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Invariant Basis Number and Basis Types for C*-Algebras

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INVARIANT BASIS NUMBER AND BASIS TYPES FOR C^*-ALGEBRAS

by

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INVARIANT BASIS NUMBER AND BASIS TYPES FOR C^*-ALGEBRAS

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We develop the property of Invariant Basis Number (IBN) in the context of C^*-algebras and their Hilbert modules. A complete K-theoretic characterization of C^*-algebras with IBN is given. A scheme for classifying C^*-algebras which do not have IBN is given and we prove that all such classes are realized. We investigate the invariance of IBN, or lack thereof, under common C^*-algebraic construction and perturbation techniques. Finally, applications of Invariant Basis Number to the study of C^*-dynamical systems and the classification program are investigated.

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Introduction and Overview

It is a natural question arising in many branches of mathematics to consider when objects display self-similarity. The structure necessary to permit self-similarity engenders some of the most beautiful mathematical objects, such as fractals, while at the same time exposes troubling possibilities, such as the Banach-Tarski paradox. Understanding when, how, and why self-similarity occurs within certain operator algebras is the object of this dissertation.

When working with C^*-algebras we can consider self-similarity in a number of senses. Perhaps most generally, a C^*-algebra A could be termed self-similar if there is a non-surjective embedding $A \hookrightarrow A$. Of course, it may well be natural to ask that the embedding be a $*$-homomorphism, a completely isometric $*$-homomorphism, or any number of other reasonable restrictions and each situation yields potentially different sorts of self-similarity. We will not consider arbitrary embeddings with certain properties but instead very particular embeddings which arise from algebraic constructions.

Given a C^*-algebra A and the n-fold direct-sum C^*-algebra A^n there are many possible embeddings $A \hookrightarrow A^n$, but perhaps the most obvious are the simple coordinate embeddings $a \mapsto (0, ..., 0, a, 0, ..., 0)$. Now, should there be an embedding $A^n \hookrightarrow A$ we will have a high degree of self-similarity, as A will "contain" n "disjoint" copies of itself, each of which contains another n copies, etc. Also, if A^n embeds into A then

A^{n^2} embeds into A^n and so on, hence a more general consideration of self-similarity would be when $A^n \hookrightarrow A^m$ for some $n > m$.

Motivated by the property of Invariant Basis Number studied in noncommutative ring theory, we will not consider our embeddings (soon, only isomorphisms) in the category of C^*-algebras and $*$-homomorphisms, but rather in the category of C^*-modules and adjointable linear maps. In order for A^m and A^n to be isomorphic as C^*-modules requires more structure than in the category of C^*-algebras, and hence we may obtain sharper results both for when an isomorphism is present and when it is not.

This dissertation will follow the general plan of appropriating Invariant Basis Number and related terminology from the ring theory, exploiting C^*-algebraic structure to obtain sharper results than possible otherwise, and then applying our results to uniquely operator-theoretic problems. A detailed overview is as follows:

Chapter 1 is an review of the literature about Invariant Basis Number in noncommutative ring theory. These results will serve as both motivation for our work and as a contrast to the sharper results possible using C^*-algebraic techniques.

Chapter 2 is devoted to background material necessary for our main results. Of primary consideration will be the theory of C^*-modules, linear homomorphisms, free modules, and orthogonal bases. Several results are folklore and, when necessary, pertinent proofs have been provided.

Chapter 3 contains the main results of our work. In Section 3.1 we define the property of Invariant Basis Number (IBN) for a C^*-algebra, give examples of algebras which have IBN, and provide a complete K-theoretical characterization of algebras with IBN. In Section 3.2 we develop the notion of *Basis Type* for C^*-algebras without IBN, give various examples, and prove that arbitrary basis types are realized in particular C^*-algebras. In Section 3.3 we demonstrate that the Basis Types are preserved under small perturbations of the C^*-algebra. In Section 3.4 we briefly formulate a definition of IBN for non-unital C^*-algebras. In Section 3.5 we consider the representation theory of free modules over a C^*-algebra with a given Basis Type. In Section 3.6 we discuss two "finiteness" conditions which are related to Invariant Basis Number. We also provide examples demonstrating that these are distinct properties.

In Chapter 4 we apply the theories of Invariant Basis Number and Basis Types to two C^*-algebraic situations: dynamical systems and the classification program. For the first, we will show that Basis Type considerations greatly impact the structure for certain classes of dynamical systems, and that certain results in the literature are simplified when seen through the lens of Basis Type. For the second, we'll find that Basis Types distinguish some C^*-algebras more readily than K-theoretic data.

Chapter 1

Roots in Ring Theory

Much of modern ring theory has been built upon the consideration not of the rings themselves but rather by study of those objects upon which rings may act, viz. modules.

We shall not be concerned with modules in their full generality, but rather those modules with a particularly rigid structure. We will always have the action of a ring be on the right, hence *right modules*, but all will be referred to simply as modules hereafter. Recall that an R-module X is *finitely generated* if there is a family of elements $\{x_1, ..., x_n\}$ such that for each $x \in X$ there are coefficients $r_1, ..., r_n \in R$ such that $x = x_1r_1 + ... + x_nr_n$. If the family $\{x_1, ..., x_n\}$ is linearly independent, in the sense that $x_1r_1 + ... + x_nr_n = 0$ if and only if $x_ir_i = 0$ for all i, then we say that X is a *finitely generated free R-module*. The finite direct sum R^n is a finitely generated free R-module in the natural way, and it is an easy exercise to show that all finitely generated free R-modules arise in this manner, i.e. are isomorphic to a finite direct sum of copies of R.

In the case when R is a field, a module is a vector space and the finitely generated free R-modules are the finite dimensional vector spaces over R. Hidden within the previous sentence is the (true) assertion that *dimension*, defined as the size of a generating family, is unique for a given vector space. When R is not a field it happens that this assertion may be false; that no proper definition of "dimension" is possible for the finitely generated free modules over certain rings. The following is an example of when this behavior occurs.

Example 1.1. This example appears in [6, §2]. Consider the $\mathbb{Z}$-module $V = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ and $R = \mathrm{End}(V) = \mathrm{Hom}(V, V)$. Obviously id_V is a singleton generating set for R when considered as an R-module. However, noticing that

$$V = \left(\bigoplus_{i \geq 1,\ odd}^{\infty} \mathbb{Z} \right) \oplus \left(\bigoplus_{i \geq 2,\ even}^{\infty} \mathbb{Z} \right) = V_{odd} \oplus V_{even}$$

we see that $id_V = (id_{V_{odd}} \oplus 0) + (0 \oplus id_{V_{even}})$. It is routine to show that $\{id_{V_{odd}} \oplus 0, 0 \oplus id_{V_{even}}\}$ is a linearly independent generating set for R. Thus R (considered as an R-module) is isomorphic to R^2. Similar decompositions of id_V can show that $R = R^n$ for any $n > 0$.

Hence we have a dichotomy: rings which are like fields in the sense that their finitely generated free modules admit a reasonable definition of dimension; and those rings R for which $R^n = R^m$ for at least one pair $n \neq m$. The former we will say have the property of *Invariant Basis Number (IBN)* and the latter, of course, lack it.

Leavitt's Work. In a series of papers from the 1950s [17–20] Leavitt explored the property of IBN (or, using his terminology, *dimensionality*) for a unital ring and gave a characterization of such rings as follows.

Theorem 1.2 (Corollary 1 in [17])**.** A ring R has IBN if [and only if] there exists a unital homomorphism $\psi : R \to R'$ for some ring R' which has IBN.

This result may be used to greatly enlarge the class of rings which have IBN (by using extensions, for example) but is not particularly useful if one wishes to prove a ring does *not* have IBN, as it is an external, rather than internal, property of the ring.

The structure of non-dimensional rings is described in [17, Theorem 1] by assigning to R an integer pair (n, k) such that

- if X is an R module with basis of size $N < n$ then X has dimension, and
- if X is an R module with basis of size $N \geq n$ then there exist integers h and m, with $n \leq h < n + k$ and $m \geq 0$, for which $N = h + mk$. Thus $R^N = R^{h+mk}$.

The pair (n, k) is termed the *module type* of the ring.

Examples are explicitly constructed in [19] and [18] of rings with module types $(n, 1)$ and $(1, k)$ for arbitrary $n, k \geq 1$. To construct the examples, Leavitt employs a universal construction from generators satisfying relations which precisely correlate with the desired type. Key to this construction is the fact that these rings are integral domains.

There is precedent for fruitful connection between Leavitt's work and C^*-algebraic theory. In [18] Leavitt constructed a family of rings defined by generators and relations nearly identical to those of the much-known Cuntz C^*-algebras. Operator theorists generalized the Cuntz algebras to a much broader class known as graph C^*-algebras. More recently, great success has been had, e.g. [1, 29] among many others, in translating many results for graph C^*-algebras, such as the Gauge-Invariant Uniqueness theorems, to the purely algebraic theory of so-called *Leavitt Path Algebras*.

Cohn's Work. In [6] Cohn improves upon Leavitt's constructions of rings without IBN and in particular gives much simplified proofs using two invariants: the *trace* and the *dependence number*. The trace is defined as follows: consider a ring R as an abelian group $(R,+)$, the commutator subgroup $C(R) = \{xy - yx : x, y \in R\} \subset R$, the quotient group $T(R) := (R,+)/C(R)$, and let $tr : R \to T(R)$ be the natural group homomorphism. Cohn proves several matricial properties of the trace, for instance that $T(R) \cong T(M_n(R))$, and remarks as a Corollary to [6, Proposition 3.1] that if $tr(1) \in T(R)$ has finite order then R has IBN. We will prove a stronger version of this result as our main characterization of C^*-algebras with Invariant Basis Number.

Lam's Discussion. In his book [14], Lam discusses IBN within the larger context of various "finiteness" properties of rings. For example, a ring is said to be *finite* if every left-invertible element is also right-invertible and *stably finite* if the matrix rings $M_n(R)$, $n > 0$, are all finite. A stably finite ring always has IBN, but the converse is not true [14, Proposition 1.8]. In fact, IBN is demonstrated to be the most easily satisfied of the many conditions Lam considers.

Chapter 2

C^*-Module Background

The most basic object of our study will be C^*-algebras, i.e. complex Banach algebras A with involution $* : A \to A$ satisfying the C^*-condition $||a^*a|| = ||a||^2$. Many of our results will require our C^*-algebras to be unital, that is they possess multiplicative units (also called identities). It is a deep truth that all C^*-algebras can be faithfully represented as a selfadjoint algebra of operators acting on some Hilbert space, although for our purposes this is not essential. Instead, we will view our C^*-algebras as acting on highly structured complex vector spaces known as Hilbert C^*-modules.

We will use as much standard notation as possible. Generic C^*-algebras will usually be A, B, etc. and their Hilbert modules X, Y, and possibly E. The algebra of compact operators on a separable Hilbert space will be denoted by $\mathbb{K}$. We will use the word *ideal* to refer exclusively to two-sided, closed ideals unless specifically noted otherwise. The cone of *positive* elements of a C^*-algebra A will be denoted A^+.

2.1 Hilbert Modules

We shall summarize the results of the theory of Hilbert C^*-modules which are pertinent to our discussion. For more comprehensive treatment we recommend the books by Lance [15] and Wegge-Olsen [30].

For the remainder, let A be a C^*-algebra. We will place no conditions on A such as amenability, nuclearity, etc. but for our main results we will only be interested in unital algebras.

Definition 2.1. A *(right) A-module*, X, is a complex vector space with the following additional structure:

1. a right-action of A, i.e. $\mathbb{C}$-bilinear map $X \times A \to X : (x, a) \mapsto xa$ satisfying $(xa)b = x(ab)$ and $(\lambda x) \cdot a = x \cdot (\lambda a) = \lambda(x \cdot a)$,

2. an A-valued inner-product, i.e. a map $\langle \cdot, \cdot \rangle : X \times X \to A$ which satisfies the following:

   - $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in X$ and $\lambda \in \mathbb{C}$,
   - $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in X$,
   - $\langle x, y \cdot a \rangle = \langle x, y \rangle a$ for all $x, y \in X$ and $a \in A$,
   - $\langle x, x \rangle \in A^+$ for all $x \in X$ and $\langle x, x \rangle = 0$ iff $x = 0$.

A few comments are in order. Note that the A-valued inner-product is *conjugate linear* in the *first* variable while linear in the second. This is a reversal of the common inner product for Hilbert spaces but is standard in the literature of C^*-modules. We will almost always write the action as "xa" instead of "$x \cdot a$". To distinguish inner products for different modules we will utilize subscripts, e.g. $\langle \cdot, \cdot \rangle_X$.

The simplest example of a C^*-module is when $A = \mathbb{C}$ and X is a Hilbert space. In this case the action is simple scalar multiplication and the inner-product is the classical one. Some basic properties of the Hilbert spaces generalize quite readily to the setting of C^*-modules, as we shall see, but others do not hold in this more general setting. We will summarize several results in the next proposition.

Proposition 2.2 (From Chapter 1 of [15])**.** Let A be a C^*-algebra and X an A-module.

1. (The Cauchy-Schwarz Inequality) For any $x, y \in X$ we have

$$||\langle x, y\rangle||^2 \leq ||\langle x, x\rangle||\ ||\langle y, y\rangle||.$$

2. The assignment $x \mapsto ||\langle x, x\rangle||^{\frac{1}{2}}$ defines a norm on X.

3. If $\{e_\lambda\}$ is an approximate unit for A then $xe_\lambda \to x$ in norm for all $x \in X$. Consequently $XA = \{x \cdot a : x \in X, a \in A\} = X$.

Definition 2.3. If an A-module is complete with respect to this norm then it is known as a *Hilbert A-module*.

Example 2.4. With $A = \mathbb{C}$ and $X = H$ a Hilbert space we see that the norm $||\langle x, x\rangle||^{\frac{1}{2}}$ is precisely the Hilbert space norm and so H is a Hilbert $\mathbb{C}$-module.

Example 2.5. Let H be a separable Hilbert space and consider the subalgebra F of finite-rank operators within $B(H)$. Then F is a $B(H)$-module under the inner product $\langle T, S\rangle = T^*S$ and right multiplication, but it is not a Hilbert $B(H)$-module as its completion would be the compacts $\mathbb{K}$ (which *is* a Hilbert $B(H)$-module).

Example 2.6. A key example comes from considering A itself as an A-module when equipped with right-multiplication as the action and an inner-product given by $\langle a, b\rangle := a^*b$. Of course $||\langle a, a\rangle||^{\frac{1}{2}} = ||a^*a||^{\frac{1}{2}} = ||a||$ and so the norm arising from the inner-product is the same as the C^*-norm on A. Since this norm is *a priori* complete we have that A is a Hilbert module over itself.

A standard way to construct new Hilbert A-modules is through direct sums. If X and Y are Hilbert A-modules then $X \oplus Y := \{(x, y) : x \in X,\ y \in Y\}$ is an A-module with coordinate-wise right action and linear structure. The inner product on $X \oplus Y$ is given by

$$\langle (x, y), (x', y')\rangle := \langle x, x'\rangle_X + \langle y, y'\rangle_Y.$$

To prove that $X \oplus Y$ is complete is a routine exercise.

Standard Modules. Because of their importance to our future discussion, we shall carefully describe what we shall term as "standard" A-modules. Consider a C^*-algebra A and the n-fold algebraic direct sum $A^n := A \oplus A \oplus ... \oplus A$. The map $A^n \times A \ni ((a_i), a) \mapsto (a_i a) \in A^n$ is a right action of A on A^n. The assignment $\langle (a_i), (b_i)\rangle = \sum_{i=1}^n a_i^* b_i$ is an A-valued inner product on A^n. It is a straightforward exercise to show that A^n is an complete A-module. We will call A^n the *standard A-module* of size n, though the term "free A-module of rank n" is also used in the literature. The question of when these modules are distinct is central to our investigation.

Module Maps

In the following we will define the correct mappings to be considered between Hilbert C^*-modules.

Definition 2.7. An *A-module homomorphism* is an A-linear map. That is to say, if X and Y are A-modules then $\phi : X \to Y$ is a homomorphism if:

- $\phi(\lambda x_1 + x_2) = \lambda\phi(x_1) + \phi(x_2)$ for all $x_1, x_2 \in X$ and $\lambda \in \mathbb{C}$,

- $\phi(xa) = \phi(x)a$ for all $x \in X$ and $a \in A$.

An A-module *isomorphism* is simply a bijective homomorphism. An A-module homomorphism $\phi : X \to Y$ is *bounded* if

$$||\phi|| := \sup_{x \in X\setminus\{0\}} \left\{ \frac{||\phi(x)||_Y}{||x||_X} \right\} < \infty.$$

An A-module homomorphism $\phi : X \to Y$ is *adjointable* if there is another A-module homomorphism $\phi^* : Y \to X$ satisfying

$$\langle \phi(x), y \rangle_Y = \langle x, \phi^*(y) \rangle_X$$

for all $x \in X$ and $y \in Y$.

In the case of Hilbert $\mathbb{C}$-modules these notions coincide with bounded and adjointable operators, respectively. However the relationship between general bounded and adjointable homomorphisms is a departure from the Hilbert space theory.

Example 2.8. We shall exhibit a bounded homomorphism which is not adjointable. Consider $C[0,1]$ and $C_0(0,1)$ as $C[0,1]$-modules with the obvious actions and inner-product(s) $\langle f, g \rangle := \overline{f}g$. The inclusion $i : C_0(0,1) \hookrightarrow C[0,1]$ is certainly $C[0,1]$-linear and bounded. However, if it were adjointable then for all $f \in C_0(0,1)$ we would have $f = \langle i(\overline{f}), 1 \rangle = \langle \overline{f}, i^*(1) \rangle = fi^*(1)$ and so $i^*(1)$ would be a unit for $C_0(0,1)$, a contradiction since $C_0(0,1)$ has no unit.

In fact, if A is a unital C^*-algebra and $B \subset A$ a proper ideal then the inclusion $i : B \hookrightarrow A$ is always bounded but never adjointable as an A-module homomorphism.

Luckily, the reverse implication still holds in the more general setting of Hilbert modules: an adjointable homomorphism is bounded [30, Lemma 15.2.3]. Even when we restrict our attention to adjointable homomorphisms, things may not behave as expected. For example, arbitrary adjointable homomorphisms may not have a "polar decomposition." It turns out that such decompositions occur precisely when another nice property, $\ker \phi^* = \phi(X)^\perp$ holds.

Theorem 2.9 (The Polar Decomposition, Prop. 15.3.7 in [30])**.** For a Hilbert A-module X and $\phi \in L(X)$ the following are equivalent:

1. T has a polar decomposition $T = V|T|$ where $V \in L(X)$ is a partial isometry on X and $|T| = (T^*T)^{\frac{1}{2}}$.

2. $X = \ker |T| \oplus \overline{|T|X}$ and $X = \ker T^* \oplus \overline{TX}$.

Further, when either of these conditions hold,

$$\ker T = \ker V \qquad \ker T^* = \ker V^*$$

$$VX = \overline{TX} \qquad V^*X = \overline{|T|X}$$

In particular, note than when T is surjective $X = \ker T^* \oplus \overline{TX}$ is automatically satisfied and T^* is injective.

The collection of adjointable homomorphisms between two A-modules X and Y is a (complex) vector space in the obvious way and will be denoted by $L(X, Y)$. In the case $X = Y$ we shall write $L(X)$ and, as a matter of fact, $L(X)$ is a C^*-algebra under the "operator norm."

Example 2.10. As mentioned before, if H is a Hilbert $\mathbb{C}$-module then adjointable homomorphisms are bounded linear operators and vice-versa. The C^*-algebra they form is, of course, $B(H)$. In particular, we have that for the Hilbert $\mathbb{C}$-modules $\mathbb{C}^n$, $n < \infty$, $L(\mathbb{C}^n) = B(\mathbb{C}^n) = M_n(\mathbb{C})$.

Example 2.11. If A is a unital C^*-algebra, considered as a Hilbert module over itself, then $A = L(A)$. Certainly for $a \in A$ the map $\psi_a : x \mapsto ax$ is A-linear and adjointable (with adjoint ψ_{a^*}). Conversely, for $\phi \in L(A)$ and $x \in A$ we have $\phi(x) = \phi(1_A)x$ and so $\phi = \psi_{\phi(1_A)}$. It is routine to check that $||\psi_a||_{L(A)} = ||a||_A$ and so the map $a \mapsto \psi_a$ is an isometric isomorphism of C^*-algebras.

Example 2.12. Consider a standard A-module A^n, A unital, and view its elements as column vectors. Thus, via matrix multiplication, elements of $M_n(A)$ may be viewed as A-linear maps from A^n to itself. Each matrix $[a_{ij}]$ is adjointable with adjoint $[a_{ji}^*]$ and hence we have $M_n(A) \subseteq L(A^n)$. Given a homomorphism $\phi \in L(A^n)$, define the coordinate maps $\pi_j : A^n \to A$ by $pi_j((a_1, ..., a_n)) = a_j$. These are bounded (in fact, norm decreasing) and adjointable with adjoints π_j^* embedding A into the j-th coordinate. The projections $\pi_j^*\pi_j \in L(A^n)$ have the effect of eliminating all terms in a tuple except the j-th entry. Of course $\sum_{i=1}^n \pi_i^*\pi_i$ is the identity of $L(A^n)$ and so

$$\phi = I\phi I = \sum_{j=1}^n \pi_j^*\pi_j \left(\phi\left(\sum_{i=1}^n \pi_n^*\pi_n\right)\right) = \sum_{j=1}^n \sum_{i=1}^n \pi_i^*\pi_i\phi\pi_j^*\pi_j.$$

Now, the composition $\pi_i\phi\pi_j^*$ is an adjointable homomorphism on A and hence $\pi_i\phi\pi_j^* = \psi_{\alpha_{ij}}$ for some $\alpha_{ij} \in A$. Letting $U_\phi = [\alpha_{ij}] \in M_n(A)$ it is relatively obvious that $U_\phi(x) = \phi(x)$ for every $x \in A^n$ and hence $\phi = U_\phi$. Thus $L(A^n) = M_n(A)$.

Example 2.13. Logic similar to that of the previous example can be employed to show that $L(A^n, A^m) = M_{m,n}(A)$.

Definition 2.14. An adjointable homomorphism $u \in L(X, Y)$ is an *isometry* if $u^*u = I_X$, a *coisometry* if $uu^* = I_Y$, and a *unitary* if it is both.

A (not necessarily adjointable) homomorphism $\phi : X \to Y$ is *isometric* if $\langle \phi(x), \phi(x') \rangle_Y = \langle x, x' \rangle_X$ for all $x, x' \in X$. The embedding $C(0,1) \hookrightarrow C[0,1]$ is an isometric homomorphism which is not an isometry (since it is not adjointable, as shown earlier). Obviously every isometry is isometric and, surprisingly, an isometric isomorphism is automatically adjointable.

Proposition 2.15 (Theorem 3.5 in [15])**.** Every surjective isometric homomorphism is a unitary.

Definition 2.16. Two Hilbert A-modules X and Y are *unitarily equivalent*, denoted $X \simeq Y$, if there is a unitary element in $L(X, Y)$.

It is clear that $\simeq$ is an equivalence relation on the set of Hilbert A-modules. Unitary equivalence is, in general, a stronger condition than A-module isomorphism. In fact, unitaries are precisely the *isometric* A-module isomorphisms. However, in the case of standard modules every A-module homomorphism $\phi : A^n \to A^m$ may be represented as a $m \times n$ matrix with elements in A and so is automatically adjointable. Therefore if $\phi : A^n \to A^m$ is an A-module isomorphism then the Polar Decomposition (Theorem 2.9) yields a unitary in $L(A^n, A^m)$. We will rely upon unitary equivalence, rather than module isomorphism, in order to emphasize the additional structure of the standard modules.

2.2 Module Bases

Let A be a unital C^*-algebra and X a Hilbert A-module. A subset *generates* X if its A-linear span is dense (with respect to the norm topology) in X. We say that X is *finitely generated* if there exists a finite generating subset for X.

Example 2.17. Any unital C^*-algebra A is finitely (in fact, singly) generated as a Hilbert A-module. A Hilbert $\mathbb{C}$-module (i.e. a Hilbert space) is finitely generated if and only if it is finite dimensional. Consider a standard A-module A^n, $n \geq 0$, and the elements $e_i := \pi_i^*(1_A)$ for $i = 1...n$. Then $\{e_1, ..., e_n\}$ is a finite generating set for A^n.

It is important to note that a generating set need not generate the module *algebraically*. For example, consider $A = C_0(0,1]$ as a Hilbert module over itself. Then defining $f(x) := x$ we have $A = \overline{fA}$ (as a consequence of Stone-Weierstrass) but $A \neq fA$ since $f \notin fA$. Hence the singleton set $\{f\}$ generates A but does not do so algebraically.

A set $\{x_\alpha\} \subset X$ is *orthogonal* if $\langle x_\alpha, x_\beta \rangle = 0$ unless $\alpha = \beta$, and it is *orthonormal* if in addition $\langle x_\alpha, x_\alpha \rangle = 1_A$ for all α. Note that elements of an orthonormal set have norm 1.

Definition 2.18. A set $\{x_\alpha\} \subset X$ is a *basis* for X if it is an orthonormal generating set.

Example 2.19. Any unital C^*-algebra has the singleton basis $\{1_A\}$. A Hilbert $\mathbb{C}$-module (i.e. a Hilbert space) has a finite basis if and only if it is finite dimensional. The elements e_i, $i = 1, ..., n$, defined previously form a basis for the standard A-module A^n and will be known as the *standard basis* for A^n.

Existence of bases is not guaranteed even when a module is finitely generated. Recall that a Hilbert A-module X is *full* when $\langle X, X \rangle = \{\langle x, y \rangle : x, y \in X\}$ is dense in A. Note that $\langle X, X \rangle$ is a two-sided ideal of A, so if X is not full then $1 \notin \langle X, X \rangle$ and in particular $\langle x, x \rangle \neq 1$ for any $x \in X$. For example, if A is a C^*-algebra and $J \subset A$ a proper ideal then J is a non-full Hilbert A-module and cannot have any orthonormal sets.

Finite Bases

While many of the following results hold true for basis sets of arbitrary cardinality, see Landi and Pavlov's work [16] for example, we will only consider the finite case.

Proposition 2.20. If $\{x_1, ..., x_n\}$ is a finite basis for a Hilbert A-module X then for any $x \in X$ we have the "Fourier decomposition"

$$x = \sum_{i=1}^{n} x_i \langle x_i, x \rangle.$$

Proof. Since $\{x_1, ..., x_n\}$ is generating, i.e. $\overline{span_A}(x_1, ..., x_n) = X$, there is a net $\{a_{i,\lambda} : i = 1, ..., n, \lambda \in \Lambda\}$ for which $\sum_{i=1}^{n} x_i a_{i\lambda} \to x$ in norm. By an application of the Cauchy-Schwarz inequality we have

$$\|\langle x_j, x \rangle - a_{j\lambda}\| = \left\| \langle x_j, x \rangle - \left\langle x_j,\ \sum_{i=1}^{n} x_i a_{i\lambda} \right\rangle \right\|$$

$$= \left\| \left\langle x_j,\ x - \sum_{i=1}^{n} x_i a_{i\lambda} \right\rangle \right\| \leq ||x_j|| \left\| x - \sum_{i=1}^{n} x_i a_{i\lambda} \right\| = \left\| x - \sum_{i=1}^{n} x_i a_{i\lambda} \right\|$$

for any $j = 1...n$. Therefore $a_{i\lambda} \to \langle x, x_i \rangle$ for $i = 1...n$ and so

$$\sum_{i=1}^{n} x_i a_{i\lambda} \to \sum_{i=1}^{n} x_i \langle x_i, x \rangle.$$

We must then conclude that $x = \sum_{i=1}^{n} x_i \langle x_i, x \rangle$, as desired. □

Within the previous proof are the following properties:

1. X is *algebraically* generated by a finite basis,

2. if $\langle x_i, x \rangle = \langle x_i, y \rangle$ for all $i = 1...n$ then $x = y$,

3. for $x, y \in X$ we have $\langle x, y \rangle = \sum_{i=1}^{n} \langle x, x_i \rangle \langle x_i, y \rangle$.

In particular, we obtain the uniqueness of the Fourier decomposition and that the basis elements "separate points."

The existence of a finite basis imposes a very particular structure on the Hilbert module. In fact, modules which admit finite basis "look like" standard modules in a very strong sense.

Theorem 2.21 (Folklore)**.** If X is a Hilbert A-module which admits a basis $\{x_1, ..., x_n\}$ then the map

$$X \ni x \mapsto (\langle x_1, x\rangle, ..., \langle x_n, x\rangle) \in A^n$$

is a unitary A-module homomorphism. Hence $X \simeq A^n$.

Of course a converse is also true: if $u \in L(X, A^n)$ is a unitary then $u^*e_1, ..., u^*e_n$ is a basis for X. As a consequence of this equivalence, all questions and results involving modules with bases will be posed in terms of the standard A-modules and their standard bases.

It is a natural question to ask if the size of a finite basis is a unique feature of a Hilbert module. In the case of Hilbert $\mathbb{C}$-modules the answer is "yes" since the dimension (i.e. basis size) of a finite dimensional Hilbert space is unique. We shall see that the answer in general depends on the structure of the C^*-algebra. The property of C^*-algebras which allows for well-defined module "dimension" is the primary interest of this dissertation.

2.3 Cuntz Algebras

The following family of algebras are extremely important not just in our current discussion but in many fields of C^*-algebraic theory. These C^*-algebras were investigated by Cuntz in his original paper [7].

Definition 2.22. A *Cuntz family of size* n is a set $V_1, ..., V_n \in B(H)$ of isometries which satisfy the following conditions:

1. for all i, j we have $V_i^* V_j = \delta_{ij} I$, i.e. their ranges are mutually orthogonal, and
2. $\sum_{i=1}^n V_i V_i^* = I$.

A countably infinite Cuntz family will be defined as satisfying the first property but satisfying $\sum_{i=1}^N V_i V_I^* < I$ for all $N > 0$ in place of the second property.

The *Cuntz algebra* $\mathcal{O}_n$ is the C^*-algebra with generating Cuntz-family $v_1, ..., v_n$ which satisfies the following universal property: whenever $V_1, ..., V_n$ is a Cuntz family in $B(H)$ then there is a unique $*$-homomorphism $\tau : \mathcal{O}_n \to C^*(V_1, ..., V_n)$ satisfying $\tau(v_i) = V_i$. The Cuntz algebra $\mathcal{O}_\infty$ is similarly defined.

The Cuntz algebras have a great deal of structure and have been exhaustively analyzed. A particularly useful feature is that $\mathcal{O}_n$ is always simple [7, Theorem 1.12]. Thus the universal map $\tau : \mathcal{O}_n \to C^*(V_1, ..., V_n)$ is always injective, hence an isomorphism, and so any C^*-algebra generated by a Cuntz family is isomorphic to a Cuntz algebra.

The Cuntz algebras have a close relation to the so-called *Toeplitz algebras* $\mathcal{E}_n := C^*(v_1, ..., v_n) \subset \mathcal{O}_{n+1}$. The Toeplitz algebras are universal objects for families of pairwise orthogonal isometries whose range projections satisfy $\sum_{i=1}^n V_i V_i^* \leq I$. The following is key to our analysis of the Cuntz (and Toeplitz) algebras.

Proposition 2.23 (Proposition 3.1 in [7])**.** Let $v_1, ..., v_n$ be the generators for $\mathcal{E}_n$. If we let $p = I - \sum_{i=1}^n V_i V_i^*$ then the ideal $p\mathcal{E}_n p$ is isomorphic to $\mathbb{K}$. As a consequence we have the short exact sequence $0 \to \mathbb{K} \to \mathcal{E}_n \to \mathcal{O}_n \to 0$.

Example 2.24. When considered as Hilbert modules over themselves, the Cuntz algebras exhibit the basis behavior we are most interested in. To be precise, consider the set of generators $v_1, ..., v_n$ of $\mathcal{O}_n$. These form an orthonormal set and for any $x \in \mathcal{O}_n$ we have

$$x = Ix = \sum_{i=1}^{n} v_i v_i^* x = \sum_{i=1}^{n} v_i \langle v_i, x \rangle$$

hence they are a basis for $\mathcal{O}_n$. But since $\mathcal{O}_n$ is unital it also has the basis consisting of just the unit. Thus $\mathcal{O}_n \simeq \mathcal{O}_n^n$.

In fact, the existence of a unital subalgebra isomorphic to $\mathcal{O}_n$ guarantees multiple basis sizes. This is essentially the content of the following proposition.

Proposition 2.25 (Compare to a remark on p24 in [15])**.** $A \simeq A^n$ if and only if there is a unital embedding $\mathcal{O}_n \hookrightarrow A$.

Proof. If $A \simeq A^n$ then A has a basis $a_1, ..., a_n$. The definition of a basis gives that $a_1, ..., a_n$ is a Cuntz family and hence generates a (unital) subalgebra of A isomorphic to $\mathcal{O}_n$. Conversely, if A has $\mathcal{O}_n$ as a unital subalgebra then it contains a Cuntz family of size n which, again from the definition, acts as a basis of size n for A. Thus $A \simeq A^n$. $\square$

Although in general it is not obvious when such a unital embedding exists, for some cases it is easy to see one cannot exist. Recall that a C^*-algebra is *properly infinite* if it contains two isometries with orthogonal ranges. The Cuntz algebras are, for $n > 1$, properly infinite and this, combined with the above proposition, gives the following corollary.

Corollary 2.26. If A is not properly infinite then $A \not\simeq A^n$ for any $n > 1$.

K-theory for Cuntz Algebras

We now turn our attention to the group $K_0(\mathcal{O}_n)$.

Lemma 2.27 (Prop. 2.2 in [8])**.** The map $\phi_n : x \mapsto \sum_{i=1}^{n} v_i x v_i^*$ is a unital endomorphism of $\mathcal{O}_n$ which is homotopic to the identity map.

Recalling that homotopic projections give rise to equivalent elements in the K_0 group [30, Remark 6.1.2], we have that since $\phi_n(p) \sim p$ for all $p \in P_\infty(\mathcal{O}_n)$ so

$$[p]_0 = [\phi_n(p)] = \sum_{i=1}^{n} [v_i p v_i^*] = n[p]_0$$

and consequently $(n-1)[p]_0 = 0$. Thus the group $K_0(\mathcal{O}_n)$ exhibits torsion.

Theorem 2.28 (Cuntz, Theorem 3.7 in [8])**.** $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$.

The Cuntz algebras have torsion in their K-theory and are, so far, our only examples of C^*-algebras which satisfy the module equivalences $A \simeq A^n$. These two properties of Cuntz algebras are linked, as we shall see in our main results.

Chapter 3

Invariant Basis Number and Basis Type

3.1 Invariant Basis Number

Definition 3.1. A unital C^*-algebra A has *Invariant Basis Number* if it satisfies the property:

$$\text{for all } m, n \geq 1,\ A^m \simeq A^n \Leftrightarrow m = n. \tag{IBN}$$

Conversely, A does not have Invariant Basis Number if there are positive integers $m \neq n$ for which $A^m \simeq A^n$.

The motivation for the terminology is as follows: suppose that X is a Hilbert A-module with finite basis sets of sizes j and k. Since every Hilbert A-module with basis of size n is unitarily equivalent to the standard A-module A^n, we would conclude that $A^k \simeq X \simeq A^j$. If A has Invariant Basis Number (hereafter, "has IBN") we must conclude that $j = k$, i.e. the size of a basis is unique for X.

The following is an alternative characterization of IBN which we will frequently use.

Proposition 3.2. A C^*-algebra A has IBN if and only if every unitary matrix over A is square. Conversely, if A does not have IBN then there exists a non-square unitary matrix over A.

The proof is a simple application of the identification $L(A^n, A^m) = M_{n,m}(A)$.

Example 3.3. The following are C^*-algebras with IBN:

$\mathbb{C}$. This is simple to see since a Hilbert $\mathbb{C}$-module with finite basis is nothing but a finite dimensional Hilbert space H. The dimension of H matches the size of the basis.

$M_n(\mathbb{C})$. We note that since $M_{n',m'}(M_n(\mathbb{C})) = M_{nn',nm'}(\mathbb{C})$ the fact that $\mathbb{C}$ has IBN combines with the above proposition to have us conclude that $M_n(\mathbb{C})$ has IBN.

$C([0,1])$. Suppose that $\{e_1, ..., e_n\}$ is a basis for $C([0,1])$ (as a module over itself. Since $1 = \langle e_i, e_i\rangle(x) = |e_i(x)|^2$ we have that e_i is strictly nonzero for all $i = 1, ..., n$. In particular $\overline{e_j(x)}e_i(x) \neq 0$ for all i, j and $x \in [0,1]$. If $n > 1$ this contradicts the requirement that $\langle e_i, e_j\rangle \equiv 0$ when $i \neq j$. Thus $C([0,1])$ only admits single-element bases. Similar arguments show that $C([0,1])^n$ admits only bases of size n, hence $C([0,1])$ has IBN.

Example 3.4. These C^*-algebras do not have IBN:

$B(H)$ for H infinite. This follows from the fact that $B(H) \oplus B(H) \cong B(H \oplus H) \cong B(H)$ where the equivalence is as C^*-algebras, not modules.

$\mathcal{O}_2$. As discussed in Example 2.24, the sets $\{I\}$ and $\{v_1, v_2\}$ are both bases for $\mathcal{O}_2$, hence $\mathcal{O}_2 \simeq \mathcal{O}_2 \oplus \mathcal{O}_2 = \mathcal{O}_2^2$.

$\mathcal{O}_n$. Similarly to $\mathcal{O}_2$, we saw $\mathcal{O}_n \simeq \mathcal{O}_n^n$.

Our main concern for this section is to identify C^*-algebras which have IBN. Several classes of C^*-algebras are well-suited for direct proofs and we will present these first. At the conclusion we will provide a complete characterization of C^*-algebras with IBN.

3.1.1 Particular Cases

We suspect that these results will not surprise experts and that perhaps, stripped of the language of IBN, they may be folkloric. However, we have not found literature on the subject and we consider them to be original results.

If A is a commutative C^*-algebra with unit (i.e. $A = C(X)$ for a compact Hausdorff space X) then the situation, at least in terms of bases, is similar to that of Hilbert $\mathbb{C}$-modules.

Proposition 3.5. Commutative C^*-algebras have Invariant Basis Number.

Proof. Let $f_1, ..., f_m$ be a basis of A^n. We have that for all $i = 1...n$

$$1_A = \langle e_i, e_i \rangle = \sum_{j=1}^{m} \langle e_i, f_j \rangle \langle f_j, e_i \rangle$$

were the last equality relies on $f_1, ..., f_m$ being a basis, see Proposition 2.20 and following remarks. Similarly

$$1_A = \langle f_j, f_j \rangle = \sum_{i=1}^{n} \langle f_j, e_i \rangle \langle e_i, f_j \rangle.$$

Thus

$$n1_A = \sum_{i=1}^{n} \langle e_i, e_i \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle e_i, f_j \rangle \langle f_j, e_i \rangle$$

and

$$m1_A = \sum_{j=1}^{m} \langle f_j, f_j \rangle = \sum_{j=1}^{m} \sum_{i=1}^{n} \langle f_j, e_i \rangle \langle e_i, f_j \rangle$$

but A is commutative and so $\langle e_i, f_j \rangle \langle f_j, e_i \rangle = \langle f_j, e_i \rangle \langle e_i, f_j \rangle$ for all i, j. Therefore

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \langle e_i, f_j \rangle \langle f_j, e_i \rangle = \sum_{j=1}^{m} \sum_{i=1}^{n} \langle f_j, e_i \rangle \langle e_i, f_j \rangle$$

and so $n1_A = m1_A$, i.e. $m = n$. □

Recall that a C^*-algebra is *finite* if it does not contain a proper isometry and *stably finite* if $M_n(A)$ does not contain a proper isometry for any $n \geq 1$. Equivalently, A is stably finite if $A \otimes \mathbb{K}$ is finite.

Theorem 3.6. Stably finite C^*-algebras have Invariant Basis Number.

Proof. Suppose that A is stably finite and does not have IBN. Then there are integers $k > j \geq 1$ for which $A^j \simeq A^k$, i.e. there is a unitary element $u \in L(A^k, A^j)$. Now since $L(A^k, A^j) = M_{j,k}(A)$, u is a $j \times k$ unitary matrix. If we write $u = [u_1 \ u_2]$ with $u_1 \in M_j(A)$ and $u_2 \in M_{j,k-j}(A)$ then

$$I_{A^j} = I_j = uu^* = u_1 u_1^* + u_2 u_2^*$$

and

$$I_{A^k} = I_k = \begin{bmatrix} I_j & 0 \\ 0 & I_{k-j} \end{bmatrix} = u^* u = \begin{bmatrix} u_1^* u_1 & u_2^* u_1 \\ u_1^* u_2 & u_2^* u_2 \end{bmatrix}.$$

Thus we have $u_1^* u_1 = I_j = u_1 u_1^* + u_2 u_2^*$, but since $u_2^* u_2 = I_{k-j}$ we have $u_2 \neq 0$ so that $u_2 u_2^* \neq 0$. Thus $u_1 u_1^* = I_j - u_2 u_2^* < I_j$ and so u_1 is a proper isometry in $M_j(A)$. This, of course, contradicts our assumption that A is stably finite. □

To exhibit a C^*-algebra which has IBN but is not stably finite requires some preparation. Recall that if A, B, and C are C^*-algebras we say B is an *extension* of A (by C) if there is a short exact sequence

$$0 \longrightarrow C \xrightarrow{\iota} B \xrightarrow{\pi} A \longrightarrow 0\cdot$$

If B is unital it is a *unital extension* of A.

Theorem 3.7. If A is an C^*-algebra which has IBN and B is a unital extension of A then B also has IBN.

Proof. By definition there is a C^*-algebra C such that we have the short exact sequence

$$0 \longrightarrow C \xrightarrow{\iota} B \xrightarrow{\pi} A \longrightarrow 0\cdot$$

Suppose to the contrary that B does not have IBN, hence has a unitary matrix $U = [u_{ij}] \in M_{n,m}(B)$ for some $n \neq m$.

By defining $\pi^{(p,q)} : M_{p,q}(B) \to M_{p,q}(A)$ as $\pi([b_{ij}]) = [\pi(b_{ij})]$ it is a simple exercise to see that

$$\pi^{(p,q)}(V)\pi^{(q,r)}(W) = \pi^{(p,r)}(VW)$$

for all $V \in M_{p,q}(B)$ and $W \in M_{q,r}(B)$. In particular we have that

$$\pi^{(n,m)}(U)\pi^{(m,n)}(U^*) = \pi^{(n,n)}(UU^*) = \pi^{(n,n)}(I_n) = I_n$$

$$\pi^{(m,n)}(U^*)\pi^{(n,m)}(U) = \pi^{(m,m)}(U^*U) = \pi^{(m,m)}(I_m) = I_m$$

and so $\pi^{(n,m)}(U) \in M_{n,m}(A)$ is a unitary. Consequently A does not have IBN, a contradiction. $\square$

Corollary 3.8. If $\pi : B \to A$ is a surjective $*$-homomorphism and A has IBN then B has IBN as well.

Example 3.9. Clarke [5] constructed a C^*-algebra A which is finite but not stably finite. A key ingredient was realizing A as a particular extension of $C(\mathbb{T}^3)$ by the compacts. Since $C(\mathbb{T}^3)$ is commutative it has IBN and hence A does as well.

We may also use Theorem 3.7 to find infinite C^*-algebras with IBN. If we consider the Toeplitz algebra $\mathfrak{T}$ (the universal C^*-algebra generated by a single non-unitary isometry) then it is well known to be realized as an extension of $C(\mathbb{T})$ by the compacts. Thus $\mathfrak{T}$ has IBN.

3.1.2 Characterization of Invariant Basis Number

We have exhausted the low-hanging fruit, as it were, and are ready to give a complete characterization of those algebras with Invariant Basis Number. We will be using K-theoretic tools and have included the necessary background and details within Appendix A. Far more expert exposition on the application of K-theory to C^*-algebras may be found in [4, 26, 30].

Theorem 3.10. A unital C^*-algebra A has Invariant Basis Number if and only if the element $[1]_0 \in K_0(A)$ has infinite order.

Proof. We will prove the contrapositive.
($\Rightarrow$) Suppose that $[1]_0$ has finite order N. Then $[1_N] = N[1]_0 = 0$ and so, by Proposition A.3 there is a projection $p \in M_{n_p}(A)$ such that $1_N \oplus p \sim p$. Since $1_{n_p} \sim (1_{n_p} - p) \oplus p$ by Proposition A.1 we have $1_N \oplus 1_{n_p} \sim 1_{n_p}$ and thus there is an element $U \in M_{N,N+n_p}(A)$ for which $UU^* = 1_N$ and $U^*U = 1_{N+n_p}$. Let x_i be the i-th column of U, thought of as a vector in A^N, i.e. $U = [x_1\ x_2\ \dots\ x_{N+n_p}]$. Since

$[x_i^* x_j] = U^*U = I_{N+n_p}$ we have that $\{x_1, ..., x_{N+n_p}\}$ is an orthonormal set in A^N. Similarly $\sum_{i=1}^{N+n_p} x_i x_i^* = I_N$ and so for any $a \in A^N$

$$a = I_N a = \sum_{i=1}^{N+n_p} x_i x_i^* a = \sum_{i=1}^{N+n_p} x_i \langle x_i, a \rangle$$

demonstrating that $x_1, ..., x_{N+n_p}$ is a basis for A^N, whence $A^N \simeq A^{N+n_p}$. Since $n_p > 0$, $N \neq N + n_p$ and so A cannot have IBN.

($\Leftarrow$) Suppose that A does not have IBN. Then there are integers $k > j \geq 1$ for which $A^k \simeq A^j$. Thus there is a unitary $u \in L(A^k, A^j) = M_{j,k}(A)$ and so $1_j \sim 1_k$. It follows that $[1_k]_0 = [1_j]_0$ and so

$$[1_{k-j}]_0 + [1_j]_0 = [1_{k-j} \oplus 1_j]_0 = [1_k]_0 = [1_j]_0$$

which by Proposition A.3 means $j[1]_0 = [1_j]_0 = 0$. $\square$

Thus, to determine if a particular C^*-algebra has IBN it is enough to compute the order of a single element of its K_0 group. The K_0 groups for wide classes of C^*-algebras are known and this allows us to determine many algebras with (and many without) IBN.

Example 3.11.

- We have $K_0(\mathbb{C}) = K_0(M_n(\mathbb{C})) = \mathbb{Z}$ (and $[1]_0 \neq 0$) for all $n \geq 0$ confirming our previous results.

- For unital stably finite C^*-algebras, K_0 is totally ordered (with order unit $[1]_0 \neq 0$) [26, Prop. 5.1.4]and so cannot have torsion, confirming again our previous results.

- Any unital C^*-algebra with trivial K_0 group must necessarily not have IBN. Of particular note: $K_0(B(H))$ is trivial.

- The Cuntz algebras have $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$ (see Theorem 2.28).

We would like to remark that the stronger statement "a C^*-algebra A has IBN if and only if $K_0(A)$ is torsion-free" is false, as evidenced by the following example.

Example 3.12. Consider the *Moore space* Y_n which is obtained from the unit disc $\mathbb{D}$ by identifying points on the boundary for which $z_1^n = z_2^n$. It is shown in [26, Example 12.2] that $K_0(C(Y_n)) = \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ and so has torsion even though $C(Y_0)$ is commutative and must have IBN. The key, of course, is that $[1]_0$ still has infinite order due to the first summand.

Finally, Theorem 3.10 allows us to conclude that a wide class of C^*-algebras does *not* have IBN. Recall that a C^*-algebra A is *properly infinite* if it contains projections p and q such that $pq = qp = 0$, and $p \sim q \sim p + q \sim 1_A$.

Corollary 3.13. If a C^*-algebra is properly infinite then it does not have IBN.

Proof. Suppose that A is properly infinite with projections p and q satisfying the relations $pq = qp = 0$, $p \sim q \sim p + q \sim 1_A$. Since $pq = qp = 0$ we have that $p + q \sim p \oplus q \in M_2(A)$ and so

$$[1_A]_0 = [p \oplus q]_0 = [p]_0 + [q]_0 = [1_A]_0 + [1_A]_0.$$

Hence $[1_A]_0$ does not have infinite order and so A cannot have IBN. $\square$

3.1.3 Permanence Properties of IBN

In this section we will demonstrate that the IBN property is preserved under several common algebraic constructions. We have already demonstrated, for example, that IBN is preserved under unital extensions, see Theorem 3.7. These next results are corollaries to our characterization, Theorem 3.10.

Corollary 3.14. If A has IBN then $A \oplus B$ has IBN for any (unital) C^*-algebra B.

The proof is immediate given that $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$ and $1_{A \oplus B} = (1_A, 1_B)$.

Corollary 3.15. If A is a C^*-algebra with IBN then $M_n(A)$ has IBN for all $n \geq 1$.

As $K_0(M_n(A)) = K_0(A)$, we are done.

We should note that $A \otimes \mathbb{K}$ is non-unital and thus cannot have IBN according to our definition. See Example 3.50 in Section 3.4 for further discussion. Consequently it would be misleading to term IBN a "stable property" of a C^*-algebra.

A converse to the previous proposition is also true.

Proposition 3.16. If $M_n(A)$ has IBN for some $n \geq 1$ then A has IBN.

Proof. Suppose to the contrary that A does not have IBN. By Proposition 3.2 there is a unitary $M_{j,k}(A)$ matrix for some $j > k \geq 1$. It follows that there is a unitary $M_{nj,nk}(A)$ matrix (obtained by placing n copies of the $j \times k$ unitary down the "diagonal" of a $nj \times nk$ matrix) and hence a unitary $M_{j,k}(M_n(A))$ matrix. We must thus conclude that $M_n(A)$ does not have IBN. $\square$

Combining the two previous results we have the following

Theorem 3.17. The following are equivalent:

1. A has IBN,
2. $M_n(A)$ has IBN for all $n \geq 1$,
3. $M_n(A)$ has IBN for some $n \geq 1$.

Our next result concerns inductive systems and limits of C^*-algebras. See Appendix A.0.6 for definitions.

Proposition 3.18. Let $\{A_i, \phi_{ij}\}$ be an inductive system of C^*-algebras such that each A_i has IBN and each ϕ_{ij} is unital, then the inductive limit A is unital and has IBN.

Proof. The fact that A is unital is well known. The continuity of K_0 (Theorem A.9) gives us an inductive system $\{K_0(A_i), K_0(\phi_{ij})\}$ of abelian groups with inductive limit $K_0(A)$. Since each A_i has IBN the subgroup $\mathbb{Z}[1_{A_i}]_0$ is isomorphic to $\mathbb{Z}$. Since each ϕ_{ij} is unital, the maps $K_0(\phi_{ij})$ take $[1_{A_i}]_0$ to $[1_{A_j}]_0$, hence are isomorphisms on the subgroups they generate. Thus the universal group homomorphisms $K_0(\phi_i)$ must be isomorphisms onto $\mathbb{Z}[1_A]_0$, hence $[1_A]$ has infinite order and, by Theorem 3.10, A has IBN. □

Finally, we will demonstrate that the property Invariant Basis Number is unfortunately *not* preserved under Morita equivalence. A good reference for the theory of Morita equivalence is [23].

Proposition 3.19. Let A be a infinite simple unital C^*-algebra, then there is a C^*-algebra B Morita equivalent to A which does not have IBN.

Proof. If A is infinite then there exists a proper isometry $v \in A$. As $vv^* \sim v^*v = 1_A$ we have

$$[1_A]_0 = [1_A - vv^*]_0 + [vv^*]_0 = [1_A - vv^*]_0 + [1_A]_0$$

and so $[1_A - vv^*]_0 = 0$ in $K_0(A)$. Now consider the full corner $B = (1_A - vv^*)A(1_A - vv^*)$, which is Morita-equivalent to A [23, Example 3.6], and note that $1_B = 1_A - vv^*$. Thus $[1_B]_0 = 0$ in $K_0(B)$ and so B does not have IBN. $\square$

Example 3.20. The infinitely generated Cuntz algebra $\mathcal{O}_\infty$ is a unital simple infinite C^*-algebra with IBN (because $K_0(\mathcal{O}_\infty) = \mathbb{Z}$) but, by the above Proposition, it contains a full corner $(1 - v_1 v_1^*)\mathcal{O}_\infty(1 - v_1 v_1^*)$ which does not have IBN.

3.2 Basis Type

We have given a complete characterization of C^*-algebras which have Invariant Basis Number and now we shall turn our attention to algebras without it. The next Theorem gives us the means to group such algebras into manageable classes and may be compared to Leavitt's work [17, Theorem 1]

Theorem 3.21. If A is a C^*-algebra which does not have IBN then there are unique largest positive integers N and K for which

a) if $n < N$ and $A^n \simeq A^j$ for some $j \geq 1$ then $j = n$,

b) if $A^j \simeq A^k$ for some $j, k \geq 1$ then $j \equiv k \mod K$.

We will say that A is of *basis type* (N, K) and we will often write *type*$(A) = (N, K)$.

Put another way, N is the smallest integer for which $A^N \simeq A^{N+k}$ for some $k > 0$, and K is the smallest such k.

Proof. Since A does not have IBN we know there exist $j > k \geq 1$ for which $A^j \simeq A^k$. Thus $N := \min\{n : A^n \simeq A^k \text{ for some } k \neq n\}$ exists and a) follows immediately.

Set $K := \min\{i > 0 : A^N \simeq A^{N+i}\}$ and suppose that $A^j \simeq A^k$ for some $j, k \geq 1$.

Claim: There are $j', k' \leq N+K$ for which $A^{j'} \simeq A^j \simeq A^k \simeq A^{k'}$. Further $j' \equiv j \mod K$ and $k' \equiv k \mod K$.

Proof of claim: The proof is identical for j or k. If $j > N+K$ then

$$A^j = A^{j-(N+K)} \oplus A^{N+K} \simeq A^{j-(N+K)} \oplus A^N = A^{j-K}.$$

Iterate this process if necessary to obtain a j' for which $N+K \geq j' \geq N$ and $A^j \simeq A^{j'}$. Note that $j' = j - nK$ for some $n > 0$ and so $j' \equiv j \mod K$. This proves our claim.

As a result of the claim, it is enough to prove that if $N \leq k < j \leq N+K$ and $A^j \simeq A^k$ then $j - k = K$. To that end, observe that

$$A^N \simeq A^{N+K} = A^{N+K-k} \oplus A^k \simeq A^{N+K-k} \oplus A^j = A^{N+K+j-k} \simeq A^{N+(j-k)}$$

and conclude by the minimality of K that $K \leq j-k$. But $N \leq k < j < N+K$ and so $j - k \leq K$ as well, thus $j - k = K$ as desired. $\square$

Corollary 3.22. If A is of basis type (N, K) then $A^n \simeq A^{n+K}$ for all $n \geq N$.

Corollary 3.23. If A is of basis type (N, K) then there are precisely $N + K - 1$ equivalence classes of standard A-modules.

Previously we exhibited several C^*-algebras which do not have IBN. Now it is possible to assign them a basis type.

Example 3.24. The Cuntz algebra $\mathcal{O}_2$ is of basis type $(1, 1)$ since $\mathcal{O}_2 \simeq \mathcal{O}_2^2$. The relationship $A \simeq A^2$ is characteristic for algebras with basis type $(1, 1)$.

Example 3.25. The Cuntz algebra $\mathcal{O}_n$ is of basis type $(1, n-1)$. As $\mathcal{O}_n \simeq \mathcal{O}_n^n$ (see Example 2.24) and so $N = 1$ and $K \leq n-1$. That $\mathcal{O}_n \not\simeq \mathcal{O}_n^j$ for $j < n$ will follow from the next Theorem.

Much as we were able to characterize C^*-algebras which have IBN in terms of their K-theory, we can give K-theoretical descriptions of the basis types as well.

Theorem 3.26. If A is a C^*-algebra of basis type (N, K) then

1. $K = |[1]_0|$ (the additive order of $[1_A]_0$ in $K_0(A)$) and

2. $N = \min\{n : [1_{n+K}]_0 = [1_n]_0\} = \min\{n : 1_{n+K} \sim 1_n\}$.

Proof. Let $j = |[1]_0|$. Then by definition there is n such that $[1_{n+j}]_0 = [1_j]_0$, i.e. there is a unitary in $M_{n,n+j}(A)$. Thus we have $j \equiv 0 \mod K$. Similarly we have that there is a unitary in $M_{N,N+K}(A)$ and so $[1_{N+K}]_0 = [1_K]_0$ whence $K \equiv 0 \mod j$. We must conclude that $|[1]_0| = j = K$.

Since $1_{n+K} \sim 1_n$ if and only if $[1_{n+K}]_0 = [1_n]_0$ we have equality of the two minimum terms. When $1_{n+K} \sim 1_n$ there is a unitary $(n+K) \times n$ matrix and hence $A^{n+K} \simeq A^n$, thus by definition $N \leq \min\{n : 1_{n+K} \sim 1_K\}$. As $A^{N+K} \simeq A^N$ by definition we have equality. □

3.2.1 Lattice Structure

We will give the Basis Types $\{(N, K) : N, K \in \mathbb{N}\}$ a lattice structure as follows:

$$(N_1, K_1) \leq (N_2, K_2) \Leftrightarrow N_1 \leq N_2 \textit{ and } K_2 \equiv 0 \mod K_1$$

$$(N_1, K_1) \vee (N_2, K_2) := (\max(N_1, N_2), \mathrm{lcm}(K_1, K_2))$$

$$(N_1, K_1) \wedge (N_2, K_2) := (\min(N_1, N_2), \gcd(K_1, K_2)).$$

This structure corresponds to several algebraic operations in a pleasing way. These results are comparable those of Leavitt [20] for noncommutative rings.

Proposition 3.27. If A does not have IBN and $\phi : A \to B$ is a unital $*$-homomorphism then B does not have IBN. Further, $type(B) \leq type(A)$.

Proof. Recall our alternative definition of IBN, Proposition 3.2. If A does not have IBN then there is a non-square rectangular unitary matrix u over A. Then $\phi(u) := [\phi(u_{ij})]$ is a unitary rectangular matrix over B, hence B cannot have IBN.

Denote $type(A) = (N_A, K_A)$ and $type(B) = (N_B, K_B)$. By the definition of basis type we conclude that $M_{n,k}(B)$ does not have any unitary matrices when $n < N_B$. As mentioned before, every unitary matrix $u \in M_{n,k}(A)$ has an image $\phi(u) \in M_{n,k}(B)$ which is also unitary. Hence we may conclude that there are no unitaries in $M_{n,k}(A)$ when $n < N_B$. Thus by definition of N_A we have $N_A \geq N_B$.

That $K_A \equiv 0 \mod K_B$ follows from the induced group homomorphism $K_0(\phi) : K_0(A) \to K_0(B)$ (see Proposition A.7) and the following easy fact: if G and H are groups with $\phi : G \to H$ a group homomorphism then $|g|_G \equiv 0 \mod |\phi(g)|_H$ for all $g \in G$ with finite order.

As $N_B \leq N_A$ and $K_A \equiv 0 \mod K_B$ we have by definition $(N_B, K_B) \leq (N_A, K_A)$. $\square$

For example, consider that the Cuntz algebra $\mathcal{O}_n$ admits a unital embedding $\mathcal{O}_{k(n-1)+1} \hookrightarrow \mathcal{O}_n$ for all $k \geq 1$ [9, Exercise V.16]. We've seen $type(\mathcal{O}_n) = (1, n-1)$ and $type(\mathcal{O}_{k(n-1)+1}) = (1, k(n-1))$ and so by definition $type(\mathcal{O}_n) \leq type(\mathcal{O}_{k(n-1)+1})$.

One application of Proposition 3.27 is in the context of inductive limits.

Corollary 3.28. If $\{A_i, \phi_{ij}\}$ is an inductive system of C^*-algebras and each ϕ_i is unital, then the direct limit C^*-algebra A of the system does not have IBN if at least one of the A_i does not have IBN.

The proof of this corollary is Proposition 3.27 applied to the canonical map $\phi_i : A_i \to A$, which is unital. This result is the counterpart to Proposition 3.18.

Proposition 3.29. If A and B are C^*-algebras without IBN then $type(A \oplus B) = type(A) \vee type(B)$.

Proof. Denote $type(A) = (N_A, K_A)$, $type(B) = (N_B, K_B)$, $type(A \oplus B) = (n, k)$, $N = \max(N_A, N_B)$, and $K = lcm(K_A, K_B)$.

Since there are natural unital $*$-homomorphisms from $A \oplus B$ to A and B we conclude via Proposition 3.27 that $A \oplus B$ does not have IBN, $type(A) \leq type(A \oplus B)$, and $type(B) \leq type(A \oplus B)$. In particular, $N_A \leq n$ and $N_B \leq n$ so $N := \max(N_A, N_B) \leq n$. As $A^N \simeq A^{N+iK_A}$ and $B^N \simeq B^{N+jK_B}$ for any positive integers i, j we have

$$(A \oplus B)^N = A^N \oplus B^N \simeq A^{N+K_AK_B} \oplus B^{N+K_AK_B} = (A \oplus B)^{N+K_AK_B}$$

and so $N \geq n$, hence $n = N$.

As $type(A) \leq type(A \oplus B)$ we have $k \equiv 0 \mod K_A$ and $k \equiv 0 \mod K_B$ and so $k \equiv 0 \mod K$. Now because K is a multiple of K_A and K_B we have

$$(A \oplus B)^{N+K} = A^{N+K} \oplus B^{N+K} \simeq A^N \oplus B^N = (A \oplus B)^N.$$

By the minimality of k we must conclude that $K \geq k$ and that, combined with $k \equiv 0 \mod K$, requires $k = K$. $\square$

Theorem 3.30. If A and B are C^*-algebras without IBN then

$$type(A \otimes B) \leq type(A) \wedge type(B).$$

Proof. Observe that we have the unital $*$-homomorphisms $a \mapsto a \otimes 1_B$ and $b \mapsto 1_A \otimes b$. Proposition 3.27 has us conclude that $type(A \otimes B) \leq type(A)$ and $type(A \otimes B) \leq$

type(B), hence the result. $\square$

Two comments are in order. First, at this time we do not know under what conditions, if any, inequality could occur. All examples to date have seen equality of the types. Second, the argument works for any cross norm on $A \otimes B$ and it would be quite interesting if $type(A \otimes_\lambda B) \neq type(A \otimes_\kappa B)$ for different cross-norms λ and κ. We also always have $type(A \otimes_\lambda B) \leq type(A \otimes_{\max} B) \leq type(A) \wedge type(B)$ for any cross norm λ.

3.2.2 Existence of Basis Types

In [18] Leavitt proved that for an arbitrary module type there is a ring with that type. We shall do the same for the basis type of C^*-algebras.

Theorem 3.31. For Basis Type (N, K) there exists a C^*-algebra A with $type(A) = (N, K)$.

Due to Theorem 3.29 it is enough to exhibit C^*-algebras of basis types $(N, 1)$ and $(1, K)$ for arbitrary $N, K \geq 1$. We have already seen that the Cuntz algebra $\mathcal{O}_{K+1}$ has basis type $(1, K)$ and so it is enough now to find algebras of types $(N, 1)$. To do so we will first make note of two results due to Rørdam.

Theorem 3.32. [25, Theorem 3.5] Let A be a simple, σ-unital C^*-algebra with stable rank one. Then the multiplier algebra of A is finite if A is non-stable and is properly infinite if A is stable.

Theorem 3.33. [24, Theorem 5.3] For each integer $n \geq 2$ there exists a C^*-algebra B such that $M_n(B)$ is stable and $M_k(B)$ is non-stable for $1 \leq k < n$. Moreover, B may be chosen to be σ-unital and with stable rank one.

Combining Theorem 3.32 and Theorem 3.33 we obtain the following result.

Lemma 3.34. Compare to [25, Example 4.3]. For each $n \geq 2$ there is a unital C^*-algebra A such that $M_k(A)$ is finite for $1 \leq k < n$ and $M_n(A)$ is properly infinite.

Proof. Given $n \geq 2$ let B be the C^*-algebra obtained from Theorem 3.33 which is σ-unital and has stable rank one. Note that $M_n(B)$ is simple, stable, σ-unital, and has stable rank one. Hence by Theorem 3.32 $M(M_n(B)) = M_n(M(B))$ is properly infinite. Similarly for each $1 \leq k < n$, $M_k(B)$ is simple, non-stable, σ-unital, and has stable rank one. Theorem 3.32 has us conclude that $M(M_k(B)) = M_k(M(B))$ is finite.

Thus setting $A = M(B)$ we have that A is a unital C^*-algebra for which $M_k(B)$ is finite precisely when $1 \leq k < n$ and $M_n(A)$ is properly infinite. □

In fact, using yet another result of Rørdam [24, Proposition 2.1] we may conclude that $M_k(A)$ is properly infinite for all $k \geq n$. This is more than is necessary, however, and we now have the tools we need to prove existence of Basis Types.

Theorem 3.35. For each $n \geq 1$ there exists a C^*-algebra of basis type $(n, 1)$.

Proof. The case $n = 1$ is satisfied by the Cuntz algebra $\mathcal{O}_2$. Given $n \geq 2$, let A be the C^*-algebra obtained from Lemma 3.34. Recalling from the construction that $M_n(A) = M(M_n(B))$ and $M_n(B)$ is stable, we conclude (see [4, Prop. 12.2.1]) that $K_0(A) = K_0(M_n(A)) = 0$. Thus A does not have IBN and is of basis type $(N, 1)$ for some $N \geq 1$ ($K = 1$ by Theorem 3.26). It remains to show that $N = n$.

Since $K_0(M_n(A)) = 0$ and $M_n(A)$ is properly infinite we conclude by [27, Prop. 4.2.3] that there is a unital embedding of $\mathcal{O}_2$ into $M_n(A)$. By Proposition 2.25 we conclude that $M_n(A) \simeq M_n(A)^2$, i.e. there is a unitary

$$u \in L(M_n(A), M_n(A)^2) = M_{1,2}(M_n(A)) = M_{n,2n}(A).$$

But thus we have $A^n \simeq A^{2n}$ and so $N \leq n$.

Suppose that $N < n$. By definition there is $j > 0$ for which $A^N \simeq A^{N+j}$. By Corollary 3.22 we may find $J > 1$ for which $A^N \simeq A^{JN}$, hence there is a unitary in

$$M_{N,JN}(A) = M_{1,J}(M_N(A)) = L(M_N(A), M_N(A)^J).$$

Thus, by Proposition 2.25, $\mathcal{O}_J$ embeds unitally into $M_N(A)$. However, by construction $M_N(A)$ is finite when $N < n$ and thus no such embedding is possible. Thus $N = n$ as desired. $\square$

3.2.3 Universal Algebras for Basis Types

In this section we will construct a family of C^*-algebras which are "universal" for the Basis Types in a particular sense. For positive integers n and m we will define $U^{nc}_{m,n}$ to be the C^*-algebra generated by the family $\{u_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq m\}$ with the relations necessary to make $U = [u_{ij}]$ an $m \times n$ unitary matrix, i.e. $UU^* = I_m$ and $U^*U = I_n$.

Proposition 3.36 (Noted by McClanahan [21])**.** For each n and m, the C^*-algebra $U^{nc}_{m,n}$ enjoys the following universal property: whenever A is a C^*-algebra with elements $\{a_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ such that $[a_{ij}] \in M_{m,n}(A)$ is a unitary matrix then there exists a unital $*$-homomorphism $\phi : U^{nc}_{m,n} \to A$ which sends each u_{ij} to a_{ij}.

There is a natural $*$-isomorphism of $U^{nc}_{m,n}$ and $U^{nc}_{n,m}$ ($u_{ij} \mapsto u^*_{ji}$) and so we will only consider the cases when $n > m$. The cases when $m = 1$ are precisely the Cuntz algebras, $U^{nc}_{1,n} = \mathcal{O}_n$, since elements of a "row unitary" satisfy precisely the Cuntz relations.

As we have already seen, the relationship $A^n \simeq A^m$ guarantees that there is a unitary in $M_{m,n}(A)$ (and vice-versa) and so any C^*-algebra with Basis Type $(m, n-m)$ has such a unitary matrix.

Theorem 3.37. If $type(A) = (m, n-m)$ then there is a unital $*$-homomorphism $\phi : U^{nc}_{m,n} \to A$.

The proof is simply applying the universal property of $U^{nc}_{n,m}$.

Since $U^{nc}_{m,n}$ itself has a unitary $m \times n$ matrix we conclude that $(U^{nc}_{m,n})^m \simeq (U^{nc}_{m,n})^m$ and so $U^{nc}_{m,n}$ does not have Invariant Basis Number.

Theorem 3.38. $U^{nc}_{m,n}$ has Basis Type $(m, n-m)$.

Proof. The relationship $(U^{nc}_{m,n})^m \simeq (U^{nc}_{m,n})^n$ guarantees that $type(U^{nc}_{m,n}) \leq (m, n-m)$. Theorem 3.37 gives a unital $*$-homomorphism $\phi : U^{nc}_{m,n} \to A$. By Proposition 3.27 we thus have $type(U^{nc}_{m,n}) \geq type(A) = (m, n-m)$ and so equality is achieved. $\square$

3.3 Perturbations of Algebras

Our goal for this section is to prove that the Basis Types are preserved under small perturbations. We shall begin with a lemma.

Lemma 3.39. Let H be a Hilbert space. For $T \in B(H^n, H^m)$ we have $T = [T_{ij}] \in M_{m,n}(B(H))$ and

$$||T||_{B(H^n,H^m)} \leq n\sqrt{m} \cdot \max_{i,j} ||T_{ij}||.$$

Proof. That $B(H^n, H^m) = M_{m,n}(B(H))$ is obvious, so the content of the lemma is the norm estimate. Fix a unit vector $h = (h_j) \in H^n$ and note that

$$\begin{aligned}
||Th||^2 &= \sum_{i=1}^{m} \left\| \sum_{j=1}^{n} t_{ij} h_j \right\|^2 \\
&\leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} ||t_{ij} h_j|| \right)^2 \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} ||t_{ij} h_j|| \cdot ||t_{ik} h_k|| \\
&\leq \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} ||t_{ij}|| \cdot ||h_j|| \cdot ||t_{ik}|| \cdot ||h_k|| \\
&\leq \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} ||t_{ij}|| \cdot ||t_{ik}|| \\
&\leq \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\max_{i,j} ||t_{ij}|| \right)^2 \\
&= mn^2 \left(\max_{i,j} ||t_{ij}|| \right)^2 .
\end{aligned}$$

Hence $||Th|| \leq n\sqrt{m} \cdot \max_{i,j} ||t_{ij}||$ for any unit vector, giving the result. □

Lemma 3.40. If $T = \left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right] \in B(H \oplus K)$ is invertible then $a \in B(K, H)$ and $b \in B(H, K)$ are invertible.

Proof. Let $T^{-1} = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$. Then

$$TT^{-1} = \begin{bmatrix} az & aw \\ bx & by \end{bmatrix} = \begin{bmatrix} I_H & 0 \\ 0 & I_K \end{bmatrix} = T^{-1}T = \begin{bmatrix} yb & xa \\ wb & za \end{bmatrix}.$$

Hence $az = yb = I_H$ and $za = by = I_K$ so $z = a^{-1}$ and $y = b^{-1}$. In particular $T^{-1} = \left[\begin{smallmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{smallmatrix}\right]$.

□

For two subspaces X and Y of a Banach space $\mathcal{X}$ the *Hausdorff distance* is defined as

$$\Delta(X,Y) = \max\left(\sup_{x\in X_1}\inf_{y\in Y_1}||x-y||, \sup_{y\in Y_1}\inf_{x\in X_1}||x-y||\right)$$

where $X_1 := X \cap \{z \in \mathcal{X} : ||z|| = 1\}$ and similarly for Y_1. This forms a metric on the powerset $\mathcal{P}(\mathcal{X})$.

For our purposes X and Y will always be C^*-subalgebras of $\mathcal{X} = B(H)$. Note that for $a \in A$ with $||a|| = \alpha \in (0,1)$ then we have $a\alpha^{-1} \in A \cap B(H)_1$. So there is $b \in B \cap B(H)_1$ with $||a\alpha^{-1} - b|| \leq \Delta(A,B)$ and then

$$||a - \alpha b|| = \alpha||a\alpha^{-1} - b|| \leq \alpha\Delta(A,B) < \Delta(A,B).$$

Thus the interior behaviors of the unit balls are also well behaved.

We are now ready to prove the main theorem for this section.

Theorem 3.41. Let A and B be unital C^*-subalgebras of $B(H)$ for some Hilbert space H. If A has Basis Type (N,K) and $\Delta(A,B) < (2N+K)^{-\frac{3}{2}}$ then B has Basis Type (N,K) as well.

Proof. As *type*$(A) = (N,K)$ we have the existence of a unitary $U = [u_{ij}] \in M_{N,N+K}(A)$. Note that because U is unitary we have $||u_{ij}|| \leq 1$ for all i,j hence there are elements $w_{ij} \in B$ such that $||u_{ij} - w_{ij}|| \leq \Delta(A,B)$ for all i,j. Set $W = [w_{ij}] \in M_{N,N+K}(B)$,

$$U' = \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix} \in M_{2N+K}(A),$$

and

$$W = \begin{bmatrix} 0 & W \\ W^* & 0 \end{bmatrix} \in M_{2N+K}(B).$$

By Lemma 3.39 we conclude that

$$||U' - W'|| \leq (2N + K)^{\frac{3}{2}} \Delta(A, B) < 1.$$

Recall the standard fact that if x is an invertible element of a Banach algebra X and $y \in X$ is such that $||y - x|| < ||x||^{-1}$ then y is invertible. Thus, since U' is unitary and $||U' - W'|| < 1$ we conclude that W' is invertible in $B(H^{2N+K})$. Since $M_{2N+K}(B)$ is a unital sub-C^*-algebra of $B(H^{2N+K})$ we conclude that W' is invertible in $M_{2N+K}(B)$. An application of Lemma 3.40 yields that $W'^{-1} \in M_{2N+K}(B)$ has the form $W'^{-1} = \begin{bmatrix} 0 & (W^*)^{-1} \\ W^{-1} & 0 \end{bmatrix}$ and so, in particular, W is invertible in $M_{N,N+K}(B) \subset B(H^{N+K}, H^N)$.

The polar decomposition of W in $M_{N,N+K}(B)$ yields a unitary in $M_{N,N+K}(B)$, hence $B^N \simeq B^{N+K}$ and B lacks IBN. Denoting *type*$(B) = (N_B, K_B)$, we have from the equivalence $B^N \simeq B^{N+K}$ that $N_B \leq N$ and $(N + K) - N \equiv 0 \mod K_B$, i.e. K_B divides K. This is precisely what's required for *type*$(B) = (N_B, K_B) \leq (N, K) =$ *type*(A).

Suppose for the sake of contradiction that *type*$(B) = (N_B, K_B) < (N, K)$. Then in particular $N_B < N$ and $N_B + K_B < N + K$, hence $(2N + K)^{-\frac{3}{2}} < (2N_B + K_B)^{-\frac{3}{2}}$. Since $\Delta(A, B) = \Delta(B, A)$ we may, remarkably, take the above arguments and apply them to conclude that *type*$(A) \leq$ *type*(B), whence *type*$(B) =$ *type*(A). □

The greatest benefit of the above theorem is that the measure of "closeness" required to preserve Basis Type does not depend on anything about A except its Basis Type. Note that we may only conclude that *lack* of IBN is preserved under small perturbations. As of this moment it seems quite possible that given $\epsilon > 0$ we might find a C^*-algebra A with IBN and a C^*-algebra B without IBN such that $\Delta(A, B) < \epsilon$. Note that in such a situation the Basis Type of B would "grow" with small ϵ.

3.4 IBN for non-unital C^*-algebras

So far our considerations with IBN have only dealt with unital C^*-algebras. This mirrors the classical theory of IBN in noncommutative ring theory where, per usual, rings with unit are of primary importance. However, in the theory of C^*-algebras there is no particular reason to restrict our attention this way. In this section we consider non-unital C^*-algebras and formulate an appropriate notion of Invariant Basis Number for them. We shall recall two ways in which a non-unital C^*-algebra may be "given" a unit and determine which is best for IBN considerations.

3.4.1 The Unitization

Given a C^*-algebra A the *unitization* of A is constructed as follows. Consider the set $\tilde{A} := \mathbb{C} \oplus A$ endowed with coordinate-wise vector space structure and the following operations:

1. $(\lambda, a) \cdot (\tau, b) = (\lambda\tau, \lambda b + \tau a + ab)$
2. $(\lambda, a)^* = (\overline{\lambda}, a^*)$
3. $||(\lambda, a)|| := \sup\{||ab + \lambda b||_A : ||b||_A \leq 1\}$.

One may check that these give $\tilde{A}$ the structure of a Banach $*$-algebra and that $||\cdot||$ is a norm which satisfies the C^*-condition. Noticing that $(1, 0) \cdot (\lambda, a) = (\lambda, a)$ we conclude that $\tilde{A}$ is a unital C^*-algebra.

Example 3.42. When $A = C(0, 1)$ we obtain $\tilde{A} = C(\mathbb{T})$. In general, if $A = C(X)$ with X non-compact then $\tilde{A} = C(X \cup \{\infty\})$ (the one-point compactification) where the "copy" of $\mathbb{C}$ plays the role of constant functions.

If A is non-unital then $\tilde{A}$ is $*$-isomorphic to $A \oplus \mathbb{C}$. However, if A is unital then $||(-\lambda, \lambda I_A)|| = \sup ||\lambda b \lambda b|| = 0$ and so $(\lambda, a) = (0, \lambda I_A + a)$, i.e. $\tilde{A}$ is isomorphic to A.

Proposition 3.43. If A is non-unital then $\tilde{A}$ has IBN.

Proof. It is not hard to check that the map $(\lambda, a) \mapsto \lambda$ is a unital $*$-homomorphism from $\tilde{A}$ to $\mathbb{C}$. Since $\mathbb{C}$ has IBN we conclude by Corollary 3.8 that $\tilde{A}$ has IBN as well.

Note that the lack of a unit is necessary, else the homomorphism is not well defined as, e.g., $(1, 0) = (0, I_A)$. $\square$

We therefore conclude that the unitization $\tilde{A}$ is not the proper C^*-algebra to consider when defining IBN for non-unital A.

3.4.2 The Multiplier Algebra

For any C^*-algebra A we will define the *multiplier algebra* of A, denoted $M(A)$ as the unique unital C^*-algebra containing A as an essential ideal (i.e. has nontrivial intersection with all other ideals) and which satisfies the following universal property: if B is a C^*-algebra containing A as an essential ideal then there is a unique $*$-homomorphism $M(A) \to B$ which restricts to the identity on the copies of A.

It is a wonderful fact that when A is represented faithfully and irreducibly on some Hilbert space H the set (known as the the *idealizer*)

$$M = \{x \in B(H) : xA \subset A, Ax \subset A\}$$

is isomorphic to $M(A)$.

If A is unital then $M(A) = A$, but if A is nonunital then $M(A)$ is in general far, far larger. For example, the multiplier of the compacts $\mathbb{K}$ is all of $B(H)$.

That there is a connection between the theory of multiplier algebras and the theory of Hilbert C^*-modules is surprising but highly useful for our purposes. Recall that $L(X)$ is the collection of adjointable A-module endomorphisms of X and has the structure of a unital C^*-algebra. For $x, y \in X$ define $\theta_{x,y} \in L(X)$ by $\theta_{x,y}(z) = x\langle y, z\rangle$. In the literature $\theta_{x,y}$ is referred to as a "rank one operator" and the subalgebra $\mathbb{K}(X) := \{\theta_{x,y} : x, y \in X\} \subset L(X)$ is itself a C^*-algebra known as the "compact homomorphisms" on X. Now considering A (unital or nonunital) as a Hilbert A-module we can always make the identification $A = \mathbb{K}(A)$ where $\theta_{a,b}$ is paired with (left) multiplication by ab^*. If A were unital then $A = \mathbb{K}(A) \subseteq L(A) = A$ and so $\mathbb{K}(A) = L(A)$. It turns out that in the nonunital case we can identify $L(A)$ as a multiplier algebra.

Proposition 3.44 (Theorem 2.4 in [15])**.** If X is a Hilbert A-module X then $L(X) = M(\mathbb{K}(X))$.

With $X = A$ we have that $L(A) = M(\mathbb{K}(A)) = M(A)$. Following identical arguments to Example 2.12 we have the following corollary.

Corollary 3.45. $L(A^n, A^m) = M_{m,n}(M(A))$ for any n, m.

Thus the question of equivalence of standard modules over a non-unital C^*-algebra A can, as in the unital case, be reduced to matrix considerations.

Corollary 3.46. $A^n \simeq A^m$ if and only if there is a unitary matrix in $M_{m,n}(M(A))$.

Since $M(A)$ is unital, presence (or lack thereof, technically) of unitary matrices is enough to determine if $M(A)$ has IBN or not. Thus we have our characterization of non-unital Invariant Basis Number.

Theorem 3.47. Let A be a non-unital C^*-algebra. The following are equivalent:

1. for all $n, m \geq 1$ we have $A^n \simeq A^m$ if and only if $n = m$

2. $M(A)$ has IBN.

Thus we will extend our definition of Invariant Basis Number to include non-unital C^*-algebras by stipulating that a non-unital C^*-algebra has IBN if its multiplier algebra has IBN.

Example 3.48. If $A = C_0(X)$ is a non-unital commutative C^*-algebra then $M(A) = C(\beta X)$ where βX is the Stone-Čech compactification of X. In particular $M(A)$ is commutative and so has IBN. Thus A has IBN.

Example 3.49. The compact operators $\mathbb{K}$ have $M(\mathbb{K}) = B(H)$. As $B(H)$ does not have IBN we conclude that K does not have IBN. Further, since $B(H)^n \simeq B(H)^m$ for any $n, m \geq 1$ we conclude that $\mathbb{K}^n \simeq \mathbb{K}^m$ for all $n, m \geq 1$.

Example 3.50. For any given C^*-algebra A consider the stabilization $A \otimes \mathbb{K}$. This is a non-unital algebra so consider $M(A \otimes \mathbb{K})$, which "contains" a copy of $M(\mathbb{K}) = B(H)$. Thus we should not be surprised that $K_0(M(A \otimes \mathbb{K}))$ is trivial [4, Proposition 12.2.1] and so the stabilized C^*-algebra $A \otimes \mathbb{K}$ does not have IBN.

We may generalize the previous example by recalling that a C^*-algebra A is *stable* if $A \cong A \otimes \mathbb{K}$.

Proposition 3.51. Any stable C^*-algebra cannot have IBN.

The proof is entirely the identification of $K_0(M(A)) = K_0(M(A \otimes \mathbb{K})) = \{0\}$.

We would like to conclude this section with the observation that some of our results for unital C^*-algebras and IBN do not carry over with this new definition for non-unital IBN. For example, it is not true that a stably finite non-unital C^*-algebra must have IBN. To see this simply consider the compacts $\mathbb{K}$: this is a stably finite

C^*-algebra but does not have IBN since $M(\mathbb{K}) = B(H)$ does not have IBN. This particular discrepancy is partially due to the fact that the property of being stably finite is not always preserved in a multiplier algebra.

3.5 Covariant Representations and Basis Type

We'll now consider how C^*-modules can be concretely realized as operators on Hilbert space. To begin, let A be a unital C^*-algebra and X a Hilbert A-module.

Definition 3.52. A *covariant representation* of X is a pair (σ, π) consisting of

- a nondegenerate $*$-representation $\pi : A \to B(H)$
- a linear map $\sigma : X \to B(H)$

which together satisfy the covariance relation

$$\sigma(xa) = \sigma(x)\pi(a)$$

for all $x \in X$ and $a \in A$.

Since $\sigma(x) = \sigma(x1_A) = \sigma(x)\pi(1_A)$ we have that $\overline{\pi(A)H}$ is reducing for $\sigma(X)$, hence the nondegeneracy condition on π is not too restrictive.

Definition 3.53. A covariant representation (σ, π) of a Hilbert A-module X is a *Toeplitz representation* if $\pi(\langle x, y\rangle) = \sigma(x)^*\sigma(y)$ for all $x, y \in X$.

Recall the "rank one" operator $\theta_{x,y} \in \mathbb{K}(X) \subset L(X)$ defined by $\theta_{x,y}(z) = x\langle y, z\rangle$. If (σ, π) is a Toeplitz representation for X then for all $z \in X$

$$\sigma(\theta_{x,y}(z)) = \sigma(x\langle y, z\rangle) = \sigma(x)\pi(\langle y, z\rangle) = \sigma(x)\sigma(y)^*\sigma(z)$$

and we can view $\theta_{x,y} \mapsto \sigma(x)\sigma(y)^*$ as a representation of $\mathbb{K}(X)$ on H. By results of Fowler and Raeburn [10, Prop. 1.6] there is a unique $*$-representation $\rho^{\sigma,\pi} : L(X) \to B(H)$ such that

- $\rho^{\sigma,\pi}(T)\sigma(x) = \sigma(Tx)$ for all $T \in L(X)$ and $x \in X$, and
- $\rho^{\sigma,\pi}(\theta_{x,y}) = \sigma(x)\sigma(y)^*$.

Definition 3.54. A representation (σ, π) is *completely coisometric* if it is Toeplitz and $\rho^{\sigma,\pi}(id_X) = I$.

Proposition 3.55. Let (σ, π) be a covariant representation of a Hilbert A-module X and $U \in L(X)$ a unitary. Then $(\sigma \circ U, \pi)$ is a covariant representation of X which is Toeplitz (resp. completely coisometric) if (σ, π) is Toeplitz (resp. completely coisometric).

Proof. To show that $(\sigma \circ U, \pi)$ is a covariant representation of X is a simple exercise.

Suppose that (σ, π) is Toeplitz. Then for any $x, y \in X$ we have

$$[(\sigma \circ U)(x)]^*[(\sigma \circ U)(y)] = \sigma(Ux)^*\sigma(Uy) = \pi(\langle Ux, Uy\rangle) = \pi(\langle x, y\rangle)$$

where the last equality is because U is unitary, hence isometric. Thus $(\sigma \circ U, \pi)$ is Toeplitz.

Supposing that (σ, π) is completely coisometric, we have that

$$\rho^{\sigma\circ U,\pi}(\theta_{x,y}) = \sigma(Ux)\sigma(Uy)^* = \rho^{\sigma,\pi}(U)\sigma(x)\sigma(y)^*\rho^{\sigma,\pi}(U^*) = \rho^{\sigma,\pi}(U\theta_{x,y}U^*)$$

for all $\theta_{x,y} \in \mathbb{K}(X)$. By the uniqueness of ρ we have $\rho^{\sigma\circ U,\pi}(T) = \rho^{\sigma,\pi}(UTU^*)$ for all $T \in L(X)$. Thus

$$\rho^{\sigma\circ U,\pi}(id_X) = \rho^{\sigma,\pi}(UU^*) = \rho^{\sigma,\pi}(id_X) = I$$

and so $(\sigma \circ U, \pi)$ is completely coisometric. $\square$

Definition 3.56. Two covariant representations (σ, π) and (τ, π) are *equivalent*, written $(\sigma, \pi) \sim_u (\tau, \pi)$ if there is a unitary $U \in L(X)$ such that $\sigma = \tau \circ U$.

That this defines an equivalence relation on covariant representations of X is easy to check. By the previous proposition $\sim_u$ preserves Toeplitz and completely coisometric representations.

3.5.1 Representations of Standard Modules

Fix a unital C^*-algebra A for the remainder of this section. Consider a standard module A^n and its standard basis $\{e_1, ..., e_n\}$.

For Toeplitz representations of A^n it is enough to focus entirely on the map σ, as if (σ, π_1) and (σ, π_2) were both Toeplitz then

$$\pi_1(a) = \pi_1(\langle e_1, e_1 a\rangle) = \sigma(e_1)^* \sigma(e_1 a) = \pi_2(\langle e_1, e_1 a\rangle) = \pi_2(a)$$

for all $a \in A$. This calculation also shows that if $\sigma_1 \sim_u \sigma_2$ for Toeplitz representations (σ_1, π_1) and (σ_2, π_2) then $\pi_1 = \pi_2$. Thus, for the rest of this section we will fix a nondegenerate $*$-representation $\pi : A \to B(H)$.

For a Toeplitz representation (σ, π) of A^n we have

$$\sigma(e_i)^* \sigma(e_j) = \pi(\langle e_i, e_j\rangle) = \pi(\delta_{ij} 1_A) = \delta_{ij} I$$

hence $\{\sigma(e_1), ..., \sigma(e_n)\}$ is a family of mutually orthogonal isometries (i.e. a Toeplitz-Cuntz family) in $B(H)$.

If (σ, π) is a completely coisometric representation of A^n then, recalling that $id_{A^n} = I_n = \sum_{i=1}^n \theta_{e_i,e_i}$, we have

$$I = \rho^{\sigma,\pi}(I_n) = \sum_{i=1}^n \rho^{\sigma,\pi}(\theta_{e_i,e_i}) = \sum_{i=1}^n \sigma(e_i)\sigma(e_i)^*$$

and so the family $\{\sigma(e_1), ..., \sigma(e_n)\}$ is in fact a Cuntz family.

Conversely, if $\omega : \mathcal{E}_n \to B(H)$ is a $*$-representation then $\omega(v_1), ..., \omega(v_n)$, where $v_1, ..., v_n$ are the generators of $\mathcal{E}_n$, is a Toplitz-Cuntz family in $B(H)$. The assignment

$$\sigma_\omega(x) := \sum_{i=1}^n \omega(v_i)\pi(\langle e_i, x\rangle)$$

defines a linear map $\sigma_\omega : A^n \to B(H)$ such that (σ_ω, π) is Toeplitz. If $\ker\omega = \mathbb{K} \subset \mathcal{E}_n$ then ω may be thought of as a representation of $\mathcal{O}_n$ and consequently (σ_ω, π) is completely coisometric.

If $\{f_1, ..., f_n\}$ is another basis (specifically of size n) of A^n then there is a unitary $U \in M_n(A)$ with $U^*e_i = f_i$. For a fixed representation ω of $\mathcal{E}_n$ we then have

$$\sum_{i=1}^n \omega(v_i)\pi(\langle f_i, x\rangle) = \sum_{i=1}^n \omega(v_i)\pi(\langle e_i, Ux\rangle) = \sigma_\omega(Ux)$$

Thus we can justify our use of the standard basis in the definition of σ_ω as, up to $\sim_u$ equivalence, it makes no difference what basis of size n is used.

Suppose that ω and τ are two representations of $\mathcal{E}_n$ such that (σ_ω, π) and (σ_τ, π) are $\sim_u$ equivalent. Then

$$\sum_{i=1}^n \omega(v_i)\pi(\langle e_i, x\rangle) = \sum_{i=1}^n \tau(v_i)\pi(\langle e_i, Ux\rangle)$$

and when $x = e_j$

$$\begin{aligned}
\omega(v_j) &= \sum_{i=1}^{n} \tau(v_i)\pi(\langle e_i, Ue_j\rangle) \\
&= \sum_{i=1}^{n} \tau(v_i)\tau(v_i)^*\sigma_\tau(Ue_j) \\
&= \rho^{\sigma_\tau,\pi}(I_n)\rho^{\sigma_\tau}(U)\tau(v_j) \\
&= \rho^{\sigma_\tau,\pi}(U)\tau(v_j)
\end{aligned}$$

This brings us to define a new, to our knowledge, notion of equivalence for representations of $\mathcal{E}_n$.

Definition 3.57. Two representations ω and τ of $\mathcal{E}_n$ are *A-free equivalent* if there is a nondegenerate $*$-representation $\pi : A \to B(H)$ and a unitary $U \in M_n(A)$ such that

$$\omega(v_j) = \rho^{\sigma_\tau,\pi}(U)\tau(v_j)$$

for all $j = 1, ..., n$.

By the previous calculation, ω and τ are A-free equivalent if and only if $(\sigma_\omega, \pi) \sim_u (\sigma_\tau, \pi)$. The calculations factor through $\mathcal{O}_n$ if σ is completely coisometric, so A-free equivalence is also well defined for representations of $\mathcal{O}_n$.

Theorem 3.58. Each $\sim_u$ equivalence class of Toeplitz (resp. completely coisometric) covariant representations of A^n corresponds to precisely one A-free equivalence class of representations of $\mathcal{E}_n$ (resp. $\mathcal{O}_n$).

Note that if X and Y are Hilbert A-modules and $U \in L(Y, X)$ a unitary then a covariant representation (σ, π) of X gives a covariant representation $(\sigma \circ U, \pi)$ of Y and vice versa. It is an easy exercise, mimicking Proposition 3.55, to show that

$(\sigma \circ U, \pi)$ is Toeplitz (resp. completely coisometric) if and only if (σ, π) is Toeplitz (resp. completely coisometric). Thus the $\sim_u$ equivalence classes of X are in one-to-one correspondence with those for Y. We'll extend the use of $\sim_u$ to include unitarily equivalent modules, i.e. $(\sigma, \pi) \sim_u (\tau, \pi)$ if (σ, π) is a covariant representation of X, (τ, π) a covariant representation of Y, $X \simeq Y$, and $\sigma = \tau \circ U$ for some unitary $U \in L(X, Y)$.

When A has Basis Type (N, K) every standard module is unitarily equivalent to a standard module A^n for $n \in \{1, ..., N + K - 1\}$, hence the representation theory of standard modules is reduced to a finite number of cases. This is made precise in our next theorem, with which we will end this section.

Theorem 3.59. Let A be a C^*-algebra of Basis Type (N, K) and (σ, π) a Toeplitz (resp. completely coisometric) representation of A^n on H. Then there is a unique positive integer $L \leq N + K - 1$ and a representation $\omega : \mathcal{E}_L \to B(H)$ such that $(\sigma, \pi) \sim_u (\sigma_\omega, \pi)$ where (σ_ω, π) is the Toeplitz (resp. completely coisometric) representation of A^L induced by ω. The representation ω is unique up to A-free equivalence.

Proof. Because $type(A) = (N, K)$ we have that $A^n \simeq A^L$ for a unique $L \leq N+K-1$. Let $U \in M_{L,n}(A)$ be a unitary, then $(\sigma \circ U^*, \pi)$ is a covariant representation of A^L which is Toeplitz and hence $\sigma \circ U^* = \sigma_\omega$ for some representation ω of $\mathcal{E}_L$. All that remains is to observe that $\sigma = \sigma \circ U^*U = \sigma_\omega \circ U$.

Suppose that τ is A-free equivalent to the representation ω above. Then, as noted previously, $\sigma_\tau = \sigma_\omega \circ W$ for some $W \in M_L(A)$. Then all we need observe is that $W^*U \in M_{L,n}(A)$ is unitary and

$$\sigma = \sigma_\omega \circ U = \sigma_\omega \circ WW^*U = \sigma_\tau \circ W^*U.$$

□

3.6 Rank Condition and Stable Finite-ness

The property of Invariant Basis Number is one of a family of "finite-ness" conditions in noncommutative ring theory. In [6] Cohn defines and investigates several of these notions, and in [14, Chapter 1] Lam gives a detailed account of Cohn's properties and several more. In this section we will reformulate and develop several of these notions in the context of C^*-modules.

Definition 3.60. A C^*-algebra satisfies the *rank condition* if whenever $A^n \simeq A^m \oplus X$ is satisfied for some positive integers n, m and A-module X we necessarily have $n \geq m$.

The term "rank condition" follows the exposition of Lam. Cohn terms this property "IBN_1." First, we shall prove an analogue of Theorem 3.7.

Proposition 3.61. If A and B are C^*-algebras, $\phi : A \to B$ is a unital $*$-homomorphism, and B satisfies the rank condition, then A also satisfies the rank condition.

Proof. Suppose that A does not satisfy the rank condition, i.e. $A^n \simeq A^m \oplus X$ for some $m > n > 0$ and A-module X. Denote by ψ the unitary map implementing the above equivalence and consider the natural inclusion $i_m : A^m \hookrightarrow A^m \oplus X$ defined by $i_m((a_i)) = ((a_i), 0)$. Then $\psi \circ i_m : A^m \to A^n$ is a A-module isometry. As such, $\psi \circ i_m$ is implemented by a matrix $V \in M_{m,n}(A)$ which satisfies $V^*V = I_m$. As ϕ is a unital $*$-homomorphism the entry-wise image $\phi(V) \in M_{m,n}(B)$ also satisfies $\phi(B)^*\phi(B) = I_m$ and so corresponds to an B-module isometry $\beta : B^m \to B^n$. Now the range of β (unitarily equivalent to B^m) is a submodule of B^n and, as the range of an adjointable homomorphism, is complementable with orthogonal complement $\ker \beta^*$. By the Polar

Decomposition (Theorem 2.9) $B^n \simeq B^m \oplus \ker \beta^*$ contradicting our hypothesis that B satisfies the rank condition. $\square$

Within the proof of Proposition 3.61 lies a key insight: the equivalence $A^n \simeq A^m \oplus X$ gives rise to a proper isometry in $M_{m,n}(A)$. As a consequence, we conclude that all commutative C^*-algebras must satisfy the rank condition as any rectangular matrix over such an algebra cannot be right-invertible.

Theorem 3.62. A C^*-algebra A is stably finite if and only if whenever X is an A-module such that $A^n \simeq A^n \oplus X$ (for some $n \geq 1$) it is necessary that $X = \{0\}$.

Proof. Suppose that $A^n \simeq A^n \oplus X$ for some $n \geq 1$ and nontrivial A-module X with the equivalence implemented by a unitary $\phi \in L(A^n \oplus X, A^n)$. The coordinate embedding $i_n : A^n \hookrightarrow A^n \oplus X$ is isometric and adjointable, hence the composition $\phi \circ i_n \in L(A^n)$ is isometric and not surjective. Now $\phi \circ i_n$ has a matricial representation $U \in M_n(A)$ and U is a proper isometry. Hence A is not stably finite.

Similarly, if there is a proper isometry $V \in M_n(A)$ for some $n \geq 1$ then V corresponds to an isometric homomorphism $\phi_V \in L(A^n)$. Since ϕ_V is proper we have that $\ker \phi_V^*$ is a nontrivial submodule of A^n, hence the decomposition $A^n = \phi_V(A^n) \oplus \ker \phi_V^* \simeq A^n \oplus \ker \phi_V^*$ contradicts our hypotheses. $\square$

The module condition in the above Theorem is how Lam defines a stably finite ring. Cohn refers to this as "IBN_2."

That stable finite-ness and the rank condition are related both to IBN and to each other comes as no surprise.

Theorem 3.63. All stably finite unital C^*-algebras satisfy the rank condition; and all unital C^*-algebras which satisfy the rank condition have IBN.

Proof. Suppose that A is a nontrivial stably finite C^*-algebra. If $m > n \geq 0$ and $A^n \simeq A^m \oplus X$ for some A-module X then $A^n \simeq A^n \oplus A^{m-n} \oplus X$ as well and we would conclude that $A^{m-n} \oplus X = 0$, a contradiction as $m - n > 0$.

Suppose that A is a C^*-algebra which satisfies the rank condition. If A did not have IBN then $A^n \simeq A^m$ for some $m > n > 0$, hence $A^n \simeq A^m \oplus 0$ a contradiction. $\square$

The following examples demonstrate that the three properties are distinct.

Example 3.64. The Toeplitz algebra $\mathcal{T}$ has been demonstrated to have IBN. As $\mathcal{T}$ contains a non-unitary isometry it is not finite and hence is not stably finite. Since $\mathcal{T}$ is an extension of the commutative C^*-algebra $C(\mathbb{T})$ we conclude via Proposition 3.61 that $\mathcal{T}$ satisfies the rank condition.

Example 3.65. Consider the higher-order Toeplitz algebra $\mathcal{E}_2$ which is the C^*-algebra generated by a pair of isometries, v_1 and v_2, with mutually orthogonal ranges.

The map $\phi : \mathcal{E}_2^2 \to \mathcal{E}_2$ defined by $\phi(x, y) = v_1 x + v_2 y$ is an $\mathcal{E}_2$-module homomorphism with adjoint $\phi^* : \mathcal{E}_2 \to \mathcal{E}_2^2$ given by $\phi^*(z) = (v_1^* z, v_2^* z)$. It is easily seen to be an isometry. A consequence of the Polar Decomposition (Theorem 2.9) is that

$$\mathcal{E}_2 \simeq \ker \phi^* \oplus \overline{\phi(\mathcal{E}_2^2)} \simeq \ker \phi^* \oplus \mathcal{E}_2^2$$

Now $\ker \phi^*$ is nontrivial as it contains $I - v_1 v_1^* - v_2 v_2^*$ and in fact coincides with the unique maximal ideal (a.k.a. submodule) $\mathbb{K}$ of $\mathcal{E}_2$. In any case, we have demonstrated that $\mathcal{E}_2$ does not satisfy the rank condition. Cuntz [8] has shown that $K_0(\mathcal{E}_2) = \mathbb{Z}$ and is generated by $[1]_0$ which has us conclude that $\mathcal{E}_2$ does have IBN.

Chapter 4

Applications: Classification and Dynamical Systems

4.1 Classification

Any attempt to describe the importance of the classification program in C^*-algebras would almost surely end up understating its impact. The program was initiated by Elliot with the goal of classifying C^*-algebras using K-theoretic invariants. For a general overview see [27]. At first the invariant for a C^*-algebra A was the group $K_0(A) \oplus K_1(A)$, but ready counterexamples required additional data to be added. Most commonly the *Elliot Invariant* is defined (for unital algebras) as the 4-tuple

$$Ell(A) := ((K_0(A), K_0(A)^+, [1_A]_0), K_1(A), TA, \rho_A)$$

where $K_0(A)^+$ is the image of the semigroup $V(A)$ (see A.0.4), TA the trace space, and ρ_A the pairing of $K_0(A)$ and TA given by evaluation of a trace at a class in K_0.

One of the most celebrated results of the classification program is the Kirchberg-

Phillips Theorem which utilizes the Elliot Invariant to classify C^*-algebras which are separable, amenable, simple, purely infinite, and satisfy the so-called Universal Coefficient Theorem.

Generally speaking, the classification program is interested in *simple* C^*-algebras. Among the many reasons for this is that it is fairly straightforward to construct non-simple C^*-algebras with identical K-theory. One use of the theory of Invariant Basis Number, and Basis Types in particular, is to distinguish these C^*-algebras which K-theory cannot.

Proposition 4.1 (Comment in §3 of [21])**.** For $m > n > 1$ the C^*-algebras $U^{nc}_{n,m}$ are not simple.

Proof. Note that it is possible to create a unitary in $M_{n,m}(\mathcal{O}_{m-n+1})$ with the form

$$\begin{bmatrix} I_{n-1} & 0 & 0 & \dots & 0 \\ 0 & V_1 & V_2 & \dots & V_{m-n+1} \end{bmatrix}.$$

The universal property of $U^{nc}_{n,m}$ thus guarantees a $*$-homomorphism $\phi : U^{nc}_{n,m} \to \mathcal{O}_{m-n+1}$ with $\phi(u_{n,n+i-1}) = V_i$. Thus ϕ is surjective but, as $\mathcal{O}_{m-n+1} \not\cong U^{nc}_{n,m}$ (as they have differing Basis Types!) ϕ is not injective, i.e. $\ker \phi$ is a nontrivial ideal. $\square$

Recently, Ara and Goodearl have proven a conjecture of McClanahan as to the K-theory of these C^*-algebras.

Theorem 4.2 (Comment after Theorem 5.2 in [2])**.** $K_0(U^{nc}_{n,m}) = \mathbb{Z}/(m-n)\mathbb{Z}$ and $K_1(U^{nc}_{n,m}) = \{0\}$

As a consequence, the family $\{U^{nc}_{n+k,m+k} : k > \max(-n, -m)\}$ shares a common K-theory and so any classification using only those invariants is impossible. However, we may naturally distinguish every $U^{nc}_{n,m}$ (with $m > n$) by examining its Basis Type, as we have shown that $type(U^{nc}_{n,m}) = (n, m-n)$ in Theorem 3.38.

4.2 Implementation of Dynamical Systems

An area of investigation in which C^*-module techniques have been particularly fruitful has been that of C^*-dynamical systems.

Definition 4.3. A C^**-dynamical system* is a pair (A, σ) consisting of a C^*-algebra A and a $*$-endomorphism σ of A.

Let (A, σ) be a C^*-dynamical system and $\pi : A \to B(H)$ a nondegenerate $*$-representation. Consider the following space

$$E_\pi = \{T \in B(H) : T\pi(a) = \pi(\sigma(a))T \ for \ all \ a \in A\}.$$

Proposition 4.4 (See [22] among others.)**.** E_π is a C^*-module over the relative commutant $\pi(A)' := \{S \in B(H) : S\pi(a) = \pi(a)S \ for \ all \ a \in A\}$.

Proof. Certainly E_π is a complex vector space. The right action of $\pi(A)'$ on X will be simple multiplication. For $T \in X$ and $S \in \pi(A)'$ we see $TS\pi(a) = T\pi(a)S = \pi(\sigma(a))TS$ and so $TS \in X$. For $T, R \in E_\pi$ and $a \in A$ we have

$$T^*R\pi(a) = T^*\pi(\sigma(a))R = (\pi(\sigma(a))^*T)^*R = (\pi(\sigma(a^*))T)^*R$$

$$= (T\pi(a^*))^*R = \pi(a^*)^*T^*R = \pi(a)T^*R$$

and so $T^*R \in \pi(A)'$. Thus $\langle T, R\rangle := T^*R$ is a $\pi(A)'$-valued inner product on E_π.

Of course the induced norm $||T||_E = ||\langle T, T\rangle||^{\frac{1}{2}}_{B(H)}$ is the operator norm $||T||_{B(H)}$ and so, as one might expect, E_π is not always complete, i.e. not always a Hilbert $\pi(A)'$-module. □

Definition 4.5. A *covariant representation of multiplicity* n of aC^*-dynamical system (A, σ) is a pair $(\pi, \{T_i : i = 1, ..., n\})$ consisting of:

- a nondegenerate $*$-representation $\pi : A \to B(H)$,
- family $T_1, ..., T_n \in B(H)$ of isometries with pairwise orthogonal ranges which satisfy the covariance relation

$$\pi(\sigma(a)) = \sum_{i=1}^{n} T_i \pi(a) T_i^*$$

  for all $a \in A$.

Since π is nondegenerate we have that the family $T_1, ..., T_n$ is a Toeplitz-Cuntz family in $B(H)$.

Example 4.6. Let $A = \ell^\infty(\mathbb{N}) \subset B(\ell^2(\mathbb{N}))$ and σ the "forward shift" defined by $\sigma(f)(1) = f(0)$ and, for $n \geq 2$, $\sigma(f)(n) = f(n-1)$. Then if V is the unilateral shift in $B(\ell^2(\mathbb{N}))$ we can see immediately that $\sigma = \mathrm{Adj}\, V$ and so (A, σ) is implemented by a Toeplitz-Cuntz family of size 1.

Note that if (A, σ) is unital, i.e. A is unital and $\sigma(1_A) = 1_A$, then the implementing Toeplitz-Cuntz family in a covariant representation is in fact a Cuntz family, i.e. $\sum_{i=1}^n T_i T_i^* = I$.

Proposition 4.7. Let (A, σ) be a unital C^*-dynamical system. Then (A, σ) has a multiplicity n covariant representation $(\pi, \{T_i : i = 1, ..., n\})$ on a Hilbert space H if and only if $E_\pi \subset H$ is unitarily equivalent to the standard module $(\pi(A)')^n$.

Proof. As remarked above, the implementing Toeplitz-Cuntz family $V_1, ..., V_n$ is a proper Cuntz family with all the consequent relations. Note that for each V_j and every $a \in A$ we have

$$\pi(\sigma(a))V_j = \sum_{i=1}^{n} V_i \pi(a) V_i^* V_j = V_j \pi(a)$$

and so $V_j \in E_\pi$. Since $\langle V_i, V_j \rangle = V_i^* V_j = \delta_{ij} I$ we have that $V_1, ..., V_n$ is an orthonormal set in E_π.

The submodule of E_π generated by $V_1, ..., V_n$ is unitarily equivalent to $(\pi(A)')^n$. Consider the map $\phi : E_\pi \to (\pi(A)')^n$ defined by

$$\phi(T) = (V_1^* T, ..., V_n^* T)$$

which is certainly surjective and $\pi(A)'$-linear. It is also adjointable with $\phi^* : (\pi(A)')^n \to E_\pi$ defined by

$$\phi^*(x_1, ..., x_n) = \sum_{i=1}^{n} V_i x_i.$$

Note then that

$$\langle \phi(T), \phi(T) \rangle = \langle (V_1^* T, ..., V_n^* T), (V_1^* T, ..., V_n^* T) \rangle = \sum_{i=1}^{n} T^* V_i V_i^* T$$

$$= T^* \left(\sum_{i=1}^{n} V_i V_i^* \right) T = T^* T = \langle T, T \rangle$$

and so ϕ is isometric. By Proposition 2.15 we conclude that ϕ is unitary, hence $E_\pi \simeq (\pi(A)')^n$.

Conversely, suppose that (A, σ) is a unital C^*-dynamical system and $E_\pi \simeq (\pi(A)')^n$ for some $*$-representation π and natural number n. Since, nearly by definition,

$(\pi(A)')^n$ has an orthonormal basis thus so too does E_π. Denote this basis by $V_1, ..., V_n$. Note for $a \in A$ we have

$$\pi(\sigma(a)) = \pi(\sigma(a)) \sum_{i=1}^{n} V_i V_i^* = \sum_{i=1}^{n} \pi(\sigma(a)) V_i V_i^* = \sum_{i=1}^{n} V_i \pi(a) V_i^*.$$

Thus $(\pi, \{V_i : i = 1, ..., n\})$ is a covariant representation of (A, σ). $\square$

4.2.1 Invariance of the Multiplicity

For this section we will consider a C^*-dynamical system (A, σ) with concrete representation $A \subseteq B(H)$ which allows at least one covariant representation $(id, \{T_i\})$. In this particular situation we'll say that (A, σ) is "implemented by a Toeplitz-Cuntz family." We will write E for E_{id} for the remainder.

As (A, σ) is implemented by a Toeplitz-Cuntz family of some multiplicity we have that $E \simeq (A')^n$ for some n, but this n is not necessarily unique! Indeed, if $E \simeq (A')^n \simeq (A')^m$ then (A, σ) is implemented by two Toeplitz-Cuntz families of differing sizes.

Example 4.8. Consider $A := \mathbb{C}I \subset B(H)$ and $\sigma = id_A$. Then $E = B(H) = A'$. $B(H)$ lacks IBN, in fact $B(H) \simeq B(H)^n$ for all n, so σ is implemented by Cuntz families of every size. This is unsurprising, as for any Cuntz family $V_1, ..., V_n \in B(H)$ we see

$$\lambda = \lambda I = \lambda \sum_{i=1}^{n} V_i I V_i^* = \sum_{i=1}^{n} V_i \lambda I V_i^*.$$

In general, given a C^*-algebra $A \subset B(H)$ calculating A' is a highly nontrivial task. However, A' is a von Neumann algebra and, as such, has relatively well-behaved K-theory. In particular, factors (von Neumann algebras B for which $B \cap B' = \mathbb{C}$) have very precise K_0 groups: $\mathbb{Z}$ for Type I_n, $\mathbb{R}$ for Type II_1, and trivial for Types I_∞, II_∞, or III. See [4, Example 5.3.2]. Using Theorem 3.10 and Theorem 3.26, the possibilities for a factor A' are thus limited to two cases: A' having IBN or A' having Basis Type $(N, 1)$ for some $N > 0$. If A' has IBN then the standard modules $(A')^n$ are distinct, giving the following theorem.

Theorem 4.9. Let $A \subset B(H)$ be such that A' is a factor and such that (A, σ) is implemented by a Toeplitz-Cuntz family. If A' has IBN then the size of the implementing family is unique.

Proof. If (A, σ) is implemented by two Toeplitz-Cuntz families $V_1, ..., V_n$ and $W_1, ..., W_m$ then by Proposition 4.7 $E \simeq (A')^n$ and $E \simeq (A')^m$, whence $(A')^n \simeq (A')^m$. Since A' has IBN we conclude that $n = m$. □

Theorem 4.10. Let $A \subset B(H)$ be such that A' is a factor and (A, σ) be a C^*-dynamical system which can be implemented by two Toeplitz-Cuntz families $\{V_i : i = 1, ..., n\}$ and $\{W_j : j = 1, ..., m\}$ with $n \neq m$. Then there exists an integer N such that (A, σ) has an implementing Toeplitz-Cuntz family of size n for all $n \geq N$.

Proof. Suppose that there are two implementing families for (A, σ) with sizes n and m, $n \neq m$. Then $E \simeq (A')^n$, $E \simeq (A')^m$, and so $(A')^n \simeq (A')^m$ from which it follows that A' does not have IBN. Now as A' is a factor we know that $K_0(A')$ is torsion-free. Hence if $[1_{A'}]_0$ has finite order then it must be the case that $[1_{A'}]_0 = 0$. By Theorem 3.26 we have that $type(A') = (N, 1)$ for some $N > 0$. By definition then $(A')^N \simeq (A')^j$ for all $j \geq N$, and in particular $(A')^N \simeq (A')^n \simeq E$ and hence (A, σ) is implemented by families of all sizes greater than or equal to N. □

4.2.2 Endomorphisms of $B(H)$

The consideration of Invariant Basis Number allows us to recover several known results in C^*-dynamical systems with significantly reduced effort.

In [13] Laca, expanding upon an observation by Arveson [3], determines that all $*$-endomorphisms of $B(H)$ arise from implementing Toeplitz-Cuntz families (of finite or infinite size) and that the size of such an implementing family is invariant for a given endomorphism. He goes about proving this fact by constructing $E_\sigma = \{T \in B(H) : TX = \sigma(X)T \ for\ all\ X \in A\}$ and noticing that the inner-product $\langle X, Y\rangle := X^*Y$ gives E the structure of a Hilbert space. Any orthonormal basis of E is a Toeplitz-Cuntz family inside $B(H)$ which, for reasons identical to those of Proposition 4.7, implements σ. Since the size of the implementing family corresponds to the dimension of the Hilbert space E_σ it is necessarily unique. Our consideration of IBN recovers this result in a similar fashion.

Theorem 4.11 (Theorem 2.1 in [13])**.** If σ is a $*$-endomorphism of $B(H)$ then σ is of the form

$$\sigma(T) = \sum_{i=1}^{n} V_i T V_i^*$$

for some family $V_1, ..., V_n$ ($n = \infty$ is possible) of mutually orthogonal isometries. The size of this family is unique for a given endomorphism.

We will only prove the uniqueness portion for the case when implementation is through a finite family.

Proof of uniqueness. We shall use the fact that $\mathbb{C}$ has IBN to significantly simplify the uniqueness result. The family of isometries forms a basis for the $B(H)'$-module $E = \{T \in B(H) : TX = \sigma(X)T \ for\ all\ X \in B(H)\}$. Since $B(H)' = \mathbb{C}$ and $\mathbb{C}$ has IBN we conclude that the size of this basis is unique. $\square$

4.2.3 Stacey's Crossed Products

In [11] Peters and Kakariadis explicitly consider endomorphisms of C^*-algebras which are implemented by Toeplitz-Cuntz families. Many of their results deal with various universal (C^*- or operator) algebras for such implementing families and the relations between them. Of current interest to us are their results concerning a construction originally due to Stacy [28]: the *crossed product of multiplicity* n is the C^*-algebra $A \times_\sigma^n \mathbb{N}$ which is universal for all covariant representations of multiplicity n. That crossed products exist for all multiplicities was originally proven by Stacy. Note that a representation of $A \times_\sigma^n \mathbb{N}$ induces a covariant representation of (A, σ) with multiplicity n. Note further that covariant representations of (A, σ) with multiplicity n themselves give a representation of $\mathcal{E}_n$ and, as seen in Section 3.5, these generate covariant representations of the standard modules A^n.

In [12] Peters and Kakariadis focus on the particular case of endomorphisms σ of $L^\infty(X, \mu)$, thought of as multiplication operators in $B(L^2(X, \mu))$. When we consider a representation $(id, \{S_i\})$ of $L^\infty(X, \mu) \times_\sigma^n \mathbb{N}$ on $L^2(X, \mu)$ we have that $\{S_i\}$ forms a basis for the $L^\infty(X, \mu)$-module (of course $L^\infty(X, \mu)' = L^\infty(X, \mu)$)

$$E = E(X, \mu) = \{T \in B(L^2(X, \mu)) : TA = \sigma(a)T \ for \ all \ a \in A\}$$

and so $E = (L^\infty(X, \mu))^n$. We shall re-investigate one of Peters and Kakariadis's results using the fact that $L^\infty(X, \mu)$, being commutative, has IBN.

Theorem 4.12 (Corollary 4.6 in [12])**.** Let α be a unital weak*-continuous isometric endomorphism of $L^\infty(X, \mu)$ and suppose that there is a representation $(id, \{S_i : i = 1, ..., n\})$ of Stacey's crossed product $L^\infty(X, \mu) \times_\alpha^n \mathbb{N}$ on $L^2(X, \mu)$. Then the following are equivalent:

1. $L^\infty(X,\mu) \times_\alpha^n \mathbb{N} \cong L^\infty(X,\mu) \times_\alpha^m \mathbb{N}$ via a $*$-isomorphism which fixes $L^\infty(X,\mu)$ elementwise;

2. There is a representation $(id, \{Q_i : i = 1, ..., m\})$ of $L^\infty(X,\mu) \times_\alpha^m \mathbb{N}$ on $L^2(X,\mu)$;

3. $n = m$.

In the original statement there is a fourth equivalence involving nonselfadjoint algebras which we will not discuss.

Proof. The conditions on α imply that it is a $*$-homomorphism. By the above discussion, 2) $\Leftrightarrow$ 3) is precisely because $L^\infty(X,\mu)$ has IBN. That 1) $\Rightarrow$ 2) is immediate by defining $Q_i = \Phi^{-1}(S_i')$ where Φ is the $*$-isomorphism and $\{S_i' : i = 1...m\}$ the generators of $L^\infty(X,\mu) \times_\alpha^m \mathbb{N}$. Of course 3) $\Rightarrow$ 1) is obvious. $\square$

Appendix A

K-Theoretical Necessities

The homological methods of algebraic K-theory give us powerful tools to analyze the structure of C^*-algebras. The techniques of the theory can be used to differentiate and, in some cases, completely characterize algebras up to isomorphism. The theory also has surprising connections to the structure of Hilbert modules and their morphisms. In this appendix we will give a quick overview of the foundations of K-theory for C^*-algebras. We will also work through examples which are of importance for the main body of our work. Important results are given with specific citation, but most of this exposition may be found with greater detail in [4, 26, 30].

Remark: We will consider only *unital* C^*-algebras in the following discussion. This greatly simplifies the development of the K-theory and poses no restrictions since we will only be interested in unital algebras for our main results.

A.0.4 The Semigroup of Projections

An element p of a C^*-algebra A is a *projection* if $p = p^2 = p^*$. Two projections p and q are *(Murray-von Neumann) equivalent*, written $p \sim_v q$, if there is $v \in A$ such that $vv^* = p$ and $v^*v = q$. In other words, $p \sim_v q$ if there is a partial isometry with range projection p and source projection q.

The $n \times n$ matrices with entries in A form a C^*-algebra $M_n(A)$ where the norm is inherited from the representations $\phi^{(n)} : M_n(A) \to B(H^n)$ which are induced by representations $\phi : A \to B(H)$. Two projections $p \in M_n(A)$ and $q \in M_m(A)$ are *equivalent*, denoted $p \sim q$, if there exists $v \in M_{n,m}(A)$ such that $vv^* = p$ and $v^*v = q$. Of course when $n = m$ this reduces to Murray-von Neumann equivalence in the algebra $M_n(A)$. The set of projections in $M_n(A)$ will be denoted $P_n(A)$ and $P_\infty(A) := \bigcup P_n(A)$. The equivalence class of a projection p will be denoted $[p]_0$, $[1_A]_0$ will be the unit class, and $[I_n]$ the class of the unit for $M_n(A)$. We briefly remark that homotopy equivalence of projections is a strictly weaker notion than $\sim$.

We may define an "addition" on $P_\infty(A)$ as follows: for $p \in P_n(A)$ and $q \in P_m(A)$ set

$$p \oplus q := \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \in P_{n+m}(A).$$

Proposition A.1. Let $p \in P_n(A)$ and $q \in P_m(A)$, then the following hold.

1. $p \oplus 0 \sim p$
2. if $p \sim p' \in P_{n'}(A)$ and $q \sim q' \in P_{m'}(A)$ then $p \oplus q \sim p' \oplus q'$
3. $p \oplus q \sim q \oplus p$
4. if $pq = qp = 0$ then $p + q \sim p \oplus q$
5. $I_n \sim (I_n - p) \oplus p$.

If we denote $V(A) := P_\infty(A)/\sim\, = \{[p]_0 : p \in P_\infty(A)\}$ then the above properties give $V(A)$ the structure of an abelian additive semigroup with unit. To be explicit, the addition in $V(A)$ is defined by $[p]_0 + [q]_0 = [p \oplus q]_0$.

A.0.5 K_0

Recall that when S is an abelian semigroup the *Grothendieck group* of S, denoted $G(S)$, is the universal enveloping group of S. For a detailed construction consult any standard text. In light of the ideas from previous sections, our course becomes clear.

Definition A.2. Let A be a unital C^*-algebra. The abelian group $K_0(A)$ is defined as

$$K_0(A) := G(V(A))$$

i.e. it is the Grothendieck group of the semigroup $V(A)$ consisting of equivalent matrix projections.

Note that we have only defined K_0 for *unital* C^*-algebras. To formulate a proper notion of K_0 for non-unital C^*-algebras is a trickier process than it may seem, see [26].

We shall abuse notation and identify elements $[p]_0 \in V(A)$ with their images in $K_0(A)$ under the Grothendieck map. This will cause little confusion as we will henceforth always be working with elements of $K_0(A)$ and not $V(A)$. We shall restate several of the more useful properties of the Grothendieck construction in the context of $K_0(A)$.

Proposition A.3.

1. $K_0(A) = \{[p]_0 - [q]_0 : p, q \in P_\infty(A)\}$

2. $[0]_0$ is the additive identity for $K_0(A)$.

3. $[p]_0 = [q]_0$ (equality in $K_0(A)$) if and only if there exists r for which $p \oplus r \sim q \oplus r$.

Example A.4. Consider $A = M_n(\mathbb{C})$. Recall the canonical traces $\tau = \tau^{(n)} : M_n(\mathbb{C}) \to \mathbb{C}$ and its properties:

- $p \sim q \Leftrightarrow \tau(p) = \tau(q)$
- $\tau(p) = \dim(p(\mathbb{C}^k))$

Then $K_0(\tau) : K_0(M_n(\mathbb{C})) \ni [p]_0 - [q]_0 \to \tau^{n_p}(p) - \tau^{n_q}(q) \in \mathbb{Z}$ is well defined and injective. If p is a one-dimensional projection then $K_0(\tau)([p]_0) = 1$ and so we obtain an isomorphism $K_0(M_n(\mathbb{C})) = \mathbb{Z}$. In particular, $K_0(\mathbb{C}) = \mathbb{Z}$.

Example A.5. Consider a separable infinite dimensional Hilbert space H. The dimension map for projections $p \mapsto \dim pH$ is surjective onto $\{0, 1, ..., \infty\} = \mathbb{Z}^+ \cup \{\infty\}$. Since von Neumann equivalence preserves dimension and $\dim(p \oplus q) = \dim p + \dim q$ we conclude that dim is a semigroup isomorphism between $V(B(H))$ and $\mathbb{Z}^+ \cup \{\infty\}$. The Grothendieck group of this semigroup is trivial, hence $K_0(B(H)) = 0$.

One of the important properties of K_0 groups is that they are a *stable property* of a C^*-algebra in the following ways.

Proposition A.6. For a unital C^*-algebra A we have $K_0(A) = K_0(M_n(A))$ for all $n \geq 1$.

The proof is technical, see [26, Prop. 4.3.8], but boils down to the (intuitively obvious) claim that $P_\infty(A)$ and $P_\infty(M_n(A))$ are "the same" under the equivalence relation $\sim$.

Although it may seem obvious, the fact that if $K_0(A) \neq K_0(B)$ then $A \neq B$ is extremely useful for distinguishing many sorts of C^*-algebras. For example, $K_0(\mathcal{O}_n) \neq K_0(\mathcal{O}_m)$ for $n \neq m$ and so the Cuntz algebras are distinct from one another.

The assignment $A \mapsto K_0(A)$ is a functor from the category of unital C^*-algebras to the category of abelian groups. The next few propositions, which we shall not prove, demonstrate that it is in fact a particularly nice covariant functor.

Proposition A.7 (Functoriality of K_0)**.** If $\pi : A \to B$ is a $*$-homomorphism then there exists a unique group homomorphism $K_0(\pi) : K_0(A) \to K_0(B)$ making the following diagram commute.

$$\begin{array}{ccc} P_\infty(A) & \xrightarrow{\pi^\infty} & P_\infty(B) \\ \Big\downarrow{\scriptstyle [\cdot]_0} & & \Big\downarrow{\scriptstyle [\cdot]_0} \\ K_0(A) & \xrightarrow{K_0(\pi)} & K_0(B) \end{array}$$

Here π^∞ acts on each $P_n(A)$ by $\pi[a_{ij}] = [\pi(a_{ij})]$. The proof is straightforward with $K_0(\pi)[p]_0 := [\pi(p)]_0$.

Proposition A.8 (Half-exactness of K_0)**.** If

$$0 \longrightarrow A \xrightarrow{\sigma} B \xrightarrow{\rho} C \longrightarrow 0$$

is a short exact sequence of C^*-algebras then

$$K_0(A) \xrightarrow{K_0(\sigma)} K_0(B) \xrightarrow{K_0(\rho)} K_0(C)$$

is an exact sequence of abelian groups.

If

$$0 \longrightarrow A \xrightarrow[\sigma]{} B \xrightarrow[\rho]{} C \longrightarrow 0$$

is a split exact sequence (i.e. there is $\lambda : C \to B$ for which $\rho \circ \lambda = id_C$) of C^*-algebras then

$$0 \longrightarrow K_0(A) \xrightarrow[K_0(\pi)]{} K_0(B) \xrightarrow[K_0(\rho)]{} K_0(C) \longrightarrow 0$$

is a split exact sequence (with splitting map $K_0(\lambda) : K_0(C) \to K_0(B)$) of abelian groups.

Unfortunately, there are known examples for when the latter sequence is not short exact. However, if the sequence of C^*-algebras splits then so does the sequence in the K-theory [26, Prop. 4.3.3].

A.0.6 Inductive Limits and Continuity of K_0

First let us recall some facts about inductive systems and inductive limits. Fix a category of algebraic objects such as groups, rings, C^*-algebras, etc and a index set I with the structure of a join-semilattice, i.e. for $i, j \in I$ there is $k \in I$ such that $i \leq k$ and $j \leq k$. An *inductive system* is a family $\{A_i : i \in I\}$ of objects in the category together with a family of morphisms $\{\phi_{ij} : i, j \in I\}$ such that $\phi_{ij} : A_i \to A_j$ when $i < j$ and $\phi_{ij} \circ \phi_{ki} = \phi_{kj}$ when $k < i < j$. We'll denote the inductive system by $\{A_i, \phi_{ij}\}$.

The *inductive limit* of an inductive system $\{A_i, \phi_{ij})$ is an object $A = \varinjlim\{A_i, \phi_{ij}\}$ within the same category as the A_i which satisfies the following universal property: there are *canonical morphisms* $\phi_i : A_i \to A$ such that $\phi_j \circ \phi_{ij} = \phi_i$ when $i < j$ and whenever there is another object N and morphisms $\{\psi_i : A_i \to N : i \in I\}$ also satisfying $\psi_j \circ \phi_{ij} = \psi_i$ then there is a unique morphism $\Theta : A \to N$ such that $\psi_i = \Theta \circ \phi_i$.

Theorem A.9 (Continuity of K_0 (Prop. 6.2.9 in [30]))**.** If $\{A_i, \phi_{ij}\}$ is an inductive system of C^*-algebras then $\{K_0(A_i), K_0(\phi_{ij})\}$ is an inductive system of abelian groups. If $A = \varinjlim\{A_i, \phi_{ij}\}$ then $K_0(A) = \varinjlim\{K_0(A_i), K_0(\phi_{ij})\}$.

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