A Sensitivity Analysis of the Premiums for a Permanent Health Insurance (PHI) Model

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A Sensitivity Analysis of the Premiums for a Permanent Health Insurance (PHI) Model

Ben D. Rickayzen*

Abstract†

This paper presents an analysis of the parameters used in a multi-state model for permanent health insurance (PHI). The model is a simplification of that used in the United Kingdom. To avoid using duration dependent probabilities, the model splits the sick state into several sub-states to act as a proxy for duration spent in a particular state. This enables a Markov approach to be adopted. Lapses are incorporated within the model, and the net premium for a particular policy is tested for sensitivity to the various parameters used, including their interaction with the lapse rate. One of our conclusions is that the net premium is insensitive to changes in the lapse rate.

Key words and phrases: PHI benefits, force of transition, Markov chain, lapses

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1 Introduction

1.1 Overview of the U.K. PHI Business

Permanent health insurance (PHI) has been written in the U.K. for over 100 years. The business was a natural extension of the fraternal (Friendly Society) weekly sickness benefit paid to its members. The rise of the welfare state in the early part of the twentieth century saw the state assume some of the responsibilities of the fraternal societies. Consequently, the amount of business written by private insurers was limited.

The PHI business has increased since World War II, with individual and group business being written by a number of insurers. The market consists of a few specialist direct insurers and reinsurers to support their operations.

The U.K. government still provides a small long-term disability benefit. Recovery rates of state claimants are low; the benefit is a substitute for unemployment benefits. Anyone earning more than national average earnings needs to insure, but there is considerable underinsurance. Increasingly the PHI business is being referred to as income protection insurance.

PHI benefits are built around the U.K. pension system and are often expressed in amounts per week or per month. These benefits cease at state pension age, which is currently age 65 for males and age 60 for females. Some limited benefit period business also is written.

The contracts are similar to those issued in North America, but the terminology differs. For example, the elimination period is referred to as the deferred period in the U.K. There are similar exclusions, but benefits are paid in full for behavioral health problems. In addition, benefits are paid whether the cause of disability is due to an accident or to sickness. The major change in the last 20 years has been the switch in individual business from non-cancelable individual business to guaranteed renewable.

The primary difference between group and individual PHI business is the impact of tax on premiums and benefits paid. Under group business, the employer generally pays the premiums, which are tax deductible, and the benefits paid to the employees are taxed as salary. Under the individual business, there is no tax relief on the premiums paid, but the benefits are paid free of income tax. Waiver of premium is included as a benefit provision. The most common deferred periods are one week, four weeks, 13 weeks, 26 weeks, and 52 weeks.
Benefit limitations apply related to pre-disability income. Benefits from all sources are taken into account, including other group and individual insurances and pensions received. Various disability definitions are offered, including inability to follow any occupation.

1.2 Objectives

The objective of this paper is to introduce a practical mathematical model of a U.K. style PHI system. Specifically, the PHI system is modeled using a multi-state process in which, as a healthy individual ages, he or she may become sick then recover, become sick again, etc., until death.\(^1\) Thus the individual’s health fluctuates between two states (sickness and health) until death. If healthy, sick, and dead are viewed as separate states, the probability that a policyholder moves from the sick state to the dead state or to the healthy state depends on the time spent in the sick state. In other words, the transition probabilities depend on duration in a particular state as well as the age of the policyholder.

It is possible to incorporate the duration-dependence aspect in the model, which leads to a much more complicated model. This is the approach used in the 1991 *Continuous Mortality Investigation Report No. 12* (CMIR 12). To obtain numerical values for the transition forces within the PHI model, CMIR 12 splits the sick states into 781 sub-states, each relating to a different duration of sickness. CMIR 12 then calculates probabilities at every \(1/156\)th of a year of age for duration of sickness up to 5 years in all (making 780 sub-states) and all sickness periods beyond 5 years are aggregated. CMIR 12 (Part D) shows how it is possible to obtain numerical values for probabilities, annuities, etc. Clearly, CMIR 12 provides a thorough and complex model.

The approach taken in this paper is to develop a simpler model, one with only three (healthy, sick, and dead) states, then split the sick state into a small number of sub-states. We adopt the approach based on Jones (1994). Though the CMIR 12 technique of splitting the sick states into sub-states pre-dates Jones, Jones’ approach is simpler because it uses constant forces of transition assumption for transition from state to state. This maintains the Markov property of the model. Increasing the number of states makes the state space more complicated, but maintaining the Markov process keeps the calculations tractable.

One advantage of using the simpler model described in this paper is that it can easily be used by actuaries who do not have access to complex models such as CMIR 12 or the detailed data required to use such

\(^1\)For a detailed discussion on the use of multi-state models in disability insurance, see, for example, Haberman and Pitacco (1999).
models. It also can be used as an initial practical model for actuaries who are interested in rough estimates for net premiums for PHI models.

The paper is organized as follows: Section 2 introduces the model of the various transition probabilities. Expressions are derived for the transition probabilities required to obtain actuarial present values. Section 3 explains the connection between the parameters used in the model and those that are derived using data contained in CMIR 12. The data contained in CMIR 12 are used to test the sensitivity of the net premium to some of the parameters involved in the transition probabilities. Section 4 describes the results, while Section 5 provides a summary and conclusions.

2 The Model

2.1 The States and Transition Probabilities

The PHI model has six states labeled one to six.

- State 1 (Super Healthy): This is the state in which new policyholders enter the model when their policy commences. Because they have provided satisfactory medical evidence, new policyholders are deemed to be select lives and therefore healthier than other insured lives of the same age. We describe these lives as super healthy.

- State 2 (Ultimate Healthy): It is likely that, in time, the selection effect will disappear and that the super healthy lives will move to the ultimate form of the healthy state from which they may become sick enough to make a claim under the PHI policy.

- State 3 (Short-Term Sick): It is possible to recover from the short-term sick state 3 and, therefore, to return to state 2.

- State 4 (Long-Term Sick): It is not possible to recover from the long-term sick state. Death is the only mode of exit from this state.

- State 5 (Lapse): We assume that only super healthy policyholders will lapse their policy because policyholders in any other state would find it worthwhile to continue their PHI policy.

- State 6: Death.
A diagrammatic representation of the multi-state model adopted in this paper is displayed in Figure 1.

It is possible to introduce more sickness states as a proxy to a greater number of durations of sickness. This has not been done, however, because it is difficult to choose parameter values for the transition forces between the different sick states. In addition, having more states would increase the computational problems, albeit not insurmountably.

The forces of transition between states in PHI are continuous functions that depend on many factors including such factors as age, sex, income, and the time spent in a state. Though the exact mathematical form of these functions is unknown, we are sure that they are not constant.

Figure 1
Outline of PHI Model
Due to the mathematical difficulties inherent in using continuously varying forces, however, we will adopt the general methodology described in Jones (1994), i.e., we assume that the forces of transition are piecewise constant over each age interval instead.

Suppose there are $n$ states labeled $1, 2, \ldots, n$. Let $\mu_{ij}(x + t)$ denote the force of transition from state $i$ to state $j$ at age $x + t$, for $i, j = 1, 2, 3, \ldots, n$, $x = 0, 1, 2, \ldots$, and $0 \leq t \leq 1$. If state $j$ is not linked directly to state $i$ then $\mu_{ij}(x + t) = 0$. It is convenient also to define, for each $i$,

$$
\mu_{ii}(x + t) = - \sum_{j=1}^{n} \mu_{ij}(x + t), \quad (1)
$$

where $i = 1, 2, 3, \ldots, n$, $x = 0, 1, 2, \ldots$, and $0 \leq t \leq 1$.

The piecewise constant force of transition implies that

$$
\mu_{ij}(x + t) = \mu_{ij}(x) \quad \text{for } x = 0, 1, 2, \ldots \text{ and } 0 \leq t < 1. \quad (2)
$$

One implication of the piecewise constant transition intensities assumption is that the length of time already spent in the current state has no effect on the future length of time that the policyholder will remain in the state, i.e., a memoryless property exists. [See Haberman (1992) for more on the memoryless property of multi-state processes with constant transition intensities.]

Next, let $p_{ij}(t, x)$ be the probability that a life currently exact age $x$ in state $i$ will be in state $j$ in $t$ years time. The common approach\(^\text{2}\) to deriving an expression for $p_{ij}(t, x)$ is to use the Chapman-Kolmogorov backward system of difference-differential equations as contained in Cox and Miller (1965, Chapter 4). The backward system of equations is derived by considering the interval $(0, t]$ as comprising subintervals $(0, h]$ and $(h, t + h]$ and letting $h \to 0$.

$$
\frac{d}{dt} p_{ij}(t, x) = \sum_{k=1}^{n} \mu_{ik}(x)p_{kj}(t, x) \quad (3)
$$

for $i, j = 1, \ldots, n$, $x = 0, 1, \ldots$, and $0 \leq t \leq 1$. These equations lead to a set of difference-differential equations. For illustration purposes, some of the differential equations are presented below:

\(^2\)See, for example, Ramsay (1989), Jones (1994), and Haberman (1995).
\[
\begin{align*}
\frac{d}{dt} p_{11}(t) &= - (\mu_{12} + \mu_{15} + \mu_{16}) p_{11}(t) \\
\frac{d}{dt} p_{12}(t) &= - (\mu_{12} + \mu_{15} + \mu_{16}) p_{12}(t) + \mu_{12} p_{22}(t) \\
\frac{d}{dt} p_{22}(t) &= - (\mu_{23} + \mu_{26}) p_{22}(t) + \mu_{23} p_{32}(t) \\
\frac{d}{dt} p_{23}(t) &= - (\mu_{23} + \mu_{26}) p_{23}(t) + \mu_{23} p_{33}(t) \\
\frac{d}{dt} p_{33}(t) &= \mu_{32} p_{23}(t) - (\mu_{32} + \mu_{34} + \mu_{36}) p_{33}(t) \\
\frac{d}{dt} p_{32}(t) &= \mu_{32} p_{22}(t) - (\mu_{32} + \mu_{34} + \mu_{36}) p_{32}(t) \\
\frac{d}{dt} p_{44}(t) &= - \mu_{46} p_{44}(t) \\
\frac{d}{dt} p_{46}(t) &= - \mu_{46} p_{46}(t) + \mu_{46} p_{66}(t) \\
\frac{d}{dt} p_{55}(t) &= - \mu_{56} p_{55}(t) \\
\frac{d}{dt} p_{56}(t) &= - \mu_{56} p_{56}(t) + \mu_{56} p_{66}(t) \\
\frac{d}{dt} p_{66}(t) &= 0
\end{align*}
\]

The easiest way to solve the system of differential equations given in equation (3) is to follow the method outlined by Cox and Miller (1965), which involves matrix manipulation. First define the following \(n \times n\) matrices

\[
\begin{align*}
M(x) &= \{\mu_{ij}(x)\}_{i,j=1}^{n} = \text{The forces of transition matrix;}
\end{align*}
\]

\[
\begin{align*}
P(t,x) &= \{p_{ij}(t,x)\}_{i,j=1}^{n} = \text{The transition probability matrix; and}
\end{align*}
\]

\[
\begin{align*}
P'(t,x) &= \{\frac{d}{dt} p_{ij}(t,x)\}_{i,j=1}^{n}.
\end{align*}
\]

The Chapman-Kolmogorov backward system of equations may be written as

\[
P'(t,x) = M(x)P(t,x)
\]

for \(x = 0, 1, \ldots,\) and \(0 \leq t \leq 1,\) with boundary condition \(P(0,x) = I\) (where \(I\) is the identity matrix).

It is easily seen that equation (5) has the solution

\[
P(t,x) = e^{tM(x)} = I + \sum_{k=1}^{\infty} \frac{t^k}{k!} (M(x))^k.
\]

If it is known that \(M(x)\) has distinct eigenvalues \(d_1(x), d_2(x), \ldots, d_n(x),\) then
\[ M(x) = A(x)D(x)A(x)^{-1} \]  

(7)

where \( D \) is the diagonal matrix

\[ D = \text{diag}(d_1(x), d_2(x), \ldots, d_n(x)) \]

and the \( i^{th} \) column of \( A(x) \) is the right-eigenvector associated with \( d_i(x) \) (Cox and Miller 1965, Chapter 4.5). Equations (6) and (7) lead to the following expression for \( P(t, x) \):

\[ P(t, x) = A(x)\text{diag}(e^{td_1(x)}, \ldots, e^{td_n(x)})A(x)^{-1}. \]  

(8)

In this paper equation (8) is used to compute \( P(t, x) \).

Once \( P(t, x) \) is known for \( x = 0, 1, \ldots \) and \( 0 \leq t \leq 1 \), we must develop an expression to compute \( p_{ij}(t, x) \) for \( x = 0, 1, \ldots \) and \( t > 1 \). Suppose \( t = k + s \) where \( k = 1, 2, \ldots \) and \( 0 \leq s < 1 \). It follows that

\[ P(k + s, x) = \left( \prod_{r=1}^{k} P(1, x + r - 1) \right) P(s, x + k). \]  

(9)

Next, as premiums and benefits are paid \( m \) times per year, we need expressions for transition probabilities at \( m \)-thly intervals. Consider the form of \( p_{ij}(1/m, x + h/m) \) where \( h = 0, 1, \ldots, m - 1 \). Under the piecewise constant assumption of equation (2) \( p_{ij}(1/m, x + h/m) \) is independent of \( h \) for \( h = 0, 1, \ldots, m - 1 \). Let us define \( y_{ij}^{(m)}(x) \) as

\[ y_{ij}^{(m)}(x) = p_{ij}(\frac{1}{m}, x + \frac{h}{m}). \]  

(10)

In other words, \( y_{ij}^{(m)}(x) \) is the probability that a person currently age \( x + h/m \) and in state \( i \) will be in state \( j \) at age \( x + (h + 1)/m \) where \( h = 0, 1, \ldots, m - 1 \). We now define the \( n \times n \) matrix

\[ \Gamma_{x}^{(m)} = \left\{ y_{ij}^{(m)}(x) \right\}_{i,j=1}^{n}. \]  

(11)

It follows that, for \( t = k + h/m, k = 0, 1, \ldots, \) and \( h = 0, 1, \ldots, m - 1 \),
2.2 Determination of the Net Premium

Premiums are assumed to be payable weekly in advance. A premium is only payable if the policyholder is either in state 1 (super healthy) or state 2 (ultimate healthy) at the start of the week in the policy year under consideration if premiums are waived during periods of sickness.

The annual net premium $P$ is determined by equating the actuarial (expected) present value of future net premiums and the actuarial (expected) present value of future benefits at policy inception. To determine the net premium we need an expression for an $m$thly annuity due payable for $z$ years whenever $x$ is in state $j$, which is:

$$ i_j a^{(m)}_{x;z} = \frac{1}{m} \sum_{r=0}^{zm-1} u^{r/m} p_{ij} \left( \frac{r}{m}, x \right) $$

and an expression for an $m$thly annuity immediate payable for $z$ years whenever $x$ is in state $j$, which is:

$$ i_j a^{(m)}_{x,z} = \frac{1}{m} \sum_{r=1}^{zm} u^{r/m} p_{ij} \left( \frac{r}{m}, x \right). $$

It follows that the actuarial present value (APV) of the future premium is

$$ \text{APV of Future Premiums} = P \left( 11 \bar{a}^{(m)}_{x;r} + 12 \bar{a}^{(m)}_{x;r} \right). $$
The PHI benefit is assumed to be paid weekly during periods of sickness at the rate of $B$ per year. The PHI benefit is only payable if the policyholder is in either state 3 (short-term sick) or state 4 (long-term sick) at the end of the week in the policy year under consideration. Hence, the actuarial present value of the PHI benefits is

$$\text{APV of Future Benefits} = B \left( 13a_{x:z}^{(m)} + 14a_{x:z}^{(m)} \right).$$

Therefore, we can find $P$ from

$$P = \frac{B \left( 13a_{x:z}^{(m)} + 14a_{x:z}^{(m)} \right)}{ \left( 11i_{x:z}^{(m)} + 12\bar{a}_{x:z}^{(m)} \right)}. \quad (16)$$

### 3 PHI Data and Parameter Values

The parameter values used in this model have been influenced by the data contained in CMIR 12. As the data used in CMIR 12 are somewhat outdated, it is not necessary to input into our model precisely the output values emanating from CMIR 12. Therefore CMIR 12 is simply used as a guide to choosing parameter values for this paper.

For convenience the ages are grouped into 5-year age bands with the forces of transition assumed to be constant over each 5-year age band. The age bands are 30-34, 35-39, ..., 60-64. Next we describe the way in which each parameter value has been chosen.

$\mu_{23}(x)$ (Unstable Healthy → Short-Term Sick): This parameter is based on the sickness inception rate, $\sigma_x$, described in Part C of CMIR 12. We use the values of $\sigma_x$ for a deferred period of 13 weeks because the data sets for the shorter deferred periods (i.e., one week and four weeks) may be less typical of the general insured population. The values for the deferred period of 13 weeks are found in Table C16 of CMIR 12 (p. 74).

The force of sickness, $\sigma_x$, in CMIR 12 should be applied to the whole of the healthy population (i.e., states 1 and 2 combined) whereas $\mu_{23}(x)$ is a force that operates only on lives in state 2 (i.e.,

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3CMIR 12 is based on data collected between 1975 and 1978. Subsequent work by Clark and Dullaway (1995), Haberman and Walsh (1998), and Renshaw and Haberman (2000) have suggested that PHI experience has changed since 1978.
the healthy state). It could be argued, therefore, that the values of \( \sigma_x \) taken from CMIR 12 should be adjusted. Because CMIR 12 is being used merely as a guide, no adjustments have been made, i.e., \( \mu_{23}(x) = \sigma_x \).

\( \mu_{16}(x) \) (Super Healthy \rightarrow Dead): Under CMIR 12 the mortality rate for healthy lives is assumed to be that of male permanent assurances 1979-82, duration 0. The rates are shown in Table E17 (p. 132) under the column headed \( m(x) \). In our model, we have divided healthy lives into super healthy and ultimate healthy states. Because lives in the latter state will experience higher mortality rates than those in the former, we have decided to assume: \( \mu_{16}(x) = 0.80m(x) \), i.e., 80 percent of the mortality rates for male permanent assurances of 1979-82, duration 0.

\( \mu_{26}(x) \) (Ultimate Healthy \rightarrow Dead): We assume \( \mu_{26}(x) = 1.20m(x) \), i.e., 120 percent of the mortality rates for male permanent assurances of 1979-82, duration 0.\(^4\)

\( \mu_{32}(x) \) (Short-Term Sick \rightarrow Ultimate Healthy): Recovery rates are described in Section 3, Part B of CMIR 12. On page 34 of CMIR 12 various values of \( \rho_{y+z,z} \), the transition intensity from sick to healthy at current age \( y+z \) and current duration of sickness \( z \), are displayed. These recovery rates vary markedly by duration of sickness (measured in weeks). In view of the relatively simple approach adopted in our model, we will use a constant parameter value, i.e., \( \mu_{32}(x) = 2.5 \) at all ages.

\( \mu_{36}(x) \) (Short-Term Sick \rightarrow Dead): These mortality intensities are described in Section 6, Part B of CMIR 12. On page 39 of CMIR 12 the values of \( \nu_{y+z,z} \) at various ages are displayed where \( \nu_{y+z,z} \) is the transition intensity from sick to dead at current age \( y+z \) and current duration of sickness \( z \) measured in weeks. For our calculations, we will use the values at 15 weeks duration of sickness, which is when the transition intensities reach their peak, i.e., \( \mu_{36}(x) = \nu_{x,15} \). Interpolated values have been used where necessary.

\( \mu_{34}(x) \) (Short-Term Sick \rightarrow Long-Term Sick): CMIR 12 does not provide explicit parameter values for \( \mu_{34}(x) \). Having considered the or-

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\(^4\)The overall effect of the mortality assumptions for \( \mu_{16}(x) \) and \( \mu_{26}(x) \) can be considered to be broadly consistent with CMIR 12. As suggested by Cordeiro (1995), net premium values are likely to be less sensitive to the parameter values chosen for the forces of mortality.
der of magnitude of all the other forces in the model, we assume $\mu_{34}(x) = 0.1$ at all ages.

$\mu_{46}(x)$ (Long-Term Sick → Dead): We can again consider the mortality intensities $\nu_{y+z,z}$ that were described under $\mu_{36}(x)$ above. It seems appropriate to use these intensities at a suitably long sickness duration. We will use the values at duration five years (260 weeks) that are shown on page 39 of CMIR 12, i.e., $\mu_{46}(x) = \nu_x,260$.

$\mu_{56}(x)$ (Lapse → Dead): Because only super healthy policyholders lapse their policies, we will assume that $\mu_{56}(x) = \mu_{16}(x)$.

$\mu_{12}(x)$ (Super Healthy → Ultimate Healthy): CMIR 12 is not able to provide explicit parameter values for $\mu_{12}(x)$. It seems reasonable, however, to ensure that our estimates of $\mu_{12}(x)$ should be such that the aggregate mortality rates implied within our model approximately reflect the U.K. Male Permanent Assurances 1979–82 (duration 0) mortality table. The values for $\mu_{12}(x)$ that meet this constraint are, for simplicity, chosen by inspection.

$\mu_{15}(x)$ (Super Healthy → Lapse): Finally, having set the other parameters, $\mu_{15}(x)$ is varied in order to investigate its effect on the net premium rate.

Table 1 displays the parameter values. Table 2 shows the number of lives in each state at various sample ages given 100 super healthy lives entering state 1 at age 30, using the data in Table 1 and assuming $\mu_{15}(x) = 0.05$ for all $x$.\footnote{The assumption $\mu_{15}(x) = 0.05$ is consistent with the assumption of Sanders and Silby (1986) who use a lapse rate of 5 percent per annum for policy duration greater than two years.} For example, Table 2 shows that, by age 65, 12.0 percent of the lives would have died, 50.6 percent would have lapsed, and none of the lives would still be in the super healthy state.

The next step is to calibrate the model, i.e., to check if the model can produce the expected proportions of lives that are healthy, sick, or dead at various ages similar to those shown in CMIR 12 (Table E14, page 126). Table 3 displays these comparisons. The proportions are similar, particularly up to age 55. In Section 4.1 we will make another reasonableness check by comparing the net premium implied by our model with that implied by CMIR 12.
Table 1
Summary of Parameters

<table>
<thead>
<tr>
<th>Age x</th>
<th>$\mu_{16}(x)$</th>
<th>$\mu_{26}(x)$</th>
<th>$\mu_{46}(x)$</th>
<th>$\mu_{36}(x)$</th>
<th>$\mu_{23}(x)$</th>
<th>$\mu_{12}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30-34</td>
<td>0.0003</td>
<td>0.0005</td>
<td>0.0172</td>
<td>0.1108</td>
<td>0.1982</td>
<td>0.0270</td>
</tr>
<tr>
<td>35-39</td>
<td>0.0004</td>
<td>0.0006</td>
<td>0.0190</td>
<td>0.1180</td>
<td>0.1766</td>
<td>0.0150</td>
</tr>
<tr>
<td>40-44</td>
<td>0.0006</td>
<td>0.0010</td>
<td>0.0215</td>
<td>0.1251</td>
<td>0.1560</td>
<td>0.0480</td>
</tr>
<tr>
<td>45-49</td>
<td>0.0011</td>
<td>0.0017</td>
<td>0.0239</td>
<td>0.1379</td>
<td>0.1408</td>
<td>0.1100</td>
</tr>
<tr>
<td>50-54</td>
<td>0.0019</td>
<td>0.0028</td>
<td>0.0271</td>
<td>0.1507</td>
<td>0.1337</td>
<td>1.1000</td>
</tr>
<tr>
<td>55-59</td>
<td>0.0031</td>
<td>0.0046</td>
<td>0.0303</td>
<td>0.1694</td>
<td>0.1375</td>
<td>1.5000</td>
</tr>
<tr>
<td>60-65</td>
<td>0.0049</td>
<td>0.0073</td>
<td>0.0343</td>
<td>0.1880</td>
<td>0.1576</td>
<td>2.0000</td>
</tr>
</tbody>
</table>

Notes: We have assumed (i) constant forces of transition over successive 5-year age bands (i.e., age 30-34, 35-39, ..., 60-64); and (ii) $\mu_{56}(x) = \mu_{16}(x)$, $\mu_{32}(x) = 2.5$, and $\mu_{34}(x) = 0.1$ for all $x$.

Table 2
Percent of Lives in Each State at Sample Ages

<table>
<thead>
<tr>
<th>State</th>
<th>Age</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>30</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>31</td>
<td>92.6</td>
<td>2.5</td>
<td>0.1</td>
<td>0</td>
<td>4.8</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>85.7</td>
<td>4.7</td>
<td>0.3</td>
<td>0</td>
<td>9.3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>13.4</td>
<td>30.3</td>
<td>1.5</td>
<td>1.5</td>
<td>50.0</td>
<td>3.3</td>
</tr>
<tr>
<td></td>
<td>65</td>
<td>0</td>
<td>32.4</td>
<td>1.9</td>
<td>3.1</td>
<td>50.6</td>
<td>12.0</td>
</tr>
</tbody>
</table>

Notes: Using the data from Table 1 and $\mu_{15}(x) = 0.05$. 
Table 3
Comparing Percentages of Healthy, Sick and Dead Lives Under CMIR 12 (Table E14) with Our Model

<table>
<thead>
<tr>
<th>Age</th>
<th>Healthy</th>
<th>Sick</th>
<th>Dead</th>
<th>Healthy</th>
<th>Sick</th>
<th>Dead</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>98.4</td>
<td>1.1</td>
<td>0.5</td>
<td>98.8</td>
<td>0.9</td>
<td>0.3</td>
</tr>
<tr>
<td>40</td>
<td>97.3</td>
<td>1.4</td>
<td>1.3</td>
<td>97.6</td>
<td>1.4</td>
<td>1.0</td>
</tr>
<tr>
<td>45</td>
<td>95.8</td>
<td>1.9</td>
<td>2.3</td>
<td>96.0</td>
<td>2.1</td>
<td>1.9</td>
</tr>
<tr>
<td>50</td>
<td>93.2</td>
<td>2.8</td>
<td>4.0</td>
<td>93.7</td>
<td>3.0</td>
<td>3.3</td>
</tr>
<tr>
<td>55</td>
<td>88.9</td>
<td>4.4</td>
<td>6.7</td>
<td>90.4</td>
<td>4.1</td>
<td>5.5</td>
</tr>
<tr>
<td>60</td>
<td>81.6</td>
<td>7.4</td>
<td>11.0</td>
<td>87.1</td>
<td>4.5</td>
<td>8.4</td>
</tr>
</tbody>
</table>

Notes: Our model uses the data from Table 1 and $\mu_{15}(x) = 0.05$.

4 The Main Results

The PHI policy under consideration here is a 35-year term policy issued to a life age 30. The sickness benefit is paid weekly during periods of sickness at the rate of £1,000 per annum. Premiums are paid weekly and are waived during periods of sickness. Benefits are paid on a weekly basis. There is no deferred period, and the benefits and premiums cease at the age of 65. The valuation rate of interest is set to 6 percent per year. The forces of transition used are given in Table 1.

4.1 Sensitivity of Net Premiums to Various Parameters

Sensitivity of $P$ to $\mu_{15}(x)$: Figure 2 shows how the net premium varies as the lapse rate $\mu_{15}(x)$ takes values between 0 and 1. The net premium is relatively insensitive to the lapse rate. For example, the net premium decreases from £33.79 per annum to £26.36 per annum as the lapse rate increases from 0 to 0.2. This relative insensitivity is due to the fact that only super healthy lives lapse their policies, and their reserves are relatively small. Lapse rates of more than 0.4 would be unrealistic. For example, it can be shown that if $\mu_{15}(x) = 0.4$, over 83 percent of the insured population age 30 at the outset would have lapsed their policy during the first five years of the policy.
It is surprising that the net premium decreases rather than increases as the lapse rate increases, which is counter-intuitive. Standard actuarial logic suggests that the net premium should increase, because when the lapse rate is small, there are large numbers of lives in the system who are in the super healthy state and therefore continue to pay premiums without receiving any PHI benefit payments. This tends to suppress the net premium averaged over all the policyholders in the system. As the lapse rate increases, more of the super healthy lives leave the system by lapsing, which will tend to increase the average premium payable in respect of the remaining, relatively unhealthy, insured population.

So why does the net premium decrease as the lapse rate increases? Figure 3 shows how the numerator and the denominator of the right side of equation (16) vary as the lapse rate increases. We show scaled versions of the numerator and the denominator in order to fit them on the same graph. Both numerator and the denominator decrease, as would be expected, because the effect of lapses is to remove lives from state 1 before they have an opportunity to enter states 2, 3, or 4. The rate of decrease is the result of the complicated interaction between the different forces within the model. It can be seen that the numerator decreases at a faster rate than the denominator, and, therefore, the overall effect is that the net premium decreases.

Finally, before discussing other sensitivity issues, it is worth comparing the net premiums calculated using the model described in this paper with those derived from the data in eMIR 12. The data contained in Table F1 on page 228 of CMIR 12 suggest that the net premium for a policy similar to that described earlier in this section, but with premium and benefit payments made continuously and with a deferred period of one week, should be £24.24 per annum. The net premium figures shown in Figure 2 are of the same magnitude and hence provide some comfort that our model (including the parameter values chosen) is consistent with the model described in CMIR 12.

**Sensitivity of \( P \) to \( \mu_{12}(x) \):** Figure 4 shows how the net premium changes when the parameter values for \( \mu_{12}(x) \) given in Table 1 are increased or decreased 10 percent. If \( \mu_{12}(x) \) is increased 10 percent, the net premium increases between 4.9 percent (when the lapse rate, \( \mu_{15}(x) = 0 \)) and 8.4 percent (when \( \mu_{15}(x) = 1.0 \)). If \( \mu_{12}(x) \) is reduced 10 percent, the net premium decreases between 5.1 percent (when \( \mu_{15}(x) = 0 \)) and 8.7 percent (when \( \mu_{15}(x) = 1.0 \)).
Figure 2
Sensitivity of Net Premium to Lapse Rate, $\mu_{15}$

Figure 3
Variations in the Numerator and Denominator Of the Net Premium as the Lapse Rate Increases
The net premium is expected to move in the same direction as \( \mu_{12}(x) \). An increase in \( \mu_{12}(x) \) causes more lives to move from the super healthy to the ultimate healthy state where they are exposed to the risk of sickness inception, which, in turn, will lead to an increase in the premium required.

**Sensitivity of \( P \) to \( \mu_{23}(x) \):** Figure 5 shows how net premiums change when the parameter values for \( \mu_{23}(x) \), the sickness inception rate, are altered 10 percent. The net premium increases approximately 8.6 percent when the \( \mu_{23}(x) \) values are increased 10 percent and decreases approximately 8.9 percent when the \( \mu_{23}(x) \) values are decreased 10 percent. These results (in terms of relative sensitivities) are largely unaffected by the level of lapse rate assumed. As expected, an increase in the sickness inception rate causes an increase in the net premium required.

Cordeiro (1995) extends the work described in CMIR 12 by considering the effect on net premiums in changes in the sickness inception rates for various deferred periods and entry ages. Cordeiro finds that, for the CMIR 12 model and data, if the sickness inception rate is doubled, the net premium is approximately doubled. The results of this paper are therefore consistent with those of Cordeiro (1995).

**Sensitivity of \( P \) to \( \mu_{32}(x) \):** Figure 6 shows how net premiums change when the parameter value for \( \mu_{32}(x) \), the recovery rate, is increased or decreased 10 percent (i.e., changed from 2.5 at all ages to 2.75 or 2.25, respectively).

The net premium increases approximately 8.3 percent when the recovery rate is reduced 10 percent and decreases approximately 7.2 percent when it is increased 10 percent. Again, the level of lapse rate has little effect on these relative sensitivities. It is to be expected that an increase in the recovery rate should lead to a reduction in the amount of PHI premium required.

Cordeiro (1995) has investigated the effect that changes in the recovery rates have on net premiums based on the CMIR 12 model and data. Cordeiro discovers that a 10 percent increase in the recovery intensity leads to a 27.6 percent reduction in the net premium for entry age 30 and deferred period one week. Therefore, the net premium is less sensitive to a change in the recovery intensity under the model described in this paper than under the model used by Cordeiro (1995).
Figure 4
Net Premium Sensitivity to a ±10% Change in $\mu_{12}(x)$

Figure 5
Net Premium Sensitivity to
A ±10% Change in the Sickness Inception Rate $\mu_{23}(x)$
Sensitivity of $P$ to $\mu_{34}(x)$: Figure 7 shows the changes in net premiums when the parameter values for $\mu_{34}(x)$ are increased or decreased 10 percent. It can be seen that the net premium is relatively insensitive to changes in $\mu_{34}(x)$ because a 10 percent increase/decrease in the latter causes only a 4.0 percent increase/decrease in the net premium. As expected, an increase in the long-term sickness inception rate leads to an increase in the net premium required.

4.2 The Relationship Between $\mu_{12}(x)$ and $\mu_{32}(x)$

In Section 3, we explain how the parameter values for $\mu_{12}$ are chosen so that the aggregate mortality rates within the model broadly reflect the male permanent assurances 1979–82, duration 0. We now analyse how sensitive the values of $\mu_{12}(x)$ are to a change in the other parameters, in particular to a 50 percent increase in the recovery rate, $\mu_{32}(x)$. In other words, we retain all the parameter values summarized in Table 1 except for $\mu_{32}(x)$, which we increase from 2.5 at all ages to 3.75, and $\mu_{12}(x)$, which we need to recalibrate in order to ensure that the aggregate mortality rates still reflect the mortality table mentioned above. The results are summarized in Table 4.

<table>
<thead>
<tr>
<th>Age</th>
<th>$\mu_{12}(x)$ Values when $\mu_{32}(x) = 2.5$</th>
<th>$\mu_{12}(x)$ Values when $\mu_{32}(x) = 3.75$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30–34</td>
<td>0.027</td>
<td>0.045</td>
</tr>
<tr>
<td>35–39</td>
<td>0.015</td>
<td>0.025</td>
</tr>
<tr>
<td>40–44</td>
<td>0.048</td>
<td>0.074</td>
</tr>
<tr>
<td>45–49</td>
<td>0.110</td>
<td>0.180</td>
</tr>
<tr>
<td>50–54</td>
<td>1.100</td>
<td>1.500</td>
</tr>
<tr>
<td>55–59</td>
<td>1.500</td>
<td>1.900</td>
</tr>
<tr>
<td>60–64</td>
<td>2.000</td>
<td>2.400</td>
</tr>
</tbody>
</table>

A 50 percent increase in $\mu_{32}(x)$ requires an increase in $\mu_{12}(x)$ of approximately the same order of magnitude up to age 50 in order to leave the aggregate mortality rates within the model unaltered.
Figure 8
Impact on Net Premium of Increasing $\mu_{32}(x)$
(From $\mu_{32}(x) = 2.50$ to $\mu_{32}(x) = 3.75$)

Figure 9
Impact on Net Premium of Increasing $\mu_{34}(x)$
(From $\mu_{34}(x) = 0.10$ to $\mu_{34}(x) = 0.15$)
This result involving changes to $\mu_{34}(x)$ and $\mu_{12}(x)$ contrasts with the result in Section 4.2 where increasing $\mu_{32}(x)$ and recalibrating $\mu_{12}(x)$ has a neutral effect on the net premium. This feature further illustrates how complicated the interaction between the transition intensities is within the model.

5 Closing Comments

An objective of this paper is to develop a simple, practical U.K. style PHI model that can be used by actuaries who do not have access to complex models such as CMIR 12 or the detailed data required to use such models or who are interested in rough estimates for net premiums for PHI models.

One of the main difficulties that needs to be overcome in maintaining the simplicity of the model, however, is that the forces of transition between different states may depend not only on the age of the policyholder, but also on the time spent in the current state. For example, the longer a policyholder remains in the sick state, the less likely he or she is to recover. That is, there is duration-dependence. This factor usually leads to a semi-Markov model being used. However, convenient expressions for the transition probabilities are then hard to obtain.

The problem of duration-dependence is handled, in part, by increasing the number of states to differentiate between short-term and long-term stays in a particular status. This enables the model to be Markov rather than semi-Markov and therefore leads to tractable solutions. The model also includes lapses.

Using a particular policy, we test the sensitivity of the net premium to changes in the most significant model parameter values ($\mu_{12}(x)$, $\mu_{15}(x)$, $\mu_{23}(x)$, $\mu_{32}(x)$, and $\mu_{34}(x)$). Not surprisingly, the net premium is relatively insensitive to changes in the the lapse rate ($\mu_{15}(x)$) because only the most healthy lives are assumed to lapse their policies and they have small reserves. We also find that when any of the forces of transition, $\mu_{23}(x)$, $\mu_{32}(x)$, or $\mu_{34}(x)$, are increased, the resultant change in the level of net premium depends little on the level of the lapse rate. As a result, actuaries may initially ignore lapse rates when considering rough estimates for net premiums for PHI models.

By contrast, however, when the force of transition from the super healthy to the ultimate healthy state ($\mu_{12}(x)$) is increased, the extent to which the net premium increases depends on the level of the lapse rate. This shows that actuaries should probably spend more of their energies trying to obtain accurate estimates of $\mu_{12}(x)$. 
Let $p_{ir} = \mathbb{E}[X_{ij}^r]$ for $r = 1, 2, \ldots$. The first three cumulants of $S(t)$ are

$$
\kappa_1 = t \sum_{i=1}^{m} \lambda_i \mu_{i1}, \quad \kappa_2 = t \sum_{i=1}^{m} \lambda_i \mu_{i2}, \quad \text{and} \quad \kappa_3 = t \sum_{i=1}^{m} \lambda_i \mu_{i3}.
$$

Using the SDS principle, the accumulated risk premium received in $(0, t)$ (ignoring interest) is $\Pi^{SDS}[S(t)] = \Pi^{SDS}(t)$, where

$$
\Pi^{SDS}(t) = \kappa_1 + \alpha_1 \kappa_2^\frac{1}{2} + \alpha_2 \kappa_3^\frac{1}{3}.
$$

It must be pointed out that although $\Pi^{SDS}(t)$ is the accumulated risk premium received in $(0, t)$, it does not specify the amount of premium received in an intermediate period $(0, s)$ for $0 < s < t$. Let $\Pi^{SDS}(s|t)$ denote the accumulated risk premium received in $(0, s)$ for $0 < s < t$. All that is known is $\Pi^{SDS}(0|t) = 0$ and $\Pi^{SDS}(t|t) = \Pi^{SDS}(t)$. How must $\Pi^{SDS}(s|t)$ be defined for fixed $t$? There are several possibilities, for example,

$$
\Pi^{SDS}(s|t) = \int_0^s d\Pi^{SDS}(y) \quad 0 < s < t
$$

or

$$
\Pi^{SDS}(s|t) = c_t s \quad 0 < s < t.
$$

where $c_t$ is a constant for fixed $t$. As premiums are usually collected at a constant rate, we propose the second approach with

$$
c_t = \frac{\Pi^{SDS}(t)}{t}.
$$

Let $\theta(t)$ denote the relative security loading in $\Pi^{SDS}(t)$ so that

$$
\Pi^{SDS}(t) = (1 + \theta(t)) \kappa_1
$$

and

$$
\theta(t) = \frac{\alpha_1 \kappa_2^\frac{1}{2} + \alpha_2 \kappa_3^\frac{1}{3}}{\kappa_1} = \frac{\alpha_1 (\sum_{i=1}^{m} \lambda_i \mu_{i2})^{\frac{1}{2}}}{t^{1/2} \sum_{i=1}^{m} \lambda_i \mu_{i1}} + \frac{\alpha_2 (\sum_{i=1}^{m} \lambda_i \mu_{i3})^{\frac{1}{3}}}{t^{2/3} \sum_{i=1}^{m} \lambda_i \mu_{i1}}.
$$
Notice that for fixed $\alpha_1$ and $\alpha_2$, $\theta(t) \to 0$ as $t \to \infty$. This property of $\theta(t)$, i.e., converging to zero for long-term contracts, also exists for other premium calculation principles such as the standard deviation principle and makes these premium calculation principles unsuitable for long-term contracts.

Consider a time horizon of $t$ years. Let $U(\tau)$ denote the surplus at time $\tau$ ($0 < \tau < t$), then

$$U(\tau) = u + c_t \tau - S(\tau)$$

with $U(0) = u \geq 0$ being the initial surplus. The ruin probability within $t$ years given an initial surplus of $u$, $\psi(u, t)$, is defined as

$$\psi(u, t) = \mathbb{P}[T(u) \leq t]$$ (8)

where $T(u) = \min\{\tau : \tau > 0 \text{ and } U(\tau) < 0\}$. It is evident that the function $\psi$ depends on the size of $u$, $c_t$, and the time horizon $t$.

For a compound Poisson process with a fixed relative security loading on the risk premium, two well-known results are that the probability of ruin depends only on the size of the relative security loading, and that it increases as the size of the loading decreases. These results are used to determine $c_t$.

Specifically, to determine the premium rate $c_t$, we set $\psi$ of equation (8) at an acceptable level and then solve the resulting equation for $c_t$. If $\epsilon$ is our acceptable probability of ruin (typically, $\epsilon < 0.05$), we must solve the equation

$$\psi(u, t) = \epsilon.$$

As $\psi(u, t)$ is a complicated function of the premium rate, $c_t$ is determined directly through simulations. Note that for fixed $u$ and $t$, $\psi(u, t)$ decreases as the relative security loading increases, i.e., as $c_t$ increases. This inverse relationship enables us to search for solutions using the bisection method.

4 The Determination of Parameters $\alpha_1$ and $\alpha_2$

The $c_t$ obtained using simulations is actually the premium rate needed to cover $m$ classes of risks at the acceptable level of the probability of
ruin. Hence, the value of $c_t t$ is an aggregate of $m$ classes of premiums collected over $t$ years. The question here is how do we allocate $c_t t$ among these $m$ classes? Though there are several approaches that can be used, we opt for the one that allows us to set the $m$ premiums via the SDS premium calculation principle, i.e., we choose the parameters so that the $\alpha_1$s are the same for each class and the $\alpha_2$s are the same for each class ($\alpha_1$ and $\alpha_2$ may be different). This means that the premium for each class satisfies the SDS premium calculation principle.

Let $c_{it}$ denote the premium allocated to the $i$th class. Set

$$\begin{align*}
c_t t &= \sum_{i=1}^{m} c_{it} \\
&= \sum_{i=1}^{m} \left( \lambda_i p_{i1} t + \alpha_1 (\lambda_i p_{i2} t)^{1/2} + \alpha_2 (\lambda_i p_{i3} t)^{1/3} \right). \quad (9)
\end{align*}$$

Because we only have one equation but two unknown parameters, we need to impose a relation between $\alpha_1$ and $\alpha_2$. We assume that

$$\alpha_1 = \gamma \alpha_2 \quad (10)$$

where $\gamma > 0$ is a known constant. In practice, $\gamma$ can be chosen in accordance with the insurers' preferences and claim experiences.

Combining equations (9) and (10), we get

$$\begin{align*}
c_t t &= \sum_{i=1}^{m} \left( \lambda_i p_{i1} t + \gamma \alpha_2 (\lambda_i p_{i2} t)^{1/2} + \alpha_2 (\lambda_i p_{i3} t)^{1/3} \right). \quad (11)
\end{align*}$$

For a given $\gamma$, we can easily solve equation (11) for $\alpha_2$. Then, $\alpha_1$ can be obtained using equation (10).

### 4.1 Simulation Assumptions

The following assumptions are used:

- There are two classes, i.e., $m = 2$.
- The time horizons used are $t = 10, 50, 100$.
- The $t$-year ruin probability is set to be 0.05.
- The initial reserves used are $u = 10, 20, 30$. 
• The premium is paid continuously at a constant rate of $c_t$ per year.

• For $i = 1, 2$, $N_i(t)$ is a Poisson process with $\lambda_i = 10$. Hence, the claim number process $N(t)$ is a Poisson process with $\lambda = 20$. This implies that the inter-occurrence time random variables (i.e., the times between successive claims) are exponential with mean $1/\lambda$; see Bowers et al., (1997, Chapter 13.3).

• Two pairs of claim size distributions are used. They are specified in two cases:

  Case 1: (Exponential-Lognormal Pair) The claim size $X_{1j}$ has an exponential distribution with density $f_1(x) = e^{-x}$, and $X_{2j}$ has a lognormal distribution, i.e., $\ln X_{2j} \sim N(\mu, \sigma^2)$, where $\mu = -\ln(2)/2$ and $\sigma^2 = \ln 2$. In this case, $p_{11} = 1$, $p_{12} = 2$, $p_{13} = 6$, and $p_{21} = 1$, $p_{22} = 2$, $p_{23} = 8$; and

  Case 2: (Gamma-Pareto Pair) The claim size $X_{1j}$ has a gamma distribution with density

$$f_1(x) = \frac{\eta x^{\eta-1} e^{-\eta x}}{\Gamma(\eta)}$$

where $\eta = 4$. The claim size $X_{2j}$ has a Pareto distribution with density

$$f_2(x) = \frac{\beta + 1}{\beta} \left( \frac{\beta}{\beta + y} \right)^{\beta+2}$$

where $\beta = 3$. In this case, $p_{11} = 1$, $p_{12} = 1.25$, $p_{13} = 1.875$, and $p_{21} = 1$, $p_{22} = 3$, $p_{23} = 27$.

The simulation is performed as follows. Let $T_k$ denote the occurrence time of the $k^{th}$ claim ($Z_k$) and define $V_k = T_k - T_{k-1}$ with $T_0 = 0$. The $V_k$s are called the inter-occurrence time random variables. Define $W_k$ as

$$W_k = u + \sum_{r=1}^{k} (c_t V_r - Z_r).$$

Ruin occurs if $W_k$ is ever negative for any $k = 1, 2, \ldots, N(t)$ where $N(t)$ is the total number of claims generated by the two classes in $(0, t)$. 
Step 1: As $T_n = V_1 + \cdots + V_n$ for $n = 1, 2, \ldots$, generate the sequence of inter-occurrence time random variables $V_k$s until the condition

$$T_n \leq t < T_{n+1}$$

occurs, then stop; see Ross (1990) for more on generating pseudo-random variables;

Step 2: Assign $N(t) = n$ and $W_0 = u$;

Step 3: For $k = 1$ to $N(t)$, do the following:

1. Generate a uniform $(0,1)$ random number $U$. If $U < \lambda_1 / \lambda$, then generate $Z_k$ from the claim distribution of class 1 (i.e., the distribution of $X_{1j}$), else generate $Z_k$ from the claim distribution of class 2 (i.e., the distribution of $X_{2j}$);

2. Compute $W_k = W_{k-1} + c_t V_k - Z_k$;

3. If $W_k < 0$, then ruin occurs. Return to Step 1 to start another simulation;

4. If $W_k \geq 0$, then go back to Step 3.1 above to continue the loop;

Step 4: If $W_k \geq 0$ for $k = 1$ to $N(t)$, then ruin does not occur. Return to Step 1 for another simulation.

For each of the two cases and for each $u$ and $t$, we perform 10,000 simulations. We choose the value of $c_t$ that yields 500 ruins out of the 10,000 simulations (as the ruin probability is set to be 0.05). Then, based on equation (11), we use

$$c_t t = \left(10t + \gamma \alpha_2(20t)^{\frac{1}{2}} + \alpha_2(60t)^{\frac{1}{3}}\right) + \left(10t + \gamma \alpha_2(20t)^{\frac{1}{2}} + \alpha_2(80t)^{\frac{1}{3}}\right)$$

for Case 1, and

$$c_t t = \left(10t + \gamma \alpha_2(12.5t)^{\frac{1}{2}} + \alpha_2(18.75t)^{\frac{1}{3}}\right) + \left(10t + \gamma \alpha_2(30t)^{\frac{1}{2}} + \alpha_2(270t)^{\frac{1}{3}}\right)$$

for Case 2, with $\gamma$ varying from 0 to 5 in steps of 0.1, to calculate $\alpha_2$. Once $\alpha_2$ is obtained, we compute $\alpha_1$ using equation (10).
4.2 Numerical Results

The results are summarized in Figures 1 to 4 and Tables 1 and 2. Figures 1 and 2 show that $\alpha_1$ decreases as $u$ increases, while Figures 3 and 4 show that $\alpha_1$ increases as $t$ increases. Similar observations also hold for $\alpha_2$ because of equation (10). Notice that in the first row of Table 1, the $C_t$ values for $t = 10, 50, 100$ are the same. This suggests that in both cases, the $C_t$ value with $u = 10$ and $t = 10$ is close to the largest premium for a probability of ultimate ruin of 0.05. The second observation is that for fixed $t$, the larger the value of $u$, the smaller the value of $C_t$. This is consistent with Figures 1 and 2.

Table 1

<table>
<thead>
<tr>
<th>$u$</th>
<th>$t = 10$</th>
<th>$t = 50$</th>
<th>$t = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>27.40</td>
<td>27.40</td>
<td>27.40</td>
</tr>
<tr>
<td>20</td>
<td>23.18</td>
<td>23.30</td>
<td>23.30</td>
</tr>
<tr>
<td>30</td>
<td>21.39</td>
<td>22.16</td>
<td>22.18</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\alpha_1$</th>
<th>$c_{1t}$</th>
<th>$c_{2t}$</th>
<th>$\alpha_1$</th>
<th>$c_{1t}$</th>
<th>$c_{2t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0104</td>
<td>13.5535</td>
<td>13.8468</td>
<td>1.2435</td>
<td>13.0559</td>
<td>16.8853</td>
</tr>
<tr>
<td>0.5</td>
<td>2.9879</td>
<td>13.6134</td>
<td>13.7869</td>
<td>3.7965</td>
<td>13.3845</td>
<td>16.5567</td>
</tr>
<tr>
<td>1.0</td>
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<td>13.7576</td>
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<td>13.5532</td>
<td>16.3880</td>
</tr>
<tr>
<td>2.5</td>
<td>4.9097</td>
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<td>6.4412</td>
<td>13.7249</td>
<td>16.2163</td>
</tr>
<tr>
<td>4.5</td>
<td>5.2876</td>
<td>13.6831</td>
<td>13.7172</td>
<td>6.9816</td>
<td>13.7945</td>
<td>16.1467</td>
</tr>
</tbody>
</table>
Figure 1
\( \alpha_1 \) Vs. \( \gamma \) for Exponential-Lognormal with \( t = 50 \)

Figure 2
\( \alpha_1 \) Vs. \( \gamma \) for Gamma-Pareto with \( t = 50 \)
Figure 3
\( \alpha_1 \) Vs. \( \gamma \) for Exponential-Lognormal with \( u = 10 \)

Figure 4
\( \alpha_1 \) Vs. \( \gamma \) for Gamma-Pareto with \( u = 10 \)
Table 2 displays $c_{1t}$ and $c_{2t}$ for $u = 10$ and $t = 50$. In both cases $c_{2t}$ exceeds $c_{1t}$. In the exponential-lognormal case, the third cumulant of the lognormal is slightly larger than that of the exponential so $c_{1t}$ and $c_{2t}$ differ only by a small margin. Moreover, $c_{2t}$ exceeds $c_{1t}$ because the lognormal is riskier (i.e., has a heavier right tail) than the exponential. In the gamma-Pareto case, the differences are much larger because the Pareto has a larger second cumulant and a much larger third cumulant, i.e., the Pareto is much riskier than the gamma. In both cases, $c_{2t} - c_{1t}$ decreases as $\gamma$ increases because $\alpha_1 (\alpha_2)$ becomes larger (smaller) when $\gamma$ increases, so a heavier (lighter) weight is put on the standard deviation (skewness) term.

5 Closing Remarks

There are three important points that must be addressed:

1. Ruin probabilities are difficult to obtain because they do not usually have closed-form solutions, so the method of simulations is a natural way to deal with the problem. One advantage of simulation is flexibility. It can be used in practical situations with real insurance data as well as more complex models that include factors such as correlated risks and investment performance.

2. From the practical point of view, the value of $t$ should not be set too large because it leads to lower risk loading factors. If the insurance market is such that one can split the time horizon into smaller time periods, then the insurer may receive higher risk loadings over each period. For example, a 10-year horizon may be split into five 2-year horizons.

3. The question of allocating premiums among the $m$ classes has no unique solution. For example, we can allocate the premiums according to their proportion of the total risk loadings. Specifically, using equation (2), we define

$$c_{it} = \lambda_i p_{1i} t + \frac{(\alpha_1 + \alpha_2 \varphi_i^{1/3}) \sigma_i}{\sum_{i=1}^{m} (\alpha_1 + \alpha_2 \varphi_i^{1/3}) \sigma_i} \left( \alpha_1 \kappa_2^{1/2} + \alpha_2 \kappa_3^{1/3} \right)$$

where $\sigma_i^2 = \text{Var} [X]$ and $\varphi_i = \mathbb{E} \left[ (X_{ij} - \mathbb{E} [X_{ij}])^3 \right] / \sigma_i^3$ is the coefficient of skewness of $X_{ij}$. As before, we set $c_{1t} = \sum_{i=1}^{m} c_{it}$. 
References


