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A Mathematical Commentary on Xenophon, Hellenika 2.1.25, 5.3.5 and Thucydides 3.2

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In Greek antiquity, gravity, actuated by archers on city walls, turned many a tide, as victorious besiegers routed a city's land forces, but in the excitement of pursuit, got too close to the city walls! Xenophon presents one such instance as but the most recent of many cases, in relating the death of Teleutias.

Teleutias was an enterprising Spartan general. The Ephors of Sparta, having complete faith in him, sent him to take charge of the war against Olynthus, on the northern shores of the Aegean Sea. Part of his forces was engaged in destroying the Olynthian gardens, farms, and orchards, as the others warded off Olynthian cavalry and light-armed troops who would try to prevent them.

Angered by the success of some Olynthian rangers against his own, he commanded his more mobile forces to pursue the Olynthians and to keep pursuing them, backing them up with an advance of hoplites marching in their line.

Their hot pursuit took them within shooting distance of the city walls. The defending archers, seeing their chance, waited for a bit of committing space, and then decimated the Spartan forces. As the Spartans withdrew, the Olynthians then sent out a counter-attack, and turned the withdrawal of the Spartan forces into a rout.

Xenophon, introducing the episode, observes that a "force pursuing up to a city often has difficulty getting away" (5.3.5). His epilogue to the story is that "a commander must never issue an order in the heat of anger" (5.3.7).

The foundation for the "difficulty getting away" when you have come too close to a city's walls is visible only through math.

Where the ancient Greeks best perceived the need for archers was when an expeditionary force came to them: if an ancient city knew a siege was facing them, what preparations would they make? As Mytilene prepares to secede from the Athenian Empire (428 BC), we see the city taking three preparations to undergo a siege: one is to buy grain, second is to raise the height of the walls, and the third was to bring in archers from Thrace (Thuc. 3.2).

The advantage in raising a city wall's height is also visible only through math. In a siege, the defenders always have the height advantage. They are throwing or shooting from the city walls, the offense is shooting from the ground. What is the height advantage? What does it mean in the real world?

Given that it was dangerous in Greek antiquity for an army to come too close to the walls of a city, and that, in fact, sometimes when an army came too close to a city, it could not get away, but was destroyed, we ask: given equal artillery on each side, how much farther would an arrow or missile go from the city wall than from, say, shoulder height?

Suppose an arrow is shot 5 feet off the ground, and another from the top of a wall at a total height of 25 feet. How much farther does the shot from the height go?

1. We shoot an arrow straight out. Suppose we are high enough it can fall as far as we want.
2. For mathematical convenience, it goes a constant speed, *n* feet per second (i.e. we ignore air resistance.)
3. On our planet, it falls with an acceleration of ~32 feet per second.

The arrows in our problem travel on two vectors, the horizontal at say 192 feet per second, and the vertical at 32 feet per second. So the range of the two arrows is simply a matter of how many seconds they get to go straight out at 192 feet per second before the acceleration of gravity brings them to the ground.

From any physics book, the formula for distance of fall is 0.5g *elapsed time squared, i.e.

s = 0.5g(t2) (it is traditional to write the distance as "s".)

Isolating time squared, we divide both sides by 16 (half g),

s/16 = t2

then take the square root on both sides to isolate the time:

$\sqrt{s/16} = t$

Application to Our Problem

We now have the tool to answer our original question: suppose an arrow is shot from 5 feet off the ground, and then from the top of a wall at a total height of 25 feet. How much farther does the shot from the height go?

Where s = 5, we solve

$\sqrt{5/16} = t$

and find the drop takes 0.559 seconds. Times 192 feet per second, the arrow of the archer on the ground goes 107 feet. Unfortunately for him, where s = 25 feet, the drop takes 1.25 seconds, and the arrow straight out from the city wall goes 1.24*192, i.e. 240 feet!

Thus in our example, there would be a belt around the city 133 feet wide in which the besieger is defenceless but at risk!

For a general rule about the height advantage, it is simpler to do the ratio first, and solve once. The 0.5 g is the same in both, unchanged, and the only thing affecting the varying magnitude under the radical is *s*, which in our problem was the height of the shooter's bow. *Hence, given identical "muzzle velocity", the ratio of the two ranges is the ratio of the square root of their heights:*

$r : R :: \sqrt{s/16} \; ; \sqrt{S/16}$

Cancelling out the 16,

$r : R : \sqrt{s} : \sqrt{S}$

$r/R = \sqrt{s/S}$

Examples: If you are shooting from twice as high, your arrow goes 1.414 times as far; from five times as high, 2.24 times as far; ten times as high, 3.16 times as far. That is, if you are on a battlement 50 feet high, and your opponent is shooting from five feet high, your arrow goes 3.16 (=square root of 10) times farther than his. This is purely mathematical, ignoring aerodynamics. It is the consequence of the simple fact that the acceleration due to the force of gravity is exponential. A final illustration: if the defender is shooting from 50 feet high, his straight-out arrow would go 338 feet, and the besieger loosing arrows from shoulder height would have to go 231 feet inside this range before his arrow would reach the wall. It is now plain to see why a "force pursuing up to a city often has difficulty getting away."

Three consequences, among many, are the loss of a war, the development of the next generation of military technology, and the most famous wound in Classical Greek history.

Everyone knows how Athens lost the Peloponnesian War: Lysander took Lampsacus, a city in the Dardanelles Straits; the Athenian navy took a position on open beach opposite Lampsacus, and was surprised and destroyed one afternoon when they thought the campaigning was done for the day. But there is more. Xenophon likes to present a military situation in a speech, and then let the event happen in accordance with the speech. Here Xenophon assigns the situation to Alcibiades:

"Alcibiades, looking down from his walls, saw the Athenians settling on open beach, and *not in front of a city*, yet supplying themselves from Sestus, 15 stades from their ships; the enemy in a harbor, *before a city*, having everything, so he told them they were not settled in a fair spot, and should shift to Sestus, up to a harbor, *up to a city*, from which they could fight whenever they wanted." (2.1.25. Translation mine, emphasis added.) Of course the Athenian commanders, Tydeus and Menander, told Alcibiades to go away. But Xenophon, experienced general himself, has told us how they lost the entire war: by throwing away the advantage of covering archery from the heights of a city wall. The Spartan fleet, pursuing all the way to a city wall instead of all the way to the open beach, would have suffered the same fate as Teleutias, and for the same reason.

The bow, among the Greeks, was the principal weapon for the city besieged. The bow being so effective in this situation explains why the an early advance in ancient siege machinery was the movable tower. It is an anti-gravity machine! Its purpose was to zero out the gravity advantage of arrows from heights! According to Vitruvius (10.13) Diades, Alexander's engineer, claimed the invention of the movable tower. You build it out of range, as high as the city walls, or even higher, armor the front with hides, move it up, and give your archers a fair chance to clear the city walls. Here, for once, is a situation where archers fighting archers is the main event in ancient Greece.

Though siege-towers and other siege-works were constructed out of range, the range of an arrow from the height of a city wall could always be surprising. One instance is Philip II, king of Macedon (359-336 BC) and father of Alexander the Great. He was besieging Methone in 354 when he got his most famous wound: an arrow from the city walls knocked an eye out (Diod. 16.34). He survived, disfigured. Had he known the ratio of height to range, he would, presumably, have been standing farther away.

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