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Extremal Results for the Number of Matchings and Independent Sets

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by

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EXTREMAL RESULTS FOR THE NUMBER OF MATCHINGS AND INDEPENDENT SETS

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This dissertation answers several questions in extremal graph theory, each concerning the maximum or minimum number of certain substructures a graph can have, given that it must satisfy certain properties. In recent years there has been increased interest in such problems, which are extremal problems for "counting" parameters of graphs. The results in this dissertation focus on graphs that have n vertices and e edges and 3-uniform hypergraphs that have n vertices and e edges.

We first observe in the preliminaries chapter that for graphs with a fixed number of vertices and edges there is a threshold graph attaining the minimum number of matchings. The first two major results develop this fact in two different directions. In Chapter 3 we consider the problem of maximizing the number of matchings in the class of threshold graphs. We solve the problem completely, concluding that a graph in this class has the maximum number of matchings if and only if it is almost alternating. The second and more fundamental question is the problem of which threshold graph, and hence which graph, has the minimum number of matchings. Ahlswede and Katona determined which graph has the fewest matchings of size 2. In Chapter 4 we extend this result to all sizes of matchings and to the total number of matchings. We prove that either the lex graph or the colex graph minimizes the number of matchings. We further prove that the lex bipartite graph has the fewest matchings among all bipartite graphs with parts of fixed sizes.

Finally, in Chapter 5, we answer an extremal question about independent sets in hypergraphs. In a graph $G = (V, E)$, a set $A \subseteq V$ is independent if $|A \cap e| < 2$ for all $e \in E$. The graph with n vertices and e edges achieving the maximum number of independent sets is the colex graph. In a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, a set A is s-independent if $|A \cap E| < s$ for all $E \in \mathcal{E}$. The final chapter discusses results about maximizing s-independent sets in r-uniform hypergraphs, the most significant result finds the 3-uniform hypergraph maximizing the number of 2-independent sets for certain numbers of vertices and edges.

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Chapter 1

Introduction

This dissertation focuses on extremal problems in graphs and hypergraphs. One classic result in extremal graph theory is Turán's theorem, which gives the maximum number of edges possible in a graph on n vertices having no complete subgraph on r vertices. In general, extremal graph theorists seek to maximize or minimize a parameter (such as the number of edges) while fixing other properties (such as having n vertices and no complete subgraph on r vertices). For many years, there has been interest in finding the maximum size of a variety of sub-structures (such as independent sets or matchings). In recent years, there has been increased interest in extremal questions about the *number* of these sub-structures. That is, rather than asking for the size of the largest independent set, one could ask which graph has the most independent sets, given some set of conditions. A classic example is the Kahn-Zhao theorem, proved initially by Kahn [15] in the bipartite case, and then extended to the general case by Zhao [24].

Theorem 1 (Kahn-Zhao)**.** *If G is a d-regular graph then* $\mathrm{ind}(G)$*, the number of* *independent sets in G, satisfies*

$$\operatorname{ind}(G) \leq \left(2^{d+1}-1\right)^{\frac{n}{2d}} = (\operatorname{ind}(K_{d,d}))^{\frac{n}{2d}}.$$

where $K_{d,d}$ is the complete balanced bipartite graph on $2d$ vertices.

In particular, if $2d$ divides n, the d-regular graph with the most independent sets is a disjoint union of complete balanced bipartite graphs.

One counting parameter for which extremal problems have been studied (extensively) is perfect matchings: see for instance [2, 4, 5, 13] and others. A natural next step is to think about matchings of all sizes. Letting $m_k(G)$ be the number of matchings of size k in a graph G, the Upper Matching Conjecture of Friedland, Krop, and Markström [11] claims that for all d-regular graphs on $2n$ vertices such that d divides n

$$m_k(G) \leq m_k\left(\frac{n}{d}K_{d,d}\right)$$

for all k where $\frac{n}{d}K_{d,d}$ is $\frac{n}{d}$ disjoint copies of the complete bipartite graph on $2d$ vertices. The Upper Matching Conjecture remains open; however, the Lower Matching Conjecture [11], which says that if G is a d-regular bipartite graph on $2n$ vertices then

$$m_k(G) \geq \binom{n}{k}^2 \left(\frac{d-p}{d}\right)^{n(d-p)} (dp)^{np}$$

where $p = \frac{k}{n}$, was recently proven by Péter Csikvári [3].

Many of the extremal results about matchings concern regular graphs or graphs with a given degree sequence. In another vein there has been work on determining which graph with given numbers of vertices and edges maximizes or minimizes a given counting parameter. (We write $\mathcal{G}_{n,e}$ for the class of graphs having n vertices and e edges.) For instance Cutler and Radcliffe [6, 7] determined which graphs in

$\mathcal{G}_{n,e}$ have the largest number of homomorphisms to various fixed image graphs H. A classic paper of Ahlswede and Katona [1] determines the maximum number of pairs of incident edges a graph in $\mathcal{G}_{n,e}$ can have. This is, of course, the same as minimizing the number of pairs of non-incident edges or matchings of size 2. In particular, they prove that either the lex or the colex graph has the minimum number of matchings of size 2.

To answer the question about which graphs in $\mathcal{G}_{n,e}$ minimize $m(G)$ and $m_k(G)$ for all k we first show in the preliminaries chapter that there is a threshold graph that attains the minimum. Threshold graphs appear as the answer to many extremal questions in graph theory [19]. We develop the fact that there is a threshold graph minimizing $m(G)$ in two directions in the first two primary chapters.

In Chapter 3 we discuss maximizing matchings in $\mathcal{T}_{n,e}$, the family of threshold graphs with n vertices and e edges. There are many equivalent definitions of threshold graphs [19] and here we rely heavily on the definition that says a graph is *threshold* if it can be constructed in stages from a single vertex by adding either an isolated vertex or a dominating vertex at each stage. The major result in Chapter 3 is Theorem 40 which says that the threshold graph attaining the maximum number of matchings is an "almost alternating" threshold graph. An almost alternating threshold graph is the threshold graph that comes as close as possible to alternately adding a dominating vertex and an isolated vertex at each stage. We also prove that any threshold graph on n vertices and e edges that is not almost alternating does not attain the maximum number of matchings in $\mathcal{T}_{n,e}$.

In Chapter 4 we solve the problem of determining which graph with n vertices and e edges has the fewest matchings. It turns out that, following the general approach of Ahlswede and Katona, we need to consider the class $\mathcal{B}_{\ell,r,e}$ of all bipartite graphs with ℓ vertices in the left part, r vertices in the right part, and having e edges.

Theorem 53 says that the lex bipartite graph is the graph in $\mathcal{B}_{\ell,r,e}$ having the fewest matchings. Theorem 52 states that either the lex graph or the colex graph attains the minimum number of matchings in $\mathcal{G}_{n,e}$, matching the result for matchings of size 2. Our techniques also allow us to determine the graphs minimizing the number of matchings of size k, for all values of k, in $\mathcal{B}_{\ell,r,e}$ and in $\mathcal{G}_{n,e}$.

Another counting parameter for which many extremal problems have been studied is the number of independent sets. For example, the maximization problem in d-regular graphs was solved by Kahn and Zhao (Theorem 1). Cutler and Radcliffe [8] showed that the Kruskal-Katona Theorem [18, 16] implies that the lex graph has the greatest number of independent sets among graphs in $\mathcal{G}_{n,e}$.

It is natural to try to extend these extremal results for counting parameters to hypergraphs. The results in Chapter 5 concern maximizing the number of independent sets in hypergraphs. In a graph an independent set is a subset of vertices containing at most one vertex from each edge. In a hypergraph there are a variety of types of independent sets in hypergraphs. For a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ we say a set $A \subseteq V$ is *s-independent* if $|A \cap E| < s$ for all $E \in \mathcal{E}$. There has been some research on independent sets in hypergraphs, most has focused on determining algorithms for finding independent sets in hypergraphs (see, e.g., [23]) or on finding the independent set of largest size (see, e.g., [17]).

In [9] Cutler and Radcliffe addressed some extremal questions concerning independent sets in hypergraphs. First they prove that the Kruskal-Katona Theorem implies the following:

Theorem 2. *Let $i_r(\mathcal{H})$ be the number of r-independent sets in $\mathcal{H}$. If $\mathcal{H}$ is an r-uniform hypergraph with n vertices and e edges then*

$$i_r(\mathcal{H}) \leq i_r(\mathcal{L}_r(n, e))$$

where $\mathcal{L}_r(n, e)$ *is the lex* r*-graph.*

In the same paper they give an asymptotically best possible upper bound on the number of j-independent sets in an r-uniform hypergraph of fixed size and order. Since they use a version of the hypergraph regularity lemma, their results only apply to graphs with a large number of vertices.

Let $\mathcal{H}_r(n, e)$ be the family of r-uniform hypergraphs with n vertices and e edges. In Chapter 5 we will show that the colex r-graph attains the maximum number of 1-independent sets in $\mathcal{H}_r(n, e)$ (this is Theorem 68). Since, by Theorem 2 we know that the lex graph maximizes 3-independent sets in 3-uniform hypergraphs the only case remaining for 3-uniform hypergraphs is that of 2-independent sets. In Theorem 70 we determine which 3-uniform hypergraphs have the maximum number of 2-independent sets in $\mathcal{H}_3(n, e)$ for large e.

Chapter 2

Preliminaries

2.1 Graphs and Hypergraphs

A *graph* G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a *vertex set* and $E(G)$ is an *edge set* where each edge is an unordered pair of vertices. All graphs in this dissertation will be *simple*: there will be no multiple edges (so $E(G)$ is not a multiset) and no loops (no edges of the form vv for $v \in V(G)$). A graph is *complete* if all pairs of vertices are in the edge set. A complete subgraph of G will be called a *clique*. A graph is *bipartite* if its vertices can be partitioned into two sets U and V such that every edge connects a vertex in U to a vertex in V.

The *neighborhood* of a vertex v is the set of vertices adjacent to v. Let $N_G(v)$ be the neighborhood of a vertex v in a graph G and let $N_G[v] = N_G(v) \cup \{v\}$. We will often suppress the subscript and use $N(v)$ and $N[v]$ when the associated graph is clear. A vertex $v \in V(G)$ is *isolated* if $N(v) = \emptyset$ and *dominating* if $N[v] = V(G)$.

For our extremal questions we will be maximizing or minimizing a certain parameter over a certain family of graphs. The notation we use for each of the families is as follows:

- $\mathcal{G}_{n,e}$ is the collection of all simple graphs with n vertices and e edges
- $\mathcal{T}_{n,e}$ is the collection of all threshold graphs with n vertices and e edges
- $\mathcal{B}_{\ell,r,e}$ is the collection of all bipartite graphs with parts of size ℓ and r and e edges
- $\mathcal{H}_r(n,e)$ is the collection of all r-uniform hypergraphs with n vertices and e edges

The results in Chapters 3 and 4 will concern matchings in graphs in $\mathcal{G}_{n,e}$, $\mathcal{T}_{n,e}$, and $\mathcal{B}_{\ell,r,e}$. A *matching* in a graph G is a set of independent edges. The set of matchings in a graph G will be denoted by $\mathcal{M}(G)$ and the set of matchings with k edges will be denoted $\mathcal{M}_k(G)$. Let $m(G) = |\mathcal{M}(G)|$ and $m_k(G) = |\mathcal{M}_k(G)|$.

A *hypergraph* $\mathcal{H}$ is an ordered pair $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ where $\mathcal{V}(\mathcal{H})$ is a vertex set and $\mathcal{E}(\mathcal{H})$ is a set of edges where each edge is a subset of $\mathcal{V}(\mathcal{H})$ of any size. A hypergraph is *r-uniform* if all edges have size r. We will often abbreviate and call an r-uniform hypergraph an *r-graph*.

The results in Chapter 5 will concern s-independent sets. In a graph $G = (V, E)$ the set $A \subset V$ is an independent set if $|A \cap e| < 2$ for all $e \in E$. When we generalize the notion of independent sets in graphs to hypergraphs there are several options depending on the number of vertices of an independent set we allow to be in an edge. In a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ we call a set A *s-independent* if $|A \cap E| < s$ for all $E \in \mathcal{E}$. Note that our usual notion of independent sets in graphs corresponds to 2-independent sets in 2-graphs.

2.2 Threshold Graphs

Threshold graphs appear as an answer to many extremal questions, especially those in which it is advantageous to have all the edges "bunched together". For instance in the class $\mathcal{G}_{n,e}$ threshold graphs maximize the number of independent sets and minimize homomorphisms into the Widom-Rowlinson graph [6].

We will make use of several equivalent definitions of threshold graphs. The first definition gives threshold graphs their name.

Definition 3. *A* threshold graph *is a graph such that there exists a number $c \in \mathbb{R}$ and a function $f : V(G) \to \mathbb{R}$ such that $uv \in E(G)$ if and only if $f(u) + f(v) \geq c$.*

Example 4. *The graph shown in Figure 2.1 is threshold. In Figure 2.2 weights have been assigned to each vertex. Note $uv \in E(G)$ if and only if $f(u) + f(v) \geq 4$. Therefore, G is threshold.*

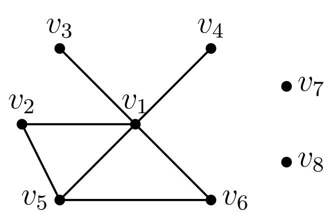

Figure 2.1: A threshold graph G.

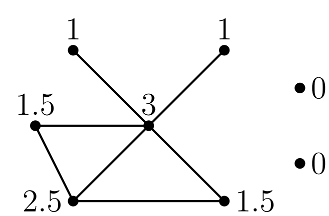

Figure 2.2: The weight function on G.

In Chapter 3 we will rely heavily on the following equivalent characterization of threshold graphs.

Theorem 5 ([19])**.** *A* threshold graph *is a graph that can be constructed from a single vertex by adding vertices one at a time that are either isolated or dominating.*

Example 6. *The graph shown in Figure 2.3 is threshold. In Figure 2.4 we demonstrate how we build the threshold graph starting with the single vertex v_6. Starting with v_6, add the vertices one at a time moving to the left with v_1 and v_5 added as dominating vertices and all others added as isolates.*

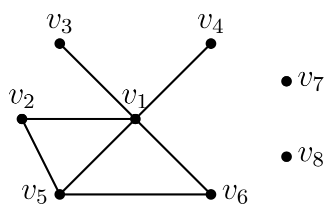

Figure 2.3: A threshold graph G.

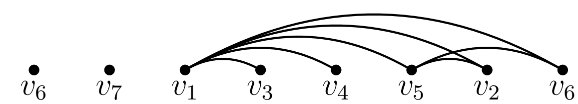

Figure 2.4: Adding isolated and dominating vertices to G.

Another characterization is that a threshold graph is a split graph with an additional condition.

Definition 7. *A* split graph *is a graph in which the vertices can be partitioned into a clique and an independent set.*

Theorem 8 ([19])**.** *A* threshold graph *is a split graph such that the vertices in the clique satisfy $N(v_1) \subseteq N(v_2) \subseteq \cdots \subseteq N(v_k)$ for some labeling of the vertices.*

Example 9. *The graph shown in Figure 2.5 is threshold. In Figure 2.6 we show that G is a split graph where $\{v_1, v_2, v_3\}$ make up the clique and $N(v_6) \subseteq N(v_5) \subseteq N(v_1)$.*

The decomposition into a clique and an independent set of a given threshold graph need not be unique. However, at most one vertex can be moved from the clique to the independent set or vice versa.

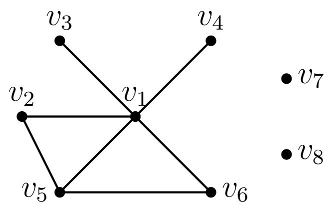

Figure 2.5: A threshold graph G.

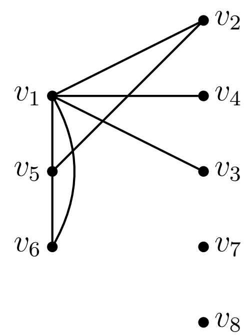

Figure 2.6: Split graph with nested neighborhoods.

There is an analogous concept of threshold graphs for bipartite graphs which we will use to find the bipartite graph minimizing the number of matchings in $\mathcal{B}_{\ell,r,e}$. In [19] threshold bipartite graphs are defined with a similar vertex weighting.

Definition 10. *A graph $G = (V, E)$ is said to be* threshold bipartite *if there exists a threshold t and a function $w : V(G) \to \mathbb{R}$ such that $|w(v)| < t$ for all $v \in V$ and distinct vertices u and v are adjacent if and only if $|w(u) - w(v)| \geq t$.*

Threshold bipartite graphs are called difference graphs in [19] and chain graphs in [22]. As with threshold graphs, there are many equivalent definitions of threshold bipartite graphs. The following lemma describes the definition that will be most useful to us.

Lemma 11 ([14]). *A graph is threshold bipartite if and only if G is bipartite and the neighborhoods of vertices in one of the parts can be linearly ordered by inclusion.*

This gives a definition for threshold bipartite graphs that is analogous to the one for threshold graphs provided in Theorem 8. Note that a threshold graph can be obtained from a threshold bipartite graph by adding all possible edges in one of the parts (on either side).

The following equivalent characterization of threshold graphs will help us define a way to move a graph towards being threshold.

Lemma 12 ([19])**.** *A graph G is threshold if and only if for all $x, y \in V(G)$ we have $N(x) \subseteq N[y]$ or $N(y) \subseteq N[x]$.*

We now define a compression move that makes a graph "more threshold". We will use this move later to show that we can find a graph that minimizes $m_k(G)$ in $\mathcal{G}_{n,e}$ that is threshold.

Definition 13. *Let G be a graph and x and y two vertices in G. Define*

$$N_G(x, \overline{y}) = \{v \in V(G) \setminus \{x, y\} : v \sim x, v \not\sim y\}.$$

Let $G_{x \to y}$ be the graph formed by deleting all edges between x and $N_G(x, \overline{y})$ and adding all edges from y to $N_G(x, \overline{y})$. This is called the compression of G from x to y*. It is clear that $G_{x \to y}$ has the same number of edges as G. In fact, $G_{i \to j}$ is a graph where we have replaced i with j whenever possible.*

To show that a graph minimizing the number of matchings can be found among the threshold graphs, we repeatedly compress a graph that minimizes the number of matchings. The following lemma will allow us to be sure that we are making progress, and not compressing round and round in a circle. The variance of the degree sequence, or (essentially equivalently) the quantity

$$d_2(G) = \sum_{v \in V(G)} d(v)^2$$

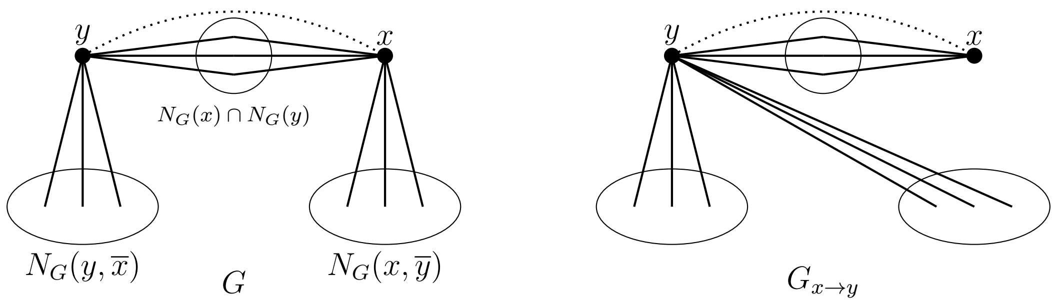

Figure 2.7: A graph G on the left and the compression of G from x to y ($G_{x\to y}$) on the right.

strictly increases whenever we do a non-trivial compression.

Lemma 14 ([6]). *Given $x, y \in V(G)$, $e(G_{x\to y}) = e(G)$ and $d_2(G_{x\to y}) \geq d_2(G)$. If $N(x) \nsubseteq N[y]$ and $N(y) \nsubseteq N[x]$ then $d_2(G_{x\to y}) > d_2(G)$.*

Corollary 15. *Suppose that $\mathcal{G}$ is a family of graphs on a fixed vertex set V such that for any $G' \in \mathcal{G}$ and $x, y \in V$ we also have $G'_{x\to y} \in \mathcal{G}$. In addition suppose that G satisfies*

$$d_2(G) = \max\left\{d_2(G') \ : \ G' \in \mathcal{G}\right\}.$$

Then G is threshold.

Proof. Suppose $x, y \in V$ and $N(x) \nsubseteq N[y]$ and $N(y) \nsubseteq N[x]$. By hypothesis, $G_{x\to y} \in \mathcal{G}$ and by Lemma 14 we know $d_2(G_{x\to y}) > d_2(G)$. This contradicts the assumption that G attains the maximum value of d_2 in $\mathcal{G}$. Thus, $N_G(x) \subseteq N_G[y]$ or $N_G(y) \subseteq N_G[x]$ and so G is threshold by Lemma 12. $\square$

Lemma 16. *Let G be a bipartite graph with bipartition (X, Y). Given $u, v \in X$, $e(G_{u\to v}) = e(G)$ and $d_2(G_{u\to v}) \geq d_2(G)$. If $N(u) \nsubseteq N(v)$ and $N(u) \nsubseteq N(v)$ then $d_2(G_{u\to v}) > d_2(G)$.*

Proof. Same calculations as in the proof of Lemma 14. □

Corollary 17. *Suppose that $\mathcal{G}$ is a family of bipartite graphs on a fixed vertex set V with a fixed bipartition (X, Y) such that for any $G' \in \mathcal{G}$ and $u, v \in X$ we also have $G'_{u\to v} \in \mathcal{G}$. In addition suppose that G satisfies*

$$d_2(G) = \max\{d_2(G') : G' \in \mathcal{G}\}.$$

Then G is bipartite threshold.

Proof. Suppose that $u, v \in X$ such that $N_G(u) \not\subseteq N_G(v)$ and $N_G(v) \not\subseteq N_G(u)$. Then $G_{u\to v} \in \mathcal{G}$ by assumption and $d_2(G_{u\to v}) > d_2(G)$ by Lemma 16, a contradiction. Thus, $N_G(u) \subseteq N_G(v)$ or $N_G(v) \subseteq N_G(u)$ and so by Lemma 11 the graph G is threshold bipartite. □

2.3 Matchings and Threshold Graphs

In the next lemma we show that $G_{x\to y}$ has at most as many matchings as G. First, for an edge e such that $r \in e$ and $s \notin e$ we write $e\Delta\{r, s\}$ to mean the edge where the vertex r is replaced with the vertex s. This notation will be used in several proofs. Also, for A and B disjoint subsets of $V(G)$ we write $E(A, B)$ to mean the set of edges that have one vertex in A and the other in B.

Lemma 18. *For all graphs G, all $x, y \in V(G)$, and all $k \in \mathbb{N}$*

$$m_k(G_{x\to y}) \leq m_k(G).$$

Proof. Let $H := G_{x\to y}$. We will construct an injection ϕ from $\mathcal{M}_k(H) \setminus \mathcal{M}_k(G)$ to $\mathcal{M}_k(G)\setminus\mathcal{M}_k(H)$ from which it follows that $m(H) \leq m(G)$. Let $A = E(x, N_G(x, \overline{y})) \subset$

$E(G)$ and $B = E(y, N_G(x, \overline{y})) \subset E(H)$. Then $H = G - A + B$. So

$$\mathcal{M}_k(H) \setminus \mathcal{M}_k(G) = \{M \in \mathcal{M}_k(H) : M \cap B \neq \emptyset\}$$

and similarly

$$\mathcal{M}_k(G) \setminus \mathcal{M}_k(H) = \{M \in \mathcal{M}_k(G) : M \cap A \neq \emptyset\}$$

Define a replacement function $r : E(H) \to E(G)$. Let

$$r(e) := \begin{cases} e\Delta\{x, y\} & \text{if } e \in B \\ yz & \text{if } e = xz, z \neq y \cdot \\ e & \text{otherwise} \end{cases}$$

For each $e \in E(H)$ note that $r(e) \in E(G)$. If $e \in B$ then $r(e) = e\Delta\{x, y\} \in A \subset E(G)$. If $e = xz \in H$ and $z \neq y$ then $z \in N_G(x) \cap N_G(y)$ and so $\phi(e) = yz \in E(G)$. Finally, if $e \notin B$ then $e \in E(G) \cap E(H)$.

Now define $\phi : \mathcal{M}_k(H) \setminus \mathcal{M}_k(G) \to \mathcal{M}_k(G) \setminus \mathcal{M}_k(H)$ by

$$\phi(M) := \{r(e) : e \in M\}.$$

Suppose $M \in \mathcal{M}_k(H) \setminus \mathcal{M}_k(G)$. We claim $\phi(M) \in \mathcal{M}_k(G) \setminus \mathcal{M}_k(H)$. Since $r : E(H) \to E(G)$ we know $\phi(M) \subset E(G)$. To show that $\phi(M) \in \mathcal{M}_k(G)$ we suppose to a contradiction that $r(e)$ is incident to $r(f)$ for some $e, f \in M$. For the first case, suppose that $r(e) \cap r(f) = x$. Note that $x \in r(e)$ for any edge e if and only if $e \in B$ or $e = xy$. Since $M \cap B \neq \emptyset$ we know that $xy \notin M$. Thus, both e and f are in B and $e \cap f = y$, a contradiction since e and f are in the matching M.

Next suppose that $r(e) \cap r(f) = y$. In H we know neither e nor f are incident to y

since $M \cap B \neq \emptyset$ and if e or f are in B then we replaced them with edges incident to x. Thus, it must be the case that both e and f are incident to x in M, a contradiction since M is a matching. Finally, if $r(e) \cap r(f) = z$ for $z \neq x$ and $z \neq y$ then r acted as the identity on e and f. So $e \cap f = z$, a contradiction. Thus, $\phi(M) \in \mathcal{M}_k(G)$. Finally, $\phi(M) \notin \mathcal{M}_k(H)$ since $\phi(M)$ has an edge from A and $A \cap E(H) = \emptyset$.

To complete the proof we show ϕ is an injection. Define $r' : E(G) \to E(H)$ by

$$r'(e) := \begin{cases} e\Delta\{x,y\} & \text{if } e \in A \\ xz & \text{if } e = yz, z \neq x \\ e & \text{otherwise} \end{cases} \cdot$$

Define $\phi' : \operatorname{im}(\phi) \to \mathcal{M}_k(H)$ by $\phi'(M) = \{r'(e) : e \in M\}$. Given $M \in \mathcal{M}_k(H) \setminus \mathcal{M}_k(G)$ it is easy to check that $\phi'(\phi(M)) = M$. Therefore ϕ has a left inverse and so ϕ is injective.

$\square$

By Corollary 15 and Lemma 18, a graph minimizing the total number of matchings among all graphs in $\mathcal{G}_{n,e}$ can be found among the threshold graphs.

The following lemma will show that compression in bipartite graphs also reduces the number of matchings. This Lemma is equivalent to Theorem 3.2 in Gross, Kahl, and Saccoman [13]. We present a proof for completeness.

Lemma 19. *If G is any graph and $u, v \in V(G)$, then $m_k(G_{u\to v}) \leq m_k(G)$ for every $k \geq 0$. In particular, if G is a bipartite graph with bipartition (X, Y) and $u, v \in X$ then $G_{u\to v}$ has at most as many k-matchings as G and again has bipartition (X, Y).*

Proof. Let $H = G_{u\to v}$. As in the proof of Lemma 18, we will construct an injection from $\mathcal{M}(H) \setminus \mathcal{M}(G)$ to $\mathcal{M}(G) \setminus \mathcal{M}(H)$ that preserves size. It then follows that

$m_k(H) \leq m_k(G)$ for all k. We define a replacement function $r : E(H) \to E(G)$ by

$$r(e) = \begin{cases} uy & \text{if } e = vy \text{ for } y \in N_G(u, \overline{v}) \\ vz & \text{if } e = uz \\ e & \text{otherwise} \end{cases}.$$

Given e in the edge set of H, we claim that $r(e)$ is an edge in G. If $y \in N_G(u, \overline{v})$, then $uy \in E(G)$. Also, if $uz \in H$ then $z \in N_G(u) \cap N_G(v)$ and so $vz \in E(G)$. Finally, if $e \neq vy$ for $y \in N_G(u, \overline{v})$ then $e \in E(G) \cap E(H)$.

Now define $\phi : \mathcal{M}(H) \setminus \mathcal{M}(G) \to \mathcal{M}(G) \setminus \mathcal{M}(H)$ by

$$\phi(M) = \{r(e) : e \in M\}$$

Given $M \in \mathcal{M}(H) \setminus \mathcal{M}(G)$ note that $\phi(M) \subseteq E(G)$ since $r(e) \in E(G)$ for all $e \in E(H)$. We claim that in fact $\phi(M) \in \mathcal{M}(G) \setminus \mathcal{M}(H)$. For the first case, suppose that for some $e, f \in M$ we have $u \in r(e) \cap r(f)$. Note that if $u \in r(e)$ then $v \in e$ since edges in H containing u are replaced by edges in G containing v instead. So if $r(e) \cap r(f) = u$ then $e \cap f = v$, a contradiction since $e, f \in M$, a matching.

Now suppose that $e, f \in M$ and $r(e) \cap r(f) = v$. There are two possible ways for v to be in $r(e)$ for some $e \in M$. The first is for u to be in e and the second is for $e = va$ for some $a \notin N_G(u, \overline{v})$. However, since $M \in \mathcal{M}(H) \setminus \mathcal{M}(G)$ it must be the case that $vy \in M$ for some $y \in N_G(u, \overline{v})$. Since M is a matching, vy is the only edge incident to v. Therefore, e and f must both contain u, a contradiction since M is a matching.

Finally suppose that $z \in r(e) \cap r(f)$ for some $z \neq u, v$. Then $z \in e \cap f$, a contradiction. Thus each vertex has at most one incident edge and $\phi(M)$ is a matching.

Note that $\phi(M) \notin \mathcal{M}(G)$ since $uy \in \phi(M)$ for some $y \in N_G(u, \overline{v})$ and no such edge is in $E(G)$. So $\phi(M) \in \mathcal{M}(H) \setminus \mathcal{M}(G)$.

To finish the proof of the lemma we need only show that ϕ is an injection. we will show that ϕ has a left inverse defined similarly to ϕ. Consider $r' : E(G) \to E(H)$ defined by

$$r'(e) = \begin{cases} vy & \text{if } e = uy \text{ for any } y \in N_G(u, \overline{v}) \\ uz & \text{if } e = vz \\ e & \text{otherwise} \end{cases}.$$

Define $\phi' : \mathcal{M}(G) \setminus \mathcal{M}(H) \to \mathcal{M}(H) \setminus \mathcal{M}(G)$ by $\phi'(M) = \{r'(e) : e \in M\}$. It is straightforward to check that $\phi'(\phi(M)) = M$. Thus ϕ has a left inverse and so ϕ is injective. $\square$

By Corollary 17 and Lemma 19, a bipartite graph minimizing the total number of matchings among bipartite graphs can be found among the threshold bipartite graphs.

2.4 Shifted Hypergraphs

Since threshold graphs appear as an answer to many extremal questions in graphs, the concept of a "threshold hypergraph" should be useful when answering similar questions in hypergraphs. As we saw there are many equivalent definitions of threshold graphs. What happens when we try to extend these definitions to hypergraphs? In [20] Reiterman, Rödl, Šiňajová, and Tůma show that the extensions of three of the equivalent definitions of threshold graphs are not equivalent in r-graphs for $r > 2$. One particular extension will be useful to us, the version known as *shifted*. In Chapter 5 we will show that s-independent sets in r-graphs are maximized by shifted hypergraphs.

Definition 20. *Given a set $A \subset [n]$ and $i, j \in [n]$ such that $A \cap \{i, j\} = \{i\}$ define $A_{i\to j} = (A \setminus \{i\}) \cup \{j\}$.*

Definition 21. *Consider a hypergraph $\mathcal{H}$ with vertex set $[n]$ and edge set $\mathcal{E}$. For $0 \le j < i \le n-1$ define the (i, j)-*shift *$S_{i\to j}$ as follows:*

- *for each $E \in \mathcal{E}$,*

$$S_{i\to j}(E) = \begin{cases} E_{i\to j} & \text{if } E \cap \{i, j\} = \{i\} \\ E & \text{otherwise} \end{cases}.$$

- *let $S_{i\to j}(\mathcal{E}) = \{S_{i\to j}(E) : E \in \mathcal{E}\} \cup \{E : E, S_{i\to j}(E) \in \mathcal{E}\}$.*

For a hypergraph $\mathcal{H}$ on vertex set $[n]$, we will write $\mathcal{H}_{i\to j}$ to mean the hypergraph on vertex set $[n]$ and with edge set $S_{i\to j}(\mathcal{E}(\mathcal{H}))$.

Thus, $\mathcal{H}_{i\to j}$ is a hypergraph with the same number of edges as $\mathcal{H}$ with the same sizes, but where we have replaced i with j whenever possible.

Definition 22. *A hypergraph $\mathcal{H} = ([n], \mathcal{E})$ is (left-)shifted if and only if $\mathcal{H}_{i\to j} = \mathcal{H}$ for all $0 \le j < i \le n-1$.*

We will extend the definition of $\mathcal{H}_{i\to j}$ slightly and set $\mathcal{H}_{i\to i} = \mathcal{H}$ for all $i \in [n]$. In the next definition we extend again to apply a number of shifts at once.

Definition 23. *Given a set E and vectors $\boldsymbol{a} = (a_i)_{i=1}^{\ell}$ and $\boldsymbol{b} = (b_i)_{i=1}^{\ell}$ with $a_i \ge b_i$ for all i, define $S_{\boldsymbol{a}\to\boldsymbol{b}}(E)$ to be the set in which we've applied the shifts $S_{a_i\to b_i}$ in* *increasing order of* $\boldsymbol{a}$*. That is, we apply* $S_{a_i \to b_i}$ *before* $S_{a_j \to b_j}$ *if* $a_i < a_j$*. Similarly, for a collection of sets* $\mathcal{E}$*, define*

$$S_{\boldsymbol{a} \to \boldsymbol{b}}(\mathcal{E}) = \{S_{\boldsymbol{a} \to \boldsymbol{b}}(E) : E \in \mathcal{E}\} \cup \{E : E, S_{\boldsymbol{a} \to \boldsymbol{b}} \in \mathcal{E}\}.$$

For a hypergraph $\mathcal{H}$ *define* $\mathcal{H}_{\boldsymbol{a} \to \boldsymbol{b}}$ *to be the hypergraph on the same vertex set with edge set* $S_{\boldsymbol{a} \to \boldsymbol{b}}(\mathcal{E}(\mathcal{H}))$*.*

We will use this definition in Section 5.3.2. In particular, we will use the fact that if we apply a shift from all the vertices in one edge to another r-set of vertices, E', then E' will be in the edge set. We prove this in the next lemma.

Lemma 24. *Let* $\mathcal{H}$ *be an* r*-graph and consider vectors* $\boldsymbol{a} = (a_i)_{i=1}^r$ *and* $\boldsymbol{b} = (b_i)_{i=1}^r$ *such that* $a_i \geq b_i$ *for all* i*. If* $\{a_i : 1 \leq i \leq r\} \in \mathcal{E}(\mathcal{H})$ *then* $\{b_i : 1 \leq i \leq r\} \in \mathcal{E}(\mathcal{H}_{\boldsymbol{a} \to \boldsymbol{b}})$*.*

Proof. Suppose $E = \{a_i : 1 \leq i \leq r\} \in \mathcal{E}(\mathcal{H})$. We claim $S_{\mathbf{a} \to \mathbf{b}}(E) = \{b_i : 1 \leq i \leq r\}$. we will show $b_i \in S_{\mathbf{a} \to \mathbf{b}}(E)$ for each i. Let $\mathbf{x}$ be a vector of those entries in $\mathbf{a}$ that are less than a_i and let $\mathbf{z}$ be a vector of those entries in $\mathbf{a}$ that are greater than a_i. Rearrange $\mathbf{b}$ similarly to form vectors $\mathbf{x'}$ and $\mathbf{z'}$: if $(\mathbf{x})_j = a_k$ define $(\mathbf{x'})_j = b_k$ and if $(\mathbf{z})_\ell = a_m$ define $(\mathbf{z'})_\ell = b_m$. Note

$$S_{\mathbf{a} \to \mathbf{b}}(E) = S_{\mathbf{z} \to \mathbf{z'}}(S_{a_i \to b_i}(S_{\mathbf{x} \to \mathbf{x'}}(E))).$$

First $a_i \in S_{\mathbf{x} \to \mathbf{x'}}(E)$ since we have only potentially removed elements that are strictly smaller than a_i. So $b_i \in S_{a_i \to b_i}(S_{\mathbf{x} \to \mathbf{x'}}(E))$. Suppose that $b_i \notin S_{\mathbf{z} \to \mathbf{z'}}(S_{a_i \to b_i}(S_{\mathbf{x} \to \mathbf{x'}}(E)))$. Then b_i is equal to some entry in $\mathbf{z}$. But any entry in $\mathbf{z}$ is strictly greater than a_i and $a_i \geq b_i$. Thus $b_i \in S_{\mathbf{z} \to \mathbf{z'}}(S_{a_i \to b_i}(S_{\mathbf{x} \to \mathbf{x'}}(E))) = S_{\mathbf{a} \to \mathbf{b}}(E)$ for all i, and so $\{b_i : 1 \leq i \leq r\} \in \mathcal{E}(\mathcal{H}_{\mathbf{a} \to \mathbf{b}})$. ∎

2.5 π-lex Graphs

In order to state our results we need to describe π-lex graphs. First, let's start with three collections of graphs: lex graphs, colex graphs, and lex bipartite graphs. Lex and colex graphs have appeared as the answer to extremal questions (see, e.g. [8, 6]). Of particular relevance is that lex, colex, and lex bipartite graphs appear in [1], Ahlswede and Katona's paper concerning the minimum number of matchings of size 2 in $\mathcal{G}_{n,e}$.

To define the lex and colex graphs, we first define the lex and colex orderings. Throughout this section, given sets A and B, $A\Delta B$ is the symmetric difference of A and B.

Definition 25. *The* lexicographic order*, $<_L$, on finite subsets of* $\mathbb{N}$ *is defined by* $A <_L B$ *if* $\min(A\Delta B) \in A$*. The* colexigraphic order*, $<_C$, is defined by* $A <_C B$ *if* $\max(A\Delta B) \in B$*.*

Restricting these orderings to 2-subsets of $[n]$ results in the lex and colex orderings on $E(K_n)$. The first few edges in the lex ordering on $E(K_n)$ are

$$\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3\}, \{2,4\}, \ldots, \{2,n\}, \{3,4\}, \ldots$$

and the first few edges in the colex ordering on $E(K_n)$ are

$$\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,5\}, \{2,5\}, \ldots$$

Note that initial segments of colex do not depend on the size of the ground set, unlike those of the lex ordering.

Definition 26. *The* lex graph $\mathcal{L}(n,e)$ *is the graph with vertex set* $[n]$ *and edge set* *consisting of the first e edges in the lex order on $E(K_n)$. Similarly, the* colex graph *$\mathcal{C}(n,e)$ is the graph with vertex set $[n]$ and edge set consisting of the first e edges in the colex order on $E(K_n)$.*

Example 27. *The graph in Figure 2.8 is $\mathcal{L}(7,8)$.*

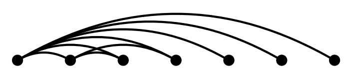

Figure 2.8: The lex graph with 7 vertices and 8 edges, $\mathcal{L}(7,8)$.

Example 28. *The graph in Figure 2.9 is $\mathcal{C}(7,8)$.*

Figure 2.9: The colex graph with 7 vertices and 8 edges, $\mathcal{C}(7,8)$.

Additionally, when we discuss the matchings minimization problem in bipartite graphs we will need the definition of a lex bipartite graph.

Definition 29. *Suppose $n = \ell + r$ with $\ell \leq r$ and $e \leq \ell r$. Write $e = qr + c$ where $0 \leq c < r$. The* lex bipartite graph *with e edges and parts L and R of size ℓ and r, respectively, is the bipartite graph in which q vertices in L have degree r and one vertex in L has degree c. We will denote this graph by $L_{\ell,r}(e)$. Note that the edges of the lex bipartite graph consist of the first e edges of $E(L,R)$ in lex order.*

The lex and colex graphs are threshold and the lex bipartite graph is bipartite threshold. This is easy to see using the definition of threshold that says a graph a threshold if it is a split graph with an additional neighborhood condition (Definition 8). In the colex case at most one vertex in the independent set has any neighbors at all. In the lex graph case the clique consists (with one possible exception) of dominant vertices, so all the vertices in the independent set are joined either to all the vertices in the clique or all but one (and the one missing vertex is the same in each case).

We will also want to think about the lex and colex 3-graphs.

Definition 30. *Using the lex order, $<_L$, we define $\mathcal{L}_r(n,e)$ to be the r-graph that has vertex set $[n]$ and edge set the first e edges in the lex order on $\binom{[n]}{r}$. Using the colex order, $<_C$, we define $\mathcal{C}_r(n,e)$ to be the r-graph that has vertex set $[n]$ and edge set the first e edges in the colex order on $\binom{[n]}{r}$.*

Example 31. *The first few 3-edges in the lex ordering are*

$$\{1,2,3\},\{1,2,4\},\ldots,\{1,2,n\},\{2,3,4\},\{2,3,5\},\ldots,\{2,3,n\},\{3,4,5\},\ldots,$$

The first few 3-edges in the colex ordering are

$$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\},\ldots$$

However, in r-graphs for $r>2$ we can define other natural orders on $\binom{[n]}{r}$ leading to other (shifted) r-graphs. In fact, we can define $r!$ orderings that generate orderings like the lex and the colex orders. While these orderings seem very natural we have not seen them introduced elsewhere.

Definition 32. *Consider a permutation $\pi=(\pi_1,\ldots,\pi_k)$ and let $A=\{a_1,a_2,\ldots,a_k\}$ and $B=\{b_1,b_2,\ldots,b_k\}$ be sets in $\binom{[n]}{k}$ where $a_1<a_2<\cdots<a_k$ and $b_1<b_2<\cdots<b_k$. Define the* π-lex order *on $\binom{[n]}{k}$ by $A<_\pi B$ if the least i for which $a_{\pi_i}\neq b_{\pi_i}$ we have $a_{\pi_i}<b_{\pi_i}$.*

Definition 33. *Given a permutation π define the* π-lex r-graph with n vertices and e edges *to be the r-graph on vertex set $[n]$ with edge set consisting of the first e edges in the π-lex order on $\binom{[n]}{r}$.*

Example 34. *The colex ordering on $\binom{[n]}{3}$ is π-lex for $\pi = (3, 2, 1)$ and the lex ordering on $\binom{[n]}{3}$ is π-lex for $\pi = (1, 2, 3)$.*

The π-lex ordering that will be particularly important to us is the $(2, 3, 1)$-lex ordering.

Example 35. *In this example, and in Chapter 5, we let $[n] = \{0, \ldots, n-1\}$. The first few edges in the $(2, 3, 1)$-lex ordering on $\binom{[n]}{3}$ are*

$$\begin{aligned}\{0,1,2\},\{0,1,3\},\ldots,\{0,1,n-1\},\{0,2,3\},\{1,2,3\},\{0,2,4\},\{1,2,4\},\ldots\\ \{0,2,n-1\},\{1,2,n-1\},\{0,3,4\},\{1,3,4\},\{2,3,4\},\{0,3,5\},\\ \{1,3,5\},\{2,3,5\},\ldots,\{0,3,n-1\},\{1,3,n-1\},\{2,3,n-1\},\\ \{0,4,5\},\{1,4,5\},\{2,4,5\},\{3,4,5\},\ldots,\{0,4,n-1\},\\ \{1,4,n-1\},\{2,4,n-1\},\{3,4,n-1\},\ldots\end{aligned}$$

Chapter 3

Maximizing Matchings in Threshold Graphs

In this chapter we answer an extremal question concerning matchings in $\mathcal{T}_{n,e}$, the family of all threshold graphs on n vertices and e edges. Recall from Section 2.3 that compressions reduce the number of matchings and hence we can find a graph minimizing the number of matchings among the threshold graphs. In this chapter we will prove that the graph in $\mathcal{T}_{n,e}$ with the *maximum* number of matchings is almost alternating, and all other graphs in the family have strictly fewer matchings.

3.1 Definitions and Results

Recall that a threshold graph is a graph that can be constructed from a single vertex by adding vertices one at a time that are either isolated or dominating. Using this definition any binary string of finite length can be used as instructions to construct a threshold graph. Letting 1 be code for a dominating vertex and 0 be code for an isolated vertex, construct the threshold graph by reading the binary string from right to left. The threshold graph with binary string σ will be denoted $T(\sigma)$. We will refer to σ as the code of the graph.

Example 36. *The following graph is threshold with code* 0010010*.*

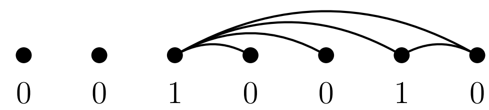

Figure 3.1: $T(0010010)$

Note that $T(0010010)$ and $T(0010011)$ are the same graph. To deal with this lack of uniqueness, a $*$ will be used to denote the first (rightmost) vertex as it can be read as either a 0 or a 1. Using this, the graph in Figure 3.1 will be denoted $T(001001*)$.

Example 37. *Consider the graph in Figure 3.2. The bold edges v_5v_3 and v_2v_1 form a matching of size 2. The empty set and single edges are also matchings. Note $m(T(001001*)) = 8$.*

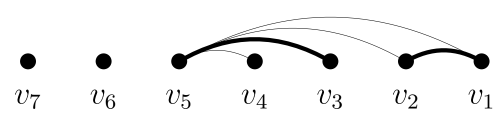

Figure 3.2: A matching in $T(001001*)$.

Definition 38. *We will use a to denote* 01 *and b to denote* 10*. A threshold graph will be called* almost alternating *if it can be written as a block of 1's or 0's followed by a string of a's and b's. A threshold graph will be considered almost alternating if it can be written this way either using the $*$ or not using the $*$. In the event that we use the $*$ we may consider it to be a* 0 *or a* 1*.*

Example 39. *The following are examples of almost alternating threshold graphs.*

(i) $00001011001{*} = 000aaba{*}$

(ii) $11101100{*} = 111aba$

(iii) $0010101{*} = 0aaa{*} = 00bbb$

The last of these three examples demonstrates that representations of almost alternating threshold graphs in terms of a's and b's are not unique when the starting block is followed by a strictly alternating string.

The main result of this chapter is the following.

Theorem 40. *A threshold graph on n vertices and e edges has the maximum number of matchings if and only if it is almost alternating. Moreover, for G a threshold graph on n vertices having e edges and A an almost alternating graph*

$$m_k(G) \leq m_k(A).$$

The proof of Theorem 40 appears in Section 3.4.

Remark. *It is not the case that $G \in \mathcal{T}_{n,e}$ attains the maximum number of matchings of size k in $\mathcal{T}_{n,e}$* if and only if *G is almost alternating. For example, there are n and e for which both the almost alternating graph and other threshold graphs have no perfect matchings. Consider $G = T(1000111{*})$ and $G' = T(1010100{*})$. Note that G' is almost alternating and that G is not. However, these each have 8 vertices, 13 edges, and no matchings of size 4.*

3.2 Almost Alternating Graphs

There are two goals for this section: to establish an alternative characterization for almost alternating graphs that will help us prove Theorem 40 and to show that there exists an almost alternating graph for any feasible number of vertices and edges. We first define two obstacles to being almost alternating.

Definition 41. *We will define a* bracketed 0-string *to be a string of at least three 0's with 1's on both ends (where the 1 on the right end may actually be the* $*$*). Similarly, a* bracketed 1-string *is a string of at least three 1's with 0's on both ends (where the* 0 *on the right end may actually be the* $*$*). A* bracketed string *refers to either a bracketed 1-string or a bracketed 0-string.*

Definition 42. *We will say that a graph* $G = T(\sigma)$ *has a* separation issue *if* σ *has a two pairs of repeated digits separated by a substring of odd length, with the first pair preceded by the opposite digit and and the last pair not ending the code.*

In the next lemma we prove that these are the only obstacles to a graph being almost alternating.

Lemma 43. *A graph that has neither a separation issue nor a bracketed string is almost alternating.*

Proof. Let $G = T(\sigma)$ be a threshold graph and suppose that σ does not have a bracketed string or a separation issue. If σ is strictly alternating then clearly σ is almost alternating. Otherwise $\sigma_0 = 0^k$ or $\sigma_1 = 1^k$ appears somewhere in σ for some $k \geq 2$. If there is no opposite digit to the left of σ_i for $i = 1$ or 2 then σ_i can be considered part of the beginning block. So suppose there is an opposite digit to the left of σ_i. If $k > 2$ then we have the string 01^k0 or 10^k1 possibly using the $*$. This is a contradiction as we assumed there is no bracketed string. Thus, if 0^k or 1^k appears in the code with an opposite digit to the left, $k = 2$. Assuming $k = 2$, if it is not possible to write the code as a beginning block of 0's or 1's followed by a's and b's then it must be the case that there are two pairs of repeated digits separated by a string of odd length. Additionally, the first pair of repeated digits must be preceded by the opposite digit, else we could consider it part of the beginning block. Also, the last pair of the repeated digits can not end the code since we can use the $*$ or not when finding the representation in a's and b's. That is, G has a separation issue, which is a contradiction. Thus, G has been written as a starting block followed by a string of a's and b's so G is almost alternating. $\square$

Given integers $n \geq 0$ and $e \leq \binom{n}{2}$, it is not immediately obvious that there exists an almost alternating threshold graph on n vertices with e edges. The proof of the next lemma gives a way to construct almost alternating graphs on n vertices with e edges for any such n and e.

Lemma 44. *Given integers n and e such that $n \geq 0$ and $0 \leq e \leq \binom{n}{2}$, there exists an almost alternating threshold graph on n vertices and e edges.*

Proof. Fix $n \geq 0$ and e with $0 \leq e \leq \binom{n}{2}$. Consider the almost alternating threshold graphs

$$G(\alpha, \beta) = T(\underbrace{aa\cdots a}_{\alpha}\underbrace{bb\cdots b}_{\beta}*) \text{ and } H(\alpha, \beta) = T(\underbrace{aa\cdots a}_{\alpha}\underbrace{bb\cdots b}_{\beta})$$

for some $\alpha, \beta \geq 0$. These graphs have

$$\begin{aligned} e(G(\alpha, \beta)) &= (2 + 4 + 6 + \cdots + 2\beta) + ((2\beta + 1) + (2\beta + 3) + \cdots + (2\beta + 2\alpha - 1)) \\ &= 2\beta\alpha + \beta(\beta + 1) + \alpha^2 \\ &= (\alpha + \beta)^2 + \beta \end{aligned}$$

and

$$\begin{aligned} e(H(\alpha,\beta)) &= (1+3+\cdots+2\beta-1)+((2\beta)+(2\beta+2)+\cdots+(2\beta+2\alpha-2)) \\ &= 2\beta\alpha+\alpha(\alpha-1)+\beta^2 \\ &= (\alpha+\beta)^2-\alpha \end{aligned}$$

respectively.

When $e \le (n/2)^2$ we can use these formulas to determine the almost alternating threshold graph that maximizes matchings. To do this, write $e = k^2 \pm l$ for nonnegative integers k, l with $l \le k$. If $e = k^2 + l$ then the graph $G(k-l, l)$ has e edges. If $e = k^2 - l$ the graph $H(l, k-l)$ has e edges. The restriction $e \le (n/2)^2$ ensures that $k = \alpha + \beta \le (n/2)$. Thus using α a's and β b's does not use more than n vertices. If $2\alpha + 2\beta < n$ then we add the appropriate number of isolates to the graph, which will form a beginning block of 0's. For $e = k^2 + l$ an almost alternating threshold graph on n vertices with e edges is

$$\underbrace{00\cdots0}_{n-2k-1}\underbrace{aa\cdots a}_{k-l}\underbrace{bb\cdots b}_{l}*.$$

Similarly, for $e = k^2 - l$ an almost alternating threshold graph on n vertices with e edges is

$$\underbrace{00\cdots0}_{n-2k}\underbrace{aa\cdots a}_{l}\underbrace{bb\cdots b}_{k-l}.$$

It is interesting to note here that when $e = k^2$ for some k, either case can be used. This reinforces the non-unique representation in a's and b's for perfectly alternating strings.

When $e > (n/2)^2$ consider $e' = \binom{n}{2} - e$, the number of edges in the complement.

Note that

$$e' = \binom{n}{2} - e < \binom{n}{2} - \left(\frac{n}{2}\right)^2 \leq \left(\frac{n}{2}\right)^2.$$

Using the algorithm above one can find an almost alternating threshold graph on n vertices and e' edges. To take the complement of a threshold graph is to simply change each 0 to a 1 and each 1 to a 0. Thus, taking the complement changes each a to a b and each b to an a. So taking the complement of this almost alternating graph on (n, e') will not only give a threshold graph on n vertices with e edges, but also an almost alternating graph. If an almost alternating threshold graph corresponding to (n, e') is

$$\underbrace{00\cdots0}_{k}\underbrace{aa\cdots a}_{\alpha}\underbrace{bb\cdots b}_{\beta}$$

then an almost alternating threshold graph on (n, e) is

$$\underbrace{11\cdots1}_{k}\underbrace{bb\cdots b}_{\alpha}\underbrace{aa\cdots a}_{\beta}.$$

Therefore, given any feasible n and e, there exists an almost alternating graph on n vertices with e edges.

□

3.3 Lemmas on Local Moves

To prove Theorem 40 we will make local switches in the code of a threshold graph that result in a graph having at least as many matchings, without changing the number of vertices or edges. The first move, the ab-switch, will preserve the total number of matchings and the number of matchings of each size. The other two local moves in this section will show that threshold graphs that have codes that include bracketed strings or have separation issues do not achieve the maximum number of matchings. Once we've shown this, we will have proved Theorem **??** by Lemma 43.

First, we present a simple lemma that will be used repeatedly.

Lemma 45. *Let σ, τ, and ρ be (possibly empty) binary strings. Suppose that $G = T(\sigma 01 \tau 10 \rho)$ has n vertices and e edges. The graph $G' = T(\sigma 10 \tau 01 \rho)$ is also a threshold graph on n vertices having e edges.*

Proof. Clearly the number of vertices remains the same under this switch. When a 01 becomes a 10, exactly one edge is added. Similarly, when a 10 is switched to a 01 exactly one edge is lost. Thus, G' is a threshold graph with n vertices and e edges. □

3.3.1 The ab-switch

The next lemma will show that if we replace a 0110 with a 1001 in the code of a threshold graph the number of matchings of each size are preserved. Note this is like switching an adjacent a and b as $0110 = ab$ and $1001 = ba$.

Definition 46. *If G' is obtained from G by replacing ab with ba or vice versa, we will say that we have performed an ab-*switch.

Lemma 47. *Let σ and ρ be (possibly empty) binary strings. Consider $G = T(\sigma 0110 \rho)$ and $G' = T(\sigma 1001 \rho)$. Then G and G' have the same number of vertices and edges. Moreover, $m(G) = m(G')$ and $m_k(G) = m_k(G')$ for all k.*

Proof. By Lemma 45 we know that G and G' have the same number of vertices and edges. Figure 3.3 demonstrates the difference between G and G'. On the left the subgraph of G induced by the vertices associated to the 0110 is shown and on the right is the subgraph of G' induced by the same subset of the vertices. Note G' can be obtained from G by removing edge yw and adding edge xv.

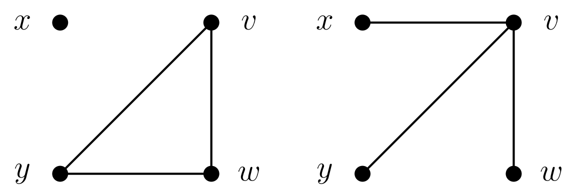

Figure 3.3: The subgraph induced by 0110 and the subgraph induced by 1001.

In G, the code 0110 corresponds to the vertices $xvwy$, i.e., x is the left 0, y is the right 0 and v and w are the left and right 1's, respectively. In G', on the other hand, 1001 corresponds to $vxyw$. That is, the vertex labels in the diagram are associated to a digit in the code and move with the digit when we make switches in the code.

We will construct two injections, one from the matchings in G to the matchings in G' and the other from the matchings in G' to the matchings in G. This will show that the number of matchings in G is equal to the number of matchings in G'.

To construct this injection we first define a replacement function $r : E(G) \to E(G')$. Let

$$r(e) = \begin{cases} xv & \text{if } e = yw \\ e\Delta\{x, y\} & \text{if } x \in e \\ e\Delta\{v, w\} & \text{if } v \in e, w \notin e, y \notin e \\ e & \text{otherwise} \end{cases}.$$

First we claim that any for edge e in $E(G)$ we have $r(e)$ in $E(G')$. Clearly $xv \in G'$. Suppose xc is some edge in $E(G)$. If c is adjacent to x in G then c must correspond to some 1 to the left of x in the code and so c is also adjacent to y in G. Since yc is an edge in G and $c \neq w$ (because w is to the right of x in the code), we know $yc \in E(G')$. Similarly, if $e = vc \in E(G)$ then c is either a dominating vertex added later than both v and w or c is a vertex added earlier than v in the construction. If

c is added later as a dominating vertex then $cw \in E(G)$ and so $cw \in E(G')$ as $c \neq y$. If c is a vertex added earlier than v in the construction then c is also adjacent to w in G as the code of w is a 1 and $c \neq w$ by assumption. So $cw \in E(G')$ since $cw \in E(G)$ and $c \neq y$ by assumption. Therefore $r(e) \in E(G')$ for all $e \in E(G)$.

Now define $\phi : \mathcal{M}(G) \to \mathcal{M}(G')$ by

$$\phi(M) = \begin{cases} M & \text{if } yw \notin M \\ \{r(e) : e \in M\} & \text{if } yw \in M \end{cases}.$$

We need to show two things: that $\phi(M)$ is a matching in G' and that ϕ is an injection. Note that $\phi(M) \subseteq E(G')$, by the fact that $yw \notin M$ in the first case and by the argument that $r(e) \in E(G')$ for all $e \in E(G)$ for the second case. To prove $\phi(M)$ is a matching we are only concerned about the case where we switch edges, i.e., when $yw \in M$. If $yw \in M$ then we replace yw with xv and resolve conflicts at x and v resulting in a matching. Note that r only changes an edge e incident to v if $w \notin e$ and $y \notin e$, but this is not a problem since vw and yv can not be in the matching if yw is.

Suppose M_1 and M_2 are matchings such that $\phi(M_1) = \phi(M_2)$. For any matching M the edge xv in $\phi(M)$ if and only if yw is in M. Thus, if $xv \notin \phi(M_1) = \phi(M_2)$ then $yw \notin M_1$ and $yw \notin M_2$ putting us in the first case so that $M_1\phi(M_1) = \phi(M_2) = M_2$. Suppose $xv \in \phi(M_1) = \phi(M_2)$. Then $yw \in M_1 \cap M_2$ putting us in the second case. If $yc \in \phi(M_1) = \phi(M_2)$ for some c then $xc \in M_1 \cap M_2$ and if $wd \in \phi(M_1) = \phi(M_2)$ for some d then $vd \in M_1 \cap M_2$. Additionally, all other edges were not moved. Thus, $M_1 = M_2$.

Therefore, $\phi : \mathcal{M}(G) \to \mathcal{M}(G')$ is an injection and so $m(G) \leq m(G')$. Moreover, the injection preserves the size of the matching and so $m_k(G) \leq m_k(G')$ for all k.

We define $\phi' : \mathcal{M}(G') \to \mathcal{M}(G)$ similarly. Define $r' : E(G') \to E(G)$ by

$$r'(e) = \begin{cases} yw & \text{if } e = xv \\ e\Delta\{x, y\} & \text{if } y \in e \\ e\Delta\{v, w\} & \text{if } w \in e \text{ and } v \notin e \\ e & \text{otherwise} \end{cases}$$

and $\phi' : \mathcal{M}(G') \to \mathcal{M}(G)$ by

$$\phi'(M) = \begin{cases} M & \text{if } xv \notin M \\ \{r(e) : e \in M\} & \text{if } xv \in M \end{cases}.$$

Since ϕ' is an injection, $m(G') \leq m(G)$ and $m_k(G') \leq m_k(G)$. Therefore, $m(G) = m(G')$ and $m_k(G) = m_k(G')$. □

Corollary 48. *Every almost alternating threshold graph on n vertices and e edges has the same number of matchings in total and the same number of matchings of each size.*

Proof. The proof of Lemma 44 shows that given n and e, the number of a's and b's and the length of the starting block is determined modulo the cases where e is a perfect square. (When e is a perfect square, there are two ways to represent the same threshold graph, one using only a's and the other using only b's. This does not affect this proof as they represent the same underlying code.) Lemma 47 shows that a's and b's in the code of the threshold graph commute. Thus, all almost alternating graphs that have the same beginning block (both in length and in digit) and have the same number of a's and b's have the same number of matchings. So, all almost alternating graphs with n vertices and e edges have the same number of matchings and the same number of matchings of each size. $\square$

3.3.2 Bracketed Strings

The proof of the next lemma gives us our first local move that will strictly increase the number of matchings, both in total and of size k for $k \geq 2$.

Lemma 49. *Any graph on n vertices with e edges that has a bracketed string appearing in its code does not have the maximum number of matchings in $\mathcal{T}_{n,e}$.*

Proof. Let G be a graph on n vertices with e edges that has a bracketed 1-string, say $G = T(\sigma 0 \underbrace{11 \cdots 1}_{l} 01\tau)$ for σ and τ (possibly empty) binary strings. Let $G' = T(\sigma 10 \underbrace{11 \cdots 1}_{l-2} 01\tau)$. We claim G' has n vertices, e edges and strictly more matchings than G. By Lemma 45, G' has the same number of vertices and edges as G.

Our attention will be focused on the subgraph associated to the bracketed 1-string. Figure 3.4 shows the subgraph of G induced by the vertices in the bracketed 1-string on the left. On the right is the subgraph of G' induced by the same subset of the vertices.

Let the vertex associated to the first 0 in the bracketed 1-string in G be called x and the vertex associated to the second 0 in the bracketed 1-string be called y. The vertices associated to the 1's in the bracketed 1-string in G form a clique. In Figure 3.4 the vertices inside the oval form this clique. Let w be the vertex associated to the rightmost 1 in the bracketed 1-string and z be the vertex associated to the leftmost 1. In G the vertex y is adjacent to every vertex in the clique and in the graph G' the vertex y is adjacent to all of those vertices except w.

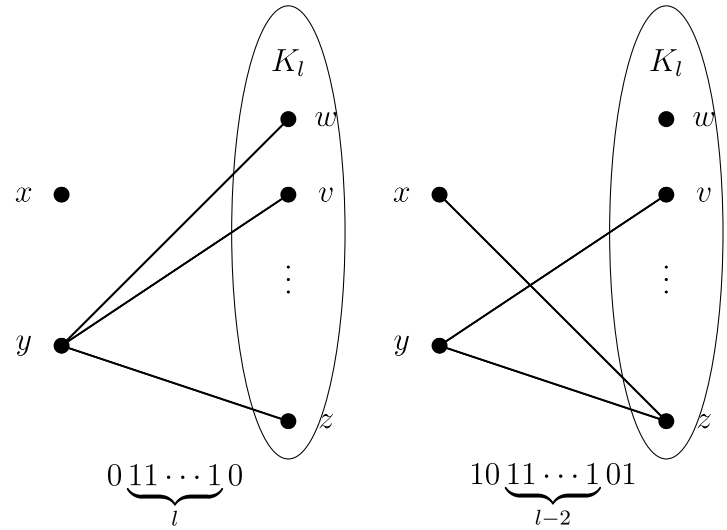

Figure 3.4: Removing a bracketed 1-string

Note that $V(G) = V(G')$ and $E(G') = E(G) - yw + xz$. To prove that G' has strictly more matchings we will construct an injection from the matchings in G to the matchings in G' that is not a surjection.

First define $r(e) : E(G) \to E(G')$ by

$$r(e) := \begin{cases} xz & \text{if } e = yw \\ e\Delta\{z, w\} & \text{if } z \in e \text{ and } y \notin e \\ e\Delta\{x, y\} & \text{if } x \in e \text{ and } w \notin e \\ e & \text{otherwise} \end{cases}.$$

First we claim that $r(e) \in E(G')$ for all $e \in E(G)$.

Define

$$\phi(M) = \begin{cases} M & \text{if } yw \notin M \\ \{r(e) : e \in M\} & \text{if } yw \in M \end{cases}.$$

Suppose that M_1 and M_2 are two matchings in G such that $\phi(M_1) = \phi(M_2)$. Note that $xz \in \phi(M)$ if and only if $yw \in M$. If $xz \notin \phi(M_1) = \phi(M_2)$ then $yw \notin M_1$ and $yw \notin M_2$ which means $M_1 = \phi(M_1) = \phi(M_2) = M_2$. Suppose that $xz \in \phi(M_1) = \phi(M_2)$. Then $yw \in M_1 \cap M_2$ and we use the second case. We can "undo" $r(e)$ to determine $M_1 = M_2$. In more detail, if $wc \in \phi(M_1) = \phi(M_2)$ for some c, then $zc \in M_1 \cap M_2$ and if $yd \in \phi(M_1) = \phi(M_2)$ for some d then $xd \in M_1 \cap M_2$. All other edges stay the same. Thus $M_1 = M_2$. Consider the matching $M = \{xz, yv\}$. Note that M is not in the image of ϕ and so $m(G') > m(G)$.

Now consider a threshold graph on n vertices with e edges that has a bracketed 0-string, say $G = T(\sigma 1 \underbrace{00\ldots0}_{k} 1\tau)$ for σ and τ binary strings. Consider $G' = T(\sigma 01 \underbrace{00\ldots0}_{k-2} 10\tau)$. Figure 3.5 demonstrates the difference between G and G' by showing the subgraph of G induced by the bracketed 0-string on the left and the subgraph of G' induced by the same vertices.

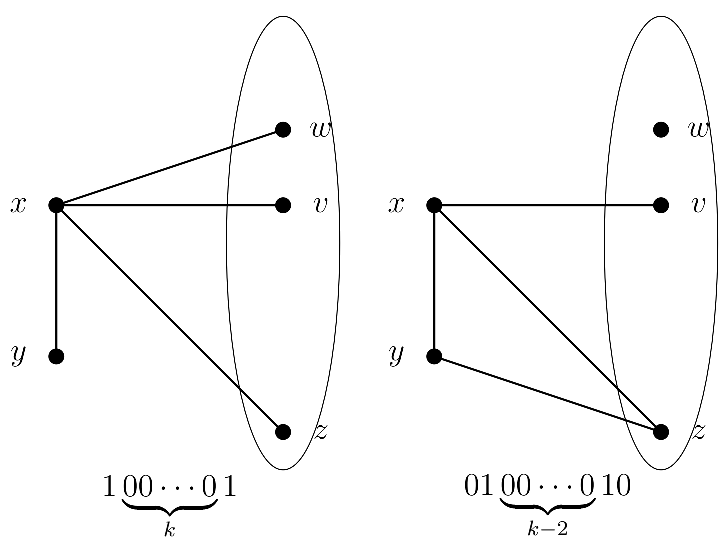

Figure 3.5: Removing a bracketed 0-string

This time, the vertices in the oval represent the 0's in this segment of the threshold graph and therefore are an independent set. Let w be the vertex associated with the rightmost 0 in the bracketed 0-string and z be the vertex associated with the leftmost 0. The x represents the first 1 and the y is the second 1. Note $V(G) = V(G')$ and $E(G') = E(G) - xw + yz$.

For a bracketed 0-string, a similar injection from the matchings in G to the matchings in G' that is not a surjection can be defined. Define $r : E(G) \to E(G')$ by

$$r(e) = \begin{cases} yz & \text{if } e = xw \\ e\Delta\{y,x\} & \text{if } y \in e \text{ and } x \notin e \\ e\Delta\{z,w\} & \text{if } z \in e \text{ and } x \notin e \\ e & \text{otherwise} \end{cases}.$$

Define $\phi : \mathcal{M}(G) \to \mathcal{M}(G')$ by

$$\phi(M) = \begin{cases} M & \text{if } xw \notin M \\ \{r(e) : e \in M\} & \text{if } xw \in M \end{cases}.$$

By a similar argument ϕ is an injection and so $m(G) \leq m(G')$ and $m_k(G) \leq m_k(G')$ for all k.

Now consider $M = \{xv, yz\}$. Note that M is not in the image of ϕ and so G' has strictly more matchings then G. Therefore, any graph that has a bracketed string does not have the maximum number of matchings. $\square$

3.3.3 Separation Issues

Recall that a code has a separation issue if it has two pairs of repeated digits separated by a substring of an odd length with the first pair preceded by the opposite digit and the last pair not ending the code. If there is a separation issue we can not write the code using a's and b's because we must separate each of the two sets of repeated digits to be an a and a b, but we are left with a substring of odd length in the middle. So any graph with a separation issue is not almost alternating.

Example 50. *The following graphs have separation issues.*

1. $011011*$
2. $11110101001010100*$

The first example is minimal in the sense that is the shortest it could be and still have a separation issue. In the second example the last ten digits (including the $*$*) have a separation issue with both repeated pairs being* 00.

There are 4 possible types for the code of a threshold graph that has a separation issue depending on the value of the pairs of repeated digits. They are:

- $\cdots 011 \underbrace{\cdots}_{odd} 11 \cdots *$
- $\cdots 100 \underbrace{\cdots}_{odd} 00 \cdots *$
- $\cdots 011 \underbrace{\cdots}_{odd} 00 \cdots *$
- $\cdots 100 \underbrace{\cdots}_{odd} 11 \cdots *.$

In the next lemma we will show that threshold graphs that have a separation issue do not maximize matchings. The proof is by induction on the length of the substring in between the repeated pairs. In the base case we will use an ab-switch to produce a bracketed string and then use Lemma 49 to conclude that the graph does not achieve the maximum.

Lemma 51. *A threshold graph with a separation issue does not have the maximum number of matchings.*

Proof. We will only consider threshold graphs that have a separation issue and do not have a bracketed string since we have already proven that threshold graphs with bracketed strings do not have the maximum number of matchings. Recall that $\{a,b\}^*$ is the set of all words formed with a's and b's. By extension, we will talk about 0,1 strings being in $\{a,b\}^*$ if they can written as a's and b's. We write, for example, $\{a,b\}^*a$ to mean a word in $\{a,b\}^*$ followed by an a. Also, let ϵ be the empty word. We say $\{a,b\}^*$ has *length* n if there are α a's and β b's and $\alpha+\beta=n$.

For the first case, suppose $G = T(\sigma 011\underbrace{\cdots}_{odd}11\tau)$ where σ and τ are binary strings and τ is not the empty string. Assume that the separation issue in the code of G is minimal. We claim that G has a substring that looks like

$$0110\ w\ 11$$

for some $w \in \{a,b\}^*b$ and the last digit shown here is not the last digit (the $*$) in the code.

There are a couple of characteristics of this string that do not follow simply from the definition of a separation issue. If there were not another 0 following the first pair of repeated digits we would have a bracketed 1-string. By Lemma 43 $w \in \{a,b\}^*$ since we are considering a minimal separation issue with no bracketed strings. Note that w possibly has length 0 in this case. The string must end with a b to avoid a bracketed 1-string with the second pair of repeated 1's.

Extending this idea to all cases, G must have one of the following four strings in its binary representation, where $w \in \{a,b\}^*$,

1. A 0110 followed by $w \in \{a,b\}^*b \cup \epsilon$ followed by 11.

$$0110\ w\ 11$$

2. A 1001 followed by $w \in \{a,b\}^*a \cup \epsilon$ followed by 00.

$$1001\ w\ 00$$

3. A 0110 followed by $w \in \{a,b\}^*a$ followed by 00.

$$0110\ w\ 00$$

4. A 1001 followed by $w \in \{a,b\}^*b$ followed by 11.

$$1001\ w\ 11$$

In all cases, there must be at least one more digit (or a $*$) to the right. The fact that w can not be the empty word in the third and fourth cases is also necessary to avoid a bracketed string, as is the condition that w end with a certain code.

We claim that if these strings are part of the binary representation of the threshold graph with n vertices and e edges then the graph does not have the maximum number of matchings in $\mathcal{T}_{n,e}$.

The proof is by induction. The proof of each case is similar, we will do cases 1 and 3 to illustrate the induction.

1. A 0110 followed by $w \in \{a,b\}^*b \cup \epsilon$ followed by 11.

$$0110\ w\ 11$$

For the base case $\langle a,b \rangle$ has length 0 and so $G = T(\sigma 011011 \tau)$ for strings σ and

τ with τ not empty. Applying Lemma 45

$$m(T(\sigma\underline{01}\ \underline{10}\ 11\tau) = m(\sigma 100111\tau).$$

Since τ is not empty we have a bracketed 1-string and can conclude the graph does not maximize matchings in $\mathcal{T}_{n,e}$. Assume that a graph that has a substring of the form 0110 w 11 is not maximal for all $w \in \{a, b\}^*$ of length at most n. Suppose $G = T(\sigma 0110\ w\ 11\tau)$ where σ and τ are binary strings with τ not empty and w has length $n + 1$.

Suppose w is a word of length $n + 1$ that starts with a. Let w' be such that $w = 01w'$. So w' has length n. Then

$$\begin{aligned} m(T(\sigma\ 01\ 10\ w\ 11\ \tau)) &= m(T(\sigma\ 01\ \underline{10}\ \underline{01}\ w'\ 11\ \tau)) \\ &= m(T(\sigma\ 01\ 01\ 10\ w'\ 11\ \tau)) \end{aligned}$$

using an ab-switch. Now we have a separation issue of shorter length and so by the inductive hypothesis G does not have the maximum number of matchings in $\mathcal{T}_{n,e}$.

Now consider the case where w starts with a b. Let w' be such that $w = 10w'$. So w' has length n. Then

$$\begin{aligned} m(T(\sigma\ 01\ 10\ w\ 11\ \tau)) &= m(T(\sigma\ \underline{01}\ \underline{10}\ 10\ w'\ 11\ \tau)) \\ &= m(T(\sigma\ 10\ 01\ 10\ w'\ 11\ \tau)). \end{aligned}$$

Again we have reduced to a separation issue of shorter length. Thus, any graph that contains the string in case 1 is not maximal.

3. A 0110 followed by $w \in \{a,b\}^*a$ followed by 00.

$$0110\ w\ 00$$

For the base case w has length 1. Since w must end in a, we know that $w = a$. Consider $G = T(\sigma 01100100\tau)$ where σ and τ are binary strings and τ is not empty. Using an ab-switch,

$$m(T(\sigma\ 01\ \underline{10}\ \underline{01}\ 00\ \tau)) = m(T(\sigma\ 01\ 01\ 10\ 00\ \tau)).$$

Recalling that τ is not empty we get a bracketed 0-string and thus the graph does not have the maximum number of matchings. Now assume that any threshold graph of the form $T(\sigma\ 0110\ w\ 00\ \tau)$ where σ and τ are binary strings, τ is not empty, and w has length at most n is not maximal. Suppose $G = T(\sigma\ 0110\ w\ 00\ \tau)$ where $w \in \{a,b\}^*$ has length $n+1$. If w starts with an a, let $w = aw'$ where $w' \in \{a,b\}^*$ of length n. Then

$$\begin{aligned} m(T(\sigma\ 01\ 10\ w\ 00\ \tau)) &= m(T(\sigma\ 01\ \underline{10}\ \underline{01}\ w'\ 00\ \tau)) \\ &= m(T(\sigma\ 01\ 01\ 10\ w'\ 00\ \tau)) \end{aligned}$$

which has a shorter separation issue. Similarly, if w begins with a b write $w = bw'$ where w' has length n. Then

$$\begin{aligned} m(T(\sigma\ 01\ 10\ w\ 00\ \tau)) &= m(T(\sigma\ \underline{01}\ \underline{10}\ 10w'\ 00\ \tau)) \\ &= m(T(\sigma\ 10\ 01\ 10\ w'\ 00\ \tau)) \end{aligned}$$

which has a shorter separation issue.

□

3.4 Proof of Theorem

Recall that our main theorem in this chapter states that a graph G in $\mathcal{T}_{n,e}$ has the maximum number of matchings in the family if and only if G is almost alternating.

Proof. Suppose that a threshold graph G is not almost alternating. Then by Lemma 43, G has a separation issue or a bracketed string. By Lemma 49 and Lemma 51, G does not have the maximum number of matchings of size k. Thus we conclude that almost alternating graphs maximize the number of matchings of size k. Since any graph that is not almost alternating has strictly fewer matchings and, by Corollary 48, all almost alternating graphs have the same number of matchings we conclude that a threshold graph has the maximum number of matchings if and only if it is almost alternating. □

Chapter 4

Graphs with the Fewest Matchings

4.1 Introduction and Statement of Results

Recall that $m(G)$ is the number of matchings in a graph G and $m_k(G)$ is the number of matchings in G that have k edges. Our main results in this chapter are that in $\mathcal{G}_{n,e}$ the parameters $m(G)$ and $m_k(G)$ are each minimized by either the lex graph or the colex graph and in $\mathcal{B}_{\ell,r,e}$ the parameters $m(G)$ and $m_k(G)$ are each minimized by the bipartite lex graph.

Theorem 52. *For all graphs G with n vertices and e edges, and for all k,*

$$m(G) \geq \min\{m(\mathcal{L}(n,e)), m(\mathcal{C}(n,e)\}$$

and

$$m_k(G) \geq \min\{m_k(\mathcal{L}(n,e)), m_k(\mathcal{C}(n,e)\}.$$

A core part of the proof of Theorem 52 is to establish that, for all $k \geq 0$, $L_{\ell,r}(e)$ minimizes $m_k(G)$ in the class $\mathcal{B}_{\ell,r,e}$. It is key that there is a unique minimizer in the bipartite case.

Theorem 53. *Suppose* $1 \leq k \leq \ell \leq r$ *and* $B \in \mathcal{B}_{\ell,r,e}$. *Then*

$$m(B) \geq m(L_{\ell,r}(e))$$

and

$$m_k(B) \geq m_k(L_{\ell,r}(e)).$$

To prove Theorem 52, we first recall that there is a graph attaining the minimum that is threshold (see Lemma 18). In Section 4.3 we will prove Theorem 53, i.e., that there is a bipartite graph that simultaneously minimizes the number of matchings of each size. Before we do that, we will describe the connections between bipartite threshold graphs and partitions and matchings and rook placements in Section 4.2. In the last section of this chapter, we use the bipartite case to show that the lex or colex graph minimizes $m(G)$ and $m_k(G)$ in the family $\mathcal{G}_{n,e}$.

4.2 Partitions and Young Diagrams

Recall that we can find a bipartite minimizing the number of matchings among the threshold bipartite graphs. We can relate threshold bipartite graphs to partitions and matchings in threshold bipartite graphs to rook placements in the Young diagram of the partition. We will make use of this connection heavily in the proof of Theorem 53.

Given (ℓ, r, e), threshold bipartite graphs G with vertex classes of size ℓ and r and $|E(G)| = e$ are in bijective correspondence with partitions of the integer e with at most ℓ parts each of size at most r. From a partition λ we can construct the associated bipartite graph G_λ by letting $E(G_\lambda) = \{x_i y_j : j \leq \lambda_i\}$. Given a threshold bipartite graph we get a partition of $|E(G)|$ by letting the degree of each vertex on the left be the size of a part. In Figure 4.1 is an example of a threshold bipartite graph with degree sequence $(4,3,2,2)$ and in Figure 4.2 is the associated Young diagram.

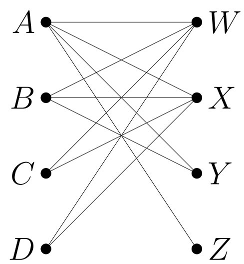

Figure 4.1: A threshold bipartite graph

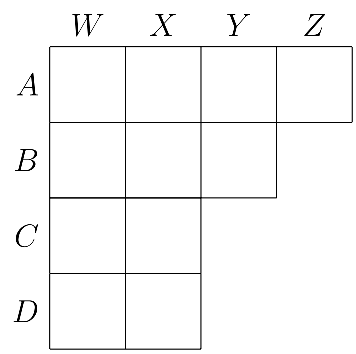

Figure 4.2: Associated Young diagram

We will use the correspondence to represent threshold bipartite graphs as Young diagrams.

Definition 54. *Let B be a subset of $[\ell]\times[r]$. If $(i,j)\in B$ we call (i,j) a* box *of B. We call B a* Young diagram *if for all $(i,j)\in B$ with $i>1$ we have $(i-1,j)\in B$ and for all $(i,j)\in B$ with $j>1$ we have $(i,j-1)\in B$. We call $[\ell]\times[r]$ the* frame *of the Young diagram and we will say that the Young diagram has* dimensions *$\ell\times r$. A* matching *in a Young diagram B is a subset M of B such that for all $(a_1,b_1),(a_2,b_2)\in M$ we have $a_1\neq a_2$ and $b_1\neq b_2$.*

Equivalently, a matching is a placement of non-attacking rooks on B. (A placement of non-attacking rooks is a placement of rooks such that no two rooks are in the same row or column.) The total number of ways to place non-attacking rooks is called the rook number. There is extensive literature on rook numbers, for example see [21]. We will use the language of rook placements in some of the proofs.

Matchings in a Young diagram B correspond to a matchings in the bipartite graph associated with B by equating the box $(i,j)\in B$ with the edge x_iy_j in the associated bipartite graph.

Notation. *Let the set of k-matchings in a Young diagram B be denoted $\mathcal{M}_k(B)$ and let $\mathcal{M}(B) = \bigcup_{k \geq 0} \mathcal{M}_k(B)$. Also define $m_k(B) = |\mathcal{M}_k(B)|$ and $m(B) = |\mathcal{M}(B)|$.*

Example 55. *Figure 4.3 is an example of a Young diagram in a 4×6 frame and Figure 2 is a matching of size 3 in that Young diagram. In all our diagrams we label rows and columns using matrix numbering.*

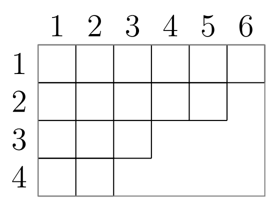

Figure 4.3: Young diagram of (6,5,3,2).

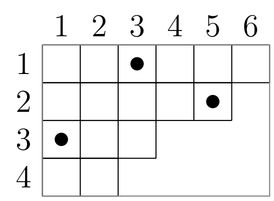

Figure 4.4: $\{(1,3),(2,5),(3,1)\}$.

It is worth noting that there are two very similar representations of partitions: Young diagrams and Ferrers diagrams. In a Ferrers diagram dots are used instead of boxes. We choose to use Young diagrams here so that, in pictures, the rooks may be represented by dots. There has been much study of Young tableaux which are Young diagrams in which numbers are placed in the boxes. Young tableaux have been useful in representation theory, see, for example, [12].

4.3 Bipartite Minimizer

In this section we prove Theorem 53. We are motivated to look at the bipartite case by the following lemma.

Lemma 56. *Given a threshold graph G with vertex set V, let $V = K \cup (V \setminus K)$ be a partition of the vertex set such that $G[K]$ forms a clique and $G[V \setminus K]$ forms an* *independent set. Suppose $|K| = s$. Let B be the bipartite graph with partite sets K and $V \setminus K$ and edge set $E(K, V \setminus K)$. Then*

$$m(G) = \sum_{k \geq 0} m_k(B) \cdot m(K_{s-k}).$$

Proof. Fixing a matching M in B of size k, there are exactly $m(K_{s-k})$ matchings in G that contain M. So there are $m_k(B) \cdot m(K_{s-k})$ matchings in G that contain a k-matching in B. Summing over all possible sizes of matchings in B we count every matching in G exactly once. $\square$

Theorem 53 states that for a given k there is a graph that simultaneously minimizes every $m_k(G)$ in $\mathcal{B}_{\ell,r,e}$. We will first show that there is a threshold bipartite graph that minimizes $m_k(G)$ in $\mathcal{B}_{\ell,r,e}$. We will then use moves on the associated Young diagrams to show that the lex bipartite graph $L_{\ell,r}(e)$ minimizes $m_k(G)$ for all k.

Definition 57. *Say P is an* out-corner *of a Young diagram B if $P \in B$ and there is no box in the diagram to its right or beneath it. Say Q is an* in-corner *if a box can be added there to create an out-corner.*

In the context of Young tableaux, the out-corners defined above to are referred to as corners.

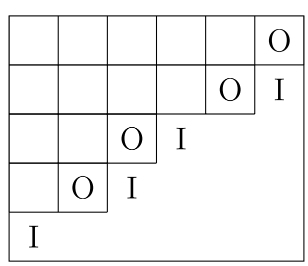

Figure 4.5: In-corners and out-corners of (6,5,3,2) in a 5×6 frame

Example 58. *In Figure 4.5 the in-corners are labeled with I and the out-corners are labeled with O. Notice that the dimensions of the frame matter; for example, in a* 4×6 *frame* $(5, 1)$ *would not be an in-corner.*

We now define a move that removes a box that is an out-corner and puts a box at an in-corner that is "further out". Let $s(P)$ be the sum of the coordinates of P.

Lemma 59. *Let B be a Young diagram in an $m \times n$ frame. Suppose P is an out-corner of B, P' is an in-corner of B, and P and P' are not adjacent. If $s(P) < s(P')$ then $B' := B + P' - P$ is a Young diagram in an $m \times n$ frame and has at most as many k-matchings as B for all $k \geq 0$. Moreover, $m(B') < m(B)$.*

Proof. Suppose $P = (i, j)$ and $P' = (i', j')$. The hypothesis states that $i + j < i' + j'$. Define $B^+ = B + P' = B' + P$. Note that B^+ is a Young diagram since P' is an in-corner. Showing that B' has fewer matchings than B is equivalent to showing that there are fewer matchings in B^+ that contain P' and not P than matchings in B^+ that contain P and not P'. To do this we will define an injection from the collection of k-matchings of B^+ that contain P' to the collection of k-matchings of B^+ that contain P.

Define $S := \{(a, j) \,:\, i' < a < i\}$ and $T := \{(i', b) \,:\, j < b < j'\}$. The injection will be defined in two parts: firstly on those matchings of B^+ that don't intersect S and secondly on those that do.

In Figure 4.6 we have drawn P' up and to the right of P. It is also possible that P' is down and to the left of P. In this case the same proof will work by using the transpose of B^+.

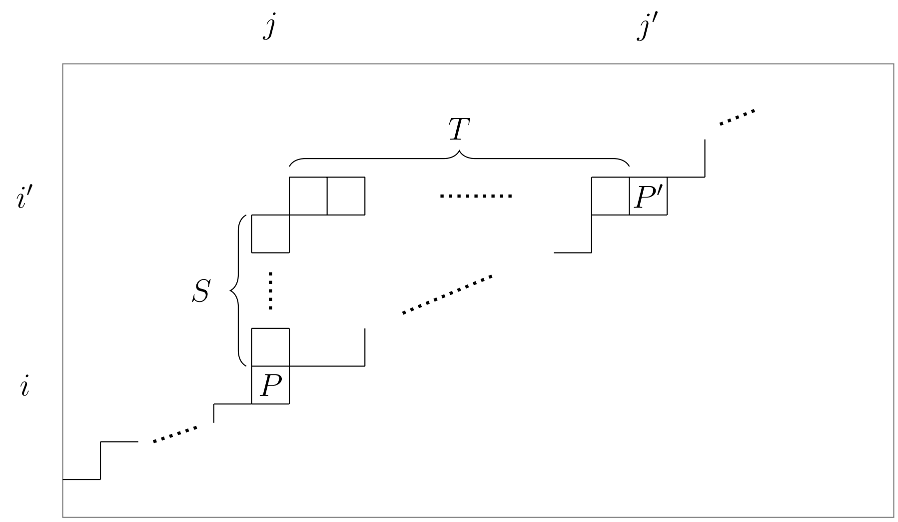

Figure 4.6: The Young diagram B^+.

Let

$$A_{\overline{S}} := \{M \in \mathcal{M}(B^+) : P' \in M, P \notin M, S \cap M = \emptyset\},$$

$$A_{\overline{T}} := \{M \in \mathcal{M}(B^+) : P \in M, P' \notin M, T \cap M = \emptyset\},$$

$$A_S := \{M \in \mathcal{M}(B^+) : P' \in M, P \notin M, S \cap M \neq \emptyset\},$$

$$A_T := \{M \in \mathcal{M}(B^+) : P \in M, P' \notin M, T \cap M \neq \emptyset\}.$$

We first define a bijection between $A_{\overline{S}}$ and $A_{\overline{T}}$ and then we define an injection from A_S to A_T. For each case, we will define a replacement function r_i on the blocks of B^+ and then an injection f_i on the appropriate matchings. In the end we will have

$$A_{\overline{S}} \xleftrightarrow{f_1} A_{\overline{T}}$$

$$A_S \overset{f_2}{\hookrightarrow} A_T.$$

Since each map will send matchings of size k to matchings of size k, we conclude $m_k(B') \leq m_k(B)$ for all $k \geq 0$.

Case 1: Suppose $M \in A_{\overline{S}}$, that is, M is a matching in B^+ such that $P' \in M$, $P \notin M$, and $S \cap M = \emptyset$. Define $r_1 : B^+ \to B^+$ by

$$r_1(a,b) := \begin{cases} (i,j) & \text{if } a = i', b = j' \\ (i',b) & \text{if } a = i \\ (a,j') & \text{if } b = j \\ (a,b) & \text{otherwise} \end{cases}.$$

We can think of r_1 as sending the rook at P' to P, and then projecting rooks in row i and column j onto row i' and column j', respectively. We claim $r_1(a,b) \in B^+$ for all $(a,b) \in B^+$. Since $P = (i,j)$ is an out-corner of B, if $(i,b) \in B^+$ then $b \leq j < j'$. Because B^+ is a Young diagram and $P' = (i',j') \in B^+$, we have $r_1(i,b) = (i',b) \in B^+$. In this case $M \cap S = \emptyset$ so a rook of the form (a,j) with $a \neq i$ must have $a < i'$. Since B^+ is a Young diagram and $P' = (i',j') \in B^+$, we know $r_1(a,j) = (a,j')$ is in B^+. Therefore, $r_1(a,b) \in B^+$ for all $(a,b) \in B^+$.

Define $f_1 : A_{\overline{S}} \to A_{\overline{T}}$ by

$$f_1(M) = \{r_1(a,b) : (a,b) \in M\}.$$

First, we show that given $M \in A_{\overline{S}}$, $f_1(M)$ is in fact in $A_{\overline{T}}$. Sending the rook in P' to P causes conflicts only for rooks in row i and column j. We have solved the problem of a rook in row i since we changed the row of this rook to i'. Note that row i' is otherwise unoccupied since $P' \in M$ and M is a matching. Similarly, we have solved the problem of a rook in column j since we changed the column of this rook to j'. This column is otherwise unoccupied since M is a matching and $P' \in M$. Thus, $f_1(M)$ is a matching in B. Moreover, $f_1(M)$ has no rooks in T and so $f_1(M) \in A_{\overline{T}}$.

Note also, of course, that $f_1(M)$ has the same size as M.

In addition, f_1 is a bijection. Define $r_1' : B^+ \to B^+$ by

$$r_1'(a,b) := \begin{cases} (i',j') & \text{if } (a,b) = (i,j) \\ (i,b) & \text{if } a = i', b \neq j \\ (a,j) & \text{if } a \neq i, b = j' \\ (a,b) & \text{otherwise} \end{cases}$$

and define $f_1' : A_{\overline{T}} \to A_{\overline{S}}$ by

$$f_1'(M) = \{r_1'(a,b) : (a,b) \in M\}.$$

It is straightforward to check that $f_1'(f_1(M)) = M$ for all $M \in A_{\overline{S}}$ and $f_1(f_1'(M)) = M$ for all $M \in A_{\overline{T}}$. Thus, there is a bijection between the matchings in B^+ with P' and no rooks in S and matchings in B^+ with P and no rooks in T.

Case 2: Suppose $M \in A_S$. That is, M is a matching in B^+ with a rook in P' and a rook in S. Define $E := \{(a,b) : (a,b) \in B^+, a > i', b > j\}$. In Figure 4.6 this is collection of blocks that are both to the right of P and below P'. Let $S^*(M) \subset \{i'+1, \ldots, i-1\}$ be the rows of blocks in S that do not share a row with any rooks of M that are in E. Similarly, let $T^*(M) \subset \{j+1, \ldots, j'-1\}$ be the columns of blocks in T that do not share a column with any rooks of M that are in E. We claim that $|S^*(M)| < |T^*(M)|$. Note that $|S| = i - i' - 1$ and $|T| = j' - j - 1$. Since $i + j < i' + j'$ we have $i - i' < j - j'$ and so $|S| < |T|$. Letting a be the number of rooks in E, then $|S^*(M)| = |S| - a$ and $|T^*(M)| = |T| - a$. So $|S^*(M)| < |T^*(M)|$. Thus, there is an injection $s : S^*(M) \to T^*(M)$. We fix some arbitrary injection s

and let (a_0, j) be the location of the rook in S. Define $r_2 : B^+ \to B^+$ by

$$r_2(a,b) := \begin{cases} (i,j) & \text{if } (a,b) = (i',j') \\ (i', s(a)) & \text{if } (a,b) \in S \\ (a_0, b) & \text{if } a = i \\ (a, j') & \text{if } b = s(a_0) \\ (a,b) & \text{otherwise} \end{cases} .$$

In Figure 4.7 the gray boxes are the images of the black boxes under the map r_2. We can think of r_2 as sending the rook in P' to P (arrow 1), sending the rook in S to a place in T via s (arrow 2), and then projecting rooks in conflicting rows and columns to rows and columns that are known to be unoccupied (arrows 3 and 4).

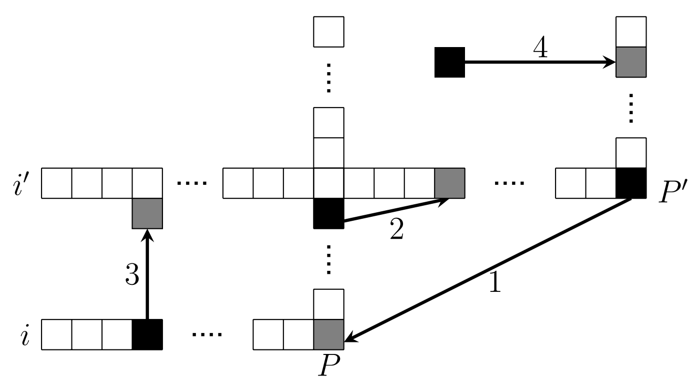

Figure 4.7: A sketch of the map r_2

Again we need to show that $r_2(a,b) \in B^+$ for all $(a,b) \in B^+$. If $(a,b) \in S$ then $(i', s(a)) \in B^+$ as $(i', s(a)) \in T$. If the rook (a,b) is in row i then $r_2(a,b) = (a_0, b) \in B^+$ as $a_0 < i$ and $(i,b) \in B^+$. Finally, if $(a,b) = (a, s(a_0))$, then $r_2(a,b) = (a, j') \in B$ since $a < i'$ and (i', j') is in B^+.

For $M \in A_S$, define

$$f_2(M) := \{r_2(a,b) : (a,b) \in M\}.$$

We claim that if M is a matching in A_S then $f_2(M)$ is a matching in A_T. First we will show that no two rooks are in the same row. There are only three rooks that change rows when we apply r_2. These rooks originally have rows i', i and a_0. After applying r_2 these rooks are in rows i, a_0 and i', respectively. Thus the rooks of $f_2(M)$ occupy the same collection of rows as those of M. Now we must show that no two rooks occupy the same column. Similarly, there are only three rooks that change columns. These rooks originally occupy columns j', j, and $s(a_0)$. After applying r_2, these rooks occupy $j, s(a_0)$, and j', respectively. So the rooks of $f_2(M)$ occupy the same collection of columns as those of M and so no two rooks are in the same column. Finally, $T \cap f_2(M) \neq \emptyset$ since the rook in S got sent to a rook in T. Thus $f_2(M)$ is a matching in A_T of the same size as M.

To show that f_2 is an injection we can define r_2' to "undo" r_2, similarly to the way that r_1 is defined. Then one can show $f_2' := \{r_2'(a,b) : (a,b) \in M\}$ is the left inverse of f_2.

Using f_1 and f_2 we know where to send all matchings in B^+ that have a rook in P' and not P. Moreover, the images of f_1 and f_2 are disjoint and thus there is an injection from matchings in B^+ with a rook in P' and not P to matchings in B^+ that have a rook in P and not P'. Since this injection preserves the size of the matching, $m_k(B') \leq m_k(B)$.

It remains to prove that B' has strictly fewer matchings than B. Consider the set of all matchings $M \in B^+$ with no rooks in E. For these M, in the definition of f_2 we may take the same fixed injection $s : S^*(M) \to T^*(M)$. Since $|S^*(M)| < |T^*(M)|$ there exists $Q \in T^*(M)$ such that $Q \neq s(R)$ for any $R \in S^*(M)$. Then $\{Q, P\} \in A_T$ and there does not exist $M \in A_S$ such that $f_2(M) = \{Q, P\}$. Thus, $m(B') < m(B)$. $\square$

Definition 60. *Given a Young diagram B with an out-corner P and an in-corner P' with $s(P) < s(P')$ we call the move that results in $B - P + P'$ an* out-block move.

There are examples of Young diagrams that are not the Young diagrams associated to $L_{\ell,r}(e)$ that have no out-block moves. For example, see Figure 4.8. For this reason we are forced to introduce an additional move. It is clear that taking the transpose of a Young diagram preserves $m_k(B)$ for all k. The following definition describes a way that we can transpose a piece of a Young diagram.

Definition 61. *Let B be a Young diagram in an $\ell \times r$ frame and let $(i, j) \in B$. Define the* transpose of B at $P = (i, j)$ *to be the diagram*

$$B_P^* := \{(a, b) \in B : a < i \text{ or } b < j\} \cup \{(a, b)^* : (a, b) \in B, a \geq i, \text{ and } b \geq j\}$$

where $(a, b)^* = (b - j + i, a - i + j)$. Call transposing at P legal *if B_P^* is a Young diagram in an $\ell \times r$ frame.*

Note that the definition of $(a, b)^*$ depends on (i, j), the place we are transposing, but we suppress this in the notation.

Example 62. *Figures 4.8 and 4.9 are two Young diagrams in a 4×5 frame. To get the Young diagram in Figure 4.9 from the Young diagram in Figure 4.8 we transpose at $P = (1, 2)$.*

We want to show that legally transposing at $P \in B$ preserves m_k for all $k \geq 0$. The following result appears as Lemma 9 in Foata and Schützenberger [10]. In their paper, they prove the interesting fact that every Young diagram is rook equivalent (meaning

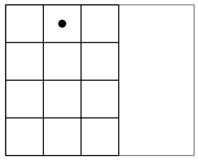

Figure 4.8: The Young diagram B of (3,3,3,3)

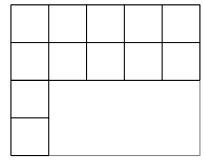

Figure 4.9: The Young diagram of $B^*_{(1,2)} = (5,5,1,1)$

it has the same number of rook placements of each size) to a unique increasing Young diagram. We include a proof here for completeness.

Lemma 63. *Let B be a Young diagram and let $P \in B$. Performing a legal transpose at P preserves the number of matchings of all sizes.*

Proof. Suppose that the sub-board to be transposed has dimensions $x \times y$. Write $P = (i,j)$. Without loss of generality, let $\max\{x,y\} = x$. We single out the following pieces of B as sketched in Figure 4.10 :

$T = \{(a,b) \ : \ (a,b) \in B, a \geq i, b \geq j\}$	the **T**ransposed portion
$U = \{(a,b) \ : \ a < i, j \leq b \leq j+x\}$	the portion **U**p from T
$L = \{(a,b) \ : \ i \leq a \leq i+x, b < j\}$	the portion to the **L**eft of T

Fix a matching in T, call it M_T. Define $T^* := \{(a,b)^* : (a,b) \in T\}$ and let M_T^* be the matching in T^* obtained by transposing the location of each of the rooks. We will define an injection from matchings in B that contain M_T to matchings in B_P^* that contain M_T^*. Doing this for every matching in T will show that there are at most as many matchings in B as in B_P^*. A similarly defined injection will work to conclude that B_P^* has at most as many matchings as B. Since these injections will preserve the size of each matching we will conclude that $m_k(B) = m_k(B_P^*)$.

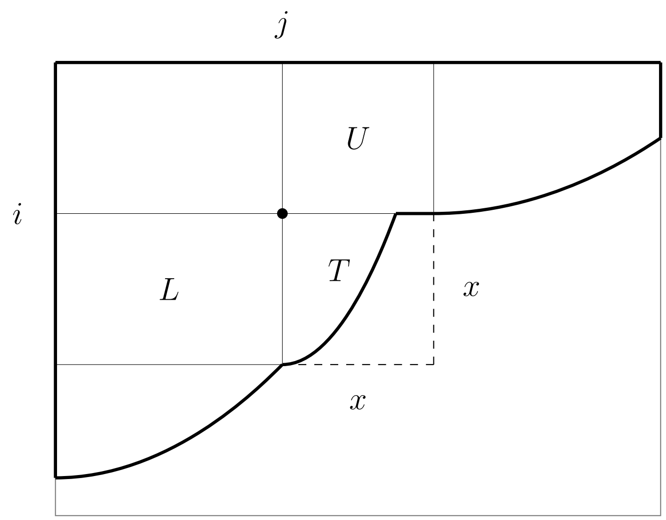

Figure 4.10: Pieces of B.

Let $U_1 \subseteq \{j, \ldots, j+x\}$ be the set of column indices between j and $j+x$ that are unoccupied by a rook in M_T and similarly let $U_2 \subseteq \{j, \ldots, j+x\}$ be the set of column indices between j and $j+x$ that are unoccupied by a rook in M_T^*. If $|M_T| = t$ then $|U_1| = |U_2| = x+1-t$. Thus, there is a bijection $u : U_1 \to U_2$.

Similarly, let $L_1 \subseteq \{i, \ldots, i+x\}$ be the set of row indices between i and $i+x$ that are not occupied by a rook in M_T and let $L_2 \subseteq \{i, \ldots, i+x\}$ be the set of row indices between i and $i+x$ of rows that are not occupied by a rook in M_T^*. Again, $|L_1| = |L_2| = x+1-t$ where t is the size of M_T and hence there is a bijection $l : L_1 \to L_2$.

Define $r : B \to B_P^*$ by

$$r(a,b) := \begin{cases} (a,b)^* & \text{if } (a,b) \in T \\ (l(a),b) & \text{if } (a,b) \in L, a \in L_1 \\ (a,u(b)) & \text{if } (a,b) \in U, b \in U_2 \\ (a,b) & \text{otherwise} \end{cases}.$$

Define f from matchings in B containing M_T to matchings in B_P^* containing M_T^* by

$$f(M) = \{r(a,b) : (a,b) \in M\}.$$

First we note that given $M \in \mathcal{M}(B)$ we have $f(M) \in \mathcal{M}(B_P^*)$. We know $f(M) \subset B_P^*$ since $r(a,b) \in B_P^*$ for all $(a,b) \in B$. For each rook, $(a,b) \in M_T$ we send (a,b) to $(a,b)^*$. Conflicts are only caused in rows $i, i+1, \ldots, i+r$ and columns $j, j+1, \ldots j+r$. These conflicts are resolved using the injections l and u which send all rooks in L and U to rows and columns unoccupied by M_T^*. This causes no additional conflicts since no additional rows or columns are changed. Thus $f(M) \in \mathcal{M}(B_P^*)$. Moreover, for a matching $M \in \mathcal{M}(B)$ containing M_T, $f(M)$ contains M_T^*.

We claim that f is a bijection. Define

$$r'(a,b) := \begin{cases} (a,b)^* & \text{if } (a,b) \in T \\ (l^{-1}(a), b) & \text{if } (a,b) \in L, a \in L_2 \\ (a, u^{-1}(b)) & \text{if } (a,b) \in U, b \in U_2 \\ (a,b) & \text{otherwise} \end{cases}.$$

Define f' from matchings in B_P^* containing M_T^* to matchings in B containing M_T by

$$f'(M) = \{r'(a,b) : (a,b) \in M\}.$$

It is straightforward to check that f' is actually the inverse of f. Thus, f is a bijection and $m_k(B) = m_k(B^*)$. $\square$

The next lemma shows how we piece together the out-block move and the transpose move. First we define the lex order on Young diagrams.

Definition 64. *To define the lex order on Young diagrams, we first define an ordering on ordered pairs. Say* $(a, b) \lesssim (c, d)$ *if* $a < c$ *or* $a = c$ *and* $b < d$*. Then the* lex order $<_L$ *on Young diagrams is defined by* $B <_L B'$ *if and only if* $\min_{\lesssim}(B \Delta B') \in B$*.*

Note that by this definition, $L_{\ell,r}(e)$ is least among all Young diagrams in $\ell \times r$ frames. The next lemma states that if we can't find an out-block move we can find a legal transpose that moves a board to one earlier in lex order.

Lemma 65. *Consider a Young diagram* B *in an* $\ell \times r$ *frame where* $\ell \leq r$ *that has no out-block moves and is not the Young diagram of* $L_{\ell,r}(e)$*. There exists* $P \in B$ *such that the transpose at* P *is legal and* $B_P^* <_L B$*.*

Proof. First we set up some notation. For $P = (i, j) \in B$, let $\rho(P) = i$ (so $\rho(P)$ is the row of P) and let $c(P) = j$ (so $c(P)$ is the column of P). For $P, Q \in B$, define $v(P, Q) = |\rho(P) - \rho(Q)|$, so $v(P, Q)$ is the vertical distance between P and Q. In the Young diagram B let P be the out-corner with $\rho(P)$ greatest among all out-corners and Q be the in-corner of B with $\rho(Q)$ least among all in-corners. If legal, transpose at $S = (\rho(Q), c(P))$. If transposing at S is not legal then we "count back" from the right hand limit of the partition so that the vertical distance between P and Q will fit. In more detail, transpose at $S' = (\rho(Q), r - v(P, Q))$ whenever transposing at S is not legal. We claim that this gives a place to transpose legally and that it results in a board that is earlier in lex order than B. Call the result of the performed transpose B^*.

First suppose that transposing at S is legal. Note that $S \in B$ as B is a Young diagram and $P \in B$. If we transpose at S then it is because the transpose at S is legal. The first row where B^* is different from B is $\rho(Q)$. In B this row has $c(Q) - 1$ blocks. In B^* this row has $c(P) + v(P, Q) = c(P) + \rho(P) - \rho(Q)$ blocks. (Since B is not the Young diagram of $L_{\ell,r}(e)$ we know $\rho(P) > \rho(Q)$ and so may remove the absolute value signs in $v(P,Q)$.) The board B^* does not exceed ℓ rows because there are no out-block moves. Moreover, $s(P) > s(Q)$, i.e, $\rho(P) + c(P) > \rho(Q) + c(Q)$. Thus $c(Q) < c(P) + \rho(P) - \rho(Q)$ and so $B^* <_L B$.

Now suppose that transposing at S is not legal. We claim that that $S' = (\rho(Q), r - v(P,Q)) \in B$. Because transposing at S is not legal, we know $c(Q) + v(P,Q) > r$. Thus, $r - v(P,Q) < c(Q)$. Moreover $r - v(P,Q) \geq 1$ as $\ell \leq r$ and $v(P,Q) < \ell$. Using that Q is an in-corner of B, we get that $S' \in B$. We need to show that the transpose is legal. Intuitively, the transposed piece fits horizontally because we "counted back" far enough to make it so. Algebraically, the length of the row $\rho(Q)$ in B^* is $r - v(P,Q) + v(P,Q) = r$ which is exactly the length that can fit. we will show that the transposed piece fits vertically after we show that $B_S^* >_L B$. Note the first place B and B^* differ is in row $\rho(Q)$ and row $\rho(Q)$ in B^* has r blocks while row $\rho(Q)$ in B has strictly fewer than r blocks since Q is an in-corner. In particular, this means, if the dimensions of the transposed section are $s \times t$, we have $s \geq t$. Thus the transposed piece will fit vertically. Therefore, transposing at one of S or S' will result in a legal transpose that moves B earlier in lex order.

$\square$

We are now ready to prove Theorem 53. The proof follows easily from Lemmas 19 and 65.

Proof of Theorem 53. Suppose G is a bipartite graph that is not equal to $L_{\ell,r}(e)$. If G is not bipartite threshold then we can apply Lemma 19 to get another graph with the same number of vertices and edges, but at most as many matchings. If G is bipartite threshold, consider the associated Young diagram B_G. If B_G has no out-block moves and $G \neq L_{\ell,r}(e)$ then by Lemma 65 there is a legal transpose at (i,j) that moves the associated board earlier in lex order. This move preserves the number of matchings by Lemma 63. Thus, $L_{\ell,r}(e)$ attains the minimum number of matchings of all sizes. This concludes the proof of Theorem 53. $\square$

4.4 Proof of Theorem 52

Recall that our main result states that either the lex or colex graph minimizes the number of matchings (of size k) in $\mathcal{G}_{n,e}$. In this section we will prove this by using the bipartite case.

Definition 66. *For a threshold graph G write $V(G) = K \cup I$ where $G[K]$ is a clique and $G[I]$ is an independent set. Let B be the bipartite graph with partite sets K and I and edge set $E(K, I)$. If B is $L_{|K|,|I|}(e)$ for some e, we will say that G is* lex-across*. We say that a parameter P is* lex-across optimized *if for all n and e there is a P-optimal threshold graph in $\mathcal{G}_{n,e}$ and moreover, for all $s, t,$ and e there is a lex-across threshold graph that optimizes P over the collection of threshold graphs, T, with $|K| = s, |I| = t,$ and $e(T) = e$.*

We will now prove a lemma that states that if some parameter is lex-across optimized, then that parameter is maximized or minimized by the lex or colex graph.

Lemma 67. *If P is a parameter that is lex-across optimized then either the lex graph $\mathcal{L}(n, e)$ or the colex graph $\mathcal{C}(n, e)$ is P-optimal.*

Proof. Given n, e let $\mathcal{O}$ be the collection of all triples (G, K, I) such that $V(G) = K \cup I$, $G[K]$ is complete, $G[I]$ is independent, and G is a P-optimal lex-across graph. Define $\overline{s}$ to be the maximum size of K among all triples in $\mathcal{O}$ and let $\underline{s}$ be the minimum size of K among all triples in $\mathcal{O}$. We consider two cases: when $\overline{s} \geq \frac{n}{2}$ and when $\overline{s} < \frac{n}{2}$.

Suppose first that $\overline{s} \geq \frac{n}{2}$. Let $(G, K, I) \in \mathcal{O}$ such that $|K| = \overline{s}$. In this case K has at least as many vertices as I. If $v \in I$ has $N(v) = K$ then $(G, K \cup \{v\}, I \setminus \{v\}) \in \mathcal{O}$

and $|K \cup \{v\}| \geq \overline{s}$, a contradiction. Since $(G, K, I) \in \mathcal{O}$ we know G is lex-across, and so it must be the case that some vertex in I has fewer than $\overline{s}$ neighbors and all other vertices in I are isolated. Thus G is the colex graph $\mathcal{C}(n, e)$.

Suppose now that $\overline{s} < \frac{n}{2}$. In this case, $\underline{s} < \frac{n}{2}$ as well. Let $(G, K, I) \in \mathcal{O}$ such that $|K| = \underline{s}$. Here $|K| < |I|$. Suppose that there is a vertex $k \in K$ such that $N(k) \cap I = \emptyset$. Then $(G, K \setminus \{k\}, I \cup \{k\}) \in \mathcal{O}$ and $|K \setminus \{k\}| < \underline{s}$, a contradiction. So every vertex in K is adjacent to some vertex in I. Since G is lex across all vertices in K except for one possible exception have closed neighborhood all of G. Thus G is the lex graph $\mathcal{L}(n, e)$. □

We are now ready to prove our main theorem, Theorem 52.

Proof of Theorem 52. By Lemma 67 we need only show that m and m_k are lex-across optimized. By Lemma 19 and Corollary 15 there is a graph minimizing m (and m_k) that is threshold in $\mathcal{G}_{n,e}$ for all n and e. By Theorem 53 and Lemma 56 there is a lex-across threshold graph that minimizes m (and m_k) over the collection of threshold graphs with $|K| = s, |I| = t$ and with e edges for all s, t, and e. Thus, m and m_k are each lex-across optimized and the lex or colex graph minimizes m and m_k in $\mathcal{G}_{n,e}$.

□

4.5 Further Directions

There are many open problems remaining in this area. For instance the Upper Matching Conjecture of Friedland, Krop, and Markström [11] claims that for all d-regular graphs G on $2n$-vertices such that d divides n we have

$$m_k(G) \leq m_k\left(\frac{n}{d} K_{d,d}\right)$$

for all k.

In a different direction, we know that the lex or colex graph doesn't necessarily minimize $m_k(G)$ for all k simultaneously. For example, consider the family $\mathcal{G}_{18,87}$. Then

	m_2	m_7
$\mathcal{L}(18,87)$	2745	0
$\mathcal{C}(18,87)$	2739	93,555

While $m_2(\mathcal{C}(18,87)) < m_2(\mathcal{L}(18,87))$, the lex graph has no 7-matchings and the colex graph has many. This indicates that it is a non-trivial problem to determine the graph $G \in \mathcal{G}_{n,e}$ that minimizes the matching polynomial

$$m_G(\lambda) = \sum_{k\geq 0} m_k(G)\lambda^k$$

for a given value of $\lambda > 0$. Theorem 52 covers the case $\lambda = 1$. By Lemma 19 the extremal graph can be taken to be threshold.

Chapter 5

Maximizing s-Independent Sets in r-Uniform Hypergraphs

This chapter will discuss some results in extremal hypergraph theory about s-independent sets. Cutler and Radcliffe [8] proved that among all r-graphs of a fixed size and order the hypergraph with the most r-independent sets is the lex r-graph. In the same paper they give an asymptotically best upper bound on the number of j-independent sets in an r-uniform hypergraph. The first result in this chapter, concerning 1-independent sets in r-uniform hypergraphs, easily follows from the definition of the colex r-graph. The most significant result is Theorem 70 which determines which graph in $\mathcal{H}_3(n, e)$, the family of 3-uniform hypergraphs on n vertices and e edges, has the maximum number of 2-independent sets for large e.

5.1 Definitions and Results

Recall that for an r-graph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and an integer s with $1 \leq s \leq r$, a set $I \subset \mathcal{V}$ is *s-independent* if $|I \cap E| < s$ for all $E \in \mathcal{E}$. Let $\mathcal{I}_s(\mathcal{H})$ denote the set of s-independent sets of a hypergraph $\mathcal{H}$ and $i_s(\mathcal{H}) = |\mathcal{I}_s\mathcal{H})|$. The first theorem says that the colex r-graph maximizes the number of 1-independent sets. (Recall the colex r-graph from Definition 30.)

Theorem 68. *If $\mathcal{H}$ is an r-graph on n vertices with e edges then*

$$i_1(\mathcal{H}) \leq i_1(\mathcal{C}_r(n,e))$$

This theorem follows from the fact that 1-independent sets are sets of isolated vertices and $\mathcal{C}_r(n,e)$ maximizes the number of isolated vertices. The proof appears in Section 5.2. The next theorem solves a more interesting problem, that of maximizing 2-independent sets in 3-graphs. Recall Definition 33 which defines the π-lex hypergraph. Below we define the $(2,3,1)$-lex like graph. (Note: for ease of notation later, we will use $[n] = \{0, 1, \ldots, n-1\}$ throughout this chapter as we did in Example 35.)

Definition 69. *We say a 3-graph $\mathcal{H}$ is* $(2,3,1)$-lex like *if:*

- *$\mathcal{H}$ is $(2,3,1)$-lex or*
- *letting k be the middle vertex (when the vertices of the edge are written in increasing order) of the $(2,3,1)$-lex greatest edge in the initial segment, the edges of $\mathcal{H}$ form an initial segment of the $(2,3,1)$-lex order except for missing edges $\{0, k-1, n-1\}, \{1, k-1, n-1\}, \ldots, \{k-2, k-1, n-1\}$*

Theorem 70. *Let $\mathcal{H}$ be a 3-uniform hypergraph on n vertices with e edges and $e \geq \max\{1.4 \times 10^6, 78n - 728\}$. Let $\mathcal{G}$ be a $(2,3,1)$-lex like 3-graph. Then*

$$i_2(\mathcal{H}) \leq i_2(\mathcal{G})$$

5.2 Maximizing 1-independent Sets in r-uniform Hypergraphs

Proposition 71. *For a hypergraph $\mathcal{H}$ let $S(\mathcal{H})$ be the set of isolated vertices in $\mathcal{H}$, and let $s(\mathcal{H}) = |S(\mathcal{H})|$. Then*

$$i_1(\mathcal{H}) = 2^{s(\mathcal{H})}.$$

Proof. A set A is 1-independent in a hypergraph $\mathcal{H}$ if and only if $|A \cap E| < 1$ for all $E \in \mathcal{E}(\mathcal{H})$. That is A is 1-independent if and only if $A \subseteq S(\mathcal{H})$ and so $i_1(\mathcal{H}) = 2^{s(\mathcal{H})}$. □

Proposition 72. *The colex r-graph $\mathcal{C}_r(n, e)$ has the maximum number of isolated vertices in $\mathcal{H}_r(n, e)$.*

Proof. The colex graph $\mathcal{C}_r(n, e)$ only introduces a vertex if all possible edges on the collection of earlier vertices are used. In more detail, if $j \in E$ for some $E \in \mathcal{E}(\mathcal{C}_r(n, e))$ then $A \in \mathcal{C}_r(n, e)$ if $A = \{a_1, \ldots, a_r\}$ has $a_i < j$ for all i. This is because for such an A we have

$$\max(A \Delta E) \geq j \in E$$

and so $A <_C E$. So, if it is possible to leave a vertex isolated, $\mathcal{C}_r(n, e)$ does. Therefore, $\mathcal{C}_r(n, e)$ has the maximum number of isolates. □

Remark. *If e is not of the form $\binom{k}{r}$ for any k then there are many graphs having the same number of isolated vertices as the colex graph. In fact, if $\binom{k-1}{r} < e < \binom{k}{r}$ then any e-subset of $\binom{K}{r}$ for K a k-set has the maximum number of isolated vertices.*

Proof of Theorem 68. By Proposition 71 to maximize the number of 1-independent sets we need only to maximize the number of isolated vertices. By Proposition 72 the colex graph $\mathcal{C}_r(n, e)$ maximizes the number of isolated vertices. Thus we are done. □

5.3 Maximizing 2-independent Sets in 3-uniform Hypergraphs

By Theorem 68 we know that $\mathcal{C}_3(n,e)$ has the most 1-independent sets of any 3-graph in the family $\mathcal{H}_3(n,e)$ and by [9] we know that $\mathcal{L}_3(n,e)$ has the most 3-independent sets of any graph in the family $\mathcal{H}_3(n,e)$. So, for 3-graphs, only the question of maximizing 2-independent sets remains.

To prove Theorem 70 we will first show that there is a 3-graph in $\mathcal{H}_3(n,e)$ maximizing the number of 2-independent sets that is shifted. Once we restrict to shifted 3-graphs we introduce a way to draw a 3-graph as a "nice" subset of a 3-dimensional cube. Using a way to count the number of 2-independent sets lost when an edge is added to a shifted 3-graph we are able to reduce the problem to 2 dimensions. Finally, we apply a set of local moves that do not decrease the number of 2-independent sets. From here we conclude that a $(2,3,1)$-lex like graph has the maximum number of 2-independent sets in $\mathcal{H}_3(n,e)$ for large e.

5.3.1 Shifted Hypergraphs Maximize s-independent Sets

In this section we will show that for any r, s, n, and e we can find a r-graph maximizing the number of s-independent sets in $\mathcal{H}_r(n,e)$ among the shifted hypergraphs. In the next proof we will construct an injection from the set of s-independent sets in some hypergraph $\mathcal{H}$ to the set of s-independent sets in the shift $\mathcal{H}_{i\to j}$. Note that in the next lemma we need not assume that the hypergraph is uniform.

Lemma 73. *Let $\mathcal{H}$ be a hypergraph with vertex set $[n]$ and let $0 \leq j < i < n$. Then for all s,*

$$i_s(\mathcal{H}_{i\to j}) \geq i_s(\mathcal{H}).$$

Proof. We will define an injection from $\mathcal{I}_s(\mathcal{H})\backslash\mathcal{I}_s(\mathcal{H}_{i\to j})$ to $\mathcal{I}_s(\mathcal{H}_{i\to j})\backslash\mathcal{I}_s(\mathcal{H})$. Let I be an independent set in $\mathcal{I}_s(\mathcal{H})\backslash\mathcal{I}_s(\mathcal{H}_{i\to j})$. If $j \notin I$ we have $|I \cap S_{i\to j}(E)| \leq |I \cap E|$ for all $E \in \mathcal{E}$ and so $j \in I$. Similarly, $i \notin I$, because if I is s-independent in $\mathcal{H}$ and $i, j \in I$ then I is s-independent in $\mathcal{H}_{i\to j}$. Thus, it makes sense to consider the map $I \mapsto I_{j\to i}$. This is clearly an injection so we need only show that $I_{j\to i} \in \mathcal{I}_s(\mathcal{H}_{i\to j}) \setminus \mathcal{I}_s(\mathcal{H})$. Let $F \in \mathcal{E}(\mathcal{H}_{i\to j})$ and consider $|I_{j\to i} \cap F|$.

Recall $\mathcal{E}(\mathcal{H}_{i\to j}) = \{S_{i\to j}(E) : E \in \mathcal{E}(\mathcal{H})\} \cup \{E : E, S_{i\to j}(E) \in \mathcal{E}(\mathcal{H})\}$. Suppose $F \in \{S_{i\to j}(E) : E \in \mathcal{E}(\mathcal{H})\}$. Then either

- $F = E$ for some $E \in \mathcal{E}(\mathcal{H})$ because $E \cap \{i, j\} \neq \{i\}$ and so $S_{i\to j}(E) = E$ or
- $F = E_{i\to j}$ for some $E \in \mathcal{E}(\mathcal{H})$

Suppose $F \in \{E : E, S_{i\to j}(E) \in \mathcal{E}(\mathcal{H})\}$. It's possible that E and $S_{i\to j}(E)$ are in $\mathcal{E}(\mathcal{H})$ for two reasons:

- $S_{i\to j}(E) = E$ because $E\cap\{i, j\} \neq \{i\}$ (which is the same as the first case above) or
- $S_{i\to j}(E) = E_{i\to j}$ but $E_{i\to j} \in \mathcal{E}(\mathcal{H})$

So the proof will be in three cases.

1. Suppose that $F = E$ for some $E \in \mathcal{E}(\mathcal{H})$ such that $E \cap \{i, j\} \neq \{i\}$. If $E \cap \{i, j\} = \emptyset$ then

   $$|I_{j\to i} \cap F| = |I_{j\to i} \cap E| = |I \cap E| < s.$$

   If $E \cap \{i, j\} = \{j\}$ then

   $$|I_{j\to i} \cap F| = |I_{j\to i} \cap E| < |I \cap E| < s.$$

If $E \cap \{i,j\} = \{i,j\}$ then

$$|I_{j\to i} \cap F| = |I_{j\to i} \cap E| = |I \cap E| < s.$$

2. Suppose that $F = E_{i\to j}$ for some $E \in \mathcal{E}(\mathcal{H})$. Then

$$|F \cap I_{j\to i}| = |E_{i\to j} \cap I_{j\to i}| = |E \cap I| < s.$$

3. Suppose that $F = E$ for some $E \in \mathcal{E}(\mathcal{H})$ such that $E \cap \{i,j\} = \{i\}$ and $E_{i\to j} \in \mathcal{E}(\mathcal{H})$. Then

$$|F \cap I_{j\to i}| = |E \cap I_{j\to i}| = |E_{i\to j} \cap I| < s.$$

Therefore $I_{j\to i} \in \mathcal{I}_s(\mathcal{H}_{i\to j})$. It remains to show that $I_{j\to i} \notin \mathcal{I}_s(\mathcal{H})$. Since $I \notin \mathcal{I}_s(\mathcal{H}_{i\to j})$ there exists $E \in \mathcal{E}(\mathcal{H}_{i\to j})$ such that $|I \cap E| \geq s$. It must be the case that $E = F_{i\to j}$ for some $F \in \mathcal{H}$. Then

$$s \leq |I \cap E| = |I_{j\to i} \cap E_{j\to i}| = |I_{j\to i} \cap F|.$$

Thus, $I_{j\to i} \notin \mathcal{I}_s(\mathcal{H})$. So, $|\mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}_{i\to j})| \leq |\mathcal{I}_s(\mathcal{H}_{i\to j}) \setminus \mathcal{I}_s(\mathcal{H})|$. Therefore,

$$|\mathcal{I}_s(\mathcal{H})| \leq |\mathcal{I}_s(\mathcal{H}_{i\to j})|.$$

$\square$

Corollary 74. *A hypergraph maximizing the number of s-independent sets among all hypergraphs with n vertices and e edges can be found among the shifted hypergraphs.*

Proof. Let $t(\mathcal{H}) = \sum_{E \in \mathcal{E}(\mathcal{H})} \sum_{i \in E} i$. Pick $\mathcal{H}$ with the maximal number of s-independent sets and $t(\mathcal{H})$ minimal. Note $\mathcal{H}_{i \to j}$ has the same number of vertices and edges as $\mathcal{H}$ and $i_s(\mathcal{H}_{i \to j}) \geq i_s(\mathcal{H})$ by Lemma 73. Thus, we must have $\mathcal{H}_{i \to j} = \mathcal{H}$, else $t(\mathcal{H}_{i \to j}) < t(\mathcal{H})$ contradicting the definition of $\mathcal{H}$. So $\mathcal{H}$ is a shifted hypergraph maximizing the number of s-independent sets. $\square$

5.3.2 Counting 2-independent Sets in Shifted 3-graphs

In this section we will develop a way to count 2-independent sets in shifted 3-graphs. This will result in a translation of the problem to an optimization problem that is easier to visualize.

Definition 75. *Let $\mathcal{H}$ be an r-graph with vertex set $[n]$ and let $I \subseteq [n]$ with $|I| \geq s$. Write $I = \{a_1, a_2, \ldots, a_m\}$ where $a_1 < a_2 < \cdots < a_m$ and let $B = [n] \setminus \{a_1, \ldots, a_s\} = \{b_1, \ldots, b_{n-s}\}$ where $b_1 < b_2 < \cdots < b_{n-s}$. Define the* minimal edge of I *to be $E_0(I) = \{a_1, \ldots, a_s\} \cup \{b_1, \ldots, b_{r-s}\}$. Note that $E_0(I)$ is the minimal set in $\binom{[n]}{r}$ under the lex ordering such that $|E \cap I| \geq s$.*

Remark. *For $\mathcal{H}$ a shifted 3-graph and $I \subset [n]$ of size at least 2 the minimal edge of I is $E_0(I) = \{a_1, a_2, b\}$ where a_1 and a_2 are the two smallest elements of I and $b = \min\{i \in [n] : i \neq a_1, a_2\}$.*

The purpose of defining the minimal edge of a set I is that I is 2-independent exactly when $E_0(I)$ is not in the hypergraph.

Lemma 76. *Let $\mathcal{H}$ be a left shifted r-graph and consider a set $I \subseteq [n]$ with $|I| \geq s$. The set I is s-independent in $\mathcal{H}$ if and only if $E_0(I) \notin \mathcal{E}(\mathcal{H})$.*

Proof. Suppose that I is an s-independent set. Then $E_0(I) \notin \mathcal{E}(\mathcal{H})$ since $|I \cap E| \geq s$. For the reverse direction we will prove the contrapositive. Suppose I is not an s-

independent set. So there exists an edge $E \in \mathcal{E}(\mathcal{H})$ such that $|E \cap I| \geq s$, say $E \cap I = \{e_1, \ldots, e_\ell\}$ where $\ell \geq s$ and $e_1 < e_2 < \cdots < e_\ell$. Suppose $E \setminus \{e_1, \ldots, e_s\} = \{f_1, \ldots, f_{r-s}\}$ where $f_1 < \cdots < f_{r-s}$. So $E = \{e_1, \ldots, e_s, f_1, \ldots, f_{r-s}\}$. Let $I = \{a_1, a_2, \ldots, a_m\}$ where $a_1 < a_2 < \cdots < a_m$ and let $B = [n] \setminus \{a_1, \ldots, a_s\} = \{b_1, \ldots, b_{n-s}\}$ where $b_1 < b_2 < \cdots < b_{n-s}$. So $E_0(I) = \{a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_{r-s}\}$. Note $e_i \geq a_i$ for all i, since $e_i \in I$ for all i and $a_1, \ldots, a_s$ are least in I. Also, $f_i \geq b_i$ for all i since the b_i's were chosen to be least in $[n]$ not equal to $a_1, \ldots, a_s$ and $f_i \in [n]$ and not equal to $a_1, \ldots, a_s$. Write $\mathbf{c} = (e_1, \ldots, e_s, f_1, \ldots, f_{r-s})$ and $\mathbf{d} = (a_1, \ldots, a_s, b_1, \ldots, b_s)$.

Since $\mathcal{H}$ is left shifted, $S_{\mathbf{c}\to\mathbf{d}}(\mathcal{E}(\mathcal{H})) = \mathcal{E}(\mathcal{H})$. Since $E = \{e_1, \ldots, e_s, f_1, \ldots, f_{r-s}\} \in \mathcal{E}(\mathcal{H})$, by Lemma 24, $E_0(I) = \{a_1, \ldots, a_s, b_1, \ldots, b_{r-s}\} \in S_{\mathbf{c}\to\mathbf{d}}(\mathcal{E}(\mathcal{H})) = \mathcal{E}(\mathcal{H})$. $\square$

Corollary 77. *Let $I \subset [n]$ with $|I| \geq s$. Suppose $\mathcal{H}' = \mathcal{H} + E$ and that $\mathcal{H}'$ and $\mathcal{H}$ are shifted r-graphs. Then $I \in i_s(\mathcal{H}) \setminus i_s(\mathcal{H}')$ if and only if $E_0(I) = E$.*

Proof. By Lemma 76, $I \in i_s(\mathcal{H})$ if and only if $E_0(I) \notin \mathcal{H}$ and $I \notin i_s(\mathcal{H}')$ if and only if $E_0(I) \in \mathcal{H}'$. Thus, $I \in i_s(\mathcal{H}) \setminus i_s(\mathcal{H}')$ if and only if $E_0(I) = E = \mathcal{H}' \setminus \mathcal{H}$. $\square$

Lemma 78. *Let $\mathcal{H}$ be a shifted 3-graph on vertex set $[n]$ and let $E = \{i, j, k\}$ with $i < j < k$ and suppose that $\mathcal{H}' = \mathcal{H} + E$ is also shifted. Then*

$$i_2(\mathcal{H}') = i_2(\mathcal{H}) - c_{ijk}$$

where

$$c_{ijk} = \begin{cases} 2^{n-1} & \text{if } \{i, j.k\} = \{0, 1, 2\} \\ 2^{n-k} & \text{if } i = 0, j = 1 \text{ and } k \neq 2 \\ 2^{n-k-1} & \text{if } i = 0 \text{ and } j \neq 1 \\ 0 & \text{if } i \neq 0 \end{cases}.$$

Remark. *We will refer to* c_{ijk} *as the* cost *of the edge* $\{i, j, k\}$*.*

Proof. By Corollary 77, $I \in i_2(\mathcal{H}) \setminus i_2(\mathcal{H}')$ if and only if $E_0(I) = E$. Thus, to determine the cost of an edge E we must count the number of sets I such that $E_0(I) = E$.

If $E = \{0, 1, 2\}$ we are counting sets such that $E_0(I) = \{0, 1, 2\}$. These are exactly those sets having two smallest elements 0 and 1, 0 and 2, or 1 and 2. The number of sets with this property is $2^{n-2} + 2^{n-3} + 2^{n-3} = 2^{n-1}$. Thus, $c_{012} = 2^{n-1}$.

Suppose that $\{0, 1, k\}$ is added to a hypergraph where $k \neq 2$. Here we count sets I such that $E_0(I) = \{0, 1, k\}$. These are the sets with smallest elements 0 and k or 1 and k. The number of sets with this property is $2^{n-k-1} + 2^{n-k-1} = 2^{n-k}$. Thus $c_{01k} = 2^{n-k}$ for $k \neq 2$.

Suppose now $E = \{0, j, k\}$ with $j \neq 1$. Here, $E_0(I) = E$ if and only if the two smallest elements of I are j and k. There are 2^{n-k} of these meaning $c_{0jk} = 2^{n-k}$ when $j \neq 1$.

Finally, if $0 \notin E$ then it is not one of the edges of the form $E = \{a_1, a_2, b\}$ where $b = \min\{i \in [n] : i \neq a_1, a_2\}$. Thus, the cost of $\{i, j, k\}$ where $i \neq 0$ is 0. $\square$

Note that $\sum_{i<j<k} c_{ijk} = 2^n - (n + 1)$ meaning that $i_2(\mathcal{K}_n^3) = n + 1$ where $\mathcal{K}_n^3$ is the complete 3-graph on n vertices. The 2-independent sets in $\mathcal{K}_n^3$ are the empty set and all the singletons.

Let $\mathcal{H}$ be a 3-graph with vertex set $[n]$. We will visualize $\mathcal{H}$ by letting its edges be $1 \times 1 \times 1$ cubes labeled by the vertices in the edge in increasing order. Then we can think of these $1 \times 1 \times 1$ cubes inside an $(n - 2) \times (n - 2) \times (n - 2)$ cube labeled as in Figure 5.1. Figure 5.2 shows the edges of the complete hypergraph on 7 vertices inside a $5 \times 5 \times 5$ cube with the visible cubes labeled.

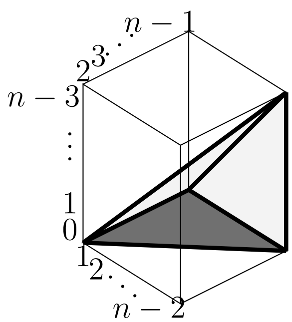

Figure 5.1: The labeling of the cube. The shaded tetrahedron represents the collection of $1 \times 1 \times 1$ cubes that have labels in increasing order.

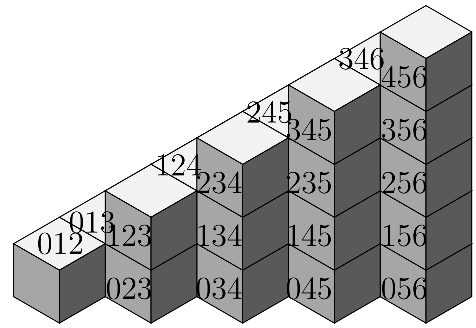

Figure 5.2: Edges of the complete hypergraph on 7 vertices

Lemma 78 says that, assuming the hypergraph is shifted, any edge that does not contain 0 is "free", i.e., adding such an edge does not cost us any independent sets. More rigorously, if $E = \{i, j, k\}$ with $i < j < k$ and $i \neq 0$ we have $i_2(\mathcal{H}) = i_2(\mathcal{H} + E)$. In the cube picture this means that any edge that is not in the bottom layer is free. For this reason, we focus on the shape of the base layer, which we will call B_n for a hypergraph with vertex set $[n]$. Figure 5.3 shows the cube where we've suppressed the first dimension and show only the edges with non-zero costs.

016	026	036	046	056
015	025	035	045	
014	024	034		
013	023			
012				

Figure 5.3: Edges in base layer, B_7.

We will call each of the squares in B_n a *cell* and label it (a, b) if the edge associated to that square is $\{0, a, b\}$. This structure gives rise to a poset.

Definition 79. *Let* $B_n = \{(a, b) : 1 \le a < b \le n - 1\}$*. Given* (a, b) *and* (c, d) *in* B_n *we say* $(a, b) \le (c, d)$ *if* $a \le c$ *and* $b \le d$*.*

Lemma 80. $(B_n, \le)$ *is a poset.*

Proof. Note $(B_n, \le)$ is a sub-poset of $[n-1]^2$. □

Recall that we are restricting ourselves to shifted hypergraphs as we can find a minimizer in that collection. By definition, a shifted hypergraph $\mathcal{H}$ on vertex set $[n]$ satisfies the following condition: if $\{a, b, c\} \in \mathcal{E}(\mathcal{H})$ for $a < b < c$ then $\{i, j, k\} \in \mathcal{E}(\mathcal{H})$ for $i < j < k$ whenever $i \le a$, $j \le b$, and $k \le c$. In the base layer B_n, this says that if $\{0, b, c\} \in \mathcal{E}(\mathcal{H})$ for $b < c$ then $\{0, j, k\} \in \mathcal{E}(\mathcal{H})$ for all $j \le b$ and $k \le c$. That is, if we include a cell (b, c) in our hypergraph, we must also include all cells that are to the left and below. If a collection of cells satisfies this condition we will call it a *downset*. This definition of a downset corresponds exactly to the notion of downset in the poset B_n.

Each cell has an associated cost as given in Lemma 78 and an associated amount of *space*: the number of edges we could get for that cost, given that taking those edges results in a shifted hypergraph.

2	1	1	1	1
4	2	2	2	
8	4	4		
16	8			
64				

Figure 5.4: Cost of each cell in B_7.

1	2	3	4	5
1	2	3	4	
1	2	3		
1	2			
1				

Figure 5.5: Space in each cell in B_7.

Definition 81. *For A, a collection of cells, let $C(A)$ be the cost of those cells and $S(A)$ be the amount of room in those cells.*

Our goal, finding the 3-graph on n vertices having e edges with the maximum number of 2-independent sets, can be rephrased as follows: find the downset A in B_n such that $C(A)$ is minimized subject to the condition that $S(A) \geq e$.

For the rest of the paper we will only be concerned with the shape of the bottom layer. Given a downset in B_n that has enough room to accommodate the number of edges we need, we can arrange the edges in higher layers to get a shifted 3-graph (often in several ways).

The shape of a $(2, 3, 1)$-lex graph in B_n is shown in Figure 5.6. The possible shapes for $(2, 3, 1)$-lex like graphs are shown in Figure 5.7.

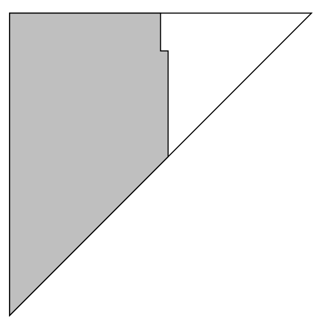

Figure 5.6: A $(2, 3, 1)$-lex 3-graph in B_n.

5.3.3 Maximizing 2-independent Sets in $\mathcal{H}_3(n, e)$ When e is Large

In this section we present a collection of moves on downsets in B_n that will show that 3-graphs that are not $(2, 3, 1)$-lex like have at most as many 2-indsets as a $(2, 3, 1)$-lex

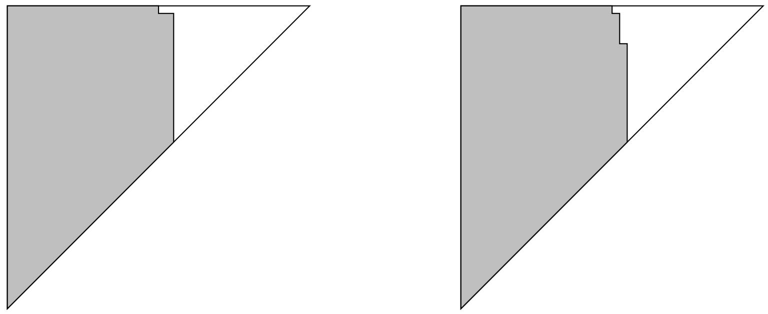

Figure 5.7: $(2,3,1)$-lex like 3-graphs in B_n

like 3-graph. To talk about the local moves, we first need the definitions of out-corner and in-corner.

Definition 82. *Given a downset A the cell (a,b) is an* out-corner *of A if it is a maximal element of A. Similarly, (a,b) is an* in-corner *if it is a minimal element of $B_n \setminus A$. Equivalently, (a,b) is an in-corner if $(a,b) \notin A$ and $A + (a,b)$ is still a downset.*

5.3.3.1 Local Moves

First we will consider some local moves where we exchange one cell of a downset A for two cells in $B_n \setminus A$.

Definition 83. *For a downset A define the* out-corner sequence of A *to be the sequence of out-corners written in increasing order of the first coordinate. We will denote this $\mathcal{O}(A) = (O_1, O_2, \ldots, O_k)$. (This is not the same as a chain in the poset. Since out-corners are maximal elements they are not comparable.)*

Lemma 84. *Let A be a downset with out-corner sequence $\mathcal{O}(A) = (O_1, O_2, \ldots, O_k)$. Consider $O_i = (a,b)$ and $O_j = (c,d)$. If $3 \leq c - a \leq \frac{c+3}{2}$ then there exists a downset A' such that $C(A') = C(A)$ and $S(A') \geq S(A)$ with edges that are at least as early in* *the* $(2,3,1)$*-lex order. Moreover, if* $j \neq i+1$ *then there exists a downset* A' *such that* $C(A') < C(A)$ *and* $S(A') \geq S(A)$*.*

Proof. Suppose first that $j = i + 1$ and $3 \leq c - a \leq \frac{c+3}{2}$. Since $3 \leq c - a$ and the previous out-corner is (a, b) we can remove cell (c, d) and replace it with cells $(a+1, d+1)$ and $(a+2, d+1)$ and still have a downset. Let $A' = A - (c, d) + (a + 1, d+1) + (a+2, d+1)$. The move from A to A' is illustrated in Figure 5.8.

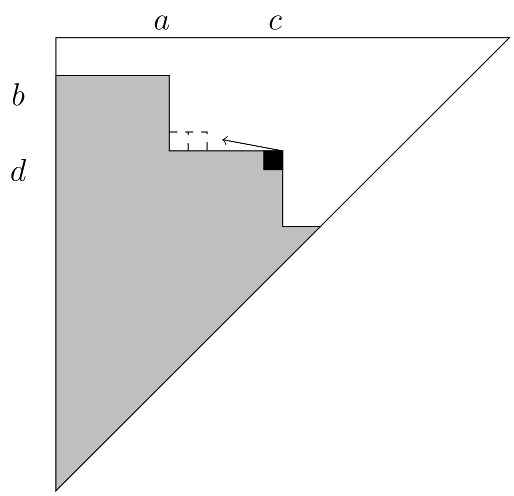

Figure 5.8: Move occurring in the proof of Lemma 84 for consecutive out-corners

Note the room of cell (c, d) is c and the room in the replacement cells is collectively $2a+3$. Since $c - a \leq \frac{c+3}{2}$ we have $c \leq 2a + 3$ and so there is at least much space in A'. Moreover, the cost of each of the replacement cells is half the cost of (c, d) and so $C(A) = C(A')$.

Suppose that there are two out-corners $O_i = (a, b)$ and $O_j = (c, d)$ such that $j \neq i + 1$ and $3 \leq c - a \leq \frac{c+3}{2}$. Let (x, y) be an out-corner in between O_i and O_j. Then consider the downset $A' = A - (c, d) + (x + 1, d + 1) + (a + 1, y + 1)$. We will still have enough space for (c, d) and the total cost of the two replacement cells will be strictly less than the cost of (c, d). Thus, $C(A') < C(A)$ and so A does not achieve the maximum number of 2-independent sets.

□

Lemma 84 says that in an optimizing downset the horizontal distance (the difference between the first coordinates) between two out-corners is either small (less than 3) or is large (about half the larger amount of space). Let's consider first when the horizontal distance between out-corners is small. If the horizontal distance between two out-corners is 1 we will say there is a *short stair* and if the horizontal distance between two out-corners is 2 we will say there is a *long stair*. What types of staircases can appear in an optimizing partition? To state the next lemma we first define the horizontal distance vector.

Definition 85. *For a downset A, let $(o_1, o_2, \dots, o_k)$ be the sequence of the first coordinates of the out-corners written in increasing order and let the* horizontal distance vector *be $D(A) = (o_2 - o_1, o_3 - o_2, \dots, o_k - o_{k-1})$.*

Lemma 86. *Consider a downset A with horizontal distance vector $D(A)$. If $D(A)$ has three consecutive 1's, two consecutive 2's, or an adjacent 1 and 2 then A does not maximize the number of 2-independent sets.*

Proof. Suppose that the horizontal distance vector has three consecutive 1's as shown in Figure 5.9. Name the corresponding out-corners (i, a), $(i+1, b)$, $(i+2, c)$, and $(i+3, d)$. Suppose we remove cell $(i+3, d)$ and add cells $(i+1, b+1)$ and $(i+2, c+1)$ to get another downset A'. Note that $(i+1)+(i+2) = 2i+3 > i+3$ and so $S(A') > S(A)$. Moreover, since $a > b > c$, the cost of $(i+2, c)$ is at most half the cost of the cell $(i+3, d)$ and the cost of the cell $(i+1, b+1)$ is at most a quarter of the cost of the cell $(i+3, d)$. Therefore $C(A') < C(A)$.

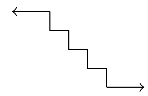

Figure 5.9: 3 short stairs

Suppose that the horizontal distance vector has two consecutive 2's as shown in Figure 5.10. Suppose the corresponding out-corners are (i, a), $(i+2, b)$, and $(i+4, c)$. Consider the downset $A' = A - (i+4, c) + (i+1, b+1) + (i+2, b+1)$. Then $C(A') < C(A)$ since each of the new cells cost at most a quarter of the removed and $S(A') > S(A)$ since $(i+1) + (i+2) = 2i + 3 > i + 4$ for $i > 1$.

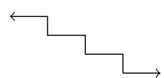

Figure 5.10: 2 long stairs

Suppose that the horizontal distance vector has a 2 followed immediately by a 1 as shown in Figure 5.11. Suppose the corresponding out-corners are (i, a), $(i+2, b)$, and $(i+3, c)$. Consider $A' = A - (i+3, c) + (i+1, b+1) + (i+2, b+1)$. Then $C(A') < C(A)$ since each of the replacement cells cost at most $\frac{1}{4}$ of the removed cell and $S(A') > S(A)$ since $(i+1) + (i+2) = 2i + 3 > i + 3$.

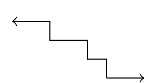

Figure 5.11: 1 long stair and 1 short stair

Finally, suppose that the horizontal distance vector has a 1 immediately followed by a 2 as shown in Figure 5.12. Name the corresponding out-corners (i, a), $(i+1, b)$, and $(i+3, c)$. Consider $A' = A - (i+3, c) + (i+1, b+1) + (i+2, c+1)$. Note that

$C(A') < C(A)$ since the cost of $(i+1, b+1)$ is at most half of the cost of $(i+3, c)$ and $(i+2, c+1)$ is at most a quarter of the cost of $(i+3, c)$. In addition, $S(A') > S(A)$ since $(i+1)+(i+2) = 2i+3 > i+2$.

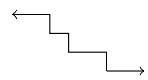

Figure 5.12: 1 short stair and 1 long stair

$\square$

5.3.3.2 Downsets That Do Not End With Stairs

In this section we will apply moves to downsets based on their last out-corner. By Lemma 84 we know that out-corners in an optimizing downset are either close or far apart. For the lemmas in this part we will assume that the second to last out-corner is not near the last out-corner. That is, we will consider downsets that do not end in stairs. These moves will differ from those in the previous section as we will consider moving larger pieces of the downset.

The decision about which move to make will be made based on the coordinates of the last out-corner. In the next lemma we consider when the last out-corner of a downset is (i, j) and $j - i \geq \lfloor \log_2(i) \rfloor$.

Lemma 87. *Suppose the last out-corner of a downset A is (i, j) where $j - i \geq \lfloor \log_2(i) \rfloor$ and $i \geq 6$. If there is a previous out-corner with space less than $\frac{i-3}{2}$ then A is not an optimizing shape.*

Proof. Let (k, m) be an earlier out-corner such that $k < \frac{i-3}{2}$.

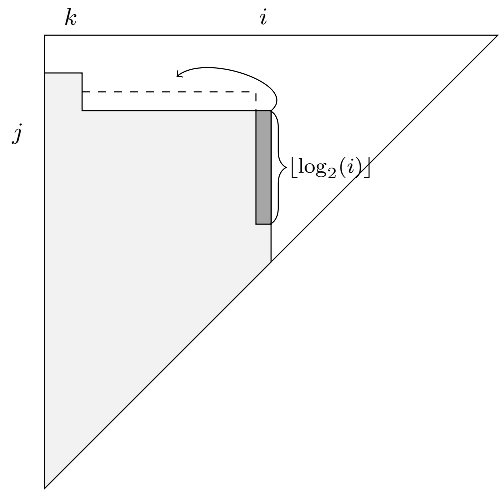

Figure 5.13: Column move in the proof of Lemma 87

For now assume (k, m) is the previous out-corner. Let $t = \lfloor \log_2(i) \rfloor$ and consider $A' = A - \{(i, h) : j - t + 1 \leq h \leq j\} + \{(h, j + 1) : k + 1 \leq h \leq i - 1\}$. That is, we consider the downset A' in which we remove t cells from the last column and replace them with the available cells in the row at height $j + 1$. This move is shown in Figure 5.13. We will show that the cost of the row is less than the cost of the column and the space in the row is at least the space in the column to conclude that $C(A') < C(A)$ and $S(A') \geq S(A)$. The cost of the column is

$$2^{n-j-1} + \cdots + 2^{n-j-1+t-1} = 2^{n-j-1}(2^t - 1) = 2^{n-j-2}(2^{\lfloor \log_2(i) \rfloor + 1} - 2) > 2^{n-j-2}(i - 2).$$

Note there are at most $i - 2$ cells in the row (since there is a previous out-corner) and each cell costs 2^{n-j-2}. Thus, the cost of the row is strictly less than the cost of the column.

Next we deal with space. Note the space in the column is exactly $i \lfloor \log_2(i) \rfloor$.

Letting $R = \{(h, j+1) : k+1 \leq h \leq i-1\}$ we get

$$\begin{aligned} S(R) &\geq \left(\frac{i-3}{2}\right) + \left(\frac{i-3}{2}+1\right) + \cdots + (i-1) \\ &= \left(i - \frac{i-3}{2}\right)\left(\frac{i-3}{2}\right) + \frac{\left(i-1-\frac{i-3}{2}\right)\left(i-\frac{i-3}{2}\right)}{2} \\ &= \frac{3}{8}i^2 + \frac{1}{2}i - \frac{15}{8} \end{aligned}$$

We compare this lower bound for $S(R)$ with $S(C) = i\lfloor \log_2(i) \rfloor$ and find that $S(R) \geq S(C)$ when $i \geq 6$. To see this note that $6 \cdot \lfloor \log_2(6) \rfloor \leq \frac{3}{8} \cdot (6)^2 + \frac{1}{2} \cdot 6 - \frac{15}{8}$ and that $\frac{d}{di}\left(i \log_2(i)\right) < \frac{d}{di}\left(\frac{3}{8}i^2 + \frac{1}{2}i - \frac{15}{8}\right)$ for $i \geq 6$.

Suppose that there is a previous out-corner (k, m) with space at most $\frac{i-3}{2}$, but there are also out-corners with more space than that. Then replacing the last out-corner (i, j) with one cell from each column between $k+1$ and $i-1$. This downset has more 2-independent sets since the cost is at most the cost of the cells in the row at height j and the space is the same as that in R. □

The previous lemma says that if the last column is tall and there is a previous out-corner with not too much space then there is a downset with lesser cost. In the next lemma we will prove that given a little more height in the last column and no previous out-corner we can still find a downset with lesser cost.

Lemma 88. *Suppose that the last out-corner of a downset A is (i, j) where $i \geq 7$ and $(j - i) \geq \lceil \log_2(i) \rceil$. Suppose any previous out-corners have space at least $i - 2$. Additionally, assume the highest out-corner has height less than $n - 1$. Then A is not an optimizing shape.*

Proof. Suppose first that (i, j) is the only out-corner. Let $t = \lceil \log_2(i) \rceil$ and consider

$$A' = A - \{(i, h) : j - t + 1 \leq h \leq j\} + \{(h, j+1) : 1 \leq h \leq i-1\}.$$

The cost of $A \setminus A'$ is

$$C(\{(i,h) : j-t+1 \leq h \leq j\}) = 2^{n-j-1}(2^t - 1)$$

and the cost of $A' \setminus A$ is

$$C(\{(h,j+1) : 1 \leq h \leq i-1\}) = 2^{n-(j+1)-1}(i-2) + 2^{n-(j+1)} = 2^{n-j-2}(i).$$

Since $t = \lceil \log_2(i) \rceil$,

$$2^{n-j-1}(2^t - 1) = 2^{n-j-1}(2^{\lceil \log_2(i) \rceil} - 1) \geq 2^{n-j-1}(i-1) > 2^{n-j-2}(i)$$

when $i \geq 3$. Thus $C(A) > C(A')$. Moreover, the space in $A \setminus A'$ is

$$S(\{(i,h) : j-t+1 \leq h \leq j\}) = ti$$

and the space in $A \setminus A'$ is

$$S(\{(h,j+1) : 1 \leq h \leq i-1\}) = \frac{(i-1)i}{2}.$$

Since $\frac{(i-1)i}{2} \geq i\lceil \log_2(i) \rceil$ for $i \geq 7$ we have $S(A') \geq S(A)$. Suppose that A has previous out-corners with space at least $i-2$. This time, replace the column with one cell from each column (which is possible since the highest out-corner has height less than $n-1$). These replacement cells have the same amount of space and at most the cost as $\{(h,j+1) : 1 \leq h \leq i-1\}$. Therefore, such an A is not an optimizing shape. $\square$

We summarize the previous two lemmas in the following proposition.

Proposition 89. *Suppose A is a downset with last out-corner (i,j) that is not the shape of a $(2,3,1)$-lex like 3-graph, $i \geq 7$, and $j-i \geq \lceil \log_2(i) \rceil$. Then A is not the optimizing shape with $(2,3,1)$-lex least edges.*

Proof. If there exists a previous out-corner with space less than $\frac{i-3}{2}$ then A is not an optimizing shape by Lemma 87. If A has a previous out-corner with space k where $\frac{i-3}{2} \leq k \leq i-3$ then A is not the optimizing shape with the $(2,3,1)$-lex least edges by Lemma 84. Finally, if (i,j) is the only out-corner or all previous out-corners have space at least $i-2$ note that the highest out-corner has height less than $n-1$. Then A is not an optimizing shape by Lemma 88. □

Remark. *If A is a downset with space greater than $21n-112$ then the space of the last out-corner of A is at least 7.*

If the last out-corner of some down-set is (i,j) and $j-i \leq \lfloor \log_2(i) \rfloor$ then we will make a similar move but this time we will trade in several columns for the row. The shape of the columns we move is a trapezoid. In the the next lemma we compute the space and cost of a triangle in B_n for some n.

Lemma 90. *Consider $T(i,j) = \{(a,b) : i \leq a \leq j-1, i+1 \leq b \leq j\} \cap B_n$, the sub-triangle of B_n with vertices (i,j), $(j-1,j)$ and $(i,i+1)$. Let $\ell = j-i$ (notice this is the side length of this "equilateral triangle"). Then*

$$S(T) = \frac{(\ell)(\ell+1)(j+2i-1)}{6}$$

and

$$C(T) = 2^{n-j-1}\left(2^{\ell+1} - 2 - \ell\right)$$

Proof. Figure 5.14 shows $T(i,j)$.

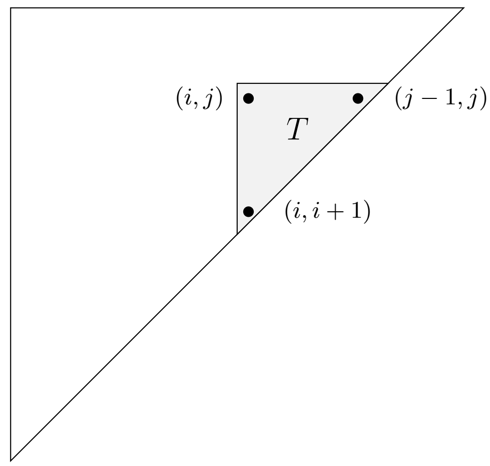

Figure 5.14: Triangle $T(i, j)$ with vertices (i, j), $(j-1, j)$ and $(i, i+1)$.

To compute the space in the triangle, we sum the space in each column:

$$\begin{aligned}
S(T(i,j)) &= \sum_{k=1}^{\ell} k \cdot (j-k) \\
&= \sum_{k=1}^{\ell} kj - k^2 \\
&= j \sum_{k=1}^{\ell} k - \sum_{k=1}^{\ell} k^2 \\
&= j \cdot \frac{\ell(\ell+1)}{2} - \frac{\ell(\ell+1)(2\ell+1)}{6} \\
&= \frac{\ell(\ell+1)(j+2i-1)}{6}
\end{aligned}$$

as desired.

For cost we first note that the cost of each cell has a factor of 2^{n-j-1}. Factoring 2^{n-j-1} out of the cost of each cell if we consider the diagonals that are parallel to the hypotenuse (that is, parallel to the line between $(j-1, j)$ and $(i, i+1)$) and note that summing along the k^{th} diagonal (where $k = 1$ is the top left) gives that

$2^0 + 2^1 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$ is the cost in each diagonal. Putting this together

$$C(T(i,j)) = (2^{n-j-1}) \left(\sum_{k=1}^{\ell} 2^k - 1 \right) = 2^{n-j-1} \left(2^{\ell+1} - 2 - \ell \right)$$

$\square$

In the next lemma we deal with downsets that have an out-corner $(j-1, j)$, that is an out-corner all the way to the right in B_n.

Lemma 91. *Suppose A is a downset in B_n, the last out-corner of A is $(j-1, j)$ and the previous out-corner, if there is one, has space less than $\frac{(j-1)-3}{2}$. If $n-1 > j \geq 200$ then A is not an optimizing shape.*

Proof. Suppose that $(j-1, j)$ is the last out-corner and that k is the amount of space in the previous out-corner, if there is one.

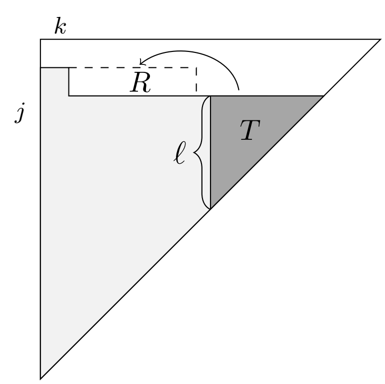

Figure 5.15: Triangle move occurring in the proof of Lemma 91

Since $j \geq 200$, there exists $\ell \in \mathbb{N}$ such that $\log_2(j) \leq \ell \leq \sqrt{\frac{j}{2}} - 1$. Choose such an ℓ. Consider the set of cells

$$T = \{(a,b) : j - \ell \leq a \leq j-1, j - \ell + 1 \leq b \leq j\} \cap B_n$$

and the set of cells

$$R = \{(a, j+1) : 1 \leq a \leq j - \ell - 1\} \cap (B_n \setminus A).$$

We claim the downset $A' = A - T + R$ has $S(A') \geq S(A)$ and $C(A') < C(A)$. By Lemma 90,

$$S(T) = \frac{\ell(\ell+1)(j + 2(j-\ell) - 1)}{6} \leq \frac{(\ell+1)^2(3j - 2\ell)}{6} \leq \frac{(\ell+1)^2 j}{2}.$$

We will compute a lower bound on the amount of space in the rectangle. By assumption, if there is a previous out-corner, it has space less than $\frac{j}{2}$ and so

$$\begin{aligned} S(R) &\geq \frac{(j-\ell-1)(j-\ell)}{2} - \frac{j^2}{8} \\ &\geq \frac{j^2}{4} \end{aligned}$$

since $\ell \leq \sqrt{\frac{j}{2}} - 1$ and $j \geq 200$.

Since $\ell \leq \sqrt{\frac{j}{2}} - 1$ we have $S(T) \leq \frac{(\ell+1)^2 j}{2} \leq \frac{j^2}{4} \leq S(R)$ and thus $S(A) \leq S(A')$.

To show that $C(A') < C(A)$ we will show that $C(R) < C(T)$ provided $\ell \geq \log_2(j)$. By Lemma 90,

$$C(T) = 2^{n-j-1}(2^{\ell+1} - \ell - 2)$$

Assuming that $R \subseteq B_n \setminus A$ gives an upper bound on the cost of R. In particular,

$$\begin{aligned} C(R) &\leq (2^{n-j-2})(j - \ell - 1) \\ &\leq 2^{n-j-2}(j) \end{aligned}$$

So $C(R) < C(T)$ if $j < 2^{\ell+2} - 2\ell - 4$. If $\ell = \log_2(j)$ then

$$2^{\ell+2} - 2\ell - 4 = 4j - 2\log_2(j) - 4 > j$$

and the inequality $j < 2^{\ell+3} - 4\ell - 8$ continues to hold for $\ell > \log_2(j)$. Thus $C(A') < C(A)$.

Since $S(A') \geq S(A)$ and $C(A') < C(A)$ we conclude that such an A does not attain the maximum number of 2-independent sets.

$\square$

Lemma 92. *Suppose A is a downset in B_n, the last out-corner of A is (i,j) such that $j - i \leq \lfloor \log_2(i) \rfloor$, $n - 1 > j \geq 200$ and the previous out-corner, if one exists, has space less than $\frac{i-3}{2}$. Then A is not an optimizing shape.*

Proof. Suppose that (i,j) is the last out-corner and that k is the amount of space in the previous out-corner. Again choose $\ell \in \mathbb{N}$ such that $\log_2(j) \leq \ell \leq \sqrt{\frac{j}{2}} - 1$.

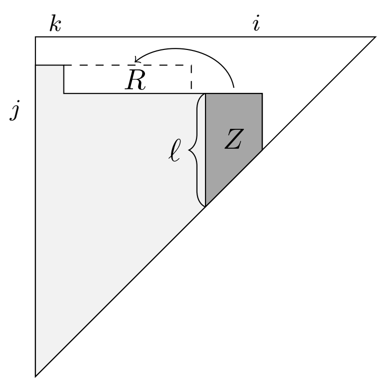

Figure 5.16: Trapezoid move

Note that $j - i \leq \log_2(i) \leq \log_2(j) \leq \ell$ and so $j - \ell \leq i$. This allows us to consider

$$Z = \{(a,b) : j - \ell \leq a \leq i, j - \ell + 1 \leq b \leq j\} \cap A.$$

As before, let

$$R = \{(a, j+1) : 1 \leq a \leq j - \ell - 1\} \cap (B_n \setminus A).$$

To show $S(A') \geq S(A)$ we will show $S(R) \geq S(Z)$. Let $T \subset B_n$ be the triangle of side length ℓ that contains Z, that is,

$$T = \{(a, b) : j - \ell \leq a \leq j - 1, j - \ell + 1 \leq b \leq j\} \cap B_n.$$

Clearly $S(Z) \leq S(T)$ and by the proof of Lemma 91 $S(T) \leq S(R)$. Therefore, $S(A) \leq S(A')$.

Let $I = \{(j - \ell, b) : j - \ell + 1 \leq b \leq j\}$, the longest column in Z. Then $C(Z) \geq C(I) = 2^{n-j-1}(2^\ell - 1)$. Recall from the proof of Lemma 91 that $C(R) \leq 2^{n-j-2} \cdot j$ so it suffices to show that $j \leq 2^{\ell+1} - 2$. Since $\ell \geq \log_2(j)$ we have

$$2^{\ell+1} - 2 \geq 2j - 2 > j.$$

Thus $C(A') < C(A)$.

Therefore, such an A does not maximize the number of 2-independent sets.

□

Remark. *If A is a downset with space at least 1,333,300 then the last out-corner of A, call it (i, j) has $j \geq 200$.*

5.3.3.3 Downsets That End With Stairs

In this section we will apply moves similar to ones we have already discussed, but this time we will accommodate when a downset ends with stairs.

Lemma 93. *Suppose that A is a downset in B_n that is not $(2,3,1)$-lex like. Additionally, given that the last out-corner of A is (i,j), suppose that A has at least one out-corner with space at least $i-2$ and at least one out-corner with space less than $\frac{i-3}{2}$. If $i \geq 9$ and the height of the top stair is at least 200, then A is not an optimizing shape.*

Proof. By Lemma 86 we know that A ends with 2 short stairs, 1 short stair, or 1 long stair. Suppose the downset ends with 2 short stairs or 1 long stair, as shown in Figures 5.17 and 5.18. In each of these cases we can replace the last out-corner with two earlier in-corners which cost strictly less and have at least as much space.

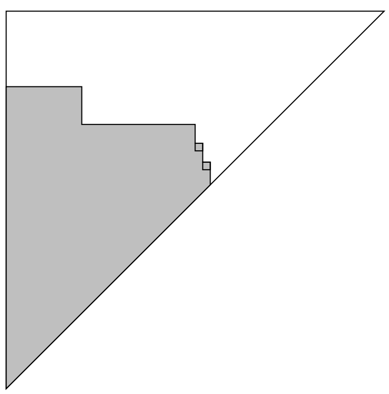

Figure 5.17: Downset ending in 2 short stairs

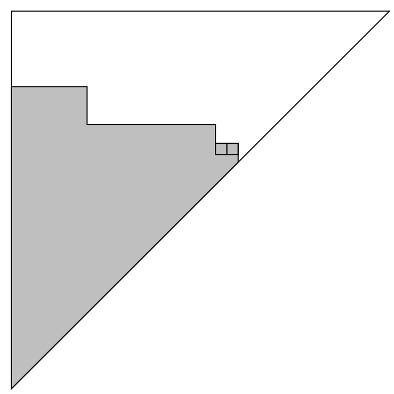

Figure 5.18: Downset ending in 1 long stair

For the last case, suppose the downset ends with one short stair as shown in Figure 5.19. Let (i,j) be the last out-corner and let $(i-1,k)$ be the previous out-corner.

If $k-(i-1) > \lfloor \log_2(i) \rfloor$ then we can apply a move similar to the column move described in Lemma 87. Let

$$R = \{(a, k+1) : 1 \leq i \leq i-2\} \cap (B_n \setminus A)$$

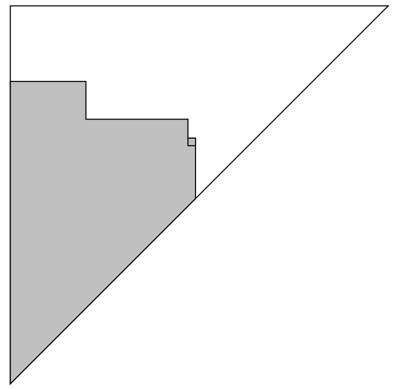

Figure 5.19: Downset ending in 1 short stair

and

$$I = \{(i-1, b) : j+1 \leq b \leq k\} \cup \{(i, b) : k-t+1 \leq b \leq j\}$$

where $t = \lfloor \log_2(i) \rfloor$. Note that these cells are in A since $k-(i-1) > \lfloor \log_2(i) \rfloor$ Define $A' = A - I + R$. Then $C(I) = 2^{n-k-1}(2^t - 1)$ and $C(R) \leq (i-2)(2^{n-k-2})$. Since

$$C(I) = 2^{n-k-2}(2^{\lfloor \log_2(i) \rfloor + 1} - 2) > 2^{n-k-2}(i-2) = C(R)$$

we know $C(A') < C(A)$.

Note that $S(R) \geq \frac{i-3}{2} + \left(\frac{i-3}{2} + 1\right) \cdots + i - 2 \geq \frac{3}{8}i^2 + \frac{1}{2}i - \frac{15}{8} - i + 1$ by the proof of Lemma 87. Also, $S(I) \leq i(\lfloor \log_2 i \rfloor - 1) + i - 1 = i\lfloor \log_2 i \rfloor - 1$. Provided $i \geq 9$, $S(I) \geq S(R)$.

Suppose $k - (i-1) \leq \lfloor \log_2(i) \rfloor$. Assuming $k \geq 200$ we can apply the trapezoid move as described in Lemma 92. Choose $\ell \in \mathbb{N}$ such that $\log_2(k) \leq \ell \leq \sqrt{\frac{k}{2}} - 1$. Let

$$Z = \{(a, b) : k - \ell \leq a \leq i, k - \ell + 1 \leq b \leq k\} \cap A$$

and

$$R = \{(a, k+1) : 1 \leq a \leq k - \ell - 1\} \cap (B_n \setminus A).$$

Letting T be the triangle of side length ℓ containing Z, that is

$$T = \{(a, b) : k - \ell \leq a \leq k - 1, k - \ell + 1 \leq b \leq k\} \cap B_n$$

we know $S(Z) \leq S(T)$. By the same arguments as in Lemma 91,

$$S(T) = \frac{\ell(\ell+1)(k + 2(k-\ell) - 1)}{6} \leq \frac{(\ell+1)^2 k}{2}$$

and

$$S(R) \geq \frac{(k-\ell-1)(k-\ell)}{2} - \frac{k^2}{8} \geq \frac{k^2}{4}.$$

Since $\ell \leq \sqrt{\frac{k}{2}} - 1$ we have $S(Z) \leq S(T) \leq S(R)$. Moreover, the cost of Z is at least the cost of the first column of Z, that is, $C(Z) \geq 2^{n-k-1}(2^\ell - 1)$. Also $C(R) \leq 2^{n-k-2}(k - \ell - 1)$. Therefore,

$$C(Z) \geq 2^{n-k-1}(2^\ell - 1) > 2^{n-k-2}(k-2) \geq C(R).$$

Therefore such an A is not an optimizing shape. $\square$

Remark. *If A is a downset in B_n with space greater than $36n - 240$ then the last out-corner of A has space at least 9.*

Lemma 94. *Suppose that A is a downset in B_n that is not $(2, 3, 1)$-lex like. Suppose that A ends with a staircase and otherwise has no out-corners. If the first out-corner is (m, k) has $k \geq 200$ and the last out-corner is (i, j) with $i \geq 13$ then A is not an optimizing shape.*

Proof. Let (i, j) be the last out-corner of A and let (m, k) be the first out-corner. Note that $m = i - 1$ or $m = i - 2$ and that $k < n - 1$ since A is not $(2, 3, 1)$-lex like. If $k - m > \lceil \log_2(m) \rceil + 1$ then we remove $t = \lceil \log_2(m) \rceil$ cells, one from each height between $k - \lceil \log_2(m) \rceil + 1$ and k and replace them with the row at height $k + 1$. Call the new downset A'. Let I be the collection of cells we remove and R the collection of cells we add. Then

$$C(I) = 2^{n-k-1}(2^t - 1)$$

and

$$C(R) \leq 2^{n-k-2}(m - 2) + 2^{n-k-1} = 2^{n-k-2}(m).$$

So

$$C(I) = 2^{n-k-1}(2^t - 1) \geq 2^{n-k-1}(m - 1) > 2^{n-k-2}(m) \geq C(R)$$

and thus $C(A') < C(A)$. Note that

$$S(I) \leq (i - 1) + (\lceil \log_2(i - 1) \rceil - 1) \cdot i = i\lceil (\log_2(i - 1) \rceil - 1$$

and

$$S(R) \geq \frac{(i - 3)(i - 2)}{2}.$$

Since $S(R) \geq S(I)$ for $i \geq 13$ we know that $S(A') \geq S(A)$. Therefore such an A is not optimal.

If $k - m \leq \lceil \log_2(m) \rceil + 1$ then we apply a trapezoid move. Choose $\ell \in \mathbb{N}$ such that $\log_2(k) \leq \ell \leq \sqrt{\frac{k}{2}} - 1$ which we can do because $k \geq 200$. Since the space of this trapezoid is at most the space in the triangle and the cost is at least as much as the cost in one column, the same argument applies. $\square$

Remark. *If A is a downset with space greater than $78n - 728$ then the last out-corner* *of A has space at least* 13.

Lemma 95. *Suppose that A is a downset in B_n with the first out-corner being $(a, n-1)$ for some a. If $H(A) = (1, 1)$ then $\mathcal{O}(A) = ((a, n-1), (a+1, n-2), (a+2, k))$ for some k. If $H(A) = (2)$ then $\mathcal{O}(A) = ((a, n-1), (a+2, n-2))$.*

Proof. In either case, if the height of the second out-corner were not $n-2$ then we could trade the last out-corner and replace it with 2 cells in the previous column. □

5.3.3.4 Proof of Theorem 70

Proof of Theorem 70. Suppose that $\mathcal{H}$ is a 3-graph with n vertices having e edges that is not $(2, 3, 1)$-lex like and let $\mathcal{G}$ be a 3-graph on n vertices having e edges that is $(2, 3, 1)$-lex like. If the shape of $\mathcal{H}$ in B_n is the same as that of $\mathcal{G}$ then $i_2(\mathcal{H}) = i_2(\mathcal{G})$. So, assume that the shape of $\mathcal{H}$ in B_n is not the same as the shape of $\mathcal{G}$. Let A be the shape of $\mathcal{H}$ in B_n.

Suppose A has only one out-corner (i, j). Suppose $j - i \geq \lceil \log_2(i) \rceil$. Since $e \geq 78n - 728$ we know $i \geq 7$ and so by Proposition 89 A is not a maximizing shape. On the other hand, suppose $j - i \leq \lfloor \log_2(i) \rfloor$. Then $n - 1 > j \geq 200$ since A is not $(2, 3, 1)$-lex like and $e \geq 1{,}333{,}300$. So by Lemma 92, A is not an optimizing shape. Therefore, if A has only one out-corner (i, j) then $j = n - 1$ and the 3-graph is $(2, 3, 1)$-lex like.

Suppose A has more than one out-corner. By Lemma 84 any two out-corners (a, b) and (c, d) with $a < c$ satisfy $a \geq c - 2$ or $a < \frac{c-3}{2}$.

Consider the last out-corner of A, call it (i, j). Since $e \geq 1{,}333{,}300$ we know that $j \geq 200$. Since $e \geq 78n - 728$ we know that $i \geq 13$.

Suppose that any previous out-corner (a, b), if there is one, has $a < \frac{i-3}{2}$. If $j - i \geq \lceil \log_2(i) \rceil$ then A is not an optimizer by Proposition 89. If $j - i \leq \lfloor \log_2(i) \rfloor$

then A is not an optimizer by Lemma 92. So, there must be a previous out-corner (a, b) with $a \geq i - 2$.

If there is a previous out-corner (a, b) with $a \geq i - 2$ then A ends with 1 short stair, 2 short stairs, or 1 long stair by Lemma 86. By Lemmas 93 and 94 we know that A is not optimal.

Finally, we know that the first drop is 1 by Lemma 95. Therefore, if A is an optimal shape then it is $(2, 3, 1)$-lex like.

□

5.3.4 Maximizing 2-independent Sets in $\mathcal{H}_3(n, e)$ When e is Small

When e is small, Lemmas 84 and 86 still apply. These lemmas give us restrictions on what the horizontal distance vector can look like. Suppose A is an optimizing downset with $\mathcal{O}_1(A) = (o_1, \ldots, o_k)$ and $D(A) = (d_1, \ldots, d_{k-1})$. The k^{th} entry of $D(A)$ can be $1, 2$ or at least $\lfloor \frac{o_{k+1}+4}{2} \rfloor$. Moreover, we have a restriction on the placements of 1's and 2's: 1 and 2 can not be adjacent, and there cannot be 3 or more consecutive 1's or 2 or more consecutive 2's. Finally, the sum of the entries of the horizontal distance vector is at most $n - 3$.

There are still examples that are $(2, 3, 1)$-lex like, such as those in the next four figures (Figures 5.20, 5.21, 5.22, and 5.23).

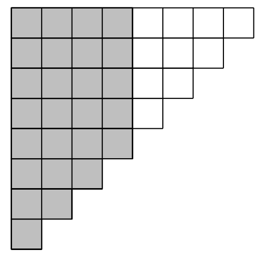

Figure 5.20: A maximizer for 10 vertices and 60 edges having 26 independent sets.

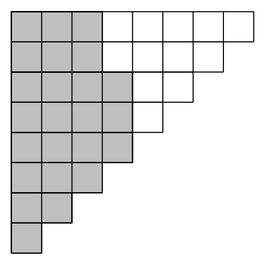

Figure 5.21: A maximizer for 10 vertices and 51 or 52 edges having 29 independent sets.

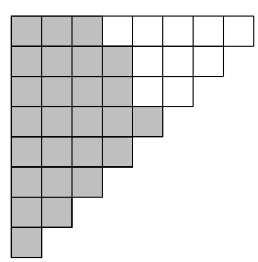

Figure 5.22: A maximizer for 10 vertices and 61 edges having 19 independent sets.

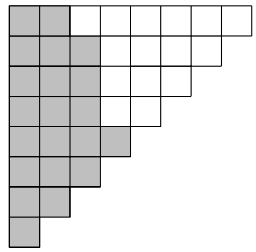

Figure 5.23: A maximizer for 10 vertices and 41 edges having 42 independent sets.

However, there are also examples in which the maximizer is not $(2,3,1)$-lex like. Figure 5.24 shows an optimal shape that is pushed as far up and left as possible and is not (2,3,1)-lex like. This particular example is disrupted by the prohibitive cost of the first column. Recall that cells in the first column are twice as expensive as their same height neighbors. This problem is removed once e is large.

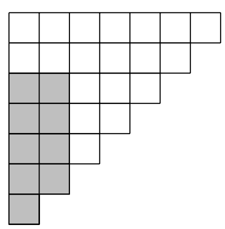

Figure 5.24: A maximizer for 9 vertices and 13 edges having 66 independent sets.

Figures 5.25, 5.26, and 5.27 show more optimal examples that are not (2,3,1)-lex. The optimizer for 9 vertices and 57 edges is prevented from being $(2,3,1)$-lex like by the width of B_9. Given a large enough e (and hence n) there is enough width to move several columns from the end of the shape.

These examples show that finding the maximizers for small e is more complicated than finding the maximizers for large e.

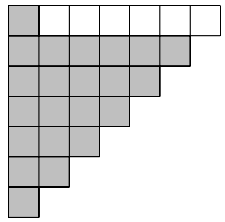

Figure 5.25: A maximizer for 9 vertices and 57 edges having 6 independent sets.

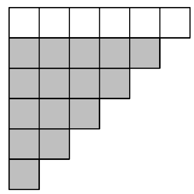

Figure 5.26: A maximizer for 8 vertices and 35 edges having 7 independent sets.

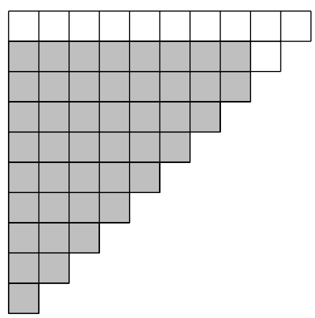

Figure 5.27: A maximizer for 12 vertices and 156 edges having 13 independent sets.

5.4 Further Directions

There are many open problems remaining in this area. Using a mix of computation and stronger lemmas we expect to solve the maximization problem for 2-independent sets in $\mathcal{H}_3(n, e)$ for all n and e. Additionally, our methods do not lead to any results about maximizing the number of 2-independent sets of each size. There is, of course, the more general problem of maximizing the number of s-independent sets in r-uniform hypergraphs. Since we know the colex graph maximizes the number of 1-

independent sets in r-graphs for any r and the lex graph maximizes the number of 3-independent sets in r-uniform hypergraphs for any r, the remaining maximization problems are those with $r > 3$ and $1 < s < r$.

Furthermore, many of the extremal questions about the number of substructures in graphs that have been answered have not been extended to hypergraphs. One particularly interesting question is that of minimizing the number of matchings in $\mathcal{H}_r(n, e)$.

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