May 2001

Scattering of elastic waves in heterogeneous media with local isotropy

Joseph A. Turner
University of Nebraska - Lincoln, jaturner@unl.edu

Phanidhar Anugonda
University of Nebraska - Lincoln

Scattering of elastic waves in heterogeneous media with local isotropy

Joseph A. Turner and Phanidhar Anugonda
Department of Engineering Mechanics, W317.4 Nebraska Hall, University of Nebraska–Lincoln, Lincoln, Nebraska 68588-0526

(Received 14 August 2000; revised 20 February 2001; accepted 23 February 2001)

The scattering of elastic waves in heterogeneous media is discussed. Explicit expressions are derived for the attenuation of longitudinal and transverse elastic waves in terms of the statistics of the density and Lamé parameter fluctuations. The derivation is based upon diagrammatic methods with the problem posed in terms of the Dyson equation. The Dyson equation is solved for the mean field response. The results are given here in a straightforward manner, in which the attenuations reduce to simple integrals on the unit circle. The medium is assumed statistically homogeneous and statistically isotropic. This model, with assumed local isotropic properties, is expected to apply to many materials. © *2001 Acoustical Society of America.* [DOI: 10.1121/1.1367245]

PACS numbers: 43.20.Bi, 43.20.Gp, 43.35.Cg [DEC]

I. INTRODUCTION

The study of wave propagation and scattering of elastic waves in heterogeneous media is related to nondestructive testing, materials characterization, acoustic emission, and seismic wave analysis. An improved understanding of the effects of scattering provides insight into the microstructure of a variety of materials such as polycrystalline metals and ceramics, composites, geophysical materials, and concrete. Elastic waves which propagate through such media lose energy due to scattering from the heterogeneous structure of the material. The scattering effects may be characterized by the attenuation. Previous research on scattering problems of this nature has been dominated by studies of polycrystalline materials.[1–4] In these models, it is assumed that density fluctuations are negligible and that the material is locally anisotropic. The grains are usually assumed to be randomly oriented, such that the medium is statistically isotropic. More general cases have also been examined[5–7] in which the grains have some prescribed alignment (texture) such that the medium is statistically anisotropic. This research, in which expressions for attenuation were derived, was also extended to derivations of elastic radiative transfer equations (RTE) and diffusion equations which describe the evolution of diffuse elastic energy.[3,4,8,9] More recent developments include the derivation of the attenuations and elastic RTE using an asymptotic approach.[10,11] In those articles, a different microstructural model was used. The elastic moduli were assumed to be locally isotropic and the density was considered to vary spatially. The medium was assumed to be statistically homogeneous and statistically isotropic as well. Such a model is expected to be reasonable for geophysical materials and concrete. They derived the differential scattering cross sections, elastic radiative transfer equations, and the elastic diffusion equation.

In this article, the same microstructural model based on local material isotropy is used to derive elastic wave attenuations. The derivation is based upon the diagrammatic approach[3,12] in which the mean response is governed by the Dyson equation. The Dyson equation is easily solved in spatial Fourier transform domain within the limits of the first-order smoothing approximation (FOSA),[12] or Keller[13] approximation. A further approximation is also made which restricts the results to frequencies below the high-frequency geometric optics limit. This high-frequency limit, in which refracted ray analysis must be used,[2] is above the range of most ultrasonic experiments. With this approximation, the attenuations for the longitudinal and transverse elastic waves reduce to simple integrations on the unit circle. The results here are in basic agreement with those of Ryzhik *et al.*[10] Therefore, their asymptotic method is presumed to be equivalent to the FOSA with the additional frequency limitation.

In the next section, the theoretical model is presented in terms of the Dyson equation. The Dyson equation is solved and expressions for the attenuations derived. Then expressions for the mean free paths and elastic diffusivity are presented. Finally, further assumptions of the form of the fluctuations are made and example calculations are presented.

II. MEAN RESPONSE

The equation of motion for the elastodynamic response of a linear, elastic material to deformation is given in terms of the Green's dyadic by

$$\left\{ -\delta_{jk}\rho(\mathbf{x})\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial x_i}C_{ijkl}(\mathbf{x})\frac{\partial}{\partial x_l}\right\} G_{k\alpha}(\mathbf{x},\mathbf{x}';t) = \delta_{j\alpha}\delta^3(\mathbf{x}-\mathbf{x}')\,\delta(t), \tag{1}$$

where $\delta^3(\mathbf{x}-\mathbf{x}')$ is the three-dimensional spatial Dirac delta function. The Green's dyadic, $G_{ij}(\mathbf{x},\mathbf{x}';t)$, is the response at location $\mathbf{x}$ in the ith direction due to an impulsive force applied at $\mathbf{x}'$ in the jth direction. In Eq. (1), $\rho(\mathbf{x})$ and $C_{ijkl}(\mathbf{x})$ define the material density and elastic modulus tensor, respectively. These material properties are assumed to vary spatially.

A spatio-temporal Fourier transform pair is defined as

$$\tilde{f}(\mathbf{p},\omega)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\mathbf{x},t)e^{i\omega t}e^{-i\mathbf{x}\cdot\mathbf{p}}\,d^3x\,dt, \tag{2}$$

$$f(\mathbf{x},t)=\frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{f}(\mathbf{p},\omega)e^{-i\omega t}e^{i\mathbf{x}\cdot\mathbf{p}}\,d^3p\,d\omega. \tag{3}$$

This transform pair defines the relation between space–time variables ($\mathbf{x}$ and t) and wave vector-angular frequency variables ($\mathbf{p}$ and ω).

The temporal transform of the equation of motion, Eq. (1), is then

$$\left\{\omega^2\delta_{jk}\rho(\mathbf{x})+\frac{\partial}{\partial x_i}C_{ijkl}(\mathbf{x})\frac{\partial}{\partial x_l}\right\}G_{k\alpha}(\mathbf{x},\mathbf{x}';\omega)$$
$$=\delta_{j\alpha}\delta^3(\mathbf{x}-\mathbf{x}'). \tag{4}$$

The material properties of the medium are assumed to vary spatially. The density is written

$$\rho(\mathbf{x})=\overline{\rho}(1+\delta\rho(\mathbf{x})), \tag{5}$$

where $\overline{\rho}$ is the average density and $\delta\rho(\mathbf{x})$ is a dimensionless measure of the density fluctuations.

Previous wave propagation studies of polycrystalline materials have used a locally anisotropic model which accounts for the crystal anisotropy.[2,3,7] Here, the elastic modulus tensor C_{ijkl} is assumed to be locally isotropic. It is written as

$$C_{ijkl}(\mathbf{x})=\overline{\lambda}(1+\delta\lambda(\mathbf{x}))\delta_{ij}\delta_{kl}+\overline{\mu}(1+\delta\mu(\mathbf{x}))$$
$$\times(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{6}$$

where $\overline{\lambda}$ and $\overline{\mu}$ are the average Lamé parameters. The elastic moduli fluctuations are defined by the dimensionless measures $\delta\lambda(\mathbf{x})$ and $\delta\mu(\mathbf{x})$. The material properties are assumed to be centered random processes such that $\langle\delta\rho(\mathbf{x})\rangle=\langle\delta\lambda(\mathbf{x})\rangle=\langle\delta\mu(\mathbf{x})\rangle=0$, where the brackets $\langle\ \rangle$ denote an ensemble average.

The derivation which follows also requires second-order statistics of the fluctuations. The covariance of the density is defined as

$$R_{\rho\rho}(\mathbf{y}-\mathbf{z})=\langle\delta\rho(\mathbf{y})\delta\rho(\mathbf{z})\rangle. \tag{7}$$

Similar definitions for the covariance (autocorrelation) of Lamé parameters and the cross correlations between different parameters are also made. The average medium is assumed statistically isotropic and statistically homogeneous. These assumptions imply that the correlation functions depend only on the magnitude of the difference of the two positions. Mathematically, these assumptions imply that $R(\mathbf{y}-\mathbf{z})=R(r)$, where $r=|\mathbf{y}-\mathbf{z}|$.

The bare Green's dyadic, $\mathbf{G}^0$, is defined as the solution to Eq. (4) when the fluctuations of all material properties are zero. It is the solution to

$$\left\{\omega^2\delta_{jk}\overline{\rho}+(\overline{\lambda}+\overline{\mu})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}+\overline{\mu}\delta_{jk}\frac{\partial^2}{\partial x_l^2}\right\}G^0_{k\alpha}(\mathbf{x},\mathbf{x}';\omega)$$
$$=\delta_{j\alpha}\delta^3(\mathbf{x}-\mathbf{x}'). \tag{8}$$

A spatial Fourier transform, as defined in Eq. (2), allows Eq. (8) to be reduced to

$$\{\hat{p}_j\hat{p}_k(\overline{\rho}\omega^2-p^2(\overline{\lambda}+2\overline{\mu}))+(\delta_{jk}-\hat{p}_j\hat{p}_k)$$
$$\times(\overline{\rho}\omega^2-p^2\overline{\mu})\}G^0_{k\alpha}(\mathbf{p})=\delta_{j\alpha}. \tag{9}$$

The solution for $\mathbf{G}^0$ is given by inspection as

$$\mathbf{G}^0(\mathbf{p})=g^0_L(p)\hat{\mathbf{p}}\hat{\mathbf{p}}+g^0_T(p)(\hat{\mathbf{p}}_2\hat{\mathbf{p}}_2+\hat{\mathbf{p}}_3\hat{\mathbf{p}}_3), \tag{10}$$

for propagation in the $\hat{\mathbf{p}}$ direction. The unit vectors $\hat{\mathbf{p}}_2$ and $\hat{\mathbf{p}}_3$ are transverse to the direction $\hat{\mathbf{p}}$, and form an orthonormal basis with $\hat{\mathbf{p}}$. The bare longitudinal and transverse propagators, which appear in Eq. (10), are

$$g^0_L(p)=[\overline{\rho}\omega^2-p^2(\overline{\lambda}+2\overline{\mu})]^{-1}=[\overline{\rho}\omega^2-p^2\overline{\rho}c_L^2]^{-1}, \tag{11}$$

$$g^0_T(p)=[\overline{\rho}\omega^2-p^2\overline{\mu}]^{-1}=[\overline{\rho}\omega^2-p^2\overline{\rho}c_T^2]^{-1}, \tag{12}$$

where average wave speeds are defined in terms of the average material properties ($\overline{\rho}c_L^2=\overline{\lambda}+2\overline{\mu}$, $\overline{\rho}c_T^2=\overline{\mu}$). The imaginary parts of the propagators, which are used below, are given by

$$\operatorname{Im} g^0_L(p)=-\frac{\pi}{\overline{\rho}}\operatorname{sgn}(\omega)\,\delta(\omega^2-p^2c_L^2), \tag{13}$$

$$\operatorname{Im} g^0_T(p)=-\frac{\pi}{\overline{\rho}}\operatorname{sgn}(\omega)\,\delta(\omega^2-p^2c_T^2). \tag{14}$$

Studies of wave propagation in heterogeneous materials do not lend themselves to solution by perturbation methods. Solutions of this sort do not converge.[12] Instead, Frisch used diagrammatic methods for solution of the mean response. The mean response, $\langle\mathbf{G}\rangle$, is governed by the Dyson equation which is given by[3,7,12]

$$\langle G_{i\alpha}(\mathbf{x},\mathbf{x}')\rangle=G^0_{i\alpha}(\mathbf{x},\mathbf{x}')+\int\int G^0_{i\beta}(\mathbf{x},\mathbf{y})M_{\beta j}(\mathbf{y},\mathbf{z})$$
$$\times\langle G_{j\alpha}(\mathbf{z},\mathbf{x}')\rangle d^3y\,d^3z. \tag{15}$$

In Eq. (15), the quantity $\mathbf{G}^0$ is the bare Green's dyadic which was defined in Eq. (10). The second-rank tensor $\mathbf{M}$ is the mass or self-energy operator.[12] The Dyson equation, Eq. (15), is easily solved in Fourier transform space under the assumption of statistical homogeneity. The spatial Fourier transform pair for $\mathbf{G}^0$ is given by

$$G^0_{i\alpha}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{q})=\frac{1}{(2\pi)^3}\int\int G^0_{i\alpha}(\mathbf{x},\mathbf{x}')$$
$$\times e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{x}'}\,d^3x\,d^3x', \tag{16}$$

$$G^0_{i\alpha}(\mathbf{x},\mathbf{x}')=\frac{1}{(2\pi)^3}\int\int G^0_{i\alpha}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{q})$$
$$\times e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{x}'}\,d^3p\,d^3q. \tag{17}$$

The Fourier transforms which define $\langle\mathbf{G}(\mathbf{p})\rangle$ and $\tilde{\mathbf{M}}(\mathbf{p})$ are given by expressions similar to that defining $\mathbf{G}^0(\mathbf{p})$. The assumption of statistical homogeneity ensures that they are functions of a single wave vector in Fourier space.

The Dyson equation, Eq. (15), is then spatially Fourier transformed and solved for $\langle\mathbf{G}(\mathbf{p})\rangle$. The result is

$$\langle\mathbf{G}(\mathbf{p})\rangle=[\mathbf{G}^0(\mathbf{p})^{-1}-\tilde{\mathbf{M}}(\mathbf{p})]^{-1}. \tag{18}$$

The Dyson equation is exact and describes the mean response of the medium. The main difficulty in the solution of Eq. (18) is the representation of $\tilde{\mathbf{M}}$. Approximations of $\tilde{\mathbf{M}}$ are often necessary for closed-form solutions of Eq. (18) to be obtained. The self-energy, $\mathbf{M}$, can be written as an expansion in powers of material property fluctuations. Approximation of $\mathbf{M}$ can then be made to first order using the first term in such an expansion. Frisch discusses the equivalence of this technique, which he called the first-order smoothing approximation (FOSA),[12] and the Keller approximation.[13] Such an approximation is valid if the fluctuations of the material properties are small $(\delta\rho(\mathbf{x})\ll 1,\ \delta\lambda(\mathbf{x})\ll 1,\ \delta\mu(\mathbf{x})\ll 1)$. To this level of approximation, the self-energy is given by

$$M_{\beta j}(\mathbf{y},\mathbf{z})\approx\langle \mathfrak{L}^1_{\beta\gamma}(\mathbf{y})G^0_{\gamma k}(\mathbf{y},\mathbf{z})\mathfrak{L}^1_{kj}(\mathbf{z})\rangle, \tag{19}$$

where $\mathfrak{L}^1$ is the first-order operator.[12]

The first-order operator for the problem studied here follows from Eqs. (1), (5), and (6). It is given by

$$\mathfrak{L}^1_{\beta\gamma}(\mathbf{y})=\omega^2\delta_{\beta\gamma}\bar{\rho}\delta\rho(\mathbf{y})+\bar{\lambda}\frac{\partial}{\partial y_\beta}\delta\lambda(\mathbf{y})\frac{\partial}{\partial y_\gamma}+\bar{\mu}\frac{\partial}{\partial y_\gamma}\delta\mu(\mathbf{y})\frac{\partial}{\partial y_\beta}+\bar{\mu}\delta_{\beta\gamma}\frac{\partial}{\partial y_l}\delta\mu(\mathbf{y})\frac{\partial}{\partial y_l}. \tag{20}$$

The spatial transform of the self-energy and the mean Green's dyadic will have the same form as the bare Green's dyadic. They are written

$$\tilde{\mathbf{M}}(\mathbf{p})=m_L(p)\hat{\mathbf{p}}\hat{\mathbf{p}}+m_T(p)(\hat{\mathbf{p}}_2\hat{\mathbf{p}}_2+\hat{\mathbf{p}}_3\hat{\mathbf{p}}_3), \tag{21}$$

and

$$\langle\tilde{\mathbf{G}}(\mathbf{s})\rangle=g_L(s)\hat{\mathbf{s}}\hat{\mathbf{s}}+g_T(s)(\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2+\hat{\mathbf{s}}_3\hat{\mathbf{s}}_3). \tag{22}$$

The propagators for the mean response are then given by the solution of the Dyson equation, Eq. (18), as

$$\begin{aligned}g_\beta(\mathbf{p})&=[g^0_\beta(\mathbf{p})^{-1}-m_\beta(\mathbf{p})]^{-1}\\&=[\bar{\rho}\omega^2-p^2\bar{\rho}c^2_\beta-m_\beta(\mathbf{p})]^{-1},\end{aligned} \tag{23}$$

for each wave type, β (L or T).

The expressions for the propagators of the mean response define the phase velocity and attenuation of each wave type. The solution of

$$\bar{\rho}\omega^2-p^2\bar{\rho}c^2_\beta-m_\beta(\mathbf{p})=0, \tag{24}$$

for the wave vector p, is required, given $\tilde{\mathbf{M}}$ defined in Eq. (19).

The inverse Fourier transform of $\langle\mathbf{G}(\mathbf{p})\rangle$ will be dominated by the zeros of the propagators. The phase velocity is given by the real part of p and the attenuation by the imaginary part. Such solutions of Eq. (24) are often done numerically using root finding techniques.[2] However, explicit expressions for the attenuation can be determined using an approximation valid below the high-frequency geometric optics limit. In this case, the wave vector, $\mathbf{p}$, within the self-energy is approximated as being equal to the bare wave vector. Such an approximation, $m(\mathbf{p})\approx m(\omega\hat{\mathbf{p}}/c_\beta)$, is sometimes called a Born approximation.[2,3] In essence, the phase velocity is assumed to remain unchanged by the heterogeneities. This approximation allows the imaginary part of $\mathbf{p}$ to be calculated directly from Eq. (24). The attenuation of each wave type, to this level of approximation, is given by

$$\alpha_\beta(\hat{\mathbf{p}})=-\frac{1}{2\bar{\rho}\omega c_\beta}\operatorname{Im} m_\beta\left(\frac{\omega}{c_\beta}\hat{\mathbf{p}}\right), \tag{25}$$

where the subscript β refers to either wave type (L or T). Thus the determination of the attenuations requires the imaginary part of the components of the spatial transform of the self-energy. The self-energy, given by Eq. (19), is a product of the bare Green's dyadic, $\mathbf{G}^0$, and two multiples of the first-order operator, $\mathfrak{L}^1$, defined in Eq. (20). This operator contains terms related to the density and Lamé parameter fluctuations. Therefore, the entire self-energy is a sum of six terms—a $\rho\rho$ term and five others ($\lambda\lambda$, $\mu\mu$, $\rho\lambda$, $\rho\mu$, and $\lambda\mu$).

The spatial double Fourier transform of the self-energy is given by

$$\tilde{M}_{\beta j}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{q})=\frac{1}{(2\pi)^3}\int\int d^3y\,d^3z e^{-i\mathbf{p}\cdot\mathbf{y}}\langle\mathfrak{L}^1_{\beta\gamma}(\mathbf{y})\times G^0_{\gamma k}(\mathbf{y},\mathbf{z})\mathfrak{L}^1_{kj}(\mathbf{z})\rangle e^{i\mathbf{q}\cdot\mathbf{z}}. \tag{26}$$

Example calculations for two of the terms of $\tilde{\mathbf{M}}$ are given here explicitly. The other terms follow from similar derivations. The first term comes from the density terms in each of the first-order operators. It is given by

$$\tilde{M}^{\rho\rho}_{\beta j}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{q})=\frac{\bar{\rho}^2\omega^4}{(2\pi)^3}\int\int d^3y\,d^3z e^{-i\mathbf{p}\cdot\mathbf{y}}G^0_{\beta j}(\mathbf{y},\mathbf{z})\times\langle\delta\rho(\mathbf{y})\delta\rho(\mathbf{z})\rangle e^{i\mathbf{q}\cdot\mathbf{z}}. \tag{27}$$

The spatial Fourier transforms of correlation functions (the power spectra of the fluctuations) are defined as

$$\tilde{R}(\mathbf{p})=\frac{1}{(2\pi)^3}\int d^3r R(\mathbf{r})e^{-i\mathbf{p}\cdot\mathbf{r}}. \tag{28}$$

This definition allows the $\rho\rho$ term of the self-energy, Eq. (27), to reduce to

$$\tilde{M}^{\rho\rho}_{\beta j}(\mathbf{p})=\bar{\rho}^2\omega^4\int d^3s\tilde{G}^0_{\beta j}(\mathbf{s})\tilde{R}_{\rho\rho}(\mathbf{p}-\mathbf{s}). \tag{29}$$

Thus, the terms of the transform of the self-energy may be written as convolutions between the bare Green's dyadic and the power spectra.

The term related to the $\lambda\lambda$ covariance is given by

$$\begin{aligned}&\tilde{M}^{\lambda\lambda}_{\beta j}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{q})\\&=\frac{\bar{\lambda}^2}{(2\pi)^3}\int\int d^3y\,d^3z e^{-i\mathbf{p}\cdot\mathbf{y}}\\&\times\left\langle\frac{\partial}{\partial y_\beta}\delta\lambda(\mathbf{y})\frac{\partial}{\partial y_\gamma}G^0_{\gamma k}(\mathbf{y},\mathbf{z})\frac{\partial}{\partial z_j}\delta\lambda(\mathbf{z})\frac{\partial}{\partial z_k}\right\rangle e^{i\mathbf{q}\cdot\mathbf{z}}.\end{aligned} \tag{30}$$

Integration by parts yields

$$\tilde{M}^{\lambda\lambda}_{\beta j}(\mathbf{p})=\bar{\lambda}^2\int d^3s\tilde{G}^0_{\gamma k}(\mathbf{s})\tilde{R}_{\lambda\lambda}(\mathbf{p}-\mathbf{s})p_\beta p_j s_\gamma s_k. \tag{31}$$

The remaining calculations follow in a similar manner. The total self-energy is finally expressed as

$$\tilde{M}_{\beta j}(\mathbf{p})=\int d^3s\,\tilde{G}^0_{\gamma k}(\mathbf{s})\{\overline{\rho}^2\omega^4\delta_{jk}\delta_{\beta\gamma}\tilde{R}_{\rho\rho}(\mathbf{p}-\mathbf{s})+\overline{\lambda}^2\tilde{R}_{\lambda\lambda}(\mathbf{p}-\mathbf{s})p_\beta p_j s_\gamma s_k+\overline{\mu}^2(s_\beta s_j p_\gamma p_k+s_\beta p_\gamma p_l s_l\delta_{jk}+\delta_{\beta\gamma}p_l s_l s_j p_k$$
$$+\delta_{\beta\gamma}\delta_{jk}p_l s_l p_m s_m)\tilde{R}_{\mu\mu}(\mathbf{p}-\mathbf{s})+\overline{\rho\lambda}\omega^2(\delta_{\beta\gamma}p_j s_k+\delta_{jk}p_\beta s_\gamma)\tilde{R}_{\rho\lambda}(\mathbf{p}-\mathbf{s})+\overline{\rho\mu}\omega^2[\delta_{\beta\gamma}(s_j p_k+p_l s_l\delta_{jk})$$
$$+\delta_{jk}(s_\beta p_\gamma+p_l s_l\delta_{\beta\gamma})]\tilde{R}_{\rho\mu}(\mathbf{p}-\mathbf{s})+\overline{\lambda\mu}(p_\beta s_j p_k s_\gamma+p_\beta s_\gamma p_l s_l\delta_{jk}+s_\beta p_j s_k p_\gamma+p_j s_k p_l s_l\delta_{\beta\gamma})\tilde{R}_{\lambda\mu}(\mathbf{p}-\mathbf{s})\}. \quad (32)$$

The imaginary part of this final expression for the transform of the self-energy allows the attenuations to be determined. The forms for the self-energy, Eq. (21), and the bare Green's dyadic, Eq. (10), are used in simplifying the attenuations. These expressions Eqs. (13)–(14) are substituted into Eq. (32). Appropriate inner products of the resulting equation allow the required components, $m_L(\mathbf{p})$ and $m_T(\mathbf{p})$, to be determined. The required quantity for the longitudinal attenuation is given by

$$\mathrm{Im}\,m_L(\mathbf{p})=-\frac{\pi}{\overline{\rho}}\frac{1}{2\omega}\hat{p}_\beta\hat{p}_j\int d^3s\left(\frac{1}{c_L}\delta(s-\omega/c_L)\hat{s}_\gamma\hat{s}_k+\frac{1}{c_T}\delta(s-\omega/c_T)(\hat{s}_{2_\gamma}\hat{s}_{2_k}+\hat{s}_{3_\gamma}\hat{s}_{3_k})\right)\{\overline{\rho}^2\omega^4\delta_{jk}\delta_{\beta\gamma}\tilde{R}_{\rho\rho}(\mathbf{p}-\mathbf{s})$$
$$+\overline{\lambda}^2 p_\beta p_j s_\gamma s_k\tilde{R}_{\lambda\lambda}(\mathbf{p}-\mathbf{s})+\overline{\mu}^2(s_\beta s_j p_\gamma p_k+s_\beta p_\gamma p_l s_l\delta_{jk}+\delta_{\beta\gamma}p_l s_l s_j p_k+\delta_{\beta\gamma}\delta_{jk}p_l s_l p_m s_m)\tilde{R}_{\mu\mu}(\mathbf{p}-\mathbf{s})$$
$$+\overline{\rho\lambda}\omega^2(\delta_{\beta\gamma}p_j s_k+\delta_{jk}p_\beta s_\gamma)\tilde{R}_{\rho\lambda}(\mathbf{p}-\mathbf{s})+\overline{\rho\mu}\omega^2[\delta_{\beta\gamma}(s_j p_k+p_l s_l\delta_{jk})+\delta_{jk}(s_\beta p_\gamma+p_l s_l\delta_{\beta\gamma})]\tilde{R}_{\rho\mu}(\mathbf{p}-\mathbf{s})$$
$$+\overline{\lambda\mu}(p_\beta s_j p_k s_\gamma+p_\beta s_\gamma p_l s_l\delta_{jk}+s_\beta p_j s_k p_\gamma+p_j s_k p_l s_l\delta_{\beta\gamma})\tilde{R}_{\lambda\mu}(\mathbf{p}-\mathbf{s})\}. \quad (33)$$

For the transverse attenuation, the required result is

$$\mathrm{Im}\,m_T(\mathbf{p})=-\frac{\pi}{\overline{\rho}}\frac{1}{4\omega}(\hat{p}_{2_\beta}\hat{p}_{2_j}+\hat{p}_{3_\beta}\hat{p}_{3_j})\int d^3s\left(\frac{1}{c_L}\delta(s-\omega/c_L)\hat{s}_\gamma\hat{s}_k+\frac{1}{c_T}\delta(s-\omega/c_T)(\hat{s}_{2_\gamma}\hat{s}_{2_k}+\hat{s}_{3_\gamma}\hat{s}_{3_k})\right)$$
$$\times\{\overline{\rho}^2\omega^4\delta_{jk}\delta_{\beta\gamma}\tilde{R}_{\rho\rho}(\mathbf{p}-\mathbf{s})+\overline{\lambda}^2\tilde{R}_{\lambda\lambda}(\mathbf{p}-\mathbf{s})p_\beta p_j s_\gamma s_k+\overline{\mu}^2(s_\beta s_j p_\gamma p_k+s_\beta p_\gamma p_l s_l\delta_{jk}+\delta_{\beta\gamma}p_l s_l s_j p_k$$
$$+\delta_{\beta\gamma}\delta_{jk}p_l s_l p_m s_m)\tilde{R}_{\mu\mu}(\mathbf{p}-\mathbf{s})+\overline{\rho\lambda}(\delta_{\beta\gamma}p_j s_k+\delta_{jk}p_\beta s_\gamma)\omega^2\tilde{R}_{\rho\lambda}(\mathbf{p}-\mathbf{s})+[\delta_{\beta\gamma}(s_j p_k+p_l s_l\delta_{jk})$$
$$+\delta_{jk}(s_\beta p_\gamma+p_l s_l\delta_{\beta\gamma})]\overline{\rho\mu}\omega^2\tilde{R}_{\rho\mu}(\mathbf{p}-\mathbf{s})+\overline{\lambda\mu}(p_\beta s_j p_k s_\gamma+p_\beta s_\gamma p_l s_l\delta_{jk}+s_\beta p_j s_k p_\gamma+p_j s_k p_l s_l\delta_{\beta\gamma})\tilde{R}_{\lambda\mu}(\mathbf{p}-\mathbf{s})\}. \quad (34)$$

The two longitudinal attenuations, α_{LL} and α_{LT}, are determined first using Eqs. (33) and (25). The total longitudinal attenuation, $\alpha_L=\alpha_{LL}+\alpha_{LT}$. The first term from $\mathbf{G}^0$ involving $\hat{\mathbf{s}}\hat{\mathbf{s}}$ gives α_{LL}, while the second term containing $(\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2+\hat{\mathbf{s}}_3\hat{\mathbf{s}}_3)$ gives α_{LT}. The frequency-limiting approximation implies that $\mathbf{p}\approx\omega\hat{\mathbf{p}}/c_L$ in Eq. (33). The integral over the magnitude of the wave number is trivial, leaving only an integration over the unit sphere. The result for the LL attenuation is

$$\alpha_{LL}=\frac{\pi\omega^4}{4c_L^4}\int d^2\hat{s}\left\{(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2\tilde{R}^{LL}_{\rho\rho}+\frac{(c_L^2-2c_T^2)^2}{c_L^4}\tilde{R}^{LL}_{\lambda\lambda}\right.$$
$$+\frac{4c_T^4}{c_L^4}(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^4\tilde{R}^{LL}_{\mu\mu}+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})\frac{2(c_L^2-2c_T^2)}{c_L^2}\tilde{R}^{LL}_{\rho\lambda}$$
$$\left.+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^3\frac{4c_T^2}{c_L^2}\tilde{R}^{LL}_{\rho\mu}+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2\frac{4c_T^2(c_L^2-2c_T^2)}{c_L^4}\tilde{R}^{LL}_{\lambda\mu}\right\}. \quad (35)$$

The notation used in Eq. (35) for the $\tilde{R}$ terms is defined by

$$\tilde{R}^{ij}_{\lambda\lambda}=\tilde{R}_{\lambda\lambda}\left(\left|\frac{\omega}{c_i}\hat{\mathbf{p}}-\frac{\omega}{c_j}\hat{\mathbf{s}}\right|\right), \quad (36)$$

where the superscripts i and j refer to the possible wave types, L or T. The correlation functions depend only upon the cosine of the angle between the incident and scattered directions, $\chi=\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}$, because of the assumption of statistical isotropy. Thus, the integration in azimuthal angle is trivial. The final result for the LL attenuation is

$$\alpha_{LL}=\frac{\pi^2\omega^4}{2c_L^4}\int_{-1}^{+1}d\chi\left\{\chi^2\tilde{R}^{LL}_{\rho\rho}(\chi)+\frac{(c_L^2-2c_T^2)^2}{c_L^4}\tilde{R}^{LL}_{\lambda\lambda}(\chi)\right.$$
$$+\frac{4c_T^4}{c_L^4}\chi^4\tilde{R}^{LL}_{\mu\mu}(\chi)+\chi\frac{2(c_L^2-2c_T^2)}{c_L^2}\tilde{R}^{LL}_{\rho\lambda}(\chi)$$
$$\left.+\chi^3\frac{4c_T^2}{c_L^2}\tilde{R}^{LL}_{\rho\mu}(\chi)+\chi^2\frac{4c_T^2(c_L^2-2c_T^2)}{c_L^4}\tilde{R}^{LL}_{\lambda\mu}(\chi)\right\}. \quad (37)$$

This result is consistent with that given by Ryzhik *et al.*[10]

The mode conversion attenuation, α_{LT}, is found in a similar manner to be

$$\alpha_{LT}=\frac{\pi\omega^4}{4c_Lc_T^3}\int d^2\hat{s}\left\{((\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)^2)\tilde{R}^{LT}_{\rho\rho}\right.$$
$$+4(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2\frac{c_T^2}{c_L^2}((\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)^2)\tilde{R}^{LT}_{\mu\mu}$$
$$\left.+4(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})((\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)^2)\frac{c_T}{c_L}\tilde{R}^{LT}_{\rho\mu}\right\}. \quad (38)$$

The medium considered here is statistically isotropic. In this type of problem, the differential scattering cross-section (the integrand) must be a function of only $\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}$. It can be easily shown that

$$(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)^2=1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2. \tag{39}$$

Thus, the mode conversion attenuation may be written

$$\alpha_{LT}=\frac{\pi\omega^4}{4c_T^3c_L}\int d^2\hat{s}(1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2)\times\left\{\tilde{R}_{\rho\rho}^{LT}+\frac{4c_T^2}{c_L^2}(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2\tilde{R}_{\mu\mu}^{LT}+\frac{4c_T}{c_L}(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})\tilde{R}_{\rho\mu}^{LT}\right\}. \tag{40}$$

After the azimuthal integration we have

$$\alpha_{LT}=\frac{\pi^2\omega^4}{2c_T^3c_L}\int_{-1}^{+1}(1-\chi^2)\times\left\{\tilde{R}_{\rho\rho}^{LT}(\chi)+4\frac{c_T^2}{c_L^2}\chi^2\tilde{R}_{\mu\mu}^{LT}(\chi)+4\frac{c_T}{c_L}\chi\tilde{R}_{\rho\mu}^{LT}(\chi)\right\}d\chi, \tag{41}$$

where $\chi=\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}$. The term in brackets corresponds to σ_{ps} given by Ryzhik *et al.*, Eq. (4.56). The complete expression agrees with that given by Papanicolaou *et al.*,[11] Eq. (A2).

The calculation of the transverse wave attenuations proceeds along a similar line. The two transverse attenuations, α_{TL} and α_{TT}, are determined from Eqs. (34) and (25). The total transverse attenuation, $\alpha_T=\alpha_{TL}+\alpha_{TT}$. The first term of $\mathbf{G}^0$ involving $\hat{\mathbf{s}}\hat{\mathbf{s}}$ gives α_{TL}, while the second term containing $(\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2+\hat{\mathbf{s}}_3\hat{\mathbf{s}}_3)$ gives α_{TT}. The frequency-limiting approximation implies that $\mathbf{p}\approx\omega\hat{\mathbf{p}}/c_T$ in Eq. (34). The integral over the magnitude of the wave number is trivial, leaving only an integration over the unit sphere. The result for the TL attenuation is

$$\alpha_{TL}=\frac{\pi\omega^4}{8c_L^3c_T}\int d^2\hat{s}\left\{((\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)^2+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)^2)\tilde{R}_{\rho\rho}^{TL}+4(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2((\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)^2+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)^2)\frac{c_T^2}{c_L^2}\tilde{R}_{\mu\mu}^{TL}+4(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})((\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)^2+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)^2)\frac{c_T}{c_L}\tilde{R}_{\rho\mu}^{TL}\right\}. \tag{42}$$

Again, the differential scattering cross sections (the integrands) must be only functions of $\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}$. It can be shown that

$$(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)^2+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)^2=1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2. \tag{43}$$

Thus, we have

$$\alpha_{TL}=\frac{\pi\omega^4}{8c_L^3c_T}\int d^2\hat{s}(1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2)\times\left\{\tilde{R}_{\rho\rho}^{TL}+\frac{4c_T^2}{c_L^2}(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2\tilde{R}_{\mu\mu}^{TL}+\frac{4c_T}{c_L}(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})\tilde{R}_{\rho\mu}^{TL}\right\}, \tag{44}$$

or, after azimuthal integration,

$$\alpha_{TL}=\frac{\pi^2\omega^4}{4c_L^3c_T}\int_{-1}^{+1}(1-\chi^2)\times\left\{\tilde{R}_{\rho\rho}^{TL}(\chi)+\frac{4c_T^2}{c_L^2}\chi^2\tilde{R}_{\mu\mu}^{TL}(\chi)+\frac{4c_T}{c_L}\chi\tilde{R}_{\rho\mu}^{TL}(\chi)\right\}d\chi. \tag{45}$$

As a check of self-consistency, the above expression for α_{TL} satisfies the required relation

$$\alpha_{TL}=\frac{1}{2}\left(\frac{c_T}{c_L}\right)^2\alpha_{LT}. \tag{46}$$

Finally, the TT attenuation may be reduced to

$$\begin{aligned}\alpha_{TT}=\frac{\pi\omega^4}{8c_T^4}\int d^2\hat{s}\{&((\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_3)^2+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_3)^2)\tilde{R}_{\rho\rho}^{TT}+\tilde{R}_{\mu\mu}^{TT}[(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)^2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)^2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)^2\\&+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)^2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)^2+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)^2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)^2+2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})((\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2))+2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})((\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_3)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)\\&\times(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_3)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3))+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2((\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_3)^2+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_3)^2)]+2\tilde{R}_{\rho\mu}^{TT}[(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)\\&\times(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)+(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_3)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_3)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})((\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_2)^2\\&+(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_3)^2+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_3)^2)]\}.\end{aligned} \tag{47}$$

Once again, the differential scattering cross sections must depend only on $\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}$. The above combinations of inner products can be reduced considerably. The necessary identities are

$$(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_3)^2+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_3)^2=1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2, \tag{48}$$

$$(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)+(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{s}}_3)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)+(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{s}}_3)(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)=(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^3-(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}), \tag{49}$$

and

$$(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)^2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)^2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_2)^2+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_2)^2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)^2+(\hat{\mathbf{s}}\cdot\hat{\mathbf{p}}_3)^2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}_3)^2=1-2(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2+(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^4. \tag{50}$$

These identities allow α_{TT} to be written in terms of $\hat{\mathbf{p}}\cdot\hat{\mathbf{s}}$ only as

$$\alpha_{TT}=\frac{\pi\omega^4}{8c_T^4}\int d^2\hat{s}\{((\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2+1)\tilde{R}^{TT}_{\rho\rho}+(1-3(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^2+4(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^4)\tilde{R}^{TT}_{\mu\mu}+4(\hat{\mathbf{p}}\cdot\hat{\mathbf{s}})^3\tilde{R}^{TT}_{\rho\mu}\}. \tag{51}$$

Integration in azimuth leaves

$$\alpha_{TT}=\frac{\pi^2\omega^4}{4c_T^4}\int_{-1}^{+1}\{(\chi^2+1)\tilde{R}^{TT}_{\rho\rho}(\chi)+(1-3\chi^2+4\chi^4)\times\tilde{R}^{TT}_{\mu\mu}(\chi)+4\chi^3\tilde{R}^{TT}_{\rho\mu}(\chi)\}d\chi. \tag{52}$$

The TT attenuation, α_{TT}, depends solely on the shear wave speed as expected.

The attenuations given by Eqs. (37), (41), (45), and (52) are the main results of this section. The results here have been reduced to integrations on the unit circle in terms of the correlation power spectra. The forward-weighted attenuations, α', which quantify the amount of forward scattering, are identical to the expressions for the attenuations given above with an additional factor of $\chi=\cos\Theta$ in the integrand.[14]

In the low-frequency limit, the correlation functions are expected to be constant. Also in this limit, the scattering is nonpreferential (an equal amount of energy is scattered forward as backward). In this case, the α's are zero. However, as seen in the equations for the attenuations, Eqs. (37), (41), (45), and (52), this is not necessarily the case. The angular dependence associated with the autocorrelations are all even functions. Those associated with the cross correlations are all odd. The implications of these results are still unclear. Must the cross correlations approach zero in the low-frequency limit? In the next section, assumptions about the form of the fluctuations are made and sample calculations presented.

III. MEAN FREE PATHS AND ELASTIC DIFFUSIVITY

The above expressions for the attenuations are now used for determination of mean free paths and elastic diffusivity. The above derivation resulted in expressions for the displacement attenuations. The energy density is proportional to the square of the displacements. Thus, the energy attenuations are twice the displacement attenuations. In addition, the scattering mean free paths, l_L and l_T, are the inverse of the energy attenuation. Therefore, we have

$$l_L=\frac{1}{2(\alpha_{LL}+\alpha_{LT})}, \tag{53}$$

$$l_T=\frac{1}{2(\alpha_{TL}+\alpha_{TT})}. \tag{54}$$

In the low-frequency Rayleigh limit, the correlation functions, $\tilde{R}$, are constant. Thus, the expressions for the attenuations, Eqs. (37), (41), (45), and (52) reduce considerably. In this limit, the integrands become simple polynomials in χ. After integration, we have

$$\alpha_{LL}=\frac{\pi^2\omega^4}{c_L^4}\left(\frac{1}{3}\tilde{R}^{LL}_{\rho\rho}+(1-2B^2)^2\tilde{R}^{LL}_{\lambda\lambda}+\frac{4}{5}B^4\tilde{R}^{LL}_{\mu\mu}+\frac{4}{3}B^2(1-2B^2)\tilde{R}^{LL}_{\lambda\mu}\right), \tag{55}$$

$$\alpha_{LT}=\frac{2\pi^2\omega^4}{3c_T^3c_L}\left(\tilde{R}^{LT}_{\rho\rho}+\frac{4}{5}B^2\tilde{R}^{LT}_{\mu\mu}\right), \tag{56}$$

$$\alpha_{TL}=\frac{\pi^2\omega^4}{3c_Tc_L^3}\left(\tilde{R}^{LT}_{\rho\rho}+\frac{4}{5}B^2\tilde{R}^{LT}_{\mu\mu}\right), \tag{57}$$

$$\alpha_{TT}=\frac{2\pi^2\omega^4}{c_T^4}\left(\frac{1}{3}\tilde{R}^{TT}_{\rho\rho}+\frac{1}{5}\tilde{R}^{TT}_{\mu\mu}\right), \tag{58}$$

where $B=c_T/c_L$ is the wave speed ratio. Also, in the low-frequency limit, the transforms of the correlation functions are independent of the wave types. In other words, $\tilde{R}_{\rho\rho}\equiv\tilde{R}^{LL}_{\rho\rho}=\tilde{R}^{LT}_{\rho\rho}=\tilde{R}^{TT}_{\rho\rho}$. Similar relations hold for the λ and μ terms as well. The scattering mean free paths then reduce to

$$l_L=\frac{15c_L^4B^3/(2\pi^2\omega^4)}{5(2+B^3)\tilde{R}_{\rho\rho}+15B^3(1-2B^2)^2\tilde{R}_{\lambda\lambda}+4B^2(3B^5+2)\tilde{R}_{\mu\mu}+20B^5(1-2B^2)\tilde{R}_{\lambda\mu}}, \tag{59}$$

$$l_T=\frac{15c_T^4/(2\pi^2\omega^4)}{5(2+B^3)\tilde{R}_{\rho\rho}+2(3+2B^5)\tilde{R}_{\mu\mu}}. \tag{60}$$

These results differ slightly from those given by Ryzhik *et al.*[10]

The elastic diffusivity, D, can be written[14]

$$D=\frac{c_Tl_T^*}{3}\frac{2+B^2L}{2+B^3}, \tag{61}$$

where $L=l_L^*/l_T^*$ is the ratio of transport mean free paths. The elastic diffusivity is a weighted average of the diffusivities of individual compressional and shear components. The weighting is determined by the diffusive equipartitioning law, which states that the energy in transverse form, E_T, and the longitudinal energy, E_L, are related by $E_T=2E_L/B^3$.[15] In the low-frequency limit, the transport mean free paths reduce to the scattering mean free paths

$$l_L^*=l_L=\frac{1}{2\alpha_L}, \tag{62}$$

$$l_T^*=l_T=\frac{1}{2\alpha_T}. \tag{63}$$

Outside the long wavelength limit, the transport mean free paths are given by[14]

$$l_L^* = \frac{1}{2} \frac{\alpha_T - \alpha'_{TT} + \alpha'_{LT}}{(\alpha_L - \alpha'_{LL})(\alpha_T - \alpha'_{TT}) - \alpha'_{LT}\alpha'_{TL}}, \tag{64}$$

$$l_T^* = \frac{1}{2} \frac{\alpha_L - \alpha'_{LL} + \alpha'_{TL}}{(\alpha_L - \alpha'_{LL})(\alpha_T - \alpha'_{TT}) - \alpha'_{LT}\alpha'_{TL}}, \tag{65}$$

where the primed attenuations are the forward-weighted displacement attenuations discussed above for the respective scattering processes. In the limit of nonpreferential scattering, the primed attenuations vanish and the mean free paths are seen to reduce to the inverse of the energy scattering attenuation as in Eqs. (62)–(63). Example calculations are now presented using the above derived quantities.

IV. EXAMPLE RESULTS

Example results are now presented using additional material assumptions. The results are put in dimensionless form for the most widespread applicability. It is first assumed that the material properties are uncorrelated with one another. This implies that the cross correlations are zero ($\tilde{R}_{\mu\lambda} = \tilde{R}_{\rho\mu} = \tilde{R}_{\rho\lambda} = 0$).

In this case, the attenuations become

$$\alpha_{LL} = \frac{\pi^2\omega^4}{2c_L^4} \int_{-1}^{+1} \{\chi^2 \tilde{R}_{\rho\rho}^{LL}(\chi) + (1-2B^2)^2 \tilde{R}_{\lambda\lambda}^{LL}(\chi) + 4B^4\chi^4 \tilde{R}_{\mu\mu}^{LL}(\chi)\} d\chi, \tag{66}$$

$$\alpha_{LT} = \frac{\pi^2\omega^4}{2c_T^3 c_L} \int_{-1}^{+1} (1-\chi^2)\{\tilde{R}_{\rho\rho}^{LT}(\chi) + 4B^2\chi^2 \tilde{R}_{\mu\mu}^{LT}(\chi)\} d\chi, \tag{67}$$

and

$$\alpha_{TL} = \tfrac{1}{2} B^2 \alpha_{LT}, \tag{68}$$

$$\alpha_{TT} = \frac{\pi^2\omega^4}{4c_T^4} \int_{-1}^{+1} \{(1+\chi^2)\tilde{R}_{\rho\rho}^{TT}(\chi) + (1-3\chi^2+4\chi^4) \times \tilde{R}_{\mu\mu}^{TT}(\chi)\} d\chi. \tag{69}$$

The forward-weighted attenuations[3,14] are defined as these with an extra factor of $\chi = \cos\Theta$ within the integrand. These attenuations are denoted with a prime.

Next, it is assumed that all autocorrelation functions have the same spatial dependence. They are assumed to have the form

$$R_{\gamma\gamma} = A_\gamma^2 R(r), \tag{70}$$

where the subscript γ refers to the material parameters of ρ, λ, and μ. The magnitude of the fluctuations for each material parameter is given by A_γ.

Finally, a form for the function $R(r)$ is assumed. As discussed by Stanke,[16] an exponential function describes the correlation of continuous and discrete materials reasonably well. Thus, it is assumed that

$$R(r) = e^{-r/H}, \tag{71}$$

where H is the correlation length. Such a model, with a single length scale, is perhaps oversimplified for materials with polydispersed scatterer sizes. However, for many materials such a model is expected to describe the statistics of the material properties well. In transform space

$$\tilde{R}(p) = \frac{H^3}{\pi^2(1+H^2p^2)^2}. \tag{72}$$

With the length scale of the spatial correlation introduced, dimensionless longitudinal and transverse frequencies are defined as $x_L = \omega H/c_L$ and $x_T = \omega H/c_T$. The transform of the difference between two wave vectors can then be written

$$\tilde{R}^{ij}(\chi) = \frac{H^3}{\pi^2(1+x_i^2+x_j^2-2x_ix_j\chi)^2}, \tag{73}$$

where the superscripts, ij, correspond to the possible wave types, L or T. The functions needed for determining the attenuations are

$$\tilde{R}^{LL}(\chi) = \frac{H^3}{\pi^2(1+2x_L^2(1-\chi))^2}, \tag{74}$$

$$\tilde{R}^{LT}(\chi) = \frac{H^3}{\pi^2(1+x_L^2+x_T^2-2x_Lx_T\chi)^2}, \tag{75}$$

$$\tilde{R}^{TT}(\chi) = \frac{H^3}{\pi^2(1+2x_T^2(1-\chi))^2}. \tag{76}$$

The attenuations may then be written in dimensionless form as

$$\alpha_{LL}H = \frac{x_L^4}{2} \int_{-1}^{+1} \frac{A_\rho^2\chi^2 + A_\lambda^2(1-2B^2)^2 + 4B^4A_\mu^2\chi^4}{(1+2x_L^2(1-\chi))^2} d\chi, \tag{77}$$

$$\alpha_{LT}H = \frac{x_L^4}{2B^3} \int_{-1}^{+1} \frac{(1-\chi^2)(A_\rho^2 + 4A_\mu^2B^2\chi^2)}{(1+x_L^2+x_T^2-2x_Lx_T\chi)^2} d\chi, \tag{78}$$

$$\alpha_{TT}H = \frac{x_T^4}{4} \int_{-1}^{+1} \frac{A_\rho^2(1+\chi^2) + A_\mu^2(1-3\chi^2+4\chi^4)}{(1+2x_T^2(1-\chi))^2} d\chi. \tag{79}$$

The integrals for the final form of the attenuations, both nonprimed and primed, can be calculated in closed-form.[3] All integrations are of the form

$$\int_{-1}^{+1} \frac{\chi^n}{(a-b\chi)^2} d\chi, \tag{80}$$

with n ranging from 0 to 5. The integrals were evaluated using numerical integration. Recursive adaptive Lobatto quadrature is available through the Matlab function ''quadl.''[17]

Figure 1 is a plot of the dimensionless longitudinal and transverse attenuations, $\alpha_L H$ and $\alpha_T H$, respectively, as a function of dimensionless frequency, x_L, for density fluctuations only ($A_\lambda = A_\mu = 0$). A wave speed ratio of $c_T/c_L = 1/\sqrt{3}$ has been used for these results and those that follow. The attenuations are seen to increase with the fourth power of frequency in the low-frequency limit. After a transition region, the attenuations increase as the frequency squared. The high-frequency geometric optics limit, in which the attenuations are constant, is not predicted due to the frequency-limiting assumption used above. The transverse

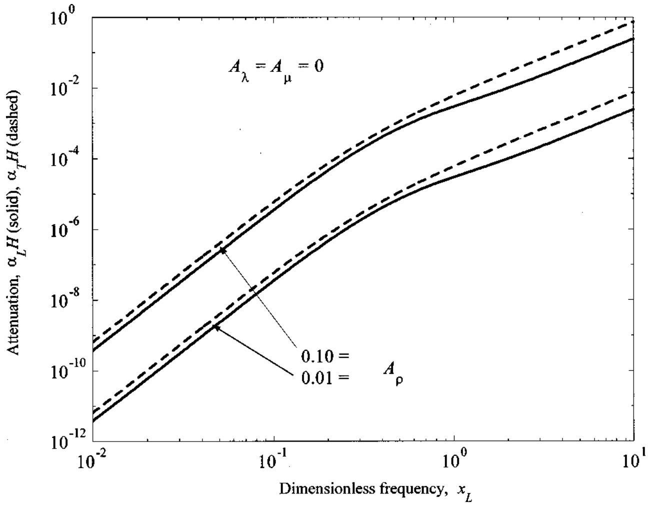

FIG. 1. Dimensionless longitudinal and transverse attenuations, $\alpha_L H$ and $\alpha_T H$, as a function of dimensionless frequency, x_L, for density fluctuations only.

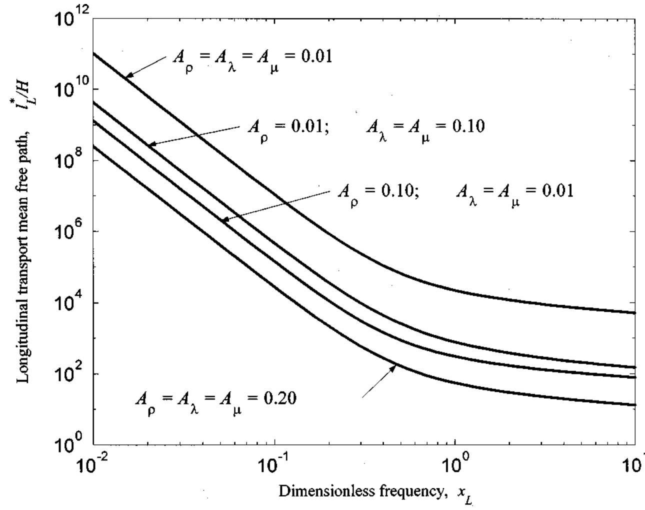

FIG. 3. Dimensionless longitudinal transport mean free path, l_L^*/H, as a function of dimensionless frequency, x_L, for different material fluctuation levels.

attenuation is larger than the longitudinal as expected. The difference between the two attenuations increases with frequency, but is a constant in each frequency regime. The actual ratio of the attenuations is a function of the fluctuations.

Figure 2 is a plot of the dimensionless longitudinal and transverse attenuations, $\alpha_L H$ and $\alpha_T H$, respectively, as a function of dimensionless frequency, x_L, for modulus fluctuations only ($A_\rho = 0$). The results have a similar form as those in Fig. 1. Combinations of both density and modulus fluctuations are simply the sum of the results from these two figures. The component of the attenuation attributed to density fluctuations (Fig. 1) is seen to be much larger than the component of attenuation attributable to modulus fluctuations for the same level of fluctuations. The importance of the density fluctuations has not been discussed previously.

The dimensionless transport mean free paths, l_L^*/H and l_T^*/H, are shown in Figs. 3 and 4. The complex forms of these terms, Eqs. (64)–(65), do not allow simple addition of factors from the different parameter fluctuations. Thus, a few different combinations of fluctuation levels have been chosen to highlight the range of these quantities. The ratio of the transport mean free paths is of interest since it appears in the definition of the diffusivity. It can be seen in Figs. 3 and 4 that the ratio, $L = l_L^*/l_T^*$, comes very near unity at the higher frequencies for many of the combinations of material fluctuations shown.

Finally, the dimensionless elastic diffusivity, D/Hc_T, is shown in Fig. 5 for various combinations of material fluctuations. Again, the low-frequency limit has the expected form, decreasing with the inverse fourth power of frequency. At higher frequencies, the diffusivity becomes nearly frequency independent as in the polycrystalline case.[3]

V. DISCUSSION

The propagation and scattering of elastic waves in heterogeneous media has been examined. Appropriate ensemble averaging of the elastic wave equation resulted in the Dyson

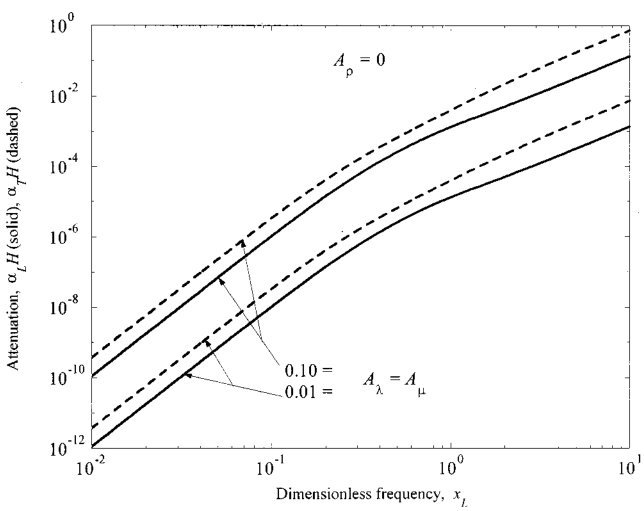

FIG. 2. Dimensionless longitudinal and transverse attenuations, $\alpha_L H$ and $\alpha_T H$, as a function of dimensionless frequency, x_L, for modulus fluctuations only.

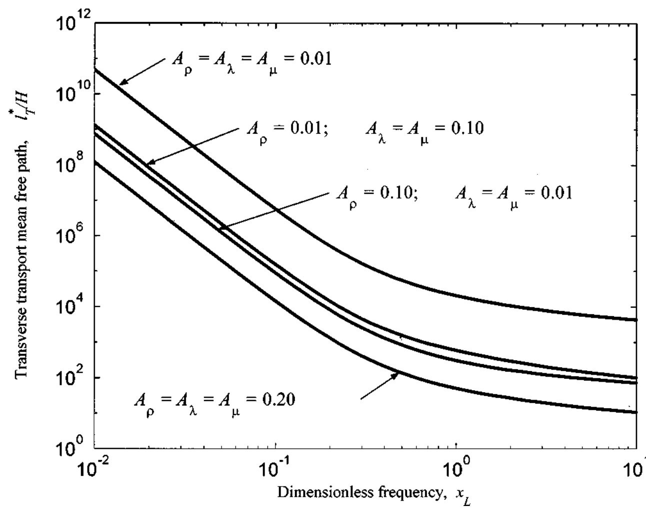

FIG. 4. Dimensionless transverse transport mean free path, l_T^*/H, as a function of dimensionless frequency, x_L, for different material fluctuation levels.

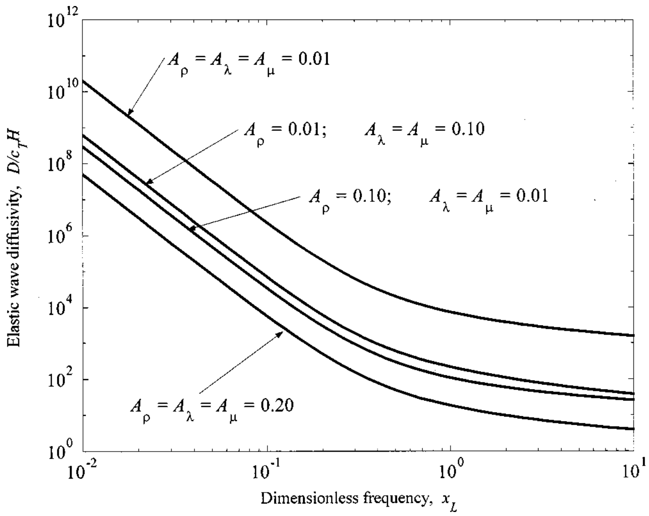

FIG. 5. Dimensionless elastic wave diffusivity, D/c_TH, as a function of dimensionless frequency, x_L, for different material fluctuation levels.

equation, governing the mean response. The problem was further specified for the case of both density and Lamé constants which varied spatially. The final forms of the attenuations are given by simple expressions involving integrations over the unit circle. The integrands are dependent upon inner products on the covariance of fluctuations of the material properties. The expressions derived here are in basic agreement with those results found using asymptotic methods.[10,11] Therefore, it is expected that the asymptotic approach would have the same restrictions as the current method. The fluctuations must be small and the frequency must not be so high that the phase velocity is appreciably altered.

The results presented here are also directly applicable to diffuse field methods such as backscatter techniques.[18] Recently, the above model has been further modified for two-phase materials. This model has been used for comparison with experiments of ultrasound diffusion in concrete.[19] The comparison between the theory and experiments is quite good despite the simplicity of many of the assumptions.

ACKNOWLEDGMENTS

The support of the Nebraska Research Initiative, the University of Nebraska Foundation Layman Fund, and the National Science Foundation (Grant No. CMS-9978707) is gratefully acknowledged. The authors also thank L. Ryzhik for helpful communication regarding this work.

[1] S. Hirsekorn, ''The scattering of ultrasonic waves by polycrystals,'' J. Acoust. Soc. Am. **72**, 1021–1031 (1982).
[2] F. E. Stanke and G. S. Kino, ''A unified theory for elastic wave propagation in polycrystalline materials,'' J. Acoust. Soc. Am. **75**, 665–681 (1984).
[3] R. L. Weaver, ''Diffusion of ultrasound in polycrystals,'' J. Mech. Phys. Solids **38**, 55–86 (1990).
[4] J. A. Turner and R. L. Weaver, ''Radiative transfer and multiple scattering of diffuse ultrasound in polycrystalline media,'' J. Acoust. Soc. Am. **96**, 3675–3683 (1994).
[5] S. Hirsekorn, ''Directional dependence of ultrasonic propagation in textured polycrystals,'' J. Acoust. Soc. Am. **79**, 1269–1279 (1986).
[6] S. Ahmed and R. B. Thompson, ''Propagation of elastic waves in equiaxed stainless-steel polycrystals with aligned [001] axes,'' J. Acoust. Soc. Am. **99**, 2086–2096 (1996).
[7] J. A. Turner, ''Elastic wave propagation and scattering in heterogeneous, anisotropic media: Textured polycrystalline materials,'' J. Acoust. Soc. Am. **106**, 541–552 (1999).
[8] J. A. Turner and R. L. Weaver, ''Radiative transfer of ultrasound,'' J. Acoust. Soc. Am. **96**, 3654–3674 (1994).
[9] J. A. Turner and R. L. Weaver, ''Time dependence of multiply scattered diffuse ultrasound in polycrystalline media,'' J. Acoust. Soc. Am. **97**, 2639–2644 (1995).
[10] L. V. Ryzhik, G. C. Papanicolaou, and J. B. Keller, ''Transport equations for elastic and other waves in random media,'' Wave Motion **24**, 327–370 (1996).
[11] G. C. Papanicolaou, L. V. Ryzhik, and J. B. Keller, ''Stability of the P-to-S energy ratio in the diffusive regime,'' Bull. Seismol. Soc. Am. **86**, 1107–1115 (1996).
[12] U. Frisch, ''Wave propagation in random media,'' in *Probalistic Methods in Applied Mathematics*, edited by A. T. Barucha-Reid (Academic, New York, 1968), Vol. I, pp. 75–198.
[13] F. C. Karal and J. B. Keller, ''Elastic, electromagnetic, and other waves in a random medium,'' J. Math. Phys. **5**, 537–547 (1964).
[14] J. A. Turner, ''Scattering and diffusion of seismic waves,'' Bull. Seismol. Soc. Am. **88**, 276–283 (1998).
[15] R. L. Weaver, ''On diffuse waves in solid media,'' J. Acoust. Soc. Am. **71**, 1608–1609 (1982).
[16] F. E. Stanke, ''Spatial autocorrelation functions for calculations of effective propagation constants in polycrystalline materials,'' J. Acoust. Soc. Am. **80**, 1479–1485 (1986).
[17] *Matlab, The Language of Technical Computing* (The Mathworks, Inc., Natick, MA).
[18] R. B. Thompson, ''A generalized model of the effects of microstructure on ultrasonic backscattering and flaw detection,'' in *Review of Progress in Quantitative NDE*, edited by D. O. Thompson and D. E. Chimenti (Plenum, New York, 1996), Vol. 15, pp. 1471–1477.
[19] P. Anugonda, J. S. Wiehn, and J. A. Turner, ''Diffusion of ultrasound in concrete,'' Ultrasonics (under review).