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BOUNDARY VALUE PROBLEMS OF NABLA FRACTIONAL DIFFERENCE EQUATIONS

by

Abigail Brackins

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BOUNDARY VALUE PROBLEMS OF NABLA FRACTIONAL DIFFERENCE EQUATIONS

Abigail Brackins, Ph.D.

University of Nebraska, 2014

Advisers: Lynn Erbe and Allan Peterson

In this dissertation we develop the theory of the nabla fractional self-adjoint difference equation,

$$\nabla_a^\nu (p\nabla y)(t) + q(t)y(\rho(t)) = f(t)$$

where $0 < \nu < 1$. We begin with an introduction to the nabla fractional calculus. In the second chapter, we show existence and uniqueness of the solution to a fractional self-adjoint initial value problem. We find a variation of constants formula for this fractional initial value problem, and use the variation of constants formula to derive the Green's function for a related boundary value problem. We study the Green's function and its properties in several settings. For a simplified boundary value problem with $p \equiv 1$, we show that the Green's function is nonnegative and we find its maximum and the maximum of its integral. For a boundary value problem with generalized boundary conditions, we find the Green's function and show that it is a generalization of the first Green's function. In the third chapter, we use the Contraction Mapping Theorem to prove existence and uniqueness of a positive solution to a forced self-adjoint fractional difference equation with a finite limit. We explore modifications to the forcing term and modifications to the space of functions in which the solution exists, and we provide examples to demonstrate the use of these theorems.

DEDICATION

This dissertation is dedicated to my father, Steven Dennis Brackins, who taught me to dream big, and to my mother, Donna Mae Brackins, who taught me the tenacity to realize those dreams.

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Contents

Chapter 1

Introduction

In this chapter we give some basic definitions and notation for the nabla discrete fractional calculus. A general overview of the nabla discrete calculus, which we summarize here, is given in [42]. We assume $a \in \mathbb{R}$, unless otherwise noted, and $b \in \mathbb{R}$ such that $b - a$ is a positive integer, and we define the sets

$$\mathbb{N}_a := \{a, a+1, a+2, \cdots\}$$

and

$$\mathbb{N}_a^b := \{a, a+1, a+2, \cdots, b\}.$$

Definition 1.0.1 (Backwards Jump Operator). *[22] The backwards jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$, is defined by*

$$\rho(t) = \max\{a, t-1\},$$

1.1 Discrete Nabla Differences

Definition 1.1.1 (Nabla Difference Operator). *[22] For any function $f : \mathbb{N}_a \to \mathbb{R}$, the nabla difference operator is defined by*

$$\nabla f(t) := f(t) - f(t-1),$$

for $t \in \mathbb{N}_{a+1}$. Higher integer order differences are defined recursively by

$$\nabla^n f(t) := \nabla(\nabla^{n-1} f(t)),$$

for $n \in \mathbb{N}_2$ and $t \in \mathbb{N}_{a+n}$. By convention, ∇^0 is taken to be the identity operator.

The nabla difference operator satisfies the following list of properties.

Theorem 1.1.2. *[42] Assume $f, g : \mathbb{N}_a \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, then for $t \in \mathbb{N}_{a+1}$,*

1. $\nabla \alpha = 0$;
2. $\nabla \alpha f(t) = \alpha \nabla f(t)$;
3. $\nabla (f(t) + g(t)) = \nabla f(t) + \nabla g(t)$;
4. $\nabla \alpha^{t+\beta} = \frac{\alpha - 1}{\alpha} \alpha^{t+\beta}$;
5. $\nabla (f(t)g(t)) = f(\rho(t)) \nabla g(t) + g(t) \nabla f(t)$
6. $\nabla \left(\frac{f(t)}{g(t)} \right) = \frac{g(t)\nabla f(t) - f(t) \nabla g(t)}{g(t) g(\rho(t))}$, *where $g(t) \neq 0$ for $t \in \mathbb{N}_a$.*

To obtain the discrete nabla analogue of the power rule, we must first define the rising function.

Definition 1.1.3 (The Rising Function)**.** *[23] For any positive integer n and any $t \in \mathbb{R}$, we define the rising function, $t^{\overline{n}}$, read t to the n rising, by*

$$t^{\overline{n}} := t(t+1)\cdots(t+n-1).$$

Theorem 1.1.4 (Nabla Power Rule)**.** *[8] For $n \in \mathbb{N}_1$, $\alpha \in \mathbb{R}$,*

$$\nabla(t+\alpha)^{\overline{n}} = n(t+\alpha)^{\overline{n-1}},$$

for $t \in \mathbb{R}$.

We are interested in generalizing the definition of the rising function to include t to the r rising, where $r \in \mathbb{R}$. To do this, we make use of the gamma function, defined as follows.

Definition 1.1.5 (Gamma Function)**.** *The gamma function is defined by*

$$\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}dt$$

for those complex numbers z for which the real part of z is positive. (It can be shown that the integral converges for all such z.)

Integration by parts is used to establish the important formula

$$\Gamma(z+1) = z\Gamma(z), \tag{1.1.1}$$

which is in turn used to extend the domain of the gamma function to complex numbers $z \neq 0, -1, -2, \cdots$. On this domain the gamma function is analytic, and using (1.1.1), it can be shown that

$$\lim_{z\to n} |\Gamma(z)| = \infty, \quad n = 0, -1, -2, \cdots . \tag{1.1.2}$$

We note that $0 < \Gamma(t) \leq 1$ for $t \in [1, 2]$, and $\Gamma(t) > 1$ for $t \in (2, \infty)$, and we can also use (1.1.1) to show that

$$\Gamma(n + 1) = n!, \quad n = 0, 1, 2, \cdots .$$

Hence we are able to use the gamma function as a generalization of the factorial function, and thus we can redefine the rising function by making the following observation.

Note that for a positive integer n,

$$\begin{aligned} t^{\overline{n}} &= t(t + 1) \cdots (t + n - 1) \\ &= \frac{(t + n - 1)(t + n - 2) \cdots t\Gamma(t)}{\Gamma(t)} \\ &= \frac{\Gamma(t + n)}{\Gamma(t)}. \end{aligned}$$

Motivated by this observation we generalize the rising function.

Definition 1.1.6 (Generalized Rising Function)**.** *[8] The generalized rising function is defined by*

$$t^{\overline{r}} = \frac{\Gamma(t + r)}{\Gamma(t)},$$

for values of t and r so that $t, t + r \notin \{0, -1, -2, \cdots\}$. We use the convention that if t is a nonpositive integer but $t + r$ is not a nonpositive integer, then $t^{\overline{r}} := 0$.

From this definition follows the generalized nabla power rules.

Theorem 1.1.7 (Generalized Nabla Power Rules). *[42] The formulas*

$$\nabla(t+\alpha)^{\overline{r}} = r(t+\alpha)^{\overline{r-1}},$$

and

$$\nabla(\alpha-t)^{\overline{r}} = -r(\alpha-\rho(t))^{\overline{r-1}},$$

hold for those values of t, r and α for which the expressions make sense.

1.2 Discrete Nabla Integrals

Definition 1.2.1 (Nabla Definite Integral). *[42] Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $c, d \in \mathbb{N}_a$, then*

$$\int_c^d f(t)\nabla t := \begin{cases} \sum_{t=c+1}^{d} f(t), & c < d \\ 0, & c \geq d. \end{cases}$$

The following theorem gives some properties of this integral, which are derived from similar properties of sums.

Theorem 1.2.2. *[42] Assume $f, g : \mathbb{N}_a \to \mathbb{R}$, $b, c, d \in \mathbb{N}_a$, $b < c < d$, and $\alpha \in \mathbb{R}$. Then*

(i) $\int_b^c \alpha f(t)\nabla t = \alpha \int_b^c f(t)\nabla t$;

(ii) $\int_b^c (f(t)+g(t))\nabla t = \int_b^c f(t)\nabla t + \int_b^c f(t)\nabla t$;

(iii) $\int_b^b f(t)\nabla t = 0$;

(iv) $\int_b^d f(t)\nabla t = \int_b^c f(t)\nabla t + \int_c^d f(t)\nabla t$;

(v) $\left|\int_b^c f(t)\nabla t\right| \leq \int_b^c |f(t)|\nabla t$;

(vi) *if* $F(t) := \int_b^t F(s)\nabla s$, *for* $t \in \mathbb{N}_b^c$, *then* $\nabla F(t) = f(t)$, $t \in \mathbb{N}_{b+1}^c$;

(vii) *if* $f(t) \geq g(t)$ *for* $t \in \{b+1, b+2, \cdots, c\}$, *then* $\int_b^c f(t)\nabla t \geq \int_b^c g(t)\nabla t$.

Definition 1.2.3. *[42] Assume* $f : \mathbb{N}_a^b \to \mathbb{R}$. *We say that* F *is a nabla antidifference of* f *on* $\mathbb{N}_a$ *provided*

$$\nabla F(t) = f(t), \quad t \in \mathbb{N}_{a+1}^b.$$

Definition 1.2.4 (Nabla Indefinite Integral)**.** *[42] If* $f : \mathbb{N}_a \to \mathbb{R}$, *then the nabla indefinite integral of* f *is defined by*

$$\int f(t)\nabla t = F(t) + C,$$

where $F(t)$ *is a nabla antidifference of* $f(t)$ *and* C *is an arbitrary constant.*

This definition leads to the following result.

Theorem 1.2.5 (Fundamental Theorem of Nabla Calculus)**.** *[42] Let* $f : \mathbb{N}_a^b \to \mathbb{R}$ *and let* F *be a nabla antidifference of* f *on* $\mathbb{N}_a^b$, *then*

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

The following formulas for indefinite integrals are derived from corresponding difference formulas.

Theorem 1.2.6. *[42] The following hold:*

(i) $\int \alpha^{t+\beta}\nabla t = \frac{\alpha}{\alpha-1}\alpha^{t+\beta} + C, \quad \alpha \neq 1$;

(ii) $\int (t-\alpha)^{\overline{r}}\nabla t = \frac{1}{r+1}(t-\alpha)^{\overline{r+1}} + C, \quad r \neq -1$;

(iii) $\int (\alpha - \rho(t))^{\overline{r}}\nabla t = -\frac{1}{r+1}(\alpha - t)^{\overline{r+1}} + C, \quad r \neq -1$.

The product rule for nabla differences leads to an integration by parts formula for nabla discrete integrals.

Theorem 1.2.7 (Integration by Parts)**.** *[42] Given two functions $u, v : \mathbb{N}_a \to \mathbb{R}$ and $b, c \in \mathbb{N}_a$, $b < c$, we have the integration by parts formula:*

$$\sum_{s=b+1}^{c} u(t)\nabla v(t) = u(t)v(t)|_b^c - \sum_{s=b+1}^{c} v(\rho(t))\nabla u(t),$$

or in integral notation,

$$\int_b^c u(t)\nabla v(t) = u(t)v(t)|_b^c - \int_b^c v(\rho(t))\nabla u(t).$$

1.3 Fractional Sums and Differences

We will use the gamma function and observations about discrete nabla sums and differences to construct fractional sums and differences. The repeated summation formula

$$\int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau_n)\nabla\tau_n \cdots \nabla\tau_2\nabla\tau_1 = \int_a^t \frac{(t-\rho(s))^{\overline{n-1}}}{(n-1)!} f(s)\nabla s$$

is derived in [42] using an integer order variation of constants formula. Motivated by this formula, we define the nabla integral order sum as follows.

Definition 1.3.1 (Integer Order Sum)**.** *[42] Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ be given and $n \in \mathbb{N}_1$, then*

$$\nabla_a^{-n} f(t) := \int_a^t \frac{(t-\rho(s))^{\overline{n-1}}}{(n-1)!} f(s)\nabla s, \quad t \in N_a.$$

We define $\nabla_a^{-0} f(t) = f(t)$.

Note that this sum depends on the values of f at all the points $a+1 \leq s \leq t$, unlike the difference $\nabla^n f(t)$ which depends on the values of f at the points $t-n \leq s \leq t$. We can use the gamma function to extend this definition to a fractional order nabla sum.

Definition 1.3.2 (Nabla Fractional Sum)**.** *[8] Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$, then the ν-th order fractional sum based at a is given by*

$$\nabla_a^{-\nu} f(t) = \int_a^t \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s) \nabla s, \quad t \in \mathbb{N}_a.$$

The ν-th order nabla fractional difference is defined by the use of the fractional sum, as follows.

Definition 1.3.3 (Nabla Fractional Difference)**.** *[8] Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$, and choose N such that $N-1 < \nu \leq N$. Then we define the ν-th order nabla fractional difference by*

$$\nabla_a^{\nu} f(t) = \nabla^N \nabla_a^{-(N-\nu)} f(t) \quad \textit{for} \quad t \in \mathbb{N}_{a+N}.$$

The following theorem extends power rules to the fractional case.

Theorem 1.3.4. *Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that μ and $\nu + \mu$ are not negative integers, then we have that*

(i) $\nabla_a^{-\nu} (t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{\overline{\mu+\nu}}$*;*

(ii) $\nabla_a^{\nu} (t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)} (t-a)^{\overline{\mu-\nu}}$*;*

for $t \in \mathbb{N}_a$.

In the paper [3], Laplace transforms are used to prove a number of interesting properties about the nabla fractional sum and difference, among them the following results. The first of these gives an alternative definition of the nabla fractional difference.

Definition 1.3.5 (Alternative Definition of the Nabla Fractional Difference)**.** *[3] Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, and $N \in \mathbb{N}$ such that $N - 1 < \nu \leq \mathbb{N}$ be given. Then the ν-th order fractional difference of f is given by*

$$\nabla_a^{\nu} f(t) = \begin{cases} \int_a^t \frac{(t-\rho(s))^{\overline{-\nu-1}}}{\Gamma(-\nu)} f(s)\nabla s, & \nu \notin \mathbb{N} \\ \nabla^N f(t), & \nu = N \in \mathbb{N} \end{cases},$$

for $t \in \mathbb{N}_{a+N}$.

Though $|\Gamma(-\nu)|$ approaches infinity as ν approaches a nonnegative integer, it can be shown that the ν^{th}-order nabla fractional difference is continuous with respect to $\nu \geq 0$. This has the interesting implication that, though a fractional nabla difference of f depends on the values of f at all the points $a + 1 \leq s \leq t$, this dependence “fades” as the order of the difference approaches a whole number.

The following theorem allows us to compose fractional order sums and differences with fractional order sums.

Theorem 1.3.6 (Composition Rule for Nabla Fractional Sums and Differences)**.** *[3] Let $\mu, \nu > 0$ and $k \in \mathbb{N}_0$ be given, and choose $N \in \mathbb{N}$ such that $N - 1 < \nu \leq N$. Then we have*

$$\nabla_a^{-\nu}\nabla_a^{-\mu} f(t) = \nabla_a^{-\nu-\mu} f(t), \quad \textit{for} \quad t \in \mathbb{N}_a,$$

and

$$\nabla_a^{\nu}\nabla_a^{-\mu} f(t) = \nabla_a^{\nu-\mu} f(t), \quad \textit{for} \quad t \in \mathbb{N}_{a+N}.$$

Compositions of fractional order sums or differences with nabla differences are generally less straightforward; for example, the composition $\nabla_a^\nu \nabla_a^\nu f(t)$ cannot be computed in such a straightforward fashion as in Theorem 1.3.6, for arbitrary $\nu > 0$. Below we give a specific example of a composition that can be computed nicely, and will prove useful to us in Chapter 2.

Theorem 1.3.7 (Composition Rule for a Fractional Difference and a First Order Nabla Difference)**.** *Let* $0 < \nu < 1$ *and* $f : \mathbb{N}_a \to \mathbb{R}$ *be given. Then*

$$\nabla_a^\nu \nabla f(t) = \nabla_a^{1+\nu} f(t).$$

Proof. Using linearity of sums, we have that

$$\begin{aligned}
\nabla_a^\nu \nabla f(t) &= \nabla_a^\nu (f(t) - f(t-1)) \\
&= \int_a^t \frac{(t-\rho(s))^{\overline{-\nu-1}}}{\Gamma(-\nu)} f(s) \nabla s - \int_a^{t-1} \frac{(t-1-\rho(s))^{\overline{-\nu-1}}}{\Gamma(-\nu)} f(s) \nabla s \\
&= \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{-\nu-1}}}{\Gamma(-\nu)} f(s) - \sum_{s=a+1}^{t-1} \frac{(t-1-\rho(s))^{\overline{-\nu-1}}}{\Gamma(-\nu)} f(s) \\
&= \frac{(t-\rho(t))^{\overline{-\nu-1}}}{\Gamma(-\nu)} f(t) \\
&\qquad + \sum_{s=a+1}^{t-1} \frac{1}{\Gamma(-\nu)} \left[(t-\rho(s))^{\overline{-\nu-1}} - (t-1-\rho(s))^{\overline{-\nu-1}} \right] f(s).
\end{aligned}$$

We also use Theorem 1.1.7 to find, where ∇_t refers to the first nabla difference with respect to t,

$$\begin{aligned}
(t-\rho(s))^{\overline{-\nu-1}} - (t-1-\rho(s))^{\overline{-\nu-1}} &= \nabla_t (t-\rho(s))^{\overline{-\nu-1}} \\
&= (-\nu-1)(t-\rho(s))^{\overline{-\nu-2}}.
\end{aligned}$$

So we have that

$$\begin{aligned}
\nabla_a^{\nu}\nabla f(t) &= \frac{(t-\rho(t))^{\overline{-\nu-1}}}{\Gamma(-\nu)}f(t) + \sum_{a+1}^{t-1}\frac{(-\nu-1)}{\Gamma(-\nu)}(t-\rho(s))^{\overline{-\nu-2}}f(s) \\
&= \frac{(t-\rho(t))^{\overline{-\nu-1}}}{\Gamma(-\nu)}f(t) + \sum_{a+1}^{t-1}\frac{(t-\rho(s))^{\overline{-\nu-2}}}{\Gamma(-\nu-1)}f(s) \\
&= \sum_{a+1}^{t}\frac{(t-\rho(s))^{\overline{-\nu-2}}}{\Gamma(-\nu-1)}f(s) = \nabla_a^{1+\nu}f(t).
\end{aligned}$$

□

1.4 Fractional Initial Value Problems

We will use the following results about fractional order nabla initial value problems to construct formulas for the solutions of boundary value problems.

Definition 1.4.1. *For $f, g : \mathbb{N}_{a+1} \to \mathbb{R}$, we define the nabla convolution product of f and g by*

$$(f * g)(t) := \int_a^t f(t-\rho(s)+a)g(s)\nabla s, \quad t \in \mathbb{N}_{a+1}.$$

Theorem 1.4.2 (Variation of Constants)**.** *[42] Let $f, g : \mathbb{N}_a \to \mathbb{R}$ and $0 < \nu < 1$. Then, for $t \in \mathbb{N}_{a+1}$, the fractional initial-value problem*

$$\begin{cases} \nabla_a^{\nu} f(t) + cf(t) = g(t), & t \in \mathbb{N}_{a+1}, \quad |c| < 1 \\ f(a+1) = A, & A \in \mathbb{R} \end{cases}$$

has the unique solution

$$f(t) = (h(\cdot, a) * g(\cdot))(t) + (A(c+1) - g(a+1))h(t, a),$$

where

$$h(t,a) := \sum_{k=0}^{\infty} (-c)^k \frac{(t-a)^{\overline{k(\nu+1)-1}}}{\Gamma(k(\nu+1))}.$$

Corollary 1.4.3. *[42] Let $f, g : \mathbb{N}_a \to \mathbb{R}$ and $0 < \nu < 1$. Then, for $t \in \mathbb{N}_{a+1}$, the fractional initial value problem*

$$\begin{cases} \nabla_a^{\nu} f(t) = g(t), & t \in \mathbb{N}_{a+1} \\ f(a+1) = A, & A \in \mathbb{R} \end{cases}$$

has the unique solution

$$f(t) = \nabla_a^{-\nu} g(t) + (A - g(a+1)) \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)}.$$

Example 1.4.4. *Use Corollary 1.4.3 to solve the fractional IVP*

$$\nabla_0^{\frac{1}{2}} y(t) = 3, \quad t \in \mathbb{N}_1$$
$$y(1) = \pi.$$

We have that the solution is given by

$$y(t) = \nabla_0^{-\frac{1}{2}} 3 + (\pi - 3) \frac{t^{\overline{-\frac{1}{2}}}}{\Gamma\left(\frac{1}{2}\right)},$$

for $t \in \mathbb{N}_1$. For the first term, we use Definition 1.3.2, Theorem 1.2.6, and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

to compute

$$\begin{aligned}\nabla_0^{-\frac{1}{2}}3 &= \int_0^t \frac{(t-\rho(s))^{\overline{-\frac{1}{2}}}}{\Gamma\left(\frac{1}{2}\right)} 3\nabla s \\ &= \frac{3}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-\rho(s))^{\overline{-\frac{1}{2}}} \nabla s \\ &= \frac{3}{\sqrt{\pi}} \left[-2(t-s)^{\overline{\frac{1}{2}}}\right]_0^t \\ &= \frac{6}{\sqrt{\pi}} t^{\overline{\frac{1}{2}}}.\end{aligned}$$

Hence the solution to the fractional IVP is given by

$$y(t) = \frac{6}{\sqrt{\pi}} t^{\overline{\frac{1}{2}}} + \left(\sqrt{\pi} - \frac{3}{\sqrt{\pi}}\right) t^{\overline{-\frac{1}{2}}}, \quad t \in \mathbb{N}_1.$$

1.5 The Contraction Mapping Theorem

In Chapter 3 we will use the Contraction Mapping Theorem to establish the existence of certain solutions to a fractional difference equation.

Theorem 1.5.1 (Contraction Mapping Theorem)**.** *[45] Let $(X, \|\cdot\|)$ be a Banach space. Assume that $T : X \to X$ is a contraction mapping, that is, there is an α, $0 \leq \alpha < 1$, such that $\|Tx - Ty\| \leq \alpha \|x - y\|$ for all $x, y \in X$. Then T has a unique fixed point z in X.*

Remark 1.5.2. *For some $L \in \mathbb{R}$, the space $\zeta := \left\{ y : \mathbb{N}_a \to \mathbb{R} \,\middle|\, \lim_{t\to\infty} y(t) = L \right\}$ together with the supremum norm, $\|\cdot\| : \zeta \to \zeta$, defined by*

$$\|y\| = \sup_{t \in \mathbb{N}_a} |y(t)|,$$

is a complete metric space. To see this, consider a Cauchy sequence $\{y_n\}$ in ζ. For *any* $t_0 \in \mathbb{N}_a$, $\{y_n(t_0)\}$ *is a Cauchy sequence in* $\mathbb{R}$, *and therefore it converges. Define its limit to be* $y_0(t_0) := \lim_{n\to\infty} y_n(t_0)$, *and since* t_0 *was arbitrary, we can construct the function* $y_0 : \mathbb{N}_a \to \mathbb{R}$ *in this fashion. Because each of the* $y_n(t)$ *has the limit* L *as* $t \to \infty$, *so does* $y_0(t)$, *and hence* $y_0 \in \zeta$.

1.6 Further Reading

Fractional calculus in the continuous setting is developed in [57], [52], [51], [1], [25], [14], [26], [16], [20], [24], [48], [2], [4], [21], [53], [56], [60], [61], [5], [15], [29], [47], [49], [50], [55], [58], [59], [62], [27], [35], [54].

Fractional calculus is extended to time scales in [17], [18].

Discrete fractional calculus in the delta setting is developed in [6], [7], [10], [28], [41], [9], [30], [31], [32], [33], [43], [44], [36], [38], [39], [11], [12], [40]. In particular, this dissertation extends the theory of the fractional self-adjoint difference equation in [13] to the nabla calculus setting.

An interesting combination of the delta and nabla operators is defined in [19] and is used to examine fractional calculus in a mixed setting.

Chapter 2

The Fractional Self-Adjoint Difference Equation

2.1 Introduction

In this section we introduce the self-adjoint linear fractional difference equation

$$\nabla_a^\nu (p\nabla y)(t) + q(t)y(\rho(t)) = f(t), \tag{2.1.1}$$

where $0 < \nu < 1$, $t \in \mathbb{N}_{a+2}^b$ for some real numbers a, b such that $b - a \in \mathbb{N}_1$, $p : \mathbb{N}_{a+1}^b \to (0, \infty)$, $q : \mathbb{N}_{a+2}^b \to \mathbb{R}$, and $f : \mathbb{N}_{a+2}^b \to \mathbb{R}$.

Note that if $\nu = 1$ we get the standard self-adjoint difference equation

$$\nabla(p\nabla y)(t) + q(t)y(\rho(t)) = f(t), \quad t \in \mathbb{N}_{a+2}^b,$$

and it is for this sole reason that we call equation (2.1.1) a fractional self-adjoint equation. Though the fractional difference operator in equation (2.1.1) cannot be said to be self-adjoint, many of the results for the self-adjoint difference equation have analogues in the fractional case.

In Section 2.2 we will prove that the solutions of equation (2.1.1) with appropriate initial conditions exist and are unique. In Section 2.3 we will establish a variation of constants formula for the nonhomogeneous initial value problem with homogeneous boundary conditions. In Section 2.4 we will derive the Green's function for a nonhomogeneous fractional boundary value problem with homogeneous boundary conditions and prove that it is nonnegative, find its maximum, and find appropriate bounds for its integral. We will generalize this Green's function to the case with general boundary conditions in Section 2.6.

2.2 Existence and Uniqueness Theorem

In this section we will prove an existence and uniqueness theorem for the nabla self-adjoint fractional initial value problem.

Theorem 2.2.1. *The fractional initial value problem*

$$\begin{cases} \nabla_a^\nu (p\nabla y)(t) + q(t)y(\rho(t)) = f(t), \quad t \in \mathbb{N}_{a+2} \\ y(a) = A, \quad y(a+1) = B \end{cases} \tag{2.2.1}$$

where $0 < \nu < 1$, $p : \mathbb{N}_{a+1} \to (0, \infty)$, $q : \mathbb{N}_{a+2} \to \mathbb{R}$, *and* $f : \mathbb{N}_{a+2} \to \mathbb{R}$ *has a unique solution* $y : \mathbb{N}_a \to \mathbb{R}$.

Proof. Consider the fractional self-adjoint equation

$$\nabla_a^\nu (p\nabla y)(t) + q(t)y(\rho(t)) = f(t).$$

We begin by rewriting the fractional difference using the summation notation given in Definition 1.3.5.

$$\frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t} [(t-\rho(s))^{\overline{-\nu-1}}(p(s)\nabla y(s))] + q(t)y(\rho(t)) = f(t)$$

Letting $t = a+2$, we get

$$\begin{aligned}
f(a+2) &= \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{a+2} [(a+2-\rho(s))^{\overline{-\nu-1}}(p(s)\nabla y(s))] + q(a+2)y(a+1) \\
&= \frac{1}{\Gamma(-\nu)} [2^{\overline{-\nu-1}} p(a+1)y(a+1) - 2^{\overline{-\nu-1}} p(a+1)y(a) \\
&\quad + 1^{\overline{-\nu-1}} p(a+2)y(a+2) - 1^{\overline{-\nu-1}} p(a+2)y(a+1)] \\
&\quad + q(a+2)y(a+1) \\
&= p(a+2)y(a+2) + A\nu p(a+1) + B[q(a+2) - \nu p(a+1) - p(a+2)].
\end{aligned}$$

Solving for $y(a+2)$ we have that

$$y(a+2) = \frac{1}{p(a+2)}[f(a+2) - A\nu p(a+1) - B(q(a+2) - \nu p(a+1) - p(a+2))].$$

Thus the value of $y(a+2)$ is uniquely determined by the initial conditions $y(a) = A$ and $y(a+1) = B$ and the values of the given functions f, p, and q. To show that $y(t)$ is uniquely determined on $\mathbb{N}_a$ we will use induction. Suppose that there exists a unique solution to the fractional initial value problem, $y(t)$, for $t \in \mathbb{N}_a^{t_0}$, where $t_0 \in \mathbb{N}_{a+2}$. We will show that the value of $y(t_0+1)$ is uniquely determined by the values of $y(t)$ on $\mathbb{N}_a^{t_0}$.

Substituting $t = t_0 + 1$ into the fractional equation, we get the following:

$$\begin{aligned} f(t_0+1) &= \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t_0+1} [(t_0+1-\rho(s))^{\overline{-\nu-1}} p(s)\nabla y(s)] + q(t_0+1)y(t_0) \\ &= \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t_0} [(t_0+1-\rho(s))^{\overline{-\nu-1}} p(s)\nabla y(s)] + q(t_0+1)y(t_0) \\ &\quad + p(t_0+1)y(t_0+1) - p(t_0+1)y(t_0). \end{aligned}$$

Solving for $y(t_0+1)$ we have that

$$\begin{aligned} y(t_0+1) = \frac{1}{p(t_0+1)} [f(t_0+1) &- \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t_0} [(t_0+1-\rho(s))^{\overline{-\nu-1}} p(s)\nabla y(s)] \\ &- q(t_0+1)y(t_0) + p(t_0+1)y(t_0)]. \end{aligned}$$

Since, by the induction hypothesis, all the values of $y(t)$ for t in $\mathbb{N}_a^{t_0}$ are known, $y(t_0+1)$ is uniquely determined and hence $y(t)$ is the unique solution of the fractional initial value problem (2.2.1) on $\mathbb{N}_a^{t_0+1}$. This combined with our base case of $t = a+2$ gives that by mathematical induction, the fractional initial value problem (2.2.1) has a unique solution that exists on $\mathbb{N}_a$. $\square$

Remark 2.2.2. *When $\nu = 1$ (non fractional case) in Theorem 2.2.1, it can be shown that for any $t_0 \in \mathbb{N}_a$ the initial conditions $y(t_0) = A$ and $y(t_0+1) = B$ determine a unique solution of the initial value problem* (2.2.1)*. However, in the fractional case $0 < \nu < 1$, it is necessary that $t_0 = a$, because the fractional difference depends on all of the values of a function back to the value at its base, a.*

2.3 Variation of Constants Formula

In this section we will establish a variation of constants formula for the fractional self-adjoint initial value problem

$$\begin{cases} \nabla_a^\nu (p\nabla y)(t) = f(t), \quad t \in \mathbb{N}_{a+2} \\ y(a+1) = \nabla y(a+1) = 0 \end{cases}$$

where $0 < \nu < 1$, $p : \mathbb{N}_{a+1} \to (0, \infty)$, $q : \mathbb{N}_{a+2} \to \mathbb{R}$, and $f : \mathbb{N}_{a+2} \to \mathbb{R}$. Our variation of constants formula will involve the Cauchy function, whose definition is given below.

Definition 2.3.1. *We define the Cauchy function $x(t, \rho(s))$ for the homogeneous fractional equation*

$$\nabla_a^\nu (p\nabla y)(t) = 0$$

to be the function $x : \mathbb{N}_{a+1} \times \mathbb{N}_{a+1} \to \mathbb{R}$ such that for each fixed $s \in \mathbb{N}_{a+1}$, $x(\cdot, \rho(s))$ is the unique solution of the fractional initial value problem

$$\begin{cases} \nabla_{\rho(s)}^\nu (p\nabla x)(t) = 0, \quad t \in \mathbb{N}_{s+1} \\ x(\rho(s)) = 0, \nabla x(s) = \frac{1}{p(s)} \end{cases} \tag{2.3.1}$$

and is given by the formula

$$x(t, \rho(s)) = \sum_{\tau=s}^{t} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu) p(\tau)}, \quad t \in \mathbb{N}_{a+1}. \tag{2.3.2}$$

Note that by our convention $x(t, \rho(s)) = 0$ for $t \leq \rho(s)$.

Theorem 2.3.2. *Let $f : \mathbb{N}_{a+2} \to \mathbb{R}$ and $p : \mathbb{N}_{a+1} \to (0, \infty)$. The solution to the fractional initial value problem*

$$\begin{cases} \nabla_a^\nu (p\nabla y)(t) = f(t), \quad t \in \mathbb{N}_{a+2} \\ y(a+1) = \nabla y(a+1) = 0 \end{cases} \tag{2.3.3}$$

is given by

$$y(t) = \sum_{s=a+2}^{t} x(t, \rho(s)) f(s)$$

where $x(t, \rho(s))$ is the Cauchy function (2.3.2).

Proof. Let $y(t)$ be the solution of the fractional initial value problem (2.3.3) and let $h(t) = p(t)\nabla y(t)$. Then $h(t)$ is a solution of the initial value problem

$$\nabla_a^\nu h(t) = f(t), \qquad h(a+1) = p(a+1)\nabla y(a+1) = 0$$

and, by Corollary 1.4.3, is given by

$$\begin{aligned} h(t) &= \nabla_a^{-\nu} f(t) - f(a+1)\frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} \\ &= \sum_{s=a+1}^{t} \left[\frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s) \right] - \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} f(a+1) \\ &= \sum_{s=a+2}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s). \end{aligned}$$

Dividing both sides by $p(t)$ we get

$$\nabla y(t) = \sum_{s=a+2}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)p(t)} f(s).$$

Summing both sides from $\tau = a + 2$ to t and using the Fundamental Theorem of

Nabla Calculus gives that

$$y(t) - y(a+1) = \sum_{\tau=a+2}^{t} \left[\sum_{s=a+2}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu) p(\tau)} f(s) \right].$$

Interchanging the order of the sums on the right hand side and using that $y(a+1) = 0$, we get

$$\begin{aligned} y(t) &= \sum_{s=a+2}^{t} \left[\sum_{\tau=s}^{t} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu) p(\tau)} f(s) \right] \\ &= \sum_{s=a+2}^{t} f(s) \left[\sum_{\tau=s}^{t} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu) p(\tau)} \right] \\ &= \sum_{s=a+2}^{t} f(s) x(t, \rho(s)). \end{aligned}$$

This completes the proof. $\square$

Corollary 2.3.3. *Assume $p : \mathbb{N}_{a+1} \to (0, \infty)$ and $u(t)$, $v(t)$ satisfy*

$$\begin{aligned} \nabla_a^{\nu}(p \nabla u)(t) &\geq \nabla_a^{\nu}(p \nabla v)(t), \quad t \in \mathbb{N}_{a+2} \\ u(a) &= v(a) \\ u(a+1) &= v(a+1). \end{aligned}$$

Then $u(t) \geq v(t)$ on $\mathbb{N}_a$.

Proof. Set $w(t) = u(t) - v(t)$, and let

$$h(t) = \nabla_a^{\nu}(p \nabla w)(t) = \nabla_a^{\nu}(p \nabla u)(t) - \nabla_a^{\nu}(p \nabla v)(t) \geq 0, \quad t \in \mathbb{N}_{a+2}.$$

Hence w solves the initial value problem

$$\nabla_a^\nu (p\nabla w)(t) = h(t), \quad t \in \mathbb{N}_{a+2}$$
$$w(a+1) = 0$$
$$\nabla w(a+1) = 0,$$

and our variation of constants formula gives that for $t \in \mathbb{N}_a$,

$$w(t) = \sum_{s=a+2}^{t} h(s) x(t, \rho(s)) = \sum_{s=a+2}^{t} h(s) \sum_{\tau=s}^{t} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu) p(\tau)} \geq 0.$$

Since $w(t) = u(t) - v(t) \geq 0$, we have that

$$u(t) \geq v(t) \quad \text{for} \quad t \in \mathbb{N}_a.$$

□

2.4 Green's Function for a Fractional Boundary Value Problem

We will now find the Green's function for a nonhomogeneous fractional boundary value problem with homogeneous boundary conditions.

Theorem 2.4.1. *The fractional boundary value problem*

$$\begin{cases} -\nabla_a^\nu (p\nabla y)(t) = h(t), \quad t \in \mathbb{N}_{a+1}^b \\ y(a) = 0, \quad y(b) = 0 \end{cases} \tag{2.4.1}$$

where $a, b \in \mathbb{R}$ *with* $b-a \in \mathbb{N}_1$, $h, p : \mathbb{N}_{a+1}^b \to \mathbb{R}$, *and* $p(t) > 0$, *has the unique solution* $y(t) = \int_a^b G(t,s)h(s)\nabla s$, *where*

$$G(t,s) = \begin{cases} \frac{x(b,\rho(s))}{x(b,a)} x(t,a), & t \leq s-1 \\ \frac{x(b,\rho(s))}{x(b,a)} x(t,a) - x(t,\rho(s)), & t \geq s \end{cases} \tag{2.4.2}$$

and $x(t, \rho(s))$ *is the Cauchy function* (2.3.2).

Proof. Let $x(t) = p(t)\nabla y(t)$, and let $A := x(a+1) = p(a+1)\nabla y(a+1)$. Then $x(t)$ solves the fractional initial value problem

$$\begin{cases} -\nabla_a^\nu x(t) = h(t) \\ x(a+1) = A \end{cases}$$

and therefore by Theorem 1.4.3,

$$x(t) = -\nabla_a^{-\nu} h(t) - (A - h(a+1)) \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)}.$$

Letting $c_0 := (A - h(a+1))$ and dividing both sides by $p(t)$ gives that

$$\begin{aligned} \nabla y(t) &= \frac{-1}{p(t)} \left(\nabla_a^{-\nu} h(t) - c_0 \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} \right) \\ &= \frac{-1}{p(t)} \left[\frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\nu-1}} h(s) + c_0 \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} \right]. \end{aligned}$$

Summing from $a+1$ to t, we get

$$y(t) = -\sum_{\tau=a+1}^{t} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} h(s) + c_0 \frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} \right].$$

Interchanging the order of sums gives that

$$
\begin{aligned}
y(t) &= -\sum_{s=a+1}^{t} h(s) \sum_{\tau=s}^{t} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} - c_0 \sum_{\tau=a+1}^{t} \frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} \\
&= -\sum_{s=a+1}^{t} h(s)x(t,\rho(s)) - c_0 x(t,a).
\end{aligned}
$$

Letting $t=b$ and solving for c_0 we get that

$$
c_0 = \frac{-\sum_{s=a+1}^{b} h(s)x(b,\rho(s))}{x(b,a)}.
$$

Substituting this value for c_0 into the formula for $y(t)$ gives us that

$$
\begin{aligned}
y(t) &= -\sum_{s=a+1}^{t} h(s)x(t,\rho(s)) + \frac{x(t,a)}{x(b,a)} \sum_{s=a+1}^{b} h(s)x(b,\rho(s)) \\
&= -\sum_{s=a+1}^{t} h(s)x(t,\rho(s)) + \frac{x(t,a)}{x(b,a)} \sum_{s=a+1}^{t} h(s)x(b,\rho(s)) + \frac{x(t,a)}{x(b,a)} \sum_{s=t+1}^{b} h(s)x(b,\rho(s)) \\
&= \sum_{s=a+1}^{t} h(s) \left[\frac{x(b,\rho(s))}{x(b,a)}x(t,a) - x(t,\rho(s))\right] + \sum_{s=t+1}^{b} h(s) \left[\frac{x(b,\rho(s))}{x(b,a)}x(t,a)\right] \\
&= \sum_{s=a+1}^{b} h(s)G(t,s),
\end{aligned}
$$

where

$$
G(t,s) = \begin{cases} \frac{x(b,\rho(s))}{x(b,a)}x(t,a), & t \le s-1 \\[2ex] \frac{x(b,\rho(s))}{x(b,a)}x(t,a) - x(t,\rho(s)), & t \ge s. \end{cases}
$$

Hence any solution to the above boundary value problem is necessarily given by the formula we have derived. Uniqueness of the solution $y(t)$ follows from Theorem 2.2.1.

$\square$

2.5 Green's Function and its Properties for the Case $p \equiv 1$

We will now find the Green's function for (2.4.1) for the case $p \equiv 1$. We will prove that this Green's function is nonnegative and we will find an upper bound for $G(t,s)$ and its integral.

Remark 2.5.1. *The Cauchy function of the fractional homogeneous equation*

$$\nabla_a^\nu y(t) = 0$$

is given by

$$x(t,\rho(s)) = \frac{(t-\rho(s))^{\overline{\nu}}}{\Gamma(\nu+1))}.$$

Using Definition 2.3.1 and Theorem 1.1.7 we have

$$\begin{aligned}
x(t,\rho(s)) &= \sum_{\tau=s}^{t} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} \\
&= \frac{1}{\Gamma(\nu)} \int_{\rho(s)}^{t} (\tau-\rho(s))^{\overline{\nu-1}} \nabla\tau \\
&= \frac{1}{\Gamma(\nu)} \left(\frac{1}{\nu}(t-\rho(s))^{\overline{\nu}} - \frac{1}{\nu} 0^{\overline{\nu}} \right) \\
&= \frac{(t-\rho(s))^{\overline{\nu}}}{\Gamma(\nu+1)}.
\end{aligned}$$

When $p \equiv 1$, the composition rule given in Theorem 1.3.7 allows us to rewrite $\nabla_a^\nu (p\nabla y)(t)$ as $\nabla_a^{1+\nu} y(t)$.

Hence the fractional boundary value problem

$$\begin{cases} -\nabla_a^{1+\nu} y(t) = h(t), \quad t \in \mathbb{N}_{a+1}^b \\ y(a) = 0, \quad y(b) = 0 \end{cases} \tag{2.5.1}$$

where $a, b \in \mathbb{R}$ *with* $b - a \in \mathbb{N}_1$, *and* $h : \mathbb{N}_{a+1}^b \to \mathbb{R}$, *has the unique solution* $y(t) = \int_a^b G(t,s)h(s)\nabla s$, *where*

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\nu+1)} \left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}} (t-a)^{\overline{\nu}} \right), & t \le s-1 \\ \frac{1}{\Gamma(\nu+1)} \left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}} (t-a)^{\overline{\nu}} - (t-s+1)^{\overline{\nu}} \right), & t \ge s. \end{cases} \tag{2.5.2}$$

Theorem 2.5.2. *The Green's function* $G(t,s)$ *for* (2.5.1) *satisfies* $G(t,s) \ge 0$ *for* $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$.

Proof. We will show that for any fixed s, $G(t,s)$ increases from $G(a,s) = 0$ to a positive value at $t = s-1$ and then decreases to $G(b,s) = 0$. Let $s \in \mathbb{N}_{a+1}^b$ be fixed but arbitrary.

First, we verify that $G(a,s) = G(b,s) = 0$.

This follows immediately from the equations

$$G(a,s) = \frac{1}{\Gamma(\nu+1)} \left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}} (a-a)^{\overline{\nu}} \right) = 0$$

and

$$G(b,s) = \frac{1}{\Gamma(\nu+1)} \left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}} (b-a)^{\overline{\nu}} - (b-s+1)^{\overline{\nu}} \right) = 0.$$

Now, we show that for each fixed s, $G(t,s)$ increases in t for values of t between $a+1$ and $s-1$. We consider the nabla difference with respect to t.

$$\begin{aligned}\nabla G(t,s) &= \nabla\left[\frac{1}{\Gamma(\nu+1)}\left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(t-a)^{\overline{\nu}}\right)\right]\\ &= \frac{\nu}{\Gamma(\nu+1)}\cdot\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(t-a)^{\overline{\nu-1}} > 0,\end{aligned}$$

as $b-s+1>0$, $b-a>0$, and though $\nu-1<0$, $t-a+\nu-1>0$. Thus $G(t,s)$ is increasing for all values of t between $a+1$ and $s-1$. Since the Green's function is zero at $t=a$ and increases for t between $a+1$ and $s-1$, $G(t,s)\geq 0$ for t values between $a+1$ and $s-1$.

We now show that $G(t,s)$ is decreasing for values of t between s and b. As above, we consider the nabla difference,

$$\begin{aligned}\nabla G(t,s) &= G(t,s)-G(t-1,s)\\ &= \frac{1}{\Gamma(\nu+1)}\left[\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(t-a)^{\overline{\nu}}-(t-s+1)^{\overline{\nu}}\right.\\ &\qquad\left.-\left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(t-a-1)^{\overline{\nu}}-(t-s)^{\overline{\nu}}\right)\right]\\ &= \frac{1}{\Gamma(\nu+1)(b-a)^{\overline{\nu}}}\left[(b-s+1)^{\overline{\nu}}(t-a)^{\overline{\nu}}-(b-a)^{\overline{\nu}}(t-s+1)^{\overline{\nu}}\right.\\ &\qquad\left.-(b-s+1)^{\overline{\nu}}(t-a-1)^{\overline{\nu}}+(b-a)^{\overline{\nu}}(t-s)^{\overline{\nu}}\right].\end{aligned}$$

We will show that the bracketed expression above is less than or equal to zero. Rearranging and factoring, the claim that the bracketed expression is less than or equal to zero is equivalent to

$$(b-a)^{\overline{\nu}}\left[(t-s)^{\overline{\nu}}-(t-s-1)^{\overline{\nu}}\right]+(b-s+1)^{\overline{\nu}}\left[(t-a)^{\overline{\nu}}-(t-a-1)^{\overline{\nu}}\right]\leq 0, \quad (2.5.3)$$

which is in turn equivalent to

$$(b-a)^{\overline{\nu}}\left[(t-s)^{\overline{\nu}}-(t-s+1)^{\overline{\nu}}\right] \leq (b-s+1)^{\overline{\nu}}\left[(t-a-1)^{\overline{\nu}}-(t-a)^{\overline{\nu}}\right].$$

Since $s-1 \geq a$, $0 < (b-a) \leq (b-(s-1))$, it remains to show that

$$(t-s)^{\overline{\nu}}-(t-s+1)^{\overline{\nu}} \leq (t-a-1)^{\overline{\nu}}-(t-a)^{\overline{\nu}}. \tag{2.5.4}$$

Note that

$$(t-s+1)^{\overline{\nu}} = \frac{(t-s+\nu)}{t-s}(t-s)^{\overline{\nu}}$$

and

$$(t-a)^{\overline{\nu}} = \frac{(t-a-1+\nu)}{t-a-1}(t-a-1)^{\overline{\nu}}.$$

Rewriting the inequality (2.5.4), simplifying, and multiplying both sides by $-1/\nu$, we see that the inequality is equivalent to

$$\begin{aligned}
(t-s)^{\overline{\nu}} - \frac{t-s+\nu}{t-s}(t-s)^{\overline{\nu}} &\leq (t-a-1)^{\overline{\nu}} - \frac{t-a-1+\nu}{t-a-1}(t-a-1)^{\overline{\nu}} \\
(t-s)^{\overline{\nu}}\left(\frac{-\nu}{t-s}\right) &\leq (t-a-1)^{\overline{\nu}}\left(\frac{-\nu}{t-a-1}\right) \\
\frac{(t-s)^{\overline{\nu}}}{t-s} &\geq \frac{(t-a-1)^{\overline{\nu}}}{t-a-1}. \qquad (2.5.5)
\end{aligned}$$

Since $s \geq a+1$, we can show that the inequality (2.5.5) holds by demonstrating that the expression $\dfrac{(t-\tau)^{\overline{\nu}}}{t-\tau}$ is increasing in τ. To show this, we will show that the nabla difference of the expression is nonnegative. Consider the nabla difference of the expression,

$$\begin{aligned}\nabla \frac{(t-\tau)^{\overline{\nu}}}{t-\tau} &= \frac{(t-\tau)^{\overline{\nu}}}{t-\tau} - \frac{(t-\tau+1)^{\overline{\nu}}}{t-\tau+1} \\ &= \frac{(t-\tau)^{\overline{\nu}}}{t-\tau} - \frac{(t-\tau+\nu)(t-\tau)^{\overline{\nu}}}{(t-\tau+1)(t-\tau)} \\ &= \frac{(t-\tau)^{\overline{\nu}}}{(t-\tau+1)(t-\tau)}(1-\nu).\end{aligned}$$

This is greater than or equal to zero because both $t-\tau$ and $1-\nu$ are greater than or equal to zero. Hence the expression $\frac{(t-\tau)^{\overline{\nu}}}{t-\tau}$ is increasing in τ, which implies that inequality (2.5.5) holds, which is equivalent to the inequality (2.5.4), and hence the bracketed expression in (2.5.3) is less than or equal to zero, so that the Green's function $G(t,s)$ is decreasing for values of t between s and b. Since the Green's function is zero at $t=b$ and is decreasing for t values between s and b, this implies that $G(t,s) \geq 0$ between s and b.

We have demonstrated that $G(t,s)$ is greater than or equal to zero for any fixed value of s, and for all t between a and b. Note that though we have $G(t,s)$ increasing to $t=s-1$ and decreasing from $t=s$, it is as yet unclear whether the maximum of $G(t,s)$ for a fixed t occurs at $(s-1,s)$ or (s,s). We will explore this question in Theorem 2.5.4. $\square$

The positivity of the Green's function allows us to prove a comparison theorem for boundary value problems.

Corollary 2.5.3. *Assume that $u(t)$ and $v(t)$ satisfy*

$$\nabla_a^{1+\nu} u(t) \geq \nabla_a^{1+\nu} v(t), \quad t \in \mathbb{N}_{a+1}^b$$
$$u(a) = v(a)$$
$$u(b) = v(b).$$

Then $u(t) \geq v(t)$ on $\mathbb{N}_a^b$.

Proof. Set $w(t) = u(t) - v(t)$ and let

$$h(t) := \nabla_a^{1+\nu} w(t) = \nabla_a^{1+\nu} u(t) - \nabla_a^{1+\nu} v(t) \geq 0, \quad t \in \mathbb{N}_{a+1}^b.$$

Hence $w(t)$ solves the fractional boundary value problem

$$\nabla_a^{1+\nu} w(t) = h(t), \quad t \in \mathbb{N}_{a+1}^b$$
$$w(a) = 0$$
$$w(b) = 0$$

and the Green's function gives that the solution of this boundary value problem is

$$w(t) = \int_a^b G(t,s) h(s) \nabla s, \quad t \in \mathbb{N}_a^b$$

where $G(t,s) \geq 0$ and $h(s) \geq 0$, therefore $w(t) \geq 0$, which implies $u(t) \geq v(t)$ for all $t \in \mathbb{N}_a^b$. $\square$

For the case $p \equiv 1$, we now derive an upper bound for $G(t,s)$.

Theorem 2.5.4. *The maximum of the Green's function $G(t,s)$ defined in Remark 2.5.1 is given by*

$$G\left(\left\lfloor\frac{b+a+3}{2}\right\rfloor-1,\left\lfloor\frac{b+a+3}{2}\right\rfloor\right)$$
$$=\begin{cases}\frac{1}{\Gamma(\nu+1)(b-a)^{\overline{\nu}}}\left(1+\frac{2\nu}{b-a-1}\right)\left[\left(\frac{b-a-1}{2}\right)^{\overline{\nu}}\right]^2, & \left\lfloor\frac{b+a+3}{2}\right\rfloor=\frac{b+a+3}{2}\\ \frac{1}{\Gamma(\nu+1)(b-a)^{\overline{\nu}}}\left[\left(\frac{b-a}{2}\right)^{\overline{\nu}}\right]^2, & \left\lfloor\frac{b+a+3}{2}\right\rfloor=\frac{b+a}{2}+1\end{cases}.$$

Here, the floor function $\lfloor\cdot\rfloor$ denotes the largest value in $\mathbb{N}_{a+1}^{b}$ that is less than or equal to its argument.

Proof. We will begin by examining the Green's function to determine whether the maximum for a fixed t will occur at $(s-1,s)$ or (s,s). We have that

$$G(s-1,s)=\frac{1}{\Gamma(\nu+1)}\left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(s-a-1)^{\overline{\nu}}\right)$$

and, for $s>a+1$,

$$\begin{aligned}G(s,s)&=\frac{1}{\Gamma(\nu+1)}\left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(s-a)^{\overline{\nu}}-1^{\overline{\nu}}\right)\\&=\frac{1}{\Gamma(\nu+1)}\left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}\left(1+\frac{\nu}{s-a-1}\right)(s-a-1)^{\overline{\nu}}-\Gamma(\nu+1)\right)\\&=\frac{1}{\Gamma(\nu+1)}\left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(s-a-1)^{\overline{\nu}}\right)+\frac{(b-s+1)^{\overline{\nu}}(s-a-1)^{\overline{\nu}}}{\Gamma(\nu)(b-a)^{\overline{\nu}}(s-a-1)}-1\\&=G(s-1,s)+\frac{(b-s+1)^{\overline{\nu}}(s-a-1)^{\overline{\nu}}}{\Gamma(\nu)(b-a)^{\overline{\nu}}(s-a-1)}-1.\end{aligned}$$

Note that the restriction $s>a+1$ simply excludes from consideration the initial value $G(a+1,a+1)=0$. From the proof of Theorem 2.5.2 we see that for certain values of s and t, $G(t,s)$ is strictly increasing from 0, so we can conclude that 0 is not the maximum of this function.

We will show that $\dfrac{(b-s+1)^{\overline{\nu}}(s-a-1)^{\overline{\nu}}}{\Gamma(\nu)(b-a)^{\overline{\nu}}(s-a-1)}-1<0$, so that $G(s,s)\le G(s-1,s)$.

We have that $\dfrac{1}{\Gamma(\nu)}<1$ because $0<\nu<1$. Also $\dfrac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}<1$ for $s>a+1$, and

$$\frac{(s-a-1)^{\overline{\nu}}}{s-a-1}=\frac{\Gamma(s-a-1+\nu)}{(s-a-1)\Gamma(s-a-1)}=\frac{\Gamma(s-a-1+\nu)}{\Gamma(s-a)}<1.$$

Hence

$$\begin{aligned}\left(\frac{1}{\Gamma(\nu)}\right)\left(\frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}\right)\left(\frac{(s-a-1)^{\overline{\nu}}}{s-a-1}\right)&<1\\ \frac{(b-s+1)^{\overline{\nu}}(s-a-1)^{\overline{\nu}}}{\Gamma(\nu)(b-a)^{\overline{\nu}}(s-a-1)}&<1\\ \frac{(b-s+1)^{\overline{\nu}}(s-a-1)^{\overline{\nu}}}{\Gamma(\nu)(b-a)^{\overline{\nu}}(s-a-1)}-1&<0,\end{aligned}$$

which demonstrates that for $s>a+1$, $G(s,s)<G(s-1,s)$.

Now we wish to maximize $G(s-1,s)$ for s values between $a+2$ and b. Consider the nabla difference with respect to s, using Theorem 1.1.7

$$\begin{aligned}\nabla[(b-s+1)^{\overline{\nu}}(s-a-1)^{\overline{\nu}}]&=(b-\rho(s)+1)^{\overline{\nu}}\nu(s-a-1)^{\overline{\nu-1}}\\ &\quad+(-\nu)(b-\rho(s)+1)^{\overline{\nu-1}}(s-a-1)^{\overline{\nu}}\\ &=\nu(b-s+2+\nu-1)(b-s+2)^{\overline{\nu-1}}(s-a-1)^{\overline{\nu-1}}\\ &\quad-\nu(b-s+2)^{\overline{\nu-1}}(s-a-1+\nu-1)^{\overline{\nu-1}}\\ &=\nu(b-s+2)^{\overline{\nu-1}}(s-a-1)^{\overline{\nu-1}}[b+a+3-2s].\end{aligned}$$

In this expression, ν, $(b-s+2)^{\overline{\nu-1}}$, and $(s-a-1)^{\overline{\nu-1}}$ are all positive. The equation $b+a+3-2s=0$ has the solution $s=\dfrac{b+a+3}{2}$, so we consider $s=\left\lfloor\dfrac{b+a+3}{2}\right\rfloor$.

If $s\le\left\lfloor\dfrac{b+a+3}{2}\right\rfloor$, the difference $b+a+3-2s$ is positive, and thus the expression

$[(b-s+1)^{\overline{\nu}}(s-a-1)^{\overline{\nu}}]$ is increasing. If $s \geq \left\lfloor \frac{b+a+3}{2} \right\rfloor$, the difference $b+a+3-2s$ is negative, and thus the expression $[(b-s+1)^{\overline{\nu}}(s-a-1)^{\overline{\nu}}]$ is decreasing. Hence the maximum of the expression $[(b-s+1)^{\overline{\nu}}(s-a-1)^{\overline{\nu}}]$ occurs at $s = \left\lfloor \frac{b+a+3}{2} \right\rfloor$.

Here, depending on whether $b+a$ is even or odd, there are two potential cases, $\left\lfloor \frac{b+a+3}{2} \right\rfloor = \frac{b+a+3}{2}$ or $\left\lfloor \frac{b+a+3}{2} \right\rfloor = \frac{b+a}{2} + 1$.

When $\left\lfloor \frac{b+a+3}{2} \right\rfloor = \frac{b+a+3}{2}$, we have

$$
\begin{aligned}
&G\left(\frac{b+a+3}{2} - 1, \frac{b+a+3}{2}\right) \\
&\qquad = \frac{1}{\Gamma(\nu)}\left[\frac{(b-\frac{b+a+3}{2}+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}\left(\frac{b+a+3}{2} - 1 - a\right)^{\overline{\nu}}\right] \\
&\qquad = \frac{1}{\nu+1}\left[\frac{\left(\frac{b-a-1}{2}\right)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}\left(\frac{b-a+1}{2}\right)^{\overline{\nu}}\right] \\
&\qquad = \frac{1}{\Gamma(\nu+1)(b-a)^{\overline{\nu}}}\left(1+\frac{2\nu}{b-a-1}\right)\left[\left(\frac{b-a-1}{2}\right)^{\overline{\nu}}\right]^2.
\end{aligned}
$$

This is a constant in terms of $b-a$ and ν.

If $\left\lfloor \frac{b+a+3}{2} \right\rfloor = \frac{b+a}{2} + 1$, then

$$
\begin{aligned}
G\left(\frac{b+a}{2}, \frac{b+a}{2}+1\right) &= \frac{1}{\Gamma(\nu+1)}\left[\frac{\left(b-\left(\frac{b+a}{2}+1\right)+1\right)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}\left(\frac{b+a}{2} - a\right)^{\overline{\nu}}\right] \\
&= \frac{1}{\Gamma(\nu+1)(b-a)^{\overline{\nu}}}\left[\left(\frac{b-a}{2}\right)^{\overline{\nu}}\right]^2.
\end{aligned}
$$

This is also a constant in terms of $b-a$ and ν. This concludes our proof. □

In certain applications of the Contraction Mapping Theorem to nonlinear boundary value problems, it is useful to have bounds for the integral of the Green's function. In Chapter 3, we will show existence of a unique solution to a fractional difference equation by showing that its solution is the fixed point of a certain map. We will find conditions for which the map is a contraction, implying the existence of a fixed point. For certain boundary value problems, the map in question involves the convolution of the Green's function and a forcing term. In this case, a bound on the integral of the Green's function allows us to find conditions on the forcing term so that the map is a contraction. For applications of this type, see [45], [46]. We establish bounds on the integral of the Green's function in the following theorem.

Theorem 2.5.5. *The following inequality holds for the Green's function $G(t,s)$ from Remark 2.5.1.*

$$\int_a^b |G(t,s)|\,\nabla s \leq \left(\frac{b-a-1}{(\nu+1)\Gamma(\nu+2)}\right)\left(\frac{\nu(b-a)+1}{1+\nu}\right)^{\overline{\nu}}$$

for all $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$.

Proof. First, note that since $G(t,s)$ is nonnegative for all $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$, we have that $\int_a^b |G(t,s)|\,\nabla s = \int_a^b G(t,s)\nabla s$. We can integrate to obtain

$$\begin{aligned}
\sum_{s=a+1}^{b} G(t,s) &= \sum_{s=a+1}^{t} G(t,s) + \sum_{s=t+1}^{b} G(t,s) \\
&= \frac{1}{\Gamma(\nu)}\left[\sum_{s=a+1}^{t} \frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(t-a)^{\overline{\nu}} - (t-s+1)^{\overline{\nu}}\right. \\
&\qquad\left. + \sum_{s=t+1}^{b} \frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(t-a)^{\overline{\nu}}\right] \\
&= \frac{1}{\Gamma(\nu)}\left[\sum_{s=a+1}^{b} \frac{(b-s+1)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}(t-a)^{\overline{\nu}} - \sum_{s=a+1}^{t}(t-s+1)^{\overline{\nu}}\right] \\
&= \frac{1}{\Gamma(\nu)}\left[\frac{(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}\int_a^b (b-\rho(s))^{\overline{\nu}}\nabla s - \int_a^b (t-\rho(s))^{\overline{\nu}}\nabla s\right].
\end{aligned}$$

Using Theorem 1.2.6, we have that

$$\begin{aligned}
\sum_{s=a+1}^{b} G(t,s) &= \frac{1}{\Gamma(\nu)}\left[\frac{(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu}}}\left[\frac{-1}{\nu+1}(b-s)^{\overline{\nu+1}}\right]_a^b - \left[\frac{-1}{\nu+1}(t-s)^{\overline{\nu+1}}\right]_a^t\right] \\
&= \frac{-(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)(b-a)^{\overline{\nu}}}(0)^{\overline{\nu+1}} + \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)(b-a)^{\overline{\nu}}}(b-a)^{\overline{\nu+1}} \\
&\quad + \frac{1}{\Gamma(\nu+2)}(0)^{\overline{\nu+1}} - \frac{1}{\Gamma(\nu+2)}(t-a)^{\overline{\nu+1}} \\
&= \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(b-a+\nu) - \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(t-a+\nu) \\
&= \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(b-t).
\end{aligned}$$

We now wish to find the maximum of this expression with respect to $t \in \mathbb{N}_{a+1}^{b}$. The maximum does not occur at $t = a$ because here the expression $\frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(b-t)$ evaluates to zero. We consider the difference of the expression with respect to t.

$$\begin{aligned}
\nabla\left[\frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(b-t)\right] &= \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(-1) + \nu\frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu+2)}(b-t+1) \\
&= \frac{-(t-a)^{\overline{\nu-1}}}{\Gamma(\nu+2)}(t-a+\nu+1) + \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu+2)}(\nu b-\nu t+\nu) \\
&= \frac{(t-a)^{\nu-1}}{\Gamma(\nu+2)}[\nu b+a+1-t(\nu+1)].
\end{aligned}$$

The fraction $\frac{(t-a)^{\nu-1}}{\Gamma(\nu+2)}$ is positive, and so we consider the bracketed expression

$$[\nu b+a+1-t(\nu+1)].$$

This expression is equal to zero when $t = \frac{\nu b+a+1}{1+\nu}$. This value is not necessarily in $\mathbb{N}_{a+1}^{b}$, but it is true that $a+1 \leq \frac{\nu b+a+1}{1+\nu} \leq b$. (In fact, this inequality is strict when $b-a > 1$.)

When $t < \frac{\nu b+a+1}{1+\nu}$, the bracketed expression $[\nu b+a+1-t(\nu+1)]$ is positive, so $\frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(b-t)$ is increasing in t. When $t > \frac{\nu b+a+1}{1+\nu}$, the bracketed expression $[\nu b + a + 1 - t(\nu + 1)]$ is negative, so $\frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(b-t)$ is decreasing in t. While $t = \frac{\nu b+a+1}{1+\nu}$ is not necessarily a value taken by the function, evaluating $\frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(b-t)$ does produce a maximum for the integral of $G(t, s)$.

When $t = \frac{\nu b+a+1}{1+\nu}$, we have

$$\begin{aligned}\frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}(b-t) &= \frac{\left(\frac{\nu b+a+1}{1+\nu} - a\right)^{\overline{\nu}}}{\Gamma(\nu+2)}\left(b - \frac{\nu b + a + 1}{1+\nu}\right) \\ &= \left(\frac{b-a-1}{(\nu+1)\Gamma(\nu+2)}\right)\left(\frac{\nu(b-a)+1}{1+\nu}\right)^{\overline{\nu}}.\end{aligned}$$

This is the desired upper bound of $\int_a^b |G(t, s)| \nabla s$. □

Remark 2.5.6. *In the non-fractional case, where $\nu = 1$, the Green's function becomes*

$$G(t,s) = \begin{cases} \frac{(b-s+1)(t-a)-(t-s+1)(b-a)}{b-a}, & s \leq t \\ \frac{(b-s+1)}{b-a}(t-a), & s \geq t+1. \end{cases}$$

The maximum of the Green's function becomes

$$G\left(\left\lfloor \frac{b+a+3}{2} \right\rfloor - 1, \left\lfloor \frac{b+a+3}{2} \right\rfloor\right) = \begin{cases} \frac{b-a}{4} - \frac{1}{4(b-a)}, & \left\lfloor \frac{b+a+3}{2} \right\rfloor = \frac{b+a+3}{2} \\ \frac{b-a}{4}, & \left\lfloor \frac{b+a+3}{2} \right\rfloor = \frac{b+a}{2} + 1. \end{cases}$$

The upper bound for the integral of the Green's function becomes

$$\int_a^b |G(t,s)| \nabla s \leq \frac{(b-a)^2 - 1}{8}.$$

2.6 Green's Function for a Nonhomogeneous Fractional Boundary Value Problem with General Boundary Conditions

In this section we generalize the Green's function from Section 2.4 to the following homogeneous fractional self-adjoint boundary value problem with general boundary conditions,

$$\begin{cases} -\nabla_a^{\nu}(p\nabla y)(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b} \\ \alpha y(a+1) - \beta \nabla y(a+1) = 0 \\ \gamma y(b) + \delta \nabla y(b) = 0 \end{cases} \tag{2.6.1}$$

where $p : \mathbb{N}_{a+1}^{b} \to (0, \infty)$, $\alpha^2 + \beta^2 > 0$, and $\gamma^2 + \delta^2 > 0$. Notice that here the left general boundary condition is based at $a+1$, in order to avoid the use of $y(a-1)$, and so we understand the fractional difference equation to be satisfied for $t \in \mathbb{N}_{a+2}^{b}$.

Lemma 2.6.1. *The homogeneous fractional self-adjoint boundary value problem* (2.6.1) *has only the trivial solution if and only if*

$$\xi = \frac{\beta\gamma}{p(a+1)} + \alpha\gamma \sum_{\tau=a+2}^{b} \frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} + \frac{\alpha\delta(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \neq 0.$$

Proof. If $-\nabla_a^{\nu}(p\nabla y)(t) = 0$, then it follows from Corollary 1.4.3 that

$$\begin{aligned} (p\nabla y)(t) &= c_0 \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} \\ \nabla y(t) &= c_0 \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(t)} \\ y(t) - y(a) &= \sum_{\tau=a+1}^{t} c_0 \frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)}. \end{aligned}$$

Letting $y(a) = c_1$, we have

$$y(t) = c_0 \sum_{\tau=a+1}^{t} \frac{(\tau - a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} + c_1. \tag{2.6.2}$$

We will use the boundary conditions to determine c_0 and c_1. Evaluating equation (2.6.2) at $t = a+1$ gives

$$\begin{aligned} y(a+1) &= \frac{c_0}{p(a+1)} + c_1 \\ \nabla y(a+1) = y(a+1) - y(a) &= \frac{c_0}{p(a+1)}. \end{aligned}$$

Using these values in the first boundary condition, $\alpha y(a+1) - \beta \nabla y(a+1) = 0$, we obtain

$$\begin{aligned} \alpha\left(\frac{c_0}{p(a+1)} + c_1\right) - \beta\left(\frac{c_0}{p(a+1)}\right) &= 0 \\ c_0\left(\frac{\alpha - \beta}{p(a+1)}\right) + c_1\alpha &= 0. \end{aligned} \tag{2.6.3}$$

Evaluating equation (2.6.2) at $t = b$ gives

$$\begin{aligned} y(b) &= \sum_{\tau=a+1}^{b} c_0 \frac{(\tau - a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} + c_1 \\ \nabla y(b) = y(b) - y(b-1) &= c_0 \frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}. \end{aligned}$$

Substituting these values into the second boundary condition, $\gamma y(b) + \delta \nabla y(b) = 0$, we obtain

$$\gamma\left(\sum_{\tau=a+1}^{b} c_0\frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)}+c_1\right)+\delta\left(c_0\frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\right)=0$$

$$c_0\left(\gamma\sum_{\tau=a+1}^{b}\frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)}+\delta\frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\right)+c_1\gamma=0. \tag{2.6.4}$$

The system of equations (2.6.3), (2.6.4) in the variables c_0, c_1 has only the trivial solution if and only if the determinant of the system is not equal to 0. The determinant of the system, which we will call $-\xi$, is

$$-\xi=\begin{vmatrix}\frac{\alpha-\beta}{p(a+1)} & \alpha\\ \gamma\sum_{\tau=a+1}^{b}\frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)}+\delta\frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} & \gamma\end{vmatrix}.$$

It follows that

$$\begin{aligned}-\xi&=\frac{\alpha\gamma}{p(a+1)}-\frac{\beta\gamma}{p(a+1)}-\alpha\gamma\sum_{\tau=a+1}^{b}\frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)}-\alpha\gamma\frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\\&=-\frac{\beta\gamma}{p(a+1)}-\alpha\gamma\sum_{\tau=a+2}^{b}\frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)}-\frac{\alpha\delta(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}.\end{aligned}$$

Hence the solution $y(t)$ is nontrivial (i.e. c_0 and c_1 are not both equal to zero) if and only if $\xi\neq 0$. □

Remark 2.6.2. *In the case that $\nu=1$, we have that*

$$\begin{cases}-\nabla(p\nabla y)(t)=0, \quad t\in\mathbb{N}_{a+2}^{b}\\ \alpha y(a+1)-\beta\nabla y(a+1)=0\\ \gamma y(b)+\delta\nabla y(b)=0\end{cases}$$

has only the trivial solution if and only if

$$\xi = \frac{\beta\gamma}{p(a+1)} + \alpha\gamma \sum_{\tau=a+2}^{b} \frac{1}{p(\tau)} + \frac{\alpha\delta}{p(b)} \neq 0.$$

Theorem 2.6.3. *If the homogeneous boundary value problem* (2.6.1) *has only the trivial solution, then the nonhomogeneous boundary value problem*

$$\begin{cases} -\nabla_a^{\nu}(p\nabla y)(t) = h(t), \quad t \in \mathbb{N}_{a+2}^{b} \\ \alpha y(a+1) - \beta\nabla y(a+1) = 0 \\ \gamma y(b) + \delta\nabla y(b) = 0 \end{cases} \tag{2.6.5}$$

has a unique solution.

Proof. Let $y_1(t)$, $y_2(t)$ be two linearly independent solutions of the difference equation $-\nabla_a^{\nu}(p\nabla y)(t) = 0$. Note that we can find two linearly independent solutions by taking $y_1(a) = 1, \quad y_1(a+1) = 0, \quad y_2(a) = 0, \quad y_2(a+1) = 1$. These initial values determine the solutions $y_1(t), y_2(t)$ for $t \in \mathbb{N}_{a+2}^{b}$ as a result of Theorem 2.2.1. The choice of initial values implies that $y_1(t)$ and $y_2(t)$ are linearly independent.

Then for arbitrary real constants c_1 and c_2, $y(t) = c_1y_1(t) + c_2y_2(t)$ is a general solution of $-\nabla_a^{\nu}(p\nabla y)(t) = 0$. Given any solution $y_0(t)$, we choose $c_1 = y_0(a)$ and $c_2 = y_0(a+1)$, and again because of Theorem 2.2.1 we know $y_0(t) = c_1y_1(t) + c_2y_2(t)$ for all $t \in \mathbb{N}_a^{b}$.

Note that for any $y(t) = c_1y_1(t) + c_2y_2(t)$, the set of boundary conditions

$$\alpha y(a+1) - \beta\nabla y(a+1) = 0$$

$$\gamma y(b) + \delta\nabla y(b) = 0$$

are satisfied if and only if c_1 and c_2 satisfy

$$\begin{aligned} c_1\left[\alpha y_1(a+1)-\beta\nabla y_1(a+1)\right]+c_2\left[\alpha y_2(a+1)-\beta\nabla y_2(a+1)\right]&=0\\ c_2\left[\gamma y_1(b)+\delta\nabla y_1(b)\right]+c_2\left[\gamma y_2(b)+\delta\nabla y_2(b)\right]&=0. \end{aligned}$$

But by assumption, y must be the trivial solution, so $c_1 = c_2 = 0$.

Since the above system of equations in c_1 and c_2 is solved only by $c_1 = c_2 = 0$, it follows that its determinant is nonzero,

$$\begin{vmatrix} \alpha y_1(a+1)-\beta\nabla y_1(a+1) & \alpha y_2(a+1)-\beta\nabla y_2(a+1) \\ \gamma y_1(b)+\delta\nabla y_1(b) & \gamma y_2(b)+\delta\nabla y_2(b) \end{vmatrix} \neq 0. \tag{2.6.6}$$

Let $y_0(t)$ be the unique solution of the initial value problem

$$\begin{cases} -\nabla_a^{\nu}(p\nabla y)(t)=h(t), \quad t\in\mathbb{N}_{a+2}^{b} \\ y(a)=A, \quad y(a+1)=B. \end{cases}$$

Then a general solution of $\nabla_a^{\nu}(p\nabla y)(t)=h(t)$ is given by

$$y(t)=c_3y_1(t)+c_4y_2(t)+y_0(t).$$

This $y(t)$ will satisfy the boundary conditions

$$\begin{aligned} \alpha y(a+1)-\beta\nabla y(a+1)&=0\\ \gamma y(b)+\delta\nabla y(b)&=0 \end{aligned}$$

if and only if c_3, c_4 are constants that solve the system of equations

$$c_3[\alpha y_1(a+1) - \beta\nabla y_1(a+1)] + c_4[\alpha y_2(a+1) - \beta\nabla y_2(a+1)] = -\alpha B + \beta(B-A)$$
$$c_3[\gamma y_1(b) + \delta\nabla y_1(b)] + c_4[\gamma y_2(b) + \delta\nabla y_2(b)] = -\gamma y_0(b) - \delta\nabla y_0(b).$$

This system of equations has a unique solution for the constants c_3 and c_4 because the determinant in equation (2.6.6) is not zero. Hence the fractional boundary value problem (2.6.5) has a unique solution. □

Theorem 2.6.4. *Assume that ξ, as defined in Lemma 2.6.1, is not zero. Then the Green's function for the boundary value problem* (2.6.1) *is given by*

$$G(t,s) = \begin{cases} u(t,s), & t \le s-1 \\ v(t,s), & t \ge s \end{cases} \tag{2.6.7}$$

where

$$u(t,s) = \frac{1}{\xi}\left(\alpha\gamma x(t,a)x(b,\rho(s)) + \alpha\delta x(t,a)\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} + \frac{\gamma(\beta-\alpha)}{p(a+1)}x(b,\rho(s)) + \frac{\delta(\beta-\alpha)}{p(a+1)}\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\right) \tag{2.6.8}$$

and

$$v(t,s) = u(t,s) - x(t,\rho(s)). \tag{2.6.9}$$

Here, $x(t,\rho(s))$ refers to the Cauchy function (2.3.2).

Proof. Suppose that $y(t)$ is a solution of the boundary value problem

$$\begin{cases} -\nabla_a^{\nu}(p\nabla y)(t) = h(t), \quad t \in \mathbb{N}_{a+2}^{b} \\ \alpha y(a+1) - \beta\nabla y(a+1) = 0 \\ \gamma y(b) + \delta\nabla y(b) = 0. \end{cases} \tag{2.6.10}$$

Then $x(t) = (p\nabla y)(t)$ solves the initial value problem

$$\begin{cases} -\nabla_a^{\nu} x(t) = h(t), \quad t \in \mathbb{N}_{a+2} \\ x(a+1) = p(a+1)[y(a+1) - y(a)]. \end{cases}$$

Corollary 1.4.3 gives that the solution of this initial value problem has the form

$$x(t) = -\nabla_a^{-\nu} h(t) - c_0 \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)}.$$

Hence we have that

$$\nabla y(t) = -\sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)p(t)} h(s) - c_0 \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(t)}.$$

We sum both sides from $a+1$ to t to get

$$y(t) - y(a) = -\sum_{\tau=a+1}^{t} \left(\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} h(s) - c_0 \frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} \right).$$

Letting $y(a) = c_1$ and interchanging sums, we obtain

$$\begin{aligned} y(t) &= -\sum_{s=a+1}^{t}\sum_{\tau=s}^{t}\left(\frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)}h(s)\right) - c_0\sum_{\tau=a+1}^{t}\frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} + c_1 \\ &= -\sum_{s=a+1}^{t} h(s)x(t,\rho(s)) - c_0x(t,a) + c_1. \end{aligned}$$

Now we will use the boundary conditions to obtain formulas for the constants c_0 and c_1. Since

$$y(a+1) = \frac{-c_0}{p(a+1)} - \frac{h(a+1)}{p(a+1)} + c_1$$

and

$$\nabla y(a+1) = \frac{-c_0}{p(a+1)} - \frac{h(a+1)}{p(a+1)},$$

we have that

$$\alpha\left(\frac{-c_0}{p(a+1)} - \frac{h(a+1)}{p(a+1)} + c_1\right) - \beta\left(\frac{-c_0}{p(a+1)} - \frac{h(a+1)}{p(a+1)}\right) = 0.$$

Since $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$, we extend the domain of h by letting $h(a+1) = 0$. Rewriting this equation to collect the terms involving c_0 and c_1, we obtain

$$c_0\left(\frac{\beta-\alpha}{p(a+1)}\right) + c_1\alpha = 0. \tag{2.6.11}$$

Since

$$y(b) = -\sum_{s=a+1}^{b} h(s)x(b,\rho(s)) - c_0x(b,a) + c_1$$

and

$$
\begin{aligned}
\nabla y(b) &= -\sum_{s=a+1}^{b} h(s)x(b,\rho(s)) - c_0 x(b,a) + c_1 \\
&\quad - \left(-\sum_{s=a+1}^{b-1} h(s)x(b-1,\rho(s)) - c_0 x(b-1,a) + c_1 \right) \\
&= -h(b)x(b,b-1) \\
&\quad - \sum_{s=a+1}^{b-1} h(s)[x(b,\rho(s)) - x(b-1,\rho(s))] - c_0[x(b,a) - x(b-1,a)] \\
&= -\frac{h(b)}{p(b)} - \sum_{s=a+1}^{b-1} h(s) \left(\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right) - c_0 \frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \\
&= -\sum_{s=a+1}^{b} h(s) \left(\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right) - c_0 \frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}
\end{aligned}
$$

we have that

$$
\begin{aligned}
0 = \gamma &\left(-\sum_{s=a+1}^{b} h(s)x(b,\rho(s)) - c_0 x(b,a) + c_1 \right) \\
&+ \delta \left(-\sum_{s=a+1}^{b} h(s) \left(\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right) - c_0 \frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right)
\end{aligned}
$$

or, rewriting to collect c_0 and c_1,

$$
\begin{aligned}
&c_0 \left(-\gamma x(b,a) - \delta \frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right) + c_1\gamma \\
&\qquad\qquad = \gamma \sum_{s=a+1}^{b} h(s)x(b,\rho(s)) + \delta \sum_{s=a+1}^{b} h(s) \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \\
&c_0 \left(-\gamma x(b,a) - \delta \frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right) + c_1\gamma = \sum_{s=a+2}^{b} h(s) \left[\gamma x(b,\rho(s)) + \delta \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right].
\end{aligned}
\tag{2.6.12}
$$

We will solve the system of equations (2.6.11), (2.6.12) by solving equation (2.6.11) for c_1 and substituting this value into equation (2.6.12). We find that

$$c_1 = \frac{(\alpha - \beta)c_0}{\alpha p(a+1)}$$

and, substituting into equation (2.6.12), we obtain

$$\begin{aligned} c_0 \left(-\gamma x(b,a) - \delta \frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} + \frac{\gamma(\alpha-\beta)c_0}{\alpha p(a+1)} \right) \\ = \sum_{s=a+2}^{b} h(s) \left[\gamma x(b,\rho(s)) + \delta \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right]. \end{aligned}$$

We simplify this equation by expanding $x(b,a)$ and multiplying both sides by α,

$$\begin{aligned} \sum_{s=a+2}^{b} h(s) \left[\alpha\gamma x(b,\rho(s)) + \alpha\delta \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right] \\ = c_0 \left(-\alpha\gamma \frac{1}{p(a+1)} - \alpha\gamma \sum_{\tau=a+2}^{b} \frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} \right. \\ \left. -\alpha\gamma \frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} + \frac{\alpha\gamma}{p(a+1)} - \frac{\beta\gamma}{p(a+1)} \right). \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{s=a+2}^{b} h(s) \left[\alpha\gamma x(b,\rho(s)) + \alpha\delta \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right] \\ = c_0 \left(- \left(\frac{\beta\gamma}{p(a+1)} + \alpha\gamma \sum_{\tau=a+2}^{b} \frac{(\tau-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} + \frac{\alpha\delta(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right) \right) = -\xi c_0. \end{aligned}$$

Hence we have that

$$c_0 = -\frac{1}{\xi} \sum_{s=a+2}^{b} h(s) \left[\alpha\gamma x(b,\rho(s)) + \alpha\delta \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \right],$$

and also that

$$
\begin{aligned}
c_1 &= \frac{(\alpha-\beta)}{\alpha p(a+1)}\left(-\frac{1}{\xi}\sum_{s=a+2}^{b} h(s)\left[\alpha\gamma x(b,\rho(s)) + \alpha\delta\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\right]\right)\\
&= \frac{(\beta-\alpha)}{\xi p(a+1)}\left(\sum_{s=a+2}^{b} h(s)\left[\gamma x(b,\rho(s)) + \delta\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\right]\right).
\end{aligned}
$$

Because $\xi \neq 0$ and $p(t) > 0$, both of these constants are well defined. Substituting both of these values into our formula for $y(t)$, we have

$$
\begin{aligned}
y(t) &= -\sum_{s=a+1}^{t} h(s)x(t,\rho(s)) - c_0 x(t,a) + c_1\\
&= -\sum_{s=a+1}^{t} h(s)x(t,\rho(s))\\
&\quad + x(t,a)\left(\frac{1}{\xi}\sum_{s=a+2}^{b} h(s)\left[\alpha\gamma x(b,\rho(s)) + \alpha\delta\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\right]\right)\\
&\quad + \frac{(\beta-\alpha)}{\xi p(a+1)}\left(\sum_{s=a+2}^{b} h(s)\left[\gamma x(b,\rho(s)) + \delta\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\right]\right).
\end{aligned}
$$

Combining the terms in the second and third sums, we obtain

$$
\begin{aligned}
y(t) = &-\sum_{s=a+1}^{t} h(s)x(t,\rho(s))\\
&+\sum_{s=a+1}^{b} h(s)\left[\frac{1}{\xi}\left(\alpha\gamma x(t,a)x(b,\rho(s)) + \alpha\delta x(t,a)\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\right.\right.\\
&\qquad\left.\left. + \frac{\gamma(\beta-\alpha)}{p(a+1)}x(b,\rho(s)) + \frac{\delta(\beta-\alpha)}{p(a+1)}\frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)}\right)\right].
\end{aligned}
$$

By rearranging the second sum by its terms, from $a+1$ to t and from $t+1$ to b, it follows that

$$
\begin{aligned}
y(t) = \sum_{s=a+1}^{t} h(s) \Bigg[\frac{1}{\xi} \Bigg(& \alpha\gamma x(t,a)x(b,\rho(s)) + \alpha\delta x(t,a) \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \\
& + \frac{\gamma(\beta-\alpha)}{p(a+1)} x(b,\rho(s)) + \frac{\delta(\beta-\alpha)}{p(a+1)} \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \Bigg) - x(t,\rho(s)) \Bigg] \\
+ \sum_{s=t+1}^{b} h(s) \Bigg[\frac{1}{\xi} \Bigg(& \alpha\gamma x(t,a)x(b,\rho(s)) + \alpha\delta x(t,a) \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \\
& + \frac{\gamma(\beta-\alpha)}{p(a+1)} x(b,\rho(s)) + \frac{\delta(\beta-\alpha)}{p(a+1)} \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)p(b)} \Bigg) \Bigg].
\end{aligned}
$$

Hence

$$
\begin{aligned}
y(t) &= \sum_{s=a+1}^{t} h(s)u(t,s) + \sum_{s=t+1}^{b} h(s)v(t,s) \\
&= \sum_{s=a+1}^{b} h(s)G(t,s).
\end{aligned}
$$

Hence $G(t,s)$ as defined in the theorem is the Green's function for the boundary value problem (2.6.1). □

Remark 2.6.5. *Recall that in Theorem 2.4.1 we had found the Green's function for the fractional boundary value problem*

$$\begin{cases} -\nabla_a^{\nu}(p\nabla y)(t) = 0, \quad t \in \mathbb{N}_{a+1}^{b} \\ y(a) = 0, \quad y(b) = 0. \end{cases}$$

This Green's function is a special case of the Green's function for general boundary conditions, as given in equation (2.6.7)*. To see this, we let* $\alpha = \beta = \gamma = 1$ *and* $\delta = 0$*. Note that for these values we have* $\alpha^2 + \beta^2 > 0$ *and* $\gamma^2 + \delta^2 > 0$*. We then obtain*

$$\xi = \frac{1}{p(a+1)} + \sum_{\tau=a+2}^{b} \frac{(\tau - a)^{\overline{\nu-1}}}{\Gamma(\nu)p(\tau)} = x(b,a)$$

and so

$$u(t,s) = \frac{x(b,\rho(s))}{x(b,a)} x(t,a)$$

and

$$v(t,s) = u(t,s) - x(t,\rho(s)) = \frac{x(b,\rho(s))}{x(b,a)} x(t,a) - x(t,\rho(s))$$

which is exactly the Green's function from Theorem 2.4.1.

Chapter 3

Applications of the Contraction Mapping Theorem to Self-Adjoint Difference Equations

3.1 Introduction

In this chapter, we use the Contraction Mapping Theorem to establish the existence of solutions to the self-adjoint fractional difference equation

$$\nabla_a^\nu(p\nabla y)(t) + q(t)y(\rho(t)) = f(t), \tag{3.1.1}$$

where $0 < \nu < 1$, $t \in \mathbb{N}_{a+1}$ for some $a \in \mathbb{R}$, $p : \mathbb{N}_{a+1} \to (0, \infty)$, $q : \mathbb{N}_{a+1} \to \mathbb{R}$, and $f : \mathbb{N}_{a+1} \to \mathbb{R}$, and the forced fractional difference equation

$$\nabla_a^\nu(p\nabla x)(t) + F(t, x(t)) = 0, \tag{3.1.2}$$

where $F : \mathbb{N}_{a+1} \times \mathbb{R} \to [0, \infty)$, that satisfy

$$\lim_{t\to\infty} y(t) = L$$

for some $L \geq 0$.

In Chapter 2, we established existence and uniqueness of solutions to a related initial value problem, but in subsequent sections, we let $q \equiv 0$ in order to be able to compute $y(t)$. This is because, as we saw in the proof of Theorem 2.2.1, a formula for $y(t_0)$ depends on all of the values $y(a), y(a+1), \ldots, y(t_0 - 1)$. This motivates the use of fixed point theorems to study the self-adjoint equation with $q \not\equiv 0$. In each of the theorems in this chapter we will define an appropriate operator, a fixed point of which will be a solution of the self-adjoint equation satisfying certain properties.

3.2 Solutions with Positive Limits

To use the contraction mapping theorem, we must determine the appropriate operator, a fixed point of which will be a solution to the fractional self-adjoint difference equation. To that aim, we prove the following lemma.

Lemma 3.2.1. *Let $p : \mathbb{N}_{a+1} \to (0, \infty)$, $q : \mathbb{N}_{a+1} \to \mathbb{R}$, and $f : \mathbb{N}_{a+1} \to \mathbb{R}$. For some $L \geq 0$, define $\zeta := \left\{ y : \mathbb{N}_a \to \mathbb{R} \,\middle|\, \lim_{t\to\infty} y(t) = L \right\}$. Suppose that for all the functions $y \in \zeta$, the series*

$$\sum_{s=a+1}^{\infty} \frac{1}{p(s)} \left[\sum_{\tau=a+1}^{s} \frac{(s - \rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (q(\tau) y(\rho(\tau)) - f(\tau)) \right]$$

converges. Then the forced fractional self-adjoint difference equation

$$\nabla_a^\nu(p\nabla y)(t) + q(t)y(\rho(t)) = f(t), \quad t \in \mathbb{N}_{a+1} \tag{3.2.1}$$

has a solution $y \in \zeta$ if and only if the summation equation

$$y(t) = L + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \left[\sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (q(\tau)y(\rho(\tau)) - f(\tau)) \right] \tag{3.2.2}$$

has a solution $y(t)$ on $\mathbb{N}_a$.

Proof. Suppose that the difference equation (3.2.1) has a solution $y \in \zeta$.

Let $x(t) = (p\nabla y)(t)$. Then $x(t)$ solves the fractional initial value problem

$$\begin{cases} \nabla_a^\nu x(t) = f(t) - q(t)y(\rho(t)), \quad t \in \mathbb{N}_{a+2} \\ x(a+1) = p(a+1)\nabla y(a+1). \end{cases}$$

From Corollary 1.4.3, $x(t)$ has the form

$$\begin{aligned} x(t) &= \nabla_a^\nu(f(t) - q(t)y(\rho(t))) \\ &\quad + [p(a+1)\nabla y(a+1) - (f(a+1) - q(a+1)y(a))] \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)}, \end{aligned}$$

and since

$$\nabla_a^\nu(p\nabla y)(a+1) = \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{a+1} (a+1-\rho(s))^{\overline{-\nu-1}}(p\nabla y)(s) = (p\nabla y)(a+1),$$

we have

$$\begin{aligned}
x(t) &= \nabla_a^{\nu}(f(t) - q(t)y(\rho(t))) \\
&\qquad + \left[\nabla_a^{\nu}(p\nabla y)(a+1) - (f(a+1) - q(a+1)y(a))\right] \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} \\
&= \nabla_a^{\nu}(f(t) - q(t)y(\rho(t))).
\end{aligned}$$

Hence

$$\nabla y(t) = \frac{1}{p(t)} \sum_{\tau=a+1}^{t} \frac{(t-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (f(\tau) - q(\tau)y(\rho(\tau))).$$

Summing from $s = t+1$ to ∞ gives that

$$\begin{aligned}
\sum_{s=t+1}^{\infty} \nabla y(t) &= \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \left[\sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (f(\tau) - q(\tau)y(\rho(\tau))) \right] \\
\lim_{s\to\infty} y(s) - y(t) &= \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \left[\sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (f(\tau) - q(\tau)y(\rho(\tau))) \right] \\
y(t) &= L + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \left[\sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (q(\tau)y(\rho(\tau)) - f(\tau)) \right].
\end{aligned}$$

So $y(t)$ is also a solution of the summation equation (3.2.2).

Conversely, we must show that if $y(t)$ is a solution of the summation equation (3.2.2), then it is also a solution of the fractional difference equation (3.2.1) with $\lim_{t\to\infty} y(t) = L$. We have that

$$y(t) = L + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} \left(q(\tau)y(\rho(\tau)) - f(\tau)\right).$$

Taking the difference with respect to t of both sides, multiplying by p, and then rewriting sums, we simplify to find

$$\nabla y(t) = \frac{1}{p(t)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} \left(f(\tau) - q(\tau)y(\rho(\tau))\right)$$
$$(p\nabla y)(t) = \frac{1}{\Gamma(\nu)} \sum_{\tau=a+1}^{s} (s-\rho(\tau))^{\overline{\nu-1}} (f(\tau) - q(\tau)y(\rho(\tau)))$$
$$(p\nabla y)(t) = \nabla_a^{-\nu}(f(t) - q(t)y(\rho(t))).$$

Taking the ν-th difference of both sides and composing using the the Composition Rule 1.3.6, we find that

$$\nabla_a^{\nu}(p\nabla y)(t) = f(t) - q(t)y(\rho(t))$$
$$\nabla_a^{\nu}(p\nabla y)(t) + q(t)y(\rho(t)) = f(t)$$

for $t \in \mathbb{N}_{a+2}$, and since the series

$$\sum_{s=a+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (q(\tau)y(\rho(\tau)) - f(\tau))$$

converges, we can take the limit as t increases to infinity of both sides of the summation equation to find that $\lim_{t\to\infty} y(t) = L$. □

We now prove the main result of this section.

Theorem 3.2.2. *Let $p : \mathbb{N}_{a+1} \to \mathbb{R}$, $f : \mathbb{N}_{a+1} \to \mathbb{R}$, and $q : \mathbb{N}_{a+1} \to \mathbb{R}$, and let $L \in [0, \infty)$ be a real number. Assume*

(1) $p(t) > 0$ for $t \in \mathbb{N}_{a+1}$ and $q(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$,

(2) $\displaystyle\sum_{s=a+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} q(\tau) < \infty,$

(3) $\displaystyle\sum_{s=a+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} f(\tau) < \infty.$

Then there exists some $t_0 \in \mathbb{N}_a$ so that the fractional self-adjoint difference equation

$$\nabla_{t_0}^{\nu}(p\nabla y)(t) + q(t)y(\rho(t)) = f(t) \tag{3.2.3}$$

has a solution $y : \mathbb{N}_{t_0} \to \mathbb{R}$ which satisfies $\lim\limits_{t\to\infty} y(t) = L$.

Proof. Because the series

$$\sum_{s=a+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} q(\tau)$$

converges, we can choose $b \in \mathbb{N}_a$ such that

$$\beta := \sum_{s=b+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} q(\tau) < 1. \tag{3.2.4}$$

Let $\zeta_b = \left\{ y : \mathbb{N}_b \to \mathbb{R} \,\middle|\, \lim\limits_{t\to\infty} y(t) = L \right\}$ and define the supremum norm, $\|\cdot\|$, on ζ_b by

$$\|y\| = \sup_{t\in\mathbb{N}_b} |y(t)|.$$

The pair $(\zeta_b, \|\cdot\|)$ defines a complete metric space. Define the operator T on ζ_b by

$$Ty(t) = L + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} \left(q(\tau)y(\rho(\tau)) - f(\tau)\right). \tag{3.2.5}$$

First, we show that $T : \zeta_b \to \zeta_b$. Let $y \in \zeta_b$ be fixed but arbitrary, so that $\lim\limits_{t\to\infty} y(t) = L$. This implies that for some $M > 0$, $|y(t)| < M$ for all $t \in \mathbb{N}_b$. Thus we have that

$$\begin{aligned}|Ty(t)| &= \left| L + \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} \left(q(\tau)y(\rho(\tau)) - f(\tau)\right) \right| \\ &\leq L + \left| \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (q(\tau)y(\rho(\tau)) - f(\tau)) \right|.\end{aligned}$$

Using our bounds on $y(t)$, we have that

$$\begin{aligned}|Ty(t)| &\leq L + \left| \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} (Mq(\tau) - f(\tau)) \right| \\ &\leq L + M \sum_{s=t+1}^{\infty} \left(\frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} q(\tau) \right) \\ &\quad + \left| \sum_{s=t+1}^{\infty} \left(\frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} f(\tau) \right) \right|.\end{aligned}$$

For the first series,

$$\begin{aligned}&M \sum_{s=t+1}^{\infty} \left(\frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} q(\tau) \right) \\ &\quad \leq M \sum_{a=t+1}^{\infty} \left(\frac{1}{p(s)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} q(\tau) \right) < \infty\end{aligned}$$

by assumption *(2)* in the statement of this theorem. For the second series,

$$\begin{aligned}&\left| \sum_{s=t+1}^{\infty} \left(\frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} f(\tau) \right) \right| \\ &\quad \leq \left| \sum_{s=a+1}^{\infty} \left(\frac{1}{p(s)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} f(\tau) \right) \right| < \infty\end{aligned}$$

by assumption *(3)* in this theorem.

Hence T is well defined. Notice that, since the series in the definition of T converges, we can take the limit as $t \to \infty$ to find that $\lim_{t\to\infty} Ty(t) = L$, and thus $Ty \in \zeta_b$.

Now, we show that T is a contraction mapping on ζ_b. Let $x, y \in \zeta_b$ and $t \in \mathbb{N}_b$ be fixed but arbitrary. Then

$$\begin{aligned}
&|Tx(t) - Ty(t)| \\
&\quad = \left| \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} \left(q(\tau)(x(\rho(\tau)) - y(\rho(\tau))) - f(\tau) + f(\tau)\right) \right| \\
&\quad \le \sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} \left(q(\tau)|x(\rho(\tau)) - y(\rho(\tau))|\right) \\
&\quad \le \left(\sum_{s=t+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} q(\tau) \right) \|x - y\| = \beta\|x - y\|.
\end{aligned}$$

Since t, x, and y are arbitrary, $\|Tx - Ty\| \le \beta\|x - y\|$, with $\beta < 1$ for all x and y in ζ_b, and therefore T is a contraction mapping. Hence T has a unique fixed point in ζ_b, call it y_0. This fixed point satisfies the summation equation (3.2.2), and therefore by Lemma 3.2.1, it is also a solution of the fractional difference equation (3.2.3) that satisfies $\lim_{t\to\infty} y_0(t) = L$. □

The following example demonstrates the use of our results.

Example 3.2.3. *For some $a > 1$, let $p(t) = (t-a)^{\overline{\nu}} t(\ln t)^2$, $q(t) = 1$ and $f(t) = 1$.*

Then $p(t) > 0$ for $t \in \mathbb{N}_{a+1}$ and $q(t) \ge 0$ for $t \in \mathbb{N}_{a+1}$.

We also have that

$$\sum_{s=a+1}^{\infty} \frac{(s-a)^{\overline{\nu}}}{p(s)} = \sum_{s=a+1}^{\infty} \frac{1}{s(\ln s)^2} < \infty,$$

and therefore

$$\sum_{s=a+1}^{\infty}\frac{1}{p(s)}\sum_{\tau=a+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu}}}{\Gamma(\nu)}q(\tau)=\sum_{s=a+1}^{\infty}\frac{1}{p(s)}\sum_{\tau=a+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)}f(\tau)$$
$$=\sum_{s=a+1}^{\infty}\frac{(s-a)^{\overline{\nu}}}{p(s)\Gamma(\nu+1)}<\infty.$$

Hence for any real number $L \geq 0$, there exists some $t_0 \in \mathbb{N}_a$ such that

$$\nabla_{t_0}^{\nu}((t-a)^{\overline{\nu}}t(\ln t)^2\nabla y(t))+y(\rho(t))=1, \quad t\in\mathbb{N}_{t_0+1} \tag{3.2.6}$$

has a solution $y : \mathbb{N}_{t_0} \to \mathbb{R}$ that satisfies $\lim_{t\to\infty} y(t) = L$.

In the above example, we required $p(t)$ to be fairly large in order for the series in the assumptions of Theorem 3.2.2 to converge. In the following theorem, we extend our results by using summation by parts.

Theorem 3.2.4. *Let $p : \mathbb{N}_{a+1} \to \mathbb{R}$, $f : \mathbb{N}_{a+1} \to \mathbb{R}$, and $q : \mathbb{N}_{a+1} \to \mathbb{R}$, and let $L \in [0,\infty)$ be a real number. Let $P(s,t) := \sum_{u=t+1}^{s}\frac{1}{p(u)}$. Assume*

(1) $p(t) > 0$ for $t \in \mathbb{N}_{a+1}$, $q(t) \geq 0$, and $f(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$,

(2) $\displaystyle\sum_{s=a+1}^{\infty}P(s-1,a)\left[\sum_{\tau=a+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}q(\tau)\right]<\infty,$

(3) $\displaystyle\sum_{s=a+1}^{\infty}P(s-1,a)\left[\sum_{\tau=a+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}f(\tau)\right]<\infty.$

(4) $\lim_{t\to\infty}\nabla_a^{-\nu}q(t)<\infty$ and $\lim_{t\to\infty}\nabla_a^{-\nu}f(t)<\infty$.

Then there exists some $t_0 \in \mathbb{N}_a$ so that the fractional self-adjoint difference equation

$$\nabla_{t_0}^{\nu}(p\nabla y)(t) + q(t)y(\rho(t)) = f(t) + \lambda(t, y(t)), \quad t \in \mathbb{N}_{t_0+1}, \tag{3.2.7}$$

where

$$\lambda(t, y(t)) = \frac{1}{\Gamma(1-\nu)}\left(\lim_{s\to\infty} \nabla_b^{-\nu}(q(s)y(\rho(s)) - f(s))\right)(t-b)^{\overline{-\nu}} \tag{3.2.8}$$

has a solution $y : \mathbb{N}_{t_0} \to \mathbb{R}$ which satisfies $\lim_{t\to\infty} y(t) = L$.

Proof. Because the series $\sum_{s=a+1}^{\infty} P(s-1,a)\left[\sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}q(\tau)\right]$ converges, we can choose $b \in \mathbb{N}_a$ such that

$$\beta := \sum_{s=b+1}^{\infty} P(s-1,a)\left[\sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}q(\tau)\right] < 1. \tag{3.2.9}$$

We will now show that the mapping defined by

$$Ty(t) = L - \sum_{s=t+1}^{\infty} P(s-1,t) \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}(q(\tau)y(\rho(\tau)) - f(\tau)) \tag{3.2.10}$$

is a contraction mapping on the set $\zeta_b := \left\{y : \mathbb{N}_b \to \mathbb{R} \,\middle|\, \lim_{t\to\infty} y(t) = L\right\}$ and that its fixed point in ζ_b is a solution of the fractional difference equation (3.2.7).

To show that $T : \zeta_b \to \zeta_b$, consider a fixed but arbitrary $y \in \zeta_b$. Since $\lim_{t\to\infty} y(t) = L$, there exists some $M > 0$ so that $|y(t)| \leq M$ for all $t \in \mathbb{N}_b$. Hence

$$\begin{aligned}
|Ty(t)| &= \left| L - \sum_{s=t+1}^{\infty} P(s-1,t) \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)} (q(\tau)y(\rho(\tau)) - f(\tau)) \right| \\
&\leq L + \left| \sum_{s=t+1}^{\infty} P(s-1,t) \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)} q(\tau)y(\rho(\tau)) \right| \\
&\quad + \left| \sum_{s=t+1}^{\infty} P(s-1,t) \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)} f(\tau) \right| \\
&\leq L + M \left| \sum_{s=t+1}^{\infty} P(s-1,t) \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)} q(\tau) \right| \\
&\quad + \left| \sum_{s=t+1}^{\infty} P(s-1,t) \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)} f(\tau) \right|.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
|Ty(t)| &\leq L + M \left| \sum_{s=a+1}^{\infty} P(s-1,t) \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)} q(\tau) \right| \\
&\quad + \left| \sum_{s=a+1}^{\infty} P(s-1,t) \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)} f(\tau) \right| \\
&< \infty.
\end{aligned}$$

So T is well defined, and thus we can take the limit of T as $t \to \infty$ to find that $\lim_{t\to\infty} Ty(t) = L$, so that $Ty \in \zeta_b$.

Now, we show that T is a contraction mapping on ζ_b. Let $x, y \in \zeta_b$ and $t \in \mathbb{N}_b$ be fixed but arbitrary. Then

$$\begin{aligned}|Tx(t)-Ty(t)| &= \left|\sum_{s=t+1}^{\infty} P(s-1,t)\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}q(\tau)(y(\rho(\tau))-x(\rho(\tau)))\right| \\ &\leq \sum_{s=t+1}^{\infty} P(s-1,t)\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}q(\tau)|x(\rho(\tau))-y(\rho(\tau))| \\ &\leq \sum_{s=t+1}^{\infty} P(s-1,t)\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}q(\tau)\|x-y\| = \beta\|x-y\|.\end{aligned}$$

Since t, x, and y are arbitrary, $\|Tx-Ty\| \leq \beta\|x-y\|$ with $\beta<1$ for all x and y in ζ_b, and therefore T is a contraction mapping. Hence T has a unique fixed point in ζ_b, call it y_0. This fixed point satisfies the equation $y_0(t) = Ty_0(t)$, which we will now show implies that y_0 is a solution of the fractional difference equation(3.2.7).

Beginning with the fixed point equation $y_0(t) = Ty_0(t)$, we first take the nabla difference with respect to t,

$$\begin{aligned}y_0(t) &= L - \sum_{s=t+1}^{\infty} P(s-1,t)\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}(q(\tau)y_0(\rho(\tau))-f(\tau)) \\ \nabla y_0(t) &= -\sum_{s=t+1}^{\infty} P(s-1,t)\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}(q(\tau)y_0(\rho(\tau))-f(\tau)) \\ &\quad+\sum_{s=t}^{\infty} P(s-1,t-1)\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}(q(\tau)y_0(\rho(\tau))-f(\tau)) \\ &= P(t-1,t-1)\sum_{\tau=b+1}^{t}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}(q(\tau)y_0(\rho(\tau))-f(\tau)) \\ &\quad+\sum_{s=t+1}^{\infty}(P(s-1,t-1)-P(s-1,t)) \\ &\qquad\sum_{\tau=b+1}^{s}\frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}(q(\tau)y_0(\rho(\tau))-f(\tau)).\end{aligned}$$

It follows that

$$y_0(t) = \sum_{s=t+1}^{\infty} \frac{1}{p(t)} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}(q(\tau)y_0(\rho(\tau)) - f(\tau)).$$

Multiplying both sides by $p(t)$, we find that

$$\begin{aligned}
(p\nabla y_0)(t) &= \sum_{s=t+1}^{\infty} \sum_{\tau=b+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)}(q(\tau)y_0(\rho(\tau)) - f(\tau)) \\
&= \sum_{s=t+1}^{\infty} \nabla_b^{1-\nu}(q(s)y_0(\rho(s)) - f(s)) \\
&= \sum + s = t + 1^{\infty} \nabla^1 (\nabla_b^{-\nu}(q(s)y_0(\rho(s)) - f(s))) \\
&= \nabla_b^{-\nu}(q(s)y_0(\rho(s)) - f(s))\Big|_{s=t}^{\infty} \\
&= \lim_{s\to\infty} \left[\nabla_b^{-\nu} q(s)y_0(\rho(s)) - f(s)\right] - \nabla_b^{-\nu}(q(t)y_0(\rho(t)) - f(t)).
\end{aligned}$$

We must verify that $\lim_{s\to\infty} \left[\nabla_b^{-\nu} q(s)y_0(\rho(s)) - f(s)\right]$ converges. Because y_0 is in ζ_b, $\lim_{s\to\infty} y_0(s) = L$. We therefore have

$$\lim_{s\to\infty} \nabla_b^{-\nu} q(s)y_0(\rho(s)) - f(s) = \left(\lim_{s\to\infty} \nabla_b^{-\nu} q(s)\right)\left(\lim_{s\to\infty} y_0(\rho(s))\right) - \left(\lim_{s\to\infty} \nabla_b^{-\nu} f(s)\right) < \infty.$$

because of assumption *(4)* in the theorem statement. Hence we have that

$$\lim_{s\to\infty} \left[\nabla_b^{-\nu} q(s)y_0(\rho(s)) - f(s)\right]$$

is a constant, and so the ν-th nabla difference based at b of this constant is given by

$$
\begin{aligned}
&\nabla_b^{\nu}\left(\lim_{s\to\infty}\nabla_b^{-\nu}q(s)y_0(\rho(s))-f(s)\right)\\
&\quad=\sum_{\tau=b+1}^{t}\frac{(t-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)}\left(\lim_{t\to\infty}\nabla_b^{-\nu}q(s)y_0(\rho(s))-f(s)\right)\\
&\quad=\left(\lim_{s\to\infty}\nabla_b^{-\nu}q(s)y_0(\rho(s))-f(s)\right)\sum_{\tau=b+1}^{t}\frac{(t-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)}\\
&\quad=\frac{1}{\Gamma(1-\nu)}\left(\lim_{s\to\infty}\nabla_b^{-\nu}(q(s)y_0(\rho(s))-f(s))\right)(t-b)^{\overline{-\nu}}\\
&\quad=\lambda(t,y_0(t)).
\end{aligned}
$$

Finally, we take the ν-th nabla difference based at b of the equation above and use $\lambda(t,y_0(t))$ to obtain

$$
\begin{aligned}
\nabla_b^{\nu}(p\nabla y_0)(t)&=\nabla_b^{\nu}\left(\lim_{s\to\infty}\nabla_b^{-\nu}q(s)y_0(\rho(s))-f(s)\right)-\nabla_b^{\nu}\nabla_b^{-\nu}(q(t)y_0(\rho(t))-f(t))\\
\nabla_b^{\nu}(p\nabla y_0)(t)&=\lambda(t,y_0(t))+f(t)-q(t)y_0(\rho(t))\\
\nabla_b^{\nu}(p\nabla y_0)(t)+q(t)y_0(\rho(t))&=f(t)+\lambda(t,y_0(t)).
\end{aligned}
$$

Therefore $y_0(t)$ satisfies the self-adjoint difference equation (3.2.7) with $t_0=b$, and $\lim_{t\to\infty}y_0(t)=L$. □

Example 3.2.5. *For some $a>0$, and some $L\geq 0$, let $p(t)=1$, $q(t)=\nabla_a^{\nu-1}\frac{1}{t^3}$, and $f(t)=\nabla_a^{\nu-1}\frac{L}{t^3}$.*

Then $p(t) > 0$ *for* $t \in \mathbb{N}_{a+1}$. *To show that* $q(t) \geq 0$, *we consider for some* $t \in \mathbb{N}_{a+1}$,

$$\begin{aligned} q(t) &= \frac{1}{\Gamma(1-\nu)} \sum_{u=a+1}^{t} (t-\rho(u))^{\overline{-\nu}} \frac{1}{u^3} \\ &= \frac{1}{\Gamma(1-\nu)} \sum_{u=a+1}^{t} \frac{\Gamma(t-u+1-\nu)}{\Gamma(t-u+1)} \frac{1}{u^3}. \end{aligned}$$

Since $1-\nu > 0$, $t-u+1-\nu > 0$, *and* $t-u+1 > 0$ *for* $t \in \mathbb{N}_a + 2$, $u \leq t$, *we can conclude that* $q(t) \geq 0$ *for* $t \in \mathbb{N}_a + 2$, *so assumption* (1) *from Theorem 3.2.4 holds.*

To demonstrate that assumptions (2) *and* (3) *hold, consider*

$$P(s-1, a) = \sum_{u=a+1}^{s-1} \frac{1}{p(u)} = s-a-2$$

and hence

$$\begin{aligned} \sum_{s=a+1}^{\infty} P(s-1,a) \left[\sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-2}}}{\Gamma(\nu-1)} q(\tau) \right] &= \sum_{s=a+1}^{\infty} (s-a-2) \nabla_a^{1-\nu} q(s) \\ &= \sum_{s=a+1}^{\infty} (s-a-2) \nabla_a^{1-\nu} \left(\nabla_a^{\nu-1} \frac{1}{s^3} \right) \\ &= \sum_{s=a+1}^{\infty} \frac{s-a-2}{s^3} \end{aligned}$$

because of the Composition Rule 1.3.6. This series converges, and so assumption (2) *holds. Assumption* (3) *will also hold as* $f(t)$ *is a scalar multiple of* $q(t)$.

Finally, to demonstrate that assumption (4) *holds, consider the limit as* $t \to \infty$ *of the* ν*-th nabla difference based at* a *of* $q(t)$.

$$\begin{aligned}\lim_{t\to\infty}\nabla_a^{-\nu}q(t) &= \lim_{t\to\infty}\nabla_a^{-\nu}\nabla_a^{\nu-1}\frac{1}{t^3}\\ &= \lim_{t\to\infty}\nabla^{-1}\frac{1}{t^3}\\ &= \lim_{t\to\infty}\sum_{s=a+1}^{t}\frac{1}{s^3}\\ &= \sum_{s=a+1}^{\infty}\frac{1}{s^3} < \infty.\end{aligned}$$

Likewise, $\lim_{t\to\infty}\nabla_a^{-\nu}f(t) < \infty$ *as* $f(t)$ *is a scalar multiple of* $q(t)$*. Notice that for this* $q(t)$*,* $f(t)$ *pair and some* $y \in \zeta_b$*,*

$$\lim_{s\to\infty}\nabla_b^{-\nu}(q(s)y(\rho(s)) - f(s)) = \lim_{s\to\infty}\nabla_b^{-\nu}\left(\nabla_a^{\nu-1}\frac{1}{t^3}\right)(y(\rho(s)) - L) = 0$$

as

$$\lim_{s\to\infty}\left(\nabla_a^{\nu-1}\frac{1}{t^3}\right) < \infty$$

and

$$\lim_{s\to\infty}(y(\rho(s)) - L) = 0.$$

Because of this, in this example $\lambda(t, y(t)) = 0$*, and since all of the assumptions are met, we conclude that for some* $b \in \mathbb{N}_a$*, the self-adjoint fractional difference equation*

$$\nabla_b^{\nu}(\nabla y(t)) + \left(\nabla_a^{\nu-1}\frac{1}{t^3}\right)y(\rho(t)) = \nabla_a^{\nu-1}\frac{L}{t^3}$$

has a solution y *that tends to* L *as* t *increases to* ∞*.*

Remark 3.2.6. *We remark that Example 3.2.5 demonstrates that Theorem 3.2.4 truly does extend the results of Theorem 3.2.2, as*

$$\sum_{s=a+1}^{\infty} \frac{1}{p(s)} \sum_{\tau=a+1}^{s} \frac{(s-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} q(\tau) = \sum_{s=a+1}^{\infty} \nabla_a^{-\nu} \nabla_a^{\nu-1} q(s) = \sum_{s=a+1}^{\infty} \sum_{\tau=a+1}^{s} \frac{1}{s^3}$$

does not converge.

3.3 Equations with Generalized Forcing Terms

In this section we will generalize the forcing term to permit $F(t,x)$, not necessarily linear in x, that satisfy a uniform Lipschitz condition with respect to x.

In the following lemma, we establish the appropriate fixed point equation, a solution of which will also be a solution of the forced fractional difference equation.

Lemma 3.3.1. *Let $p : \mathbb{N}_{a+1} \to (0,\infty)$ and $F : \mathbb{N}_{a+1} \times \mathbb{R} \to [0,\infty)$. Define ζ to be the space of all positive functions, $\zeta = \{x : \mathbb{N}_a \to [0,\infty)\}$. Suppose that for all the functions $x \in \zeta$, the series*

$$\sum_{\tau=a+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(\tau))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s,x(s)) \right]$$

converges. Then the forced fractional self-adjoint difference equation

$$\nabla_a^{\nu}(p\nabla x)(t) + F(t,x(t)) = 0, \quad t \in \mathbb{N}_{a+1} \tag{3.3.1}$$

has a solution $x \in \zeta$ with $\lim_{t\to\infty} x(t) = L$ for some $L \geq 0$, if and only if the summation equation

$$x(t) = L + \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s,x(s)) \right] \tag{3.3.2}$$

has a solution $x(t)$ on $\mathbb{N}_a$.

Proof. Suppose the fractional difference equation (3.3.1) has a solution $x \in \zeta$ that satisfies $\lim_{t\to\infty} x(t) = L$. Let $y(t) = (p\nabla x)(t)$. Then $y(t)$ solves the fractional initial value problem

$$\begin{cases} \nabla_a^\nu y(t) = -F(t, x(t)), \quad t \in \mathbb{N}_{a+2} \\ y(a+1) = p(a+1)\nabla x(a+1). \end{cases}$$

From Corollary 1.4.3, $y(t)$ has the form

$$y(t) = -\nabla_a^\nu F(t, x(t)) + [p(a+1)\nabla x(a+1) + F(a+1, x(a+1))] \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)},$$

and since

$$\nabla_a^\nu (p\nabla x)(a+1) = \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{a+1} (a+1-\rho(s))^{\overline{-\nu-1}} (p\nabla x)(s) = (p\nabla x)(a+1),$$

we have

$$\begin{aligned} y(t) &= -\nabla_a^\nu F(t, x(t)) + [\nabla_a^\nu (p\nabla x)(a+1) + F(a+1, x(a+1))] \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} \\ &= -\nabla_a^\nu F(t, x(t)). \end{aligned}$$

Multiplying by $p(t)$ and summing from $\tau = t+1$ to ∞ gives that

$$\begin{aligned} \nabla x(t) &= \frac{-1}{p(t)} \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \\ \sum_{\tau=t+1}^{\infty} \nabla x(t) &= -\sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right] \\ \lim_{\tau\to\infty} x(\tau) - x(t) &= -\sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right] \end{aligned}$$

It follows that

$$x(t) = L + \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right].$$

So $x(t)$ is also a solution of the summation equation.

Conversely, if $x(t)$ is a solution of the summation equation on $\mathbb{N}_a$, then

$$x(t) = L + \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right].$$

Take the nabla difference with respect to t of both sides, multiply by $p(t)$, and then take the ν-th difference based at a of both sides to obtain

$$\begin{aligned} \nabla x(t) &= \frac{-1}{p(t)} \sum_{s=a+1}^{t} \frac{(t - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \\ (p\nabla x)(t) &= -\nabla_a^{-\nu} F(t, x(t)) \\ \nabla_a^{\nu}(p\nabla x)(t) &= -\nabla_a^{\nu} \nabla_a^{-\nu} F(t, x(t)). \end{aligned}$$

By the composition rule given in Theorem 1.3.6, we have that

$$\nabla_a^{\nu}(p\nabla x)(t) = -F(t, x(t))\nabla_a^{\nu}(p\nabla x)(t) + F(t, x(t)) = 0.$$

Hence $x(t)$ is also a solution of the difference equation (3.3.1). Since $p(t) > 0$ and $F(t, x(t)) \geq 0$, $x(t) \geq L \geq 0$ for all t, so $x \in \zeta$.

Furthermore, since

$$\sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s))$$

converges, it follows that $\lim_{t\to\infty} x(t) = L$. $\square$

Theorem 3.3.2. *Assume* $F : \mathbb{N}_{a+1} \times [0,\infty) \to [0,\infty)$ *satisfies a uniform Lipschitz condition with respect to its second variable in* $\mathbb{N}_{a+1} \times [0,\infty)$*, i.e. there is a constant* $K > 0$ *such that if* $u, v \in \mathbb{R}$ *and* $t \in \mathbb{N}_a$*,*

$$|F(t,u) - F(t,v)| \le K|u - v|,$$

and assume $p : \mathbb{N}_{a+1} \to (0,\infty)$*. Let* $(\zeta, \|\cdot\|)$ *be the complete metric space of positive valued functions* $\zeta = \{x : \mathbb{N}_a \to [0,\infty)\}$ *together with the supremum norm. If*

(H1) the series $\displaystyle\sum_{\tau=a+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s,x(s)) \right]$ *converges for every* $x \in \zeta$*, and*

(H2) $\displaystyle\beta := \frac{K}{\Gamma(\nu+1)} \sum_{\tau=a+1}^{\infty} \frac{(\tau-a)^{\overline{\nu}}}{p(\tau)} < 1$

Then there exists a unique positive solution of the fractional difference equation (3.3.1) *with* $\lim_{t\to\infty} x(t) = L$ *for any* $L \ge 0$*.*

Proof. Let (ζ, d) be the complete metric space of positive valued functions together with the supremum norm, and let $L \ge 0$ be fixed but arbitrary. Consider the mapping T defined by

$$Tx(t) = L + \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s,x(s)) \right]. \tag{3.3.3}$$

We will use the Contraction Mapping Theorem to show that T has a unique fixed point. First, we show that $T : \zeta \to \zeta$. Since $p(t) > 0$ and $F(s,x(s)) \ge 0$, we know that $Tx(t) \ge L \ge 0$ for all t, so $Tx \in \zeta$.

Next, we show that T is a contraction mapping. Let $x, y \in \zeta$ and $t \in N_a$ be fixed but arbitrary. Then

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} |F(s, x(s)) - F(s, y(s))| \right] \\
&\leq K \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} |x(s) - y(s)| \right] \\
&\leq K\|x-y\| \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} \right] \\
&= K\|x-y\| \sum_{\tau=t+1}^{\infty} \frac{(\tau-a)^{\overline{\nu}}}{\Gamma(\nu+1)p(\tau)} \\
&= \frac{K}{\Gamma(\nu+1)} \left(\sum_{\tau=t+1}^{\infty} \frac{(\tau-a)^{\overline{\nu}}}{p(\tau)} \right) \|x-y\| = \beta\|x-y\|.
\end{aligned}$$

So

$$\|Tx - Ty\| \leq \beta\|x-y\|$$

with $\beta < 1$, and hence T is a contraction mapping. Thus T has a unique fixed point $x \in \zeta$. Because the series

$$\sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\infty} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right]$$

converges, we have that $\lim_{t\to\infty} x(t) = L$. A fixed point of T is a solution of the fractional summation equation (3.3.2), and hence by Lemma 3.3.1, there is a unique solution of the fractional difference equation (3.3.1) that tends to L as t tends to infinity. $\square$

Example 3.3.3. *For some $a > 0$, let $p(t) = (t-a)^{\overline{\nu}} t(\ln t)^2$.*

As we saw in Example 3.2.3, $p(t) > 0$ for $t \in \mathbb{N}_{a+1}$ and $\sum_{\tau=a+1}^{\infty} \frac{(\tau-a)^{\overline{\nu}}}{p(\tau)} < \infty$.

Choose K so that

$$0 < K < \frac{\Gamma(\nu+1)}{\sum_{\tau=a+1}^{\infty} \frac{(\tau-a)^{\overline{\nu}}}{p(\tau)}},$$

and let $F(t,x) = \frac{K}{1+x}$.

We must first show that $F(t,x)$ satisfies a uniform Lipschitz condition with respect to its second variable. Note that $x \geq 0$ so that $F(t,x) \geq 0$, and for $x, y \geq 0$,

$$\begin{aligned} |F(t,x) - F(t,y)| &= \left| \frac{K}{x+1} - \frac{K}{y+1} \right| \\ &= K \left| \frac{y+1-(x+1)}{(x+1)(y+1)} \right| \\ &= K \frac{|x-y|}{(x+1)(y+1)} \\ &\leq K|x-y|. \end{aligned}$$

We must also show that (H1) and (H2) hold. These conditions will follow from our choice of K, because

$$\begin{aligned} \sum_{\tau=a+1}^{\infty} \frac{1}{p(\tau)} \sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s,x(s)) &\leq \sum_{\tau=a+1}^{\infty} \frac{1}{p(\tau)} \sum_{s=a+1}^{\tau} \frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} K \\ &\leq \frac{K}{\Gamma(\nu+1)} \sum_{\tau=a+1}^{\infty} \frac{(\tau-a)^{\overline{\nu-1}}}{p(\tau)} \\ &< 1 < \infty. \end{aligned}$$

Hence for any $L \geq 0$, there exists a unique positive solution of

$$\nabla_{t_0}^{\nu}((t-a)^{\overline{\nu}} t (\ln t)^2 \nabla y(t)) + \frac{K}{1+y(t)} = 0, \quad t \in \mathbb{N}_{t_0+1}, \tag{3.3.4}$$

with $\lim_{t\to\infty} y(t) = L$.

Remark 3.3.4. *In Example 3.3.4 we chose a small* $p(t)$*, so that*

$$\sum_{\tau=a+1}^{\infty} \frac{(\tau - a)^{\overline{\nu-1}}}{p(\tau)}$$

barely converges. This meant we also needed a small enough $F(t, x)$ *so that the condition* (H1) *would hold. We could choose a larger function, such as* $F(t, x) = K$ *(with the same choice of* K *as in Example 3.3.4) but choosing the largest possible* $F(t, x)$ *is limited by our ability to compute* $\nabla_a^{-\nu} F(s, x)$*.*

We can conversely choose a large (yet still uniformly Lipschitz continuous in its second argument in $\mathbb{N}_{a+1} \times [0, \infty)$*)* $F(t, x)$*, but we will need to pair it with a similarly large* $p(t)$ *so that conditions* (H1) *and* (H2) *hold.*

3.4 Equations with Generalized Forcing Terms in a Modified Complete Metric Space

In Theorem 3.3.2, the condition (H1), that the series

$$\sum_{\tau=a+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right]$$

converges for every $x \in \zeta$, is quite strong. We can make this condition less strong by making modifications to the complete metric space $(\zeta, \|\cdot\|)$. We can choose a smaller set of functions ζ, such as $\zeta_L := \left\{ y : \mathbb{N}_a \to \mathbb{R} \,\middle|\, \lim_{t\to\infty} y(t) = L \right\}$, so that the burden of convergence can be partially borne by the choice of $x(t)$. However, this will make the uniqueness conclusion of the theorem less impactful. We also can replace the supremum norm with a weighted norm to suit the choice of the pair p, $F(t, x)$.

Lemma 3.4.1. *Assume* $M > 0$ *and*

$$\zeta_M := \{x : \mathbb{N}_a \to [M, \infty) : \nabla x(t) \leq 0\}.$$

Assume $p : \mathbb{N}_a \to (0, \infty)$ *satisfies* $\displaystyle\sum_{\tau=a}^{\infty} \frac{1}{p(t)} < \infty$ *and define* $d : \zeta_M \times \zeta_M \to [0, \infty)$ *by*

$$d(x, y) := \sup_{t \in \mathbb{N}_a} \frac{|x(t) - y(t)|}{w(t)},$$

where

$$w(t) := e^{-\left[\sum_{\tau=a}^{t} \frac{1}{p(\tau)}\right]}.$$

Note that $0 < L := \lim_{t \to \infty} w(t) \leq 1$. *Then the pair* (ζ_M, d) *is a complete metric space.*

Proof. To prove this lemma, we first show that d is a metric. It is clearly non-negative, and since $0 < w(t) < 1$ for all $t \in \mathbb{N}_a$, $d(x, y) = 0$ is only satisfied when $x(t) = y(t)$ for all $t \in \mathbb{N}_a$. It is symmetric, and satisfies the triangle inequality, as follows, for $x, y, z \in \zeta_M$:

$$\begin{aligned} d(x, z) &= \sup_{t \in \mathbb{N}_a} \frac{|x(t) - y(t) + y(t) - z(t)|}{w(t)} \\ &\leq \sup_{t \in \mathbb{N}_a} \left(\frac{|x(t) - y(t)|}{w(t)} + \frac{|y(t) - z(t)|}{w(t)} \right) \\ &\leq \sup_{t \in \mathbb{N}_a} \frac{|x(t) - y(t)|}{w(t)} + \sup_{t \in \mathbb{N}_a} \frac{|y(t) - z(t)|}{w(t)} = d(x, y) + d(y, z). \end{aligned}$$

Hence d is a metric.

To see that (ζ_M, d) is complete, consider a Cauchy sequence $\{x_n\}$ in ζ_M. For each $t_0 \in \mathbb{N}_a$, $\{x_n(t_0)\}$ is a Cauchy sequence in $\mathbb{R}$, and therefore converges. Define its limit to be $x_0(t_0) = \lim_{n \to \infty} x_n(t_0)$, and since t_0 was arbitrary, we can construct the function $x_0 : \mathbb{N}_a \to \mathbb{R}$ in this fashion. Because each of the x_n satisfies $x_n(t) \geq M$ and

$\nabla x_n(t) \leq 0$, so does $x_0(t)$, and hence $x_0 \in \zeta_M$. $\square$

Theorem 3.4.2. *Assume $F : \mathbb{N}_{a+1} \times [0, \infty) \to [0, \infty)$ satisfies a uniform Lipschitz condition with respect to its second variable in $\mathbb{N}_{a+1} \times [0, \infty)$, i.e. there is a constant $K > 0$ such that if $u, v \in \mathbb{R}$ and $t \in \mathbb{N}_a$,*

$$|F(t, u) - F(t, v)| \leq K|u - v|,$$

and assume $p : \mathbb{N}_{a+1} \to (0, \infty)$. Let (ζ_M, d) be the complete metric space as defined in Lemma 3.4.1. If

(H1) the series
$$\sum_{\tau=a+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right]$$
converges for every x in ζ_M, and

(H2) $\alpha := \dfrac{K}{L \cdot \Gamma(\nu + 1)} \displaystyle\sum_{\tau=a+1}^{\infty} \frac{(\tau - a)^{\overline{\nu}}}{p(\tau)} < 1$

Then there exists a unique positive solution of the fractional difference equation

$$\nabla_a^{\nu}(p\nabla x)(t) + F(t, x(t)) = 0$$

with $\lim_{t\to\infty} x(t) = M$.

Proof. Consider the mapping T on ζ_M defined by

$$Tx(t) = M + \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right]. \tag{3.4.1}$$

We will use the Contraction Mapping Theorem to show that T has a unique fixed point. First, we show that $T : \zeta_M \to \zeta_M$. Since $p(t) > 0$ and $F(s, x(s)) \geq 0$, we know that $Tx(t) \geq M \geq 0$ for all t. Also note that

$$\nabla Tx(t) = -\frac{1}{p(t)} \left[\sum_{s=a+1}^{t} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right] \leq 0.$$

Hence $Tx \in \zeta_M$.

Next, we show that T is a contraction mapping. Let $x, y \in \zeta_M$ and $t \in N_a$ be fixed but arbitrary. Then

$$\begin{aligned}
\frac{|Tx(t) - Ty(t)|}{w(t)} &= \frac{1}{w(t)} \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} |F(s, x(s)) - F(s, y(s))| \right] \\
&\leq \frac{K}{w(t)} \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} |x(s) - y(s)| \right] \\
&\leq \frac{K}{w(t)} \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} w(s) \right] d(x, y) \\
&\leq \frac{K}{L} \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} \right] d(x, y) \\
&= \frac{K}{L} \left[\sum_{\tau=t+1}^{\infty} \frac{(\tau - a)^{\overline{\nu}}}{\Gamma(\nu+1) p(\tau)} \right] d(x, y) = \alpha d(x, y).
\end{aligned}$$

So $d(Tx, Ty) \leq \alpha d(x, y)$ with $\alpha < 1$, and hence T is a contraction mapping. Thus T has a unique fixed point $x \in \zeta_M$. Because the series

$$\sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\infty} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right]$$

converges, we have that $\lim_{t \to \infty} x(t) = M$. A fixed point of T is a solution of the summation equation

$$x(t) = M + \sum_{\tau=t+1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=a+1}^{\tau} \frac{(\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} F(s, x(s)) \right],$$

and hence by Lemma 3.3.1, there is a unique solution of the fractional difference equation

$$\nabla_a^\nu (p\nabla x)(t) + F(t, x(t)) = 0$$

that tends to M as t tends to infinity. $\square$

Next we give an example of the above theorem.

Example 3.4.3. *Let*

$$M := \sum_{\tau=a+2}^{\infty} \frac{\Gamma(\nu+1)}{3 \cdot 2^{(\tau-a+3)}(\tau-a)^{\overline{\nu}}}.$$

This series converges because $\dfrac{\Gamma(\nu+1)}{(\tau-a)^{\overline{\nu}}} < 1$ *for* $\tau \geq a+2$*. Let* L *be the solution of the equation*

$$e^{-(\frac{1}{2}+ML)} = L.$$

It follows from the form of the equation that $0 < L < 1$*. Now let*

$$p(t) = \begin{cases} 4 & t = a, a+1 \\ \frac{3 \cdot 2^{(t-a+3)}(t-a)^{\overline{\nu}}}{\Gamma(\nu+1) \cdot L} & t \in \mathbb{N}_{a+2} \end{cases}$$

and $F(t,x) = \frac{3x}{t+2}$*, where* $x \geq 0$ *and* $t \in \mathbb{N}_{a+1}$*. First notice that* $F(t,x)$ *is uniformly Lipschitz continuous in its second variable in* $\mathbb{N}_{a+1} \times [0, \infty)$ *with Lipschitz constant* $K = 3$*. Also notice that*

$$\begin{aligned} \sum_{\tau=a}^{\infty} \frac{1}{p(\tau)} &= \frac{1}{2} + \sum_{\tau=a+2}^{\infty} \frac{1}{p(\tau)} \\ &= \frac{1}{2} + L \cdot \sum_{\tau=a+2}^{\infty} \frac{\Gamma(\nu+1)}{3 \cdot 2^{(\tau-a+3)}(\tau-a)^{\overline{\nu}}} \\ &= \frac{1}{2} + L \cdot M < \infty \end{aligned}$$

and so

$$\lim_{t\to\infty} w(t) = e^{-\left[\sum_{\tau=a}^{\infty}\frac{1}{p(\tau)}\right]}$$
$$= e^{-(\frac{1}{2}+ML)} = L,$$

as desired.

We claim that the hypotheses (H1) and (H2) of Theorem 3.4.2 hold. Let $x \in \zeta_M$ *be arbitrary but fixed. Then, using the fact that* x *is decreasing and therefore for* $t \geq a+1$*, we have that* $x(a+1) \geq x(t)$ *for* $t \geq a+1$*,*

$$\begin{aligned}
&\sum_{\tau=a+1}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=a+1}^{\tau}\frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)}F(s,x(s))\right]\\
&= \sum_{\tau=a+1}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=a+1}^{\tau}\frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)}\frac{3|x(s)|}{s+2}\right]\\
&\leq 3x(a+1)\sum_{\tau=a+1}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=a+1}^{\tau}\frac{(\tau-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)}\right]\\
&= 3x(a+1)\sum_{\tau=a+1}^{\infty}\frac{(\tau-a)^{\overline{\nu}}}{\Gamma(\nu+1)p(\tau)}\\
&= 3x(a+1)\sum_{\tau=a+1}^{\infty}\frac{(\tau-a)^{\overline{\nu}}\Gamma(\nu+1)L}{\Gamma(\nu+1)3\cdot 2^{(\tau-a+3)}(\tau-a)^{\overline{\nu}}}\\
&= x(a+1)\cdot L\sum_{\tau=a+1}^{\infty}\frac{1}{2^{(\tau-a+3)}}\\
&= \frac{L}{4}x(a+1) < \infty.
\end{aligned}$$

Hence (H1) holds.

To see that (H2) holds, consider

$$\begin{aligned}
\alpha &= \frac{K}{L \cdot \Gamma(\nu+1)} \sum_{\tau=a+1}^{\infty} \frac{(\tau-a)^{\overline{\nu}}}{p(\tau)} \\
&= \frac{K}{L \cdot \Gamma(\nu+1)} \sum_{\tau=a+1}^{\infty} \frac{(\tau-a)^{\overline{\nu}}\Gamma(\nu+1)L}{3 \cdot 2^{(\tau-a+3)}(\tau-a)^{\overline{\nu}}} \\
&= \frac{K}{3} \sum_{\tau=a+1}^{\infty} \frac{1}{2^{(\tau-a+3)}} \\
&= \frac{K}{12} = \frac{1}{4} < 1.
\end{aligned}$$

Thus the second hypothesis (H2) is also satisfied. Hence, Theorem 3.4.2 implies that with F, p as defined above, the self-adjoint fractional difference equation

$$\nabla_a^{\nu}(p\nabla x)(t) + F(t, x(t)) = 0$$

has a unique positive solution with $\lim_{t\to\infty} x(t) = M$.

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