we investigate and in certain cases can classify when a digraph has a compatible circuit when all the vertices are in S_3. Can we characterize the existence of a compatible circuit in a colored eulerian digraph G with nonfixable vertices of outdegree three?

Question 2: The BEST Theorem [62,63] provides a formula that counts the number of eulerian circuits in an eulerian digraph. Does there exist a formula to count the number of compatible circuits in a colored eulerian digraph?

Question 3: Given a graph G with no compatible circuit is there some choice for f such that $H_{G,f}$ does not have a 3-circuit traversal. If some transition function f always exists and can be found in polynomial time, then the problem of determining if a graph has a compatible circuit is in co-NP.

Question 4: For a digraph G that is not eulerian the Chinese Postman Problem [19] is to find a closed walk in G that travels each edge at least once and has the shortest length. Given an edge-colored strongly connected digraph (not necessarily eulerian), what is the minimum length of a closed walk with no monochromatic transitions?

The Chinese Postman Problem has many applications in routing problems. Introducing colors allows us to enforce additional restrictions on the routing. For instance we could color the edges of a road network such that a compatible circuit is a route for a mail truck that avoids left turns [46]. UPS [48] uses such routes to reduce the time of deliveries and number of accidents, saving millions of dollars.

Chapter 4

Edge-disjoint rainbow spanning trees in complete graphs [1]

4.1 Introduction

Let G be an edge-colored copy of K_n, where each color appears on at most $n/2$ edges (incident edges may have the same color). A rainbow spanning tree is a spanning tree of G such that each edge has a different color. Brualdi and Hollingsworth [11] conjectured that every properly edge-colored K_n ($n \geq 6$ and even) where each color class is a perfect matching has a decomposition of the edges of K_n into $n/2$ edge-disjoint rainbow spanning trees. They proved there are at least two edge-disjoint rainbow spanning trees in such an edge-colored K_n. Kaneko, Kano, and Suzuki [37] strengthened the conjecture to say that for any proper edge-coloring of K_n ($n \geq 6$) contains at least $\lfloor n/2 \rfloor$ edge-disjoint rainbow spanning trees, and they proved there are at least three edge-disjoint rainbow spanning trees. Akbari and Alipour [1] showed that each K_n that is an edge-colored such that no color appears more than $n/2$ times

[1] The following chapter is joint work with Paul Horn and Stephen Hartke. Paul and I worked together on this project during REGS in the summer of 2013.

contains at least two rainbow spanning trees.

Our main result is

Theorem 4.1. *Let G be an edge-colored copy of K_n, where each color appears on at most $n/2$ edges and $n \geq 1{,}000{,}000$. The graph G contains at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees.*

The strategy of the proof of Theorem 4.1 is to randomly construct $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint subgraphs of G such that with high probability each subgraph has a rainbow spanning tree. This result is the best known for the conjecture by Kaneko, Kano, and Suzuki. Horn [33] has shown that if the edge coloring is a proper coloring where each color class is a perfect matching then there are at least ϵn rainbow spanning trees for some positive constant ϵ, which is the best known result for the conjecture by Brualdi and Hollingsworth.

There have been many results in finding rainbow subgraphs in edge-colored graphs; Kano and Li [38] surveyed results and conjecture on monochromatic and rainbow (also called heterochromatic) subgraphs of an edge-colored graph. Related work includes Brualdi and Hollingsworth [12] finding rainbow spanning trees and forests in edge-colored complete bipartite graphs, and Constantine [14] showing that for certain values of n there exists a proper coloring of K_n such that the edges of K_n decompose into isomorphic rainbow spanning trees.

The existence of rainbow cycles has also been studied. Albert, Frieze, and Reed [2] showed that for an edge-colored K_n where each color appears at most $\lceil cn \rceil$ times then there is a rainbow hamiltonian cycle if $c < 1/64$ (Rue (see [27]) provided a correction to the constant). Frieze and Krivelevich [27] proved that there exists a c such that if each color appears at most $\lceil cn \rceil$ times then there are rainbow cycles of all lengths.

This chapter is organized as follows. Section 4.2 includes definitions, notation, and results used throughout the chapter. Section 4.3, 4.4, and 4.5 contains lemmas describing properties of the random subgraphs we generate. Section 4.6 provides the proof of our main result and Section 4.7 provides additional open questions.

4.2 Definitions

First we establish some notation that we will use throughout the chapter. Let G be a graph and $S \subseteq V(G)$. Let $G[S]$ denote the induced subgraph of G on the vertex set S. Let $[S, \overline{S}]_G$ be the set of edges between S and $\overline{S}$ in G. For natural numbers q and k, $[q]$ represents the set $\{1, \ldots, q\}$, and $\binom{[q]}{k}$ is the collection of all k-subsets of $[q]$. Throughout the chapter the logarithm function used has base e. One inequality that we will use often is the union sum bound which states that for events $A_1, \ldots, A_r$ that

$$\mathbb{P}\left[\bigcup_{i=1}^{r} A_i\right] \leq \sum_{i=1}^{r} \mathbb{P}\left[A_i\right].$$

Throughout the rest of the chapter let G be an edge-colored copy of K_n, where the set of edges of each color has size at most $n/2$, and $n \geq 1{,}000{,}000$. We assume G is colored with q colors, where $n - 1 \leq q \leq \binom{n}{2}$. Let C_j be the set of edges of color j in G. Define $c_j = |C_j|$, and without loss of generality assume $c_1 \geq c_2 \geq \cdots \geq c_q$. Note that $1 \leq c_j \leq n/2$ for all j.

Let $t = \lfloor n/(C \log n) \rfloor$ where $C = 1000$. Note that we have not optimized the constant C, and C can be slightly improved at the cost of more calculation. Since $\frac{n}{C \log n} - 1 \leq t \leq \frac{n}{C \log n}$ we have

$$\frac{-1}{t} \leq \frac{-C \log n}{n} \quad \text{and} \quad \frac{C \log n}{n} \leq \frac{1}{t} \leq \left(\frac{n}{n - C \log n}\right) \frac{C \log n}{n}. \tag{$*$}$$

We will frequently use these bounds on t.

We construct edge-disjoint subgraphs $G_1, \dots, G_t$ of G in the following way: independently and uniformly select each edge of G to be in G_i with probability $1/t$. Each G_i (considered as an uncolored graph) is distributed as an Erdős-Rényi random graph $G(n, 1/t)$. Note that the subgraphs are not independent. We will show that with high probability each of the subgraphs $G_1, \dots, G_t$ simultaneously contain a rainbow spanning tree.

To prove that a graph has a rainbow spanning tree we use Theorem 2.4 from Section 2.1.1 that provides necessary and sufficient conditions for the existence of a rainbow spanning tree.

Theorem 2.4. *A graph G has a rainbow spanning tree if and only if, for every partition π of $V(G)$ into s parts, there are at least $s-1$ different colors represented between the parts of π.*

We show that for every partition π of $V(G)$ into s parts, that there are at least $s-1$ colors between the parts for each G_i. Sections 4.3, 4.4, and 4.5 describe properties of the subgraphs $G_1, \dots, G_t$ for certain partitions π of $V(G)$ into s parts. Many of our proofs use the following variant of Chernoff's inequality [13], frequently attributed to Bernstein (see [16]).

Lemma 4.2 (Bernstein's Inequality)**.** *Suppose X_i are independently identically distributed Bernoulli random variables, and $X = \sum X_i$. Then*

$$\mathbb{P}\left[X \geq \mathbb{E}[X] + \lambda\right] \leq \exp\left(-\frac{\lambda^2}{2(\mathbb{E}[X] + \lambda/3)}\right)$$

and

$$\mathbb{P}\left[X \leq \mathbb{E}[X] - \lambda\right] \leq \exp\left(-\frac{\lambda^2}{2\mathbb{E}[X]}\right).$$

In several places in the chapter we use Jensen's inequality.

Lemma 4.3 (Jensen's Inequality (see [64])). *Let $f(x)$ be a real-valued convex function defined on an interval $I = [a, b]$. If $x_1, \ldots, x_n \in I$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, then*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

We also make use of the following upper bounds for binomial coefficients:

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k = \exp(k \log n - k \log k + k) \leq n^k.$$

4.3 Partitions with n or $n-1$ parts

In this section we show that a partition π of $V(G)$ into n or $n-1$ parts has enough colors between the parts. Since color classes can have small size, there might not be any edges of a given color in a subgraph G_i. Therefore, we group small color classes together to form larger pseudocolor classes. Recall that c_j is the size of the color class C_j, and $c_1 \geq c_2 \geq \cdots \geq c_q$. Define the pseudocolor classes $D_1, \ldots, D_{n-1}$ of G recursively as follows:

$$D_k = \left(\bigcup_{j=1}^{\ell} C_j\right) - \left(\bigcup_{i=1}^{k-1} D_i\right),$$

where ℓ is the smallest integer such that $\left|\left(\bigcup_{j=1}^{\ell} C_j\right) - \left(\bigcup_{i=1}^{k-1} D_i\right)\right| \geq n/4$. Note that the $n-1$ pseudocolor classes might not contain all the edges of G.

Lemma 4.4. *Each of the $n-1$ pseudocolor classes $D_1, \ldots, D_{n-1}$ have size at least $n/4$ and at most $n/2$.*

Proof. Consider the pseudocolor class D_k, for $1 \leq k \leq n-1$. Since each of the pseudocolor classes $D_1 \ldots, D_{k-1}$ has size at most $n/2$, there are at least $\frac{n}{2}(n-k)$ edges not in $\bigcup_{i=1}^{k-1} D_i$. Therefore there exists ℓ' and ℓ such that $D_k = \bigcup_{i=\ell'}^{\ell} C_i$, where

$|D_k| = \sum_{i=\ell'}^{\ell} c_i \geq n/4$.

If $\ell' = \ell$ then $|D_k| = |C_\ell| \leq n/2$. Otherwise, we know $c_\ell \leq c_{\ell-1} \leq c_{\ell'} \leq n/4$. So,

$$|D_k| = \sum_{i=\ell'}^{\ell-1} c_i + c_\ell \leq \frac{n}{4} + c_\ell \leq \frac{n}{4} + \frac{n}{4} = \frac{n}{2},$$

which proves that the pseudocolor class D_k has size at most $\frac{n}{2}$. $\square$

Lemma 4.5. *For a fixed subgraph G_i and pseudocolor class D_j,*

$$\mathbb{P}\left[|E(G_i) \cap D_j| \leq \frac{|D_j|}{t} - \sqrt{3\frac{n}{t}\log n}\right] \leq \frac{1}{n^3}.$$

As a consequence, with probability at least $1 - \frac{1}{n}$ every subgraph G_i has at least one edge from each of the pseudocolor classes $D_1, \ldots, D_{n-1}$.

Proof. Fix a subgraph G_i and a pseudocolor class D_j. The expected number of edges in G_i from the pseudocolor class D_j is $\frac{|D_j|}{t}$. By Bernstein's Inequality where $\lambda = \sqrt{3\frac{n}{t}\log n}$, we have

$$\begin{aligned}\mathbb{P}\left[|E(G_i) \cap D_j| \leq \frac{|D_j|}{t} - \sqrt{3\frac{n}{t}\log n}\right] &\leq \exp\left(\frac{-3\frac{n}{t}\log n}{2\frac{|D_j|}{t}}\right) \\ &\leq \exp\left(\frac{-3n\log n}{2\frac{n}{2}}\right) = \frac{1}{n^3}.\end{aligned}$$

Since $|D_j| \geq n/4$ by Lemma 4.4, $n \geq 1,000$, and $C \geq 50$, we have

$$\frac{|D_j|}{t} - \sqrt{3\frac{n}{t}\log n} \geq \frac{n}{4t} - \sqrt{3\frac{n}{t}\log n} \geq 1.$$

The second statement of the lemma follows from the previous inequalities by using the union sum bound for the $n-1$ pseudocolor classes and t subgraphs and recalling that $t < n$. $\square$

Lemma 4.5 shows that if we consider a partition π of $V(G)$ into s parts, where $s = n$ there must be at least $n-1$ colors in G_i between the parts of π. In the case when the partition has $s = n-1$ parts there is at most one edge inside the parts of π, so there are at least $n-2$ colors in G_i between the parts of π.

4.4 Partitions where $\left(1 - \frac{14}{\sqrt{C}}\right) n \leq s \leq n-2$

In this section we consider partitions π of $V(G)$ into s parts where $\left(1 - \frac{14}{\sqrt{C}}\right) n \leq s \leq n-2$. First, we introduce a new function that will help with our calculations. The function f will be used to bound the probability that $q-(s-2)$ colors do not appear between the parts of π in G_i.

Lemma 4.6. *For an integer ℓ and real numbers $c_1, \ldots, c_q$, define*

$$f(c_1, \ldots, c_q; \ell) = \sum_{I \in \binom{[q]}{q-\ell}} \exp\left(-\frac{1}{t}\sum_{j \in I} c_j\right).$$

If $1 \leq c_j \leq \frac{n}{2}$ for each j, $\sum_{i=1}^{q} c_j = \binom{n}{2}$, and $\frac{n}{2} \leq \ell \leq n-4$, then

$$f(c_1, \ldots, c_q; \ell) \leq \exp\left(-\frac{49C}{200}(n-\ell)\log n\right).$$

Proof. For convenience we define $w(I) = \sum_{j \in I} c_j$ for a subset $I \subseteq [q]$.

Claim 4.7.

$$f(c_1, \ldots, c_q; \ell) \leq f(\underbrace{1, 1, \ldots, 1}_{k-1 \text{ times}}, x^*, \underbrace{\frac{n}{2}, \ldots, \frac{n}{2}}_{q-k \text{ times}}; \ell),$$

where $1 \leq x^* < \frac{n}{2}$, and where k and x^* are so that $(k-1) + (q-k)\frac{n}{2} + x^* = \binom{n}{2}$.

Proof of Claim 4.7. Since $f(c_1,\ldots,c_q;\ell)$ is a symmetric function in the c_j's, it suffices to show that when $c_2 \geq c_1$,

$$f(c_1,c_2,\ldots,c_q;\ell) \leq f(c_1-\epsilon,c_2+\epsilon,\ldots,c_q;\ell),$$

where $\epsilon = \min\{c_1-1,\frac{n}{2}-c_2\}$.

$$\begin{aligned} f(c_1,c_2,\ldots,c_q;\ell) = &\sum_{I\in\binom{[q]-\{1,2\}}{q-\ell}} \exp\left(-\frac{w(I)}{t}\right) + \sum_{I\in\binom{[q]-\{1,2\}}{q-\ell-2}} \exp\left(-\frac{c_1}{t}-\frac{c_2}{t}-\frac{w(I)}{t}\right) \\ &+ \sum_{I\in\binom{[q]-\{1,2\}}{q-\ell-1}} \left(\exp\left(-\frac{c_1}{t}-\frac{w(I)}{t}\right)+\exp\left(-\frac{c_2}{t}-\frac{w(I)}{t}\right)\right) \end{aligned}$$

The first two summations are unchanged in $f(c_1-\epsilon,c_2+\epsilon,\ldots,c_q;\ell)$, and hence it suffices to show that for every $I\in\binom{[q]-\{1,2\}}{\ell-1}$,

$$\begin{aligned} &\exp\left(-\frac{c_1}{t}-\frac{w(I)}{t}\right)+\exp\left(-\frac{c_2}{t}-\frac{w(I)}{t}\right) \\ &\qquad \leq \exp\left(-\frac{(c_1-\epsilon)}{t}-\frac{w(I)}{t}\right)+\exp\left(-\frac{(c_2+\epsilon)}{t}-\frac{w(I)}{t}\right). \end{aligned}$$

This follows immediately by Jensen's inequality and the convexity of $\exp(\alpha x+\beta)$ as a function in x. $\square$

Claim 4.8.

$$f(\underbrace{1,1,\ldots,1}_{k-1\ times},x^*,\underbrace{\frac{n}{2},\ldots,\frac{n}{2}}_{q-k\ times};\ell) \leq f(\underbrace{1,\ldots,1}_{k\ times},\underbrace{\frac{n}{2},\ldots,\frac{n}{2}}_{q-k\ times};\ell),$$

where $\frac{n(n-2)}{2} \le k + (q-k)\frac{n}{2} \le \binom{n}{2}$.

Proof of Claim 4.8. The function f is decreasing in each c_j, and in particular c_k. $\square$

Now consider

$$\begin{aligned}
f(\underbrace{1,\dots,1}_{k\text{ times}}, \underbrace{\frac{n}{2},\dots,\frac{n}{2}}_{q-k\text{ times}};\ell) &= \sum_{I\in\binom{[q]}{q-\ell}} \exp\left(-\frac{1}{t}w(I)\right)\\
&\le \sum_{r=\max\{0,\ell-(q-k)\}}^{\min\{\ell,k\}} \binom{k}{r}\binom{q-k}{\ell-r}\exp\left(-\frac{1}{t}\left(\frac{n(n-2)}{2}-(\ell-r)\frac{n}{2}-r\right)\right)\\
&\le \sum_{r=\max\{0,\ell-(q-k)\}}^{\min\{\ell,k\}} k^r(q-k)^{(q-k)-(\ell-r)}\exp\left(-\frac{1}{t}\left(\frac{n}{2}(n-(\ell-r)-2)-r\right)\right)\\
&\le \sum_{r=\max\{0,\ell-(q-k)\}}^{\min\{\ell,k\}} \exp\left((q-k-\ell+2r)\log n-\frac{1}{t}\left(\frac{n}{2}(n-\ell+r-2)-r\right)\right)\\
&\le \sum_{r=\max\{0,\ell-(q-k)\}}^{\min\{\ell,k\}} \exp\left(\log n\left(q-k-\ell+2r-\frac{C}{2}(n-\ell+r-2)+\frac{Cr}{n}\right)\right) \quad \text{by } (*)\\
&\le n\exp\left(\log n\left((n-\ell)\left(1-\frac{C}{2}\right)+r\left(2-\frac{C}{2}+\frac{C}{n}\right)+C\right)\right)\\
&\le \exp\left(\log n\left((n-\ell)\left(1-\frac{C}{2}\right)+C+1\right)\right).
\end{aligned}$$

Since $n-\ell \ge 4$ and $C \ge 250$, we have

$$1+\frac{1}{n-\ell} \le \frac{C}{200} \le C\left(\frac{1}{2}-\frac{1}{n-\ell}-\frac{49}{200}\right).$$

Multiplying by $(n-\ell)$ and rearranging we have

$$(n-\ell)\left(1-\frac{C}{2}\right)+1+C \le \frac{-49C}{200}(n-\ell).$$

Thus the sum above is bounded by

$$\exp\left(-\frac{49C}{200}(n-\ell)\log n\right). \qquad \square$$

Lemma 4.9. *Let Π be the set of partitions of $V(G)$ into s parts, where $\left(1-\frac{14}{\sqrt{C}}\right)n \leq s \leq n-2$. For a partition $\pi \in \Pi$, let $\mathcal{B}_{\pi,i}$ be the event that there are less than $s-1$ colors between the parts of π in G_i. Then*

$$\mathbb{P}\left[\bigcup_{i=1}^{t}\bigcup_{\pi\in\Pi}\mathcal{B}_{\pi,i}\right] \leq \frac{1}{n}.$$

Proof. Fix a subgraph G_i and a partition $\pi \in \Pi$. Recall that $C_1, \ldots, C_q$ are the color classes of G with sizes $c_1, \ldots, c_q$, respectively. Let $I_{\pi,i}$ be the set of colors that do not appear on edges of G_i between the parts of π.

The total number of edges in G that have a color indexed by $I_{\pi,i}$ is $\sum_{i\in I_{\pi,i}} c_j$. By convexity of $\binom{x}{2}$, there are at most $\binom{n-s+1}{2}$ edges inside the parts of π. Note that if $I_{\pi,i}$ does not have size $q-(s-2)$, then it contains a set $I' \subseteq I_{\pi,i}$ of size $q-(s-2)$, and the event that no edges of G_i between the parts of π have colors in $I_{\pi,i}$ is contained in the event that no edges of G_i between the parts have colors in I'. Thus,

$$\begin{aligned}
\mathbb{P}\left[\mathcal{B}_{\pi,i}\right] &\leq \sum_{I\in\binom{[q]}{q-(s-2)}} \left(1-\frac{1}{t}\right)^{\sum_{j\in I} c_j - \binom{n-s+1}{2}} \\
&\leq f(c_1, c_2, \ldots, c_q; s-2)\left(1-\frac{1}{t}\right)^{-\binom{n-s+1}{2}} \\
&\leq f(c_1, c_2, \ldots, c_q; s-2)\exp\left(\frac{1}{t}\binom{n-s+1}{2}\right) && \text{since } \left(1-\frac{1}{t}\right) \leq e^{-\frac{1}{t}} \\
&\leq \exp\left(-\frac{49C}{200}\left(n-(s-2)\right)\log n + \frac{(n-s+1)^2}{2t}\right) && \text{by Lemma 4.6.}
\end{aligned}$$

Since $s \geq \left(1-\frac{14}{\sqrt{C}}\right)n$, we know $n-s+1 \leq \frac{14n}{\sqrt{C}}+1$. Thus we can bound the previous line by

$$\le \exp\left((n-s+1)\left(-\frac{49C}{200}\log n+\frac{1}{2t}\left(\frac{14}{\sqrt{C}}n+1\right)\right)\right)$$
$$\le \exp\left((n-s+1)\log n\left(-\frac{49C}{200}+\frac{n}{n-C\log n}\left(\frac{14\sqrt{C}}{2}+\frac{C}{2n}\right)\right)\right) \qquad \text{by } (*).$$

We now perform a union bound over all partitions $\pi \in \Pi$. The number of partitions of $V(G)$ into s nonempty parts is at most

$$\binom{n}{s}s^{n-s} \le \binom{n}{n-s}n^{n-s} \le n^{2(n-s)} = \exp(2(n-s)\log n) \le \exp(2(n-s+1)\log n).$$

Therefore,

$$\mathbb{P}\left[\bigcup_{\substack{\pi\in\Pi \\ \text{with } s \text{ parts}}} \mathcal{B}_{\pi,i}\right] \le \exp\left((n-s+1)\log n\left(2-\frac{49C}{200}+\frac{n}{n-C\log n}\left(\frac{14\sqrt{C}}{2}+\frac{C}{2n}\right)\right)\right).$$

Since $C = 1000$ and $n \ge 1{,}000{,}000$, we have

$$2-\frac{49C}{200}+\frac{n}{n-C\log n}\left(\frac{14\sqrt{C}}{2}+\frac{C}{2n}\right) \le -1,$$

and since $(n-s+1) \ge 3$,

$$\mathbb{P}\left[\bigcup_{\substack{\pi\in\Pi \\ \text{with } s \text{ parts}}} \mathcal{B}_{\pi,i}\right] \le \exp\left(-3\log n\right) = \frac{1}{n^3}.$$

This gives a bound on the probability for a fixed partition size s. Using the union sum bound over all partition sizes s, where $\left(1 - \frac{14}{\sqrt{C}}\right) n \leq s \leq n - 2$, and over all t subgraphs completes the proof. □

This proves when s is large there are enough colors between the parts.

4.5 Partitions where $2 \leq s \leq \left(1 - \frac{14}{\sqrt{C}}\right) n$

Next, we prove several results that will be used to show there are enough colors in G_i between the parts of the partition when the number of parts is small. Our goal is to show that for a partition π of $V(G)$ into s parts, the number of edges between the parts in G_i is so large that there must be at least $s - 1$ colors between the parts.

Lemma 4.10. *For a fixed subgraph G_i and color j,*

$$\mathbb{P}\left[|E(G_i) \cap C_j| \geq \frac{n}{2t} + 4\sqrt{\frac{n}{t} \log n}\right] \leq \frac{1}{n^4}.$$

As a consequence, with probability at least $1 - \frac{1}{n}$, every color appears at most $\frac{n}{2t} + 4\sqrt{\frac{n}{t} \log n}$ times in every G_i.

Proof. Fix a color j and a subgraph G_i. Order the edges of C_j as $e_1, \ldots, e_{c_j}$. For $1 \leq k \leq c_j$, let X_k be the indicator random variable for the event $e_k \in E(G_i)$. For a color class with size less than $\frac{n}{2}$ we introduce dummy random variables, so we can apply Bernstein's Inequality. For $c_j + 1 \leq k \leq n/2$, let X_k be a random variable distributed independently as a Bernoulli random variable with probability $1/t$.

By construction, $|E(G_i) \cap C_j| \leq X = \sum_{k=1}^{n/2} X_k$ and $\mathbb{E}[X] = \frac{n}{2t}$. By Bernstein's

Inequality where $\lambda = 4\sqrt{\frac{n}{t}\log n}$, we have

$$\begin{aligned}
&\mathbb{P}\left[|E(G_i)\cap C_j| \geq \frac{n}{2t} + 4\sqrt{\frac{n}{t}\log n}\right] \\
\leq &\mathbb{P}\left[X \geq \frac{n}{2t} + 4\sqrt{\frac{n}{t}\log n}\right] \\
\leq &\exp\left(\frac{-\frac{16n}{t}\log n}{2\left(\frac{n}{2t} + \frac{4}{3}\sqrt{\frac{n}{t}\log n}\right)}\right) \\
= &\exp\left(\frac{-16\log n}{1 + \frac{8}{3}\sqrt{\frac{t}{n}\log n}}\right) \\
\leq &\exp\left(\frac{-16}{1+\frac{8}{3\sqrt{C}}}\log n\right) \quad \text{since } t \leq \frac{n}{C\log n}, \\
\leq &\exp\left(\frac{-16}{\frac{11}{3}}\log n\right) \leq \left(\frac{1}{n}\right)^{48/11} \leq \frac{1}{n^4} \quad \text{since } C \geq 1,
\end{aligned}$$

which proves the first statement.

The second statement of the lemma follows from the previous inequality by using the union sum bound for the q color classes and t subgraphs, and recalling that $q < n^2$ and $t < n$. $\square$

Lemma 4.11. *Fix $S \subseteq V(G)$. Let $\mathcal{B}_{S,i}$ be the event*

$$\left|[S,\overline{S}]_{G_i}\right| \leq \frac{|S|(n-|S|)}{t} - \sqrt{\frac{6|S|(n-|S|)}{t}\min\{|S|, n-|S|\}\log n}.$$

Then

$$\mathbb{P}\left[\bigcup_{i=1}^{t}\bigcup_{S\subseteq V(G)}\mathcal{B}_{S,i}\right] \leq \frac{4}{n}.$$

Proof. Fix a subgraph G_i and a set of vertices $S \subseteq V(G)$. Let $r = |S|$. The expected number of edges in G_i between S and $\overline{S}$ is $r(n-r)/t$. By Bernstein's Inequality with

$\lambda = \sqrt{6\frac{r(n-r)}{t}\min\{r, n-r\}\log n}$, we have

$$\mathbb{P}\left[\mathcal{B}_{S,i}\right] \leq \exp\left(\frac{-6\frac{r(n-r)}{t}\min\{r, n-r\}\log n}{2\frac{r(n-r)}{t}}\right) = n^{-3\min\{r,n-r\}}.$$

So

$$\begin{aligned}\mathbb{P}\left[\bigcup_{S\subseteq V(G)} \mathcal{B}_{S,i}\right] &\leq \sum_{r=1}^{n/2}\binom{n}{r}n^{-3r} + \sum_{r=n/2}^{n}\binom{n}{n-r}n^{-3(n-r)} = 2\sum_{r=1}^{n/2}\binom{n}{r}n^{-3r}\\ &\leq 2\sum_{r=1}^{n/2} n^{-2r} \leq 2n^{-2} + 2\left(\sum_{r=2}^{n/2} n^{-4}\right) \leq \frac{2}{n^2} + \frac{2}{n^3} \leq \frac{4}{n^2}.\end{aligned}$$

Applying the union sum bound for the t subgraphs gives the final statement of the lemma. □

The previous lemma gives a lower bound on the number of edges between S and $\overline{S}$. We use this lemma to find a lower bound on the number of edges between the parts for a partition $\pi = \{P_1, \ldots, P_s\}$ of $V(G)$.

Definition 4.12. For $x \in [0, n]$, let

$$f(x) = \frac{x(n-x)}{t} - \sqrt{\frac{6x(n-x)}{t}\min\{x, n-x\}\log n}.$$

If none of the bad events $\mathcal{B}_{S,i}$ from Lemma 4.11 occur, then the sum $\frac{1}{2}\sum_{\pi=\{P_1,\ldots,P_s\}} f(|P_i|)$, where $\sum_{i=1}^{s}|P_i| = n$, is a lower bound on the number of edges between the parts of the partition π. We bound this sum for all partitions. If $-f(x)$ was convex then we could immediately find a lower bound by using Jensen's Inequality 4.3. Since $-f(x)$ is not convex, we bound it with a function that is convex.

Let $h(x)$ be a function with domain $[a, b]$. We say a function h is *concave down* if for $x, y \in [a, b]$ and $\lambda \in [0, 1]$, then $h(\lambda x + (1-\lambda)y) \geq \lambda h(x) + (1-\lambda)h(y)$. First, we present two basic results about concave down functions.

Lemma 4.13. *Let $h(x)$ be a differentiable function with domain $[a, b]$. Suppose that h is concave down on $[z, b]$, where $z \in (a, b)$. Let $\ell(x)$ be the line tangent to h at the point $(z, h(z))$. Then the function*

$$h_1(x) = \begin{cases} \ell(x) & \text{if } a \leq x \leq z, \\ h(x) & \text{if } z < x \leq b \end{cases}$$

is concave down.

Proof. Let $y_1, y_2 \in [a, b]$ where $y_1 \leq y_2$, and $\lambda \in [0, 1]$. If y_1 and y_2 are both in $[a, z]$ or $[z, b]$ then

$$h_1(\lambda y_1 + (1-\lambda)y_2) \geq \lambda h_1(y_1) + (1-\lambda)h_1(y_2),$$

since ℓ and h are both concave down.

Consider the case when $y_1 \in [a, z)$ and $y_2 \in (z, b]$. Let $\lambda \in [0, 1]$ and $w = \lambda y_1 + (1-\lambda)y_2$. Let b be the y-intercept of the line $\ell(x)$, i.e. $\ell(x) = h'(z)x + b$. Since h is concave down on the interval $[z, b]$, we know that h lies below the tangent line $\ell(x)$ on the interval $[z, b]$. In particular, $h'(z)y_2 + b \geq h(y_2)$. Let $\epsilon = h'(z)y_2 + b - h(y_2) \geq 0$.

If $w \leq z$, then we want to show that $h_1(w) = h'(z)(w - y_1) + \ell(y_1) \geq \frac{h(y_2)-\ell(y_1)}{y_2-y_1}(w - y_1) + \ell(y_1)$. Note that it is enough to show that $h'(z) \geq \frac{h(y_2)-\ell(y_1)}{y_2-y_1}$. Since $\epsilon \geq 0$, we have

$$h'(z) \geq h'(z)\frac{y_2 - y_1 - \epsilon}{y_2 - y_1} = \frac{(h'(z)y_2 + b - \epsilon) - (h'(z)y_1 + b)}{y_2 - y_1} = \frac{h(y_2) - \ell(y_1)}{y_2 - y_1}.$$

Suppose $w > z$. The line between z and y_2 is given by $\frac{h(y_2)-h(z)}{y_2-z}(w - z) + h(z)$, and by the concavity of $h(x)$ on $[z, b]$ we know $h_1(w) \geq \frac{h(y_2)-h(z)}{y_2-z}(w - z) + h(z)$. We want to show $\frac{h(y_2)-h(z)}{y_2-z}(w - z) + h(z) \geq \frac{h(y_2)-\ell(y_1)}{y_2-y_1}(w - z) + h(z)$. It is enough to show that $\frac{h(y_2)-h(z)}{y_2-z} \geq \frac{h(y_2)-\ell(y_1)}{y_2-y_1}$. We know $y_1 < z$ and $\epsilon > 0$. Adding $h'(z)y_2^2 - h'(z)y_2 z - h'(z)y_1 y_2 + h'(z)zy_1 - \epsilon y_2$ to both sides to the inequality $\epsilon y_1 \leq \epsilon z$ and factoring gives us

$$\begin{aligned} \left(h'(z)y_2 - \epsilon - h'(z)z\right)(y_2 - y_1) &\leq \left(h'(z)y_2 - \epsilon - h'(z)y_1\right)(y_2 - z) \\ \frac{(h'(z)y_2 + b - \epsilon) - (h'(z)z + b)}{y_2 - z} &\leq \frac{(h'(z)y_2 + b - \epsilon) - (h'(z)y_1 + b)}{y_2 - y_1} \\ \frac{h(y_2) - h(z)}{y_2 - z} &\leq \frac{h(y_2) - \ell(y_1)}{y_2 - y_1}. \end{aligned}$$

□

Lemma 4.14. *Let h_1 and h_2 be two concave down functions. The function $h(x) = \min\{h_1(x), h_2(x)\}$ is concave down.*

Proof. For every x, y and $\lambda \in [0,1]$ we have

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= \min\{h_1(\lambda x + (1-\lambda)y), h_2(\lambda x + (1-\lambda)y)\} \\ &\leq \lambda \min\{h_1(x), h_2(x)\} + (1-\lambda)\min\{h_1(y), h_2(y)\} \\ &= \lambda h(x) + (1-\lambda)h(x). \end{aligned}$$

□

We next define several functions that will lead to a concave down lower bound for the function f. Define on $[0,n]$ the functions

$$\begin{aligned} f_1(x) &= \frac{x(n-x)}{t} - x\sqrt{\frac{6(n-x)}{t}\log n}, \\ f_2(x) &= \frac{x(n-x)}{t} - (n-x)\sqrt{\frac{6x}{t}\log n}. \end{aligned}$$

Note that

$$f(x) = \begin{cases} f_1(x) & 0 \leq x \leq n/2, \\ f_2(x) & n/2 < x \leq n. \end{cases}$$

Let $\ell(x) = f_2'(x)(x - n/2) - f_2(n/2)$ be the tangent line of $f_2(x)$ at the point $\left(\frac{n}{2}, \frac{n^2}{4t} - \frac{n}{2}\sqrt{\frac{3n}{t}\log n}\right)$. Let c be the point such that $f_1(x)$ achieves its maximum value on the interval $[0, n]$. Define

$$f_3(x) = \begin{cases} \ell(x) & 0 \leq x \leq n/2, \\ f_2(x) & n/2 < x \leq n \end{cases}$$

and

$$f_4(x) = \begin{cases} f_1(x) & 0 \leq x \leq c, \\ f_1(c) & c < x \leq n. \end{cases}$$

By Lemma 4.13 the functions f_3 and f_4 are concave down.

On the interval $[0, n]$ define $f_5(x) = \min\{f_3(x), f_4(x)\}$. The function $f_5(x)$ is concave down by Lemma 4.14, where $f(x) \geq f_5(x)$ for all $x \in [0, n]$. Figure 4.1 shows the functions $f(x)$ and $\ell(x)$ used to create $f_5(x)$.

Lemma 4.15. *The sum $\sum_{i=1}^{s} f(x_i)$, where $\sum_{i=1}^{s} x_i = n$ and $x_i \geq 1$ for all i, is bounded below by*

$$\sum_{i=1}^{s} f(x_i) \geq (s-1)f(1) + f(n-s+1).$$

Proof. The proof is broken up into two cases based on whether $s \leq n/2$, or $s > n/2$.

When $s \leq n/2$ the function $f(x) \geq f_5(x)$, so $\sum_{i=1}^{s} f(x_i) \geq \sum_{i=1}^{s} f_5(x_i)$. Since the function $f_5(x)$ is concave down the sum $\sum_{i=1}^{s} f_5(x)$ is minimized when there is one part of size $n - s + 1$ and all the other parts are of size 1. Since $n - s + 1 \geq n/2$, we have $f_5(n-s+1) = f(n-s+1)$. Note that $\ell(1) \geq f_1(1)$, which implies $f_5(1) = f(1)$. Thus

$$\sum_{i=1}^{s} f(x_i) \geq \sum_{i=1}^{s} f_5(x_i) \geq (s-1)f_5(1) + f_5(n-s+1) = (s-1)f(1) + f(n-s+1).$$

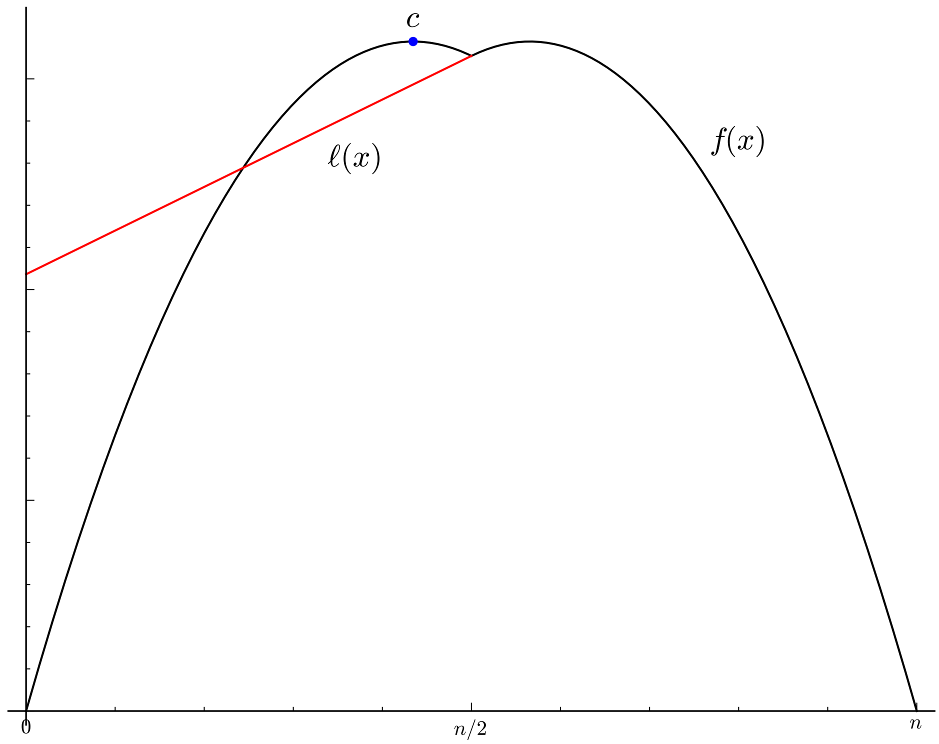

Figure 4.1: The function $f(x)$, along with the line $\ell(x)$.

When $s > n/2$, we have $x_i \leq n/2$ for all i. Therefore $f(x_i) = f_1(x_i)$ for all i. Since $f_1(x)$ is concave down the sum is minimized when one parts has size $n - s + 1$ and the rest have size 1. $\square$

Lemma 4.16. *Let π be a partition of the vertices of G into s parts. Suppose none of the events $\mathcal{B}_{S,i}$ from Lemma 4.11 hold for all $S \subseteq V(G)$ and $1 \leq i \leq t$. Then in each of the subgraphs $G_1, \ldots, G_t$, the number of edges between the parts of π is at least*

$$\frac{1}{2}\left((s-1)\left(\frac{n-1}{t}-\sqrt{6(n-1)\frac{\log n}{t}}\right)+\frac{(n-s+1)(s-1)}{t}-(s-1)\sqrt{6(n-s+1)\frac{\log n}{t}}\right)$$

when $s \leq n/2$, and

$$\frac{1}{2}\left((s-1)\left(\frac{n-1}{t}-\sqrt{6(n-1)\frac{\log n}{t}}\right)+\frac{(n-s+1)(s-1)}{t}-(n-s+1)\sqrt{6(s-1)\frac{\log n}{t}}\right)$$

when $s > n/2$.

Proof. If none of the events $\mathcal{B}_{S,i}$ hold then the sum $\frac{1}{2}\sum_{\pi=\{P_1,\ldots,P_s\}} f(x)$ where $\sum_{i=1}^{s}|P_i| = n$ is a lower bound on the number of edges between the parts of π. By Lemma 4.15 we know this sum is bounded below by $\frac{1}{2}\left((s-1)f(1)+f(n-s+1)\right)$. $\square$

Lemma 4.17. *Let π be a partition of the vertices of G into s parts, where $2 \leq s \leq \left(1-\frac{14}{\sqrt{C}}\right)n$. Suppose none of the events $\mathcal{B}_{S,i}$ from Lemma 4.11 hold for all $S \subseteq V(G)$ and $1 \leq i \leq t$, and every color appears in each G_i at most $\frac{n}{2t}+4\sqrt{\frac{n}{t}\log n}$ times (as in Lemma 4.10). Then in each of the subgraphs $G_1,\ldots,G_t$, the number of colors between the parts of π is at least $s-1$.*

Proof. Suppose there exists a subgraph G_i and a partition π into s parts where there are at most $s-2$ colors between the parts in G_i. Then by assumption there are at most

$$(s-2)\left(\frac{n}{2t}+4\sqrt{\frac{n}{t}\log n}\right)$$

edges in G_i between the parts of π. We will show that the number of edges between the parts of π can not be this small, giving a contradiction.

Suppose $\frac{n}{2} < s \leq \left(1-\frac{14}{\sqrt{C}}\right)n$. By Lemma 4.16 there are at least

$$\frac{1}{2}\left((s-1)\left(\frac{n-1}{t}-\sqrt{6(n-1)\frac{\log n}{t}}\right)+\frac{(n-s+1)(s-1)}{t}-(n-s+1)\sqrt{6(s-1)\frac{\log n}{t}}\right)$$

edges in G_i between the parts of π. If π has at most $s-2$ colors in G_i between the parts, then

$$(s-2)\left(\frac{n}{2t}+4\sqrt{\frac{n}{t}\log n}\right)\geq\frac{s-1}{2}\left(\frac{n-1}{t}-\sqrt{\frac{6(n-1)\log n}{t}}+(n-s+1)\left(\frac{1}{t}-\sqrt{\frac{6\log n}{(s-1)t}}\right)\right).$$

Rearranging we have

$$\frac{s-2}{s-1}\left(\frac{n}{t}+8\sqrt{\frac{n}{t}\log n}\right)+\frac{1}{t}+\sqrt{6(n-1)\frac{\log n}{t}}+(n-s+1)\sqrt{\frac{6\log n}{(s-1)t}}\geq\frac{n}{t}+\frac{(n-s+1)}{t}.$$

We will give an upper bound to the left side and a lower bound to the right side that give a contradiction.

Since s is an integer and $n/2 < s$, we have

$$(n-s+1)\sqrt{\frac{6}{n(s-1)}}\leq\frac{n}{2}\sqrt{\frac{12}{n^2}}=\sqrt{3}. \qquad (\dagger)$$

Therefore

$$\begin{aligned}
&\frac{s-2}{s-1}\left(\frac{n}{t}+8\sqrt{\frac{n}{t}\log n}\right)+\frac{1}{t}+\sqrt{6(n-1)\frac{\log n}{t}}+(n-s+1)\sqrt{\frac{6\log n}{(s-1)t}}\\
&\leq\left(\frac{n\sqrt{C}\log n}{n-C\log n}\right)\left(\sqrt{C}+\frac{\sqrt{C}}{n}+\sqrt{\frac{n-C\log n}{n}}\left(8+\sqrt{\frac{6(n-1)}{n}}+(n-s+1)\sqrt{\frac{6}{n(s-1)}}\right)\right)\\
&\leq\sqrt{C}\log n\left(\frac{n}{n-C\log n}\right)\left(\sqrt{C}+\frac{\sqrt{C}}{n}+\sqrt{\frac{n-C\log n}{n}}\left(8+\sqrt{6}+\sqrt{3}\right)\right)\quad\text{by }(\dagger).
\end{aligned}$$

Since $C=1000$ and $n\geq 1{,}000{,}000$, $\frac{n}{n-C\log n}\leq 1.02$ and $\sqrt{\frac{n}{n-C\log n}}\leq 1.01$. Thus the term above is bounded above by

$$\sqrt{C}\log n\left(1.02\sqrt{C}+\frac{1.02\sqrt{C}}{n}+1.01(8+\sqrt{6}+\sqrt{3})\right)\leq\sqrt{C}\log n\left(1.02\sqrt{C}+12.31\right).$$

We next bound the right side. By $(*)$ we have $\frac{1}{t} \geq \frac{C \log n}{n}$, and since $s \leq \left(1 - \frac{14}{\sqrt{C}}\right) n$, so

$$\frac{n}{t} + \frac{(n-s+1)}{t} \geq C \log n + C \log n \frac{n-s+1}{n} \geq C \log n + C \log n \frac{14}{\sqrt{C}}$$
$$= \sqrt{C} \log n (\sqrt{C} + 14).$$

When $C = 1000$ and $n \geq 1{,}000{,}000$ we have $\sqrt{C} + 14 > 1.02\sqrt{C} + 12.31$, which gives a contradiction. So, there must be at least $s-1$ colors in G_i between the parts of π when $n/2 < s \leq \left(1 - \frac{14}{\sqrt{C}}\right) n$.

Suppose $2 \leq s \leq n/2$. By Lemma 4.16 there are at least

$$\frac{1}{2}\left((s-1)\left(\frac{n-1}{t} - \sqrt{6(n-1)\frac{\log n}{t}}\right) + \frac{(n-s+1)(s-1)}{t} - (s-1)\sqrt{6(n-s+1)\frac{\log n}{t}}\right)$$

edges in G_i between the parts of π. If π has at most $s-2$ colors in G_i between the parts then

$$\frac{(s-1)}{2}\left(\frac{n-1}{t} - \sqrt{6(n-1)\frac{\log n}{t}} + \frac{(n-s+1)}{t} - \sqrt{6(n-s+1)\frac{\log n}{t}}\right)$$
$$\leq (s-2)\left(\frac{n}{2t} + 4\sqrt{\frac{n}{t}\log n}\right)$$

Rearranging we have

$$\frac{s-2}{s-1}\left(\frac{n}{t} + 8\sqrt{\frac{n}{t}\log n}\right) + \frac{1}{t} + \sqrt{6(n-1)\frac{\log n}{t}} + \sqrt{6(n-s+1)\frac{\log n}{t}}$$
$$\geq \frac{n}{t} + \frac{(n-s+1)}{t}.$$

Using $\frac{1}{t} \leq \frac{C\log n}{n-C\log n}$ from $(*)$, we have

$$\frac{s-2}{s-1}\left(\frac{n}{t}+8\sqrt{\frac{n}{t}\log n}\right)+\frac{1}{t}+\sqrt{6(n-1)\frac{\log n}{t}}+\sqrt{6(n-s+1)\frac{\log n}{t}}$$
$$\leq\left(\frac{n\sqrt{C}\log n}{n-C\log n}\right)\left(\sqrt{C}+\frac{\sqrt{C}}{n}+\sqrt{\frac{n-C\log n}{n}}\left(8+\sqrt{\frac{6(n-1)}{n}}+\sqrt{\frac{6(n-s+1)}{n}}\right)\right).$$

Since $C = 1000$ and $n \geq 1{,}000{,}000$, $\frac{n}{n-C\log n} \leq 1.02$ and $\sqrt{\frac{n}{n-C\log n}} \leq 1.01$. Thus the term above is bounded above by

$$\sqrt{C}\log n\left(1.02\sqrt{C}+\frac{1.02\sqrt{C}}{n}+1.01\left(8+2\sqrt{6}\right)\right) \leq \sqrt{C}\log n\left(1.02\sqrt{C}+13.1\right).$$

Bounding the right side using $\frac{1}{t} \geq \frac{C\log n}{n}$ from $(*)$, and $s \leq \frac{n}{2}$, we have

$$\begin{aligned}\frac{n}{t}+\frac{(n-s+1)}{t} \geq C\log n + C\log n\frac{(n-s+1)}{n} &\geq C\log n + C\log n\frac{\frac{n}{2}}{n}\\ &= \sqrt{C}\log n\left(\frac{3\sqrt{C}}{2}\right).\end{aligned}$$

Again, when $C = 1000$ and $n \geq 1{,}000{,}000$ we have $\frac{3\sqrt{C}}{2} > 1.02\sqrt{C} + 13.1$ which leads to a contradiction. Thus, there must be at least $s-1$ colors in G_i between the parts of π when $2 \leq s \leq \frac{n}{2}$. $\square$

4.6 Main result

Theorem 4.1. *Let G be an edge-colored copy of K_n, where each color appears on at most $n/2$ edges and $n \geq 1{,}000{,}000$. The graph G contains at least $\lfloor n/(1000\log n)\rfloor$ edge-disjoint rainbow spanning trees.*

Proof. Recall that $t = n/(C \log n)$ where $C = 1000$. We perform the random experiment of decomposing the edges of G into t edge-disjoint subgraphs G_i by independently and uniformly selecting each edge of G to be in the subgraph G_i with probability $1/t$. With probability at least $1 - \frac{7}{n}$ none of the bad events from Lemmas 4.5, 4.9, 4.10, and 4.11 occur in any of the subgraphs G_i. Henceforth let $G_1, \ldots, G_t$ be fixed subgraphs where none of these bad events occur.

We want to show that each G_i has a rainbow spanning tree. By Theorem 2.4 it is enough to show that for every partition π of $V(G)$ into s parts, there are at least $s - 1$ different colors appearing on the edges of G_i between the parts of π.

By Lemma 4.5, every G_i has at least one edge from each of the $n - 1$ pseudocolor classes. When $s = n$ there must be at least $n - 1$ colors in G_i between the parts of π. When $s = n - 1$ there is at most one edge inside the parts of π, so there are at least $n - 2$ colors in G_i between the parts of π.

If $\left(1 - \frac{14}{\sqrt{C}}\right) n \leq s \leq n - 2$, then by Lemma 4.9 every partition π of $V(G)$ into s parts has at least $s - 1$ colors in G_i between the parts, for every subgraph $G_1, \ldots, G_t$.

Finally, we assume that $s \leq \left(1 - \frac{14}{\sqrt{C}}\right) n$. When $s = 1$ there are zero colors between the parts, so the condition is vacuously true. So suppose $2 \leq s \leq \left(1 - \frac{14}{\sqrt{C}}\right) n$. Since Lemmas 4.10 and 4.11 hold, by Lemma 4.17 the number of colors between the parts of π is at least $s - 1$ for every subgraph $G_1, \ldots, G_t$.

Therefore all of the subgraphs $G_1, \ldots, G_t$ contain a rainbow spanning tree, and so G contains at least $t = \lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees. □

4.7 Future work

We conclude this section with some future questions and directions to consider. Our main result shows that there are at least $\left\lfloor \frac{n}{1000 \log n} \right\rfloor$ edge-disjoint rainbow spanning trees in a properly edge-colored complete graph on n vertices. Erdős-Rényi [21] proved that the threshold function for a random graph to be connected is $\frac{\log n}{n}$. Thus our approach can not hope to be improved past $n/\log n$ without considering more sophisticated constructions of the random graphs $G_1, \ldots, G_t$.

Question 1: Can one improve the bound on the number of edge-disjoint rainbow spanning trees to get $\lfloor \frac{n}{2} \rfloor$? Perhaps an easier question is to show that there are $\Omega(n)$ edge-disjoint rainbow spanning trees in every properly edge-colored complete graph.

Question 2: Are there are other graphs besides K_n that have many edge-disjoint rainbow spanning trees when properly edge-colored? Some natural examples to consider are the complete bipartite graph $K_{n,n}$, Cayley graphs, hypercube, or other dense graphs.

Question 3: Are there "nice" necessary and sufficient conditions for when an edge-colored graph G has k edge-disjoint rainbow spanning trees? This problem can be rephrased in terms of when the graphic and partition matroids have many common independent sets.

Chapter 5

Characterizing forbidden subgraphs that imply pancyclicity in 4-connected claw-free graphs[1]

5.1 Introduction

A graph G is *hamiltonian* if it contains a spanning cycle. Determining if a graph is hamiltonian is a NP-complete problem, and finding sufficient conditions for hamiltonicity has been the focus of much research. One sufficient condition we are interested in is looking at highly connected graphs that do not have certain induced subgraphs. Given a family $\mathcal{F}$ of graphs, a graph G is said to be $\mathcal{F}$*-free* if G contains no member of $\mathcal{F}$ as an induced subgraph. If $\mathcal{F} = \{K_{1,3}\}$, then G is said to be *claw-free*.

The following well known conjecture of Matthews and Sumner [45] has provided

[1]This chapter is joint work with Michael Ferrara, Tim Morris, and Michael Santana. This chapter is part of a larger paper proving that a 4-connected $\{K_{1,3}, N\}$-free graph, where $N \in \{N(2,2,2), N(3,2,1), N(4,1,1)\}$, is pancyclic. Though all authors contributed to the entire chapter, my focus was on the net $N(3,2,1)$, which is presented here.

the impetus for a great deal of research into the hamiltonicity of claw-free graphs.

Conjecture 5.1 (The Matthews-Sumner Conjecture [45]). *If G is a 4-connected claw-free graph, then G is hamiltonian.*

In [52], Ryjáček demonstrated that this is equivalent to a conjecture of Thomassen [60] that every 4-connected line graph is hamiltonian. Also in [52], Ryjáček showed that every 7-connected, claw-free graph is hamiltonian. Kaiser and Vrána [36] then showed that every 5-connected claw-free graph with minimum degree at least 6 is hamiltonian, which currently represents the best general progress towards affirming Conjecture 5.1. Recently, in [55], Conjecture 5.1 was also shown to be equivalent to the statement that every 4-connected claw-free graph is hamiltonian-connected.

The Matthews-Sumner Conjecture has also fostered a large body of research into other cycle-structural properties of claw-free graphs. In this chapter, we are specifically interested in the pancyclicity of highly connected claw-free graphs. A graph G is *pancyclic* if it contains cycles of each length from 3 to $|V(G)|$. Significantly fewer results of this type can be found in the literature, in part because it has been shown in many cases [53, 54] that closure techniques such as those in [52] do not apply to pancyclicity.

In [57], Shepherd showed the following, which extended a well-known result of Duffus, Gould and Jacobson [17]. Here $N(1,1,1)$ denotes the *net*, which is a triangle with a pendant joined to each vertex.

Theorem 5.2 (Shepherd [57]). *Every 3-connected, $\{K_{1,3}, N(1,1,1)\}$-free graph is pancyclic.*

Gould, Łuczak and Pfender [30] obtained the following characterization of forbidden pairs of subgraphs that imply pancyclicity in 3-connected graphs. Here Ł denotes the graph obtained by connecting two disjoint triangles with a single edge and $N(i,j,k)$ is the *generalized net* obtained by identifying an endpoint of each of the paths P_{i+1}, P_{j+1} and P_{k+1} with distinct vertices of a triangle. A connected, P_3-free graph is complete, which is trivially pancyclic. Therefore we consider forbidden pairs that are not P_3.

Theorem 5.3 (Gould, Łuczak, Pfender [30])**.** *Let X and Y be connected graphs on at least three vertices. If neither X nor Y is P_3 and Y is not $K_{1,3}$, then every 3-connected $\{X,Y\}$-free graph G is pancyclic if and only if $X = K_{1,3}$ and Y is a subgraph of one of the graphs in the family*

$$\mathcal{F} = \{P_7, Ł, N(4,0,0), N(3,1,0), N(2,2,0), N(2,1,1)\}.$$

Motivated by the Matthews-Sumner Conjecture and Theorem 5.3, Gould [29] posed the following problem at the 2010 SIAM Discrete Math meeting in Austin, Texas.

Problem 5.4. *Characterize the pairs of forbidden subgraphs that imply a 4-connected graph is pancyclic.*

The first progress towards this problem appears in [23].

Theorem 5.5 (Ferrara, Morris, Wenger [23])**.** *If G is a 4-connected $\{K_{1,3}, P_{10}\}$-free graph, then either G is pancyclic or G is the line graph of the Petersen graph. Consequently, every 4-connected, $\{K_{1,3}, P_9\}$-free graph is pancyclic.*

The line graph of the Petersen graph is 4-connected, claw-free and contains no cycle of length 4 (see $L(P)$ in Figure 5.1). Noting that in Theorem 5.3, all generalized nets of the form $N(i,j,0)$ with $i+j=4$ are in the family $\mathcal{F}$, Ferrara, Gould, Gehrke, Magnant, and Powell [22] showed the following.

Theorem 5.6 (Ferrara, Gould, Gehrke, Magnant, Powell [22]). *Every 4-connected $\{K_{1,3}, N(i,j,0)\}$-free graph with $i+j=6$ is pancyclic. This result is best possible, in that the line graph of the Petersen graph is $N(i,j,0)$-free for all $i,j \geq 0$ with $i+j=7$.*

In joint work with Michael Ferrara, Tim Morris, and Michael Santana we provide in this chapter another step toward providing a complete characterization. In particular, our main results are the following two theorems.

Theorem 5.7. *Let X and Y be connected graphs with at least three edges such that every 4-connected $\{X,Y\}$-free graph is pancyclic. Then, without loss of generality, X is either $K_{1,3}$ or $K_{1,4}$, and Y is an induced subgraph of one of P_9, Ł, or the generalized net $N(i,j,k)$ with $i+j+k=6$.*

Theorem 5.8. *Let Y be a connected graph with at least three edges. Every 4-connected $\{K_{1,3}, Y\}$-free graph is pancyclic if and only if Y is an induced subgraph of one of P_9, Ł, or the generalized net $N(i,j,k)$ with $i+j+k=6$.*

Theorem 5.8 follows from Theorems 5.5, 5.6, 5.7 and the following result.

Theorem 5.9. *Every 4-connected $\{K_{1,3}, N\}$-free graph, where N is one of $N(2,2,2)$, $N(3,2,1)$, or $N(4,1,1)$, is pancyclic.*

In this chapter we provide the proof of $N(3,2,1)$ only and omit the proofs for the other two generalized nets. The proof for all the nets have a similar proof technique.

In Section 5.2 we provide a proof to Theorem 5.7. In Section 5.3 we provide the proof that a 4-connected $\{K_{1,3}, N(3,2,1)\}$-free graph G is pancyclic. Section 5.3 is broken up into several subsections: Section 5.3.1 shows that there are cycles of lengths 3, 4, and 5, Section 5.3.2 gives many technical lemmas that provide a framework for

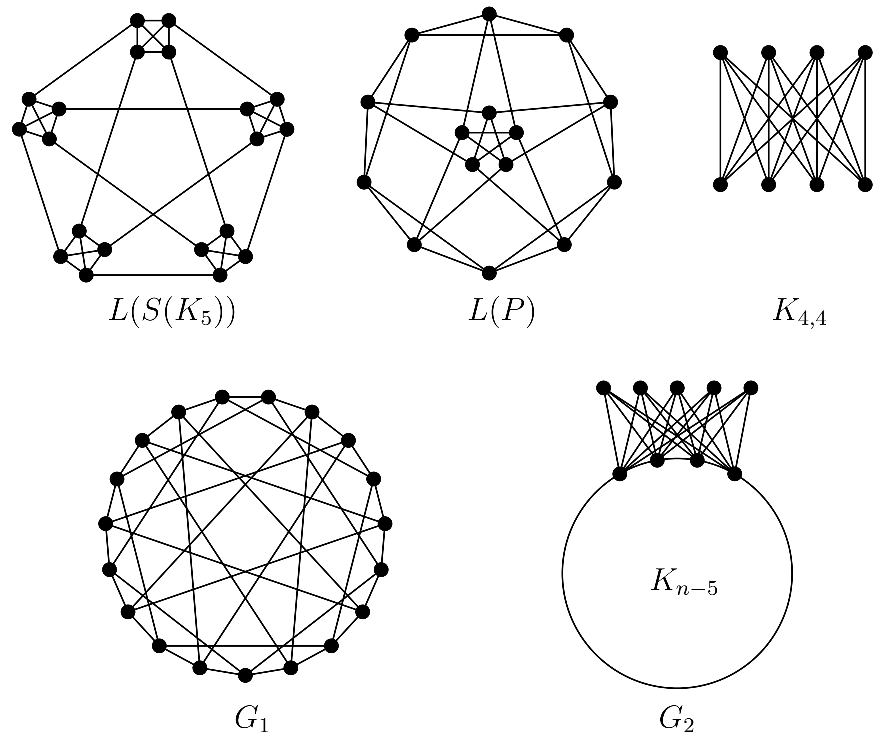

Figure 5.1: Some 4-connected claw-free graphs that are not pancyclic.

showing G has cycles of length $s \geq 6$, finally Section 5.3.3 proves that a 4-connected, $\{K_{1,3}, N(3,2,1)\}$ graph is pancyclic.

5.2 Proof of Theorem 5.7

This section determines which possible pairs of forbidden subgraphs can imply a 4-connected graph is pancylic.

Lemma 5.10. *Let X and Y be connected graphs with at least three edges. If each 4-connected, $\{X, Y\}$-free graph is pancyclic, then without loss of generality, $X \in \{K_{1,3}, K_{1,4}\}$.*

Proof. Note that the line graph $L(P)$ of the Petersen graph, $K_{4,4}$, and the graphs

G_1 and G_2 of Figure 5.1 are each 4-connected and are not pancyclic as they do not contain C_4, C_3, C_4, and C_n, respectively. In addition, $L(P)$ is $\{K_{1,3},\ K_{1,4}\}$-free.

Suppose on the contrary that $X, Y \notin \{K_{1,3}, K_{1,4}\}$. As $K_{4,4}$ is not pancyclic, we may conclude without loss of generality that X is an induced subgraph of $K_{4,4}$. As $X \notin \{P_3, K_{1,3}, K_{1,4}\}$, X must contain an induced C_4.

As G_1 does not contain C_4, it must contain Y as an induced subgraph. Therefore, Y must have girth at least 5 and maximum degree 4. Furthermore, G_2 is C_4-free so that Y must also be an induced subgraph of G_2. However, the only induced subgraphs of G_2 with girth at least 5 and maximum degree 4 are $K_{1,3}$ and $K_{1,4}$. So, Y must contain an induced $K_{1,3}$ or $K_{1,4}$.

Lastly, $L(P)$ is also C_4-free so that Y must be an induced subgraph of $L(P)$. However, neither $K_{1,3}$ nor $K_{1,4}$ is an induced subgraph of $L(P)$, the final contradiction necessary to establish the lemma. $\square$

In the remainder of this section, we will assume that X and Y are connected graphs with at least three edges such that every 4-connected, $\{X, Y\}$-free graph is pancyclic, and $X \in \{K_{1,3}, K_{1,4}\}$. To complete the proof of Theorem 5.7, we must characterize the possibilities for Y. In doing so, we will make use of the following family of graphs developed by Lubotsky, Phillips, and Sarnak [43].

Theorem 5.11. *For any numbers g and n, there exist infinitely many d such that there exists a connected, d-regular, vertex-transitive graph on at least n vertices with girth at least g. In particular, these graphs exist when $d = p + 1$, where p is an odd prime.*

These graphs, often called "Ramanujan graphs," were used by Brandt, Favaron, and Ryjáček [9] to show that for each $k \geq 2$, there exists a k-connected, claw-free graph that is not pancyclic. We will use a very similar approach to prove the following lemma, which together with Lemma 5.10, immediately implies Theorem 5.7.

Lemma 5.12. *There exists a 4-connected, claw-free, non-pancyclic graph G such that if Y is an induced subgraph of $L(P)$, $L(S(K_5))$, and G, then Y is an induced subgraph of P_9, $Ł$, or $N(i,j,k)$, with $i+j+k=6$.*

The graph $L(P)$ is the line graph of the Petersen graph and $L(S(K_5))$ is the line graph of the graph obtained by subdividing each edge of the complete graph K_5. See Figure 5.1 for picture of $L(P)$ and $L(S(K_5))$.

Proof. Let H be a connected, 4-regular, vertex-transitive graph with girth $g \geq 9$, as guaranteed by [43]. By a result of Mader [44], a connected, vertex-transitive, d-regular graph must also be d-edge-connected, implying that H is also 4-edge-connected. It follows that $L(H)$ is a 6-regular, 4-connected, claw-free graph. Note that each vertex v of H is represented by a graph $G_v \cong K_4$ in $L(H)$, where $xy \in E(H)$ corresponds to a vertex $z \in L(H)$ in exactly two K_4's.

Let H' be obtained from $L(H)$ by performing a 4-split on each vertex as follows. Let $v \in V(L(H))$ with neighbors $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, where the x_i's and y_i's are in distinct K_4's. Delete v and replace it with adjacent vertices x, y such that $N(x) = \{y, x_1, x_2, x_3\}$ and $N(y) = \{x, y_1, y_2, y_3\}$. It is well known that if a graph F is 4-connected and F' is obtained from F by performing 4-splits, then F' is 4-connected. Thus, H' is 4-connected, and it is easy to verify that H' is claw-free, as for every three neighbors of a vertex v, two must be in a common K_4. Note that H' contains 3-cycles and 4-cycles, but does not contain cycles of length t, $5 \leq t < 2g$ (recall $g \geq 9$).

For a given G_v in H', subdivide each edge of G_v exactly twice. For the sake of clarity, color these new vertices blue, and color the original vertices of H' red, and

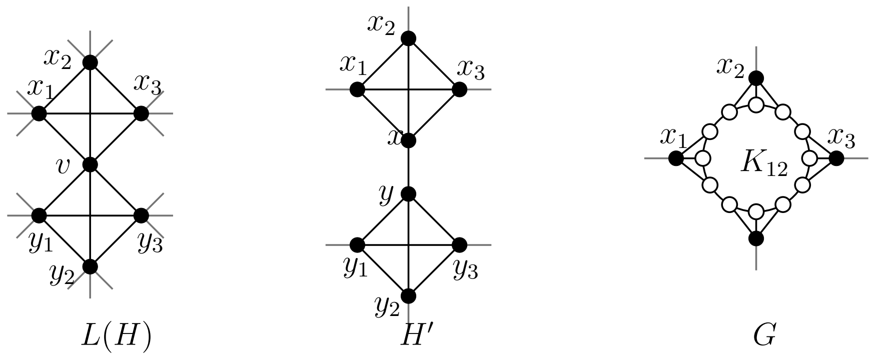

Figure 5.2: Forming the graph G.

then add edges so that the 12 new blue vertices induce a clique. Let $\widehat{G}_v$ be this new subgraph of order 16, and repeat this for each G_v in H' to obtain the graph G.

We claim first that G is claw-free. Indeed, if a red vertex is the center of a claw, then at least two of the other vertices in the claw must be blue vertices lying in a common $\widehat{G}_v$. A similar argument holds to show that no blue vertex is the center of a claw. To establish that G is 4-connected, consider a set S of at most three vertices in G. If S has any blue vertices, then it must contain three blue vertices, as removing at most two blue vertices will not disconnect any $\widehat{G}_v$, let alone G. However, deleting three blue vertices from a single $\widehat{G}_v$ cannot disconnect G, as in the worst case these three vertices would have a common red neighbor $v' \in \widehat{G}_v$. If $G - S$ is disconnected, then separating a vertex x from G_v is akin to disconnecting H' by deleting only x. As H' is 4-connected, this is not possible. So, we may assume S contains only red vertices. This directly corresponds to deleting vertices in H', which is 4-connected. Thus, in all cases, $G - S$ is connected.

We claim that G is not pancyclic. Indeed, G contains cycles of length $3, \ldots, 16$. However, any cycle C of length 17 must contain vertices from distinct modified K_4's in G. If we ignore all blue vertices in C, this corresponds to a cycle C' in H' using distinct vertices from distinct K_4's. As the smallest cycles in H' are of length 3, 4, and $2g \geq 18$, the cycle C' must have length at least $2g \geq 18$ in H', and thus C has length at least 18 in G. Consequently, G has no cycle of length 17, and so is not pancyclic.

Lastly, let Y be an induced subgraph of $L(P)$, $L(S(K_5))$, and G. We show that Y is an induced subgraph of P_9, $\L$, or $N(i,j,k)$ with $i+j+k=6$.

To begin, we claim that Y is either a tree or has girth 3. Suppose on the contrary that Y is not a tree and has girth more than 3. Since $L(P)$ is C_4-free, and $L(S(K_5))$ is C_5-free, Y must have girth at least 6. In addition, $L(S(K_5))$ implies that Y has girth at most 10, else it contains two vertices from one of the K_4's and hence contains a 3-cycle. However, every cycle of G that has length less than $2g \geq 18$ contains a 3-cycle, a contradiction.

Suppose now that Y is not a tree. If Y has two distinct cycles, then by the above argument, we may assume that Y has at least two distinct 3-cycles. Considering $L(P)$, no two 3-cycles can share two vertices, and considering $L(S(K_5))$, no two 3-cycles can share exactly one vertex. So, they must be joined by a path. By considering $L(P)$, it is clear that if two 3-cycles are joined by a path, they are joined by a single edge. That is, $\L$ is an induced subgraph of Y. While there are many induced subgraphs of $\L$ in G, it is easy to see that if $Y \neq \L$, then Y must contain a 4-cycle, a contradiction to $Y \subseteq L(P)$. So, unless $Y = \L$, Y cannot contain two distinct cycles.

Thus, if Y has a cycle, it must be a 3-cycle, and Y must be unicyclic. That is, Y is a generalized net. As noted in [22], $L(P)$ is $N(i,j,k)$-free when $k=0$ and $i+j=7$. It is also easy to note that $L(P)$ is $N(i,j,k)$-free when $i,j,k \geq 1$ and $i+j+k=7$. Thus, Y must be an induced subgraph of $N(i,j,k)$ where $i+j+k=6$.

Lastly, if Y is a tree, then as $L(P)$ is $K_{1,3}$-free, Y must be a path, and by [23], Y must be an induced subgraph of P_9. This completes the proof. $\square$

5.3 Proof of Theorem 5.9

This section provides the proof of the main contribution of this chapter towards Theorem 5.8. In section 5.3.1 we prove that for $N \in \{N(4,1,1), N(3,2,1), N(2,2,2)\}$ a $\{K_{1,3}, N\}$-free graph has cycles of lengths 3, 4, and 5. In section 5.3.2 we prove several technical lemmas. Section 5.3.3 shows that a 4-connected, claw-free, $N(3,2,1)$-free graph is pancyclic.

All graphs in this section are simple. Throughout this chapter, we assume that all cycles have an inherent orientation, which in our figures we always represent clockwise. For some vertex v on a cycle C we denote the first, second, and i^{th} predecessor of v as v^-, v^{--}, and v^{-i} respectively. Similarly we denote the first, second, and i^{th} successor of v as v^+, v^{++}, and v^{+i} respectively. We let xCy denote the path $xx^+ \ldots y$ and xC^-y denote the path $xx^- \ldots y$. Also, $xCyx$ denotes the cycle formed by adding an edge to the endpoints of the path xCy. Further, let $[u,v]_C$ denote the set of vertices on uCv, and let $(u,v)_C$ denote the set of vertices on u^+Cv^-. The intervals $(u,v]_C$ and $[u,v)_C$ are defined similarly. Let $\langle a; a_1, a_2, a_3 \rangle$ denote a $K_{1,3}$ in G with center vertex a and pendant edges aa_1, aa_2, and aa_3. Also, let $N(abc; a_1 \ldots a_i, b_1 \ldots b_j, c_1 \ldots c_k)$ denote a $N(i,j,k)$ generalized net with central triangle abc and pendant paths $aa_1 \ldots a_i$, $bb_1 \ldots b_j$, and $cc_1 \ldots c_k$.

In our proofs we frequently look at claws or nets that are subgraphs in G. These subgraphs can not be induced, and hence imply there are additional edges in the graph. For a forbidden subgraph H we use the notation $H \to S$ to denote the set of possible edges in G that imply that H is not an induced subgraph of G. Often there will be pairs of vertices in G we know are not adjacent for other reasons, and we will not include those pairs in the set S. When we use $H \to S$, we are using the fact that H is not induced as well as prior reasoning to conclude that G must contain an edge from S.

5.3.1 Short cycles

In this section we prove that for any $N \in \{N(4,1,1), N(3,2,1), N(2,2,2)\}$, a 4-connected, $\{K_{1,3}, N\}$-free graph contains cycles of length 3, 4 and 5. We use the following claim throughout this section, which we give without proof.

Claim 5.13. *If G is 4-connected, claw-free, and does not contain C_4, then G is 4-regular and for all $v \in V(G)$, $N(v)$ induces $2K_2$.*

We are now ready to prove the main result of this section.

Lemma 5.14. *If G is a 4-connected, $\{K_{1,3}, N\}$-free graph, where $N \in \{N(2,2,2), N(3,2,1), N(4,1,1)\}$, then G contains cycles of length 3, 4 and 5.*

Proof. Let G be a 4-connected, $\{K_{1,3}, N\}$-free graph. Note that as G is claw-free and has minimum degree at least four, G necessarily contains a triangle. To demonstrate the existence of 4-cycles and 5-cycles, we proceed by considering the distinct choices for N separately.

Case 1: $N = N(4,1,1)$.

By Theorem 5.6, if G is $\{K_{1,3}, N(5,1,0)\}$-free then G is pancyclic. Therefore, G must contain an induced $N(5,1,0)$, which we denote by $N_1 = N(a_0b_0c_0; a_1a_2a_3a_4a_5, b_1)$. Since G has minimum degree at least 4 and N_1 is induced, c_0 is adjacent to a pair of vertices u_1 and u_2 that lie outside of N_1. Let N_{u_i} be the $N(4,1,1)$ net given by $N(a_0b_0c_0; a_1a_2a_3a_4, b_1, u_i)$ for $i \in \{1,2\}$.

Suppose first that G does not contain a 4-cycle, so that by Claim 5.13, u_1 and u_2 are adjacent. Now, as G contains no 4-cycle, u_1 and u_2 can have no common neighbor aside from c_0, and further if u_1 and u_2 are adjacent to distinct vertices

x and y, respectively, then $xy \notin E(G)$. This is a contradiction, as for $i \in \{1, 2\}$, $N_{u_i} \to \{a_2u_i, a_3u_i, a_4u_i\}$ since all other possible edges immediately result in a C_4. If a_2u_i is an edge, then the claw $\langle a_2; a_1, a_3, u_i\rangle \to \{a_3u_i\}$. Thus u_1 and u_2 must have either a common neighbor or adjacent neighbors amongst a_2, a_3, and a_4, implying there is a 4-cycle.

Suppose then that G does not contain a 5-cycle. This implies that u_i is not adjacent to a_1, a_2 or b_1, and that if u_i is adjacent to b_0, then u_{3-i} is not adjacent to a_0 for $i \in \{1, 2\}$. Assume first that neither u_1 nor u_2 is adjacent to either of a_0 or b_0. As G is $N(4, 1, 1)$-free N_{u_i} is not induced, so u_i must have some neighbor $a_p \in \{a_3, a_4\}$. The claw $\langle a_p; u_i, a_{p-1}, a_{p+1}\rangle$ then implies that each u_i is adjacent to a pair of adjacent vertices in $\{a_3, a_4, a_5\}$. This implies that there is a 5-cycle.

Thus, we may assume that u_1 is adjacent to one of a_0 or b_0. As N is not induced and G contains no 5-cycle, the appropriate choice of $\langle a_0; u_1, a_1, b_0\rangle$ or $\langle b_0; u_1, b_1, a_0\rangle$ implies that a_0 and b_0 are both adjacent to u_1. As either u_2b_0 or u_2a_0 would create a C_5, $N_{u_2} \to \{u_2a_3, u_2a_4\}$. Suppose first that $u_2a_3 \in E(G)$. As u_2 is not adjacent to a_2, the claw $\langle a_3; u_2, a_2, a_3\rangle$ implies that $u_2a_4 \in E(G)$, so we may assume $u_2a_4 \in E(G)$. This then implies that u_1 has no neighbor in $\{a_3, a_4, a_5\}$, as any of these possible edges would complete a C_5 in G. If u_1u_2 was an edge of G, then $u_1u_2c_0a_0b_0u_1$ would be a 5-cycle, so we conclude that u_1 must have some neighbor v that lies outside of $V(N_1) \cup \{u_2\}$. However, then $N(a_0b_0u_1; a_1a_2a_3a_4, b_1, v)$ immediately forces a 5-cycle in G unless v is adjacent to a_3. However, then v is adjacent to either a_2 or a_4, which implies that G contains a 5-cycle.

Case 2: $N = N(3, 2, 1)$.

By Case 1, if G is $\{K_{1,3}, N(4, 1, 1)\}$-free then G contains cycles of length 4 and 5 so let $N_1 = N(a_0b_0c_0; a_1a_2a_3a_4, b_1, c_1)$ be an induced $N(4, 1, 1)$ net in G.

Suppose that G does not have a 4-cycle. Since G has minimum degree at least 4,

b_1 is adjacent to three vertices u_1, u_2, u_3 not in $V(N_1)$. By Claim 5.13, we may assume that $u_3b_0, u_1u_2 \in E(G)$. Additionally, c_0 has a neighbor v_1 not in $V(N_1)$; note that $v_i \neq u_i$, since G has no 4-cycle. For $i \in \{1,2\}$, let $N_{u_i} = N(a_0b_0c_0; a_1a_2a_3, b_1u_i, c_1)$. Now, $N_{u_i} \to \{u_ia_1, u_ia_2, u_ia_3, u_ic_1\}$. If both u_1 and u_2 are adjacent to vertices in $\{a_1, a_2, a_3\}$, then we obtain a 4-cycle in a manner similar to that in Case 1. So, we may assume $u_2c_1 \in E(G)$ and $u_1c_1 \notin E(G)$, else G contains a 4-cycle.

As $\delta(G) \geq 4$, c_1 is adjacent to some v_2 not in $V(N_1)$ where $v_2 \neq v_1$, and $v_2 \neq u_i$, otherwise G would have a 4-cycle. By Claim 5.13, $v_2u_2 \in E(G)$. Now we have $N(a_0b_0c_0; a_1a_2a_3, b_1u_1, c_1) \to \{u_1a_1, u_1a_2, u_1a_3\}$ and $N(a_0c_0b_0; a_1a_2a_3, c_1v_2, b_1) \to \{v_2a_1, v_2a_2, v_2a_3\}$. We obtain 4-cycles in a manner similar to that of Case 1, except when $xa_1, xa_2, ya_3 \in E(G)$, where $x \in \{u_1, v_2\}$ and $y \in \{u_1, v_2\} - \{x\}$. Note that $\langle a_3; a_2, a_4, y\rangle \to \{ya_4\}$, otherwise we get a 4-cycle. If, for instance, $x = u_1$ and $y = c_1$, then $N(a_1a_2u_1; a_0c_0c_1, a_3a_4, b_1)$ is necessarily induced, a contradiction. The other possibility for x and y are handled similarly.

Next assume that G contains no 5-cycle. By Case 1 G contains an induced $N(4,1,1)$, $N_1 = N(a_0b_0c_0; a_1a_2a_3a_4, b_1, c_1)$. Since $\delta(G) \geq 4$, c_1 has distinct neighbors u_1, u_2 and u_3 outside of $V(N_1)$ where, without loss of generality, u_1 and u_2 are adjacent. Hence neither u_1 nor u_2 are adjacent to any vertex in $\{a_0, a_1, b_0, b_1\}$.

Now $\langle c_1; c_0, u_3, u_2\rangle \to \{u_3c_0, u_2c_0, u_2u_3\}$, so assume first that $u_3c_0 \in E(G)$. Then for $i \in \{1,2\}$ the nets $N(a_0b_0c_0; a_1a_2a_3, b_1, c_1u_i) \to \{u_ia_2, u_ia_3\}$. This implies that u_1 and u_2 have adjacent neighbors in $\{a_2, a_3, a_4\}$, implying G has a 5-cycle. Consequently, $u_3c_0 \notin E(G)$, so assume instead that $u_2u_3 \in E(G)$, which implies that none of u_1, u_2 and u_3 are adjacent to c_0. Hence, as G contains no 5-cycle, for each $i \in \{1,2,3\}$ it follows that $N(a_0b_0c_0; a_1a_2a_3, b_1, c_1u_i) \to \{u_ia_2, u_ia_3\}$ so that two of the vertices $\{u_1, u_2, u_3\}$ have a common neighbor in $\{a_2, a_3\}$. This forces a 5-cycle in G, so we may assume that u_2c_0 is an edge in G, but neither c_0u_3 nor u_2u_3 are. Then

$N(a_0b_0c_0; a_1a_2a_3, b_1, c_1u_3) \to \{u_3a_2, u_3a_3\}$. Note that as u_3 is adjacent to c_1 and N_1 is induced, u_3 must have adjacent neighbors in $\{a_0, \ldots, a_4\}$. Hence, if $u_3a_4 \in E(G)$, then $N(a_0b_0c_0; a_1a_2, b_1, c_1u_3a_4)$ is induced, so we must in fact have that u_3a_2 and u_3a_3 are in $E(G)$. It then follows that $N(u_3a_2a_3; c_1u_2, a_1a_0b_0, a_4)$ is induced, the final contradiction needed to complete this case.

Case 3: $N = N(2, 2, 2)$.

By Case 2, we may assume that G is not $N(3, 2, 1)$-free, so let $N_3 = N(a_0b_0c_0; a_1a_2a_3, b_1b_2, c_1)$ be an induced subgraph of G. Since G has minimum degree at least 4 and N_3 is induced, c_1 has at least three neighbors outside of $V(N_3)$, call them u_1, u_2 and u_3. As G is claw-free, we may assume that $u_1u_2 \in E(G)$.

Assume first that G contains no 4-cycle so that u_3c_0 is an edge in G by Claim 5.13. Now, for $i \in \{1, 2\}$, $N(a_0b_0c_0; a_1a_2, b_1b_2, c_1u_i) \to \{u_ia_1, u_ia_2, u_ib_1, u_ib_2\}$. Since u_1 and u_2 have no common neighbors outside of c_1, we may conclude without loss of generality that u_1a_2 and u_2b_2 are edges in G (as G is claw-free), we also have that one of u_1a_1 or u_1a_3 is also in G, as well as possibly u_2b_1.

Now, u_3 has two neighbors aside from c_0 and c_1, call them v_1 and v_2. If $v_1 = a_3$, then v_2a_3 is an edge by Claim 5.13. If u_1a_3 is an edge, then $u_1a_3u_3c_1u_1$ is a 4-cycle so u_1a_1 is an edge of G. The net $N(a_1a_2u_1; a_0c_0, a_3v_2, u_2b_2) \to \{a_0v_2, v_2b_2\}$, but a_0v_2 is not an edge since otherwise $\langle a_o; a_1, v_2, c_0\rangle \to \{a_1v_2, c_0v_2\}$ which both give a 4-cycle. Thus we may assume v_2b_2 is an edge, and we note that $\langle b_2; b_1, v_2, u_2\rangle \to \{v_2b_1\}$. Then $N(c_1u_1u_2; c_0a_0, a_2a_3, b_2b_1)$ is an induced copy of $N(2, 2, 2)$, so we may assume that $v_1 \neq a_3$ and similarly $v_2 \neq a_3$.

By Claim 5.13, v_1 and v_2 are necessarily adjacent. Without loss of generality, the nets $N(a_0b_0c_0; a_1a_2, b_1b_2, u_3v_i)$ for $i \in \{1, 2\}$ imply that v_1a_2 and v_2b_2 are edges in G. If $v_2b_1 \notin E(G)$, then $N(u_3v_1v_2; c_0a_0, a_2u_1, b_2b_1)$ is induced. Otherwise, $N(a_0b_0c_0; a_1a_2, b_1v_2, c_1u_2)$ is induced, so G contains a 4-cycle.

Finally, suppose that G contains no 5-cycle; we proceed by considering how many of u_1, u_2, and u_3 are adjacent to c_0 and note that if all three of these vertices are adjacent to c_0, we immediately have a C_5. Thus, assume that $N(c_0) \cap \{u_1, u_2, u_3) = \emptyset$, which implies that u_1, u_2 and u_3 are pairwise adjacent since G is claw-free. Consequently, as G contains no C_5 and no u_i is adjacent to c_0, the nets $N_{u_i} = N(a_0b_0c_0; a_1a_2, b_1b_2, c_1u_i)$ imply that each u_i is adjacent to one of a_2 or b_2, resulting in a 5-cycle in G.

Next assume that two of u_1, u_2, and u_3 are adjacent to c_0. Since $u_1u_2 \in E(G)$, we must have that u_1 and u_2 are adjacent to c_0, while u_3 is not. As $\delta(G) \geq 4$ and G contains no 5-cycle, there exist distinct vertices v_1 and v_2 outside of $\{u_1, u_2, u_3, a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1\}$ such that u_1v_1 and u_2v_2 are in G. If $v_1 = a_3$, then $v_2 \neq a_3$ and, since $N(a_0b_0c_0; a_1a_2, b_1b_2, c_1u_3)$ is not induced, $u_3b_2 \in E(G)$. However, since G does not have a 5-cycle, this implies that $N(a_0b_0c_0; a_1a_2, b_1b_2, u_2v_2)$ is an induced $N(2, 2, 2)$ net. Thus $v_1 \neq a_3$ and similarly $v_2 \neq a_3$. Then, in a manner similar to the previous cases, the nets $N(a_0b_0c_0; a_1a_2, b_1b_2, c_1u_3)$ and $N(a_0b_0c_0; a_1a_2, b_1b_2, u_iv_i)$ for $i \in \{1, 2\}$ imply that G has a 5-cycle.

Thus we may assume that exactly one of u_1, u_2 and u_3 is adjacent to c_0, in particular we claim that u_3c_0 must be in G. If instead u_1 is the only vertex in $\{u_1, u_2, u_3\}$ adjacent to c_0, then $\langle c_1; u_2, u_3, c_0\rangle$ implies that $c_0c_1u_3u_2u_1c_0$ is a 5-cycle in G. Thus we have that u_3c_0 is an edge in G, and further N_{u_1} and N_{u_2} imply that at most one of u_1a_2 or u_2a_2 is an edge since u_ia_2 forces the edge u_ia_3 which would then complete a 5-cycle in G. As neither N_{u_1} nor N_{u_2} is induced, we assume first that u_1a_2 (or identically u_2a_2) is an edge of G, so that u_1a_3 and u_2b_2 are edges of G as well.

Now u_3 has two neighbors v_1 and v_2 distinct from $N_3 \cup \{u_1, u_2\}$. Neither v_1 nor v_2 is adjacent to any of $V(N_1) \cup \{u_1, u_2\} - \{c_0, c_1\}$ since G does not have a 5-cycle. If both v_1 and v_2 are adjacent to c_0, then $\langle u_3; v_1, v_2, c_1\rangle \to \{v_1v_2, v_1c_1, v_2c_1\}$, which implies that G has a 5-cycle. Thus we may assume that v_2c_0 is not an edge, but then $N(a_0b_0c_0; a_1a_2, b_1b_2, u_3v_2)$ is an induced $N(2,2,2)$ net.

Thus neither u_1a_2 nor u_2a_2 is an edge and we may assume that both u_1b_2 and u_2b_2 are edges. Note that u_3 is not adjacent to any of $\{u_1, u_2, b_0, b_1, b_2, a_0, a_1\}$ since there is no 5-cycle. Then $N(a_0b_0c_0; a_1a_2, b_1b_2, c_1u_3) \to \{u_3c_0, u_3a_2\}$, but we cannot have both u_3c_0 and u_3a_2 in G. If u_3a_2, then $\langle c_1; c_0, u_3, u_1\rangle \to \{c_0u_3, c_0u_1, u_1u_3\}$ in each case there is a 5-cycle. Thus we may assume that u_3c_0 is an edge. The vertex u_3 has 2 neighbors v_1 and v_2 other than $N_1 \cup \{u_1, u_2\} - \{a_3\}$. If $v_2 = a_3$, then $\langle u_3; c_0, v_1, v_2\rangle \to \{v_1v_2\}$, but then $N(a_0b_0c_0; a_1a_2, b_1b_2, u_3v_1)$ is an induced copy of $N(2,2,2)$, thus $v_2 \neq a_3$. Now, $N(a_0b_0c_0; a_1a_2, b_1b_2, u_3v_i) \to \{v_ic_0, v_ia_2\}$. If both v_1c_0 and v_2c_0 are edges, then $\langle u_3; v_1, v_2, c_1\rangle \to \{v_1v_2, v_1c_1, v_2c_1\}$, and in any case G has a 5-cycle. So assume that v_2c_0 is not an edge, implying that v_2a_2 and thus v_2a_3 are edges. Then $\langle c_0; c_1, b_0, v_1\rangle \to \{v_1c_1\}$. Now v_1 has a neighbor x not in N_1 and thus $N(a_0b_0c_0, a_1a_2, b_1b_2, v_1x)$ is induced so G is not $N(2,2,2)$-free. □

5.3.2 Technical lemmas

In this section, we give a number of technical lemmas that will simplify the case structure of the proof, where we demonstrate that a 4-connected, $\{K_{1,3}, N(3,2,1)\}$-free graph of order n contains cycle of length s for $6 \leq s \leq n$. The majority of these lemmas use standard techniques, so we omit or shorten many of their proofs.

Let G be a 4-connected, claw-free graph and let C be a cycle in G of length s, where $5 \leq s < |V(G)|$, and assume that G contains no $(s+1)$-cycle. Since G is 4-connected, for each vertex $v \in V(G) - V(C)$ there exist four internally disjoint $(v - C)$-paths, each containing a unique vertex from C. Let $w, x, y, z \in V(C)$ be these vertices, and let P_x denote the path containing x, P_y denote the path containing y, and so on. Assume that amongst all choices of v, w, x, y and z, $|P_x| + |P_y| + |P_z|$ is minimum.

The claw centered at v with one vertex from P_x, P_y and P_z is not induced, thus v lies on a triangle T. For $a \in \{w, x, y, z\}$ let F_a denote the (unique) $a - T$ path that is a subpath of P_a, and let a' be the endpoint of F_a in T. It is possible that $a' = a$ if v is adjacent to a, and also therefore possible that F_a is a trivial path (a path of order one). However, since v is in $V(G) - V(C)$ and v is in $x'y'z'$, at most two of x', y' or z' lie on C. Finally, let $F = T \cup \left(\bigcup_{a \in \{x,y,z\}} F_a\right)$ and note that the minimality of $|P_x| + |P_y| + |P_z|$ implies that $F - \{x, y, z\}$ is induced.

Let $xx_1 \dots x_{p+1} = x'$, $yy_1 \dots y_{q+1} = y'$, and $zz_1 \dots z_{t+1} = z'$ denote the vertices on F_x, F_y and F_z, respectively. Also, let $I_x = x_1 \dots x_p$, $I_y = y_1 \dots y_q$ and $I_z = z_1 \dots z_t$ denote the interior subpaths of F_x, F_y and F_z, and note that I_x, I_y or I_z may be empty. The assumption that G contains no $(s+1)$-cycle also yields that x^-x^+, y^-y^+ and z^-z^+ are edges in G as the claws $\langle x; x_1, x^-, x^+\rangle$, $\langle y; y_1, y^-, y^+\rangle$ and $\langle z; z_1, z^-, z^+\rangle$ are not induced.

Up to relabeling and reversing the orientation of C, assume $|I_x| \geq |I_y| \geq |I_z|$ and also that x, y and z appear on C in this order when traversing C in the clockwise direction. As a result, if v is adjacent to exactly one vertex on C then it is $z' = z$, and if v is adjacent to exactly two vertices on C they are $y' = y$ and $z' = z$.

For the remainder of this section, when convenient we let a denote an arbitrary element of $\{w, x, y, z\}$ and we will use a in a flexible manner that allows us to introduce notation relating to all of the vertices in $\{w, x, y, z\}$ without the need for tedious repetition. For instance, given the notation defined above, when unambiguous we refer to P_a, F_a, I_a and so on. We also will frequently omit the assumption that a is some vertex in $\{w, x, y, z\}$, again in order to minimize repetition.

Our first lemma follows routinely from the minimality of $|P_x| + |P_y| + |P_z|$ and the assumption that G contains no $(s+1)$-cycle.

Lemma 5.15. *If $\{x', y', z'\} \cap V(C) = \emptyset$ then there are no edges between $V(F) - \{x, y, z\}$ and $V(C)$ except for xx_1, yy_1 and zz_1. If $\{x', y', z'\} \cap V(C) = \{z\}$, i.e. $z = z'$, then there are no edges between $V(F) - \{x, y, z\}$ and $V(C)$ except xx_1, yy_1, $x'z$, and $y'z$. If $\{x', y', z'\} \cap V(C) = \{y, z\}$ and $|I_x| \geq 1$, then there are no edges between $V(F) - \{x, y, z\}$ and $V(C)$ except vy, vz, x_1x, and possibly x_1u for at most one $u \in V(C) - \{x\}$.*

Our next lemmas provide useful structural information about various intervals of vertices on C.

Lemma 5.16. *If u and v are vertices on C such that $[u, v]_C \subseteq N[a]$, then $[u, v]_C$ induces a clique in G.*

Proof. Suppose that b and c are nonadjacent vertices in $[u, v]_C$ such that u, b, c and v appear in that order on C in the positive direction. The claw $\langle a; a_1, b, c\rangle$ is not induced, so without loss of generality, ba_1 is an edge of G. This implies that $b \notin \{a^-, a^+\}$, lest G contain an $(s+1)$-cycle. Since c appears after b in $[u, v]_C$, we must have $b^+ \in [u, v]_C$ and b^+a is an edge. Then $a^-C^-b^+aa_1bC^-a^+a^-$ is an $(s+1)$-cycle in G. □

For $a \in \{w, x, y, z\}$, let $Q_C(a) = [a_\ell, a_r]_C$ be the largest interval of C such that $a \in [a_\ell, a_r]_C$ and $[a_\ell, a_r]_C \subseteq N[a]$. When the context is clear, we simply write $Q(a)$. By Lemma 5.16, $Q(a)$ induces a clique in G. Note that $Q(a)$ contains, at a minimum, the vertices a, a^- and a^+. Also, if $G[V(C)] = K_s$ we have $Q(a) = V(C)$ for all choices of a.

If $G[V(C)] \neq K_s$, then the maximality of $Q(a)$ implies that a is adjacent to neither a_ℓ^- nor a_r^+. Additionally, as $Q(a)$ is a clique, no pair of vertices in $Q(a)$ can have a common neighbor in $V(G) - V(C)$, otherwise G contain a cycle of length $s+1$. See Figure 5.3 for a picture showing the vertices x', y', and z'; the paths P_x, P_y, and P_z to the cycle C; and the cliques $Q(x)$, $Q(y)$, and $Q(z)$.

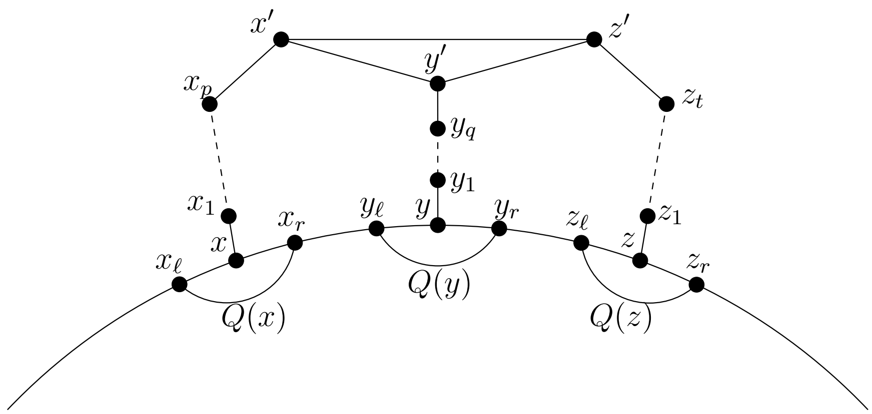

Figure 5.3: The vertices x', y', and z', along with the paths to the cycle C. The picture also shows that cliques $Q(x)$, $Q(y)$, and $Q(z)$. In this figure we have $y' \neq y$ and $z' \neq z$, but in certain cases these vertices maybe the same.

Our next lemma follows easily from the maximality of $[a_\ell, a_r]_C$ and the fact that G contains no $(s+1)$-cycle.

Lemma 5.17. *If $V(C)$ does not induce a complete graph, then a_ℓ and a_r are only adjacent to vertices in $V(C)$. In particular, neither a_ℓ nor a_r is in $\{w, x, y, z\}$.*

Proof. Suppose a_ℓ is adjacent to some vertex v' not on C, and consider the claw $\langle a_\ell; a, a_\ell^-, v'\rangle$. As mentioned above, a_ℓ and a can have no common neighbors outside of C, so either $a_\ell^- a \in E(G)$, which contradicts the maximality of $Q(a)$, or $a_\ell^- v' \in E(G)$, forming an $(s+1)$-cycle. The case where a_r has some neighbor off of C is identical. $\square$

If $Q(a) \neq Q(b)$, then we can assume a_ℓ, a and b_ℓ, b appear consecutively in C. If $Q(a) = Q(b)$, then we can assume a_ℓ, a, b appear consecutively in C. This observation simplifies the cycles we describe throughout the chapter.

Let $\mathcal{O}$ denote the set of vertices in $V(C)$ that have a neighbor off of C. By Lemma 5.17, we know that no $x \in \mathcal{O}$ can be a_r or a_ℓ for any choice of a, and further that $x^- x^+$ is an edge in G for any such x. Hence, we can replace $x^- x x^+$ on C with $x^- x^+$ to obtain a cycle in which we may continue to utilize the structure ensured by $Q(a)$ as needed. Further, suppose that $x_1, \ldots, x_m$, where $m \geq 2$, are vertices in $\mathcal{O}$ that appear consecutively on C in that order. It is not difficult to prove by induction that since G contains no $(s+1)$-cycle, $x_1^- x_m^+$ is an edge in G. Hence, for any set X of vertices in G we may define a cycle $C(X)$ in which the following hold:

1. $|V(C(X))| = |V(C) - X| = s - |X|$,
2. the vertices in $V(C(X)) = V(C) - X$ appear in exactly the same order on C and $C(X)$, and
3. for each $a \in \{w, x, y, z\} - X$, $Q(a) - X$ is a clique consisting of consecutive vertices on $C(X)$ with endpoints a_r and a_ℓ.

If $X = \{x_1, \ldots, x_m\}$, we will sometimes write $C(x_1, \ldots, x_m)$ in place of the more cumbersome $C(\{x_1, \ldots, x_m\})$.

Lemma 5.18. *Let a and b be distinct elements in $\{w, x, y, z\}$, and let P be an $(a-b)$-path of length λ with no internal vertices on C.*

1. *If $2 \leq \lambda \leq 4$, then $|(a_r, b_\ell)_C| \geq \lambda - 1$ and $|(b_r, a_\ell)_C| \geq \lambda - 1$.*
2. *If $\lambda = 5$ and $ab \notin E(G)$, then $|(a_r, b_\ell)_C| \geq 4$ and $|(b_r, a_\ell)_C| \geq 4$.*

Proof. Throughout the proof, let c and d be distinct vertices in $\{w, x, y, z\} - \{a, b\}$. First, if $G[V(C)] = K_s$, then $s \geq 5$ clearly implies that G has an $(s+1)$-cycle when the length of P is between 2 and 5. Thus, going forward we will assume that $G[V(C)] \neq K_s$.

To establish (*1*), we consider only the scenario where $\lambda = 4$ and $|(a_r, b_\ell)_C| \leq 2$, as the cases $\lambda = 3$ and $\lambda = 2$ and the symmetric case where $\lambda = 4$ and $|(b_r, a_\ell)_C| \leq 2$ are handled similarly.

First, if $1 \leq |(a_r, b_\ell)_C| \leq 2$, then the fact that $Q(a)$ and $Q(b)$ are complete allow us to replace either (a_r, b_ℓ) or $[a_r, b_\ell)$ with the interior vertices of P to obtain an $(s+1)$-cycle. In the former case $a^- a_r C^- aPbC^- b_\ell b^+ Ca^-$ is an $(s+1)$-cycle, and the later case $a^- a_r^- C^- aPbC^- b_\ell b^+ Ca^-$ is an $(s+1)$-cycle. Thus, suppose next that $a_r = b_\ell$. If $|Q(a)| \geq 4$ then either $a^- \neq a_\ell$ or $a_r^- \neq a$. If $a^- \neq a_\ell$ then $a^{-2} a_r^- C^- aPbC^- b_\ell^- b^+ Ca^{-2}$ is an $(s+1)$-cycle, and the case where $a_r^- \neq a$ is similar. Therefore by symmetry we have that $|Q(a)| = |Q(b)| = 3$ and $a_r = b_\ell$. As c has a neighbor in $V(G) - V(C)$, Lemma 5.17 implies that $c \notin Q(a) \cup Q(b)$, so that $aPbCc^- c^+ Ca$ is an $(s+1)$-cycle in G.

Suppose $a \in Q(b)$. Then, as $Q(b)$ is complete, we can assume a and b are consecutive on C. Now, if c and d are not consecutive on C, then $aPbCc^- c^+ Cd^- d^+ Ca$ is an $(s+1)$-cycle. If, instead, $c = d^-$ is an edge on C, then the claw $\langle c; c^-, c_1, d^+ \rangle$ implies that $c^- d^+ \in E(G)$, so $aPbCc^- d^+ Ca$ is an $(s+1)$-cycle. Again, the case where $b \in Q(a)$ is identical.

Finally suppose $a_r \in (b_\ell, b)_C$ and $a \in (b_r^+, b_\ell^-)_C$. Then $a^- b_\ell^- C^- aPbC^- b_\ell^{+2} b^+ Ca^-$ is an $(s+1)$-cycle that skips the vertices b_ℓ and b_ℓ^+. Note that $b_\ell^+ \neq b$ since $a_r \in (b_\ell, b)_C$.

Therefore, suppose that the length of P is 5 and that $ab \notin E(G)$, so $a \notin Q(b)$ and $b \notin Q(a)$. Suppose first that $a_r \in [b_\ell, b)_C$, and note that this must also be the case in $C' = C(c, d)$. Thus, $C" = a^- a_r^- C'^- aPbC'^- a_r^+ b^+ C' a^-$ skips a_r and, as it also omits c and d, has length $s+1$. Consequently, we may assume that a, a_r, b_ℓ and b appear in that order along C in the positive direction. If a_r and b_ℓ are consecutive on C then let $C' = C(c)$ and consider $C" = a^- a_r^- C'^- aPbC'^- b_\ell^+ b^+ C' a^-$. This cycle skips a_r and b_ℓ on C' and has length $s+1$.

It remains to consider when $|(a_r, b_\ell)_C| \in \{1, 2, 3\}$. If $|(a_r, b_\ell)_C| = 1$, then $a^- a_r^- C^- a PbC^- b_\ell^+ b^+ Ca^-$ is an $(s+1)$-cycle in G. Similarly, if $|(a_r, b_\ell)_C| = 2$ or $|(a_r, b_\ell)_C| = 3$, then $a^- a_r^- C^- aPbC^- b_\ell b^+ Ca^-$ and $a^- a_r C^- aPbC^- b_\ell b^+ Ca^-$, respectively, are $(s+1)$-cycles in G. Thus we may conclude that $|(a_r, b_\ell)_C| \geq 4$, as desired, and symmetrically, that $|(b_r, a_\ell)_C| \geq 4$. $\square$

The following lemma by Gould, Łuczak and Pfender [30] will be useful as we proceed.

Lemma 5.19 (Gould, Łuczak, Pfender [30])**.** *Let G be a claw-free graph with minimum degree $\delta(G) \geq 3$, and let C be a cycle of length t with no hops, for some $t \geq 5$. Set*

$$X = \{v \in V(C) | \textit{there is no chord incident to } v\},$$

and suppose for some chord xy of C we have $|X \cap V(xCy)| \leq 2$. Then G contains cycles C' and C'' of lengths $t-1$ and $t-2$, respectively.

Lemma 5.20. *Suppose $s = |V(C)| \geq 6$ and let a and b be distinct elements of $\{w, x, y, z\}$ where at least one of a' or b' is not on C and $a_1 b$ is not an edge. Further, let $P = aI_a a' b' I_b b$ have length λ. If $2 \leq \lambda \leq 5$ and any of the following hold*

(1) there is an edge between $\{a_\ell^-, a_\ell, a\}$ and $\{b_\ell^-, b_\ell, b\}$, or

(2) there is an edge between $\{a, a_r, a_r^+\}$ and $\{b, b_r, b_r^+\}$, or

(3) $Q(a) = Q(b)$,

then G contains an $(s+1)$-cycle unless either $P = avb$ and the only edge satisfying (1) or (2) is ab.

Proof. Note first that conditions (1) and (2) are identical up to the reversal of C, so it suffices to assume that either (1) or (3) holds.

If $Q(a) \neq Q(b)$ and $ab \in E(G)$, then since we assume that $a_1b \notin E(G)$, $\langle a; a_1, a_\ell, b\rangle \to \{a_\ell b\}$. Further, if either $a_\ell b$ or ab_ℓ is an edge of G, then $\langle b; b_1, b_\ell, a_\ell\rangle \to \{a_\ell b_\ell\}$ or $\langle a; a_1, a_\ell, b_\ell\rangle \to \{a_\ell b_\ell\}$, respectively. Therefore any edge between $\{a, a_\ell\}$ and $\{b, b_\ell\}$ implies $a_\ell b_\ell$ is an edge. Up to renaming, the case when $a_\ell^- b \in E(G)$ is symmetric to the case when $ab_\ell^- \in E(G)$. Similarly, the case $a_\ell^- b_\ell \in E(G)$ is symmetric to the case $a_\ell b_\ell^- \in E(G)$. Therefore when $Q(a) \neq Q(b)$ it is enough to suppose that one of $a_\ell b_\ell$, $a_\ell^- b$, $a_\ell^- b_\ell$, or $a_\ell^- b_\ell^-$ is an edge of G.

Case 1: $|\mathcal{O}| \geq \lambda$.

First suppose $Q(a) = Q(b)$, so we can assume a and b appear consecutively on C. Select $S \subseteq \mathcal{O} - \{a, b\}$, where $|S| = \lambda - 2$. Then $C(S)$ has length $s - \lambda + 2$ and a and b are consecutive vertices in $C(S)$, so $aPbC(S)a$ is an $(s+1)$-cycle in G. Thus, for the remainder of this case we will assume that condition (1) holds.

If $Q(a) \neq Q(b)$, then (because $Q(a)$ and $Q(b)$ are cliques so internal vertices may be reaaranged) we can assume both aa_ℓ and bb_ℓ are edges in C. We show that if there is an edge a^*b^* satisfying condition (1) (i.e. $a^* \in \{a, a_\ell, a_\ell^-\}$ and $b^* \in \{b, b_\ell, b_\ell^-\}$), then there is a set $S \subseteq \mathcal{O}$ such that $C(S)$ contains the vertices a^* and b^*.

When $a_\ell b_\ell \in E(G)$, select $S \subseteq \mathcal{O} - \{a, b\}$ with $|S| = \lambda - 2$. By Lemma 5.17 $a_\ell, b_\ell \notin \mathcal{O}$, so a, a_ℓ, b, and b_ℓ are all vertices in $C(S)$, and both aa_ℓ and bb_ℓ lie on $C(S)$. Then, the cycle $a_\ell b_\ell C(S)^- aPbC(S)a_\ell$ has length $s+1$.

Suppose then, that $a_\ell^- b \in E(G)$. If $\lambda = 2$, then $a_\ell^- bPaCb^- b^+ Ca_\ell^-$ is an $(s+1)$-cycle. When $\lambda \geq 3$ select $S \subseteq \mathcal{O} - \{a, b, a_\ell^-\}$ with $|S| = \lambda - 3$ so that a, b, and a_ℓ^- are all vertices in $C(S)$ and vertices $a_\ell^- a_\ell a$ appear consecutively in $C(S)$. The cycle $a_\ell^- bPaC(S)b^- b^+ C(S)a_\ell^-$ skips a_ℓ and has length $s+1$.

Another possibility is that $a_\ell^- b_\ell \in E(G)$. If $\lambda = 2$, then $a_\ell^- b_\ell C^- a^+ a_\ell aPbCa_\ell^-$ is an $(s+1)$-cycle. When $\lambda \geq 3$ select $S \subseteq \mathcal{O} - \{a, b, a_\ell^-\}$ with $|S| = \lambda - 3$. The cycle $a_\ell^- b_\ell C(S)^- aPbC(S)a_\ell^-$ has length $s+1$.

Finally, we assume that $a_\ell^- b_\ell^- \in E(G)$. If $\lambda = 2$ or $\lambda = 3$, then $a_\ell^- b_\ell^- C^- a^+ a_\ell a P b b_\ell b^+ C a_\ell^-$ or $a_\ell^- b_\ell^- C^- a P b b_\ell b^+ C a_\ell^-$ is an $(s+1)$-cycle. When $\lambda \geq 4$ select $S \subseteq \mathcal{O} - \{a, b, a_\ell^-, b_\ell^-\}$ with $|S| = \lambda - 4$. The cycle $a_\ell^- b_\ell^- C(S)^- a P b C(S) a_\ell^-$ has length $s+1$. This completes Case 1.

We know $\{w, x, y, z\} \subseteq \mathcal{O}$, so it follows that if $\lambda \leq 4$, then G has an $(s+1)$-cycle. Thus, we need only to consider the cases where $\lambda = 5$ and $|\mathcal{O}| = 4$ (so specifically $\mathcal{O} = \{w, x, y, z\}$), and where $\lambda = 6$ and $4 \leq |\mathcal{O}| \leq 5$.

Case 2: $\lambda = 5$ and $\mathcal{O} = \{w, x, y, z\}$.

Case 2.1: Suppose $Q(a) = Q(b)$ or $a_\ell b_\ell \in E(G)$.

Let $S = \{c, d\}$. If $Q(a) = Q(b)$, then we may assume a and b are consecutive on C and we let $C' = aPbC(S)a$, and if $a_\ell b_\ell \in E(G)$ let $C' = a_\ell b_\ell C(S)^- a P b C(S) a_\ell$. Note that in both cases C' has length $s+2$. If the cycle C' contains a hop, then G has an $(s+1)$-cycle, so we suppose that C' does not contain a hop.

Let $u \in V(C) - (\mathcal{O} \cup \{c_\ell, c_r\})$ and note $d(u) \geq 4$. Suppose first that u does not have a chord in C', which implies that u must be adjacent to both c and d (which are not in $V(C')$). As $u \notin \{c_\ell, c_r\}$, $\langle c; c_1, u, c_\ell \rangle \to \{uc_\ell\}$ and $\langle c; c_1, u, c_r \rangle \to \{uc_r\}$ implying that $c_\ell u$ and $c_r u$ are edges. As u does not have a chord in C', then $c_\ell u c_r$ appear consecutively and $c_\ell c_r$ is a hop on C'. This contradiction implies that G has an $(s+1)$-cycle. Note that when $Q(a) = Q(b)$ that ab is a chord of C', and when $a_\ell b_\ell \in E(G)$ then $a_\ell a$ and $b_\ell b$ are chords of C'. Therefore every vertex in $V(C') - \{c_\ell, c_r\}$ must have a chord in C'. Consequently, every chord of C' satisfies the conditions of Lemma 5.19, so G has an $(s+1)$-cycle.

Case 2.2: One of $a_\ell^- b$, $a_\ell^- b_\ell$ or, $a_\ell^- b_\ell^-$ is an edge.

Suppose $a_\ell^- \notin \mathcal{O}$. Then $|\mathcal{O} - \{a, b\}| = \lambda - 3$ and $|\mathcal{O} - \{a, b, b_\ell^-\}| \geq \lambda - 4$. This implies that regardless which of $a_\ell^- b$, $a_\ell^- b_\ell$, or $a_\ell^- b_\ell^-$ is an edge of G, either $S = \mathcal{O} - \{a, b, a_\ell^-\}$ or $S = \mathcal{O} - \{a, b, a_\ell^-, b_\ell^-\}$ can be used to show that G has an $(s+1)$

cycle via an identical argument as was used in Case 1.

Thus $a_\ell^- \in \mathcal{O}$; without loss of generality assume $a_\ell^- = c$. If $a_\ell^- b_\ell^- = cb_\ell^-$ is an edge and $b_\ell^- \notin \mathcal{O}$, then $\langle x; c_1, c^-, b_\ell^- \rangle \to \{x^- b_\ell^-\}$ and $aPbCc^- b_\ell^- C^- a$ is an $(s+1)$-cycle. If $b_\ell^- \in \mathcal{O}$ and $c_1 b_\ell^-$ is an edge, i.e. $b_\ell^- = d$, then we now that c_1 or d_1 lies on the subgraph F. By Lemma 5.15 we know that $c_1 = d_1$ for $c, d \in \{w, x, y, z\}$ can only occur if if $c_1 = v$ or $c_1 = x_1$. When $c_1 = x$, the path $aI_a a' b' I_b b$ has length 2 and not 5, and when $c_1 = v$, the vertex v appears along the path $aI_a a' b' I_b$, thus the path from c to a or from d to b has length strictly less than 5 and we are in in Case 1. If $a_\ell^- b_\ell = cb_\ell$ is an edge, then $\langle c; c_1, a_\ell, b_\ell \rangle \to \{a_\ell b_\ell\}$ which by Case 2.1 implies G has an $(s+1)$-cycle. Finally, if $a_\ell^- b = bc$ is an edge, then the claw $\langle b; b_1, c, b_\ell \rangle \to \{b_1 c, b_\ell c\}$ so we may assume that $b_1 c$ is an edge. Then the path $cI_b b' a' I_a a$ has length λ, where $c_\ell a_\ell \in E(G)$ (since $a_\ell \in Q(c)$). Again by Case 2.1 we know G has an $(s+1)$-cycle. $\square$

In the proof we require $s \geq 6$, yet to this point we have only demonstrated that G must contain cycles of length $3, 4$ and 5. The following observation shows that if $\lambda \leq 6$, then we may assume $s \geq 6$.

Claim 5.21. *If the path $P = aI_a a' b' I_b b$ has length 2, 3, 4 or 5, then G has a 6-cycle.*

Proof. Suppose for the sake of contradiction that G has no 6-cycle, so $s = |V(C)| = 5$. We may assume that C contains w, x, y, and z, and some fifth vertex p, in that order. As w, x, and z all have hops, we immediately get that $G[V(C)]$ is either complete or missing only the edge wz. Let p be the vertex in the set $V(C) - \{w, x, y, z\}$. By Lemma 5.17 the vertex p must be both a_ℓ and a_r for every choice of a, so $G[V(C)]$ is therefore complete. Consequently $aPbCa$ has length $s + \lambda - 1$ and we can skip any subset of vertices in $V(C) - \{a, b\}$ necessary to obtain a 6-cycle. $\square$

The next lemma shows that under certain conditions we can guarantee that G has an induced $N(3,2,1)$ net. This will be especially helpful in later cases of the proof in the next section to handle when P_x or P_w is long.

Lemma 5.22. *Suppose $y' = y$, $z' = z$, and hence $x' = v$. Let $u \in V(C)$ be a vertex where there is an induced $(u-x')$-path $P = x'u_k \dots u_1 u$ whose internal vertices are all off $V(C)$. If the length of P is at least 2 and there are no edges between $\{u_2, \dots, u_k\}$ and $V(C)$, then G contains an induced $N(3,2,2)$.*

Proof. Define $I_u = \{u_1, \dots, u_k\}$ and note that $G[V(C)]$ is not a clique since $yx'z$ has length 2 and we are assuming that G contains no $(s+1)$-cycle. We may also assume that there are no edges between $\{y, y_\ell, y_\ell^-\}$ and $\{z, z_\ell, z_\ell^-\}$, lest again G contains an $(s+1)$-cycle by Lemma 5.20.

First we show that if $|I_u| \leq 3$ then u_1 is not adjacent to $\{u_\ell, u_\ell^-, y_\ell, y_\ell^-, z_\ell, z_\ell^-\}$. By Lemma 5.17, a_ℓ is not adjacent to u_1 for $a \in \{x, y, z\}$. If $u_1 u_\ell^-$ is an edge then G immediately has an $(s+1)$-cycle. Suppose $u_1 a_\ell^-$ is an edge for $a \in \{y, z\}$. Then the cycle $a_\ell^- I_u x' y C^- a_\ell a^+ C a_\ell^-$ is an $s + |I_u|$-cycle. We can skip any of the vertices $\{a_\ell, w, x, y, z\} - \{a, a_\ell^-\}$ to form an $(s+1)$-cycle.

Note that when $|I_u| = 3$ implies that u_1 is not adjacent to $\{y_\ell, y_\ell^-, z_\ell, z_\ell^-\}$, and since $|I_u| > 0$ u_1 is not adjacent to y or z. When $|I_u| \geq 3$, the net $N(x'yz; u_k u_{k-1} u_{k-2}, y_\ell y_\ell^-, z_\ell z_\ell^-)$ is induced.

When $|I_u| \in \{1,2\}$ the net $N(x'yz; I_u u u_\ell, y_\ell y_\ell^-, z_\ell z_\ell^-)$ is either induced, or G has an $(s+1)$-cycle by Lemma 5.20. $\square$

Together the lemmas from this section provide a framework for showing a 4-connected, $\{K_{1,3}, N\}$-free graph G containing an s-cycle C contains an $(s+1)$-cycle. The proof technique for Theorem 5.9 consists of considering a claw or net denoted by H. Since G is $\{K_{1,3}, N\}$-free, H is not induced. We proceed by case analysis showing each possible adjacency within H reaches a contradiction. When applicable Lemma 5.15 provides many non-adjacencies simplifying this case analysis. Lemma 5.20 is also frequently used to show non-adjacencies between certain vertices. By Claim 5.21 if there is a path $P = aI_a a'b'I_b b$ of length at most five, then G has a 6-cycle. If P has length at most five, then $G[V(C)]$ can not be a clique. Moreover, whenever one of these paths has length 2, 3 or 4, there is a vertex between $Q(a)$ and $Q(b)$ by Lemma 5.18. All of these conclusions will be useful as we proceed.

5.3.3 Long cycles for $N(3,2,1)$

This section concludes with proving that if G is 4-connected, $\{K_{1,3}, N(3,2,1)\}$-free and contains a cycle of length s, then G has an $(s+1)$-cycle.

Theorem 5.23. *Every* 4*-connected,* $\{K_{1,3}, N(3,2,1)\}$*-free graph of order* n *contains cycles of length* s *for* $6 \leq s \leq n$*.*

Proof. We proceed by induction on s. By Lemma 5.14, G has cycles of length 3, 4, and 5. Thus we assume that C is a cycle of length s, where $5 \leq s \leq n-1$, as described at the beginning of Subsection 5.3.2. The proof is broken up into cases based on how many vertices of x', y' and z' are in $V(C)$. Throughout the proof we assume there is no $(s+1)$-cycle, and we either find an induced $N(3,2,1)$ or an $(s+1)$-cycle, leading to a contradiction in either case.

If $aI_a a'b'I_b b$ has length at most 5 for $a, b \in \{x, y, z\}$ and $G[V(C)]$ is a clique, then G contains an $(s+1)$-cycle, a contradiction. Therefore whenever one of these path lengths is short we can assume $G[V(C)]$ is not a clique. Moreover, whenever one of these paths has length 2, 3 or 4, there is a vertex between $Q(a)$ and $Q(b)$ by Lemma 5.18.

Case 1: None of the vertices x', y', z' are in $V(C)$.

Note that by Lemma 5.15 the only edges from $F - \{x, y, z\}$ to $V(C)$ are x_1x, y_1y and z_1z.

Case 1.1: $|I_x| \geq 3$.

When $|I_y| \geq 1$ and $|I_z| \geq 1$ the net $N(x'y'z'; x_px_{p-1}x_{p-2}, y_qy_{q-1}, z_t)$ is an induced $N(3,2,1)$ (by construction of F and Lemma 5.15), a contradiction. Similarly, when $|I_y| \geq 2$ and $|I_z| = 0$ the net $N(x'y'z'; x_px_{p-1}x_{p-2}, y_qy_{q-1}, z)$ is an induced $N(3,2,1)$. When $|I_y| \leq 1$ and $|I_z| = 0$ the net $N(x'y'z'; I_x, I_yyy_\ell, zz_\ell z_\ell^-)$ is either induced, so there is an $N(3,2,1)$ induced subgraph, or G has an $(s+1)$-cycle by Lemma 5.20.

Case 1.2: $|I_x| = 2$.

Let u be y_ℓ if $G[V(C)]$ is a clique, and any vertex in $V(C) - \{x, y, z\}$ otherwise. When $|I_z| \geq 1$ the net $N(y'x'z'; I_yyu, x_2x_1, z_t)$ is induced and contains an induced $N(3,2,1)$. When $|I_y| = 2$ and $|I_z| = 0$ the path $yI_yy'z'z$ has length 5, so $G[V(C)]$ is not a clique. When $z_\ell^- \neq y$, the net $N(z'x'y'; zz_\ell z_\ell^-, y_2y_1, x_2)$ is induced by Lemma 5.15, and when $z_\ell^- = y$ the net $N(z'x'y'; zz_\ell z_\ell^-, x_2x_1, y_1)$ is induced. When $|I_y| \leq 1$ and $|I_z| = 0$, the net $N(x'y'z'; I_x, I_yyy_\ell, zz_\ell z_\ell^-)$ either is induced, so there is a $N(3,2,1)$ induced subgraph, or G has an $(s+1)$-cycle by Lemma 5.20.

Case 1.3: $|I_x| \leq 1$.

First consider the case when $|I_x| = |I_y| = |I_z| = 1$. The path $aa_1a'b'b_1b$ has length 5 for any distinct $a, b \in \{x, y, z\}$, so $G[V(C)]$ is not a clique. By Claim 5.21 the graph G has a 6-cycle, so $s \geq 6$. By Lemmas 5.20 and 5.15 the net $N(x'y'z'; x_1xx_\ell, y_1y, z_1)$ is either induced or G has an $(s+1)$-cycle.

When $|I_y| = 1$ and $|I_z| = 0$, the net $N(x'y'z'; x_q, y_1yy_\ell, zz_\ell)$ is an induced $N(3,2,1)$ or G has an $(s+1)$-cycle by Lemmas 5.20 and 5.15. When $|I_x| \leq 1$ and $|I_y| = 0$, the net $N(x'y'z'; I_xxx_\ell x_\ell^-, yy_\ell y_\ell^-, zz_\ell z_\ell^-)$ is either induced, so there is an induced $N(3,2,1)$, or G has an $(s+1)$-cycle by Lemmas 5.20 and 5.15.

Case 2: The vertex z' is in $V(C)$, but the vertices x' and y' are not.

By Lemma 5.15 the only edges from $F - \{x, y, z\}$ to $V(C)$ are x_1x, y_1y, xz and yz.

Case 2.1: $|I_y| \geq 3$.

Let u be z_ℓ if $G[V(C)]$ is not a clique, and any vertex in $V(C) - \{x, y, z\}$ otherwise. The net $N(x'y'z; I_x, I_y, u)$ is induced, and contains an induced $N(3, 2, 1)$ net.

Case 2.2: $1 \leq |I_y| \leq 2$.

The net $N(x'y'z; I_x, I_y yy_\ell, z_\ell z_\ell^-)$ is induced, so $N(3, 2, 1)$ is an induced subgraph, or G has an $(s+1)$-cycle by Lemmas 5.20 and 5.15.

Case 2.3: $|I_y| = 0$.

When $|I_x| \geq 3$, the net $N(x'y'z; I_x, yy_\ell, z_\ell z_\ell^-)$ is induced, so $N(3, 2, 1)$ is an induced subgraph or G has an $(s+1)$-cycle by Lemmas 5.20 and 5.15. When $|I_x| = 2$, the net $N(y'zx'; yy_\ell y_\ell^-, z_\ell z_\ell^-, x_2)$ is induced or there is an $(s+1)$-cycle by Lemma 5.20. When $|I_x| \leq 1$, the net $N(x'y'z'; I_x x_\ell x_\ell^-, yy_\ell y_\ell^-, z_\ell z_\ell^-)$ is either induced, so $N(3, 2, 1)$ is an induced subgraph, or G has an $(s+1)$-cycle.

Case 3: The vertices y' and z' are both in $V(C)$ (and thus $x' = v$ is not in $V(C)$).

Case 3.1: $|I_x| > 0$.

Notice that $x'x_p \dots x_1x$ has length at least two, and there are no edges between $\{x_2, \dots, x_p\}$ by the minimality of the path. Therefore by Lemma 4.9 the graph G has an induced $N(3, 2, 2)$, which implies there is an induced $N(3, 2, 1)$ subgraph.

Case 3.2: $|I_x| = 0$.

Assume v and w are chosen to minimize the length of I_w, given that v is adjacent to three vertices in $V(C)$.

First we show that xyz forms a triangle. By Lemmas 5.20 and 5.15 the net $N(vyz; xx_\ell x_\ell^-, y_\ell y_\ell^-, z_\ell) \to \{xy, xz\}$. If $xy \in E(G)$, then $\langle y; y_\ell, z, x\rangle \to \{xz\}$, and if

$xz \in E(G)$, then $\langle z; z_\ell, y, x\rangle \to \{xy\}$, which proves xyz is a triangle.

Suppose $|I_w| > 0$, and let $w_i u$ be an edge with $w_i \in I_w - \{w_1\}$ and $u \in V(C)$. If $u \notin \{x, y, z\}$, then this contradicts the choice of w. If $u \in \{x, y, z\}$, then the claw $\langle u; u_\ell, b, w_i\rangle \to \{bw_i\}$ (by Lemmas 5.17 and 5.20) for any $b \in \{x, y, z\} - \{u\}$. Therefore, w_i is adjacent to $\{x, y, z\}$ and contradicts the extremal choice of v. Since $|I_w| > 0$, G contains an induced $N(3, 2, 2)$ by Lemma 5.22.

Next we consider the case when $|I_w| = 0$. The vertex v is adjacent to the vertices w, x, y and z. Let $N_C(v)$ be the set of vertices adjacent to v that are also in $V(C)$. Next we show that $G[N_C(v)]$ is complete. Let $a, b \in N_C(v) - \{x, y\}$. We show that a is adjacent to both x and y by first considering the net $N(vxy; aa_\ell a_\ell^-, x_\ell, x_\ell^-, y_\ell) \to \{ax, ay\}$. If $ax \in E(G)$, then $\langle x; x_\ell, a, y\rangle \to \{ay\}$ by Lemmas 5.17 and 5.20, and if $ay \in E(G)$, then $\langle y; y_\ell, a, x\rangle \to \{ax\}$. Similarly b is adjacent to both x and y. The claw $\langle y; y_\ell, a, b\rangle \to \{ab\}$ by Lemma 5.20, which proves that $G[N_C(v)]$ is complete.

Up to renaming, the vertices w, x, y and z appear in this order on C. By Lemma 5.18, the cliques $Q(a)$ and $Q(b)$ do not intersect for any distinct $a, b \in \{w, x, y, z\}$. If $w_r x_\ell$, $x_r y_\ell$, $y_r z_\ell$ and $z_r w_\ell$ are edges, then the net $N(w_r w_\ell w; x_\ell x_r y_\ell, z_r z_\ell, v) \to \{w_r y_\ell, w_r z_\ell, w_\ell x_r, x_\ell z_r, x_r z_\ell, y_\ell z_r\}$. Each of these edges has the form $a_\ell b_r$, where $a, b \in \{w, x, y, z\}$, and there is an edge $a_\ell c_r$ where $c \in \{w, x, y, z\} - \{a, b\}$. For each edge $a_\ell b_r$ in the previous set, the claw $\langle a_\ell; a, b_r, c_r\rangle \to \{ab_r, ac_r, b_r c_r\}$, which by Lemma 5.20 implies that G has an $(s + 1)$-cycle.

Therefore the graph G does not contain all the edges $w_r x_\ell$, $x_r y_\ell$, $y_r z_\ell$ and $z_r w_\ell$. Up to renaming we can assume $w_r x_\ell$ is not an edge of G, while the other edges may or may not be edges in G. There exists a vertex $\gamma \in (w_r, x_\ell)_C$, where $w_r\gamma \in E(G)$, but $w_r u \notin E(G)$ for all $u \in (\gamma, x_\ell]_C$. We will use the vertex γ to find an induced $N(3, 2, 1)$ net or an $(s + 1)$-cycle.

We show the vertex v is adjacent to exactly four vertices in $V(C)$. For each pair of distinct $a, b \in N_C(v) - \{w\}$, the net $N(wab; w_r\gamma\gamma^+, a_ra_r^+, b_r) \to \{\gamma a_r, \gamma^+ a_r, \gamma b_r, \gamma^+ a_r, \gamma^+ a_r^+, \gamma^+ b_r\}$. Suppose γ is adjacent to a vertex $a_r^* \in \{a_r, a_r^+\}$, for some $a \in N_C(v) - \{w\}$. Then γ^+ is adjacent to a_r^* by considering the claw $\langle \gamma; w_r, \gamma^+, a_r^* \rangle$. If v is adjacent to more than four vertices in $V(C)$, then the nets and claws in the previous three sentences imply that γ^+ must be adjacent to three vertices a_r^*, b_r^* and c_r^*, where $a_r^* \in \{a_r, a_r^+\}$, $b_r^* \in \{b_r, b_r^+\}$ and $c_r^* \in \{c_r, c_r^+\}$, for three distinct vertices $a, b, c \in N_C(v) - \{w\}$. This implies that G has an $(s+1)$-cycle by the claw $\langle \gamma^+; a_r^*, b_r^*, c_r^* \rangle$ and Lemma 5.20.

Thus v is adjacent to only the vertices w, x, y and z in $V(C)$. If γ^+ is adjacent to a vertex $a_r^* \in \{a_r, a_r^+\}$ for every $a \in \{x, y, z\}$, then G has an $(s+1)$-cycle by considering the claw $\langle \gamma^+; x_r^*, y_r^*, z_r^* \rangle$ and Lemma 5.20. Therefore γ^+ must not be adjacent to c_r and c_r^* for some $c \in \{x, y, z\}$. The net $N(wca; w_r\gamma\gamma^+, c_rc_r^+, a_r) \to \{\gamma a_r, \gamma^+ a_r\}$ for every $a \in \{x, y, z\} - \{c\}$. Recall from the previous paragraph that γa_r being an edge implies that $\gamma^+ a_r$ is an edge. Thus γ^+ is adjacent to a_r and b_r for distinct $a, b \in \{x, y, z\} - \{c\}$.

Up to renaming, the vertex a_r appears before b_r in $(\gamma, w)_C$. Next we show that if G contains certain edges, then G has an $(s+1)$-cycle. To help simplify the $(s+1)$-cycles we assume that aa_r appear consecutively on the cycle C for all $a \in \{w, x, y, z\}$. Recall the cliques $Q(w)$, $Q(x)$, $Q(y)$ and $Q(z)$ are disjoint by Lemma 5.18. If C does not have the property that aa_r appear consecutively, then we can find a new cycle C' on vertex set $V(C)$ by taking all the vertices in $Q(a) - \{a_\ell, a_r\}$ and reorder them such that a appears last (note that since $Q(a)$ is a clique C' is a cycle in G).

If $\gamma\gamma^{+2}$ is an edge, then $\gamma\gamma^{+2}CavbC^-a_r\gamma^+b_rC\gamma$ is an $(s+1)$-cycle. If $\gamma^{+2}b_r$ is an edge, then $\gamma^{+2}b_rC\gamma^+a_rC^-bvaC^-\gamma^{+2}$ is an $(s+1)$-cycle. If γb_ℓ is an edge, then $\gamma b_\ell C^-a_r\gamma^+CavbC^-b_\ell^-b_rC\gamma$ is an $(s+1)$-cycle. If γa_r is an edge, then $\gamma a_rCbvaC^-\gamma^+ b_rC\gamma$ is an $(s+1)$-cycle. Thus, $\gamma\gamma^{+2}$, $\gamma^{+2}b_r$, γb_ℓ and γa_r are not edges in G. The claws $\langle \gamma^+; \gamma, \gamma^{+2}, b_r \rangle \to \{\gamma b_r\}$ and $\langle \gamma^+; \gamma, \gamma^{+2}, a_r \rangle \to \{\gamma^{+2} a_r\}$ imply that G has the edges γb_r and $\gamma^{+2} a_r$.

If $b_\ell b_r^+$ is an edge, then $b_\ell b_r^+ C \gamma b_r b_\ell^+ C^- bva C^- \gamma^+ a_r C b_\ell$ is an $(s+1)$-cycle. The claws $\langle b_r; b_r^+, b_\ell, \gamma \rangle \to \{\gamma b_r^+\}$ and $\langle \gamma; w_r, \gamma^+, b_r^+ \rangle \to \{\gamma^+ b_r^+\}$ ($w_r b_r^+ \notin E(G)$ by Lemma 5.20, and $\gamma^+ w_r \notin E(G)$ by the choice of γ) imply that G has the edges γb_r^+ and $\gamma^+ b_r^+$.

Case 3.2.1: γ^+ is adjacent to a.

Consider the net $N = N(\gamma\gamma^+ b_r; w_r w_\ell w_\ell^-, av, b_\ell)$. We show that N is induced or there is an $(s+1)$-cycle.

We now give a rather lengthy statement showing that if G has certain edges, then G has an $(s+1)$-cycle. If γb_ℓ is an edge, then $\gamma b_\ell C^- a_r \gamma^+ C avb C^- b_\ell^+ b_r C \gamma$ is an $(s+1)$-cycle; if $\gamma^+ b_\ell \in E(G)$, then $\gamma^+ b_\ell C^- a_r \gamma^{+2} C avb C^- b_\ell^+ b_r C \gamma^+$ is an $(s+1)$-cycle; if $\gamma w_\ell \in E(G)$, then $w_\ell \gamma C^- w_r w_\ell^+ C wva C^- \gamma^+ a_r C w_\ell$ is an $(s+1)$-cycle; if $\gamma^+ w_\ell \in E(G)$, then $w_\ell \gamma^+ C^- w_r w_\ell^+ C wva C^- \gamma^{+2} a_r C w_\ell$ is an $(s+1)$-cycle; if $\gamma w_\ell^- \in E(G)$, then $w_\ell^- \gamma C^- w_r w_\ell C wva C^- u^+ a_r C w_\ell^-$ is an $(s+1)$-cycle; if $\gamma^+ w_\ell^- \in E(G)$, then $w_\ell^- \gamma^+ C^- w_r w_\ell C wva C^- \gamma^{+2} a_r C w_\ell^-$ is an $(s+1)$-cycle; if $b_r w_\ell^- \in E(G)$, then $b_r w_\ell^- C^- b_r^+ \gamma C^- w_r w_\ell C wva C^- \gamma^+ a_r C b_r$ is an $(s+1)$-cycle; if $b_r w_\ell \in E(G)$, then $b_r w_\ell C^- b_r^+ \gamma C^- w_r w_\ell^+ C wva C^- \gamma^{+2} a_r C b_r$ is an $(s+1)$-cycle.

Next we show that v is not adjacent to any vertex in $V(N) - \{a\}$. Recall that v is only adjacent to w, x, y and z on C. The vertices γ and γ^+ are in $(w_r, x_\ell]$, so they can not be adjacent to v. By Lemma 5.17 none of the vertices a_r or a_ℓ are adjacent to v for any $a \in \{w, x, y, z\}$. If v is adjacent to w_ℓ^-, then G has an $(s+1)$-cycle.

By Lemma 5.20 and recalling that $a_r \gamma$ is not an edge, we know a can not be adjacent to any vertex in $V(N) - \{\gamma^+, v\}$. By the extremal choice of γ we know $\gamma^+ w_r$ is not an edge. By Lemma 5.20 $a_r w_r$ is not an edge. If $w_r w_\ell^-$ is an edge, then $\langle w_r; w_\ell^-, w, \gamma \rangle \to \{\gamma w_\ell^-, \gamma w\}$ which implies that G has an $(s+1)$-cycle given in the paragraph above. If $w_r b_\ell$ is an edge, then $\langle w_r; w_\ell, b_\ell, \gamma \rangle \to \{\gamma b_\ell, \gamma w_\ell\}$ which implies that G has an $(s+1)$-cycle given in the paragraph above.

This proves that either N is induced or G has one of the edges discussed above, which implies G has an $(s+1)$-cycle.

Case 3.2.2: γ^+ is not adjacent to a.

Now consider the net $N = N(\gamma\gamma^+ b_r; w_r w_\ell w_\ell^-, a_r a, b_\ell)$. The vertex a is not adjacent to γ, since γa_r is not an edge, and a is not adjacent to γ^+ by assumption. Lemma 5.20 shows that a can not be adjacent to any other vertex in $V(N) - \{a_r\}$.

Notice that all the cycles in the previous case never used the edge $\gamma^+ a$. Therefore if any of the edges in the previous paragraph appear, then the $(s+1)$-cycle in the table is a cycle of G. Thus it is enough to consider the edges adjacent to a_r.

Using the fact that $\gamma^+ a$ is not an edge and by the extremal choice of a_r we have $\langle a_r; a, a_r^+, \gamma^+\rangle \to \{\gamma a_r^+\}$. Therefore in this case we can assume γa_r^+ is an edge. By Lemma 5.20 $a_r b_r$ and $a_r w_r$ are not edges. The rest of the edges between vertices of $V(N)$ incident to a_r are given in the next paragraph along with the resulting $(s+1)$-cycle in G.

If $\gamma a_r \in E(G)$, then $\gamma a_r C bva C^- \gamma^+ b_r C \gamma$ is an $(s+1)$-cycle; if $a_r b_\ell$, then $a_r b_\ell C^- a_r^+ \gamma^+ C^- b_r b_\ell^- C bva C^- \gamma^{+2} a_r$ is an $(s+1)$-cycle; if $a_r w_\ell \in E(G)$, then $a_r w_\ell C^- a_r^+ \gamma^+ C^- w_r w_\ell^- C wva C^- \gamma^{+2} a_r$ is an $(s+1)$-cycle; if $a_r w_\ell^- \in E(G)$, then $a_r w_\ell C^- a_r^+ \gamma^+ C^- w_r w_\ell C wv aC^- \gamma^{+2} a_r$ is an $(s+1)$-cycle.

This proves that either the net $N(uu^+ a_r; w_r w_\ell w_\ell^-, b_r v, a_\ell)$ is induced or G has an $(s+1)$-cycle. □

5.4 Future work

The section above proves that a 4-connected, $\{K_{1,3}, N(3,2,1)\}$-free graph is pancyclic. The proof for the nets $N(2,2,2)$ and $N(4,1,1)$ are similar in nature to the proof of

Theorem 5.3.3.

Question 1: Determine the family of subgraphs such that every 4-connected, $\{K_{1,4}, Y\}$-free graph is pancyclic. We know the family must be a subgraph of the family of graphs P_9, L, and the generalized nets $N(i, j, k)$ where $i + j + k = 6$.

Question 2: Show that a k-connected $\{K_{1,3}, N(i, j, k)\}$ graph is pancylic for $k \geq 5$ when the sum of i, j, and k is some constant.

Question 3: Find forbidden minor conditions ensuring that a graph is pancyclic or hamiltonian. Mark Ellingham [20] at CanaDAM 2013 presented a result for hamiltonicity of 3-connected planar graphs with a forbidden minor.

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