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BETTI SEQUENCES OVER LOCAL RINGS AND CONNECTED SUMS OF GORENSTEIN RINGS

by

Zheng Yang

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BETTI SEQUENCES OVER LOCAL RINGS AND

CONNECTED SUMS OF GORENSTEIN RINGS

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University of Nebraska, 2015

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This thesis consists of two parts:

1) Polynomial growth of Betti sequences over local rings (Chapter 2),

2) Connected sums of Gorenstein rings (Chapter 3).

Chapter 1 gives an introduction for the two topics discussed in this thesis.

The first part of the thesis deals with modules over complete intersections using free resolutions. The asymptotic patterns of the Betti sequences of the finitely generated modules over a local ring R reflect and affect the singularity of R. Given a commutative noetherian local ring and an integer c, sufficient conditions and necessary conditions are obtained for all Betti sequences of finitely generated modules to be eventually polynomial of degree less than c. When $c \leq 3$ this property characterizes hypersurface sections of local complete intersections with multiplicity 2^b. This is joint work with Avramov and Seceleanu.

The second part of the thesis studies a construction on the set of Gorenstein local rings, known as their connected sum. Given a Gorenstein ring, one would like to know whether it can be decomposed as a connected sum and if so, what are its components. We give a concrete description in the case of Gorenstein Artin local algebra over a field. We further investigate conditions on the decomposability of some classes of Gorenstein Artin rings. This is joint work with Hariharan and Celikbas.

DEDICATION

To my parents.

ACKNOWLEDGMENTS

First of all, I must give my deepest thanks to my thesis advisor, Professor Luchezar "Lucho" Avramov, for everything he has done. His teaching has been most important and influential through all steps in my graduate studies. Lucho first introduced me to algebra from his first year course on Galois theory and then to commutative algebra through his reading courses based on well-known textbooks by Atiyah-Macdonald and Matsumura. The first paper he handed me to read was on connected sums, which is the beginning of the story for Chapter 3 in this thesis. His deep knowledge and beautiful insights on infinite free resolutions has finally brought me into this wonderland. I hope Chapter 2 will give an idea of what we have found there and, I can only wish to present this thesis as an example to reflect his many ideas and teaching philosophies on mathematics.

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Contents

Chapter 1

Introduction

Starting from the integers, many objects taught in school mathematics, such as rational numbers, real numbers, complex numbers, functions and matrices, are examples of rings. They are sets whose elements one can add, subtract and multiply with the usual properties. A distinction is that multiplication of matrices is not commutative, while multiplication of numbers and functions is commutative. This thesis is about commutative rings.

Some rings are more special than others. Rings where one can divide by non-zero elements, as the rational numbers, are called fields. Some subsets of rings are more important than others. The subsets whose elements one can add and multiply with any element of the ring are called ideals. For example, in the rings of integers and polynomials in one variable over fields, all the elements of an ideal are multiples of a single element. This is a strong finiteness property. If every ideal is a combination of finitely many elements, then the ring is called Noetherian. This thesis is about Noetherian commutative rings.

Rings are non-linear objects, but they operate linearly on analogs of vector spaces, called modules. Thus, the elements of a module M over a ring R can be added to one another and multiplied by elements of R. As in vector spaces, we can form the span of any subset of a given module. However, there is a basic new phenomenon when passing from fields to rings: Unlike vector spaces, modules do not necessarily have bases.

If a module has a basis, then it is said to be free. When M is not free, we can try to measure the failure of freeness. This is done by mapping a free module onto it, which is the same as choosing a set of generators for M. The dependence among the chosen generators, are the solutions of a system of linear equations in R. They form a new module, called a first syzygy of M. If it is not free, we again map a free module onto it and solve the corresponding system of equations. Its solutions are called a second syzygy of M. This method of approximating M by free modules was introduced by Hilbert in his study of graded modules over polynomial rings. The sequence of R-linear maps of free modules is called a free resolution of M. The properties of resolutions reflect and affect certain special properties of rings. At present, the class of rings for which the best information is available is called complete intersection. They include polynomial rings.

The first part of the thesis deals with modules over complete intersections using free resolutions.

Rings can also be studied more directly. The model here is number theory. Some of the familiar operations on numbers have natural extensions to commutative rings. The idea of congruences is used to define quotient rings. The passage from integers to rationals has counterpart in localization. The formation of p-adic numbers has given rise to the notion of completion. All of these are procedures that construct new rings out of the old ones. Many other ways exist to produce new rings out of old ones. They include formation of polynomial rings, of products of rings, passage to subrings, and so on.

In each case, properties are transferred from old rings to new rings. One needs to know what properties are inherited and how they are linked. On the other hand, given a ring, it is important to be able to recognize it from its certain familiar properties. An instructive way is to decompose it into simpler pieces. In order to understand the given object, one needs to see how their pieces fit together. Connected sum is our main tool for composing and decomposing rings. Its name reflects properties of cohomology rings of connected sums of manifolds, known from algebraic topology. We apply it on an important class of rings. They are called Gorenstein rings and they contain complete intersections.

In the second part of the thesis, we study Gorenstein rings using connected sums. Next we give more specific descriptions for the main results in each part.

1.1 Complete Intersections

In this part rings are commutative and noetherian. They are either local or graded.

We start with

1.1.1 Example. 0) Take $P = k[[x,y]]$, $R = P$ and $M = R/(x,y)$. The free resolution of M has the following simple form:

$$\mathbf{F}\colon 0 \leftarrow M \leftarrow R^1 \xleftarrow{\begin{bmatrix} x & y \end{bmatrix}} R^2 \xleftarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R^1 \leftarrow 0$$

This is an example of a Koszul complex.

Regular rings are rings over which each module admits a finite approximation by free modules; thus, one needs to solve only a finite string of iteratively defined systems of equations. Examples of regular local rings include the formal power series rings over a field.

To illustrate some classical results and motivating examples, we take for simplicity that P a power series ring and I an ideal of P. Set $R = P/I$ and M is an R-module. These examples are not too special. Indeed, any complete local ring can be presented as a quotient of a regular local ring, due to Cohen's structure theorem.

The simplest rings over which infinite free resolutions first appear are hypersurfaces. These are the rings with I generated by a single nonzero element.

1.1.2 Example. 1) Take $P = k[[x, y]]$, $I = (x^2 + y^2)$ and $M = k$. The free resolution of M is

$$\mathbf{F}\colon 0 \leftarrow R \xleftarrow{\begin{bmatrix} x & y \end{bmatrix}} R^2 \xleftarrow{\begin{bmatrix} -y & x \\ x & y \end{bmatrix}} R^2 \xleftarrow{\begin{bmatrix} y & -x \\ -x & -y \end{bmatrix}} R^2 \xleftarrow{\begin{bmatrix} -y & x \\ x & y \end{bmatrix}} R^2 \leftarrow \cdots$$

the first map encodes the defining relation of the module M. In the 2 by 2 matrices, one column says that x and y commute, the other column comes from the defining relation of the ring R. The defining equation of R dominates asymptotic behavior because these 2 by 2 matrices repeat themselves with periodicity 2.

2) Take $P = \mathbb{Q}[[x, y, z]]$ and $I = (x^3 + 3y^3 - 2yz^2 + 5z^3)$. Take $\mathfrak{m} = (x, y, z)R$ and $M = R/\mathfrak{m}^2$. The free resolution of M:

$$\mathbf{F}\colon 0 \leftarrow R^1 \leftarrow R^6 \leftarrow R^9 \leftarrow R^9 \leftarrow R^9 \leftarrow R^9 \cdots$$

This example is taken from [12, 1.3] and computed using Macaulay2.

We note that all the matrices in the above examples have entries in the maximal ideal. In fact, it is always possible to achieve this, by using a minimal system of generators for each syzygy of the resolution. All free resolutions with such property are isomorphic. The rank of the ith module F_i in a minimal resolution is called the ith *Betti number* of M, denoted by

$$\beta_i^R(M) = \operatorname{rank}_R(F_i).$$

When R is regular, the Betti sequences of all M are finite. Indeed, a remarkable theorem of Serre says that if the Betti sequence of k is finite, then it is so for any M. In the introduction of his book [29], Matsumura states that Serre's characterization grasps the essence of regular local ring, and is also an important meeting-point of ideal theory and homological algebra. In general, for any M with a finite resolution over R, the length of its Betti sequence is equal to the number $s = \operatorname{depth} R - \operatorname{depth} M$. This is the well-known Auslander-Buchbaum formula.

When R is hypersurface, Eisenbud (1980) proved that minimal free resolution of M becomes periodic of period 2 after at most $s+1$ steps. In fact, each Betti sequence is eventually constant.

To continue, we look at examples when I is generated by f_1, f_2.

1.1.3 Example. 3) Take $P = k[[x,y]]$ and $I_2 = (x^2, xy)$. Set $R = P/I_2$ and $M = k$. The minimal free resolution of M is

$$\mathbf{F}\colon 0 \leftarrow R \xleftarrow{\partial_1} R^2 \xleftarrow{\partial_2} R^3 \xleftarrow{\partial_3} R^5 \xleftarrow{\partial_4} R^8 \leftarrow \cdots$$

4) Take $P = k[[x,y]]$ and $I_1 = (x^2, y^2)$. Set $R = P/I_1$ and $M = k$. The minimal free

resolution of M is

$$\mathbf{F}\colon 0 \leftarrow R \xleftarrow{\partial_1} R^2 \xleftarrow{\partial_2} R^3 \xleftarrow{\partial_3} R^4 \xleftarrow{\partial_4} R^5 \leftarrow \cdots$$

In these two examples, the two generators in I_1 are coprime while they share a common factor x in I_2.

Another way to express the difference is to say whether f_2 is a zero-divisor modulo f_1 or not. In general, a sequence $f_1, \dots, f_c$ such that f_i is not a zero divisor modulo $f_1, \dots, f_{i-1}$ for all i is called regular. In such a case, R is called a complete intersection (c.i) of codimension c. In general,

(1) If R is c.i of codimension c, then each one of the sequences $(\beta_{2i}^R(M))_{i\geq 0}$ and $(\beta_{2i+1}^R(M))_{i\geq 0}$ is eventually given by some polynomial in i.

(2) If R is not c.i, then there exist constants $1 < a < b$ such that $a^i \leq \beta_i^R(k) \leq b^i$ for $i \geq 0$.

The first theorem is due to Gulliksen [21], and the second is a theorem of Avramov [10].

The question that we study in this thesis are:

1.1.4 Question. For which R the Betti sequence of every R-module is eventually given by some polynomial in i?

By Gulliksen's results, to answer this question, we may assume R is c.i. and ask more precisely:

1.1.5 Question. For which c.i. the Betti sequence of every R-module is eventually given by a polynomial in i?

For $c \leq 1$, the answers are known:

(1) The Betti sequences are eventually zero if and only if R is regular.

(2) The Betti sequences are eventually constant if and only if R is a hypersurface.

The first is the well-known Auslander-Buchsbaum-Serre theorem. The second is compiled from the work of Shamash, Gulliksen and Eisenbud.

To continue, we denote $R^{\mathfrak{g}}$ to be the associated graded ring of R for the $\mathfrak{m}$-adic filtration. The positive integer $e(R)$, called multiplicity, is an important invariant of R. It is known that the multiplicity of a c.i ring of codimension c is not less than 2^c. When it is equal to 2^c, then R has minimal multiplicity. For details, see 2.2.5 and 2.2.7.

Avramov proved in [6]:

(1) When R has minimal multiplicity, then the Betti sequences of all R-modules are eventually polynomial of degree at most $c-1$.

(2) If the Betti sequence of $R/\mathfrak{m}^2$ is eventually polynomial, then $R^{\mathfrak{g}}$ has at least $c-1$ quadratic relations.

The simplest open case left out from Avramov's theorems is $e(R) = 5$. Take for example,

$$R = \frac{k[[x,y]]}{(x^2, y^3)}.$$

In joint work with Avramov and Seceleanu, we tighten the gap between the known necessary conditions and sufficient conditions. To sharpen the sufficient condition for polynomial growth, we prove (see Theorem 2.3.1):

Theorem A. *Let $(R, \mathfrak{m}, k)$ be a local ring with infinite residue field.*

If the $\mathfrak{m}$*-adic completion* $\widehat{R}$ *is isomorphic to* $Q/(g)$*, where* Q *is a local complete intersection ring of minimal multiplicity and* g *is in the square of its maximal ideal, then the Betti sequences of all finite* R*-modules are eventually polynomial.*

Theorem 2.3.1 is proved in Section 2.3. This is done by deforming $\widehat{R}$ to a codimension $c-1$ complete intersection ring with *minimal multiplicity*, usually different from the one in the hypothesis of the theorem, which determines all but finitely many Betti numbers of M. Delicate constructions are involved. They utilize the left action, by composition products, of $\mathrm{Ext}_R(k,k)$ on $\mathrm{Ext}_R(M,k)$, and draw heavily on the work in [6] and [8]. Additional input comes from properties of rings of minimal multiplicity, reviewed or proved in Section 2.2.

To strengthen the necessary condition of Avramov, we recycle the argument in [6, 2.1] for other R-modules to get the converse of 1.1 for c less than 4. See Theorem 2.5.1.

When R is homogenous, we close up the gap and characterize rings with eventual polynomial Betti sequences. We say that R is *homogeneous* of the type $(n_1,\dots,n_c)$ if its $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to $k[[x_1,\dots,x_e]]/I$, where k is a field, $k[[x_1,\dots,x_e]]$ a ring of formal power series over it, and I an ideal minimally generated by forms $f_1,\dots,f_c$ of degree at least 2 with $\deg(f_i)=n_i$ and $n_1\le\cdots\le n_c$. The result below answers, in the positive, a question raised in [6, p.32]. See Corollary 2.5.2.

Theorem B. *Let* R *be a homogeneous local ring and* c *a non-negative integer.*

Every Betti sequence over R *is eventually polynomial of degree less than* c *if and only if* R *is complete intersection of the type* $(2,\dots,2,n)$ *for some* $n\ge 2$*.*

1.2 Connected Sums

Given two rings R and S, the simplest way to combine them into a new ring is to take their product $R \times S$. However, the product ring is not local even when R and S are local because it has idempotents $(1,0)$ (or $(0,1)$). Is it possible to make a local ring out of this construction?

Let $(R, \mathfrak{m}_R, k)$ and $(S, \mathfrak{m}_S, k)$ be local rings. Since both $\mathfrak{m}_R \times S$ and $R \times \mathfrak{m}_S$ are maximal ideals of $R \times S$, we can try to glue them into one. The way to do this is to throwing away some elements. This leads to the following construction: Given local rings R and S with common residue field k, and ring homomorphisms $R \xrightarrow{\varepsilon_R} k \xleftarrow{\varepsilon_S} S$, put:

$$R \times_k S = \{(r,s) \in R \times S : \varepsilon_R(r) = \varepsilon_S(s)\}.$$

Thus $R \times_k S$ is a subring of $R \times S$ and it is local with maximal ideal $\mathfrak{m}_R \times \mathfrak{m}_S$. It is called the fiber product of R and S over k.

1.2.1 Example. Take $R = k[x]/(x^4)$ and $S = k[y]/(y^3)$. Let both ε_R and ε_S be the canonical surjections. We get

$$R \times_k S = k[x,y]/(x^4, y^3, xy).$$

An ideal is irreducible if it cannot be written as an intersection of two larger ideals. Every ideal in a Noetherian ring is a finite intersection of irreducible ideals. Thus it is important to understand the structure of rings or its quotients with the ideal (0) being irreducible. We start with 0-dimensional Noetherian local rings. They are Artinian. A quotient and localization of an Artinian ring is again Artinian. If an Artinian local ring with the property that (0) is an irreducible ideal, then it is called Gorenstein Artin.

It is useful to observe that (0) is irreducible means $\operatorname{soc}(R)$ is a 1-dimensional vector space. Recall that the socle of R is the annihilator of the maximal ideal, that is, $\operatorname{soc}(R) = \{x \in R \mid x\mathfrak{m} = 0\}$. In the above example, $\operatorname{soc}(R) = \langle x^3 \rangle$ and $\operatorname{soc}(S) = \langle y^2 \rangle$. So R and S are Gorenstein Artin. However, $\operatorname{soc}(P) = \operatorname{soc}(R) \oplus \operatorname{soc}(S)$, which is 2-dimensional. So P is not Gorenstein. Can we make a Gorenstein ring out of the fiber product?

Again, by identifying socle elements, we achieve the following construction:

Given Gorenstein Artin local rings R and S with common residue field k. Let $\operatorname{soc}(R) = \langle \delta_R \rangle$, $\operatorname{soc}(S) = \langle \delta_S \rangle$. Identifying δ_R with $(\delta_R, 0)$ and δ_S with $(0, \delta_S)$, a connected sum of R and S over k is the quotient ring

$$R \#_k S = (R \times_k S)/\langle \delta_R - \delta_S \rangle.$$

1.2.2 Example. For the rings in Example 1.2.1 we have

$$R \#_k S = k[x, y]/(x^3 - y^2, xy).$$

In general, a connected sum of two Gorenstein rings is Gorenstein. Hence, one would like to be able to recognize whether a given Gorenstein ring is a connected sum and if so, what are its components. To illustrate:

1.2.3 Example. Take $Q = k[x, y, z]/(xy, xz, yz, x^3 - y^2, y^2 - z^2)$. It is Gorenstein Artin. This ring Q can be written as a connected sum:

$$Q \cong k[x]/(x^4) \#_k k[y, z]/(yz, y^2 - z^2).$$

In general, we ask:

1.2.4 Question. Given a Gorenstein Artin ring Q, can it be decomposed as a connected sum? If we know Q is a connected sum, what is a decomposition?

For k-algebras, we are able to give answers directly in terms of their defining ideals. In particular, we prove:

1.2.5 Theorem. *(see Theorem 3.3.17) Let Q be a Gorenstein Artin local k-algebra with $\ell\ell(Q) \geq 1$. Then the following are equivalent:*

(1) Q can be decomposed nontrivially as a connected sum over k.

(2) $Q \cong k[Y_1, \ldots, Y_m, Z_1 \ldots, Z_n]/I_Q$ for m, $n \geq 1$ with $\boldsymbol{Y} \cdot \boldsymbol{Z} \subset I_Q \subset \langle \boldsymbol{Y}, \boldsymbol{Z} \rangle^2$.

If the above conditions are satisfied, then we can write $Q \cong R\#_k S$, where $R = k[\boldsymbol{Y}]/I_R$, $S = k[\boldsymbol{Z}]/I_S$ with $I_R = I_Q \cap k[\boldsymbol{Y}]$, and $I_S = I_Q \cap k[\boldsymbol{Z}]$.

In particular, this result gives the components of Q as a connected sum. In general, by definition of the fibre product, R and S can be easily identified with quotients of $R \times_k S$. It is not easy to recover the components R and S from a connected sum $R\#_k S$. In the case of Artinian k-algebras, we can use the vector space arguments to deal with the defining ideals.

One gets different corollaries from this description. We give one for the minimal number of generators of the defining ideals, denoted by $\mu(I)$.

1.2.6 Corollary. *(see Proposition 3.3.4) There is an equality*

$$\mu(I_Q) = \mu(I_R) + \mu(I_S) + mn + c, \textit{where} \quad c = \begin{cases} 1 & m, n \geq 2 \\ -1 & m = 1 = n \\ 0 & \textit{otherwise} \end{cases}$$

Some corollaries also provide answers to the question of indecomposability of connected sums. For example, corollary 1.2.6 yields for small embedding dimensions:

1.2.7 Corollary. *(see Theorem 3.3.6) Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin k-algebra. Then Q is indecomposable as a connected sum over k in the cases:*

i) $\operatorname{edim}(Q) = 3$ *and* $\mu(I_Q) \neq 5$.

ii) $\operatorname{edim}(Q) = 4$ *and* $\mu(I_Q)$ *is an even number.*

Among the applications, we single out a special structure theorem. To describe it, we first recall some definitions. The graded ring associated to the maximal ideal $\mathfrak{m}$ of T, is defined as $T^{\mathrm{g}} \cong \oplus_{i=0}^{\infty}(\mathfrak{m}^i/\mathfrak{m}^{i+1})$. We denote it by G. When T is Artinian, so is G. The socle degree is measured by the number $\ell\ell(T) = \max\{n : \mathfrak{m}^n \neq 0\}$.

When the socle of G in degree two and higher is 1-dimensional, G acquires a simple decomposition. It is a fiber product of two graded k-algebras, one is Gorenstein and the other is itself a fiber product of finitely many $k[x]/(x^2)$. We call such a G Gorenstein up to linear socle.

Here is how these constructions are connected into one

1.2.8 Theorem. *(see Theorem 3.4.7) Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin k-algebra and set $s = \ell\ell(Q)$. If $G = Q^{\mathrm{g}}$ is Gorenstein up to linear socle and $s \geq 3$, then there exist Gorenstein Artin local k-algebras R and S such that $Q \cong R\#_k S$, where*

a) $\ell\ell(S) \leq 2$, and if G is not Gorenstein, then $\ell\ell(S) = 2$.

b) $R^{\mathrm{g}} \cong G/\langle \operatorname{soc}(G) \cap G_1 \rangle$. In particular, $\ell\ell(R) = s$.

This result answers the two questions 1.2.4 when Q^{g} is Gorenstein up to linear socle. It is inspired by a construction of Iarrobino and motivated by the structure theorems for two special Gorenstein rings: they are called stretched and short rings in the literature. The ring Q in 1.2.3 is an example for both. As a consequence, our theorem 1.2.8 generalizes those structure theorems by removing the conditions on the residue fields.

Chapter 2

Polynomial Growth of Betti Sequences over Local Rings

The contents of this chapter are contained in the author's paper with Luchezar L. Avramov and Alexandra Seceleanu: *Polynomial growth of Betti sequences over local rings*.

Introduction

This paper is concerned with free resolutions of finitely generated modules over a commutative noetherian R with unique maximal ideal $\mathfrak{m}$. Each finitely generated R-module M has a unique up to isomorphism *minimal* free resolution. The rank $\beta_i^R(M)$ of the ith module in such a resolution is called the ith *Betti number* of M.

Asymptotic patterns of Betti sequences reflect and affect the singularity of R.

For example, when R is complete intersection of codimension c Gulliksen [21] proved that each one of the sequences $(\beta_{2i}^R(M))_{i\geq 0}$ and $(\beta_{2i+1}^R(M))_{i\geq 0}$ is eventually given by some polynomial in i. Eisenbud [18] for $c=1$ and Avramov [5] in general showed that these polynomials have equal degrees and leading coefficients. Conversely, if $\beta_i^R(R/\mathfrak{m})$ is bounded above by a polynomial in i of degree $c-1$, then R is complete intersection of codimension at most c, due to Gulliksen [22].

Here we study modules M with *eventually polynomial* Betti sequences; that is, those for which there exists a polynomial β_M^R such that $\beta_i^R(M) = \beta_M^R(i)$ holds for $i \gg 0$. If $\beta_{R/\mathfrak{m}}^R$ exists, then R is complete intersection by Gulliksen's theorem; let c denote its codimension. Avramov [6] showed that if $\beta_{R/\mathfrak{m}^2}^R$ exists as well, then $R^{\mathfrak{g}}$, the associated graded ring of R for the $\mathfrak{m}$-adic filtration, has at least $c-1$ quadratic relations. The multiplicity of a complete intersection ring of codimension c is not less than 2^c. When equality holds β_M^R exists for *every* finite R-module M; see [6].

The results in this paper tighten the gap between the known necessary conditions and sufficient conditions. The sample below illustrates the flavor. Standard modifications allow for a formulation that makes no hypotheses on the residue field.

Main Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring with infinite residue field.*

If the $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to $Q/(g)$, where Q is a local complete intersection ring of minimal multiplicity and g is in the square of its maximal ideal, then the Betti sequences of all finite R-modules are eventually polynomial.

The converse holds if $\operatorname{ecodim} R \le 3$, *or if $R^{\mathfrak{g}}$ is complete intersection.*

As a corollary we show that the Betti sequences of all graded modules over a *standard graded* commutative ring A are eventually polynomial if and only if A is a graded complete intersection ring with at most one non-quadratic relation. This result answers, in the positive, a question raised in [6, p.32]. We do not know whether the restriction on the codimension can be removed in the local case.

The main theorem is assembled in Section 2.5 from results in earlier sections.

Its first assertion is proved in Section 2.3. This is done by deforming $\widehat{R}$ to a codimension $c-1$ complete intersection ring with *minimal multiplicity*, usually different from the one in the hypothesis of the theorem, which determines all but finitely many Betti numbers of M. Delicate constructions are involved. They utilize the left action, by composition products, of $\operatorname{Ext}_R(k,k)$ on $\operatorname{Ext}_R(M,k)$, and draw heavily on the work in [6] and [8]. Additional input comes from properties of rings of minimal multiplicity, reviewed or proved in Section 2.2.

The (partial) converse assertion is obtained in Section 2.4, by identifying modules whose Poincaré series have poles at -1. A variety of techniques is put to use in the explicit computations of these series. Background material is presented in Section 2.1, which also introduces notation and definitions used in the entire paper.

For general information on free resolutions we routinely refer to [10].

2.1 Background

Throughout the paper $(R,\mathfrak{m},k)$ denotes a local ring; that is, R is a commutative noetherian ring with unique maximal ideal $\mathfrak{m}$, and k is the residue field $R/\mathfrak{m}$. The minimal number of generators of $\mathfrak{m}$ is called the *embedding dimension* of R, denoted $\operatorname{edim} R$. The *codimension* of R is the number $\operatorname{ecodim} R = \operatorname{edim} R - \dim R$.

Throughout the section M denotes a finite, that is, finitely generated, R-module.

We recall some notions used to describe minimal free resolutions.

2.1.1. The ith *Betti number* $\beta_i^R(M)$ of M is defined to be the rank of the ith module in a minimal free resolution of M over R. It can be computed in different ways:

$$\operatorname{rank}_k \operatorname{Tor}_i^R(M,k) = \beta_i^R(M) = \operatorname{rank}_k \operatorname{Ext}_R^i(M,k)$$

2.1.2. When $\mathcal{M}$ is a graded k-vector space with $\operatorname{rank}_k \mathcal{M}^j$ finite for each j and zero for small j, its *Hilbert series* is the formal Laurent series $H_{\mathcal{M}}(z) = \sum_j \operatorname{rank}_k \mathcal{M}^j z^j$.

The *Poincaré series* of M is defined by $P^R_M(z) = H_{\operatorname{Ext}_R(M,k)}(z)$.

Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. When k is infinite we set $\acute{R} = \widehat{R}$; otherwise, we let $\acute{R}$ denote the completion of the polynomial ring $R[x]$ at its ideal $\mathfrak{m}[x]$. Thus, $\acute{R}$ is a complete local ring with maximal ideal $\mathfrak{m}\acute{R}$ and infinite residue field $\acute{k}$.

The canonical map $R \to \acute{R}$ is flat. For the $\acute{R}$-module $\acute{M} = \acute{R} \otimes_R M$ a standard isomorphism $\operatorname{Ext}_{\acute{R}}(\acute{M}, \acute{k}) \cong \operatorname{Ext}_R(M, k) \otimes_k \acute{k}$ gives $P^{\acute{R}}_{\acute{M}}(z) = P^R_M(z)$.

2.1.3. Form the graded k-vector spaces $\mathcal{E} = \operatorname{Ext}_R(k, k)$ and $\mathcal{M} = \operatorname{Ext}_R(M, k)$.

For all $i, j \in \mathbb{Z}$ *composition products* yield k-linear maps $\mathcal{E}^i \otimes_k \mathcal{M}^j \to \mathcal{M}^{i+j}$. They turn $\mathcal{E}$ into graded algebra and $\mathcal{M}$ into a left graded $\mathcal{E}$-module; see [39, §1].

Let $\mathcal{N}$ be a graded $\mathcal{E}$-module and s an integer. A graded $\mathcal{E}$-module $\Sigma^s\mathcal{N}$ is defined by $(\Sigma^s\mathcal{N})^j = \mathcal{N}^{j+s}$ and $\xi\varsigma^s(\nu) = (-1)^{is}\varsigma^s(\xi\nu)$, where ξ is in $\mathcal{E}^i$ and $\varsigma^s(\nu)$ is the element of $(\Sigma^s\mathcal{N})^j$ corresponding to $\nu \in \mathcal{N}^{j+s}$; note that $H_{\Sigma^s\mathcal{N}}(z) = z^{-s}H_{\mathcal{N}}(z)$.

Let F be a minimal free resolution of M over R. Thus, $\partial_n(F_n) \subseteq \mathfrak{m}F_{n-1}$ for $n \geq 1$ and for every non-negative integer s we have an exact sequence of R-modules

$$0 \to L \to F_{s-1} \to \cdots \to F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0 \qquad (2.1.1)$$

Break it up into short exact sequences and compose the connecting maps in the exact sequence obtained by applying $\operatorname{Ext}_R(?, k)$ to each one. Since connecting maps are compatible with composition products, we get an isomorphism

$$\operatorname{Ext}_R(L, k) \cong \Sigma^s(\mathcal{M}^{\geqslant s}) \qquad (2.1.2)$$

of graded $\mathcal{R}$-modules. The exact sequence 2.1.1 also yields an equality

$$P^R_M(z) = \sum_{i=0}^{s-1} \beta^R_i(M) z^i + z^s P^R_L(z) \qquad (2.1.3)$$

The next result can easily be extracted from the work of Roos in [31, §1]. Formula (2.1.1) does not appear there, so we include a variant of his arguments.

2.1.4 Proposition. *Let $(R, \mathfrak{m})$ be a local ring of embedding dimension e.*

If $\mathcal{A}$ is the k-subalgebra of $\mathcal{E} = \operatorname{Ext}_R(k,k)$ generated by $\mathcal{E}^1$ and $\overline{\mathcal{E}} = \mathcal{E}/\mathcal{E}\mathcal{E}^1$, then

$$P^R_{R/\mathfrak{m}^2}(z) = \frac{1+z}{z} \cdot H_{\overline{\mathcal{E}}}(z) + \frac{ez-1}{z} \cdot P^R_k(z) \tag{2.1.1}$$

$$H_{\overline{\mathcal{E}}}(z) = \frac{P^R_k(z)}{H_{\mathcal{A}}(z)} \tag{2.1.2}$$

Proof. Set $\mathcal{M} = \operatorname{Ext}_R(R/\mathfrak{m}^2, k)$. By [13, 2.7], the exact sequence of R-modules

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}^2 \to k \to 0$$

and the isomorphism $\operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) \cong \mathcal{E}^1$ induce an exact sequence

$$\cdots \to \Sigma^{-1}(\mathcal{E} \otimes_k \mathcal{E}^1) \xrightarrow{\Sigma^{-1}\delta} \mathcal{E} \to \mathcal{M} \to \mathcal{E} \otimes_k \mathcal{E}^1 \xrightarrow{\delta} \Sigma\mathcal{E} \to \cdots$$

of graded left $\mathcal{E}$-modules with $\delta(\xi \otimes \tau) = \varsigma^{-1}(\xi\tau)$. It yields an exact sequence

$$0 \to \overline{\mathcal{E}} \to \mathcal{M} \to \mathcal{E} \otimes_k \mathcal{E}^1 \xrightarrow{\Sigma^{-1}\delta} \Sigma\mathcal{E} \to \Sigma\overline{\mathcal{E}} \to 0 \tag{2.1.3}$$

of graded left $\mathcal{E}$-modules, from which we obtain an equality

$$H_{\overline{\mathcal{E}}}(z) - H_{\mathcal{M}}(z) + eH_{\mathcal{E}}(z) - \frac{1}{z} \cdot H_{\mathcal{E}}(z) + \frac{1}{z} \cdot H_{\overline{\mathcal{E}}} = 0$$

As $H_{\mathcal{M}}(z) = P^R_{R/\mathfrak{m}^2}(z)$ and $H_{\mathcal{E}}(z) = P^R_k(z)$, it is equivalent to (2.1.1).

As $\mathcal{E}^1 = \mathcal{A}^1$, we have $\overline{\mathcal{E}} \cong \mathcal{E} \otimes_{\mathcal{A}} k$. Recall that $\mathcal{E}$ is a graded connected Hopf algebra; see [20, 2.3.4]. Since $\mathcal{A}$ is generated by elements of degree 1, and such elements are necessarily primitive, $\mathcal{A}$ is actually a Hopf subalgebra of $\mathcal{E}$. A theorem of Milnor and Moore, see [30, 4.4], then shows that $\mathcal{E}$ is free as a right $\mathcal{A}$-module. This implies $\mathcal{E} \cong \overline{\mathcal{E}} \otimes_k \mathcal{A}$, hence $H_{\mathcal{E}}(z) = H_{\mathcal{A}}(z)H_{\overline{\mathcal{E}}}(z)$, which yields (2.1.2). □

We will also need graded counterparts of some of the preceding notions.

2.1.5. A graded commutative k-algebra A is *standard* if $A_0 = k$ and A is generated over A_0 by finitely many elements of degree 1. A *standard presentation* of A is an isomorphism $A \cong S/Y$ of graded k-algebras, with S a polynomial ring in $\operatorname{rank}_k(A_1)$ indeterminates of degree 1. All such presentations are isomorphic.

When N is a finite graded A-module the following number is non-negative:

$$\operatorname{reg}_A N = \max\{j - i \mid \operatorname{Tor}_i^A(N, k)_j \neq 0\} \tag{2.1.1}$$

is known as the *Castelnuovo-Mumford regularity* of N over A.

An A-module N with $\operatorname{reg}_A N = 0$ is said to be *Koszul*.

The algebra A is said to be *Koszul* if $k = A/(A_1)$ is a Koszul A-module.

2.1.6. Set $M_n^{\mathfrak{g}} = \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$ and $M^{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} M_n^{\mathfrak{g}}$. Thus, $R^{\mathfrak{g}}$ is the *associated graded ring* of R and $M^{\mathfrak{g}}$ the *associated graded module* of M. Since $R^{\mathfrak{g}}$ is a standard graded k-algebra and $M^{\mathfrak{g}}$ an $R^{\mathfrak{g}}$-module generated by $M_0^{\mathfrak{g}}$, the Hilbert-Serre Theorem yields polynomials $h_M^R(z)$ and $\overline{h}_M^R(z)$ in $\mathbb{Z}[z]$, with $\overline{h}_M^R(1) \neq 0$, such that

$$H_{M^{\mathfrak{g}}}(z) = \frac{h_M^R(z)}{(1-z)^{\dim R}} = \frac{\overline{h}_M^R(z)}{(1-z)^{\dim M}} \tag{2.1.1}$$

We set $H_M^R(z) = H_{M^{\mathfrak{g}}}(z)$, write $H_R(z)$ for $H_R^R(z)$, and $h_R(z)$ for $h_R^R(z)$.

The number $h_M^R(1)$, called the *multiplicity* of M over R, is denoted by $e_R(M)$. It satisfies $e_R(M) \geq 0$ with equality if and only if $\dim M < \dim R$; set $e(R) = e_R(R)$.

For $m \in M \smallsetminus \{0\}$ set $v(n) = \max\{l \mid m \in \mathfrak{m}^l\}$. The image of m in $M_{v(n)}$ is called the *initial form* of m and is denoted by m^*; in addition, set $0^* = 0$.

We recall a notion introduced by Herzog and Iyengar [23].

2.1.7. Let F be a minimal free resolution of M over R. For each $i \in \mathbb{Z}$ we have

$$\partial_i(\mathfrak{m}^{p-i}F_i) = \mathfrak{m}^{p-i}(\partial_i F_i) \subseteq \mathfrak{m}^{p-i+1}F_{i-1} = \mathfrak{m}^{p-(i-1)}F_{i-1}$$

so F defines a complex $\operatorname{lin}^R(F)$ of graded $R^{\mathfrak{g}}$-modules with $\operatorname{lin}^R(F)_i = F_i^{\mathfrak{g}}(-i)$ and differentials of degree zero. In [23, 1.7] the *linearity defect* of M is defined to be

$$\operatorname{ld}_R(M) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(\operatorname{lin}^R(F)) \neq 0\} \tag{2.1.1}$$

If R is Cohen-Macaulay, $\operatorname{ld}_R(M)$ is finite, and $P_M^R(z) = p(z)/q(z)$ with coprime polynomials $p(z)$ and $q(z)$ in $\mathbb{Z}[z]$ satisfying $q(-1) \neq 0$, then Sega [36, 6.2] gives

$$p(-1)e(R) = q(-1)e_R(M) \tag{2.1.2}$$

When $\operatorname{ld}_R(M) = 0$ the R-module M is said to be *Koszul*. This means that $\operatorname{lin}^R(F)$ is a minimal graded free resolution of $M^{\mathfrak{g}}$ over $R^{\mathfrak{g}}$, and is equivalent to the $R^{\mathfrak{g}}$-module $M^{\mathfrak{g}}$ being Koszul in the sense of 2.1.5; see [23, 1.5], which yields

$$P_M^R(z) = \frac{h_M^R(-z)}{h_R(-z)} \tag{2.1.3}$$

The local ring R is said to be *Koszul* if k is a Koszul R-module.

2.2 Quadratic dimension

As before, $(R, \mathfrak{m}, k)$ denotes a local ring. The purpose of this section is to introduce invariants of R that are central to the discussions in the paper.

2.2.1. Let S be the symmetric algebra $\mathsf{S}_k(\mathfrak{m}/\mathfrak{m}^2)$, and $\pi\colon S \to R^{\mathfrak{g}}$ the (surjective) homomorphism of graded k-algebras extending the identity map of $\mathfrak{m}/\mathfrak{m}^2$.

We define the *associated quadratic ring* of R to be the graded k-algebra

$$R^{\mathfrak{q}} = S/K \quad \text{where} \quad K = S\operatorname{Ker}(\pi_2) \tag{2.2.1}$$

The *quadratic dimension* and *quadratic span* of R are defined, respectively, by

$$\operatorname{qdim} R = \dim R^{\mathfrak{q}} \quad \text{and} \quad \operatorname{qtype} R = \operatorname{rank}_k \operatorname{Ker}(\pi_2) \tag{2.2.2}$$

For ease of reference, we spell out a few formal properties. Note that quadratic dimension interpolates between Krull dimension and embedding dimension.

2.2.2 Lemma. *The ring $R^{\mathfrak{q}}$ and ideal K from* (2.2.1) *satisfy the following relations.*

$$0 \leq \operatorname{qdim} R - \dim R = \operatorname{ecodim} R - \operatorname{height} K \tag{2.2.1}$$

$$0 \leq \operatorname{edim} R - \operatorname{qdim} R = \operatorname{height} K \leq \operatorname{qtype} R \tag{2.2.2}$$

$$\operatorname{qtype} R = \binom{\operatorname{edim} R + 1}{2} - \operatorname{rank}_k \mathfrak{m}^2/\mathfrak{m}^3 \tag{2.2.3}$$

$$\operatorname{edim} R = \operatorname{qdim} R \iff S \cong R^{\mathfrak{q}} \iff \operatorname{qtype} R = 0 \tag{2.2.4}$$

A local ring $(R', \mathfrak{m}', k')$ and a flat homomorphism $\iota\colon R \to R'$ with $\mathfrak{m}' = \iota(\mathfrak{m})R'$ yield

$$\dim R' = \dim R\,, \quad \operatorname{qdim} R' = \operatorname{qdim} R\,, \quad \textit{and} \quad \operatorname{edim} R' = \operatorname{edim} R\,. \tag{2.2.5}$$

Proof. In (2.2.2) the second inequality holds by the Principal Ideal Theorem, and the other two relations hold because S is a polynomial ring and $\dim S = \operatorname{edim} R$. The same equality and $\dim R^{\mathsf{g}} = \dim R$ yield a string that proves (2.2.1):

$$\operatorname{qdim} R - \dim R = \dim S - \operatorname{height} K - \dim R^{\mathsf{g}} = \operatorname{ecodim} R - \operatorname{height} K$$

The equality in (2.2.3) and equivalences in (2.2.4) are clear from the definition.

Set $S' = \mathsf{S}_{k'}(\mathfrak{m}'/\mathfrak{m}'^2)$ and let $\pi' \colon S' \to R'^{\mathsf{g}}$ be the canonical map. The natural maps of graded k'-algebras $S \otimes_k k' \to S'$ and $R^{\mathsf{g}} \otimes_k k' \to R'^{\mathsf{g}}$ are bijective. They take $\pi \otimes_k k'$ to π', and so induce $\operatorname{Ker}(\pi_2 \otimes_k k') \cong \operatorname{Ker}(\pi_2')$. As $\operatorname{Ker}(\pi_2) \otimes_k k'$ is isomorphic to $\operatorname{Ker}(\pi_2 \otimes_k k')$, we have $R^{\mathsf{q}} \otimes_k k' \cong R'^{\mathsf{q}}$ as graded k-algebras. The equalities in (2.2.5) follow from these isomorphisms. $\square$

It is clear from (2.2.3) that $\operatorname{qdim} R$ equals $\operatorname{edim} R$, the largest possible value allowed by (2.2.1), if and only if $K = (0)$; that is, if and only if *qtypeR* $= 0$. Except in special cases, no simple description for the rings over which $\operatorname{qdim} R$ has the least possible value allowed by (2.2.2). One such case is studied in the rest of the section.

2.2.3. A *minimal Cohen presentation* of R is isomorphism $P/I \cong \widehat{R}$, with a regular local ring $(P, \mathfrak{p}, k)$ and $I \subseteq \mathfrak{p}^2$. One always exists, by Cohen's Structure Theorem.

A Cohen presentation induces isomorphisms $\mathfrak{p}/\mathfrak{p}^2 \cong \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \cong \mathfrak{m}/\mathfrak{m}^2$. We use them to identify $\mathfrak{p}/\mathfrak{p}^2$ and $\mathfrak{m}/\mathfrak{m}^2$, and then the maps $\mathsf{S}_k(\mathfrak{p}/\mathfrak{p}^2) \cong P^{\mathsf{g}} \twoheadrightarrow R^{\mathsf{g}}$ and $S = \mathsf{S}_k(\mathfrak{m}/\mathfrak{m}^2) \twoheadrightarrow R^{\mathsf{g}}$. Thus, we do not distinguish between I^*, the ideal of P^{g} generated by the initial forms of the elements in I, and the ideal $\operatorname{Ker}(S \twoheadrightarrow R^{\mathsf{g}})$.

We set $\overline{I} = I/\mathfrak{p}I$ and consider the composed map of natural k-linear maps

$$\gamma \colon \overline{I} \twoheadrightarrow I/(\mathfrak{p}^3 \cap I) \cong (I + \mathfrak{p}^3)/\mathfrak{p}^3 = I_2^* \tag{2.2.1}$$

It is given by $\gamma(\overline{f}) = f^*$, where $\overline{f}$ denotes the image in $\overline{I}$ of an element $f \in I$.

2.2.4. Assume $R = Q/(g_1, \dots, g_b)$ with $(Q, \mathfrak{q}, k)$ a local ring and $g_i \in \mathfrak{q}^{n_i}$ for $1 \leq i \leq b$.

When $\{g_1, \dots, g_b\}$ is part of a system of parameters there is an inequality

$$e(R) \geq n_1 \cdots n_b \cdot e(Q) \tag{2.2.1}$$

with equality if $g_1^*, \dots, g_b^*$ is Q^{g}-regular; see [14, VIII, §7, Proposition 4].

The sequence $g_1^*, \dots, g_b^*$ is Q^{g}-regular if and only if $g_1, \dots, g_b$ is Q-regular and $(g_1^*, \dots, g_b^*) = \operatorname{Ker}(Q^{\mathrm{g}} \to R^{\mathrm{g}})$; see Valabrega and Valla [44, 2.7 and 1.1].

When Q^{g} is Cohen-Macaulay, $g_1^*, \dots, g_b^*$ is Q^{g}-regular and $v(g_i) = n_i$ if and only if $g_1, \dots, g_b$ is Q-regular and equality holds in (2.2.1); see Rossi and Valla [32, 1.8][i].

We recall the notion of complete intersection, for rings and for graded algebras.

2.2.5. The ring R is said to be *complete intersection*, or *c.i.*, if in some minimal Cohen presentation the ideal I can be generated by a P-regular sequence. When this is the case, each minimal generating set of the defining ideal in every minimal Cohen presentation is a regular sequence of $\operatorname{ecodim} R$ elements; see [10, 7.3.3].

When R is c.i. Tate [43, Theorem 6], see also [10, 7.1.1, 7.3.3, and 7.1.5], gives

$$P_k^R(z) = \frac{(1+z)^{\operatorname{edim} R}}{(1-z^2)^{\operatorname{ecodim} R}} = \frac{(1+z)^{\dim R}}{(1-z)^{\operatorname{ecodim} R}} \tag{2.2.1}$$

When R is c.i., formula 2.2.1 applied to a minimal Cohen factorization of R gives $e(R) \geq 2^{\operatorname{ecodim} R}$; in case equality holds we say that R has *minimal multiplicity*.

---

[i]The hypothesis of that theorem includes the assumption that $g_1, \dots, g_b$ is regular. However, this condition is needed in the proof of just one of the implications, since it it is implied by the regularity of $g_1^*, \dots, g_b^*$, due to the theorem of Valla and Valabrega cited above.

2.2.6. A standard graded commutative k-algebra A is said to be *graded complete intersection* if in some (and hence, by 2.1.5, in every) standard presentation $A \cong S/Y$ the ideal Y can be generated by a regular sequence of forms. When it is, Tate's construction [43, Theorem 4] yields a minimal graded free resolution of k over A. It shows that A is Koszul if and only if Y can be generated by quadratics.

Next we characterize the complete intersections with $\operatorname{qdim} R = \dim R$.

2.2.7 Proposition. *The following conditions are equivalent.*

(1) R is complete intersection and $\operatorname{qdim} R = \dim R$*.*

(2) R is complete intersection with minimal multiplicity.

(3) R is complete intersection and is Koszul.

(4) R is complete intersection and $R^{\mathsf{g}} \cong R^{\mathsf{q}}$ as graded k-algebras.

(5) R^{g} is graded complete intersection of quadrics.

Proof. With notation as in 2.2.3, let $\{f_1, \ldots, f_c\}$ be a minimal generating set of I.

As noted in 2.2.5, in statements (1) through (4) we may replace the hypothesis *R is complete intersection* with *$f_1, \ldots, f_c$ is a P-regular sequence.*

(4) $\Longrightarrow$ (1). This implication is evident.

(1) $\Longrightarrow$ (5). From (2.2.1), (2.2.1), and the Principal Ideal Theorem we get

$$\operatorname{height} I^* = \operatorname{ecodim} R = c \geq \operatorname{rank}_k I^* \geq \operatorname{height} I^*$$

Looking back at (2.2.1) we conclude that $f_1^*, \ldots, f_c^*$ are quadratic forms that generate an ideal of height c. As P^{g} is a polynomial ring they form a regular sequence, and then by Valabrega-Valla, see 2.2.4, they generate $\operatorname{Ker}(P^{\mathsf{g}} \to R^{\mathsf{g}})$.

(5) $\Longrightarrow$ (4) & (3). If $g_1, \dots, g_c$ in $\mathfrak{p}$ are such that $\{g_1^*, \dots, g_b^*\}$ minimally generates I^*, then $g_1^*, \dots, g_b^*$ is a $P^{\mathfrak{g}}$-regular sequence of quadrics, see 2.2.6. Valabrega-Valla theorem implies that $g_1, \dots, g_b$ is P-regular and gives $I = (g_1, \dots, g_b)$, so R is c.i.

Since $I^* = SI_2^*$, we have $R^{\mathfrak{q}} = S/I^* \cong R^{\mathfrak{g}}$ by definition, so (4) holds.

Graded complete intersections of quadrics are Koszul, see 2.2.6, so (3) holds.

(3) $\Longrightarrow$ (2). We have $e(R) = 2^c$ because formulas (2.1.3) and (2.2.1) give

$$H_R(z) = 1/P_k^R(-z) = (1+z)^c/(1-z)^{\dim R}$$

(2) $\Longrightarrow$ (5). The elements $f_1^*, \dots, f_c^*$ are in I_2^* and form a regular sequence, by Rossi-Valla, so they generate I^*, by Valabrega-Valla; see 2.2.4. $\square$

2.2.8. A *codimension b deformation* of R to Q is a surjective ring homomorphism $\varkappa\colon Q \twoheadrightarrow R$, with Q local and $\operatorname{Ker}(\varkappa)$ generated by a Q-regular sequence of length b.

The deformation $\varkappa$ is said to be *embedded* if $\operatorname{edim} Q = \operatorname{edim} R$. When this is the case, there is well known, see e.g. [10, 3.3.4], coefficient-wise inequality

$$P_M^R(z) \preccurlyeq P_M^Q(z) \cdot \frac{1}{(1-z^2)^b} \tag{2.2.1}$$

Now we can state the main property of quadratic dimension that is relevant to this work. It refers to the ring $\acute{R}$, the modified completion of R described in 2.1.2.

2.2.9 Theorem. *A local ring R is c.i. with $\operatorname{qdim} R \le \dim R + 1$ if and only if $\acute{R}$ has a codimension 1 embedded deformation to a c.i. ring with minimal multiplicity.*

The "if" direction of the theorem follows easily from the definitions. More delicate arguments are needed for the converse, and they are used again in a proof in the next section. In order to avoid duplication, Theorem 2.2.9 is proved in 2.3.8, with a special case of Theorem 2.3.3 delivering the "only if" direction.

2.3 Homological reductions

In this section $(R, \mathfrak{m}, k)$ denotes a local ring and M a finite R-module. The quadratic dimension $\operatorname{qdim} R$ in the hypothesis of next result is defined in 2.2.1.

2.3.1 Theorem. *If R is c.i. and $\operatorname{qdim} R \leq \dim R + 1$, then M satisfies*

$$P_M^R(z) = \frac{p_M^R(z)}{(1-z)^{c_M^R}} \tag{2.3.1}$$

for some $p_M^R(z) \in \mathbb{Z}[z]$ with $p_M^R(1) \neq 0$ and $c_M^M \in \mathbb{Z}$ with $0 \leq c_M^R \leq \operatorname{ecodim} R$.

In view of Proposition 2.2.7, the special case $\operatorname{qdim} R = \dim R$ of the preceding theorem coincides with [6, 2.3], and also with [23, 5.10]. The proof given in [Proof of Theorem *2.3.1*] below depends on neither one of those earlier arguments.

2.3.2. Following [8, 7.1], we say that M has *critical degree* at most s, and write $\operatorname{cr\,deg}_R M \leq s$, if a minimal free resolution F of M over R admits a morphism $\mu\colon F \to F$ of negative degree q, such that $\mu(F_{i+q}) = F_i$ for all $i > s$. It is clear that $\operatorname{cr\,deg} M = -\infty$ if $M = 0$, and $\operatorname{cr\,deg} M \geq -1$ otherwise.

2.3.3 Theorem. *Assume that R is complete, k is infinite, and $\operatorname{cr\,deg}_R M = -1$.*

If R is c.i. and $\operatorname{qdim} R \leq \dim R + 1$, then there exists a codimension 1 embedded deformation $Q \twoheadrightarrow R$, where Q is a c.i. ring with minimal multiplicity and

$$P_M^R(z) = P_M^Q(z) \cdot \frac{1}{(1-z^2)} \tag{2.3.1}$$

The balance of the section is devoted to proving the theorems stated above. The arguments rely on properties of the functor $\operatorname{Ext}_R(?, k)$ when R is a c.i. ring. We start by reviewing the relevant homological machinery.

2.3.4. Let $\varkappa\colon Q \twoheadrightarrow R$ be an embedded deformation; set $J = \operatorname{Ker}\varkappa$ and $\overline{J} = J/\mathfrak{p}J$.

There exists a natural injective homomorphism of graded k-algebras

$$\mathsf{S}_k(\operatorname{Hom}_k(\Sigma^2\overline{J}, k)) \to \operatorname{Ext}_R(k,k)$$

whose image, $\mathcal{R}$, lies in the center of $\operatorname{Ext}_R(k,k)$; see [9, 3.3].

The algebra $\mathcal{R}$ is called the *subalgebra of cohomology operators* defined by $\varkappa$.

When $J = (g_1, \dots, g_b)$ and $\{\chi_1, \dots, \chi_b\}$ is the basis of $\mathcal{R}^2$ dual to the basis $\{\varsigma^2(\overline{g}_1), \dots, \varsigma^2(\overline{g}_b)\}$ of $\Sigma^2\overline{J}$, we identify $\mathcal{R}$ and the polynomial ring $k[\chi_1, \dots, \chi_b]$.

In view of 2.1.3, $\mathcal{M} = \operatorname{Ext}_R(M,k)$ becomes a graded $\mathcal{R}$-module.

The following finiteness result for the action of $\mathcal{R}$ underlies its applications.

2.3.5. In view of [9, 5.1], it follows from Gulliksen [21, 3.1] that when $\operatorname{proj\,dim}_Q M$ is finite the $\mathcal{R}$-module $\mathcal{M}$ is finitely generated; see also [10, 9.1.4].

In particular, when R is c.i. and $\widehat{R} \cong P/I$ is a minimal Cohen presentation, then the natural map $P \twoheadrightarrow R$ is an embedded deformation and $\operatorname{proj\,dim}_Q M$ is finite, and thus $\operatorname{Ext}_R(M,k)$ is a finite $\mathcal{R}$-module for *every* finite R-module M.

The critical degree of M can be described in terms of the action of $\mathcal{R}$ on $\mathcal{M}$.

2.3.6. If $\operatorname{proj\,dim}_Q M$ is finite, then [8, 7.2(1)] yields for $\mathcal{M} = \operatorname{Ext}_R(M,k)$ an equality

$$\operatorname{cr\,deg}_R M = \sup\{i \mid ((0 : \mathcal{R}^+)_{\mathcal{M}})^{\geqslant i} \neq 0\} \tag{2.3.1}$$

As $(0 : \mathcal{R}^+)_{\mathcal{M}}$ is a finite $\mathcal{R}$-module, the supremum is finite, hence so is $\operatorname{cr\,deg}_R M$.

A codimension 1 embedded deformation $\varkappa\colon Q \twoheadrightarrow R$ is said to be a *homological reduction* of M if the equality (2.3.1) holds; see [8, 6.3] and cf. (2.2.1). By [8, 6.2], this happens if and only if the subalgebra of cohomology operators defined by $\varkappa$, see

2.3.4, is generated by a non-zero-divisor on $\mathcal{M}$. If k is infinite, then [8, 7.4] gives

$$\operatorname{cr}\deg_R M = \inf\{i \mid \Omega^{i+1}_{\widehat{R}}(\widehat{M}) \text{ admits a homological reduction}\} \tag{2.3.2}$$

We insert a version of the Prime Avoidance Lemma, tailored to measure.

2.3.7 Lemma. *Let k be an infinite field and A a commutative graded k-algebra generated by finitely many elements of positive degree. Let Y be a proper ideal of A, generated in a single degree n, and T be a non-zero finite graded A-module.*

If Y contains a non-zero-divisor on T, then Y_n contains a finite union of hyperplanes whose complement is a Zariski-dense set of non-zero-divisors on T.

Proof. The set Z of elements of A that are zero divisors on T is the union of the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ of A, associated with T. They are homogeneous (because T is graded), which implies $Z \cap Y_n \subseteq \bigcup_{i=1}^s(\mathfrak{p}_i \cap Y_n)$. Since k is infinite and $Y_n \not\subseteq Z$ each subspace $\mathfrak{p}_i \cap Y_n$ of Y_n is proper, so it is contained in some proper hyperplane. Thus, $Z \cap Y_n$ is contained in a union of finitely many hyperplanes; their complement is Zariski-open in Y_n, and is different from Y_n because k is infinite. □

Proof of Theorem 2.3.3. Choose a minimal Cohen presentation $R \cong P/I$.

As in 2.2.3, identify P^{g} and $S = \mathsf{S}_k(\mathfrak{m}/\mathfrak{m}^2)$, and let I^* be the ideal of initial form of I. Thus, $R^{\mathsf{q}} = S/(I^*_2)$, see 2.2.1. From (2.2.1), (2.2.5) and the hypotheses we get

$$\operatorname{height}(I^*_2) = \operatorname{ecodim} R - (\operatorname{qdim} R - \dim R) \geq \operatorname{ecodim} R - 1$$

Set $c = \operatorname{ecodim} R$. From the preceding relations and the maps in (2.2.1) we obtain

$$c = \operatorname{ecodim} R = \operatorname{rank}_k \overline{I} \geq \operatorname{rank}_k I^*_2 \geq \operatorname{height}(I^*_2) \geq c - 1 \tag{2.3.1}$$

Let $\mathcal{R}$ be the algebra of cohomology operators defined by $P \twoheadrightarrow R$, and recall from 2.3.4 that $\mathcal{R}^2$ and $\overline{I}$ are dual vector spaces. When X is a subset of one of them, we write $X^{\perp}$ for the subspace $\{y \mid y(x) = 0 \text{ for all } x \in X\}$ of the dual space.

Since $(0 : \mathcal{R}^{+})_{\mathcal{M}} = 0$, there exist $u_1, \dots, u_r \in \overline{I}$ such that $\mathcal{U} = \mathcal{R}^2 \smallsetminus \bigcup_{i=1}^{r} u_i^{\perp}$ is Zariski-dense in $\mathcal{R}^2$ and each $\chi \in \mathcal{U}$ is a non-zero-divisor on $\mathcal{M}$; see Lemma 2.3.7.

Let's say that a generating set $\{f_1, \dots, f_c\}$ of I is *adjusted to* M if $f_1^*, \dots, f_{c-1}^*$ is an S-regular sequence of quadrics and the dual basis $\{\chi_1, \dots, \chi_c\}$ of $\mathcal{R}^2$ has $\chi_c \in \mathcal{U}$.

Finding such a set would finish the proof, by setting $Q = P/(f_1, \dots, f_{c-1})$.

Indeed, the natural map $Q \twoheadrightarrow R$ then is a codimension one embedded deformation. As $f_1^*, \dots, f_{c-1}^*$ is an S-regular sequence of quadrics, Q is a c.i. ring with minimal multiplicity by (2.2.1). On the other hand, [5, 2.3] shows that the subalgebra of cohomology operators of $\operatorname{Ext}_R(k, k)$, defined by $Q \twoheadrightarrow R$ as in 2.3.4, is the subring $k[\chi_c]$ of $k[\chi_1, \dots, \chi_c]$. Since χ_c is in $\mathcal{U}$, it is a non-zero-divisor on $\mathcal{M}$, and so $Q \twoheadrightarrow R$ is a homological reduction of M; see 2.3.6.

It remains to show that generating sets adjusted to M exits. Precisely one of the inequalities in (2.3.1) is strict, so there are three cases to consider.

Case 1. I has a generating set adjusted to M when $\operatorname{height}(I^*) > c - 1$.

Pick χ_c in $\mathcal{U}$ and complete it to a basis $\{\chi_1, \dots, \chi_c\}$ of $\mathcal{R}^2$. Let $\{\overline{f}_1, \dots, \overline{f}_c\}$ be the dual basis of $\overline{I}$. From (2.3.1) we see that γ from (2.2.1) is bijective, so $\{f_1^*, \dots, f_c^*\}$ is a set of quadrics that generates I^*. As I^* has height c, again by (2.3.1), the sequence $f_1^*, \dots, f_c^*$ is regular by the Cohen-Macaulay Theorem.

Case 2. I has a generating set adjusted to M when $\operatorname{rank}_k(I_2^*) > \operatorname{height}(I^*)$.

The hypothesis implies $\operatorname{rank}_k(\overline{I}) = \operatorname{rank}_k(I_2^*)$, so the map γ in (2.2.1) is bijective; we identify $\overline{I}$ and I_2^* through γ and do not distinguish between $\overline{f}$ and f^*.

We proceed to choose $f_1, \ldots, f_j$ in I for $1 \le j < c$, so that $f_1^*, \ldots, f_j^*$ is an S-regular sequence in I_2^* and $\operatorname{rank}_k k(f_1^*, \ldots, f_j^*, u_i) = j + 1$ holds for $1 \le i \le r$.

To start, pick $f_1 \in I \smallsetminus \mathfrak{p}I$ so that u_i is not in $k\overline{f}_1$ for $i = 1, \ldots, r$. Assume, by induction, that the desired conditions hold for some j with $1 \le j \le c - 2$. The set

$$V = \{v \in I_2^* \mid \operatorname{rank}_k k(f_1^*, \ldots, f_j^*, u_i, v) = j + 2 \text{ for } i = 1, \ldots, r\}$$

is dense in I_2^*. As $f_1^*, \ldots, f_j^*$ is regular, the associated primes of $T = S/S(f_1^*, \ldots, f_j^*)$ have height $j \le c - 2$. Since I^* has height $c - 1$, it contains a T-regular element. Lemma 2.3.7 shows that I_2^* contains a dense subset W of such elements, hence we can choose $f_{j+1} \in I$ so that f_{j+1}^* lies in $V \cap W$. This completes the induction step.

Now we have $f_1, \ldots, f_{c-1}$ in I with $f_1^*, \ldots, f_{c-1}^*$ an S-regular sequence in I_2^*.

In particular, $\{\overline{f}_1, \ldots, \overline{f}_{c-1}\}$ is k-linearly independent, hence by picking $f_c \in I$ so that $\overline{f}_c$ is not in $k(\overline{f}_1, \ldots, \overline{f}_{c-1})$ we obtain a basis $\{\overline{f}_1, \ldots, \overline{f}_c\}$ of $\overline{I}$. The dual basis $\{\chi_1, \ldots, \chi_c\}$ satisfies $\chi_c(u_i) \ne 0$ for $1 \le i \le r$, so χ_c is in $\mathcal{U}$.

Case 3. I has a generating set adjusted to M when $\operatorname{rank}_k(\overline{I}) > \operatorname{rank}_k(I_2^*)$.

Here the map γ in (2.2.1) is surjective and $\operatorname{Ker}(\gamma) = k\overline{f}_c$ for some $f_c \in I \smallsetminus \mathfrak{p}I$.

As $\mathcal{U}$ is dense in $\mathcal{R}^2$, it is not contained in $\overline{f}_c^{\perp}$, so we may pick $\chi \in \mathcal{U} \smallsetminus \overline{f}_c^{\perp}$ with $\chi(f_c) = 1$. By choosing $f_1, \ldots, f_{c-1}$ so that $\{\overline{f}_1, \ldots, \overline{f}_{c-1}\}$ is a basis of $\chi_c^{\perp}$, we get a basis $\{\overline{f}_1, \ldots, \overline{f}_{c-1}, \overline{f}_c\}$ of $\overline{I}$, whose dual basis $\{\chi_1, \ldots, \chi_c\}$ has $\chi_c = \chi \in \mathcal{U}$.

On the other hand, $\{f_1^*, \ldots, f_{c-1}^*\}$ is a basis of I_2^*. This implies $\operatorname{height}(I^*) = c - 1$, see (2.3.1), so $f_1^*, \ldots, f_{c-1}^*$ is S-regular by the Cohen-Macaulay Theorem. $\square$

2.3.8. *Proof of Theorem* 2.2.9. The various dimensions of $\acute{R}$ and R are equal, see (2.2.5), and $P_{\acute{M}}^{\acute{R}}(z) = P_M^R(z)$, see 2.1.2. Thus, for the rest of the proof we may assume that R is complete and k is infinite.

When R is c.i. with $\operatorname{qdim} R \leq \dim R + 1$, apply Theorem 2.3.3 with $M = \acute{R}$ to get a codimension 1 embedded deformation to a c.i. ring with minimal multiplicity.

Conversely, let $Q \twoheadrightarrow R$ be a codimension 1 deformation with Q c.i. of minimal multiplicity. Choose a minimal Cohen presentation $\widehat{Q} \cong P/J$. It yields a minimal Cohen presentation $R \cong P/I$ with $I \supseteq J$, whence an inclusion $I^* \supseteq J^*$ of initial ideals of P^{g}. In view of 2.2.3, formula (2.2.2) gives the first equality in the string

$$\operatorname{qdim} R - \dim R = \operatorname{ecodim} R - \operatorname{height} I^* \leq \operatorname{ecodim} Q + 1 - \operatorname{height} J^* = 1$$

The last equality holds because Q^{g} is complete intersection; see Proposition 2.2.7. $\square$

Proof of Theorem 2.3.1. We argue by induction on the codimension of R.

When $\operatorname{ecodim} R = 0$ the ring R is regular, so there is nothing to prove. By induction, we may assume that the desired assertion holds for c.i. rings Q satisfying $\operatorname{qdim} Q - \dim Q \leq 1$ and $\operatorname{ecodim} Q = c$ for some $c \geq 0$. In view of Proposition 2.2.7, this covers all c.i. rings Q of codimension c with minimal multiplicity.

Set $L = \Omega^R_{s+1}(M)$ with $s = \operatorname{cr\,deg}_R M$; thus, L has a codimension 1 homological reduction $Q \twoheadrightarrow R$, by (2.3.2). Theorem 2.3.3 shows that it can be chosen so that Q is a c.i. ring with minimal multiplicity. Formulas (2.1.1) and (2.3.2) then give

$$P^R_M(z) = \sum_{i=0}^{s-1} \beta^R_i(M) z^i + z^s P^R_L(z) = \sum_{i=0}^{s-1} \beta^R_i(M) z^i + z^s P^Q_L(z) \cdot \frac{1}{1 - z^2}$$

For the first equality in the next string, we use the polynomial $p^Q_L(z)$ with $p^Q_L(1) \neq 0$ and integer c^Q_L satisfying $0 \leq c^Q_L \leq \operatorname{ecodim} Q$, provided by the induction hypothesis:

$$P^Q_L(z) \cdot \frac{1}{1 - z^2} = \frac{p^Q_L(z)}{(1 - z)^{c^Q_L}} \cdot \frac{1}{1 - z^2} = \frac{p^Q_L(z)}{1 + z} \cdot \frac{1}{(1 - z)^{c^Q_L + 1}} \tag{2.3.1}$$

The algebra Q^{g} is Koszul, see Proposition 2.2.7, so the linearity defect of L over Q is finite, by [23, 5.10]. From (2.1.2) and (2.3.1) we get the first equality in the string

$$p_L^Q(-1)e(Q) = (1-(-1))^{c_L^Q} e_Q(L) = 2^{c_L^Q} e_Q(L) = 0$$

The last one holds because we have $\dim M \leq \dim R < \dim Q$; see 2.1.6. Thus, $p_L^Q(z) = (1+z)p_L^R(z)$ for some polynomial $p_L^R(z)$ with integer coefficients.

By concatenating the results of the preceding computations, we get an expression

$$P_M^R(z) = \frac{p_M^R(z)}{(1-z)^{c_M^Q+1}} \quad \text{with} \quad p_M^R(z) = (1-z)^{c_L^Q+1} \sum_{i=0}^{s-1} \beta_i^R(M) z^i + z^s p_L^R(z)$$

where $p_M^R(1) = p_L^R(1) = p_L^Q(1)/2 \neq 0$ and $c_L^Q + 1 \leq \operatorname{ecodim} Q + 1 = \operatorname{ecodim} R$ hold. $\square$

2.4 Poincaré series

In this section $(R, \mathfrak{m}, k)$ denotes a c.i. local ring.

The first conclusion of the following theorem is a partial converse to Theorem 2.3.1. The second conclusion concerns the invariant $\operatorname{qtype} R$ defined in (2.2.2).

2.4.1 Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring and set $c = \operatorname{ecodim} R$.*

If all quotient R-modules N of $R/\mathfrak{m}^2$ satisfy

$$(1-z)^{c_N} P_N^R(z) \in \mathbb{Z}[z] \tag{2.4.1}$$

for some $c_N \in \mathbb{Z}$, then R is c.i. and the following inequalities hold:

$$\operatorname{qdim} R \leq \dim R + \max\{1, c-2\} \tag{2.4.2}$$

$$\operatorname{qtype} R \geq c - 1 \tag{2.4.3}$$

The proof is deferred to the end of this section (before 2.4.6). It uses explicit computations of Poincaré series that present some intrinsic interest.

2.4.2 Proposition. *If $(R, \mathfrak{m}, k)$ is c.i. with* $\operatorname{ecodim} R = c$ *and* $\operatorname{qtype} R = q$*, then*

$$P^R_{R/\mathfrak{m}^2}(z) = \frac{(1-z)^q + (1+z)^{e-q-1}(ez-1)}{z(1+z)^{c-q-1}(1-z)^c} \tag{2.4.1}$$

Proof. Sjödin [39, Remark 2, p. 208] proves $\mathcal{E} \cong \mathcal{A} \otimes_k \mathcal{B}$, where $\mathcal{B}$ is a polynomial ring over k on $c - r$ indeterminates of degree 2. Thus, we have

$$H_{\overline{\mathcal{E}}}(z) = \frac{P^R_k(z)}{H_{\mathcal{A}}(z)} = H_{\mathcal{B}}(z) = \frac{1}{(1-z^2)^{c-q}} \tag{2.4.2}$$

Substituting (2.2.1) and (2.4.2) into (2.1.1) yields the desired result. □

Proposition 2.4.2 was obtained in [6, 2.1], but the computation presented above took a very different path. It is amusing to note that the techniques developed for the original argument are applied to prove Proposition 2.4.3.

Techniques developed for the proof of [6, 2.1] are used to prove the next result.

2.4.3 Proposition. *Let $(R, \mathfrak{m}, k)$ be a c.i. local ring of codimension c, let $\{y_1, \ldots, y_e\}$ be a minimal generating set of $\mathfrak{m}$, let b be an integer with $0 \leq b \leq e$, and set*

$$N = R/((y_1, \ldots, y_b) + (y_{b+1}, \ldots, y_e)^2)$$

If $b \leq c-2$ and the ideal K from (2.2.1) *satisfies $K_2 \subseteq S_1(\overline{y}_1,\dots,\overline{y}_b)$, then*

$$P_N^R(z) = \frac{1+(1+z)^{e-b-1}((e-b)z-1)}{z(1+z)^{c-b-1}(1-z)^c} \tag{2.4.1}$$

Proof. As in the proof of Theorem 2.3.1, we may assume that R is complete and choose a minimal Cohen presentation $R \cong P/I$ with $(P,\mathfrak{p},k)$ regular local. We use the notation in 2.2.3 and select $x_1,\dots,x_e$ in $\mathfrak{p}$ that map to $y_1,\dots,y_e$, respectively.

Pick a minimal generating set $\{f_1,\dots,f_c\}$ of I. For $h=1,\dots,c$ the hypothesis on K_2 implies $f_h = \sum_{i=1}^b s_{hi}x_i + g_h$ in I, with $s_{hi} \in \mathfrak{p}$ and $g_h \in \mathfrak{p}^3$. Thus, we have

$$f_h = \sum_{1\leq i\leq b} b_{hi}x_i + \sum_{b<i\leq j\leq e} c_{hij}x_ix_j \tag{2.4.2}$$

for $h=1,\dots,c$, with $b_{hi}\in\mathfrak{p}$ and $c_{hij}\in\mathfrak{p}$. In particular, I is contained in the ideal $\mathfrak{p}X$ of P, where $X=(x_1,\dots,x_b)+(x_{b+1},\dots,x_e)^2$, and $P/X \cong N$ as P-modules.

Let B be the Koszul complex on $x_1,\dots,x_b$; it is a minimal free resolution of $P/(x_1,\dots,x_b)$ over P. Set $L=P/(x_{b+1},\dots,x_e)^2$ and let C be a minimal free resolution of L over P. For each $i\in\mathbb{Z}$ we have an isomorphism

$$\mathrm{H}_n(B\otimes_P C) \cong \mathrm{H}_n(B\otimes_P L) \cong \begin{cases} N & \text{when } n=0\,; \\ 0 & \text{otherwise,} \end{cases}$$

because $x_1,\dots,x_b$ is an N-regular sequence. Since $B\otimes_P C$ is a complex of free P-modules, it is a minimal free resolution of N over P. Pick for $(B\otimes_P C)_1$ a P-basis $\{v_i\}_{1\leq i\leq b}\cup\{v_{ij}\}_{b+1\leq i\leq j\leq e}$ with $\partial(v_i)=x_i$ and $\partial(v_{ij})=x_ix_j$ for all i,j.

The Koszul complex B is naturally a graded-commutative DG P-algebra. Srinivasan [41, 3.6] has constructed such a graded-commutative DG algebra structure on C. Thus, $B\otimes_P C$ acquires a structure of graded-commutative DG P-algebra.

Let A be the Koszul complex on $f_1, \dots, f_c$, with its natural DG P-algebra structure. It is a minimal free resolution of R over P. Pick a basis $\{u_i\}_{1\leq i\leq c}$ of A_1, such that $\partial(u_h) = f_h$ for each h, and define a P-linear map $\varphi_1 \colon A_1 \to (B \otimes_P C)_1$ by

$$\varphi_1(u_h) = \sum_{1\leq i\leq b} b_{hi} v_i + \sum_{b<i\leq j\leq e} c_{hij} v_{ij} \tag{2.4.3}$$

The underlying graded algebra of A is the exterior algebra on A_1, so ϕ_1 extends uniquely to a homomorphism of graded P-algebras $\varphi\colon A \to B \otimes_P C$. This is a morphism of DG algebras. Indeed, $\partial_1\varphi_1(u_h) = f_h = \varphi_0\partial_1(u_h)$ holds for $h = 1, \dots, c$ by (2.4.2) and (2.4.3). Since $\{u_1, \dots, u_c\}$ generates A as a P-algebra and φ is a homomorphism of algebras, the Leibniz formula implies $\partial_n\varphi_n = \varphi_{n-1}\partial_n$ for all n.

We now know that R has a minimal P-free resolution A that is a DG algebra, and the R-module N has a minimal P-free resolution $B \otimes_P C$ that is a DG A-module, via the morphism φ. By [3, 3.1.1], there is a spectral sequence with

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}_p^{\overline{A}}(\overline{B \otimes_P C}, k)_q \implies \mathrm{Tor}^R_{p+q}(N, k)$$

where $\overline{?} = ? \otimes_Q k$. Furthermore, [3, 4.1.1] shows that this sequence collapses on the second page, and thus yields equalities of formal power series

$$P^R_N(z) = \sum_{n=0}^{\infty} \sum_{p+q=n} \mathrm{rank}_k(\mathrm{E}^2_{p,q}) z^n = \sum_{n=0}^{\infty} \Big(\sum_{p+q=n} \mathrm{rank}_k \, \mathrm{Tor}_p^{\overline{A}}(\overline{B \otimes_P C}, k)_q \Big) z^n$$

In (2.4.3) all b_{hi} and c_{hij} lie in $\mathfrak{p}$, so $\overline{\varphi}(\overline{u}_h) = 0$ for $\overline{u}_h = u_h \otimes 1$ and $h = 1, \dots, c$. As $\overline{A}$ is the exterior algebra generated by $\{\overline{u}_1, \dots, \overline{u}_c\}$, we have $(\overline{A}_+)(\overline{B \otimes_P C}) = 0$,

$\operatorname{Tor}_p^{\overline{A}}(k,k)_q = 0$ for $q \neq p$ and $\operatorname{rank}_k \operatorname{Tor}_p^{\overline{A}}(k,k)_p = \binom{p+c-1}{c-1}$, whence

$$\begin{aligned}
P_N^R(z) &= \Big(\sum_{p=0}^{\infty} \operatorname{rank}_k\big(\operatorname{Tor}_p^{\overline{A}}(k,k)_p\big) z^{2p}\Big)\Big(\sum_{q=0}^{\infty} \operatorname{rank}_k(\overline{B \otimes_P C}) z^q\Big) \\
&= \frac{1}{(1-z^2)^c} \cdot H_{\overline{B}}(z) \cdot H_{\overline{C}}(z)
\end{aligned}$$

As $\overline{B}$ is the exterior algebra on the k-vector space with basis $\{\overline{v}_1, \dots, \overline{v}_b\}$, we have

$$H_{\overline{B}}(z) = (1+z)^b$$

On the other hand, we have $\operatorname{rank}_P C_0 = 1$ and $\operatorname{rank}_P C_q = q\binom{e-b+1}{q+1}$ for each $q \neq 0$; see [41, 2.1]. A direct computation now yields the expression

$$H_{\overline{C}}(z) = (1+z) \cdot \frac{1 + (1+z)^{e-b-1}((e-b)z - 1)}{z}$$

The equality (2.4.1) is obtained by combining the last three equalities above. $\square$

For use in the next proof and later in the paper, we recall a measure of asymptotic behavior of Betti sequences; in the context of local rings it was introduced in [5].

2.4.4. The *complexity* of a non-zero R-module M is the number

$$\operatorname{cx}_R M = \inf\{d \in \mathbb{N} \cup \{0\} \mid \beta_i^R(M) \leq a i^{d-1} \text{ for some } a \in \mathbb{N} \text{ and } i \gg 0\}$$

In addition, we set $\operatorname{cx}_R 0 = -1$. We recall a few of general results:

(1) Every finite R-module N satisfies $\operatorname{cx}_R N \leq \operatorname{cx}_R k$; this follows from [10, 4.1.9].

(2) If $\mathfrak{m}^n \neq 0$ for some $n \geq 0$ and $\operatorname{cx}_R(R/\mathfrak{m}^n) = c < \infty$, then R is c.i. and $\operatorname{ecodim} R = c$, by [7, Corollary 5]; see also [10, 10.3.8].

2.4.5. *Proof of Theorem* 2.4.1. The hypothesis yields $\operatorname{cx}_R k < \infty$, so R is c.i. by 2.4.4(2).

Set $c = \operatorname{ecodim} R$. Let K be the ideal defined in (2.2.1), and set $h = \operatorname{height}(K)$ and $q = \operatorname{qtype} R$. In view of (2.2.1) and (2.2.2), we have to prove

$$h \geq \min\{2, c-1\} \quad \text{and} \quad q \geq c-1$$

Set $e = \operatorname{edim} R$. If $e \leq 1$, then $c = e$ holds, and when $c \leq 1$ both inequalities above are evident. Thus, for the rest of the proof we assume $e \geq 2$ and $c \geq 2$.

Comparing the expression in (2.4.1) with $N = R/\mathfrak{m}^2$ and the one in (2.4.1), we now get $c - q - 1 \leq 0$; this completes the proof of (2.4.3)

From $q \geq c-1$ we get $K \neq 0$, so (2.4.2) is obvious when $c = 2$. To finish the proof of (2.4.2), it remains to show that $c \geq 3$ implies $h \geq 2$. The contrary means that K is minimally generated by $q \geq 2$ quadrics contained in a principal prime ideal; that is, $K_2 \subseteq S_1\overline{y}$ for some $\overline{y} \in S_1$. We have $e - 2 \geq c - 2 \geq 1$, so the R-module N in (2.4.1) has $P^R_N(z) = p(z)/(1+z)^{c-2}(1-z)^c$ with $p(z) \in \mathbb{Z}[z]$ satisfying $p(-1) = -1$. This contradicts our hypothesis, and so proves $h \geq 2$. $\square$

The last result in this section invokes Castelnuovo-Mumford regularity; see 2.1.5.

2.4.6 Proposition. *Let* $(Q, \mathfrak{q}, k)$ *be a c.i. local ring, let* S *denote the symmetric algebra of* $\mathfrak{q}/\mathfrak{q}^2$*, and set* $n = \operatorname{reg}_S Q + 1$.

If g *is a* Q*-regular element in* $\mathfrak{q}^{n+1}$*, then* $R = Q/(g)$ *and* $\mathfrak{m} = \mathfrak{q}R$ *satisfy*

$$P^R_{R/\mathfrak{m}^n}(z) = \frac{p(z)}{(1+z)(1-z)^c} \tag{2.4.1}$$

with $c = \operatorname{ecodim} R$ *and* $p(z) \in \mathbb{Z}[z]$ *such that* $p(-1) = 2^{c-1} - e(Q)$.

Proof. Set $d = \dim Q$ and $c = \operatorname{ecodim} R$, and note that $\operatorname{ecodim} Q = c - 1$.

We have a string of equalities, where the first one is due to Sega [35, 3.4], the third one comes from (2.1.1) and (2.2.5), and the remaining two are clear:

$$\begin{aligned} P^Q_{\mathfrak{q}^n}(z) &= H^Q_{\mathfrak{q}^n}(-z) \cdot P^Q_k(z) \\ &= \frac{H_Q(-z) - H^Q_{Q/\mathfrak{q}^n}(-z)}{(-z)^n} \cdot P^Q_k(z) \\ &= \frac{1}{(-z)^n} \cdot \Big(\frac{h_Q(-z)}{(1+z)^d} - H_{Q/\mathfrak{q}^n}(-z) \Big) \cdot \frac{(1+z)^d}{(1-z)^{c-1}} \\ &= \frac{h_Q(-z) - (1+z)^d H^Q_{Q/\mathfrak{q}^n}(-z)}{(-z)^n (1-z)^{c-1}} \end{aligned} \tag{2.4.2}$$

The numerator of the last fraction is in $\mathbb{Z}[z]$. Its order is equal to n, as one sees by comparing the orders of the first and the last power series in the preceding string.

We have $g \in \mathfrak{q}^{n+1} = \mathfrak{q} \operatorname{Ann}_Q Q/\mathfrak{q}^n$. The first equality in the next string comes from Shamash [38, Theorem 1], see also [10, 3.3.5(2)], the second one from the isomorphism $R/\mathfrak{m}^n \cong Q/\mathfrak{q}^n$, the third one is clear, and the last one comes from (2.4.2):

$$\begin{aligned} P^R_{R/\mathfrak{m}^n}(z) &= \frac{P^Q_{R/\mathfrak{m}^n}(z)}{1-z^2} = \frac{P^Q_{Q/\mathfrak{q}^n}(z)}{1-z^2} = \frac{1 + zP^Q_{\mathfrak{q}^n}(z)}{1-z^2} \\ &= \frac{(-z)^a p(z)}{(-z)^{n-1}(1-z)^c} \end{aligned} \tag{2.4.3}$$

Here a is in $\mathbb{Z}$ and non-negative, $p(z)$ is in $\mathbb{Z}[z]$ with $p(0) \neq 0$, and also

$$\begin{aligned} (-z)^a p(z) &= (-z)^{n-1}(1-z)^{c-1} - \Big(h_Q(-z) - (1+z)^d H^Q_{Q/\mathfrak{q}^n}(-z) \Big) \\ &= (-z)^{n-1}(1 + \text{higher order terms in } z) \end{aligned} \tag{2.4.4}$$

The second equality holds because the difference in parentheses has order n, as noted earlier. This implies $a = n-1$, so (2.4.3) simplifies to (2.4.1). Finally, we get

$$p(-1) = 2^{c-1} - h_Q(1) = 2^{c-1} - e(Q)$$

by evaluating both sides of (2.4.4) at $z = -1$. $\square$

2.5 Betti numbers

When $(R, \mathfrak{m}, k)$ is a local ring and M a finite R-module, we say that the Betti sequence of M is *eventually polynomial* (*of degree* b) if there exists a polynomial $\beta^R_M(x) \in \mathbb{Q}[x]$ (with $\deg \beta^R_M = b$), such that $\beta^R_i(M) = \beta^R_M(i)$ for $i \gg 0$.

The next result, with $\acute{R}$ from 2.1.1, contains Theorem A from the introduction.

2.5.1 Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring and c a non-negative integer.*

The first one of the following conditions implies the second one.

(1) There is an isomorphism $\acute{R} \cong Q/(g)$, where $(Q, \mathfrak{q}, k)$ is a local c.i. ring of codimension $c-1$ and minimal multiplicity, and g is in $\mathfrak{q}^2$ and Q-regular.

(2) The Betti sequences of all finite R-modules are eventually polynomial, of degree less than c for all modules and of degree $c-1$ for some modules.

The conditions are equivalent when $c \leq 3$, or when R^{g} is complete intersection.

We say that a local ring $(R, \mathfrak{m}, k)$ is *homogeneous* if its $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to the completion of R^{g} at the ideal $R^{\mathsf{g}}_{\geqslant 1}$. The last assertion of Theorem 2.5.1 yields the next result that, in turn, implies Theorem B from the introduction.

2.5.2 Corollary. *When R is a homogeneous local ring, all finite R-modules have eventually polynomial Betti sequences if and only if $R^{\mathsf{g}} \cong B/(g)$, where B is a graded complete intersection of quadrics and g is a B-regular form.* $\square$

The proof of the theorem a compilation of results from the last three sections, which were stated in terms of Poincaré series. The translation into Betti sequences goes through an elementary property of generating functions; see e.g. [42, 4.3.1].

2.5.3. The sequence $(\beta_i^R(M))_i$ is eventually polynomial (of degree b) if and only if $P_M^R(z) = p(z)/(1-z)^b$ for some $p(z) \in \mathbb{Z}[z]$ and $b \in \mathbb{N}$ (with $p(1) \neq 0$ and $b = n+1$).

Proof of Theorem 2.5.1. Referring to Theorem 2.2.9 and 2.5.3, rephrase (1) and (2) as

(i′) R is c.i. of codimension c and satisfies $\operatorname{qdim} R \leq \dim R + 1$.

(ii′) $(1-z)^c P_M^R(z) = p_M^R(z) \in \mathbb{Z}[z]$ for all M and $p_M^R(1) \neq 0$ for some M.

Also, recall Tate's formula (2.2.1), which we rewrite in the form

$$(1-z)^{\operatorname{ecodim} R} P_k^R(z) = (1+z)^{\dim R} \tag{2.5.1}$$

The implication (i′) $\Longrightarrow$ (ii′) follows from Theorem 2.3.1 and (2.5.1).

When (ii′) holds R is c.i. by Theorem 2.4.1, and we have $\operatorname{ecodim} R = c$ by (2.5.1). It remains to prove that the inequality in (i′) holds under additional conditions.

When $c \leq 3$, that inequality follows directly from formula (2.4.2).

When $R^{\mathfrak{g}}$ is c.i., it is isomorphic to $S/(g_1, \ldots, g_c)$, where S is the symmetric algebra on $\mathfrak{m}/\mathfrak{m}^2$ and $g_1, \ldots, g_c$ is an S-regular sequence of forms of degree at least 2. Let K be the ideal of S generated by the g_i's of degree 2. From (2.4.3) we see that there are at least $c-1$ such forms, and this implies $\operatorname{height} K \geq c-1$. Now (2.2.1) yields $\operatorname{qdim} R - \dim R = c - \operatorname{height} K \leq 1$, as desired. $\square$

We also obtain new characterizations of complete intersection rings with minimal multiplicity, this time in terms of Betti sequences of modules.

2.5.4 Theorem. *When* $\dim R$ *is positive the following conditions are equivalent.*

(1) R is complete intersection with minimal multiplicity.

(2) $(\beta_i^{R/(g)}(N))_i$ is eventually polynomial for each R-regular element g in $\mathfrak{m}^2$ and every finite R-module N with $gN = 0$.

(3) $(\beta_i^{R/(g)}(R/\mathfrak{m}^n))_i$ is eventually polynomial for some R-regular element g in $\mathfrak{m}^{n+1}$ with $n > \operatorname{reg}_S R^{\mathfrak{g}}$.

Proof. When (1) holds we get $\operatorname{qdim} R/(g) - \dim R/(g) \leq 1$ from Proposition 2.2.7, and then Theorem 2.3.1 and 2.5.3 show that (ii) holds.

When (3) holds $R/(g)$ is c.i. by 2.4.4(2), hence so is R. Proposition 2.4.6 gives $P^R_{R/\mathfrak{m}^n}(z) = p(z)/(1+z)(1-z)^c$ with $p(-1) = 2^{\operatorname{ecodim} Q} - e(Q)$. Since the Betti sequence of $R/\mathfrak{m}^n$ is eventually polynomial, $P^R_{R/\mathfrak{m}^n}(z)$ has no pole at $z = -1$. Thus, we get $p(-1) = 0$, which means $e(Q) = 2^{\operatorname{ecodim} Q}$, and so (i) holds. $\square$

So far, we have focused on asymptotically polynomial behavior of Betti sequences. We briefly touch upon the question of estimating when it starts.

2.5.5. Let R be a local c.i. ring and set $m = \operatorname{depth} R - \operatorname{depth} M$.

(1) If $\operatorname{cx}_R M = 0$, then $\beta_i^R(M) = 0$ for $i > m$ (Auslander-Buchsbaum Equality).

(2) If $\operatorname{cx}_R M = 1$, then $\beta_i^R(M) = \beta_{i+1}^R(M)$ for $i > m$ by [18, 4.1].

On the other hand, [8, 7.3] gives $\operatorname{cr}\deg_R M = m$ if $\operatorname{cx}_R M = 0$, $\operatorname{cr}\deg_R M \leq m$ if $\operatorname{cx}_R M = 1$, and $\beta_i^R(M) < \beta_{i+1}^R(M)$ for $i \geq \operatorname{cr}\deg_R M$ when $\operatorname{cx}_R M \geq 2$. This suggests using critical degree as an indicator for the start of polynomial growth. However, bounds on $\operatorname{cr}\deg_R M$ are not available in general, except for:

(1) If $\operatorname{cx}_R M = 2$, then by [11, 7.6] there is an inequality

$$\operatorname{cr}\deg_R M < m + \max\{2\beta^R_m(M)\,,\, 2\beta^R_{m+1}(M) - 1\}$$

Because of the inequality recalled above, the next result provides a practical estimate of the start of polynomial behavior of degree 1.

2.5.6 Proposition. *Let $(R, \mathfrak{m}, k)$ be a c.i. local ring and M a finite R-module.*

If the sequence $(\beta^R_i(M))_i$ is eventually polynomial of degree 1, then

$$\beta^R_{i+1}(M) - \beta^R_i(M) \quad \textit{is constant for} \quad i \geq \operatorname{cr}\deg_R M + 3$$

Proof. After a standard reduction, we may assume that R is complete and k is algebraically closed; see [11, p.54]. The hypothesis on $(\beta^R_i(M))_i$ implies $\operatorname{cx}_R M = 2$, so by [5, 3.6] there is a codimension 2 deformation $Q \twoheadrightarrow R$ with $\operatorname{proj}\dim_Q M$ finite. The algebra of cohomology operators defined by that deformation, see 2.3.4, is a graded polynomial ring $\mathcal{R} = k[\chi_1, \chi_2]$ with indeterminates of degree 2.

It is proved in [11, 4.7] that the graded $\mathcal{R}$-module $\mathcal{M} = \operatorname{Ext}_R(M, k)$ is a direct sum of non-positive shifts of the following graded modules:

$$\begin{aligned}
\mathcal{L}(n) &= \Sigma^{2n}((\chi_1, \chi_2)^n) && \text{for} \quad n \geq 0 \\
\mathcal{L}'(n) &= \Sigma^{-2n+2}\operatorname{Hom}_k(\mathcal{R}/(\chi_1, \chi_2)^n, k) && \text{for} \quad n \leq -1 \\
\mathcal{M}(n, \chi) &= \Sigma^{2n}((\chi_1, \chi_2)^n/(\vartheta^n)) && \text{for} \quad n \geq 1 \quad \text{and} \quad \vartheta \in \mathcal{R}^2 \smallsetminus \{0\}
\end{aligned}$$

Set $t = \operatorname{cr}\deg_R M + 3$. Since $\mathcal{L}'(n)$ has finite length, no submodule of $\Sigma^t\mathcal{M}^{\geqslant t}$ is isomorphic to $\mathcal{L}'(n)$; see 2.3.6. It is remarked after the proof of [11, 7.6] that $\Sigma^t\mathcal{M}^{\geqslant t}$ is generated in degrees 0 and 1. We have $\Sigma^t\mathcal{M}^{\geqslant t} \cong \operatorname{Ext}_R(\Omega^R_t(M), k)$, see (2.1.2), so there exist integers $r(j), s(j), b_i^j \geq 0$ and $c_i^j \geq 1$, with $j = 0, 1$, such that

$$\begin{aligned}
\operatorname{Ext}_R^{\mathsf{even}}(L,k) &\cong \bigoplus_{i=1}^{r(0)} \mathcal{L}(b_i^0) \bigoplus_{i=1}^{s(0)} \mathcal{M}(c_i^0, \vartheta_i) \\
\operatorname{Ext}_R^{\mathsf{odd}}(L,k) &\cong \bigoplus_{i=1}^{r(1)} \Sigma^{-1}\mathcal{L}(b_i^1) \bigoplus_{i=1}^{s(1)} \Sigma^{-1}\mathcal{M}(c_i^1, \vartheta_i)
\end{aligned}$$

The graded vector spaces $\mathcal{L}(n)$ and $\mathcal{M}(n, \vartheta)$ are zero in odd degrees. For $u \geq 0$ they have $\operatorname{rank}_k \mathcal{L}(n)^{2u} = n + u + 1$ and $\operatorname{rank}_k \mathcal{M}(n, \vartheta)^{2u} = n$. In addition, [11, 4.9] gives $r(0) = r(1)$, so for $u \geq 0$ the isomorphisms above yield

$$\begin{aligned}
\beta_{2u}^R(L) &= r(u+1) + b(0) + c(0) \\
\beta_{2u+1}^R(L) &= r(u+1) + b(1) + c(1)
\end{aligned}$$

where $r = r(0)$, and $b(j) = \sum_{i=1}^{r(j)} b_i^j$ and $c(j) = \sum_{i=1}^{s(j)} c_i^j$ for $j = 0, 1$. Thus, for $u \geq 0$ differences between consecutive Betti numbers are given by the expressions

$$\begin{aligned}
\beta_{t+2u+1}^R(M) - \beta_{t+2u}^R(M) &= b(1) - b(0) + c(1) - c(0) \\
\beta_{t+2u+2}^R(M) - \beta_{t+2u+1}^R(M) &= b(0) - b(1) + c(0) - c(1) + r
\end{aligned} \tag{2.5.1}$$

If $(\beta_i^R(M))_{i \gg 0}$ is arithmetic, then $\beta_{i+1}^R(M) - \beta_i^R(M)$ is constant for $i \gg 0$. It follows that for $u \gg 0$ the right-hand sides in (2.5.1) are equal. Since these expressions do not depend on u, the sequence $(\beta_{t+i}^R(M))_{i \geqslant 0}$ is arithmetic. $\square$

Chapter 3

Decomposing Gorenstein Rings as Connected Sums

The contents of this chapter are joint work with H. Ananthnarayan and E. Celikbas.

The main object of study in this Chapter is a construction of Gorenstein rings, called a connected sum, defined by Ananthnarayan, Avramov and Moore in [2]. Given Cohen-Macaulay local rings R, S and k of the same dimension, and ring homomorphisms $R \xrightarrow{\varepsilon_R} k \xleftarrow{\varepsilon_S} S$, the authors consider the fibre product (or pullback) $R \times_k S = \{(r,s) \in R \times S : \varepsilon_R(r) = \varepsilon_S(s)\}$ and define a connected sum of R and S over k as an appropriate quotient of $R \times_k S$. They prove that when R and S are Gorenstein, a connected sum is also a Gorenstein local ring of the same dimension.

In this Chapter, we focus on connected sums over a field in the Artinian case, i.e., when R and S are Gorenstein Artinian local rings and k is their common residue field. These objects have been studied from different perspectives by various authors starting with Sah [33] in the graded case and, in the local case by Lescot, see Remark 3.2.4(c). A topologically influenced version was also studied by Smith and Stong [40, Section 4], and quite a few authors approach this area via *Macaulay's inverse systems*, e.g., see [15]. Completely different techniques are used in this article: we look at intrinsic properties of the ring and its defining ideal in Section 3.3 and, in Section 3.4, we focus on conditions on the associated graded ring which force it to be a connected sum.

A natural question is: Given a Gorenstein Artin ring Q, can it be decomposed as a connected sum? This question was studied from a geometric point of view for *projective bundle ideals* by Smith and Stong in [40, Section 4]. It is also known that if Q has a non-trivial decomposition as a connected sum over k and the embedding dimension of Q is greater than 2, then Q cannot be a *complete intersection*. See [2, 8.3]. The question of decomposability has also been studied using inverse systems by Buczyńska et al. in [15] using polynomials that are *direct sums* and corresponding *apolar* Gorenstein algebras, see Remark 3.3.3.

In Section 3.3, we study properties of fibre products and connected sums of algebras over a field, and give two other necessary conditions for Q to be decomposable as a non-trivial connected sum: one concerning the second Hilbert coefficient $H_Q(2)$ of Q (Theorem 3.3.8), and the other in terms of the associated graded ring Q^{g} of Q (Theorem 3.3.13). In particular, if Q is a *compressed Gorenstein algebra* over k with Loewy length at least 4, then Q is indecomposable as a connected sum.

Finally, we identify necessary conditions for a Gorenstein Artin local algebra over a field to be a connected sum in Proposition 3.3.14 and obtain a characterization for the same in terms of the defining ideals, see Theorem 3.3.17. A secondary question is: If Q is a connected sum, what are its components? In general, it is not clear how to extract the components of a connected sum, but this characterization allows us to do so in this case, see Proposition 3.3.16.

As an application, in Section 3.4, we study the associated graded ring of a Gorenstein Artin ring and look for conditions which force the given Gorenstein ring to be a connected sum. The motivation comes from the following: In [34], Sally proves a structure theorem for stretched Gorenstein rings and in [19], Elias and Rossi give a similar structure theorem for short Gorenstein rings under some assumptions on the residue field. When such a short or stretched Gorenstein ring Q is an algebra over a field, these structure theorems show that Q can be written as a connected sum of a graded Gorenstein Artin ring R with the same Loewy length as Q, and a Gorenstein Artin ring S with Loewy length less than three.

In either case, using a construction of Iarrobino, we see that the associated graded ring of Q has the property that in degrees two or higher, its socle is one-dimensional. We call such graded k-algebras *Gorenstein up to linear socle*. Using the characterization proved in Section 3.3, we show that Q is a connected sum, with one component of Loewy length less than three and the other having a Gorenstein associated graded ring. This gives us results regarding the Poincaré series of Q and minimal number of generators of its defining ideal. See Theorem 3.4.7 and its corollaries.

In Section 3.5, we use Theorem 3.4.7 to give applications to short and stretched Gorenstein k-algebras. In particular, we show that these rings, when they are not graded, are non-trivial connected sums, and derive some consequences without any restrictions on the residue field.

The first two sections contain results regarding the main tools used in the rest of the paper. In Section 3.1, we collect some properties of associated graded rings and Poincaré series. Section 3.2 contains information on fibre products and connected sums, including their interactions with the objects introduced in Section 3.1. The results in Section 3.1 are well-known and Section 3.2 includes known results rephrased in our notation.

3.1 Preliminaries

In this section, we introduce notation and terminology.

3.1.1 Notation

a) If T is a local ring and M is an T-module, $\lambda(M)$ and $\mu(M)$ respectively denote the *length* and the *minimal number of generators* of M as a T-module.

b) Let $(T, \mathfrak{m}, k)$ be an Artinian local ring. Then $\operatorname{edim}(T)$ denotes the *embedding dimension* of T which is equal to $\mu(\mathfrak{m})$. The *socle* of T is $\operatorname{soc}(T) = \operatorname{ann}_T(\mathfrak{m})$.

Moreover, the *type* of T is $\operatorname{type}(T) = \dim_k(\operatorname{soc}(T))$, and the *Loewy length* of T is $\ell\ell(T) = \max\{n : \mathfrak{m}^n \neq 0\}$. [i]

c) If k is a field, a *graded k-algebra* G is a graded ring $G = \oplus_{i\geq 0} G_i$ with $G_0 = k$. It has a unique homogeneous maximal ideal, $G_+ = \oplus_{i\geq 1} G_i$. We say G is *standard graded* if G_+ is generated by G_1.

d) Let m and n be positive integers. Then $\boldsymbol{Y}$ and $\boldsymbol{Z}$ denote the sets of indeterminates $\{Y_1, \ldots, Y_m\}$ and $\{Z_1, \ldots, Z_n\}$ respectively, and $\boldsymbol{Y} \cdot \boldsymbol{Z}$ denotes

$$\{Y_i Z_j : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

e) For an ideal J of $k[\boldsymbol{Y}]$ or $k[\boldsymbol{Z}]$, J^e denotes its extension to $k[\boldsymbol{Y}, \boldsymbol{Z}]$ via the natural inclusions $\iota_Y : k[\boldsymbol{Y}] \hookrightarrow k[\boldsymbol{Y}, \boldsymbol{Z}]$ and $\iota_Z : k[\boldsymbol{Z}] \hookrightarrow k[\boldsymbol{Y}, \boldsymbol{Z}]$ respectively.

3.1.2 Associated graded rings

3.1.1 Definition. Let $(T, \mathfrak{m}, k)$ be a Noetherian local ring.

(i) The graded ring associated to the maximal ideal $\mathfrak{m}$ of P, denoted $T^{\mathfrak{g}}$, is defined as

$$T^{\mathfrak{g}} \cong \bigoplus_{i=0}^{\infty} (\mathfrak{m}^i / \mathfrak{m}^{i+1}).$$

(ii) Let $G = T^{\mathfrak{g}}$. We define the *Hilbert function* of T as $H_T(i) = \dim_k(G_i)$ for $i \geq 0$.

(iii) When T is Artinian with $\ell\ell(T) = s$, we write the Hilbert function of T as

[i] If T is also Gorenstein, its Loewy length is also referred to as *socle degree* in the literature.

$$H_T = (H_T(0), \dots, H_T(s)).$$

Furthermore, if T is Gorenstein, we say that T is *stretched* if $\mathfrak{m}_T^2$ is principal and T is *short* if $\mathfrak{m}_T^4 = 0$.

3.1.2 Remark. With notation as above, let $G = T^{\mathfrak{g}}$ and $G_{\geq n} = \bigoplus_{i=n}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ for $n \geq 0$.

a) For each $n \geq 0$, $G_{\geq n}$ is the nth power of the homogeneous maximal ideal G_+ of G and a minimal generating set of $G_{\geq n}$ lifts to a minimal generating set of $\mathfrak{m}^n$.

In particular, if T is Artinian local, then so is $T^{\mathfrak{g}}$. Furthermore, $\lambda(T) = \lambda(T^{\mathfrak{g}})$ and $\ell\ell(T) = \ell\ell(T^{\mathfrak{g}})$.

b) For each $x \in T \setminus \{0\}$, there exists a unique $i \in \mathbb{Z}$ such that $x \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$. The *initial form* of x is the element $x^* \in G$ of degree i that is the image of x in $\mathfrak{m}^i/\mathfrak{m}^{i+1}$.

c) For an ideal K of T, K^* denotes the ideal of G defined by $\langle x^* : x \in K \rangle$. Note that, if $R \cong T/K$, then $R^{\mathfrak{g}} \cong G/K^*$.

We now recall a construction due to Iarrobino [25, 1C]:

3.1.3 Remark (Iarrobino's Construction)**.** Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin local ring with $\ell\ell(Q) = s$, and let $G = Q^{\mathfrak{g}}$ be its associated graded ring. Iarrobino (cf. [25, 1.3]) showed that

$$C = \bigoplus_{i \geq 0} \frac{(0 :_Q \mathfrak{m}_Q^{s-i}) \cap \mathfrak{m}_Q^i}{(0 :_Q \mathfrak{m}_Q^{s-i}) \cap \mathfrak{m}_Q^{i+1}}$$

is an ideal in G. He also proved that $Q_0 = G/C$ is a graded Gorenstein quotient of G with $\deg(\mathrm{soc}(Q_0)) = s$.

Note that $H_{Q_0}(i) = H_G(i)$ for $i \geq s-1$ since $C_i = 0$ for $i \geq s-1$.

3.1.3 Poincaré Series

3.1.4 Definition. For a local ring $(T, \mathfrak{m}, k)$, the *Poincaré series* of T, is the formal power series

$$\mathbf{P}^T(t) = \sum_{i \geq 0} \beta_i^T t^i, \quad \text{with } \beta_i^T = \dim_k \left(\operatorname{Tor}_i^T(k,k)\right).$$

If T is an Artinian ring, then by Cohen's Structure Theorem, it is isomorphic to a quotient of a regular local ring, say $(\widetilde{T}, \mathfrak{m}_{\widetilde{T}}, k)$, such that $T \cong \widetilde{T}/I_T$, where $I_T \subseteq \mathfrak{m}_{\widetilde{T}}^2$. In the following remark, we compute the minimal number of generators of its defining ideal I_T.

3.1.5 Remark (Minimal Number of Generators)**.** Let $(T, \mathfrak{m}, k)$ be an Artinian local ring with $\operatorname{edim}(T) = d$. By [10, Cor. 7.1.5], we have

$$\beta_1^T = d \quad \text{and} \quad \mu(I_T) = \beta_2^T - \binom{\beta_1^T}{2} = \beta_2^T - \binom{d}{2}.$$

The same formula holds if $T = k[X_1, \ldots, X_d]/I_T$, where $I_T \subseteq \langle \boldsymbol{X} \rangle^2$ is a homogeneous ideal, since we can also write $T \cong \widetilde{T}/I_T\widetilde{T}$, where $\widetilde{T} = k[[\boldsymbol{X}]]$.

Next we list some properties of the Poincaré series of a Gorenstein Artin local ring.

3.1.6 Remark (Poincaré Series of Gorenstein Rings)**.**
Let $(T, \mathfrak{m}, k)$ be a Gorenstein Artin local ring and $\overline{T}$ represent the quotient $T/\operatorname{soc}(T)$.

a) If $\operatorname{edim}(T) \leq 4$, then $\mathbf{P}^T(t)$ is rational. See [4, Thm. 6.4].

b) If $\operatorname{edim}(T) \geq 2$, then $\mathbf{P}^T(t)^{-1} = \mathbf{P}^{\overline{T}}(t)^{-1} + t^2$, by [27, Thm. 2].

c) If $\ell\ell(T) = 2$ and $\operatorname{edim}(T) = n$, then $\mathbf{P}^T(t)^{-1} = 1 - t$ if $n = 1$. If $n \geq 2$, then $\mathbf{P}^T(t)^{-1} = 1 - nt + t^2$ (for example, by (b), since $\ell\ell(\overline{T}) = 1$).

In particular, $\mathbf{P}^T(t)$ is a rational function of t.

3.2 Fibre Products and Connected Sums

We begin with the definitions and some basic properties of fibre products and connected sums. For more details, see [2, Sections 1 and 2] and [1, Chapter 4].

3.2.1 Definition. Let $(R, \mathfrak{m}_R, k)$ and $(S, \mathfrak{m}_S, k)$ be local rings. The *fibre product* R and S over k is the ring $R \times_k S = \{(r,s) \in R \times S : \pi_R(r) = \pi_S(s)\}$, where π_R and π_S are the natural projections from R and S respectively onto k.

3.2.2 Remark. With the notation as in Definition 3.2.1, set $P = R \times_k S$. Then, by identifying $\mathfrak{m}_R$ with $\{(r,0) : r \in \mathfrak{m}_R\}$ and $\mathfrak{m}_S$ with $\{(0,s) : s \in \mathfrak{m}_S\}$, we see that:

a) P is a local ring with unique maximal ideal $\mathfrak{m}_P = \mathfrak{m}_R \times \mathfrak{m}_S$.

Hence $\operatorname{edim}(P) = \operatorname{edim}(R) + \operatorname{edim}(S)$ and by [2, (1.0.3)], $P^{\mathfrak{g}} \cong R^{\mathfrak{g}} \times_k S^{\mathfrak{g}}$. This implies that for $i \geq 1$, $H_P(i) = H_R(i) + H_S(i)$.

b) If R and S are Artinian, then $\lambda(P) = \lambda(R) + \lambda(S) - 1$. Furthermore, if $\ell\ell(R)$, $\ell\ell(S) \geq 1$, then $\operatorname{soc}(P) = \operatorname{soc}(R) \oplus \operatorname{soc}(S)$, and hence, $\operatorname{type}(P) = \operatorname{type}(R) + \operatorname{type}(S)$. In particular, when $R \ncong k \ncong S$, P is not Gorenstein.

3.2.3 Definition. Let $(R, \mathfrak{m}_R, k)$ and $(S, \mathfrak{m}_S, k)$ be Gorenstein Artin local rings different from k. Let $\operatorname{soc}(R) = \langle \delta_R \rangle$, $\operatorname{soc}(S) = \langle \delta_S \rangle$. Identifying δ_R with $(\delta_R, 0)$ and δ_S with $(0, \delta_S)$, a *c*onnected sum of R and S over k, denoted $R\#_k S$, is the ring

$$R\#_k S = (R \times_k S)/\langle \delta_R - \delta_S \rangle.$$

Connected sums of R and S over k depend on the generators of the socle δ_R and δ_S chosen. For example, the connected sums $Q_1 = (R \times_k S)/\langle y^2 - z^2 \rangle$ and $Q_2 = (R \times_k S)/\langle y^2 - 5z^2 \rangle$ of $R = \mathbb{Q}[Y]/\langle Y^3 \rangle$ and $S = \mathbb{Q}[Z]/\langle Z^3 \rangle$ are not isomorphic as rings, as shown in [2, 3.1].

3.2.4 Remark. With notation as in Definition 3.2.3, set $P = R \times_k S$ and let $Q = R\#_k S$.

a) Note that $\lambda(Q) = \lambda(P) - 1 = \lambda(R) + \lambda(S) - 2$ since $0 \neq \delta_R - \delta_S \in \mathrm{soc}(P)$.

b) By the definition of Q, it is clear that $\overline{Q} \cong \overline{P} \cong \overline{R} \times_k \overline{S}$, where $^-$ denotes going modulo the socle.

c) If R and S are different from k, i.e., $\ell\ell(R)$, $\ell\ell(S) \geq 1$, then Q is a Gorenstein Artin local ring. This is proved in [28, 4.4], see [2, 2.8] for a more general result.

d) If $\mathrm{soc}(R) \subseteq \mathfrak{m}_R^2$ and $\mathrm{soc}(S) \subseteq \mathfrak{m}_S^2$, i.e., $\ell\ell(R)$, $\ell\ell(S) \geq 2$, then

$$\mathrm{edim}(Q) = \mathrm{edim}(P) = \mathrm{edim}(R) + \mathrm{edim}(S).$$

3.2.5 Remark (Trivial Fibre Products and Connected Sums)**.** Every ring is trivially a fibre product over its residue field. Indeed, if $(R, \mathfrak{m}_R, k)$ be an Artinian local ring, then $R \times_k k \cong R$.

Similarly, every Gorenstein Artin local ring, which is not a field, is trivially a connected sum over its residue field. In order to see this, consider $(R, \mathfrak{m}_R, k)$ to be a Gorenstein Artin local ring with $\ell\ell(R) \geq 1$ and S be a k-algebra of length two. Note that this forces S to be Gorenstein. One can check that $R\#_k S \cong R$.

3.2.6 Remark (Poincaré Series of Fibre Products and Connected Sums)**.**
Let $(R, \mathfrak{m}_R, k)$ and $(S, \mathfrak{m}_S, k)$ be Artinian local rings.

a) If $P = R \times_k S$, then, by [17, 1],

$$\frac{1}{\mathbf{P}^P(t)} = \frac{1}{\mathbf{P}^R(t)} + \frac{1}{\mathbf{P}^S(t)} - 1. \tag{3.2.6.1}$$

b) Furthermore, if R and S are Gorenstein Artin with $\ell\ell(R)$, $\ell\ell(S) \geq 2$, then (a) and

Remark 3.1.6(b) give $\mathbf{P}^Q(t)^{-1} = \mathbf{P}^{\overline{R}}(t)^{-1} + \mathbf{P}^{\overline{S}}(t)^{-1} - 1 + t^2$. In particular, we get

$$\frac{1}{\mathbf{P}^Q(t)} = \frac{1}{\mathbf{P}^R(t)} + \frac{1}{\mathbf{P}^S(t)} - 1 + \phi(t) = \frac{1}{\mathbf{P}^P(t)} + \phi(t), \tag{3.2.6.2}$$

where

$$\phi(t) = \begin{cases} -t^2 & \operatorname{edim}(R), \operatorname{edim}(S) \geq 2 \\ t^2 & \operatorname{edim}(R) = 1 = \operatorname{edim}(S) \\ 0 & \text{otherwise} \end{cases}$$

When $(R, \mathfrak{m}_R, k)$ and $(S, \mathfrak{m}_S, k)$ are equicharacteristic, i.e., they contain a field and hence their common residue field k, we can say more about their fibre products and connected sums over k. We begin with the following observations:

3.2.7 Remark (Fibre Products and Connected Sums in Equicharacteristic)**.** Let $R = k[\boldsymbol{Y}]/I_R$ and $S = k[\boldsymbol{Z}]/I_S$ be k-algebras with $I_R \subseteq \langle \boldsymbol{Y} \rangle^2$ and $I_S \subseteq \langle \boldsymbol{Z} \rangle^2$. Note that this implies $\operatorname{edim}(R) = m$ and $\operatorname{edim}(S) = n$.

a) Let P denote the fibre product $R \times_k S$.

Then $P \cong k[\boldsymbol{Y}, \boldsymbol{Z}]/I_P$, where $I_P = I_R^e + I_S^e + \langle \boldsymbol{Y} \cdot \boldsymbol{Z} \rangle$. In particular, if R and S are standard graded, then so is P.

b) Suppose R and S are Gorenstein and $Q = R \#_k S$ is their connected sum over k.

Then, by Definition 3.2.3, there exist $\Delta_R \in k[\boldsymbol{Y}]$ and $\Delta_S \in k[\boldsymbol{Z}]$ such that their respective images $\delta_R \in R$ and $\delta_S \in S$ generate the respective socles and

$$Q \cong (R \times_k S)/\langle \delta_R - \delta_S \rangle \cong k[\boldsymbol{Y}, \boldsymbol{Z}]/I_Q,$$

where $I_Q = I_P + \langle \Delta_R - \Delta_S \rangle = I_R^e + I_S^e + \langle \boldsymbol{Y} \cdot \boldsymbol{Z} \rangle + \langle \Delta_R - \Delta_S \rangle$. In particular, if R and S are standard graded, then Q is standard graded if and only if $\ell\ell(R) = \ell\ell(S)$.

A different point of view to study Gorenstein Artin k-algebras is via *Macaulay's inverse systems*. The next remark tells us what connected sums are in this light.

3.2.8 Remark (Connected Sums and Inverse Systems)**.** One can study Gorenstein Artin k-algebras via Macaulay's inverse systems. In this correspondence, such rings correspond to polynomials. For example, the ring $k[X_1, \ldots, X_n]/I$, where $I = \langle X_iX_j, X_i^2 - X_j^2 : 1 \leq i < j \leq n\rangle$ corresponds to the polynomial $Z_1^2 + \cdots + Z_n^2$. For more details, see [1, Section 1.4] or [19, Section 2].

If R and S are Gorenstein Artin k-algebras corresponding to polynomials $F(\boldsymbol{Y})$ and $G(\boldsymbol{Z})$ respectively, then the Gorenstein Artin k-algebra corresponding to $F + G$ is a connected sum of R and S over k. For example, see [1, 4.24].

In particular, whenever the polynomial corresponding to a Gorenstein Artin k-algebra Q is of the form $F(\boldsymbol{Y}) + Z_1^2 + \cdots + Z_n^2$, then Q is a connected sum, over k, of the Gorenstein Artin rings corresponding to the polynomials $F(\boldsymbol{Y})$ and $Z_1^2 + \cdots + Z_n^2$, say R and S respectively. As seen above, S is a Gorenstein Artin local ring with $\ell\ell(S) = 2$. This observation plays an important role in Section 5.

3.3 Connected Sums and Indecomposability

The main question we would like to address is:

Main Question: When is a given Gorenstein Artin local ring Q decomposable as a connected sum over k?

In light of Remark 3.2.5, we make the following key definition about connected sums. Note that a similar terminology is also used for fibre products in this article.

3.3.1 Definition. Let $(Q, \mathfrak{m}, k)$ be a Gorenstein Artin local ring. We say that Q can be *decomposed as a connected sum* over k if there exist Gorenstein Artin local rings R and S such that $Q \cong R\#_k S$ and $R \ncong Q \ncong S$. In this case, we call R and S the *components* in a connected sum decomposition of Q, and say that $Q \cong R\#_k S$ is a non-trivial decomposition.

If Q cannot be decomposed as a connected sum over k, we say that Q is *indecomposable* as a connected sum over k.

3.3.2 Remark. If R and S are Gorenstein Artin local rings with $\ell\ell(R)$, $\ell\ell(S) \geq 2$, then $Q = R\#_k S$ is non-trivial decomposition of Q as a connected sum over k. Indeed, $\lambda(Q) = \lambda(R) + \lambda(S) - 2$ and $\ell\ell(R)$, $\ell\ell(S) \geq 2$ forces $\lambda(Q) > \max\{\lambda(R), \lambda(S)\}$, hence $R \ncong Q \ncong S$.

3.3.3 Remark. In terms of inverse systems, answering the main question amounts to the following: Given a polynomial F corresponding to Q, write $F = F_1 + F_2$, where F_1 and F_2 are polynomials in disjoint sets of variables.

When F is homogeneous, the above property has been studied in [15]. The authors define such a polynomial to be a *direct sum* and the corresponding Gorenstein algebra to be *apolar*.

It is known (see [2, 8.3]) that if Q is a *complete intersection* with $\operatorname{edim}(Q) \geq 3$, then Q is indecomposable as a connected sum over k. The same also follows for k-algebras from Theorem 3.3.8.

Let $(Q, \mathfrak{m}, k)$ be a Gorenstein local k-algebra We now see other conditions which force indecomposability: the first concerning the second Hilbert coefficient $H_Q(2)$ of Q (Theorem 3.3.8), and the second in terms of the associated graded ring $Q^{\mathfrak{g}}$ of Q (Theorem 3.3.13). Finally, we give a characterization of connected sums of k-algebras over k in terms of the defining ideals (Theorem 3.3.17). When the embedding dimension of Q is three or four, one can give conditions in terms of the minimal number of generators of the defining ideal of Q which force indecomposability (Theorem 3.3.6). The notation in the rest of this section is as in Remark 3.2.7.

3.3.1 Minimal Number of Generators of the Defining Ideal

We begin with a result on the minimal number of generators of the defining ideals of P and Q.

3.3.4 Proposition. *With notation as in Remark 3.2.7, we have the following:*

a) $\mu(I_P) = \mu(I_R) + \mu(I_S) + mn$.

b) $\mu(I_Q) = \mu(I_R) + \mu(I_S) + mn + \psi = \mu(I_P) + \psi$,

where $\psi = 1$ *when* m, $n \geq 2$, $\psi = -1$ *when* $m = 1 = n$, *and* $\psi = 0$ *otherwise.*

Proof.

a) Comparing the coefficients of t and t^2 in Equation 3.2.6.1, we see that $\beta_1^P = \beta_1^R + \beta_1^S$ and $\beta_2^P = \beta_2^R + \beta_2^S + 2\beta_1^R\beta_1^S$. Hence (a) follows by Remark 3.1.5.

b) First of all, note that if $m = 1 = n$, then $\mu(I_R) = 1 = \mu(I_S)$. Since Q is Gorenstein Artin with $\operatorname{edim}(Q) = m + n = 2$, it is a well known result of Serre [37] that $\mu(I_Q) = 2$. Hence, without loss of generality, we may assume that $m \geq 2$.

It is clear from Equation 3.2.6.2 that $\beta_2^Q = \beta_2^P$ when $n = 1$ and $\beta_2^Q = \beta_2^P + 1$ when $n \geq 2$. Since $\beta_1^Q = \beta_1^P$ in either case, the proof of (b) is complete by Remark 3.1.5. $\square$

3.3.5 Remark. Let $Q \cong R\#_k S$ be a non-trivial decomposition of Q as a connected sum over k, and the notation be as in Remark 3.2.7.

Let $\operatorname{edim}(R) = m$ and $\operatorname{edim}(S) = n$. Then $\operatorname{edim}(Q) = m + n$.

a) Suppose $\operatorname{edim}(Q) = 3$. Then without loss of generality, $m = 2$ and $n = 1$. Hence $\mu(S) = 1$ and by the above-mentioned result of Serre, $\mu(I_R) = 2$. Thus $\mu(I_Q) = 5$.

b) If $\operatorname{edim}(Q) = 4$, then either $m = n = 2$, in which case $\mu(I_R) = 2 = \mu(I_S)$ forcing $\mu(I_Q) = 9$ or, without loss of generality, $m = 3$ and $n = 1$. In this case, $\mu(I_S) = 1$, and by [45], $\mu(I_R)$ is an odd number. Hence, in either case, by Proposition 3.3.4(b), $\mu(I_Q)$ is odd.

Thus we have proved:

3.3.6 Theorem. *Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin k-algebra. With notations as in Remark 3.2.7, Q is indecomposable as a connected sum over k when one of the following holds:*

i) $\operatorname{edim}(Q) = 3$ *and* $\mu(I_Q) \neq 5$.

ii) $\operatorname{edim}(Q) = 4$ *and* $\mu(I_Q)$ *is an even number.*

3.3.2 Hilbert Functions

We first obtain a numerical criterion satisfied by connected sums.

3.3.7 Proposition. *Let R and S be Gorenstein Artin k-algebras, with* $\operatorname{edim}(R) = m$ *and* $\operatorname{edim}(S) = n$ *and $\ell\ell(R), \ell\ell(S) \geq 2$. If $Q = R\#_k S$, then $H_Q(2) \leq \binom{m+n+1}{2} - mn$.*

Proof. With notation as in Remark 3.2.7, since $\boldsymbol{Y} \cdot \boldsymbol{Z} \subseteq I_Q$, we have $\boldsymbol{Y}^* \cdot \boldsymbol{Z}^* \subseteq I_Q^*$. Hence, $H_{k[\boldsymbol{Y},\boldsymbol{Z}]}(2) = \binom{m+n+1}{2}$ implies that

$$H_Q(2) = H_{Q^{\mathrm{g}}}(2) = \binom{m+n+1}{2} - \dim_k((I_Q^* + \langle \boldsymbol{Y} \cdot \boldsymbol{Z}\rangle^3)/\langle \boldsymbol{Y} \cdot \boldsymbol{Z}\rangle^3) \leq \binom{m+n+1}{2} - mn. \qquad \square$$

Note that for positive integers m and n, if $d = m + n$ is fixed, the minimum value of $mn = m(d - m)$ is obtained when $m = 1$ or $m = d - 1$. Hence, an immediate corollary is:

3.3.8 Theorem. *Let Q be a Gorenstein Artin local k-algebra with* $\operatorname{edim}(Q) = d$. *If $H_Q(2) \geq \binom{d}{2} + 2$, then Q is indecomposable as a connected sum over k.*

A Gorenstein Artin k-algebra Q is said to be *compressed* if it has a maximum possible Hilbert function given the embedding dimension d and Loewy length s, i.e., if the Hilbert function of Q is

$$H_Q(i) = \min\{\binom{d+i-1}{i}, \binom{d+s-i-1}{s-i}\}.$$

3.3.9 Corollary. *If Q is a compressed Gorenstein k-algebra with $\ell\ell(Q) \geq 4$, then Q is indecomposable as a connected sum over k.*

3.3.10 Remark. Since *generic Gorenstein k-algebras* are compressed (see [24]), they are indecomposable as connected sums over k. This was proved in [40, 4.4] by using different techniques.

3.3.3 Associated Graded Rings

The following basic property of the associated graded ring of a connected sum of k-algebras is used to obtain a necessary condition.

3.3.11 Proposition. *Let R and S be Gorenstein Artin k-algebras with $\ell\ell(R) \neq \ell\ell(S)$. Then the associated graded ring of $R\#_k S$ is a fibre product.*

Moreover, if $\ell\ell(R)$ and $\ell\ell(S)$ are at least two, the non-trivial connected sum $R\#_k S$ is not standard graded.

Proof. Let $P = R \times_k S$ and $Q = R\#_k S$. Let $\operatorname{soc}(R) = \langle \delta_R \rangle$ and $\operatorname{soc}(S) = \langle \delta_S \rangle$. Since $Q \cong P/\langle \delta_R - u\delta_S \rangle$ for some unit u in S, by Remark 3.1.2(c), we have

$$Q^{\mathrm{g}} \cong P^{\mathrm{g}}/\langle \delta_R - u\delta_S \rangle^*.$$

Recall that by Remark 3.2.2(a), $P^{\mathrm{g}} \cong R^{\mathrm{g}} \times_k S^{\mathrm{g}}$.

Without loss of generality, we may assume that $\ell\ell(R) > \ell\ell(S)$. Hence

$$\langle \delta_R - u\delta_S \rangle^* = \langle u\delta_S \rangle^*.$$

Thus we see that $Q^{\mathrm{g}} \cong (R^{\mathrm{g}} \times_k S^{\mathrm{g}})/\langle u\delta_S \rangle^* \cong R^{\mathrm{g}} \times_k S/\langle \delta_S \rangle^{\mathrm{g}}$.

Finally, if $\ell\ell(R) > \ell\ell(S) \geq 2$, then $R^{\mathrm{g}} \neq k \neq S/\langle \delta_S \rangle^{\mathrm{g}}$. Hence Q^{g} is not Gorenstein, by Remark 3.2.2(b). Thus $Q \not\cong Q^{\mathrm{g}}$, hence Q is not standard graded. $\square$

By Remark 3.2.7, if R and S are standard graded k algebras not isomorphic to k with $\ell\ell(R) = \ell\ell(S)$, then $Q = R\#_k S$ is non-trivial connected sum, but $Q^{\mathfrak{g}} \cong Q$ is indecomposable as a fibre product over k.

The condition $\ell\ell(R) \neq \ell\ell(S)$ is necessary in the above proposition, even if R and S are not standard graded, as is seen in the following example.

3.3.12 Example. Let $R \cong k[Y_1, Y_2]/\langle Y_1^2Y_2, Y_1^3 - Y_2^2\rangle$, $S = k[Z]/\langle Z^5\rangle$ and $Q \cong R\#_k S$. Then $G = Q^{\mathfrak{g}} \cong k[Y_1, Y_2, Z]/\langle Y_1Z, Y_2Z, Y_1^2Y_2, Y_2^2, Y_1^4 - Z^4\rangle$. Note that in this case, R is not standard graded, and $\ell\ell(R) = \ell\ell(S) = 4$. We now show that G is indecomposable as a fibre product over k.

Suppose $G \cong A \times_k B$ is a non-trivial fibre product, for some k-algebras A and B. Since $3 = \operatorname{edim}(G) = \operatorname{edim}(A) + \operatorname{edim}(B)$, we may assume that $\operatorname{edim}(A) = 2$ and $\operatorname{edim}(B) = 1$. Furthermore, $2 = \operatorname{type}(G) = \operatorname{type}(A) + \operatorname{type}(B)$ forces A and B to be Gorenstein, and $\operatorname{soc}(G) = \operatorname{soc}(A) \oplus \operatorname{soc}(B)$ implies that $\ell\ell(A) = 4$ and $\ell\ell(B) = 2$ or vice versa. Since the Hilbert function of B, $H_B(i) = 1$ for each $i \leq \ell\ell(B)$, one can check that if $\ell\ell(B) = 4$, then $\ell\ell(A) = 3$. As this cannot happen, we must have $\ell\ell(A) = 4$ and $\ell\ell(B) = 2$.

Thus, if $G \cong A \times_k B$ is a non-trivial fibre product, we may assume that A and B are Gorenstein, $\operatorname{edim}(A) = 2$, $\ell\ell(A) = 4$, and $B \cong k[V]/\langle V^3\rangle$. In particular, we can write $A \cong k[U_1, U_2]/\langle f_1, f_2\rangle$. Note that U_1, U_2 and V are indeterminates over k.

Thus $G \cong k[U_1, U_2, V]/\langle U_1V, U_2V, V^3, f_1, f_2\rangle$. Let small letters denote the respective images of the indeterminates in G and

$$\phi : k[U_1, U_2, V]/\langle U_1V, U_2V, V^3, f_1, f_2\rangle \longrightarrow k[Y_1, Y_2, Z]/\langle Y_1Z, Y_2Z, Y_1^2Y_2, Y_2^2, Y_1^4 - Z^4\rangle$$

be an isomorphism.

Write

$$\phi(u_1) = a_{11}y_1 + a_{12}y_2 + a_{13}z,$$
$$\phi(u_2) = a_{21}y_1 + a_{22}y_2 + a_{23}z,$$
$$\phi(v) = a_{31}y_1 + a_{32}y_2 + a_{33}z.$$

The fact that $\phi(v^3) = 0$ forces $a_{31} = 0 = a_{33}$.

Hence $a_{32} \neq 0$ and therefore, $a_{11} = 0 = a_{21}$ since $\phi(u_1 v) = 0 = \phi(u_2 v)$. This gives us a contradiction as $a_{i1} = 0$ for all i implies that $y_1 \notin \mathrm{im}(\phi)$. Hence G cannot be written as a non-trivial fibre product.

Thus, the above example shows that the indecomposibility of $G = Q^{\mathrm{g}}$ as a fibre product over k does not imply the indecomposibility of Q as a connected sum over k. However, it imposes some restrictions as shown by Proposition 3.3.11, which we note in the following:

3.3.13 Theorem. *Let Q be a Gorenstein Artin k-algebra such that its associated graded ring Q^{g} is indecomposable as a fibre product over k. If R and S are Gorenstein Artin k-algebras such that $Q \cong R\#_k S$ is a non-trivial connected sum over k, then:*
a) $\ell\ell(R) = \ell\ell(S)$ and
b) if Q is not standard graded, then at least one of R and S is not standard graded.

3.3.4 A Characterization in terms of the Defining Ideals

The following theorem gives us a criteria to determine whether a given Gorenstein Artin local k-algebra is a connected sum.

3.3.14 Proposition. *Let $Q = k[\boldsymbol{Y}, \boldsymbol{Z}]/I_Q$ be a Gorenstein Artin local ring. Let $R = k[\boldsymbol{Y}]/I_R$ and $S = k[\boldsymbol{Z}]/I_S$, where $I_R = I_Q \cap k[\boldsymbol{Y}]$ and $I_S = I_Q \cap k[\boldsymbol{Z}]$. Suppose*

$\boldsymbol{Y} \cdot \boldsymbol{Z} \subseteq I_Q$. Then

a) R and S are Gorenstein Artin and

b) $Q \cong R \#_k S$.

Proof. The inclusions $k[\boldsymbol{Y}]$, $k[\boldsymbol{Z}] \hookrightarrow k[\boldsymbol{Y}, \boldsymbol{Z}]$ induce inclusions $R \hookrightarrow Q$ and $S \hookrightarrow Q$. Let y and z denote the respective images of Y and Z in the quotient rings Q, R and S.

a) By symmetry, it suffices to prove (a) for R. Let $\delta_R \in \operatorname{soc}(R)$. Then $\boldsymbol{Y} \cdot \Delta_R \subseteq I_R \subseteq I_Q$, where $\Delta_R \in k[\boldsymbol{Y}]$ is a preimage in $k[\boldsymbol{Y}]$ of δ_R. Moreover, since $\boldsymbol{Y} \cdot \boldsymbol{Z} \subseteq I_Q$, we see that $\boldsymbol{Z} \cdot \Delta_R \subseteq I_Q$. Hence $\delta_R \in \operatorname{soc}(Q)$. Therefore $0 \neq \operatorname{soc}(R) \subseteq \operatorname{soc}(Q)$ which is a one-dimensional k-vector space. Thus $\dim_k(\operatorname{soc}(R)) = 1$, i.e., R is Gorenstein Artin.
b) Let $P = R \times_k S$. Then $P \cong k[\boldsymbol{Y}, \boldsymbol{Z}]/I_P$, where $I_P = I_R^e + I_S^e + \langle \boldsymbol{Y} \cdot \boldsymbol{Z} \rangle$. By the hypothesis, $I_P \subseteq I_Q$ and hence there is a natural surjective map $\pi : P \longrightarrow Q$.

Let δ_R and δ_S be the generators of $\operatorname{soc}(R)$ and $\operatorname{soc}(S)$ respectively, with their respective preimages in $k[\boldsymbol{Y}, \boldsymbol{Z}]$ being Δ_R and Δ_S. Note that if $\Delta_R \in I_Q$, then $\Delta_R \in I_R$, which is not true. Similarly, $\Delta_S \notin I_Q$, i.e., $\pi(\delta_R) \neq 0 \neq \pi(\delta_S)$ in Q. But $\operatorname{soc}(P) = \langle \delta_R, \delta_S \rangle$ maps into $\operatorname{soc}(Q)$, which is one-dimensional. Hence $\pi(\delta_R) = u\pi(\delta_S)$ for some unit $u \in k$. Thus $\delta_R - u\delta_S \in \ker(\pi)$ for some $u \in k$.

Finally, we show that $\ker(\pi) \subseteq \operatorname{soc}(P)$. Let $f \in \ker(\pi)$ and F be a preimage in $k[\boldsymbol{Y}, \boldsymbol{Z}]$. In order to prove $f \in \operatorname{soc}(P)$, it is enough to prove that Y_iF, $Z_jF \in I_P$ for all i and j.

We show that $Y_1F \in I_P$. The others follow similarly. Write $F = F_1(\boldsymbol{Y}) + F_2(\boldsymbol{Y}, \boldsymbol{Z})$, where every term of F_2 is a multiple of some Z_j. Then $Y_1F_2 \in I_P \subseteq I_Q$.

Now $f \in \ker(\pi)$ implies $F \in I_Q$. Thus $Y_1F_1 = Y_1F - Y_1F_2 \in I_Q \cap k[\boldsymbol{Y}] = I_R$. Therefore, Y_1F_1 is also in I_P, which implies that $Y_1F \in I_P$.

This proves that $\ker(\pi) \subseteq \operatorname{soc}(P)$.

Thus, we have

$$0 \subsetneq \langle \delta_R - u\delta_S \rangle \subseteq \ker(\pi) \subsetneq \langle \delta_R, \delta_S \rangle = \operatorname{soc}(P).$$

Since $\lambda(\operatorname{soc}(P)) = 2$, this forces $\ker(\pi) = \langle \delta_R - u\delta_S \rangle$. Therefore, $Q \cong P/\langle \delta_R - u\delta_S \rangle$ for some unit $u \in k$, and hence is a connected sum of R and S over k. $\square$

The requirement that $\boldsymbol{Y} \cdot \boldsymbol{Z} \subseteq I_Q$ is necessary in the above proposition, otherwise R or S may not be Gorenstein as can be seen from the following example.

3.3.15 Example. Let $Q = \mathbb{Q}[Y_1, Z_1, Z_2]/I_Q$, where $I_Q = \langle Y_1Z_1 - Z_2^2, Y_1^2, Z_1^2 \rangle$. Then $I_Q \cap \mathbb{Q}[Z_1, Z_2] = \langle Z_1^2, Z_1Z_2^2, Z_2^4 \rangle$. (We use the elimination package in Macaulay2). Note that Q is Gorenstein, but $S = \mathbb{Q}[Z_1, Z_2]/(I_Q \cap \mathbb{Q}[Z_1, Z_2])$ is not Gorenstein because its socle is two dimensional.

By definition of the fibre product, if $P = R \times_k S$, R and S can be identified with quotients of P. On the other hand, if $Q = R\#_k S$, in general it is not clear how one can recover the components R and S from Q. Proposition 3.3.16 shows that R and S can be identified with subrings in the case of k-algebras.

3.3.16 Proposition. *Let P, Q, R and S be k-algebras with $P \cong k[\boldsymbol{Y}, \boldsymbol{Z}]/I_P$, $Q \cong k[\boldsymbol{Y}, \boldsymbol{Z}]/I_Q$, $R \cong k[\boldsymbol{Y}]/I_R$ and $S \cong k[\boldsymbol{Z}]/I_S$.*

a) If $P \cong R \times_k S$, then $I_R = I_P \cap k[\boldsymbol{Y}]$ and $I_S = I_P \cap k[\boldsymbol{Z}]$.

b) If R and S are Gorenstein Artin with $Q \cong R\#_k S$, then $I_R = I_Q \cap k[\boldsymbol{Y}]$ and $I_S = I_Q \cap k[\boldsymbol{Z}]$.

Proof.

a) By Remark 3.2.7(a), since $I_P = I_R^e + I_S^e + \langle \boldsymbol{Y} \cdot \boldsymbol{Z} \rangle$, it is enough to show $I_P \cap k[\boldsymbol{Y}] \subseteq I_R$.

Let $F(\boldsymbol{Y}) \in I_P \cap k[\boldsymbol{Y}]$. We can write $F = F_1(\boldsymbol{Y}) + F_2(\boldsymbol{Y}, \boldsymbol{Z})$, where $F_1 \in I_R$ and every term of F_2 is a multiple of some Z_j. But $F - F_1 \in k[\boldsymbol{Y}]$, hence $F_2 = F - F_1 = 0$.

b) Let $I_{R'} = I_Q \cap k[\boldsymbol{Y}]$ and $I_{S'} = I_Q \cap k[\boldsymbol{Z}]$. Since $I_R^e + I_S^e \subseteq I_Q$, we have $I_R \subseteq I_{R'}$ and $I_S \subseteq I_{S'}$, which induce natural surjective maps $\pi_1 : R \longrightarrow R'$ and $\pi_2 : S \longrightarrow S'$, where $R' = k[\boldsymbol{Y}]/I_{R'}$, $S' = k[\boldsymbol{Z}]/I_{S'}$. In particular, $\lambda(R') \leq \lambda(R)$ and $\lambda(S') \leq \lambda(S)$. In order to prove that $I_R = I_{R'}$ and $I_S = I_{S'}$, it is enough to prove that π_1 and π_2 are isomorphisms, in particular, it is enough to show that $\lambda(R') = \lambda(R)$ and $\lambda(S') = \lambda(S)$.

Since $\boldsymbol{Y} \cdot \boldsymbol{Z} \subseteq I_Q$, by Theorem 3.3.14, R' and S' are Gorenstein Artin and $Q \cong R' \#_k S'$. Hence $\lambda(R') + \lambda(S') = \lambda(Q) + 2 = \lambda(R) + \lambda(S)$. Since $\lambda(R') \leq \lambda(R)$ and $\lambda(S') \leq \lambda(S)$, we get $\lambda(R') = \lambda(R)$ and $\lambda(S') = \lambda(S)$, completing the proof. $\square$

Proposition 3.3.14 and Proposition 3.3.16(b) give equivalent conditions for a Gorenstein Artin k-algebra to be a connected sum over k. We summarize the results of this section in the following:

3.3.17 Theorem. *Let Q be a Gorenstein Artin local k-algebra with $\ell\ell(Q) \geq 1$. Then the following are equivalent:*

(1) Q can be decomposed nontrivially as a connected sum over k.

(2) $Q \cong k[Y_1, \ldots, Y_m, Z_1 \ldots, Z_n]/I_Q$ for m, $n \geq 1$ with $\boldsymbol{Y} \cdot \boldsymbol{Z} \subset I_Q \subset \langle \boldsymbol{Y}, \boldsymbol{Z} \rangle^2$.

If the above conditions are satisfied, then we can write $Q \cong R \#_k S$, where $R = k[\boldsymbol{Y}]/I_R$, $S = k[\boldsymbol{Z}]/I_S$ with $I_R = I_Q \cap k[\boldsymbol{Y}]$, and $I_S = I_Q \cap k[\boldsymbol{Z}]$. Furthermore, the following hold:

(1) $\lambda(Q) = \lambda(R) + \lambda(S) - 2$ and $\operatorname{edim}(Q) = \operatorname{edim}(R) + \operatorname{edim}(S)$.

(2) For $0 < i < \min\{\ell\ell(S), \ell\ell(R)\}$, $H_Q(i) = H_R(i) + H_S(i)$.

(3) $I_Q = I_R^e + I_S^e + \langle \boldsymbol{Y} \cdot \boldsymbol{Z} \rangle + \langle \Delta_R - \Delta_S \rangle$, where $\Delta_R \in k[\boldsymbol{Y}]$ and $\Delta_S \in k[\boldsymbol{Z}]$ are such that their respective images $\delta_R \in R$ and $\delta_S \in S$ generate the respective socles.

(4) $\mu(I_Q) = \mu(I_R) + \mu(I_S) + mn + \phi(m,n)$, *and*

$$\frac{1}{\mathbf{P}^Q(t)} = \frac{1}{\mathbf{P}^R(t)} + \frac{1}{\mathbf{P}^S(t)} - 1 - \phi(m,n)t^2$$

where $\phi(m,n)$ *is* 1 *when* $m,\ n \geq 2$, $\phi(1,1) = -1$, *and* $\phi(m,n) = 0$ *otherwise.*

(5) If $\ell\ell(R) > \ell\ell(S) \geq 2$, $Q^{\mathrm{g}} \cong R^{\mathrm{g}} \times_k S/\operatorname{soc}(S)^{\mathrm{g}}$.

3.4 Associated Graded Rings and Connected Sums

In this section, we explore the connections between associated graded rings and connected sums. In particular, we study conditions on the associated graded ring of a Gorenstein Artin k-algebra which force it to be a connected sum. We note that as an immediate consequence of Proposition 3.3.11, we get the following:

3.4.1 Corollary. *Let* R *and* S *be Gorenstein Artin* k*-algebras where* $\ell\ell(S) = 2$ *and* $\ell\ell(R) \geq 3$. *If* $Q \cong R\#_k S$, *then there is a surjective map* $\pi\colon G = Q^{\mathrm{g}} \twoheadrightarrow A = R^{\mathrm{g}}$ *such that* $\ker(\pi)_i = 0$, $i \geq 2$.

The main theorem we prove in this section (Theorem 3.4.7) shows that the converse of Corollary 3.4.1 is true when A itself is a graded Gorenstein ring.

With A and G as in Corollary 3.4.1, if A is further assumed to be graded Gorenstein, one can see that $\operatorname{soc}(G) \cap (G_+)^2 \cong \operatorname{soc}(A)$, which is a one-dimensional k-vector space. In fact, as we show in Proposition 3.4.3, these properties are equivalent. Hence, we introduce the following:

3.4.2 Definition. We say that a graded Artinian k-algebra G is *Gorenstein up to linear socle* if the socle of G in degree two and higher is a one-dimensional vector space over k, i.e., $\dim_k(\mathrm{soc}(G) \cap (G_+)^2) = 1$.

In Theorem 3.4.7, we prove that if the associated graded ring of a Gorenstein Artin local ring Q, with Loewy length at least 3, is Gorenstein up to linear socle, then Q can be decomposed as a connected sum. We begin with the following proposition which characterizes the property of being Gorenstein up to linear socle.

3.4.3 Proposition. *Let G be a graded k-algebra with $\ell\ell(G) \geq 2$. Then the following are equivalent:*

i) G is Gorenstein up to linear socle.

ii) There exist graded k-algebras A and B, where A is graded Gorenstein and $B_+^2 = 0$ such that $G \cong A \times_k B$.

iii) There exists a graded Gorenstein ring A and a surjective ring homomorphism $\pi\colon G \twoheadrightarrow A$ such that $\ker(\pi) \cap (G_+)^2 = 0$.

In particular, if (ii) holds, then we have $A \cong G/\langle \mathrm{soc}(G) \cap G_1 \rangle$, $\ell\ell(A) = \ell\ell(G)$, $\lambda(G) - \lambda(A) = \mathrm{edim}(G) - \mathrm{edim}(A) = \mathrm{type}(G) - 1 = \mathrm{edim}(B)$.

Proof. First of all, note that if G itself is Gorenstein, then the above statements are all trivially true. G satisfies (i), we can take $A = G$ in (iii), and write G as the trivial fibre product $G \times_k k$ in (ii). Hence, we may assume that G itself is not Gorenstein, i.e., $(\mathrm{soc}(G) + (G_+)^2)/(G_+)^2 \neq 0$.

(i) $\Rightarrow$ (ii): Let G be Gorenstein up to linear socle. Then we can find a k-basis, say $\{\overline{z_1}, \ldots, \overline{z_n}\}$, for $(\mathrm{soc}(G) + (G_+)^2)/(G_+)^2$, where $n = \mathrm{type}(G) - 1$. Extend this to a k-basis $\{\overline{y_1}, \ldots, \overline{y_m}, \overline{z_1}, \ldots, \overline{z_n}\}$ of $G_+/(G_+)^2$, and lift it to a minimal generating set $\{\boldsymbol{y}, \boldsymbol{z}\}$ of G_+ in G_1. Since $\langle \boldsymbol{y} \rangle \cap \langle \boldsymbol{z} \rangle = 0$ and $\langle \boldsymbol{y} \rangle + \langle \boldsymbol{z} \rangle = G_+$, it is easy to see that $G \cong A \times_k B$, where $A = G/\langle \boldsymbol{z} \rangle$ and $B = G/\langle \boldsymbol{y} \rangle$ are graded k-algebras.

Since $z_i \in \operatorname{soc}(G)$ for each i, their images in B, which are the generators of B_+, are in $\operatorname{soc}(B)$, in particular, $B_+ = \operatorname{soc}(B)$. Thus $B_+^2 = 0$. Furthermore, $\operatorname{type}(B) = n$ and $\operatorname{type}(G) = \operatorname{type}(A) + \operatorname{type}(B)$ implies $\operatorname{type}(A) = 1$, i.e., A is a graded Gorenstein k-algebra, proving (ii).

(ii) $\Rightarrow$ (iii): Suppose (ii) is true. By Remark 3.2.2(a), $A \cong G/J$,

where $J = \{(0, b) : b \in B_+\}$. Since $B_+^2 = 0$, (iii) holds.

(iii) $\Rightarrow$ (i): Assuming (iii), we see that $\pi|_{(G_+)^i} : (G_+)^i \longrightarrow (A_+)^i$ is an isomorphism for each $i \geq 2$. In particular, if $\operatorname{soc}(A) = (A_+)^s$, then $s = \ell\ell(A) = \ell\ell(G) \geq 2$ and $\dim_k((G_+)^s) = 1$.

Suppose $z \in \operatorname{soc}(G) \cap (G_+)^2$. Then $\pi(z) \in \operatorname{soc}(A) = (A_+)^s$. Since $s \geq 2$, this forces $z \in (G_+)^s$. Thus $(G_+)^s \subseteq \operatorname{soc}(G) \cap (G_+)^2 \subseteq (G_+)^s$, proving (i).

The last part follows from the proof above, the properties of fibre products and the facts that $\operatorname{type}(A) = 1$ and $\operatorname{edim}(B) = \operatorname{type}(B) = \lambda(B) - 1$. $\square$

In the following lemma, we record some properties of Gorenstein rings whose associated graded rings are Gorenstein up to linear socle.

3.4.4 Lemma. *Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin k-algebra and set $s = \ell\ell(Q)$. If $G = Q^{\mathfrak{g}}$ is Gorenstein up to linear socle but not Gorenstein, and $s \geq 3$, then the following hold:*

a) $(0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2 = \mathfrak{m}_Q^{s-1}$.

b) $(0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}$ *is a k-vector space of dimension* $\operatorname{type}(G) - 1$.

c) If $w \in (0 :_Q \mathfrak{m}_Q^2) \backslash \mathfrak{m}_Q^{s-1}$, then $w \in \mathfrak{m}_Q \backslash \mathfrak{m}_Q^2$, $w \cdot \mathfrak{m}_Q = \operatorname{soc}(Q)$ and $w^* \in \operatorname{soc}(G) \backslash (G_+)^2$.

Proof. By Proposition 3.4.3, there exists a graded Gorenstein ring A and a surjective ring homomorphism $\pi : G \twoheadrightarrow A$ such that $\ker(\pi) \cap (G_+)^2 = 0$. Note that the induced map $\pi : (G_+)^i \longrightarrow (A_+)^i$ is an isomorphism for each $i \geq 2$.

a) Let $w \in (0 :_Q \mathfrak{m}_Q^2)$. Then $\pi(w^*) \in (0 :_A (A_+)^2)$. Since A is graded Gorenstein, we have $(0 :_A (A_+)^2) = (A_+)^{s-1}$ and hence $\pi(w^*) \in (A_+)^{s-1}$. Suppose further $w \in \mathfrak{m}_Q^2$. Then $\deg(w^*) \geq 2$ in G. Since $\pi : G_i \longrightarrow A_i$ is an isomorphism for $i \geq 2$, $\pi(w^*) \in (A_+)^{s-1}$ forces $w^* \in (G_+)^{s-1}$, i.e., $w \in \mathfrak{m}_Q^{s-1}$. Thus $(0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2 \subseteq \mathfrak{m}_Q^{s-1}$. The other inclusion is clear since $\mathfrak{m}_Q^s = \operatorname{soc}(Q)$.

b) By (a),

$$\frac{(0 :_Q \mathfrak{m}_Q^2)}{\mathfrak{m}_Q^{s-1}} \cong \frac{(0 :_Q \mathfrak{m}_Q^2)}{(0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2} \cong \frac{(0 :_Q \mathfrak{m}_Q^2) + \mathfrak{m}_Q^2}{\mathfrak{m}_Q^2}$$

which is a k-vector space since $\mathfrak{m}_Q^2 \neq 0$.

Let $n = \lambda(\ker(\pi)) = \operatorname{edim}(Q) - \operatorname{edim}(A)$. Note that since $(0 :_Q \mathfrak{m}_Q^2)$ is the canonical module of $Q/\mathfrak{m}_Q^2$, $\lambda(0 :_Q \mathfrak{m}_Q^2) = \lambda(Q/\mathfrak{m}_Q^2)$.

Also, $\ker(\pi)_i = 0$ for $i \geq 2$ gives $\lambda(\mathfrak{m}_Q^i) = \lambda((G_+)^i) = \lambda((A_+)^i)$ for $i \geq 2$. Hence

$$\lambda((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \lambda(Q/\mathfrak{m}_Q^2) - \lambda((A_+)^{s-1}) = \lambda(Q/\mathfrak{m}_Q^2) - \lambda(A/(A_+)^2),$$

where the last equality follows from $\lambda((A_+)^{s-1}) = \lambda(A/(A_+)^2)$, which holds since A is a graded Gorenstein ring.

Thus, by Proposition 3.4.3,

$$\dim_{\mathsf{k}}((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \lambda((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \operatorname{edim}(Q) - \operatorname{edim}(A) = \operatorname{type}(G) - 1 = n.$$

c) If $w \in \mathfrak{m}_Q^2$, then $w \in \mathfrak{m}_Q^2 \cap (0 :_Q \mathfrak{m}_Q^2) = \mathfrak{m}_Q^{s-1}$ by (a). Hence $w \notin \mathfrak{m}_Q^2$ and so $w^* \notin (G_+)^2$. Now $w \cdot \mathfrak{m}_Q^2 = 0$ implies that $w\mathfrak{m}_Q \subseteq \operatorname{soc}(Q)$. Since Q is Gorenstein and $s \geq 3$, $\operatorname{soc}(Q) \subseteq w\mathfrak{m}_Q$ proving $w\mathfrak{m}_Q = \operatorname{soc}(Q) = \mathfrak{m}_Q^s$. In particular, since $s \geq 3$, $w^* \in \operatorname{soc}(G) \setminus (G_+)^2$. □

We now prove the following proposition, which is crucial in our proof of the main theorem.

3.4.5 Proposition. *Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin k-algebra and set $s = \ell\ell(Q)$. If $G = Q^{\mathfrak{g}}$ is Gorenstein up to linear socle but not Gorenstein, and $s \geq 3$, then there* *is an ideal J in Q such that the following hold:*

a) $J\mathfrak{m}_Q^2 = 0$, *and hence* $J\mathfrak{m}_Q \subseteq \operatorname{soc}(Q)$,

b) $\mu(J) = \operatorname{type}(G) - 1$,

c) $0 :_Q J + J = \mathfrak{m}_Q$,

d) $\mathfrak{m}_Q^r = (0 :_Q J)^r$ *for* $r \geq 2$ *and*

e) $\mu(0 :_Q J) + \mu(J) = \operatorname{edim}(Q)$, *and hence* $J^2 \neq 0$.

Proof. By Lemma 3.4.4(b), there exist elements $z_1, \ldots, z_n \in (0 :_Q \mathfrak{m}_Q^2) \setminus \mathfrak{m}_Q^{s-1}$, where $n = \operatorname{type}(G) - 1$, such that their images form a basis for $(0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}$. By Lemma 3.4.4(c), $\boldsymbol{z} \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$. Now (a) follows by setting $J = \langle z_1, \ldots, z_n \rangle$.

Notice that $\boldsymbol{z}$ is a part of a minimal generating set of $\mathfrak{m}_Q$, i.e., $\boldsymbol{z}$ are linearly independent modulo $\mathfrak{m}_Q^2$. Indeed, suppose $\sum_{i=1}^n a_i z_i \in \mathfrak{m}_Q^2$. Then $\sum_{i=1}^n a_i z_i \in \mathfrak{m}_Q^{s-1}$ by Lemma 3.4.4(a), and hence $\sum a_i \overline{z_i} = 0$ in $(0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}$. Since $\{\overline{z_1}, \ldots, \overline{z_n}\}$ is linearly independent, $a_i \in \mathfrak{m}_Q$ for all i. In particular, $\boldsymbol{z}$ forms a minimal generating set for J showing that $\mu(J) = \operatorname{type}(G) - 1$, which proves (b).

Let $I = 0 :_Q J$. If $w \in J \subseteq (0 :_Q \mathfrak{m}_Q^2)$, then $w \cdot \mathfrak{m}_Q^{s-1} = 0$. Hence, if $w \in J \cap I$, then $w \in (0 :_Q (J + \mathfrak{m}_Q^{s-1})) = (0 :_Q (0 :_Q \mathfrak{m}_Q^2)) = \mathfrak{m}_Q^2$ since Q is Gorenstein Artin. Thus $w \in \mathfrak{m}_Q^{s-1}$ by Lemma 3.4.4(a). Write $w = \sum_{i=1}^n a_i z_i$. Since $w \in \mathfrak{m}_Q^{s-1}$ and $\boldsymbol{z}$ are linearly independent modulo $\mathfrak{m}_Q^{s-1}$, $a_i \in \mathfrak{m}_Q$. Thus, $w \in J \cdot \mathfrak{m}_Q \subseteq \operatorname{soc}(Q)$. Therefore $J \cap I \subseteq \operatorname{soc}(Q)$. Since Q is Gorenstein and $J \neq 0 \neq I$, the other inclusion is clear, giving $J \cap I = \operatorname{soc}(Q)$. Finally, since $0 :_Q I = (0 :_Q (0 :_Q J)) = J$, we see that $I + J = \mathfrak{m}_Q$ by taking annihilators of both sides in Q, proving (c).

In order to prove (d), let A and $\pi : G \longrightarrow A$ be as in Proposition 3.4.3. Since $z_i \in 0 :_Q \mathfrak{m}_Q^2$, $\pi(z_i^*) \in 0 :_A (A_+)^2 = (A_+)^{s-1}$. But $z_i \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$ implies that $\deg(z_i^*) = 1$ in G. Hence either $\deg(\pi(z_i^*)) = 1$ or $\pi(z_i^*) = 0$ in A.

Since $\pi(z_i^*) \in (A_+)^{s-1}$, $\deg(\pi(z_i^*)) \neq 1$, forcing $\pi(z_i^*) = 0$ in A. Counting lengths, we see that $\ker(\pi) = \langle z_1^*, \ldots, z_n^* \rangle$.

Let $\{y_1, \ldots, y_m\}$ be a minimal generating set for I. Since $A_+ = \pi(G_+)$ and $\pi(z_i^*) = 0$, we get $A_+ = \langle \pi(y_1^*), \ldots, \pi(y_m^*) \rangle$. Thus $(A_+)^r = \langle \pi(y_1^*), \ldots, \pi(y_m^*) \rangle^r$ for each r. Therefore, the fact that $\pi : (G_+)^r \longrightarrow (A_+)^r$ is an isomorphism for $r \geq 2$ forces $(G_+)^r = \langle \boldsymbol{y}^* \rangle^r$ and hence $\mathfrak{m}_Q^r = I^r$ for $r \geq 2$.

Finally, observe that $J^2 = 0$ implies $J \subseteq 0 :_Q J$ and hence $0 :_Q J = \mathfrak{m}_Q$. Now, $\mu(J) = n \neq 0$ by assumption.

Hence, $J^2 \neq 0$ follows if we prove $\mu(I) + \mu(J) = \mu(\mathfrak{m}_Q) = \operatorname{edim}(Q)$.

Now $J\mathfrak{m}_Q = \operatorname{soc}(Q)$ implies that $\lambda(J) = \lambda(J/J\mathfrak{m}_Q) + 1 = n + 1$.

Furthermore, $I^2 \subseteq I\mathfrak{m}_Q \subseteq \mathfrak{m}_Q^2 = I^2$ forces $I\mathfrak{m}_Q = \mathfrak{m}_Q^2$. Since $I \cong 0 :_Q J$ is isomorphic to the canonical module of Q/J, we see that

$$\begin{aligned}
\mu(I) = \lambda(I/\mathfrak{m}_Q^2) = \lambda(I) - \lambda(\mathfrak{m}_Q^2) &= \lambda(Q/J) - \lambda(\mathfrak{m}_Q^2) = \lambda(Q/\mathfrak{m}_Q^2) - \lambda(J) \\
&= 1 + \lambda(\mathfrak{m}_Q/\mathfrak{m}_Q^2) - (n+1) = \operatorname{edim}(Q) - n = \operatorname{edim}(A),
\end{aligned}$$

which proves (e). $\square$

3.4.6 Remark. The following is an important observation which is hidden in the proof of (d) above: With hypothesis and notation as in Proposition 3.4.5,

$$\langle \operatorname{soc}(G) \cap G_1 \rangle = \langle z_1^*, \ldots, z_n^* \rangle.$$

We are now ready to state and prove the main theorem of this section.

3.4.7 Theorem. *Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin k-algebra and set $s = \ell\ell(Q)$. If $G = Q^{\mathrm{g}}$ is Gorenstein up to linear socle and $s \geq 3$, then there exist Gorenstein Artin local k-algebras R and S such that $Q \cong R \#_k S$, where*

a) $\ell\ell(S) \leq 2$, and if G is not Gorenstein, then $\ell\ell(S) = 2$.

b) $R^{\mathfrak{g}} \cong G/\langle \operatorname{soc}(G) \cap G_1 \rangle$*. In particular,* $\ell\ell(R) = s$ *and* $H_R(i) = H_Q(i)$ *for* $2 \leq i \leq s$*.*

c)

$$\frac{1}{\mathbf{P}^R(t)} = \frac{1}{\mathbf{P}^Q(t)} - \frac{1}{\mathbf{P}^S(t)} - \phi(t) + 1,$$

where $\phi(t)$ *is given as in Remark 3.2.6(b) and, when* $\ell\ell(S) = 2$*,* $\mathbf{P}^S(t)$ *is as in Remark 3.1.6(c).*

In particular, $\mathbf{P}^Q(t)$ *is rational in* t *if and only if* $\mathbf{P}^R(t)$ *is so.*

Proof. If G is Gorenstein, then one can take $R = Q$ and $S = k[Z]/\langle Z^2 \rangle$. Then the conclusions of the theorem are trivially true. Hence, we now assume that G is not Gorenstein.

By Proposition 3.4.5, we see that $\mathfrak{m}_Q = \langle y_1, \ldots, y_m, z_1, \ldots, z_n \rangle$,

where $n = \operatorname{type}(G) - 1 \geq 1$, $\boldsymbol{y} \cdot \boldsymbol{z} = 0$ and $\langle \boldsymbol{z} \rangle^2 \neq 0$. Hence we can write

$$Q \cong \widetilde{Q}/I_Q \text{ where } \widetilde{Q} = k[Y_1, \ldots, Y_m, Z_1, \ldots, Z_n]$$
$$\boldsymbol{Y} \cdot \boldsymbol{Z} \subseteq I_Q \subseteq \langle \boldsymbol{Y}, \boldsymbol{Z} \rangle^2$$
$$\langle \boldsymbol{Z} \rangle^2 \not\subseteq I_Q \supseteq \langle \boldsymbol{Z} \rangle^3.$$

Hence, if $I_R = I_Q \cap k[\boldsymbol{Y}]$ and $I_S = I_Q \cap k[\boldsymbol{Z}]$, then by Theorem 3.3.17, $R = k[\boldsymbol{Y}]/I_R$ and $S = k[\boldsymbol{Z}]/I_S$ are Gorenstein Artin such that $Q \cong R \#_k S$.

Now, $\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle^2 \cdot Z_j \subseteq I_Q$ for each j implies $\langle \boldsymbol{Z} \rangle^3 \subseteq I_S$ proving $\mathfrak{m}_S^3 = 0$. On the other hand, $\langle \boldsymbol{Z} \rangle^2 \not\subseteq I_Q$ and hence, $\mathfrak{m}_S^2 \neq 0$. Thus $\ell\ell(S) = 2$. This proves (a). Now (c) follows from Remark 3.1.6(c) and Remark 3.2.6(b).

Since $\ell\ell(Q) \geq 3 > \ell\ell(S)$, we see that $\ell\ell(R) = \ell\ell(Q) \neq \ell\ell(S)$. Hence, by (the proof of) Proposition 3.3.11, $G \cong R^{\mathfrak{g}} \times_k S/\operatorname{soc}(S)^{\mathfrak{g}}$. In particular, $R^{\mathfrak{g}} \cong G/\langle z_1^*, \ldots, z_n^* \rangle$. By Remark 3.4.6, we get $R^{\mathfrak{g}} \cong G/\langle \operatorname{soc}(G) \cap G_1 \rangle$, proving (b). $\square$

3.4.8 Corollary. *Let the hypothesis and notation be as in Theorem 3.4.7 and its proof. If we further assume that* G *is not Gorenstein, then* $\mu(I_Q) = \mu(I_R) + \binom{n+1}{2} + mn$ *when*

$m \geq 2$ and $\mu(I_Q) = \binom{n+1}{2} + n$ when $m = 1$.

Proof. By Theorem 3.4.7, since G is not Gorenstein, we have $\ell\ell(S) = 2$. Thus, if $n = \operatorname{edim}(S) \geq 2$, then by Remark 3.1.6(c) and Remark 3.1.5, $\mu(I_S) = \binom{n+1}{2} - 1$. Hence the proof follows from Proposition 3.3.4(b). $\square$

3.4.9 Corollary. *Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin k-algebra such that $G = Q^{\mathfrak{g}}$ is Gorenstein up to linear socle. If $\ell\ell(Q) \geq 3$, $\mathbf{P}^Q(t)$ is a rational function of t in the following situations:*

i) $\operatorname{edim}(Q) - \operatorname{type}(G) \leq 3$.

ii) $\lambda(Q) - \operatorname{type}(G) \leq 10$.

Proof. Note that by Theorem 3.4.7(a), there are Gorenstein Artin local rings R and S such that $Q \cong R\#_k S$, where $\ell\ell(S) \leq 2$ and $R^{\mathfrak{g}} \cong G/\langle \operatorname{soc}(G) \cap G_1 \rangle$.

Let $A = G/\langle \operatorname{soc}(G) \cap G_1 \rangle$. By Proposition 3.4.3, A is a graded Gorenstein Artin k-algebra such that $\ell\ell(A) = \ell\ell(Q)$, $\operatorname{edim}(A) = \operatorname{edim}(Q) - \operatorname{type}(G) + 1$ and $\lambda(A) = \lambda(Q) - \operatorname{type}(G) + 1$.

(ii) The assumption that $\lambda(Q) - \operatorname{type}(G) \leq 10$ implies that $\lambda(A) \leq 11$. Since A is graded Gorenstein, and hence has a palindromic Hilbert function, the hypothesis that $\ell\ell(Q) \geq 3$ forces $\operatorname{edim}(A) \leq 4$. Thus (ii) reduces to (i).

(i) In this case, $\operatorname{edim}(R) = \operatorname{edim}(A) \leq 4$. Hence, by Remark 3.1.6(a), $\mathbf{P}^R(t)$ is rational. Thus, by Theorem 3.4.7(b), $\mathbf{P}^Q(t)$ is rational. $\square$

3.5 Short and Stretched Gorenstein Rings

In her paper on stretched Gorenstein rings, Sally proved a structure theorem [34, 1.2] for a stretched Gorenstein local ring $(Q, \mathfrak{m}_Q, k)$ when $\operatorname{char}(k) \neq 2$. Elias and Rossi proved a similar structure theorem [19, 4.1] for a short Gorenstein local ring

$(Q, \mathfrak{m}_Q, k)$ when $\operatorname{char}(k) = 0$ and k is algebraically closed. In particular, if Q is a k-algebra where $\operatorname{char}(k) = 0$, then, if Q is either a short (with k algebraically closed) or a stretched (with $\ell\ell(Q) \geq 3$) Gorenstein Artin k-algebra, the structure theorems of Elias-Rossi and Sally respectively show that Q corresponds to a polynomial of the form $F(\boldsymbol{Y}) + Z_1^2 + \cdots + Z_n^2$ in terms of Macaulay's inverse systems. Thus, in the respective cases, this shows that Q is a connected sum, where one of the components is a Gorenstein ring with Loewy length equal to two.

Theorem 3.4.7 generalizes these two results, which can be seen as follows:

3.5.1 Proposition. *Let Q be either a short or a stretched Gorenstein ring with $\ell\ell(Q) = s \geq 3$.*

Then $G = Q^{\mathfrak{g}}$ is Gorenstein up to linear socle and,

$$\operatorname{edim}(Q) = H_Q(s-1) + \operatorname{type}(G) - 1.$$

Proof. Let Q_0 be the quotient of $G = Q^{\mathfrak{g}}$ as defined by Iarrobino (see Remark 3.1.3). Since the last two Hilbert coefficients of G and Q_0 are the same by Remark 3.1.3, and the Hilbert function of a graded Gorenstein k-algebra is palindromic, we see the following:

i) Let Q be a short Gorenstein ring with $H_Q = (1, h, n, 1)$. Then $H_{Q_0} = (1, n, n, 1)$.

ii) Let Q be a stretched Gorenstein ring with $H_Q = (1, h, 1, \ldots, 1)$.

Then $H_{Q_0} = (1, 1, 1, \ldots, 1)$.

Thus if we take A to be Q_0 in Proposition 3.4.3(ii), we see that G is Gorenstein up to linear socle, and the formula for $\operatorname{edim}(Q)$ holds since $\operatorname{edim}(A) = H_Q(s-1)$. □

3.5.2 Theorem. *Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin k-algebra with $\ell\ell(Q) \geq 3$. Then Q is stretched if and only if $G = Q^{\mathfrak{g}}$ is Gorenstein up to linear socle and* $\operatorname{type}(G) = \operatorname{edim}(Q)$.

In particular, Q is a connected sum with

$$\mathbf{P}^Q(t) = \frac{1}{1 - \operatorname{edim}(Q)t + t^2}$$

when $\operatorname{edim}(Q) \geq 2$.

Proof. Note that if Q is stretched, the required properties of G hold by Proposition 3.5.1.

For the converse, assume that G is Gorenstein up to linear socle with $\operatorname{edim}(Q) = \operatorname{type}(G)$. Then by Proposition 3.4.3, we have $G \cong A \times_k B$ with $\operatorname{edim}(B) = \operatorname{edim}(Q) - 1$ and $\operatorname{edim}(A) = 1$. Since $\ell\ell(Q) \geq 3$, Theorem 3.4.7(b) and (c) show that Q is stretched, and give the formula for $\mathbf{P}^Q(t)$. □

3.5.3 Remark. It is shown in [16] if $(Q, \mathfrak{m}_Q, k)$ is a short Gorenstein Artin k-algebra with Hilbert function $H_Q = (1, h, n, 1)$, then $\mathbf{P}^Q(t)$ is rational when $n \leq 4$, with the assumption that k is an algebraically closed field of characteristic zero. In Theorem 3.5.4, we show the same is true for $n \leq 4$ without any restrictions on k.

3.5.4 Theorem. *Let $(Q, \mathfrak{m}_Q, k)$ be a short Gorenstein Artin k-algebra with Hilbert function $H_Q = (1, h, n, 1)$. Then Q is a connected sum. Furthermore, if $n \leq 4$, then $\mathbf{P}^Q(t)$ is rational.*

Proof. By Proposition 3.5.1 and Theorem 3.4.7, there exist Gorenstein Artin local k-algebras R and S such that $Q \cong R \#_k S$, where $\ell\ell(S) \leq 2$ and $\operatorname{edim}(R) = n \leq 4$. The fact that $\mathbf{P}^Q(t)$ is rational follows from Corollary 3.4.9 and Theorem 3.4.7. □

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