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Tame Filling Functions and Closure Properties

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TAME FILLING FUNCTIONS AND CLOSURE PROPERTIES

by

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TAME FILLING FUNCTIONS AND CLOSURE PROPERTIES

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Let G be a group with a finite presentation $\mathcal{P} = \langle A|R\rangle$ such that A is inverse- closed. Let $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be a nondecreasing function. Loosely, f is an intrinsic tame filling function for $(G, \mathcal{P})$ if for every word w over A^* that represents the identity element in G, there exists a van Kampen diagram Δ for w over $\mathcal{P}$ and a continuous choice of paths from the basepoint $*$ of Δ to points on the boundary of Δ such that the paths are steadily moving outward as measured by f. The isodiametric function (or intrinsic diameter function) introduced by Gersten and the extrinsic diameter function introduced by Bridson and Riley are useful invariants capturing the topology of the Cayley complex. Tame filling functions are a refinement of the diameter functions, which were introduced by Brittenham and Hermiller and are used to gain insight on how wildly maximum distances can occur in van Kampen diagrams. Brittenham and Hermiller showed that tame filling functions are a quasi-isometry invariant and that if f is an intrinsic (respectively extrinsic) tame filling function for $(G, \mathcal{P})$, then $(G, \mathcal{P})$ has an intrinsic (respectively extrinsic) diameter function equivalent to the function $n \mapsto \lceil f(n) \rceil$. In contrast to diameter functions, it is unknown if every pair $(G, \mathcal{P})$ has a finite-valued tame filling function.

In this thesis I show that two group constructions, namely graph products (a generalization of direct and free products) and certain free products with amalgamation, preserve finite-valued intrinsic tame filling functions.

DEDICATION

To my parents, Nabeeh and Zakiyyah Nu'Man.

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Contents

Chapter 1

Introduction

In 1911 Dehn [6] posed three algorithmic problems for finitely presented groups: the word problem, the conjugacy problem, and the isomorphism problem. The word problem for a group asks if there is an algorithm which, given a "word" in the generators of the group, determines whether or not the word represents the trivial element. It is known that the word problem for finitely presented groups is algorithmically unsolvable in general, and proofs can be found in [1] and [13]. One way geometric group theorists approach decision problems is by studying group properties that are invariant under quasi-isometries, and in particular filling functions. Examples of filling functions include the Dehn function (or isoperimetric function), the isodiametric function (also called intrinsic diameter function), and the extrinsic diameter function [2], [3]. Gersten showed that if any of these filling invariants are computable, then the word problem has a solution [7].

Filling functions for a finitely presented group record geometric properties of van Kampen diagrams with respect to a given presentation. Filling functions that are invariant under group presentation are of particular interest because they display underlying algorithmic and algebraic properties of a group, and in [7] Gersten showed that the isoperimetric and isodiametric (intrinsic diameter) functions are quasi-isometry invariant up to equivalence of functions. In [3], Bridson and Riley show that the extrinsic diameter function is also a quasi-isometry invariant up to equivalence of functions. Therefore, it makes sense to say a finitely presented group satisfies a linear, quadratic, polynomial, exponential, etc., isoperimetric (respectively isodiametric and extrinsic diameter) function. Furthermore, in [7] Gersten showed that if f is an isoperimetric function for $(G, \mathcal{P})$ then $g(n) = Mf(n) + n$ is an isodiametric function for a group g with presentation $\mathcal{P}$, $(G, \mathcal{P})$, where M is the maximum length of a relator in $\mathcal{P}$. Conversely, Cohen in [5] and Gersten in [7] showed that if f is an isodiametric function for $(G, \mathcal{P})$, then there exists positive constants $a, b > 0$ such that $g(n) = a^{b^{f(n)+n}}$ is an isoperimetric function for $(G, \mathcal{P})$.

Bridson and Riley [3] showed that the intrinsic diameter (isodiametric) function is an upper bound for the extrinsic diameter function. Additionally, in [3] they construct the first example of a group G, with presentation $\mathcal{P}$, for which we have positive constants $a, b > 0$ such that the extrinsic diameter function is bounded above by $n \mapsto n^b$ with the intrinsic diameter function at least $n \mapsto n^{a+b}$.

The intrinsic and extrinsic diameter functions bound maximum distances within a van Kampen diagram and Cayley complex, respectively, but do not give insight to the manner in which maximum distances can occur. In [4], Brittenham and Hermiller define tame filling functions as a refinement of diameter filling functions in an effort to understand how wildly maximum distances can occur and to gain traction on the word problem. For a finitely presented group G with presentation $\mathcal{P}$, an intuitive description of an intrinsic tame filling function (respectively, extrinsic tame filling function) is a nondecreasing function f such that for every word w that represents the identity element in G, there exists a van Kampen diagram Δ with base point $*$ and a continuous choice of paths from $*$ to points on the boundary of Δ such that the paths are steadily moving outward as measured by f. A complete definition of a tame filling function is given in Definition 2.5 on page 10. In [4], Brittenham and Hermiller show that almost convex groups, groups that admit a finite complete rewriting system, and more generally stackable groups, all have a well-defined intrinsic (respectively, extrinsic) tame filling function.

Chapter 2 provides background and definitions used throughout the paper, including a minor excursion on diameter functions and the formal definition of tame filling functions. Furthermore, in Chapter 2 we present known results about diameter functions, tame filling functions, and how they relate to each other.

Let $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be a function. We say f is *subnegative* if for all n, m in $\mathbb{N}[\frac{1}{4}]$ we have $f(n) + f(m) \leq f(n + m)$.

Definition 1.1. *Let* $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ *be any function. The* subnegative closure of f, *denoted by* $\overline{f}$, *is the least subnegative function greater than or equal to* f

The subnegative closure of f can be expressed by

$$\overline{f}(n) = \max\left\{ \sum_{i=1}^{k} f(n_i) \mid \sum_{i=1}^{k} n_i = n \right\}.$$

In Chapter 3, we show that the class of groups with finite-valued tame filling functions is closed under taking graph products. Given a finite simple graph Λ and a collection of finitely presented groups $\{G_i\}_{i=1}^{m}$ associated to the vertices of Λ, the *graph product of groups* $\{G_i\}_{i=1}^{m}$ associated to Λ is the quotient $G\Lambda$ of the free product of the groups of G_i by the normal closure of the set $\{[g_i, g_j] = 1 | (v_i, v_j) \in E(\Lambda) \text{and } g_i \in G_i, g_j \in G_j\}$. We make use of the subnegative closure in proving the following result for graph products.

Theorem 3.5 *Let* Λ *be a finite simple graph. Let* $\{G_i\}_{i=1}^m$ *be a family of finitely generated groups associated to the vertices of* Λ *with presentations* $\mathcal{P}_i = \langle A_i | R_i \rangle$*. Let* $h_i : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ *be an intrinsic tame filling function for* $(G_i, \mathcal{P}_i)$ *for* $1 \leq i \leq m$*. Then* $h : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ *defined by*

$$h(n) = \sum_{i=1}^{m} \overline{h_i}(n) + n$$

is equivalent to an intrinsic tame filling function for the graph product $G\Lambda$ *of the groups* $\{G_i\}_{i=1}^m$ *, with respect to the presentation*

$$\mathcal{P} = \langle \bigcup_{i=1}^{m} A_i \mid \bigcup_{i=1}^{m} R_i \cup \{[g_i, g_j] = 1 \mid g_i \in G_i, g_j \in G_j \text{ and } G_i, G_j$$
$$\text{are associated to adjacent vertices in } \Lambda\} \rangle.$$

The work of Meier in [11] gives a bound on the isodiametric function for a graph product in terms of the isodiametric functions of the respective vertex groups. The proof of Theorem 3.5 above extends Meier's proof on a bound for the isodiametric function using extra information contained in tame filling invariants.

Given a group $G = \langle A|R \rangle$ with subgroup H, a set S of words over A^* is a *left transversal for* H *in* G if and only if the map $s \mapsto sH$ from S to G/H is a bijection. A left transversal S is *geodesic* if all the words in S label geodesics in the associated Cayley graph of G. In Chapter 4 we show that the class of tame filling functions is closed under taking certain free products with amalgamation and prove the following result.

Theorem 4.1. *Let* f_α *and* f_β *be intrinsic tame filling functions for* $G_\alpha = \langle A_\alpha | R_\alpha \rangle$ *and* $G_\beta = \langle A_\beta | R_\beta \rangle$*, respectively. Let* $H = \langle h_1, \ldots h_m \rangle$ *be a finitely generated group and let* $\hat{\alpha} : H \hookrightarrow G_\alpha$ *and* $\hat{\beta} : H \hookrightarrow G_\beta$ *be injective homomorphisms. Let* $G = G_\alpha *_H G_\beta$ *with presentation*

$$\mathcal{P} = \langle A_\alpha, A_\beta, h_1, \ldots, h_m | \ R_\alpha \cup R_\beta \cup \{h_i = \hat{\alpha}\left(h_i\right), h_i = \hat{\beta}\left(h_i\right) \ \forall\ 1 \leq i \leq m\}\rangle.$$

Suppose that there is a prefix-closed set of geodesic normal forms for G *of the form*

$$\mathcal{N} = \{t_1 t_2 \ldots t_n y_h | \ t_i \text{ alternates between } T_\alpha \backslash \epsilon \text{ and } T_\beta \backslash \epsilon, \ y_h \in \mathcal{N}_H\},$$

where, for $\gamma \in \{\alpha, \beta\}$*,* T_γ *is a set of geodesic left transversals for* $G_\gamma / \hat{\gamma}\left(H\right)$*, containing the empty word, with respect to the following presentation for* G_γ

$$\mathcal{P}_\gamma = \langle A_\gamma, h_1, \ldots, h_m | \ R_\gamma \cup \{h_i = \hat{\gamma}\left(h_i\right) \ \forall\ 1 \leq i \leq m\}\rangle$$

and $\mathcal{N}_H$ *is a set of prefix-closed geodesic normal forms for* H*. Then* $(G, \mathcal{P})$ *has an intrinsic tame filling function equivalent to* $f(n) = \overline{f_\alpha}(n) + \overline{f_\beta}(n) + n$*.*

Chapter 2

Background

Let G be a finitely presented group with presentation $\mathcal{P} = \langle A|R\rangle$, where A is a finite, inverse closed, generating set for G and R is a finite set of defining relations closed under inverses and cyclic conjugation. We will always assume that our generating set A is closed under inversion. A *word* w over A is a finite concatenation of letters from the generating set A, and the *length* of w, denoted $\ell(w)$, is the number of letters in the word. If two words u and v represent the same group element, then we write $u =_G v$. If they are identically the same word, then we write $u = v$.

The *Cayley graph* of G with respect to the presentation $\mathcal{P}$, denoted $\Gamma(G, A)$, is the graph whose vertices are elements of G with an oriented edge from vertex g to vertex h labeled by a generator $a \in A$ if and only if $h =_G ga$ in G. By attaching a 2-cell, labeled by $r \in R$, at each vertex in $\Gamma(G, A)$ we obtain the *Cayley complex* $X(G, A)$ for G. Let $V(\Gamma(G, A))$ and $E(\Gamma(G, A))$ denote the vertex set and directed edge set of the Cayley graph $\Gamma(G, A)$, respectively. Additionally, let $e(g, a)$ denote the directed edge in $\Gamma(G, A)$ from vertex g labeled by a. Given an edge e, we denote the initial and terminal vertices of e as $\text{int}(e)$ and $\text{ter}(e)$, respectively. Identify each edge of $\Gamma(G, A)$ with the unit interval $[0, 1]$. Given elements g and h in G, the *word* *metric* on $\Gamma(G, A) = X(G, A)^{(1)}$, denoted d_Γ, is the length of the shortest path in $\Gamma(G, A)$ from vertex g to vertex h.

2.1 Diameter functions

Given a word $w \in A^*$ satisfying $w =_G \epsilon$, a *van Kampen diagram* Δ for w over the presentation $\mathcal{P}$ is a connected, simply connected, planar 2-complex with a vertex $*$ on the boundary of Δ, $\partial\Delta$, such that the $\partial\Delta$ is labeled by the word w starting at $*$and for every 2-cell σ of Δ, the label of $\partial\sigma$ is a word in R. For any van Kampen diagram Δ let d_Δ denote the path metric on the 1-skeleton of Δ. Furthermore, let $\pi_\Delta : \Delta \to X(G, A)$ be the canonical map such that $\pi_\Delta(*) = \epsilon$ and n-cells are mapped to n-cells preserving both label and direction.

To display the relationship between diameter functions and tame filling functions we provide the definition of the intrinsic diameter function (or isodiametric function, originally defined in [7]) and extrinsic diameter function defined by Bridson and Riley in [3].

Definition 2.1. *Given a group G the intrinsic (respectively, extrinsic) diameter function, with respect to the presentation $\mathcal{P}$, is the minimal nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ such that for all $w \in A^*$ with $w =_G \epsilon$, there exists a van Kampen diagram Δ for w over $\mathcal{P}$ such that for all vertices $v \in \Delta^{(0)}$ we have $d_\Delta(*, v) \leq f(\ell(w))$ (respectively, $d_X(\epsilon, \pi_\Delta(v)) \leq f(\ell(w))$).*

Since $\ell(w) \in \mathbb{N}$ and A is a finite set, there are finitely many words w such that $w =_G \epsilon$ and $\ell(w) \leq n$. This gives us that there is a minimal value for $f(n)$, and so the intrinsic and extrinsic diameter functions are well-defined. As stated in the introduction, in [3] Bridson and Riley showed that the extrinsic diameter function is a lower bound for the intrinsic diameter function and that there exists a group G for which we have positive constants $a, b > 0$ such that the extrinsic diameter is bounded above by $n \mapsto n^b$ with intrinsic diameter function at least $n \mapsto n^{a+b}$.

2.2 Tame filling functions

To reach a formal definition of a tame filling function for $(G, \mathcal{P})$, given a word $w =_G \epsilon$ and a van Kampen diagram Δ for w with respect to the presentation $\mathcal{P}$ we need a continuous choice of paths from the base point $*$ to points on the $\partial\Delta$. The continuous choice of paths used is called a 1-combing and was originally defined by Mihalik and Tschantz in [12]. We adopt the following definitions from [4].

Definition 2.2. [12] *Let Y be a combinatorial 2-complex and Z a subcomplex of $Y^{(1)}$. A* 1-combing *of the pair (Y, Z) at a basepoint $y_0 \in Y$ is a continuous map $\Psi : Z \times [0, 1] \to Y$ such that:*

(c1) $\Psi(p, 0) = y_0$ *and* $\Psi(p, 1) = p$ *for all* $p \in Y$,

(c2) When $y_0 \in Z$ *we have* $\Psi(y_0, t) = y_o$ *for all* $t \in [0, 1]$, *and*

(c3) When $p \in Z^{(0)}$, *we have* $\Psi(p, t) \in Y^{(1)}$ *for all* $t \in [0, 1]$.

Considering the unit interval $[0, 1]$ as an interval of time, a tame filling function f bounds the manner in which the continuous paths of a 1-combing are able to travel outward, away from the basepoint. In order to measure the tameness of the paths in a 1-combing, we need to have a way to measure distances to points within $\Gamma(G, A)$ and 2-cells in $X(G, A)$. This is done by using the following *coarse distance*, which was introduced in [4].

Definition 2.3. [4] *Let Y be a combinatorial 2-complex with basepoint y and let p be any point in Y. The* coarse distance*, denoted $\tilde{d}_Y(y,p)$, is defined as follows.*

- *For $p \in Y^{(0)}$, let $\tilde{d}_Y(y,p) = d_Y(y,p)$ where d_Y denotes the path metric on $Y^{(1)}$;*
- *For p in the interior of the edge e in Y with endpoints* $\mathrm{int}(e)$ *and* $\mathrm{ter}(e)$*, let*
$$\tilde{d}_Y(y,p) = \min\{d_Y(y,\mathrm{int}(e)), d_Y(y,\mathrm{ter}(e))\} + \frac{1}{2};$$
- *For p in the interior of a 2-cell σ in Y, let*
$$\tilde{d}_Y(y,p) = \max\{d_Y(y,q)\ |q \in \partial\sigma\backslash Y^{(0)}\} - \frac{1}{4}.$$

One goal for defining tame filling functions is to measure how wildly or tamely maximum distances can occur in a van Kampen diagram and Cayley complex. A sense of tameness needs to be established; this is done with the following tameness condition, introduced in [4].

Definition 2.4. [4] *A 1-combing Ψ of the pair (Y, Z) at the basepoint $y_0 \in Y$ is f-*tame *for a nondecreasing function $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ if for all $p \in Z$ and $0 \le s < t \le 1$ we have*
$$\tilde{d}_Y(y_0, \Psi(p,s)) \le f\left(\tilde{d}_Y(y_0, \Psi(p,t))\right).$$

The f-tame condition requires that if a path reaches a distance greater than $f(n)$ (from the basepoint) at a time s, then it cannot return to points within distance n (from the basepoint) at a later time t. For the pair $(G, \mathcal{P})$ a *combed filling* is a set $\mathcal{F} = \{(\Delta_w, \Psi_w) | w \in A^*, w =_G \epsilon\}$, where Δ_w is a van Kampen diagram for w over $\mathcal{P}$ and Ψ_w is a 1-combing of $(\Delta_w, \partial\Delta_w)$ at the basepoint $*$. Using combed fillings and the f-tame condition, Brittenham and Hermiller [4] defined an intrinsic (respectively, extrinsic) tame filling function as follows.

Definition 2.5. [4] *A nondecreasing function* $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ *is an* intrinsic *[respectively,* extrinsic*]* tame filling function *for the group* G *with presentation* $\mathcal{P} = \langle A|R\rangle$ *if for every word* $w \in A^*$ *with* $w =_G \epsilon$*, there exists a combed filling* $\mathcal{F}$ *such that the 1-combing* Ψ_w *of* $(\Delta_w, \partial D_w)$ *at the basepoint* $*$ *[respectively,* $\pi_\Delta \circ \Psi$ *of* $(X(G,A), \Gamma(G,A))$ *at the basepoint* ϵ*] is* f*-tame.*

Tame filling functions are quasi-isometry invariant up to equivalence [4], where we say two functions $f, g : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ are *equivalent*, $f \sim g$, if there exists a constant C such that for all $n \in \mathbb{N}[\frac{1}{4}]$ we have $f(n) \leq Cg(Cn+C)+C$ and $g(n) \leq Cf(Cn+C)+C$. If $f \sim g$ then we also have $\overline{f} \sim \overline{g}$, where $\overline{f}, \overline{g}$ are the subnegative closures of f and g, respectively.

Since tame filling functions are a refinement of the intrinsic and extrinsic diameter function, it is important to point out a few observations. Tame filling functions are defined on $\mathbb{N}[\frac{1}{4}]$ to account for how distances are measured under the coarse distance in Definition 2.2. Furthermore, tame filling functions do not depend on the length of w, as diameter functions do, and therefore it is not known whether or not every group G with finite presentation $\mathcal{P}$ admits a finite-valued intrinsic (respectively, extrinsic) tame filling function. It is known from [[4], Corollary 4.5] that if $(G, \mathcal{P})$ has a well-defined extrinsic tame filling function then $(G, \mathcal{P})$ is tame combable as defined by Mihalik and Tschantz in [12]. Lastly, Brittenham and Hermiller in [[4], Proposition 3.2] show that if f is an intrinsic (respectively extrinsic) tame filling function for $(G, \mathcal{P})$, then the function $\hat{f}(n) = \lceil f(n) \rceil$ is an upper bound for the intrinsic (respectively extrinsic) diameter function for $(G, \mathcal{P})$.

2.3 Edge combings

Tame filling functions require a combed filling in which the provided 1-combings are f-tame. The following definitions and results, from [4], allow us to reduce the type of van Kampen diagrams used to ensure a finitely presented group admits an intrinsic (respectively, extrinsic) tame filling function. This enables us to establish the existence of a tame filling function by looking at fewer van Kampen diagrams.

For a group G with finite generating set A and Cayley graph $\Gamma(G, A)$, a set of *normal forms* for G over A is a choice of one reduced word y_g in A^* for each group element g of G with $y_g =_G g$. Let $\mathcal{N} = \{y_g | g \in G\} \subseteq A^*$ be a set of normal forms (including the empty word) for $G = \langle A|R\rangle$ that label simple paths in $X(G, A)$. An *$\mathcal{N}$-diagram* is a van Kampen diagram Δ for the word $y_g a y_{ga}^{-1}$ for some $g \in G$ and $a \in A$. Let v_g denote the terminal vertex on the boundary of Δ of the path on the boundary of Δ starting at the basepoint $*$ labeled by y_g. Let $\hat{e}$ denote the edge in $\partial\Delta$ with initial vertex v_g labeled by a and let u_g be the terminal vertex of $\hat{e}$.

Definition 2.6. [4] *Let Δ be an $\mathcal{N}$-diagram for $G = \langle A|R\rangle$. An* edge 1-combing *of Δ is a continuous function $\Theta : \hat{e} \times [0, 1] \to \Delta$ such that*

(e1) $\Theta(p, 0) = *$ *and* $\Theta(p, 1) = p$ *for all* $p \in \hat{e}$,

(e2) if $*$ *is in* $\hat{e}$ *then* $\Theta(*, t) = *$ *for all* $t \in [0, 1]$*, and*

(e3) $\Theta((v_g), \cdot)$, $\Theta((u_g), \cdot) : [0, 1] \to \Delta$ *follow the paths in $\partial\Delta$ labeled by y_g and y_{ga} from $*$, respectively.*

Definition 2.7. [4] *A* combed $\mathcal{N}$-filling *is a collection* $\mathcal{E} = \{(\Delta_e, \Theta_e) | e \in E(X(G,A))\}$ *such that for each* $e = e(g,a)$ *for some* $g \in G$ *and* $a \in A$*,* Δ_e *is an* $\mathcal{N}$*-diagram and* Θ_e *is an edge combing of* Δ_e *that satisfies the following gluing condition:*

(g1) For every pair of edges e *and* e' *in* $E(X(G,A))$ *with a common endpoint* g *and for all* $t \in [0,1]$ *we have* $\pi_{\Delta_e} \circ \Theta_e (\hat{g}_e, t) = \pi_{\Delta_{e'}} \circ \Theta_{e'} (\hat{g}_{e'}, t)$*, where* $\hat{g}_e$ *and* $\hat{g}_{e'}$ *are the vertices of* Δ_e *and* $\Delta_{e'}$*, respectively, that map to* g *in* X*.*

A combed $\mathcal{N}$-filling is *geodesic* if each word in the set of normal forms $\mathcal{N}$ labels a geodesic path in the associated Cayley graph. The following result from [4] connects combed $\mathcal{N}$-fillings and combed fillings.

Lemma 2.8. [[4], Proposition 4.3] *Let* G *be a finitely presented group with presentation* $\mathcal{P} = \langle A | R \rangle$*, and let* $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ *be a nondecreasing function. The following are equivalent, up to equivalence of functions:*

1. f *is an intrinsic [respectively, extrinsic] tame filling function for* $(G, \mathcal{P})$*.*
2. $(G, \mathcal{P})$ *has a geodesic combed* $\mathcal{N}$*-filling* $\mathcal{E} = \{(\Delta_e, \Theta_e) | e \in E(X(G,A))\}$ *such that each edge 1-combing* Θ_e *[respectively,* $\pi_{\Delta_e} \circ \Theta_e$*] is* f*-tame.*

Remark 2.9. *In the proof of Lemma 2.8, (1) implies (2), uses shortlex normal forms.*

We utilize Lemma 2.8 in the proofs of Theorem 3.5 and Theorem 4.1 in the following sections.

Chapter 3

Graph products

In this section I prove that the tame filling property is preserved under the graph product construction.

Let Λ be a finite graph with no loops and no multiple edges. Let $V(\Lambda) = \{v_1, \dots v_m\}$ and $E(\Lambda)$ denote the vertex set and edge set of Λ, respectively. Let $e(v_i, v_j)$ denote the edge in Λ between vertices v_i and v_j. For each vertex v_i we associate a finitely presented group G_i. The *graph product of groups* $\{G_i\}_{i=1}^m$ associated to Λ is the quotient, $G\Lambda$, of the free product of the groups G_i by the normal closure of the set $\{[g_i, g_j] = 1 \mid e(v_i, v_j) \in E(\Lambda) \text{ and } g_i \in G_i, g_j \in G_j\}$. Graph products were originally defined by Green in [8] and are a generalization of direct products (in which Λ is a complete graph) and free products (in which Λ is a graph with no edges).

For each vertex group G_i, let $\mathcal{P}_i = \langle A_i | R_i \rangle$ be a presentation for G_i. We may assume that each set R_i does not contain any relations of the form $a_j = a_k$ or $a_j = 1_{G_i}$, for generators $a_j, a_k \in A_i$. If it does, we can use Tietze transformations to obtain a presentation for G_i with either a_j or a_k removed. Let $A = \bigcup_{i=1}^m A_i$ and $R = \bigcup_{i=1}^m R_i$.

We can present $G\Lambda$ as

$$G\Lambda = \langle A \mid R \cup \{[a_i, a_j] = 1 \ \forall \ a_i \in A_i, a_j \in A_j \text{ such that } (v_i, v_j) \in E(\Lambda)\}\rangle \,. \qquad (3.1)$$

For the family of groups $\{G_i\}_{i=1}^m$, let $<_G$ be a total ordering on the finite set of groups. The following definitions and results of Hermiller and Meier are in [[9], pg. 9 and 10].

Definition 3.1. [9] *Given a word w in A^*, a subword w' of w is a* local word *if it is written in letters coming from a single vertex group and there is no longer subword of w, containing w', written in letters coming from a single vertex group.*

Definition 3.2. [9] *Given sets of normal forms $\mathcal{N}_i$ for G_i over A_i, a word w in A^* is* proper *if it satisfies the following local condition and obstruction condition:*

(L) Each local word w_i over A_i^* of w is in the normal form set $\mathcal{N}_i$.

(O) If $w = \ldots w_i \ldots w_j \ldots$, where w_i and w_j are local words in A_i^* and A_j^*, respectively, with $G_j \leq_G G_i$ and v_i is adjacent to v_j in Λ, then there is a local word w_k such that $w = \ldots w_i \ldots w_k \ldots w_j \ldots$, where v_k and v_j are non-adjacent in Λ.

Lemma 3.3. [[9], Proposition 3.2] *Let $G\Lambda$ be the graph product of a finite set of groups $\{G_i\}_{i=1}^m$ each with normal forms $\{\mathcal{N}_i\}_{i=1}^m$. Then the set of proper words in A^* is a set of normal forms for $G\Lambda$.*

Lemma 3.4. [[9], Proposition 3.3] *Let $G\Lambda$ be the graph product of a finite set of groups, all having geodesic normal forms. Then the normal forms of $G\Lambda$ given by the proper words are geodesic.*

Theorem 3.5. *Let Λ be a finite simple graph. Let $\{G_i\}_{i=1}^m$ be a family of finitely generated groups associated to the vertices of Λ with presentations $\mathcal{P}_i = \langle A_i | R_i \rangle$. Let $h_i : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ be an intrinsic tame filling function for $(G_i, \mathcal{P}_i)$ for $1 \leq i \leq m$. Then $h : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ defined by*

$$h(n) = \sum_{i=1}^{m} \overline{h_i}(n) + n$$

is equivalent to an intrinsic tame filling function for the graph product $G\Lambda$ of the groups $\{G_i\}_{i=1}^m$, with respect to the presentation

$$\begin{aligned}\mathcal{P} = \langle \cup_{i=1}^m A_i \mid \cup_{i=1}^m R_i \cup \{[g_i, g_j] = 1 \mid g_i \in G_i, g_j \in G_j \textit{ and } G_i, G_j \\ \textit{are associated to adjacent vertices in } \Lambda\}\rangle.\end{aligned}$$

Proof. Let $G\Lambda$ be a graph product of the family of groups $\{G_i\}_{i=1}^m$. Since h_i is an intrinsic tame filling function for $(G_i, \mathcal{P}_i)$ by Lemma 2.8 $(G_i, \mathcal{P}_i)$ has a geodesic combed $\mathcal{N}_i$-filling, $\mathcal{E}_i = \{(\Sigma_e, \Omega_e) \,| e \in E_{X_i}\}$, such that each edge 1-combing Ω_e is $\tilde{h}_i$-tame, where $\tilde{h}_i$ is equivalent to h_i, for $1 \leq i \leq m$. Since $f \sim g$ implies $\overline{f} \sim \overline{g}$, by a slight abuse of notation, we will simply write the function h_i, instead of $\tilde{h}_i$. Let $\mathcal{N}$ be the set of proper words as defined in Definition 3.2. Since the normal forms $\mathcal{N}_i$ are geodesic for $1 \leq i \leq m$, by Lemma 3.4 the set $\mathcal{N}$ is a set of geodesic normal forms for $G\Lambda$.

Using the normal form set $\mathcal{N}$ we will construct a geodesic combed normal $\mathcal{N}$-filling $\mathcal{E} = \{(\Delta_e, \Psi_e) | e \in E_X\}$ such that each edge 1-combing in Ψ_e is h-tame. For $g \in G\Lambda$, let y_g be the normal form for g in $\mathcal{N}$. Let $x \in A = \cup_{i=1}^m A_i$. This proof is divided into five parts: (1) factor the normal form y_g (2) compute the normal form for $y_g x$, (3) construct a van Kampen diagram Δ for $y_g x y_{yx}^{-1}$, (4) construct an edge 1-combing Ψ, and (5) show that Ψ is h-tame.

Fix $g \in G\Lambda$ and express $y_g = w_{\alpha_1} \dots w_{\alpha_k}$ where each w_{α_i} is a local word in $\mathcal{N}_{\alpha_i}$ for $1 \le \alpha_i \le m$. Without loss of generality, let $x \in A_{\alpha_j}$ for some $1 \le \alpha_j \le m$.

Part 1: Factorization of y_g.

First we will express y_g as $y_g = w_\rho w_\tau w_\sigma$. In $y_g x = w_{\alpha_1} \dots w_{\alpha_k} x$ shuffle the letter x to the left (past a local word w_{α_i}) whenever the corresponding vertex v_{α_i} is adjacent to v_{α_j} in the graph Λ. Once the letter x has been shuffled as far left as possible in $y_g x$ we then shuffle the letter x to the right of a local word w_{α_i} whenever the corresponding vertices are adjacent and $G_{\alpha_i} <_G G_{\alpha_j}$. This process will stabilize when we have $y_g x =_{G\Lambda} w_{\alpha_1} \dots w_{\alpha_s} x w_{\alpha_{s+1}} \dots w_{\alpha_k}$ for some $1 \le s \le k$ in which $G_{\alpha_s} <_G G_{\alpha_j} <_G G_{\alpha_{s+1}}$.

Let $w_\rho = w_{\alpha_1} \dots w_{\alpha_{s-1}}$, $w_\tau = w_{\alpha_s}$ and $w_\sigma = w_{\alpha_{s+1}} \dots w_{\alpha_k}$. It is possible to have w_ρ,w_τ, or w_σ be the empty word ϵ and so we have the following possible factorizations of y_g as $y_g = w_\rho w_\tau w_\sigma$.

(i.) $w_\rho = \epsilon$, $w_\tau = \epsilon$, and $w_\sigma = \epsilon$

(ii.) $w_\rho = \epsilon$, $w_\tau = \epsilon$, and $w_\sigma = w_{\alpha_1} \dots w_{\alpha_k} \neq \epsilon$

(iii.) $w_\rho = \epsilon$, $w_\tau = w_{\alpha_1} \neq \epsilon$, and $w_\sigma = \epsilon$

(iv.) $w_\rho = \epsilon$, $w_\tau = w_{\alpha_1} \neq \epsilon$, and $w_\sigma = w_{a_2} \dots w_{\alpha_k} \neq \epsilon$

(v.) $w_\rho = w_{\alpha_1} \dots w_{\alpha_{k-1}} \neq \epsilon$, $w_\tau = w_{\alpha_k} \neq \epsilon$, and $w_\sigma = \epsilon$

(vi.) $w_\rho = w_{\alpha_1} \dots w_{\alpha_{s-1}} \neq \epsilon$, $w_\tau = w_{\alpha_s} \neq \epsilon$, and $w_\sigma = w_{\alpha_{s+1}} \dots w_{\alpha_k} \neq \epsilon$

This completes our factorization of y_g as $y_g = w_\rho w_\tau w_\sigma$.

Part 2: Computation of y_{gx}

Recall y_g is in $\mathcal{N}$, where $\mathcal{N}$ is the set of proper words in A^*. Having written $y_g = w_\rho w_\tau w_\sigma$, we now compute the normal form for $y_g x$, where we recall $x \in A_{\alpha_j}$. We will explicitly compute y_{gx} using the factorization of y_g in case (vi) from Part 1 and note that computing the normal form for $y_g x$ in the other cases is done by allowing at least one of w_ρ, w_τ, or w_σ be the empty word.

Suppose we have $w_\rho = w_{\alpha_1} \dots w_{\alpha_{s-1}} \neq \epsilon$, $w_\tau = w_{\alpha_s} \neq \epsilon$, and $w_\sigma = w_{\alpha_{s+1}} \dots w_{\alpha_k} \neq \epsilon$ for some $1 \leq s \leq k$. By construction we have $[w_\sigma, x] =_{G\Lambda} 1$ and in particular $[w_{\alpha_i}, x] =_{G\Lambda} 1$ for all $s+1 \leq i \leq k$. Let $y_{w_\tau x}$ be the normal form for $w_\tau x$ in $\mathcal{N}$. Then we have

$$\begin{aligned} y_g x = w_\rho w_\tau w_\sigma x =_{G\Lambda} w_\rho w_\tau x w_\sigma =_{G\Lambda} w_\rho y_{w_\tau x} w_\sigma \qquad (3.2) \\ =_{G\Lambda} w_{\alpha_1} \dots w_{\alpha_{s-1}} y_{w_\tau x} w_{\alpha_{s+1}} \dots w_{\alpha_k}. \qquad (3.3) \end{aligned}$$

We claim that $w_\rho y_{w_\tau x} w_\sigma$ is a proper word in $\mathcal{N}$ and so $y_{gx} = w_\rho y_{w_\tau x} w_\sigma$.

Suppose $w_\tau \in \mathcal{N}_{\alpha_j}$ and $y_{w_\tau x}$ in $\mathcal{N}_{\alpha_j}$ is the empty word. Since v_{α_j} is adjacent to $v_{\alpha_{s+1}}$ in Λ, we have $[w_\tau, w_{\alpha_{s+1}}] =_{G\Lambda} 1$. From line (3.3) above, if $\alpha_{s-1} = \alpha_{s+1}$ then $[w_{\alpha_{s-1}}, w_\tau] =_{G\Lambda} 1$, contradicting y_g being in normal form. Therefore, we have that $\alpha_{s-1} \neq \alpha_{s+1}$ and in $w_\rho w_\sigma = w_{\alpha_1} \dots w_{\alpha_{s-1}} w_{\alpha_{s+1}} \dots w_{\alpha_k}$ each local word is in normal form with consecutive local words representing elements in distinct vertex groups. Furthermore, in $w_\rho w_\sigma = w_{\alpha_1} \dots w_{\alpha_{s-1}} w_{\alpha_{s+1}} \dots w_{\alpha_k}$, if $G_{\alpha_{s+1}} <_G G_{\alpha_{s-1}}$ and $v_{\alpha_{s+1}}$ and $v_{\alpha_{s-1}}$ are adjacent in Λ, then applying condition (O) in Definition 3.2 to the normal form y_g we must have that v_{α_s} and $v_{\alpha_{s+1}}$ are not adjacent in Λ. Since $w_\tau = w_{\alpha_s}$, this contradicts $[w_\tau, w_{\alpha_{s+1}}] =_{G\Lambda} 1$; therefore, we either have $G_{\alpha_{s+1}} \not<_G G_{\alpha_{s-1}}$ or $v_{\alpha_{s-1}}$ and $v_{\alpha_{s+1}}$ are not adjacent in Λ.

To show $w_\rho w_\sigma$ satisfies the obstruction condition in Definition 3.2, suppose we have local words w_i and w_ℓ of $w_\rho w_\sigma$ such that $w_\rho w_\sigma = \ldots w_i \ldots w_\ell \ldots$ with $\ell \leq i$ and v_i and v_ℓ adjacent in Λ. If $w_i \ldots w_\ell$ is a subword of w_ρ or w_σ, then there exists a local word w_t such that we have $w_i \ldots w_\ell = w_i \ldots w_t \ldots w_\ell$ and the vertices v_t and v_ℓ are non-adjacent in Λ. The local word w_t is also a local word of $w_\rho w_\sigma$ and so the obstruction condition in Definition 3.2 holds for $w_\rho w_\sigma$.

If $w_i \ldots w_\ell$ is not a subword of w_ρ or w_σ, then w_i and w_ℓ are local words of w_ρ and w_σ, respectively, such that we have $y_g = \ldots w_i \ldots w_{\alpha_{s-1}} w_{\alpha_s} w_{\alpha_{s+1}} \ldots w_\ell \ldots$. If there exists a local word w_t such that $y_g = \ldots w_{\alpha_{s+1}} \ldots w_t \ldots w_\ell$ with v_t and v_ℓ are non-adjacent in Λ, then again w_t is a local word of $w_\rho w_\sigma$ and the obstruction condition holds for $w_\rho w_\sigma$. If there does not exists such a local word, then in y_g we have v_ℓ is adjacent to v_m in Λ for all $\alpha_{s+1} \leq m \leq \ell - 1$. Recall by construction we have $[w_{\alpha_i}, x] = 1$ for all local words in w_σ. Furthermore, since $x \in A_{\alpha_j}$ and $w_\tau = w_{\alpha_z} \in \mathcal{N}_{\alpha_j}$ we have $[w_{\alpha_i}, w_\tau] =_{G\Lambda} 1$ for all local words of w_σ. This gives us v_ℓ is adjacent to v_m for all $\alpha_s \leq m \leq \ell - 1$. Since y_g is in normal form there must exist a local word w_t such that $w_{\alpha_i} \ldots w_{\alpha_{s-1}} = w_i \ldots w_t \ldots w_{\alpha_{s-1}}$ where v_t and v_ℓ are non-adjacent in Λ. The local word w_t is also a local word in $w_\rho w_\sigma$ and again the obstruction condition holds for $w_\rho w_\sigma$. Since conditions (L) and (O) in Definition 3.2 hold for $y_g x = w_\rho w_\sigma$, we have the normal form for $y_g x$ in $\mathcal{N}$ is $y_{gx} = w_\rho y_{w_\tau x} w_\sigma = w_\rho w_\sigma$.

Suppose $w_\tau \in \mathcal{N}_{\alpha_j}$ and $y_{w_\tau x}$ in $\mathcal{N}_{\alpha_j}$ is not the empty word. Since $w_{\alpha_{s-1}}$ and $w_\tau = w_{\alpha_s}$ are consecutive local words the associated vertex groups are distinct and therefore the associated vertex groups for $y_{w_\tau x}$ and $w_{\alpha_{s-1}}$ are also distinct. Similarly, we have the associated vertex groups for $y_{w_\tau x}$ and $w_{\alpha_{s+1}}$ are distinct. Since consecutive local words represent group elements in distinct vertex groups and are in the prescribed normal forms, condition (L) in Definition 3.2 holds. Lastly, since y_g satisfies condition (O) in Definition 3.2 and $y_{w_\tau x}, w_\tau = w_{\alpha_s} \in \mathcal{N}_{\alpha_j}$ we have $w_\rho y_{w_\tau x} w_\sigma$ also satisfies condition (O) and therefore the normal form for $y_g x$ in $\mathcal{N}$ is $y_{gx} = w_\rho y_{w_\tau x} w_\sigma$.

Now, suppose $w_\tau \notin \mathcal{N}_{\alpha_j}$. Let y_x be the normal form for x in $\mathcal{N}_{\alpha_j}$. Since $x \in A_{\alpha_j}$, we have w_τ and y_x represent group elements in different vertex groups. By construction of w_τ in Part (1) we have $w_\tau = w_{\alpha_s}$ where $G_{\alpha_s} <_G G_{\alpha_j}$. Thus, by Definition 3.2 we have the normal form for $w_\tau x$ in $\mathcal{N}$ is $y_{w_\tau x} = w_\tau y_x$. Furthermore, this gives us $y_g x = w_\rho w_\tau w_\sigma x =_{G\Lambda} w_\rho w_\tau x w_\sigma =_{G\Lambda} w_\rho y_{w_\tau x} w_\sigma =_{G\Lambda} w_\rho w_\tau y_x w_\sigma$.

By construction of $w_\sigma = w_{\alpha_{s+1}} \ldots w_{\alpha_k}$, we have w_σ does not have any local words in $\mathcal{N}_{\alpha_j}$ as a subword since we shuffled the letter x past a local word whenever the corresponding vertex was adjacent to v_{α_j} and Λ is a simple graph. In particular, the local word $w_{\alpha_{s+1}}$ is not in $\mathcal{N}_{\alpha_j}$ and in $w_\rho w_\tau y_x w_\sigma = w_{\alpha_1} \ldots w_{\alpha_{s-1}} w_{\alpha_s} y_x w_{\alpha_{s+1}} \ldots w_{\alpha_k}$ each local word w_{α_i} is in $\mathcal{N}_{\alpha_i}$ with consecutive local words representing group elements in distinct vertex groups. To show $w_\rho w_\tau y_x w_\sigma$ satisfies condition (O) suppose we have $w_\rho w_\tau y_x w_\sigma = \ldots w_i \ldots w_\ell \ldots$ with $i \geq \ell$ and v_{α_i} and v_{α_ℓ} adjacent in Λ. That is, based on the ordering $<_G$ on $\{G_i\}_{i=1}^m$, the local word w_ℓ should come before the local word w_i. If w_i and w_ℓ are local words of y_g, since y_g is in normal form there exists a local word w_j of y_g such that v_j and v_ℓ are non-adjacent in Λ. We then have w_j is also local word of $w_\rho w_\tau x w_\sigma$ and so condition (O) still holds. If w_ℓ is y_x then $w_\rho w_\tau x w_\sigma = \ldots w_i \ldots w_{\alpha_{s-1}} w_{\alpha_s} y_x \ldots$ and $v_{\alpha_{s-1}}$ and v_{a_j} are non-adjacent in Λ. Since conditions (L) and (O) hold for $w_\rho w_\tau y_x w_\sigma$, we have the normal form for $y_g x$ in $\mathcal{N}$ is $y_{gx} = w_\rho w_\tau y_x w_\sigma$. We note that in the case where the normal form for x in $\mathcal{N}_{\alpha_j}$ is x, we simply have $y_{gx} = w_\rho w_\tau x w_\sigma$.

What we have done in Part (2) is show that the normal form for $y_g x$ in $\mathcal{N}$ is $y_{gx} = w_\rho y_{w_\tau x} w_\sigma$, where $y_{w_\tau x}$ is the normal form for $w_\tau x$.

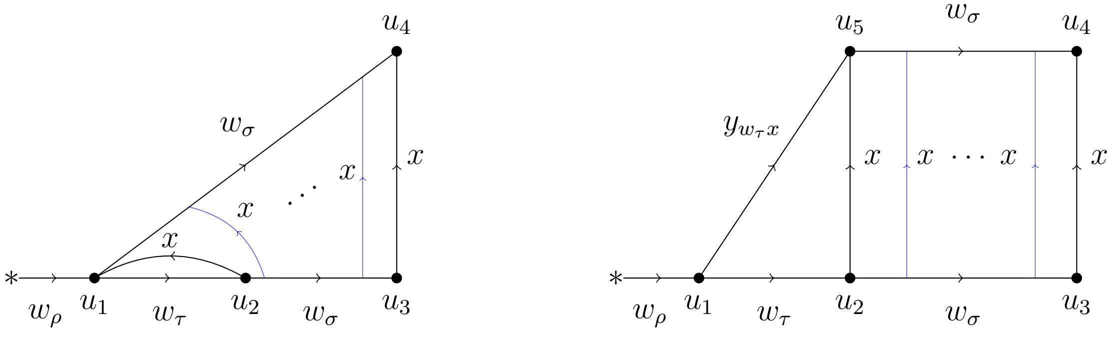
Part 3: Construction of a van Kampen diagram Δ for $y_g x y_{gx}^{-1}$

Recall in Part (1) we factored y_g as $y_g = w_\rho w_\tau w_\sigma$. Having written $y_{gx} = w_\rho y_{w_\tau x} w_\sigma$ in Part(2), we now construct a van Kampen diagram Δ for $y_g x y_{gx}^{-1}$.

Express w_σ as $w_\sigma = a_{i_1} \dots a_{i_s}$, where each $a_{i_t} \in A$. By construction we have $[w_\sigma, x] = 1$, and in particular $[a_{i_t}, x] = 1$ for all $i_1 \leq i_t \leq i_s$. Let Σ_{i_t} be a van Kampen diagram with boundary word $a_{i_t} x a_{i_t}^{-1} x^{-1}$, read counterclockwise starting at the basepoint v_{i_t}, for each $i_1 \leq i_t \leq i_s$. Additionally, starting at the basepoint v_{i_t} and reading counterclockwise around the boundary of Σ_{i_t}, label the vertices on $\partial\Sigma_{i_t}$ as u_{i_t}, u'_{i_t} and v'_{i_t}, respectively. Let Θ_{i_2} be the van Kampen diagram obtained by gluing the van Kampen diagrams Σ_{i_1} and Σ_{i_2} together along the edge $e(u_{i_1}, x)$ in Σ_{i_1} and $e(v_{i_2}, x)$ in Σ_{i_2}. Inductively define the van Kampen diagram Θ_{i_t} by gluing the van Kampen diagrams $\Theta_{i_{t-1}}$ and Σ_{i_t} together along the edge $e(u_{i_{t-1}}, x)$ in $\Theta_{i_{t-1}}$ and the edge $e(v_{i_t}, x)$ in Σ_{i_t}, for $i_3 \leq i_t \leq i_s$. Let $\Theta_\sigma = \Theta_{i_s}$. We note that Θ_σ has boundary word $w_\sigma x {w_\sigma}^{-1} x^{-1}$.

Recall we have $y_g = w_\rho w_\tau w_\sigma$, $x \in A_{\alpha_j}$, and $y_{gx} = w_\rho y_{w_\tau x} w_\sigma$, where $y_{w_\tau x}$ is the normal form for $w_\tau x$. When $w_\tau \in \mathcal{N}_{\alpha_j}$ and $x \in A_{\alpha_j}$ there exists a pair $(\Delta_\tau, \Omega_\tau)$ in $\mathcal{E}_{\alpha_j}$ such that Δ_τ is a van Kampen diagram for the word $w_\tau x y_{w_\tau x}^{-1}$ over the presentation $\mathcal{P}_{\alpha_j}$ of G_{α_j}, and the edge 1-combing Ω_τ is h_{α_j}-tame. Here it is possible to have $y_{w_\tau x}$ being the empty word, in which case the boundary of Δ_τ is $w_\tau x^{-1}$. When $w_\tau \notin \mathcal{N}_{\alpha_j}$, since $\epsilon \in \mathcal{N}_{\alpha_j}$ and $x \in A_{\alpha_j}$ there exists a pair (Δ_x, Ω_x) in $\mathcal{E}_{\alpha_j}$ such that Δ_x is a van Kampen diagram for the word $x y_x^{-1}$ over the presentation $\mathcal{P}_{\alpha_j}$ of G_{α_j}, and the edge 1-combing Ω_x is h_{α_j}-tame. We note when the normal form y_x is x, the van Kampen diagram Δ_x is a single edge labeled by x with no 2-cells.

We now construct the van Kampen diagram Δ for $y_g x y_{gx}^{-1}$. When $w_\tau \in \mathcal{N}_{\alpha_j}$ let μ be a simple edge path labeled by w_ρ with boundary vertices $*$ and u_1. Adjoin the basepoint of the van Kampen diagram Δ_τ to the vertex u_1 and call the resulting diagram $\tilde{\Delta}$. When $w_\tau \notin \mathcal{N}_{\alpha_j}$ let μ be a simple edge path labeled by $w_\rho w_\tau$ with boundary vertices $*$ and u_2 and adjoin the basepoint of the van Kampen diagram Δ_x to the vertex u_2; call the resulting diagram $\tilde{\Delta}$.



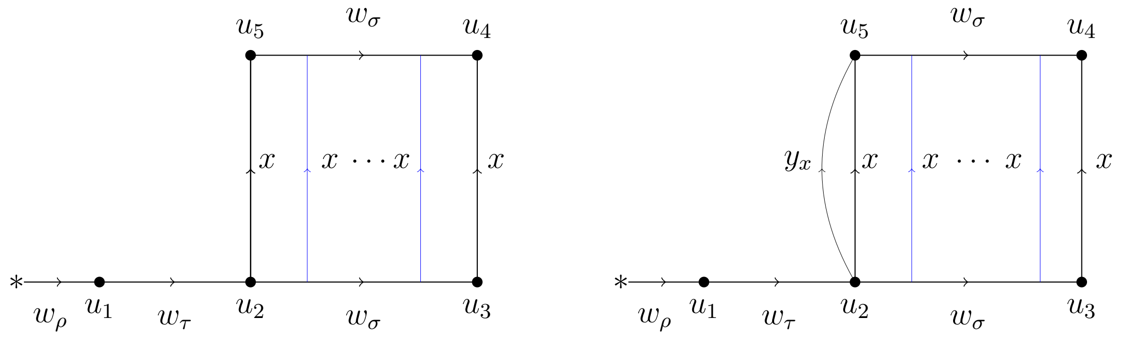
(a): $y_{w_\tau x} = \epsilon$ (b): $y_{w_\tau x} \neq \epsilon$

Figure 3.1: van Kampen diagram Δ when $w_\tau \in \mathcal{N}_{\alpha_j}$.

Next, in both cases when $w_\tau \in \mathcal{N}_{\alpha_j}$ and when $w_\tau \notin \mathcal{N}_{\alpha_j}$, attach the van Kampen diagram Θ_σ onto $\tilde{\Delta}$, by gluing the edge $e(v_{i_1}, x)$ on the boundary of Θ_σ to the edge labeled by x on the boundary of $\tilde{\Delta}$; call the resulting diagram Δ. To simplify notation, in Δ label the vertices v_{i_1}, u_{i_s}, $u'_{i,s}$, v'_{i_1} on the boundary of Δ as u_2, u_3, u_4, and u_5, respectively. When $w_\tau \in \mathcal{N}_{\alpha_j}$, the van Kampen diagram Δ is illustrated in Figures 3.1 (a) and 3.1 (b). When $w_\tau \notin \mathcal{N}_{\alpha_j}$, the van Kampen diagram Δ is illustrated in Figures 3.2 (a) and 3.2 (b).

Part 4: Construction of an edge 1-combing Ψ for Δ

Let $(\Delta_\tau, \Omega_\tau)$, (Δ_x, Ω_x), Θ_σ, and Δ be as defined in Part (3). With a slight abuse of notation, we can think of Δ_τ, Δ_x, and Θ_σ as subcomplexes of Δ with the basepoints u_1, u_2, and u_2 respectively. Let $e_1 = (u_2, x)$ and $e_2 = e(u_3, x)$ be the edges in Δ with initial vertex u_2 and u_3 labeled by x, respectively. Let $J : e_2 \to [0,1]$ and $K : \Theta_\sigma \to [0,1] \times [0,1]$ be homeomorphisms with $J(u_3) = 0$ and $J(u_4) = 1$ and $K(u_2) = (0,0)$, $K(u_3) = (1,0)$, $K(u_4) = (1,1)$ and $K(u_5) = (0,1)$. Observe, for $p \in e_2$ we have $K(p) = (1, J(p))$.



(a): $y_x = x$ (b): $y_x \neq x$

Figure 3.2: van Kampen diagram Δ when $w_\tau \notin \mathcal{N}_{\alpha_j}$.

For $p \in \Theta_\sigma$, let $(x_p, z_p) = K^{-1}(p)$. Let $r : \Theta_\sigma \to e_1$ be defined by $r(p) = K^{-1}\left((0, z_p)\right)$. Then r is a retraction of Θ_σ onto the edge e_1. Note that the edge e_1 is on the boundary of Δ_τ and Δ_x (as a subcomplex of Δ). For $p \in e_2$, define the map $\eta_p : [0,1] \to \Theta_\sigma$ by $\eta_p(s) = K^{-1}\left((s, z_p)\right)$. Then η_p is a path in Θ_σ from $r(p)$ to p. Lastly, let $\phi_{w_\rho}, \phi_{w_\tau} : [0,1] \to \Delta$ be the continuous maps that follow the paths labeled by w_ρ and $w_\rho w_\tau$ in Δ from the basepoint $*$ to u_1 and u_2, respectively, at a constant speed.

Construction of Ψ when $w_\tau \in \mathcal{N}_{\alpha_j}$ or $w_\tau \notin \mathcal{N}_{\alpha_j}$ and $y_x \neq x$: Let Δ be the van Kampen diagram shown in Figure 3.1 (b). We will construct an explicit edge 1-combing $\Psi : e_2 \times [0,1] \to \Delta$ of Δ. Note that the construction of Ψ for Δ in Figure 3.1 (a) and Figure 3.2 (b) are the special cases in which $y_{w_\tau x} = \epsilon$ and $w_\tau = \epsilon$, respectively. Before our construction, we define a few auxiliary maps.

Notation 3.6. Let i, j, k and ℓ be the lengths of w_ρ, w_τ, w_σ, and $y_{w_\tau x}$, respectively. To aid with computation we introduce the following notation:

- $m_g = \dfrac{i}{i+j+k}$
- $n_g = \dfrac{j}{i+j+k}$
- $p_g = \dfrac{k}{i+j+k}$
- $m_{gx} = \dfrac{i}{i+\ell+k}$
- $n_{gx} = \dfrac{\ell}{i+\ell+k}$
- $p_{gx} = \dfrac{k}{i+\ell+k}$

For $p \in e_2$, let

$$\begin{aligned} a_p &= m_g + J(p)\,(m_{gx} - m_g) \;\; \textit{and} \\ b_p &= (m_g + n_g) + J(p)\,(m_{gx} + n_{gx} - m_g - n_g)\,. \end{aligned}$$

Let $Y_1 = \{(p,s)|0 \leq s \leq a_p\}$, $Y_2 = \{(p,s)|a_p \leq s \leq b_p\}$, and $Y_3 = \{(p,s)|b_p \leq s \leq 1\}$; all closed subsets of $e_2 \times [0,1]$ with $e_2 \times [0,1] = Y_1 \cup Y_2 \cup Y_3$.

Define the following continuous maps:

$$\begin{aligned} &q_1 : e_2 \times [0,1] \to Y_1 \text{ by } q_1((p,s)) = (p, a_p s) \\ &q_2 : e_2 \times [0,1] \to Y_2 \text{ by } q_2((p,s)) = (p, a_p + s(b_p - a_p)) \\ &q_3 : e_2 \times [0,1] \to Y_3 \text{ by } q_3((p,s)) = (p, b_p + s(1 - b_p)). \end{aligned}$$

Let $i \in \{1,2,3\}$. Define the equivalence relation $\sim_i$ on $e_2 \times [0,1]$ where $(p,s) \sim_i (p',t)$ if and only if $q_i((p,s)) = q_i((p',t))$. Then the map $\pi_i : e_2 \times [0,1] \to (e_2 \times [0,1])/\sim_i$ is a quotient map for $i = 1,2,3$. Since $q_i : e_2 \times [0,1] \to Y_i$ is a continuous map such that $(p,s) \sim_i (p',t)$ implies $q_i((p,s)) = q_i((p',t))$ for all (p,s) and (p',t) in $e_2 \times [0,1]$, by the universal property of quotient maps there exists a unique continuous map $f_i : (e_2 \times [0,1])/\sim_i \to Y_i$ such that $q_i = f_i \pi_i$. Note that $e_2 \times [0,1]$ is a compact space, Y_i is a Hausdorff space, and so k_i is in fact a homeomorphism with $f_i\left(\overline{(p,s)}^i\right) = q_i((p,s))$ and

$$f_1^{-1}(p,s) = \begin{cases} \overline{\left(p, \dfrac{s}{a_p}\right)}^1, & a_p \neq 0 \\ \overline{(p,0)}^1 & a_p = 0 \end{cases}$$

$$f_2^{-1}(p,s) = \begin{cases} \overline{\left(p, \dfrac{s-a_p}{b_p-a_p}\right)}^2, & a_p \neq b_p \\ \overline{(p,a_p)}^2 & a_p = b_p \end{cases} \qquad f_3^{-1}(p,s) \;=\; \begin{cases} \overline{\left(p, \dfrac{s-b_p}{1-b_p}\right)}^3, & b_p \neq 1 \\ \overline{(p,1)}^3 & b_p = 1 \end{cases}$$

where $\overline{(p,s)}^i$ denotes the equivalence class of (p,s) under $\sim_i$.

Define the following maps:

$$g_1 : e_2 \times [0,1] \to \Delta \text{ by } g_1((p,s)) = \phi_{w_\rho}(s)$$
$$g_2 : e_2 \times [0,1] \to \Delta \text{ by } g_2((p,s)) = \Omega_\tau\left(r(p), s\right)$$
$$g_3 : e_2 \times [0,1] \to \Delta \text{ by } g_3((p,s)) = \eta_{r(p)}(s).$$

Observe g_1, g_2 and g_3 are continuous since $\phi_{w_\rho}, \Omega_\tau$, and η_p are continuous maps. Next we show given $(p,s) \sim_i (p',t)$ we have $g_i\left((p,s)\right) = g_i\left((p',t)\right)$, for $i = 1,2,3$. We explicitly show this in the case $i=2$, and note that the cases in which $i = 1,3$ are similar.

Suppose $(p,s) \sim_2 (p',t)$. Then $q_2\left((p,s)\right) = q_2\left((p',t)\right)$, which implies $(p, a_p + s(b_p - a_p)) = (p', a_{p'} + t(b_{p'} - a_{p'}))$. Therefore $p = p'$ and $a_p + s(b_p - a_p) = a_{p'} + t(b_{p'} - a_{p'})$. Since the map $J : e_2 \to [0,1]$ is injective, having $p = p'$ implies $a_p = a_{p'}$ and $b_p = b_{p'}$. Furthermore,

$$a_p + s(b_p - a_p) = a_{p'} + t(b_{p'} - a_{p'}) = a_p + t(b_p - a_p)$$

implies $0 = (s-t)(a_p - b_p)$. Therefore, $(p,s) \sim_2 (p',t)$ when $(p,s) = (p',t)$ or when $p = p'$ and $a_p = b_p$.

Applying the definition of a_p and b_p above to the equation $a_p = b_p$ gives us $0 = n_g + J(p)\left(n_{gx} - n_g\right) = n_g\left(1 - J(p)\right) + J(p)n_{gx}$. Since $n_g, J(p), n_{gx} \geq 0$ and $J(p) \leq 1$ we must have $n_g(1 - J(p)) = 0$ and $J(p)n_{gx} = 0$. We can then conclude

$(p,s) \sim_2 (p',t)$ when we either have: (1) $p = p'$ and $s = t$, (2) $p = p'$, $n_g = 0$, and $J(p) = 0$, (3) $p = p'$, $n_g = 0$, and $n_{gx} = 0$, or (4) $p = p'$, $J(p) = 1$ and $n_{gx} = 0$.

Suppose $(p,s) = (p',t)$; then we immediately have $g_2((p,s)) = g_2((p',t))$. Next, suppose $p = p'$, $n_g = 0$, and $J(p) = 0$. Observe, $n_g = 0$ implies $w_\tau = \epsilon$. Also $J(p) = 0$ implies that $p = u_3$ and so $r(p) = u_2$. Since $\Omega_\tau(u_2, \cdot)$ is the path in Δ from u_1 that follows along $w_\tau = \epsilon$ we have $\Omega_\tau(u_2, \cdot)$ is the constant path at u_1. Since $p = p'$ we also have $r(p') = u_2$ and $\Omega_\tau(r(p'), \cdot)$ is also the constant path at u_1, hence $g_2\left((p,s)\right) = g_2\left((p',t)\right)$.

Next, suppose $p = p'$, $n_g = 0$, and $n_{gx} = 0$. Observe $n_g = 0$ and $n_{gx} = 0$ implies $w_\tau = \epsilon$ and $y_{w_\tau x} = \epsilon$, respectively. Thus $w_\tau x y_{w_\tau x}^{-1} =_{G\Lambda} \epsilon$ implies that x represents the identity element in $G\Lambda$. Since the empty word ϵ is the unique normal form for the identity element of $G\Lambda$ in the normal form set $\mathcal{N}$, we cannot have $x = \epsilon$ and therefore this case cannot occur.

Finally, suppose we have $p = p'$, $J(p) = 1$ and $n_{gx} = 0$. Observe, $J(p) = 1$ implies that $p = u_4$ and therefore $r(p) = u_5$. Additionally, $n_{gx} = 0$ implies $y_{w_\tau x} = \epsilon$ and so the vertex u_1 equals the vertex u_5. Since $\Omega_\tau(u_5, \cdot)$ is the path in Δ from u_1 to u_5 that follows along $y_{w_\tau x} = \epsilon$ we have $\Omega_\tau(u_5, \cdot)$ is the constant path at u_1. Furthermore, since $p = p'$ we have $r(p') = u_5$ and $\Omega_\tau(r(p'), \cdot)$ is also the constant path at u_1, and hence $g_2\left((p,s)\right) = g_2\left((p',t)\right)$.

Thus we have shown in all four cases that for $(p,s) \sim_2 (p',t)$ we have $g_2\left((p,s)\right) = g_2\left((p',t)\right)$. A similar argument shows that for $(p,s) \sim_i (p',t)$ we have $g_i\left((p,s)\right) = g_i\left((p',t)\right)$, for $i = 1,3$. Applying the universal property of quotient maps to $g_i : e_2 \times [0,1] \to \Delta$ there exists a unique continuous map $\bar{g}_i : (e_2 \times [0,1])/\sim_i \to \Delta$ such that $g_i = \bar{g}_i \circ \pi_i$ for $i = 1,2,3$. In particular, for $i \in \{1,2,3\}$ we have $\bar{g}_i\left(\overline{(p,s)}\right) = g_i((p,s))$. This gives us a commutative diagram of continuous maps shown in Figure 3.3.

$$\begin{tikzcd}
\Delta & e_2 \times [0,1] \arrow[l, "g_i"'] \arrow[r, "q_i"] \arrow[d, "\pi_i"', "\pi_i"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowleft"] & Y_i \\
 & \dfrac{e_2 \times [0,1]}{\sim_i} \arrow[ul, "\bar{g}_i"] \arrow[ur, "f_i"'] &
\end{tikzcd}$$

Figure 3.3: Induced quotient maps for $i = 1, 2, 3$.

Definition 3.7. Now we define the edge 1-combing $\Psi : e_2 \times [0,1] \to \Delta$ by

$$\Psi(p,s) = \begin{cases} \left(\bar{g}_1 \circ {f_1}^{-1}\right)(p,s) & (p,s) \in Y_1 \\ \left(\bar{g}_2 \circ {f_2}^{-1}\right)(p,s) & (p,s) \in Y_2 \\ \left(\bar{g}_3 \circ {f_3}^{-1}\right)(p,s) & (p,s) \in Y_3 \end{cases}$$

Recall that $e_2 \times [0,1] = Y_1 \cup Y_2 \cup Y_3$. Additionally, since f_i^{-1} and $\bar{g}_i$ are continuous, we have $(\bar{g}_i \circ {f_i}^{-1})$ is continuous for $i = 1, 2, 3$. Next, we need to show that these functions agree where both are defined. For $(p,s) \in Y_1 \cap Y_2$ we have $s = a_p$.

When $a_p = 0$ we have $(p,0) \sim_1 (p,1)$ and so $\overline{(p,0)}^{1} = \overline{(p,1)}^{1}$. This gives us

$$\left(\bar{g}_1\left({f_1}^{-1}(p,a_p)\right)\right) = \bar{g}_1\left(\overline{(p,0)}^{1}\right) = \bar{g}_1\left(\overline{(p,1)}^{1}\right) = g_1\left((p,1)\right) = \phi_{w_\rho}(1) = u_1$$

$$\left(\bar{g}_2\left({f_2}^{-1}(p,a_p)\right)\right) \overset{a_p = b_p}{=} \bar{g}_2\left(\overline{(p,a_p)}^{2}\right) = g_2((p,a_p)) = \Omega_\tau\left(r(p),0\right) = u_1$$

$$\left(\bar{g}_2\left({f_2}^{-1}(p,a_p)\right)\right) \overset{a_p \neq b_p}{=} \bar{g}_2\left(\overline{(p,0)}^{2}\right) = g_2((p,a_p)) = \Omega_\tau\left(r(p),0\right) = u_1$$

When $a_p \neq 0$ and $a_p = b_p$ we have $(p,0) \sim_2 (p,a_p)$ thus $\overline{(p,0)}^2 = \overline{(p,a_p)}^2$. Using this information we have

$$\begin{aligned}
&\left(\bar{g}_1\left({f_1}^{-1}(p,a_p)\right)\right) = \bar{g}_1\left(\overline{(p,1)}^1\right) = g_1\left((p,1)\right) = \phi_{w_p}(1) = u_1 \\
&\left(\bar{g}_2\left({f_2}^{-1}(p,a_p)\right)\right) \overset{a_p=b_p}{=} \bar{g}_2\left(\overline{(p,a_p)}^2\right) = \bar{g}_2\left(\overline{(p,0)}^2\right) = g_2\left((p,0)\right) = \Omega_\tau(r(p),0) = u_1 \\
&\left(\bar{g}_2\left({f_2}^{-1}(p,a_p)\right)\right) \overset{a_p\neq b_p}{=} \bar{g}_2\left(\overline{(p,0)}^2\right) = g_2\left((p,0)\right) = \Omega_\tau(r(p),0) = u_1
\end{aligned}$$

For $(p,s) \in Y_2 \cap Y_3$ we have $s = b_p$. When $a_p = b_p$ we have $(p,1) \sim_2 (p,a_p)$ and so $\overline{(p,1)}^2 = \overline{(p,a_p)}^2$. Furthermore, when $b_p = 1$ we have $(p,1) \sim_3 (p,0)$ giving us $\overline{(p,1)}^3 = \overline{(p,0)}^3$. Using this information we have

$$\begin{aligned}
&\left(\bar{g}_2\left({f_2}^{-1}(p,b_p)\right)\right) = \bar{g}_2\left(\overline{(p,a_p)}^2\right) = \bar{g}_2\left(\overline{(p,1)}^2\right) = g_2\left((p,1)\right) = \Omega_\tau(r(p),1) = r(p) \\
&\left(\bar{g}_3\left({f_3}^{-1}(p,b_p)\right)\right) \overset{b_p=1}{=} \bar{g}_3\left(\overline{(p,1)}^3\right) = \left(\overline{(p,0)}^3\right) = g_3((p,0)) = \eta_{r(p)}(0) = r(p) \\
&\left(\bar{g}_3\left({f_3}^{-1}(p,b_p)\right)\right) \overset{b_p\neq 1}{=} \bar{g}_3\left(\overline{(p,0)}^3\right) = g_3((p,0)) = r(p)
\end{aligned}$$

When $a_p \neq b_p$ we have

$$\begin{aligned}
&\left(\bar{g}_2\left({f_2}^{-1}(p,b_p)\right)\right) = \bar{g}_2\left(\overline{(p,1)}^2\right) = g_2\left((p,1)\right) = \Omega_\tau(r(p),1) = r(p) \\
&\left(\bar{g}_3\left({f_3}^{-1}(p,b_p)\right)\right) \overset{b_p=1}{=} \bar{g}_3\left(\overline{(p,1)}^3\right) = \bar{g}_3\left(\overline{(p,0)}^3\right) = g_3\left((p,0)\right) = r(p) \\
&\left(\bar{g}_3\left({f_3}^{-1}(p,b_p)\right)\right) \overset{b_p\neq 1}{=} \bar{g}_2\left(\overline{(p,0)}^3\right) = g_3\left((p,0)\right) = \eta_{r(p)}(0) = r(p)
\end{aligned}$$

Since Y_1, Y_2 and Y_3 are closed subsets with $e_2 \times [0,1] = Y_1 \cup Y_2 \cup Y_3$ and we have continuous functions $\left(\bar{g}_1 \circ {f_1}^{-1}\right), \left(\bar{g}_2 \circ {f_2}^{-1}\right)$, and $\left(\bar{g}_3 \circ {f_3}^{-1}\right)$ that agree on points in $Y_1 \cap Y_2$ and $Y_2 \cap Y_3$, by the Pasting Lemma we have Ψ is continuous. We define Ψ in terms of a_p and b_p to ensure the gluing condition in the definition of a combed

$\mathcal{N}$-filling is satisfied.

It is worth noting that using (Δ_x, Ω_x) and ψ_{w_τ} in the construction above instead of $(\Delta_\tau, \Omega_\tau)$ and ϕ_{w_ρ} will give you an edge 1-combing for Δ in Figure 3.2 (b).

Construction of Ψ when $w_\tau \notin \mathcal{N}_{\alpha_j}$ and $y_x = x$: Let Δ be the van Kampen diagram illustrated in Figure 3.2 (a). For $r \in [0,1]$ define $\xi_r : [0,1] \to [0,1] \times [0,1]$ by $\xi_r(s) = (0,0)(1-s) + (0,r)s$ and define $\omega_r : [0,1] \to [0,1] \times [0,1]$ by $\omega_r(s) = (0,r)(1-s) + (1,r)s$. Next, for any $r \in [0,1]$ define $\Xi_r : [0,1] \to [0,1] \times [0,1]$ by

$$\Xi_r(s) = \begin{cases} \xi_r(2s) & s \in [0, \frac{1}{2}] \\ \omega_r(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

For p in e_2, recall $K(p) = (1, z_p) \in [0,1] \times [0,1]$ for some $z_p \in [0,1]$. Abusing notation slightly, let Ξ_p denote the function Ξ_{y_p}.

Let i, j, k, ℓ, m_g, n_g, and p_g be as defined in 3.6 on page 22. Since $y_{w_\tau x} = w_\tau x$, we have $\ell = \ell(y_{w_\tau x}) = j + 1$. Let

- $m_{gx} = \dfrac{i}{i+j+k+1}$
- $n_{gx} = \dfrac{j}{i+j+k+1}$
- $p_{gx} = \dfrac{k}{i+j+k+1}$

For $p \in e_2$, let

$$a_p = m_g + n_g + J(p)\left(m_{gx} + n_{gx} - m_g - n_g\right).$$

Similar to the construction of Ψ in Definition 3.7, let $Y_1 = \{(p,s) | 0 \le s \le a_p\}$ and $Y_2 = \{(p,s) | a_p \le s \le 1\}$ and define the following continuous maps:

$$q_1 : e_2 \times [0,1] \to Y_1 \text{ by } q_1((p,s)) = (p, a_p s)$$

$$q_2 : e_2 \times [0,1] \to Y_2 \text{ by } q_2((p,s)) = (p, a_p + s(1-a_p)).$$

Then for $i \in \{1,2\}$, $\pi_i : e_2 \times [0,1] \to (e_2 \times [0,1])/\sim_i$ is a quotient map where $(p,s) \sim_i (p',t)$ if and only if $q_i((p,s)) = q_i((p',t))$. By the universal property of quotient maps there exists a unique continuous map $f_i : (e_2 \times [0,1])/\sim_i \to Y_i$ such that $q_i = f_i\pi_i$ for $i = 1,2$. Again, since $e_2 \times [0,1]$ is a compact space and Y_i is a Hausdorff space, f_i is a homeomorphism with $f_i\left(\overline{(p,s)}\right) = q_i((p,s))$ and

$$f_1^{-1}(p,s) = \begin{cases} \overline{\left(p, \dfrac{s}{a_p}\right)}^1, & a_p \neq 0 \\ \overline{(p,0)}^1 & a_p = 0 \end{cases} \qquad f_2^{-1}(p,s) = \begin{cases} \overline{\left(p, \dfrac{s-a_p}{1-a_p}\right)}^2, & a_p \neq 1 \\ \overline{(p,1)}^2 & a_p = 1 \end{cases}$$

Now, define the following maps:

$$g_1 : e_2 \times [0,1] \to \Delta \text{ by } g_1((p,s)) = \phi_{w_\tau}(s)$$
$$g_2 : e_2 \times [0,1] \to \Delta \text{ by } g_2((p,s)) = \left(K^{-1} \circ \ \Xi_p\right)(s).$$

Then g_i is a continuous map such that $(p,s) \sim_i (p',t)$ implies $g_i((p,s)) = g_i((p',t))$ for $i = 1,2$. By the universal property of quotient maps there exists a unique continuous map $\bar{g}_i : (e_2 \times [0,1])/\sim_i \to \Delta$ with $\bar{g}_i\left(\overline{(p,s)}\right) = g_i((p,s))$.

Definition 3.8. Define the edge 1-combing $\Psi : e_2 \times [0,1] \to \Delta$ by

$$\Psi(p,s) = \begin{cases} \left(\bar{g}_1 \circ f_1{}^{-1}\right)(p,s) & (p,s) \in Y_1 \\ \left(\bar{g}_2 \circ f_2{}^{-1}\right)(p,s) & (p,s) \in Y_2 \end{cases}$$

To show that Ψ is well-defined, for $(p,s) \in Y_1 \cap Y_2$ we have $s = a_p$. When $a_p = 0$ we have $(p,0) \sim_1 (p,1)$ and so $\overline{(p,0)}^1 = \overline{(p,1)}^1$. Similarly, when $a_p = 1$ we have $(p,0) \sim_2 (p,1)$ and we have $\overline{(p,0)}^2 = \overline{(p,1)}^2$. Using this information we have,

$$\left(\bar{g}_1\left({f_1}^{-1}(p,a_p)\right)\right) = \bar{g}_1\left(\overline{(p,0)}^1\right) = \bar{g}_1\left(\overline{(p,1)}^1\right) = g_1\left((p,1)\right) = \phi_{w_\tau}(1) = u_2.$$

$$\begin{aligned}\left(\bar{g}_2\left({f_2}^{-1}(p,a_p)\right)\right) \overset{a_p\neq 1}{=} \bar{g}_2\left(\overline{(p,0)}^2\right) = g_2((p,0)) = K^{-1}\left(\Xi_p(0)\right) &= K^{-1}\left(\xi_p(0)\right)\\ &= K^{-1}\left((0,0)\right)\\ &= u_2.\end{aligned}$$

When $a_p \neq 0$ we have

$$\begin{aligned}&\left(\bar{g}_1\left({f_1}^{-1}(p,a_p)\right)\right) = \bar{g}_1\left(\overline{(p,1)}^1\right) = g_1\left((p,1)\right) = \phi_{w_\tau}(1) = u_2\\ &\left(\bar{g}_2\left({f_2}^{-1}(p,a_p)\right)\right) \overset{a_p\neq 1}{=} \bar{g}_2\left(\overline{(p,0)}^2\right) = g_2\left((p,0)\right) = u_2\\ &\left(\bar{g}_2\left({f_2}^{-1}(p,a_p)\right)\right) \overset{a_p=1}{=} \bar{g}_2\left(\overline{(p,1)}^2\right) = \bar{g}_2\left(\overline{(p,0)}^2\right) = g_2\left((p,0)\right) = u_2\end{aligned}$$

Since Y_1 and Y_2 are closed subsets with $e_2 \times [0,1] = Y_1 \cup Y_2$ and we have continuous functions $\left(\bar{g}_1 \circ {f_1}^{-1}\right)$ and $\left(\bar{g}_2 \circ {f_2}^{-1}\right)$ that agree on points in $Y_1 \cap Y_2$, by the Pasting Lemma we have Ψ is continuous.

To show that Ψ is a 1-combing of the pair (Δ, e_2) at the basepoint $*$ let p in e_2. Then at time $s = 0$ we have

$$\Psi(p,0) = (\bar{g}_1 \circ {f_1}^{-1})(p,0) = \bar{g}_1\left(\overline{(p,0)}^1\right) = g_1((p,0)) = \phi_{w_\tau}(0) = *.$$

Similarly, at time $s = 1$ we have

$$\begin{aligned}\Psi(p,1) = (\bar{g_2} \circ {f_2}^{-1})(p,1) = \bar{g_2}\left(\overline{(p,1)}^2\right) = g_2((p,1)) &= K^{-1}\left(\Xi_p(1)\right) \\ &= K^{-1}(\omega p(1)) \\ &= K^{-1}((1, y_p) \\ &= p.\end{aligned}$$

Furthermore, we note in the case when $*$ is a vertex on the edge e_2, then we have $u_1 = u_2 = u_3 = *$, $u_4 = u_5$. Therefore, $J(p) = J(u_3) = 0$, $K(*) = K(u_2) = (0,0)$ and for $s \in [0,1]$ we have,

$$\begin{aligned}\Psi(*,s) = \bar{g_2}\left(f_2^{-1}\left(*, \frac{s-a_p}{1-a_p}\right)\right) = g_2\left(\overline{\left(*, \frac{s-a_p}{1-a_p}\right)}^2\right) &= \left(K^{-1} \circ \Xi_*\right)\left(\frac{s-a_p}{1-a_p}\right) \\ &= K^{-1}\left((0,0)\right) \\ &= *.\end{aligned}$$

Lastly, for u_3 in e_2 we have $\Psi(u_3, \cdot) : [0,1] \to \Delta$ follows the path labeled by w_τ on the interval $[0, a_p]$ and the path labeled by w_σ on the interval $[a_p, 1]$. Similarly, $\Psi(u_5, \cdot)$ follows the path $w_\tau x$ on the interval $[0, a_p]$ and the path labeled by w_σ on the interval $[a_p, 1]$. Since $(e_2)^{(0)} = \{u_3, u_5\}$ we have $\Psi(p,t) \subseteq \Delta^{(1)}$ whenever $p \in \{u_3, u_4\}$ and $t \in [0,1]$. By construction Ψ follows y_g and y_{gx} at a constant speed and so the gluing condition holds. This completes the construction of the edge 1-combing Ψ.

Part 5: f-tameness of (Δ, Ψ)

The final step is to show that the function Ψ is h-tame, for the function $h : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ defined by $h(n) = \displaystyle\sum_{i=1}^{m} \overline{h}_i(n) + n$, where $\overline{h}_i$ is the subnegative closure of h_i for $1 \leq i \leq m$ as defined in Definition 1.1. Note, for $n \in \mathbb{N}[\frac{1}{4}]$ we have $n \leq h(n)$.

Case I: Suppose we have Δ as illustrated in Figure 3.1 (b) and Ψ as defined on page 26. Let p be a point in e_2; we proceed with the following subcases.

- **Subcase (a):** Suppose $0 \leq s < t \leq a_p$. Then

$$\Psi(p, s) = \left(\bar{g_1} \circ {f_1}^{-1}\right)\left(p, \frac{s}{a_p}\right) = \phi_{w_\rho}\left(\frac{s}{a_p}\right).$$

Since $n \leq h(n)$ for all n and ϕ_{w_ρ} is the path from the basepoint $*$ to the vertex u_1 labeled by w_ρ we have,

$$\tilde{d}_\Delta\left(*, \Psi(p, s)\right) \leq \tilde{d}_\Delta\left(*, \Psi(p, t)\right) \leq h\left(\tilde{d}_\Delta\left(*, \Psi(p, t)\right)\right).$$

- **Subcase (b):** Suppose $a_p \leq s < t \leq b_p$. For $q \in \Delta_\tau$ (as a subcomplex of Δ), every path from the basepoint $*$ to q must go through the vertex u_1 and so we have $\tilde{d}_\Delta(*, q) = \tilde{d}_\Delta(*, u_1) + \tilde{d}_\Delta(u_1, q)$. Furthermore, since $(\Delta_\tau, \Omega_\tau)$ is in the geodesic combed $\mathcal{N}_{\alpha_j}$-filling $\mathcal{E}_{\alpha_j}$, we have the edge 1-combing Ω_τ is h_{α_j} -tame.

Combining these two facts together with the fact that $n \leq h(n)$ gives us,

$$\begin{aligned}
\tilde{d}_\Delta\left(*, \Psi(p,s)\right) &= \tilde{d}_\Delta\left(*, (\bar{g_2} \circ {f_2}^{-1})\left(r(p), \frac{s-a_p}{b_p-a_p}\right)\right) \\
&= \tilde{d}_\Delta\left(*, \Omega_\tau\left(r(p), \frac{s-a_p}{b_p-a_p}\right)\right) \\
&= \tilde{d}_\Delta\left(*, u_1\right) + \tilde{d}_{\Delta_\tau}\left(*', \Omega_\tau\left(r(p), \frac{s-a_p}{b_p-a_p}\right)\right) \\
&\leq \tilde{d}_\Delta\left(*, u_1\right) + h_{\alpha_j}\left(\tilde{d}_{\Delta_\tau}\left(*', \Omega_\tau\left(r(p), \frac{t-a_p}{b_p-a_p}\right)\right)\right) \\
&\leq h_{\alpha_j}\left(\tilde{d}_\Delta\left(*, u_1\right)\right) + h_{\alpha_j}\left(\tilde{d}_{\Delta_\tau}\left(*', \Omega_\tau\left(r(p), \frac{t-a_p}{b_p-a_p}\right)\right)\right) \\
&\leq \overline{h}_{\alpha_j}\left(\tilde{d}_\Delta\left(*, u_1\right)\right) + \overline{h}_{\alpha_j}\left(\tilde{d}_{\Delta_\tau}\left(*', \Omega_\tau\left(r(p), \frac{t-a_p}{b_p-a_p}\right)\right)\right) \\
&\leq \overline{h}_{\alpha_j}\left(\tilde{d}_\Delta\left(*, u_1\right) + \tilde{d}_{\Delta_\tau}\left(*', \Omega_\tau\left(r(p), \frac{t-a_p}{b_p-a_p}\right)\right)\right) \\
&= \overline{h}_{\alpha_j}\left(\tilde{d}_\Delta\left(*, \Omega_\tau\left(r(p), \frac{t-a_p}{b_p-a_p}\right)\right)\right) \\
&= \overline{h}_{\alpha_j}\left(\tilde{d}_\Delta\left(*, (\bar{g_2} \circ {f_2}^{-1})\left(r(p), \frac{t-a_p}{b_p-a_p}\right)\right)\right) \\
&= \overline{h}_{\alpha_j}\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right) \\
&= \overline{h}\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right).
\end{aligned}$$

- **Subase (c):** Suppose $b_p \leq s < t \leq 1$. Recall that Θ_σ, thought of as a subcomplex of Δ, is the van Kampen diagram for the word $w_\sigma x {w_\sigma}^{-1} x^{-1}$ with basepoint u_2, and there are no 2-cells strictly in the interior of Θ_σ. Additionally, recall the map η_p is the straight line path in Θ_σ from $r(p)$ to p, and so for $s < t$ we have $\tilde{d}_\Delta\left(u_2, \eta_p(s)\right) \leq \tilde{d}_\Delta\left(u_2, \eta_p(t)\right)$.

Therefore, for $b_p \leq s < t \leq 1$ we have

$$\begin{aligned}\tilde{d}_\Delta\left(*, \Psi(p,s)\right) = \left(\bar{g_3} \circ {f_3}^{-1}\right)\left(p, \frac{s-b_p}{1-b_p}\right) = \eta_{r(p)}\left(\frac{s-b_p}{1-b_p}\right) &\leq \eta_{r(p)}\left(\frac{t-b_p}{1-b_p}\right) \\ &= \tilde{d}_\Delta\left(*, \Psi(p,t)\right) \\ &\leq h\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right).\end{aligned}$$

• **Subcase (d):** Suppose $a_p \leq s < b_p < t \leq 1$. Observe for any point $q \in \Theta_\sigma$ that any path in Δ from $*$ to q must cross the edge e_1 in Δ. Furthermore, from case (c), for $b_p \leq t \leq 1$ we know $\tilde{d}_\Delta\left(*, \Psi(p,b_p)\right) \leq \tilde{d}_\Delta\left(*, \Psi(p,t)\right)$. Putting these two facts together with the fact that h is nondecreasing we have

$$\tilde{d}_\Delta\left(*, \Psi(p,s)\right) \leq h\left(\tilde{d}_\Delta\left(*, \Psi(p,b_p)\right)\right) = h\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right).$$

• **Subcase (e):** Now, suppose $0 \leq s \leq a_p \leq t \leq b_p$. Recall from case (a) we know on the interval $[0, a_p]$ the distance $\tilde{d}_\Delta\left(\Psi(p,\cdot)\right)$ is an nondecreasing function. Furthermore, for $r \in [a_p, 1]$ any path in Δ from $*$ to $\Psi(p,r)$ must go through the vertex u_1 and therefore we have $\tilde{d}_\Delta\left(*, \Psi(p,r)\right) \geq \tilde{d}_\Delta\left(*, u_1\right)$. We are then able to conclude, using case (b), that

$$\tilde{d}_\Delta\left(*, \Psi(p,s)\right) \leq \tilde{d}_\Delta\left(*, \Psi(p,a_p)\right) = \tilde{d}_\Delta\left(*, u_1\right) \leq \tilde{d}_\Delta\left(*, \Psi(p,t)\right) \leq h\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right).$$

In the above subcases we have shown that the edge 1-combing Ψ as defined in Definition 3.7 is h-tame.

Case II: Suppose we have Δ as illustrated in Figure 3.2 (a) and Ψ as defined on page 29. Recall, for any $r \in [0,1]$ the function $\Xi_r : [0,1] \to [0,1] \times [0,1]$ is defined by $\Xi_r(s) = \xi_r(2s)$ for $s \in [0, \frac{1}{2}]$ and $\Xi_r(s) = \omega_r(2s-1)$ for $s \in [\frac{1}{2}, 1]$. Let $p \in e_2$; we proceed with the following subcases.

- **Subcase (a):** Suppose $0 \leq s < t \leq a_p$. Then

$$\Psi(p,s) = \left(\bar{g_1} \circ {f_1}^{-1}\right)\left(p, \frac{s}{a_p}\right) = \phi_{w_\tau}\left(\frac{s}{a_p}\right).$$

Since ϕ_{w_τ} is the path from the basepoint $*$ to the vertex u_2 labeled by $w_\rho w_\tau$ we have

$$\tilde{d}_\Delta\left(*, \Psi(p,s)\right) \leq \tilde{d}_\Delta\left(*, \Psi(p,t)\right) \leq h\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right).$$

- **Subcase (b):** Suppose $a_p \leq s < t \leq 1$. Then $\Psi(p,s) = \left(\bar{g_2} \circ {f_2}^{-1}\right)\left(p, \frac{s-a_p}{1-a_p}\right) = (K^{-1} \circ\ \Xi_p)\left(\frac{s-a_p}{1-a_p}\right)$. Note, for any point $p \in \Theta_\sigma$ we have $\tilde{d}_\Delta(*, p) = \tilde{d}_\Delta(*, u_2) + \tilde{d}_\Delta(u_2, p)$, since any path in Δ from $*$ to p must go through the vertex u_2. Since the map $(K^{-1} \circ\ \Xi_p)$ follows the straight line path in Θ_σ from the basepoint u_3 to $r(p)$ on e_1 and from $r(p)$ to p in e_2. Since Θ_σ has no interior 2-cells, for $s < t$ we have

$$\tilde{d}_\Delta\left(u_3, K^{-1} \circ\ \Xi_p\left(\frac{s-a_p}{1-a_p}\right)\right) \leq \tilde{d}_\Delta\left(u_3, K^{-1} \circ\ \Xi_p\left(\frac{t-a_p}{1-a_p}\right)\right),$$

Therefore we can conclude

$$\begin{aligned}
\tilde{d}_\Delta\left(*, \Psi(p,s)\right) &= \tilde{d}_\Delta\left(*, (K^{-1} \circ\ \Xi_p)\left(\frac{s-a_p}{1-a_p}\right)\right)\\
&= \tilde{d}_\Delta(*, u_2) + \tilde{d}_\Delta\left(u_2, (K^{-1} \circ\ \Xi_p)\left(\frac{s-a_p}{1-a_p}\right)\right)\\
&\leq \tilde{d}_\Delta(*, u_2) + \tilde{d}_\Delta\left(u_2, (K^{-1} \circ\ \Xi_p\left(\frac{t-a_p}{1-a_p}\right)\right)\\
&\leq \tilde{d}_\Delta\left(*, (K^{-1} \circ\ \Xi_p)\left(\frac{t-a_p}{1-a_p}\right)\right)\\
&= \tilde{d}_\Delta\left(*, \Psi(p,t)\right)\\
&\leq h\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right).
\end{aligned}$$

- **Subcase (c):** For the last case, suppose $0 \leq s < a_p < t \leq 1$. For $q \in \Theta_\sigma$ we have any path in Δ from the basepoint $*$ to q must go through vertex u_2 and so $\tilde{d}_\Delta(*, q) \geq \tilde{d}_\Delta\left(*, u_2\right)$.Therefore we have

$$\tilde{d}_\Delta\left(*, \Psi(p,s)\right) \leq \tilde{d}_\Delta\left(*, u_2\right) \leq \tilde{d}_\Delta\left(*, \Psi(p,t)\right) \leq h\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right).$$

In the above two cases we have shown that the edge 1-combing Ψ is h-tame. That is, we have know shown that the pair (Δ, Ψ), where Δ was constructed in Part (3) and Ψ was constructed in Part (4), is h-tame.

In summary, for $e \in E(X(G\Lambda))$ with initial vertex g and label $x \in A$, let y_g be the normal form for g in $\mathcal{N}$. Use Part (1) to factor y_g as $y_g = w_\rho w_\tau w_\sigma$. Let (Δ_e, Ψ_e) be the van Kampen diagram and edge 1-combing for $y_g x y_{gx}^{-1}$ obtained from following Parts (2)-(4). Let $\mathcal{E} = \{(\Delta_e, \Psi_e) | e \in E(X(G\Lambda))\}$. Then $\mathcal{E}$ is a geodesic combed $\mathcal{N}$-filling for $(G\Lambda, \mathcal{P})$ such that each edge 1-combing Ψ_e is h-tame. Applying Lemma 2.8, we have that h is equivalent to an intrinsic tame filling function for graph product $(G\Lambda, \mathcal{P})$. $\square$

Chapter 4

Free Products with Amalgamation

Here we study the behavior of tame filling functions under a special case of amalgamated products. The main result in this chapter is Theorem 4.1. Before we state Theorem 4.1, we recall a few facts about free products with amalgamation.

Let $G = \langle A|R \rangle$ and $H = \langle B|S \rangle$ be finitely presented groups, and let K be a group with injective homomorphisms $\alpha : K \hookrightarrow G$ and $\beta : K \hookrightarrow H$. Then the free product with amalgamation, $G *_K H$, can be presented by

$$G *_K H = \langle A \cup B |\ R \cup S \cup \{\alpha(k) = \beta(k)\ \forall\ k \in K\} \rangle.$$

Given a group $G = \langle A|R \rangle$ with subgroup H, a set S of words over A^* is a *left transversal for* H *in* G if and only if the map $s \mapsto sH$ from S to G/H is a bijection. A left transversal S is *geodesic* if all the words in S label geodesics in the associated Cayley graph of G. From the normal form theorem for amalgamated free products [[10], page 187] we have the following normal forms for elements in $G *_K H$.

$$\mathcal{N} = \{t_1 t_2 \ldots t_n k|\ t_i \text{ alternates between } T_1 \backslash \epsilon \text{ or } T_2 \backslash \epsilon,\ k \in K\},$$

where T_1 and T_2 are left transversal for G/K and H/K, respectively.

Theorem 4.1. *Let* f_α *and* f_β *be intrinsic tame filling functions for* $G_\alpha = \langle A_\alpha | R_\alpha \rangle$ *and* $G_\beta = \langle A_\beta | R_\beta \rangle$*, respectively. Let* $H = \langle h_1, \ldots h_m \rangle$ *be a finitely generated group and let* $\hat{\alpha} : H \hookrightarrow G_\alpha$ *and* $\hat{\beta} : H \hookrightarrow G_\beta$ *be injective homomorphisms. Let* $G = G_\alpha *_H G_\beta$ *with presentation*

$$\mathcal{P} = \langle A_\alpha, A_\beta, h_1, \ldots, h_m |\ R_\alpha \cup R_\beta \cup \{h_i = \hat{\alpha}(h_i), h_i = \hat{\beta}(h_i)\ \ \forall\ 1 \leq i \leq m\}\rangle.$$

Suppose that there is a prefix-closed set of geodesic normal forms for G *of the form*

$$\mathcal{N} = \{t_1 t_2 \ldots t_n y_h |\ t_i \text{ alternates between } T_\alpha \backslash \epsilon \text{ and } T_\beta \backslash \epsilon,\ \ y_h \in \mathcal{N}_H\},$$

where, for $\gamma \in \{\alpha, \beta\}$, T_γ *is a set of geodesic left transversals for* $G_\gamma / \hat{\gamma}(H)$*, containing the empty word, with respect to the following presentation for* G_γ

$$\mathcal{P}_\gamma = \langle A_\gamma, h_1, \ldots, h_m |\ R_\gamma \cup \{h_i = \hat{\gamma}(h_i)\ \ \forall\ 1 \leq i \leq m\}\rangle$$

and $\mathcal{N}_H$ *is a set of prefix-closed geodesic normal forms for* H*. Then* $(G, \mathcal{P})$ *has an intrinsic tame filling function equivalent to*

$$f(n) = \overline{f_\alpha}(n) + \overline{f_\beta}(n) + n.$$

Before we prove Theorem 4.1, we make an observation. If the set of normal forms $\mathcal{N}$ described in the statement of Theorem 4.1 are geodesic normal forms for $G = G_\alpha *_H G_\beta$ then

$$\mathcal{N}_\alpha = \{t y_h |\ t \in T_\alpha, y_h \in \mathcal{N}_H\} \text{ and } \mathcal{N}_\beta = \{t y_h |\ t \in T_\beta, y_h \in \mathcal{N}_H\}, \tag{4.1}$$

are geodesic normal forms for G_α and G_β, respectively.

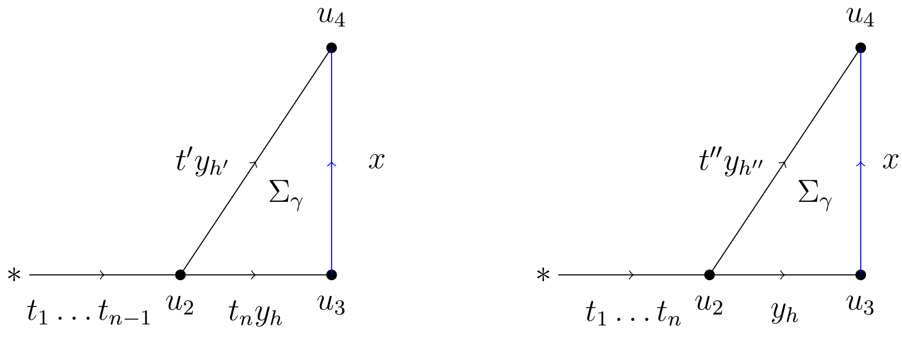
Proof of Theorem 4.1. Let $\mathcal{N}$ be the set of geodesic normal forms for $(G, \mathcal{P})$ described in Theorem 4.1. Since elements of $\mathcal{N}$ are geodesic, from line (4.1), we have a set of geodesic normal forms, $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$, for G_α and G_β, respectively. Let $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ be the presentations for G_α and G_β given in Theorem 4.1. Since f_α is an intrinsic tame filling functions for G_α by Lemma 2.8 $(G_\alpha, \mathcal{P}_\alpha)$ has a geodesic combed $\mathcal{N}_\alpha$-filling, $\mathcal{E}_\alpha$, such that each edge 1-combing is $\tilde{f}_\alpha$-tame where $\tilde{f}_\alpha$ is equivalent to f_α. Similarly, $(G_\beta, \mathcal{P}_\beta)$ has a geodesic combed $\mathcal{N}_\beta$-filling, $\mathcal{E}_\beta$, such that each edge 1-combing is $\tilde{f}_\beta$-tame where $\tilde{f}_\beta$ is equivalent to f_β. Again, since $f \sim g$ implies $\overline{f} \sim \overline{g}$ with a slight abuse of notation, we will simply denote $\tilde{f}_\alpha$ and $\tilde{f}_\beta$ as f_α and f_β.

We will construct a geodesic combed $\mathcal{N}$-filling for $(G, \mathcal{P})$ such that each edge 1-combing is f-tame. Similar to the proof of Theorem 3.5 we divide this proof into four parts: (1) compute the normal form for $y_g x$, (2) construct a van Kampen diagram Δ for $y_g x y_{yx}^{-1}$, (3) construct an edge 1-combing Ψ, and (4) show that Ψ is f-tame, where f is defined in the statement of Theorem 4.1.

Part 1: Computation of y_{gx}

Let $A = A_\alpha \cup A_\beta \cup \{h_1, \ldots, h_m\}$ and let $x \in A$. Let $\bar{A}_\alpha = A_\alpha \cup \{h_1 \ldots h_m\}$ and let $\bar{A}_\beta = A_\beta \cup \{h_1 \ldots h_m\}$. For g in G, let $y_g = t_1 t_2 \ldots t_n y_h$ be the normal form for g in $\mathcal{N}$. We consider the word $y_g x = t_1 \ldots t_n y_h x$.

Suppose $t_n \in T_\gamma$, $x \in \bar{A}_\gamma$, and $y_h x \notin \bar{A}_\gamma$ for some $\gamma \in \{\alpha, \beta\}$. Then $t_n y_h \in \mathcal{N}_\gamma$ and there exists $t' y_{h'} \in \mathcal{N}_\gamma$ such that $t_n y_h x =_{G_\gamma} t' y_{h'}$. Therefore, the normal form for gx in $\mathcal{N}$ is $y_{gx} = t_1 \ldots t_{n-1} t' y_{h'}$. Next, suppose $t_n \in T_\eta$ and $x \in \bar{A}_\mu$ for $(\eta, \mu) \in \{(\alpha, \beta), (\beta, \alpha)\}$. Then $y_h \in \mathcal{N}_\mu$ and so $y_h x \in G_\mu$. Therefore, there exist $t'' y_{h''} \in \mathcal{N}_\mu$ such that $y_h x =_{G_\mu} t'' y_{h''}$. This implies that the normal form for gx in $\mathcal{N}$ is $y_{gx} =_{G_\beta} t_1 \ldots t_n t'' y_{h''}$. Lastly, suppose $t_n \in T_\gamma$ and $x \in \{h_1, \ldots, h_m\}$ for some $\gamma \in \{\alpha, \beta\}$. Observe, $y_h \in \mathcal{N}_\alpha \cap \mathcal{N}_\beta$ and therefore $y_h \in T_\gamma$. We then have $t_n y_h \in \mathcal{N}_\gamma$ and x is in $\bar{A}_\alpha$. The normal form for y_{gx} is then obtained by applying the previous two cases. In summary, we have shown that given $y_g \in \mathcal{N}$ and $x \in A$, we have the following possible normal forms for y_{gx}: (1) $y_{gx} = t_1 \ldots t_{n-1} t' y_{h'}$ or (2) $y_{gx} = t_1 \ldots t_n t'' y_{h''}$.



(I) (II)

Figure 4.1: van Kampen diagram Δ.

Part 2: Construction of a van Kampen diagram Δ for $y_g x y_{gx}^{-1}$

Next, we construct a van Kampen diagram for $y_g x y_{gx}^{-1}$ for each case above. In Case I, we have $t_n y_h \in \mathcal{N}_\gamma$ and $x \in A_\gamma$ for some $\gamma \in \{\alpha, \beta\}$ and so there exists a pair $(\Sigma_\gamma, \Omega_\gamma)$ in $\mathcal{E}_\gamma$ such that Σ_γ is a van Kampen diagram for $t_n y_h x (t' y_{h'})^{-1}$ and Ψ_γ is an f_γ-tame 1-edge combing of Σ_γ. Let Φ be a simple edge path labeled by $t_1 \dots t_{n-1}$ with initial vertex $*$ and terminal vertex u_2. Adjoin the basepoint of Σ_γ to the vertex u_2 and let Δ be the resulting van Kampen diagram, illustrated in Figure 4.1 (I).

In Case II, we have $t_n \in T_\eta$, $y_h \in \mathcal{N}_\mu$ and $x \in A_\mu$ for $(\eta, \mu) \in \{(\alpha, \beta), (\beta, \alpha)\}$. Therefore, there exists a pair (Σ_μ, Ω_μ) in $\mathcal{E}_\eta$ for the word $y_h x (t'' y_{h''})^{-1}$. Let Φ be a simple edge path labeled by $t_1 \dots t_n$ with boundary $*$ and u_2. Let Δ be the diagram obtained by adjoining the basepoint of Σ_η to vertex u_2 as illustrated in Figure 4.1 (II). It is worth noting that Case II also covers the case when $n = 0$.

Part 3: Construction of an edge 1-combing Ψ for Δ

Next, we construct an edge 1-combing Ψ for the van Kampen diagram Δ. We will construct Ψ for Case I, in which Δ as illustrated in Figure 4.1 (I), and note that the construction of Ψ for the other case is similar.

Let i, j and k be the lengths of $t_1 \dots t_{n-1}$, $t_n y_h$ and $t' y_{h'}$, respectively. Let $(\Sigma_\gamma, \Omega_\gamma)$ and Δ be as defined in Case I of Part 2. With a slight abuse of notation, we can think of Σ_γ as a subcomplex of Δ with the basepoint u_2. Let e be the edge on the boundary of Δ labeled by x in $y_g x y_{gx}^{-1}$. Furthermore, label the initial and terminal vertices of the edge e as u_3 and u_4, as illustrated in Figure 4.1. Let $J : e \to [0,1]$ be a homeomorphism with $J(u_3) = 0$ and $J(u_4) = 1$.

To aid with computation we introduce the following notation:

$$m_g = \frac{i}{i+j}\ , \qquad m_{gx} = \frac{i}{i+k}\ , \qquad n_g = \frac{j}{i+j}\ , \qquad n_{gx} = \frac{k}{i+k}$$

For $p \in e_2$, let $a_p = m_g + J(p)\,(m_{gx} - m_g)$. Let $\phi : [0,1] \to \Delta$ be the continuous constant speed map in Δ_1 from vertices $*$ to u_2 such that $\phi(0) = *$ and $\phi(1) = u_2$. Recall the pair $(\Sigma_\gamma, \Omega_\gamma) \in \mathcal{E}_\gamma$ is associated to the word $w' = (t_n y_h) x (t' y_{h'})^{-1}$ and $\gamma \in \{\alpha, \beta\}$. Let $Y_1 = \{(p,s) | 0 \le s \le a_p\}$ and $Y_2 = \{(p,s) | a_p \le s \le 1\}$ and define the following continuous maps:

$$q_1 : e \times [0,1] \to Y_1 \text{ by } q_1((p,s)) = (p, a_p s)$$

$$q_2 : e \times [0,1] \to Y_2 \text{ by } q_2((p,s)) = (p, a_p + s(1 - a_p)).$$

Let $i \in 1,2$. Let $\pi_i : e \times [0,1] \to (e \times [0,1])/\sim_i$ be the quotient map, where $(p,s) \sim_i (p',t)$ if and only if $q_i((p,s)) = q_i((p',t))$. By the universal property of quotient maps there exists a unique continuous map $k_i : (e \times [0,1])/\sim_i \to Y_i$ such that $q_i = k_i \pi_i$ with $k_i\left(\overline{(p,s)}^i\right) = q_i((p,s))$, where $\overline{(p,s)}^i$ denotes the equivalence class of (p,s) under $\sim_i$. Since $e \times [0,1]$ is a compact space and Y_i is a Hausdorff space we have k_i is in fact a homeomorphism and we have

$$k_1^{-1}(p,s) = \begin{cases} \overline{\left(p, \dfrac{s}{a_p}\right)}^1, & a_p \neq 0 \\ \overline{(p,0)}^1 & a_p = 0 \end{cases} \qquad k_2^{-1}(p,s) = \begin{cases} \overline{\left(p, \dfrac{s-a_p}{1-a_p}\right)}^2, & a_p \neq 1 \\ \overline{(p,1)}^2 & a_p = 1 \end{cases}$$

Now, define the following maps:

$$g_1 : e \times [0,1] \to \Delta \text{ by } g_1((p,s)) = \phi_(s)$$
$$g_2 : e \times [0,1] \to \Delta \text{ by } g_2((p,s)) = \Omega_\gamma\left((p,s)\right).$$

Observe g_1 and g_2 are continuous since ϕ and Ω_γ are continuous maps. Next, we show given $(p,s) \sim_i (p',t)$ we have $g_i\left((p,s)\right) = g_i\left((p',t)\right)$, for $i = 1,2$. We explicitly show this in the case $i = 2$, and note the case $i = 1$ is similar.

Suppose $(p,s) \sim_2 (p',t)$. Then $q_2\left((p,s)\right) = q_2\left((p',t)\right)$, which implies $(p, a_p + s(1-a_p)) = (p', a_{p'} + t(1-a_{p'}))$. Therefore $p = p'$ and $a_p + s(1-a_p) = a_{p'} + t(1-a_{p'})$. Since the map $J : e \to [0,1]$ is injective, having $p = p'$ implies that $a_p = a_{p'}$. Furthermore,

$$\begin{aligned} a_p + s(1-a_p) &= a_{p'} + t(1-a_{p'}) \\ &= a_p + t(1-a_p), \end{aligned}$$

implies $0 = (s-t)(1-a_p)$. Therefore, $(p,s) \sim_2 (p',t)$ when $(p,s) = (p',t)$ or when $p = p'$ and $a_p = 1$.

Applying the definition of a_p above to the equation $a_p = 1$ gives us $0 = n_g + J(p)\,(n_{gx} - n_g) = n_g\,(1 - J(p)) + J(p)n_{gx}$. Since $n_g, J(p), n_{gx}, (1 - J(p)) \geq 0$, we must have $n_g(1 - J(p)) = 0$ and $J(p)n_{gx} = 0$. We can then conclude $(p, s) \sim_2 (p', t)$ when we either have: (1) $p = p'$ and $s = t$, (2) $p = p'$, $n_g = 0$, and $J(p) = 0$, (3) $p = p'$, $n_g = 0$, and $n_{gx} = 0$, or (4) $p = p'$, $J(p) = 1$ and $n_{gx} = 0$.

When $(p, s) = (p', t)$, then we immediately have $g_2((p, s)) = g_2((p', t))$. When $p = p'$, $n_g = 0$, and $J(p) = 0$ we have $n_g = 0$ which implies $t_n y_h = \epsilon$. Also $J(p) = 0$ implies that $p = u_3$, and so we have that $\Omega_\gamma(u_3, \cdot)$ is the constant path at u_3. Since $p = p'$ we also have $\Omega_\gamma(p', \cdot)$ is the constant path at u_2, hence $g_2\left((p, s)\right) = g_2\left((p', t)\right)$. Next, suppose $p = p'$, $n_g = 0$, and $n_{gx} = 0$. Since $n_g = 0$ and $n_{gx} = 0$, then $t_n y_h = \epsilon$ and $t' y_{h'} = \epsilon$, respectively. Thus $t_n y_h x {t' y_{h'}}^{-1} = \epsilon$ implies that x is the identity element in G. Since the empty word ϵ is the unique normal form for the identity element of G in $\mathcal{N}$, we can not have $x = \epsilon$, so this case cannot occur.

Finally, suppose we have $p = p'$, $J(p) = 1$ and $n_{gx} = 0$. Since $J(p) = 1$ we have $p = u_4$. Additionally, since $n_{gx} = 0$ then $t' y_{h'} = \epsilon$ and we have the path $\Omega_\gamma(u_4, \cdot)$, which starts at u_3 and follows along $t' y_{h'} = \epsilon$, is the constant path at u_3. Furthermore, since $p = p'$ we have $\Omega_\gamma(p', \cdot)$ is also the constant path at u_2, and hence $g_2\left((p, s)\right) = g_2\left((p', t)\right)$. This completes the proof that for $(p, s) \sim_2 (p', t)$ we have $g_2\left((p, s)\right) = g_2\left((p', t)\right)$.

Since $g_i : e \times [0, 1] \to \Delta$ is a continuous map such that $(p, s) \sim_i (p', t)$ implies $g_i((p, s)) = g_i((p', t))$ for all (p, s) and (p', t) in $e \times [0, 1]$, by the universal property of quotient maps there exists a unique continuous function $\bar{g_i} : (e \times [0, 1])/\sim_i \to \Delta$ such that $g_i = \bar{g_i} \circ \pi_i$ for $i = 1, 2$. In particular, we have $\bar{g_i}\left(\overline{(p, s)}^{i}\right) = g_i((p, s))$.

We are now able to define the edge 1-combing $\Psi : e \times [0,1] \to \Delta$ by

$$\Psi(p,s) = \begin{cases} \left(\bar{g}_1 \circ {k_1}^{-1}\right)(p,s) & (p,s), \in Y_1 \\ \\ \left(\bar{g}_2 \circ {k_2}^{-1}\right)(p,s) & (p,s), \in Y_2 \end{cases}$$

To show Ψ is well-defined, for $(p,s) \in Y_1 \cap Y_2$ we have $s = a_p$. If $a_p = 0$, then observe we have $q_1((p,0)) = (p,0)$ and $q_1((p,1)) = (p,a_p) = (p,0)$ therefore $(p,0) \sim_1 (p,1)$ and so $\overline{(p,0)}^1 = \overline{(p,1)}^1$. Furthermore, when $a_p = 0$ we have

$$\left(\bar{g}_1\left({k_1}^{-1}(p,a_p)\right)\right) = \bar{g}_1\left(\overline{(p,0)^1}\right) = \bar{g}_1\left(\overline{(p,1)}^1\right) = g_1((p,1)) = \phi(1) = u_2$$

$$\begin{aligned}\left(\bar{g}_2\left({k_2}^{-1}(p,a_p)\right)\right) = \bar{g}_2\left(\overline{\left(p, \frac{a_p - a_p}{1-a_p}\right)}^2\right) = \bar{g}_2\left(\overline{(p,0)}^2\right) &= g_2\left((p,0)\right) \\ &= \Omega_\gamma\left((p,0)\right) \\ &= u_2.\end{aligned}$$

If $a_p = 1$, we have $(p,0) \sim_2 (p,1)$ and so $\overline{(p,0)}^2 = \overline{(p,1)}^2$. We additionally have

$$\left(\bar{g}_1\left({k_1}^{-1}(p,a_p)\right)\right) = \bar{g}_1\left(\overline{(p,1)}^1\right) = g_1((p,1)) = \phi(1) = u_2$$

$$\begin{aligned}\left(\bar{g}_2\left({k_2}^{-1}(p,a_p)\right)\right) = \bar{g}_2\left(\overline{(p,1)}^2\right) = \bar{g}_2\left(\overline{(p,0)}^2\right) &= g_2\left((p,0)\right) \\ &= \Omega_\gamma\left((p,0)\right) \\ &= u_2.\end{aligned}$$

Lastly, if $a_p \neq 0$ and $a_p \neq 1$, we have

$$\left(\bar{g_1}\left({k_1}^{-1}(p, a_p)\right)\right) = \bar{g_1}\left(\overline{(p,1)}^1\right) = g_1((p,1)) = \phi(1) = u_2$$
$$\left(\bar{g_2}\left({k_2}^{-1}(p, a_p)\right)\right) = \bar{g_2}\left(\overline{(p,0)}^2\right) = g_2\left((p,0)\right) = \Omega_\gamma\left((p,0)\right) = u_2.$$

Since Y_1 and Y_2 are closed subsets with $e_2 \times [0,1] = Y_1 \cup Y_2$ and we have continuous functions $\left(\bar{g_1} \circ {k_1}^{-1}\right)$ and $\left(\bar{g_2} \circ {k_2}^{-1}\right)$ that agree on points in $Y_1 \cap Y_2$, then by the Pasting Lemma the map Ψ is continuous. We have constructed Ψ to be a constant speed path on the endpoints of the edge e, and so the gluing condition (g1) of Definition 2.7 is satisfied.

Part 4: f-tameness of (Δ, Ψ)

The final step is to analyze the f-tameness of Ψ for $f : \mathbb{N}[\frac{1}{4}] \to \mathbb{N}[\frac{1}{4}]$ defined by $f(n) = \overline{f_\alpha}(n) + \overline{f_\beta}(n) + n$. Note we have $n \leq f(n)$ for $n \in \mathbb{N}[\frac{1}{4}]$.
Additionally for $n, m \in \mathbb{N}[\frac{1}{4}]$ we have

$$\begin{aligned} f(n) + f(m) &= \overline{f_\alpha}(n) + \overline{f_\beta}(n) + n + \overline{f_\alpha}(m) + \overline{f_\beta}(m) + m \\ &= \overline{f_\alpha}(n) + \overline{f_\alpha}(m) + \overline{f_\beta}(n) + \overline{f_\beta}(m) + n + m \\ &\leq \overline{f_\alpha}(n+m) + \overline{f_\beta}(n+m) + n + m \\ &= f(n+m), \end{aligned}$$

So f is subnegative. In the following 3 cases we analyze the tameness of Ψ.

- **Case (a).** Suppose $p \in e$ and $0 \leq s < t \leq a_p$. Since the map $(\bar{g_1} \circ {k_1}^{-1})(p,s) = \phi(s)$ follows the geodesic edge path labeled by $t_1 \ldots t_{n-1}$ on the interval $[0, a_p]$ we have $\tilde{d}_\Delta\left(*, \Psi(p,s)\right) = \tilde{d}_\Delta\left(*, \phi(s)\right) \leq \tilde{d}_\Delta\left(*, \phi(t)\right) = \tilde{d}_\Delta\left(*, \Psi(p,t)\right)$ for $s < t$.

Therefore, since $n \leq f(n)$

$$\tilde{d}_\Delta\left(*, \Psi(p,s)\right) \leq \tilde{d}_\Delta\left(*, \Psi(p,t)\right) \leq f\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right). \tag{4.2}$$

- **Case (b).** Suppose $p \in e$ and $0 \leq s < a_p < t \leq 1$. Since $0 \leq s < a_p$ and $\Psi(p, a_p) = u_2$ in Δ, from case (a) we have

$$\tilde{d}_\Delta\left(*, \Psi(p,s)\right) \leq f\left(\tilde{d}_\Delta\left(*, \Psi(p,a_p)\right)\right) = f\left(\tilde{d}_\Delta\left(*, u_2\right)\right). \tag{4.3}$$

Furthermore, for $q \in \Sigma_\gamma$ every path from the basepoint $*$ to q must go through the vertex u_2 giving us $d_\Delta(*, q) = d_\Delta(*, u_2) + d_\Delta(u_2, q)$. In particular, since $a_p < t \leq 1$ we have $\Psi(p,t) \in \Sigma_\gamma$. This gives us

$$\tilde{d}_\Delta\left(*, \Psi(p,t)\right) = \tilde{d}_\Delta\left(*, u_2\right) + \tilde{d}_\Delta\left(u_2, \Psi(p,t)\right). \tag{4.4}$$

Thus, when $p \in e$ we have

$$\begin{aligned}
\tilde{d}_\Delta\left(*, \Psi(p,s)\right) &\leq f\left(\tilde{d}_\Delta\left(*, u_2\right)\right) && \text{from (4.10)} \\
&\leq f\left(\tilde{d}_\Delta\left(*, u_2\right)\right) + f\left(\tilde{d}_\Delta\left(u_2, \Psi(p,t)\right)\right) && \\
&\leq f\left(\tilde{d}_\Delta\left(*, u_2\right) + \tilde{d}_\Delta\left(u_2, \Psi(p,t)\right)\right) && \text{from (4.8)} \\
&= f\left(\tilde{d}_\Delta\left(*, \Psi(p,t)\right)\right) && \text{from (4.11)}
\end{aligned}$$

- **Case (c).** Suppose $p \in e$ and $a_p \leq s < t \leq 1$. Recall $(\Sigma_\gamma, \Omega_\gamma) \in \mathcal{E}_\gamma$ and therefore the edge 1-combing Ω_γ is f_γ-tame. Additionally, on the interval $[a_p, 1]$, we have the path $\Psi(p, \cdot)$ in contained in Σ_γ. Since every path in Δ from $*$ to any point $q \in \Sigma_\gamma$ must go through the vertex u_2, for $s \in [a_p, 1]$ we have

$\tilde{d}_\Delta(*, \Omega_\gamma(q, s)) = \tilde{d}_\Delta(*, u_2) + \tilde{d}_\Delta(u_2, \Omega_\gamma(q, s))$. Observe the embedding of Σ_γ into Δ maps the basepoint of Σ_γ, $*'$, to the vertex u_2 and maps n-cells to n-cells preserving edge labels and orientation. Therefore, for $q \in \Sigma_\gamma$ we have $\tilde{d}_\Delta(u_2, q) = \tilde{d}_{\Sigma_\gamma}(*', q)$. Thus we have

$$\begin{aligned}
\tilde{d}_\Delta\left(*, \Psi(p, s)\right) &= \tilde{d}_\Delta\left(*, \left(\bar{g_2} \circ {k_2}^{-1}\right)(p, s)\right) \\
&= \tilde{d}_\Delta\left(*, g_2\left(p, \frac{s - a_p}{1 - a_p}\right)\right) \\
&= \tilde{d}_\Delta\left(*, \Omega_\gamma\left(p, \frac{s - a_p}{1 - a_p}\right)\right) \\
&= \tilde{d}_\Delta\left(*, u_2\right) + \tilde{d}_\Delta\left(u_2,\ \Omega_\gamma\left(p, \frac{s - a_p}{1 - a_p}\right)\right) \\
&\le \tilde{d}_\Delta\left(*, u_2\right) + f_\gamma\left(\tilde{d}_\Delta\left(u_2,\ \Omega_\gamma\left(p, \frac{t - a_p}{1 - a_p}\right)\right)\right) \\
&\le f_\gamma\left(\tilde{d}_\Delta\left(*, u_2\right)\right) + f_\gamma\left(\tilde{d}_\Delta\left(u_2,\ \Omega_\gamma\left(p, \frac{t - a_p}{1 - a_p}\right)\right)\right) \\
&\le \overline{f_\gamma}\left(\tilde{d}_\Delta\left(*, u_2\right)\right) + \overline{f_\gamma}\left(\tilde{d}_\Delta\left(u_2,\ \Omega_\gamma\left(p, \frac{t - a_p}{1 - a_p}\right)\right)\right) \\
&\le \overline{f_\gamma}\left(\tilde{d}_\Delta\left(*, u_2\right) + \tilde{d}_\Delta\left(u_2,\ \Omega_\gamma\left(p, \frac{t - a_p}{1 - a_p}\right)\right)\right) \qquad \text{subnegativity of } \overline{f_\gamma} \\
&= \overline{f_\gamma}\left(\tilde{d}_\Delta\left(*, \Omega_\gamma\left(\tilde{p}, \frac{t - a_p}{1 - a_p}\right)\right)\right) \\
&= \overline{f_\gamma}\left(\tilde{d}_\Delta\left(*, g_2\left(p, \frac{t - a_p}{1 - a_p}\right)\right)\right) \\
&= \overline{f_\gamma}\left(\tilde{d}_\Delta\left(*, \Psi(p, t)\right)\right) \\
&\le f\left(\tilde{d}_\Delta\left(*, \Psi(p, t)\right)\right).
\end{aligned}$$

In the above three cases, we showed that the edge 1-combing Ψ of Δ is f-tame. Therefore, $(G, \mathcal{P})$ has a geodesic combed $\mathcal{N}$-filling $\mathcal{E}$ such that each edge 1-combing is f-tame, and by Lemma 2.8 the function f is equivalent to an intrinsic tame filling function for $(G, \mathcal{P})$. □

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