2000

Modeling Corporate Bond Default Risk: A Multiple Time Series Approach

Wai-Sum Chan
University of Hong Kong, chanws@hku.hk

Modeling Corporate Bond Default Risk: A Multiple Time Series Approach

Wai-Sum Chan*

Abstract†

A multiple time series approach is used to forecast the short-term U.S. corporate bond default level. These time series have two auxiliary economic variables: U.S. price inflation and U.S. GNP growth rate. Actual U.S. data from the turn of the century to the present are used to estimate the parameters of multivariate time series model. Diagnostic checks are performed to examine adequacy of the model. The model's forecast for the aggregate U.S. bond default level in 2000-2001 are 0.42% and 0.56%, respectively, while the forecast for the speculative-grade default rate in 2000 is 3.6%, which is more pessimistic than some other forecasts available in the market.

Key words and phrases: *autoregressive, moving average, stationary, forecasting, high-yield bonds, vector time series*

1 Introduction

A bond is said to be in default when the bond issuer has missed a payment of interest, filed for bankruptcy, or announced a distressed-creditor restructuring. The default rate is measured on an initial population of bonds for a finite period of time, such as one year.[1]

*Wai-Sum Chan, Ph.D., F.S.A., is an associate professor of actuarial science in the University of Hong Kong. Dr. Chan has a B.B.A. (Honours) in accounting from the Chinese University of Hong Kong and a Ph.D. in applied statistics from Temple University, U.S.A. He is a Fellow of both the Society of Actuaries and the Royal Statistical Society.

Dr. Chan's address is: Department of Statistics & Actuarial Science, The University of Hong Kong, Pokfulam Road, HONG KONG. Internet address: *chanws@hku.hk*

†This research is supported by a CRCG grant from the University of Hong Kong. The author is indebted to the anonymous referees for their helpful comments on an earlier version of this paper.

[1]For example, Caouette et al., (1998, p. 195) computed the annual default rates for all domestic corporate U.S. bonds from 1971 to 1997.

The incidence of default by U.S. corporate bond issuers is spread unevenly over this century, with high rates of default in the 1910s, the Great Depression of the 1930s, and again in the late 1980s and early 1990s. Figure 1 shows the aggregate default rate for all U.S. corporate domestic bond issuers as an annual time series from 1900 to 1999 and is derived using data in Vanderhoof et al., (1989), Altman and Kishore (1998), and Altman et al., (2000). Notice how aggregate corporate default risk has ebbed and flowed since 1900. Though the risk of asset default has not been a real threat to life insurance companies over the last 50 years, this can easily change. As most life insurance companies in U.S. hold a significant portion of corporate bonds[2] in their investment portfolios, it is important for actuaries to watch for movements of bond default levels.

This paper investigates the possibility of using a multiple (vector) time series model to provide short-term forecasts of the future level of aggregate bond defaults. In addition to the bond default rate, two other economic variables are incorporated into the vector model: the U.S. price inflation rate and the U.S. gross national product (GNP) growth rate. The inflation rate is the most important driving force of some commonly used actuarial stochastic models (Wilkie, 1995). On the other hand, the GNP growth rate is an important leading indicator of economic stability (Vanderhoof et al., 1989). The price inflation time series and the GNP growth rate time series are shown in Figures 2 and 3, respectively.

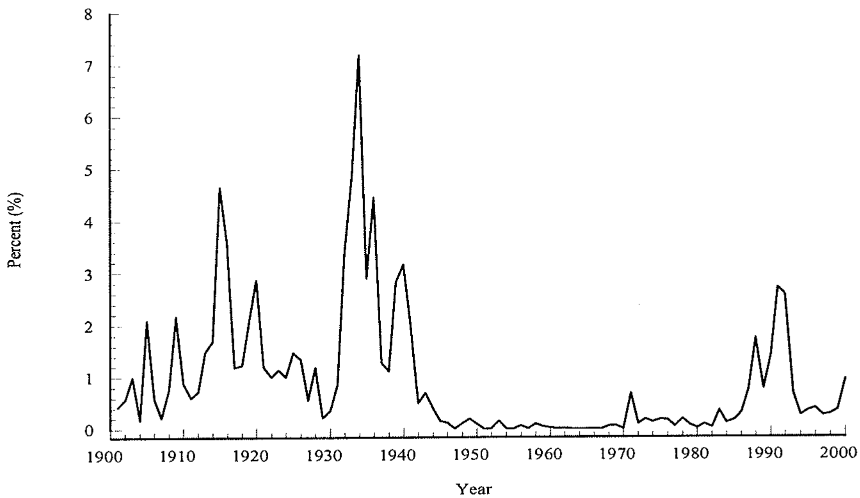
The procedure suggested by Tiao and Box (1981) is used to build a multivariate stochastic model for the three variables. This procedure has the advantage of being more direct and transparent, as compared with alternatives due to Granger and Newbold (1977) and Sims (1980). The sequential and iterative steps of tentative specification, estimation, and diagnostic checking parallel those of the well-known Box-Jenkins method in the univariate time series case. The model is completely determined by the data. Actuarial applications of the Tiao and Box approach can be found in Frees et al., (1997). Unfortunately, detailed model building information was not given in Frees' paper.

The main objectives of this paper are:

- To introduce actuaries to some of the advanced multiple time series analysis techniques used in building vector stochastic models;

[2]See, Vanderhoof et al. (1989, p.547).

Figure 1
Aggregate Default Rate for U.S. Corporate Domestic Bonds (B_t)

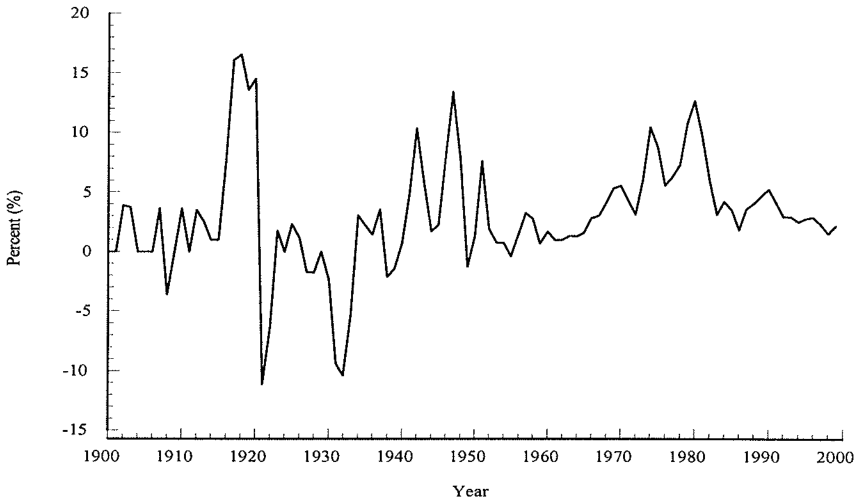


Data Sources: Vanderhoof et al., (1989), Altman and Kishore (1998), and Altman et al., (2000).

- To illustrate the Tiao and Box multiple time series model building procedure in a step-by-step manner so that actuaries who are not expert in this area can still follow the procedure;
- To provide actuaries with a tool for determining whether or not a block of business of an entire company has enough surplus to withstand a possible catastrophic event (Zurcher, 1993);
- To provide actuaries with a tool for determining whether or not a leading economic indicator may serve an an alarm signal for future possible jumps in the bond default levels.

The paper is organized as follows. Section 2 provides a review of the multiple time series modeling approach due to Tiao and Box (1981). Discussion is restricted to points necessary for describing the applications in this paper. Further details can be found in Tiao and Box (1981), Lütkepohl (1993), and Reinsel (1997). Section 3 describes the data while

Figure 2
U.S. Price Inflation (I_t)



Data Sources: Liesner (1989), ftp://ftp.bls.gov/pub/special.requests/cpi/cpiai.txt and http://www.bea.doc.gov/bea/ARTICLES/NATIONAL/NIPA/1998/0898nip3.pdf

Section 4 deals with the process of fitting the model. Section 5 provides an analysis of high-yield bonds. Section 6 concludes the paper.

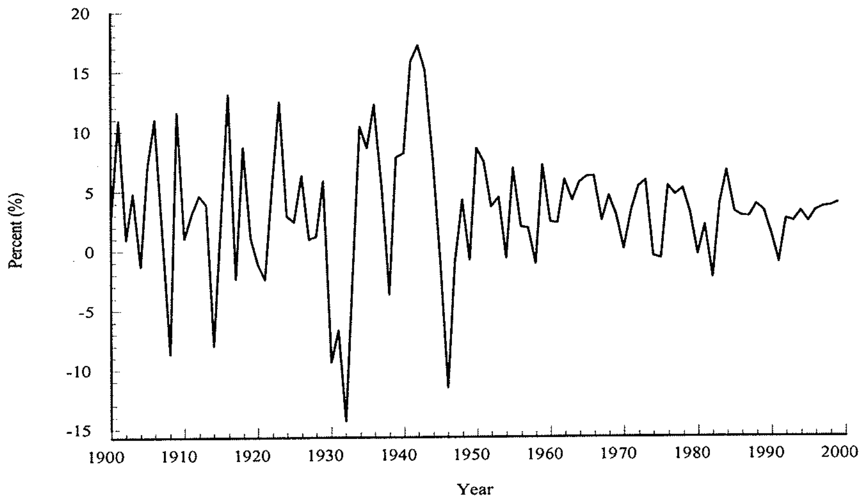
2 Multiple Time Series Analysis

Consider an m-element stationary column vector time series $\mathbf{Y}_t$ with mean $\boldsymbol{\mu}$ for $t = \ldots, -1, 0, 1, \ldots$. $\mathbf{Y}_t$ follows a vector autoregressive moving-average (ARMA) process of order p and q if

$$\boldsymbol{\Phi}(B)\mathbf{Y}_t = \mathbf{c} + \boldsymbol{\Theta}(B)\boldsymbol{\varepsilon}_t \tag{1}$$

where B is the backward shift operator such that $B\mathbf{Y}_t = \mathbf{Y}_{t-1}$, $\boldsymbol{\Phi}(B)$ and $\boldsymbol{\Theta}(B)$ are are $m \times m$ matrix polynomials in B of finite degrees p and q respectively,

Figure 3
U.S. Gross National Product (GNP) Growth Rate (G_t)



Data Sources: Liesner (1989), ftp://ftp.bls.gov/pub/special.requests/cpi/cpiai.txt and http://www.bea.doc.gov/bea/ARTICLES/NATIONAL/NIPA/1998/0898nip3.pdf

$$\boldsymbol{\Phi}(B) = \mathbf{I} - \boldsymbol{\varphi}_1 B - \cdots - \boldsymbol{\varphi}_p B^p$$
$$\boldsymbol{\Theta}(B) = \mathbf{I} - \boldsymbol{\theta}_1 B - \ldots - \boldsymbol{\theta}_q B^q$$

c is a m-dimensional constant column vector, and $\{\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \ldots, \varepsilon_{mt})'\}$ is a sequence of independent and identically distributed Gaussian random column vectors with mean zero and positive-definite variance-covariance matrix $\boldsymbol{\Sigma} = \{\sigma_{ij}\}$. The zeros of the determinantal polynomials $|\boldsymbol{\Phi}(B)|$ and $|\boldsymbol{\Theta}(B)|$ are all assumed to be on or outside the unit circle. It implies that the vector process is both stationary and invertible.

The cross-covariance matrix of order k, $\boldsymbol{\Gamma}(k)$, is given by

$$\begin{aligned}\boldsymbol{\Gamma}(k) &= E[(\mathbf{Y}_t - \boldsymbol{\mu})(\mathbf{Y}_{t-k} - \boldsymbol{\mu})'] \\ &= \{\gamma_{ij}(k)\}, \qquad i, j = 1, \ldots, m \end{aligned} \tag{2}$$

for all integers k. Also,

$$\boldsymbol{\rho}(k) = \{\rho_{ij}(k)\} = \frac{\gamma_{ij}(k)}{\sigma_{ij}}$$

is defined as the corresponding cross-correlation matrix (CCM). Tiao and Box (1981) define the partial autoregression matrix (PAM) at lag k, denoted by $\mathbf{\Pi}(k)$, to be the last matrix coefficient when the data are fitted to a vector autoregressive process of order k. This is a direct extension of the Box and Jenkins (1976, page 64) definition of the partial autocorrelation function for univariate time series.

When $p = 0$, that is, $\mathbf{Y}_t$ is a vector MA(q) process, $\mathbf{\Gamma}(k)$ and $\boldsymbol{\rho}(k)$ are zero for $k > q$. On the other hand, the partial autoregression matrices $\mathbf{\Pi}(k)$ of a vector AR(p) process are zero for $k > p$. These cut off properties provide useful information for identifying the order of pure vector AR or MA models. However, both CCM and PAM are not useful when dealing with mixed vector ARMA processes (i.e., both $p > 0$ and $q > 0$). They do not exhibit cut off patterns. Simple inspection of the matrices $\boldsymbol{\rho}(k)$ and $\mathbf{\Pi}(k)$ would not, in general, give clear values of p and q for mixed models.

Tiao and Tsay (1983) proposed the extended cross-correlation matrix (ECCM) based on the concept of iterated least-squares regression. The asymptotic pattern of the ECCM for a vector ARMA(p, q) model is given in Table 1. There is a remarkable zero-triangle in the table and its vertex is in (p, q) position. Hence, the ECCM can be a useful tool in model specification, particularly for a mixed vector ARMA process.

Tiao and Box (1981) suggested an iterative modeling approach consisting of tentative specification, estimation, and diagnostic checking. For tentative specification the sample cross-correlation matrix (SCCM), denoted by $\hat{\boldsymbol{\rho}}(k) = \{\hat{\rho}_{ij}(k)\}$ is used. These statistics are particularly useful in spotting low order moving average models. If the series $\boldsymbol{\varepsilon}_t$ is a white noise, the standard error of each element of the SCCM is approximately $1/\sqrt{n}$. These statistics, however, provide a crude signal-to-noise ratio guide and are not meant to give formal significant tests.

Estimates of $\mathbf{\Pi}(k)$ and their standard errors can be obtained by fitting autoregressive models of successively higher order by least squares. Tiao and Box (1981) recommended using the likelihood ratio statistic to test the null hypothesis $\boldsymbol{\varphi}_k = 0$ against the alternative $\boldsymbol{\varphi}_k \neq 0$ if an AR(k) process is fitted. To conduct such a test, Bartlett's (1938) statistic, $M(k)$, is used. $M(k)$ is asymptotically χ^2 distributed with m^2 degrees of freedom if the null hypothesis is true.

Sample ECCM can be computed using iterated least-squares regressions. One can construct a two-way table from the sample matrices.

Table 1
The Asymptotic Pattern of the Extended Cross-Correlation Matrix for a Vector ARMA(p, q) Model

	MA order							
AR order	...	...	$q-1$	q	$q+1$	...	...	...
$\vdots$	X	...	X	X	X	X	...	X
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$p-1$	X	...	X	X	X	X	...	X
p	X	...	X	O	O	O	...	O
$p+1$	X	...	X	X	O	O	...	O
$p+2$	X	...	X	X	X	O	...	O
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\vdots$	X	...	X	X	X	X	...	O

Note: X represents nonzero matrix and O represents zero matrix.

The identification is carried out by visual searching the vertex of the zero-triangle inside the sample ECCM table. It is particularly useful in specifying the order of a mixed vector ARMA process.

After the order of the vector ARMA model is tentatively selected, asymptotically efficient estimates of the parameters can be determined using the maximum likelihood approach. Approximate standard errors of the estimates of the elements of $\boldsymbol{\varphi}_i$ for $i = 1, 2, \ldots, p$ and $\boldsymbol{\theta}_j$ for $j = 1, 2, \ldots, q$ can also be obtained and used to test for the significance of the parameters. Further gains in the efficiency of the estimates may be achieved by eliminating parameters that are found to be statistically insignificant. Interested readers may refer to Reinsel (1997, Chapter 5) for a detailed discussion of the maximum likelihood estimation for vector ARMA models.

The maximization of the likelihood function can be conducted by a conditional likelihood method or an exact likelihood method. The conditional likelihood method is computationally convenient, but may be inadequate if the sample size is not sufficiently large. Thus, in this paper we estimate the parameters initially using the conditional likelihood approach and eliminate parameters that are small relative to their standard error. The model is then re-estimated using the exact likelihood method. To guard against incorrectly specifying the model, a detailed diagnostic analysis of the residuals is required. This includes an examination of the plots of standardized residuals and the ECCM table of the residuals.

3 Preliminary Data Analysis

3.1 Data Transformation

The vector time series data consist of three key variables: the U.S. corporate bond default rate (B_t), the U.S. inflation rate (I_t), and the U.S. GNP growth rate (G_t). The data are available from 1900 to 1999. Summary statistics for the observed time series are given in Table 2.

The aggregate bond default rate, on the average over the past 100 years, is less than 1%. On the other hand, the average inflation rate and growth rate are around 3%. From Table 2, we also observe that the distributions of B_t and I_t are positively skewed while the distribution of G_t is negatively skewed. Furthermore, all the observed distributions are leptokurtosis (fat tail), with the default rate distribution having the thickest tail.[3]

This suggests that a transformation of the default rate might be called for, so the square-root transformation (a special case in the class of power transformations introduced by Box and Cox, 1964) of the default rate is used, i.e.,

$$D_t = \sqrt{B_t}.$$

The square-root transformation not only stabilizes the variance and the kurtosis of the default rate, but also prevents the default rate from being negative.

The summary statistics for the transformed variable, D_t, are also given in Table 2. The coefficients of skewness and kurtosis of D_t are significantly improved (in the sense of being closer to a Gaussian distribution). It is not unexpected that (D_t, I_t) and (D_t, G_t) are negatively correlated, while (I_t, G_t) is positively correlated. A view of the possible interrelationships of the economic variables using scatter-plot diagrams is given in Figures 4 through 7. These figures show some strong contemporaneous relationships among series. It justifies the use of multiple time series model for the variables.

[3]An observed distribution is called leptokurtosis if its sample coefficient of excess kurtosis is greater than zero.

Table 2
Summary Statistics in Percent (%)

	B_t	D_t	I_t	G_t
Sample Size	100	100	100	100
Mean	0.9173	0.7587	2.9955	3.3198
Median	0.3795	0.6159	2.6635	3.2705
Standard Deviation	1.2679	0.5875	4.7830	5.2751
Minimum	0.0000	0.0000	-11.0930	-14.3160
Maximum	7.1840	2.6803	16.5250	17.1430
Skewness	2.2613	0.8702	0.1943	-0.3606
Kurtosis	6.0373	0.2410	1.6398	1.5019
			Correlation Matrix	
	D_t	1.00	-0.25	-0.20
	I_t	-0.25	1.00	0.18
	G_t	-0.20	0.18	1.00

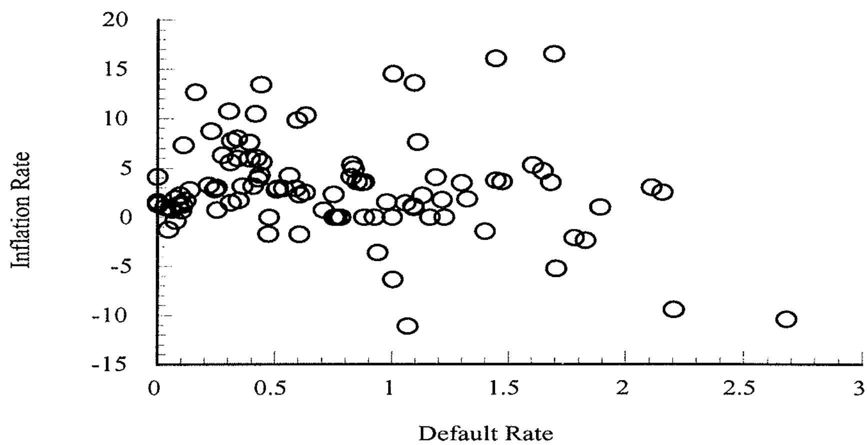
Note: B_t, I_t and G_t are expressed in percent.

For comparison purposes we fit univariate time series models to the economic variables following the orthodox Box and Jenkins (1976) approach. Table 3 gives the sample autocorrelation function (SACF) and the sample partial autocorrelation function (SPACF) of each individual variable up to order 8. The sample autocorrelation coefficients of D_t are exponentially decaying. On the other hand, the sample partial autocorrelations are cut off after lag one. It indicates an AR(1) model for the bond default series. Both the SACF and SPACF for the inflation series are decaying after lag one. It is likely that the underlying process for I_t is an ARMA(1,1). As both lag-1 and lag-4 autocorrelations are significant for the G_t series, an AR(4) model is appropriate.

3.2 Checks for Outliers

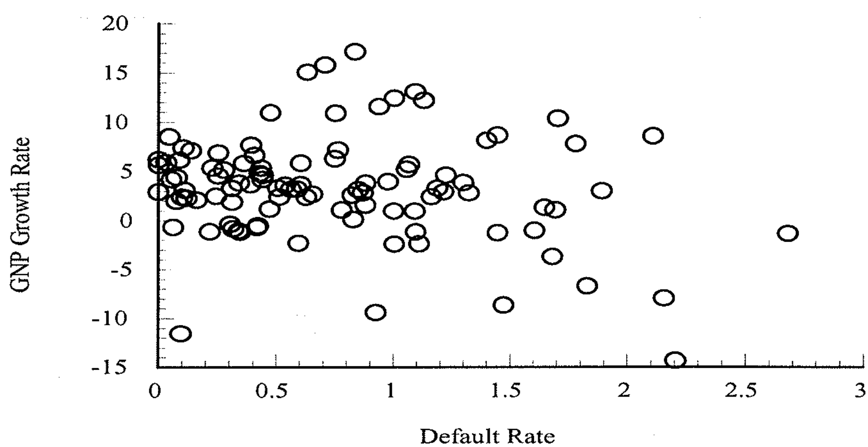
Time series observations are often influenced by interruptive events such as strikes, outbreaks of wars, sudden political or economic crises, or even unnoticed errors of typing and recording. The consequences of these interruptive events create spurious observations, which are inconsistent with the rest of the series. Such observations are usually referred to as outliers.

Figure 4
Scatter Plot of the Variables I_t and G_t



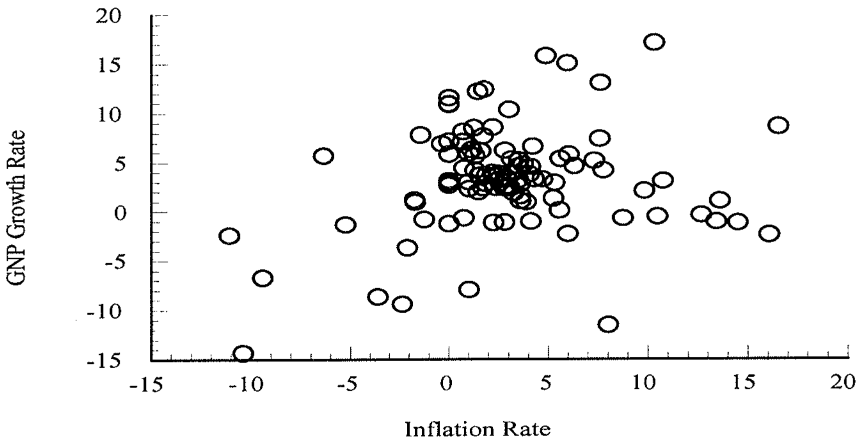
Data Sources: Liesner (1989), ftp://ftp.bls.gov/pub/special.requests/cpi/cpiai.txt and http://www.bea.doc.gov/bea/ARTICLES/NATIONAL/NIPA/1998/0898nip3.pdf

Figure 5
Scatter Plot of the Variables B_t and G_t



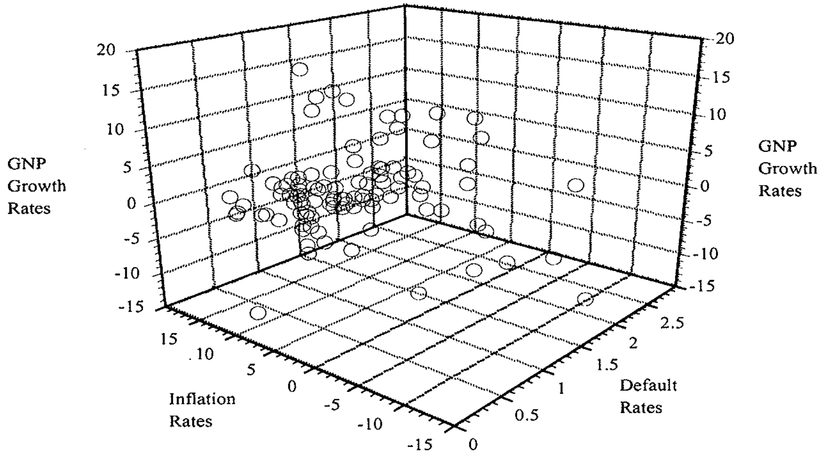
Data Sources: Liesner (1989), ftp://ftp.bls.gov/pub/special.requests/cpi/cpiai.txt and http://www.bea.doc.gov/bea/ARTICLES/NATIONAL/NIPA/1998/0898nip3.pdf

Figure 6
Scatter Plot of the Variables I_t and G_t



Data Sources: Liesner (1989), ftp://ftp.bls.gov/pub/special.requests/cpi/cpiai.txt and http://www.bea.doc.gov/bea/ARTICLES/NATIONAL/NIPA/1998/0898nip3.pdf

Figure 7
3D Scatter Plot of the Variables B_t, I_t and G_t



Data Sources: Liesner (1989), ftp://ftp.bls.gov/pub/special.requests/cpi/cpiai.txt and http://www.bea.doc.gov/bea/ARTICLES/NATIONAL/NIPA/1998/0898nip3.pdf

Table 3
Autocorrelation Coefficients
And Partial Autocorrelation Coefficients

	Lag Order							
	1	2	3	4	5	6	7	8
	(a) Autocorrelation Coefficients							
D_t	0.79	0.67	0.59	0.55	0.49	0.45	0.41	0.35
	(0.10)	(0.15)	(0.18)	(0.20)	(0.21)	(0.23)	(0.23)	(0.24)
I_t	0.62	0.26	0.14	.08	0.15	.17	0.10	0.01
	(0.10)	(0.13)	(0.14)	(0.14)	(0.14)	(0.14)	(0.14)	(0.15)
G_t	0.28	0.03	-0.09	-0.23	-0.16	0.05	.09	-0.02
	(0.10)	(0.11)	(0.11)	(0.11)	(0.11)	(0.12)	(0.12)	(0.12)
	(b) Partial Autocorrelation Coefficients							
D_t	0.79	0.11	0.07	0.12	-0.01	0.02	0.02	-0.06
	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)
I_t	0.62	-0.21	0.12	-0.04	0.19	-0.04	-0.00	-0.08
	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)
G_t	0.28	-0.05	-0.09	-0.20	-0.05	0.11	0.03	-0.13
	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)

Note: Standard errors of autocorrelations are given in parentheses.

As retaining outliers can lead to erroneous model specification and biased predictions (Chan, 1998), an outlier analysis is performed on the specified models for the time series D_t, I_t, and G_t. No outliers were found for the D_t and G_t series; a switch outlier[4] was detected in I_t at $t = 1921$. The magnitude of the outlier is estimated as 9.00. See de Jong and Penzer (1998) for more on time series outlier detection and switch outliers. The analysis in this paper is based on the outlier-adjusted series. Finally, the fitted univariate time series models for each series are summarized in Table 4.

[4]A switch outlier occurs where there are consecutive extreme values on either side of the current level of the series.

Table 4
Univariate Time Series Models for D_t, I_t and G_t

Variable	Model	Equation	$\hat{\sigma}^2$	Q_{12}
Bond Default	AR(1)	$D_t = \underbrace{0.16}_{(0.06)} + \underbrace{0.80}_{(0.06)} D_{t-1} + \varepsilon_t$	0.13	7.2
Inflation	ARMA(1,1)	$I_t = \underbrace{1.56}_{(0.55)} + \underbrace{0.52}_{(0.11)} I_{t-1} + \underbrace{0.49}_{(0.11)} \varepsilon_{t-1} + \varepsilon_t$	8.55	7.6
GNP Growth	AR(4)	$G_t = \underbrace{3.06}_{(0.68)} + \underbrace{0.27}_{(0.10)} G_{t-1} - \underbrace{0.21}_{(0.10)} G_{t-4} + \varepsilon_t$	24.38	7.4

Notes: Standard errors of estimates are given in parentheses; $\hat{\sigma}^2$ denotes the estimates of residual variance; and Q_{12} is the Box-Pierce portmanteau statistic of the residuals with lag order up to 12. Note that Q_{12} is asymptotically distributed as a χ^2 with degrees of freedom equal to 15 minus the number of parameters estimated. None of the Q_{12} statistics reported is significant at the 5% level.

3.3 Check for Cointegration

Cointegration analysis has attracted considerable research interest in recent years. Engle and Granger (1991) and Rao (1994) have described large growth in the business and economic applications of this area. A vector time series is said to be cointegrated if each of the series taken individually is nonstationary with a unit root, while some linear combination of the series is stationary. Cointegration of two (or more) time series suggests that there is a long-run, or equilibrium, relationship between them. The error correction mechanism (ECM) developed by Engle and Granger (1987) can reconcile the short-run behavior of an economic variable with its long-run behavior.

It should be noted that if the vector $(D_t, I_t, G_t)'$ process is cointegrated, then it is not correct to fit a vector ARMA model to the differenced data. Therefore, it is important to check for cointegration using the observed series. The first requirement for cointegration is that D_t, I_t, and G_t are each individually nonstationary with a unit root. We employ the augmented Dickey-Fuller (ADF) test to examine each series. For a stochastic variable Y_t, Dickey and Fuller (1981) considered the following regression model:

$$(1 - B)Y_t = \alpha_0 + \alpha_1 t + \delta Y_{t-1} + \sum_{j=1}^{m} \beta_j \left\{ (1 - B)Y_{t-j} \right\} + \varepsilon_t.$$

The null hypothesis is that $\delta = 0$; that is, a unit root exists in Y (i.e., Y is is nonstationary with a unit root). The ADF test is applied to the three observed series with $m = 2$, the results are given in Table 5. The ADF tests indicate that not all the series are nonstationary with a unit root, and hence the vector process $(D_t, I_t, G_t)'$ is not cointegrated.

4 The Fitted Model

The multiple time series modeling procedures mentioned in Section 2 of this paper can be efficiently performed using matrix-based computer languages such as S-Plus, GAUSS, MATLAB, and SCA. The computations performed in this section are carried out using the SCA system (Liu and Hudak, 1994).

Model Specification: The sample cross-correlation matrix (SCCM) and the partial autoregression matrix (PAM) are first examined. Tiao and Box (1981) suggested summarizing the SCCM and PAM using indicator symbols +, −, and ·, where + denotes a value greater than twice

Table 5
Augmented Dickey-Fuller (ADF) Tests

	ADF Test Statistic	Critical Value (at 5%)	Conclusion
D_t	-2.71	-3.45	Nonstationary with unit root
I_t	-3.99	-3.45	Stationary
G_t	-5.61	-3.45	Stationary

Notes: The ADF test statistic is simply the t-ratio for $\delta = 0$ and the critical values are obtained from MacKinnon (1991, Chapter 13).

the estimated standard error, $-$ denotes a value less than twice the estimated standard error, and $\cdot$ denotes an insignificant value based on the above criteria. The resulting indicator matrices for SCCM and PAM are given in Table 6.

Both SCCM and PAM do not provide a cut off pattern. This suggests that the underlying process could be a mixed process. Therefore, the sample ECCM table is computed. The results are presented using indicator symbols in Table 7. We find a zero-triangle in the table and its vertex is in (1,1) position. Hence, we tentatively specify a vector ARMA(1,1) model for the data.

Model Estimation: The specified ARMA(1,1) model is first estimated using conditional likelihood method. All parameters in the model are computed. We call this model a "full model" in Table 8. Imposing zero restrictions on the coefficients that are insignificant, we re-estimate the model by exact likelihood method. The final model is given in Table 8.

It should be noted that only stationary and invertible vector time series models were considered for the process $(D_t, I_t, G_t)'$. That is, it was assumed that the zeros of the determinantal polynomials $|\boldsymbol{\Phi}(B)|$ and $|\boldsymbol{\Theta}(B)|$ are all on or outside the unit circle. It is important to check these assumptions for the final fitted model in Table 8. For the vector ARMA(1,1) model, the stationarity and invertiblity conditions are equivalent to restricting all the eigenvalues of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\theta}_1$ inside the unit circle (Wei, 1990, p. 345). The eigenvalues of $\hat{\boldsymbol{\varphi}}_1$ and $\hat{\boldsymbol{\theta}}_1$ are $(0.8602, 0.3688, 0.4810)$ and $(0.3660, -0.5240, 0.0000)$, respectively, which suggests that the final fitted model in Table 8 satisfies the basic assumptions.

Table 6
Indicator Matrices for SCCM and PAM

	lag (k)				
	1	2	3	4	5
	(a) Sample Cross-Correlation Matrix (SCCM)				
	$\begin{pmatrix} + & - & - \\ \cdot & + & + \\ \cdot & \cdot & + \end{pmatrix}$	$\begin{pmatrix} + & - & \cdot \\ \cdot & + & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} + & - & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} + & - & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & - \end{pmatrix}$	$\begin{pmatrix} + & - & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
	(b) Partial Autoregression Matrix (PAM)				
	$\begin{pmatrix} + & \cdot & \cdot \\ \cdot & + & \cdot \\ \cdot & \cdot & + \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & - & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & + \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
$M(k)$	193.34	25.04	10.71	15.06	7.53

Notes: Critical values for $M(k)$: 16.9 for 5% level; 21.7 for 1% level.

Table 7
Indicator Matrices for ECCM

AR Order (p)	MA Order (q): 0	1	2	3	4
0	$\begin{pmatrix} + & - & \cdot \\ \cdot & + & + \\ \cdot & \cdot & + \end{pmatrix}$	$\begin{pmatrix} + & - & \cdot \\ \cdot & + & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} + & - & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} + & - & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & - \end{pmatrix}$	$\begin{pmatrix} + & - & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
1	$\begin{pmatrix} \cdot & \cdot & \cdot \\ - & + & + \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
2	$\begin{pmatrix} - & \cdot & + \\ - & + & + \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & + & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
3	$\begin{pmatrix} \cdot & + & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & + & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & - & \cdot \\ \cdot & - & \cdot \end{pmatrix}$	$\begin{pmatrix} - & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ - & \cdot & + \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
4	$\begin{pmatrix} + & \cdot & - \\ - & \cdot & + \\ \cdot & - & \cdot \end{pmatrix}$	$\begin{pmatrix} + & + & \cdot \\ \cdot & \cdot & \cdot \\ - & \cdot & + \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$

The full estimated model can be re-written as follows:

$$\begin{aligned}\hat{D}_t &= 0.126 + 0.853\, D_{t-1} - 0.012\, I_{t-1} + 0.007\, G_{t-1} + \varepsilon_{D,t} \\ &\quad - 0.358\, \varepsilon_{D,t-1} + 0.019\, \varepsilon_{I,t-1} - 0.024\, \varepsilon_{G,t-1} \qquad (3)\\ \hat{I}_t &= 1.789 + 0.716\, D_{t-1} + 0.519\, I_{t-1} - 0.266\, G_{t-1} + \varepsilon_{I,t} \\ &\quad - 0.887\, \varepsilon_{D,t-1} + 0.480\, \varepsilon_{I,t-1} + 0.336\, \varepsilon_{G,t-1} \qquad (4)\\ \hat{G}_t &= 1.367 + 1.825\, D_{t-1} + 0.003\, I_{t-1} + 0.160\, G_{t-1} + \varepsilon_{G,t} \\ &\quad - 0.790\, \varepsilon_{D,t-1} - 0.148\, \varepsilon_{I,t-1} + 0.158\, \varepsilon_{G,t-1}. \qquad (5)\end{aligned}$$

The final estimated model can be re-written as follows:

$$\begin{aligned}\hat{D}_t &= 0.124 + 0.898\, D_{t-1} - 0.014\, G_{t-1} + \varepsilon_{D,t} - 0.366\, \varepsilon_{D,t-1} \qquad (6)\\ \hat{I}_t &= 1.569 + 0.481\, I_{t-1} + \varepsilon_{I,t} + 0.524\, \varepsilon_{I,t-1} + 0.110\, \varepsilon_{G,t-1} \qquad (7)\\ \hat{G}_t &= 1.141 + 1.429\, D_{t-1} + 0.331\, G_{t-1} + \varepsilon_{G,t} \qquad (8)\end{aligned}$$

with

$$\hat{\Sigma} = \begin{pmatrix} 0.123 & -0.280 & -0.844 \\ -0.280 & 8.096 & 4.455 \\ -0.844 & 4.455 & 25.045 \end{pmatrix}. \qquad (9)$$

Diagnostic Checking: The indicator matrices of the residual ECCM are given in Table 9. The zero-triangle is pointing at the (0,0) position. It indicates that there is no significant serial correlation information left in the residuals. The portmanteau test of McLeod and Li (1983) is based on the squared residuals of a time series model and is a test for homoscedasticity of the residuals. The test statistics for the residuals from the fitted models (6) to (8) are 1.4799, 1.2413, and 1.6902, respectively. They should be compared with a χ_1^2 variate (critical value at the 5% level is 3.841). We conclude that the residuals are homoscedastic and the fitted model is adequate for the series.

Table 8
Estimation Results

c	$\hat{\varphi}_1$			$\hat{\theta}_1$			$\hat{\Sigma}$		
(a) Full Model									
0.126	0.853	−0.012	0.007	0.358	−0.019	0.024	0.120	−0.298	−0.847
(0.077)	(0.068)	(0.009)	(0.018)	(0.121)	(0.016)	(0.021)			
1.789	0.716	0.519	−0.266	0.887	−0.480	−0.336	−0.298	7.884	4.480
(1.222)	(1.070)	(0.127)	(0.213)	(1.197)	(0.117)	(0.188)			
1.367	1.825	0.003	0.160	0.790	0.148	−0.158	−0.847	4.480	25.094
(1.373)	(1.227)	(0.139)	(0.241)	(1.624)	(0.182)	(0.232)			
(b) Final Model									
0.124	0.898	0	−0.014	0.366	0	0	0.123	−0.280	−0.844
(0.045)	(0.046)		(0.006)	(0.109)					
1.569	0	0.481	0	0	−0.524	−0.110	−0.280	8.096	4.455
(0.541)		(0.098)			(0.103)	(0.050)			
1.141	1.429	0	0.331	0	0	0	−0.844	4.455	25.045
(0.875)	(0.797)		(0.089)						

Notes: Standard errors of estimates are given in parentheses.

Table 9
Indicator Matrices for ECCM of the Residuals

AR Order (p)	MA Order (q) 0	1	2	3	4
0	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
1	$\begin{pmatrix} + & \cdot & - \\ - & \cdot & + \\ \cdot & \cdot & + \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
2	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & - \\ \cdot & \cdot & - \\ + & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
3	$\begin{pmatrix} + & \cdot & \cdot \\ + & \cdot & \cdot \\ + & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} + & \cdot & \cdot \\ + & - & \cdot \\ + & - & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ - & \cdot & \cdot \\ - & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$

5 An Extension

From 1900 through 1945 significant increases in the default rate were typically preceded by weakness in the overall economy as reflected in the GNP growth. Since 1945, it has more often been the case that increases in the default rate occur in advance of a weakening in the general economy. For example, in the worst episode of the post-war era, the default rate began to rise in 1985 rising from 0.315% to its peak of 2.715% in 1990. The GNP growth, on the other hand, peaked in 1984 but did not fall below the zero mark until the year of 1991. The results in equations (6) to (9) are able to describe such lead-lag relationship among the variables explicitly. Furthermore, the final model also captures the correlation momentum among innovations (residuals) implicitly through $\hat{\Sigma}$.

The U.S. high-yield bond market has been developing rapidly since 1980. Many investment managers now consider high-yield bonds a separate and distinct asset class. By the end of 1996, it was estimated that insurance companies and pension funds owned more than 40% of the high-yield debt market. It is important to study the historical default rate on high-yield bonds. Unfortunately, the history of high-yield market is short. Only 40 quarterly default figures on high-yield bonds are available from 1990 to 1999 (Altman et al., 2000).

The quarterly high-yield default rate, QD_t, as well as its corresponding quarterly GNP growth rate, QG_t, are plotted in Figure 8. Using the multiple time series modeling approach as described in Section 2 yields the following model for the series:

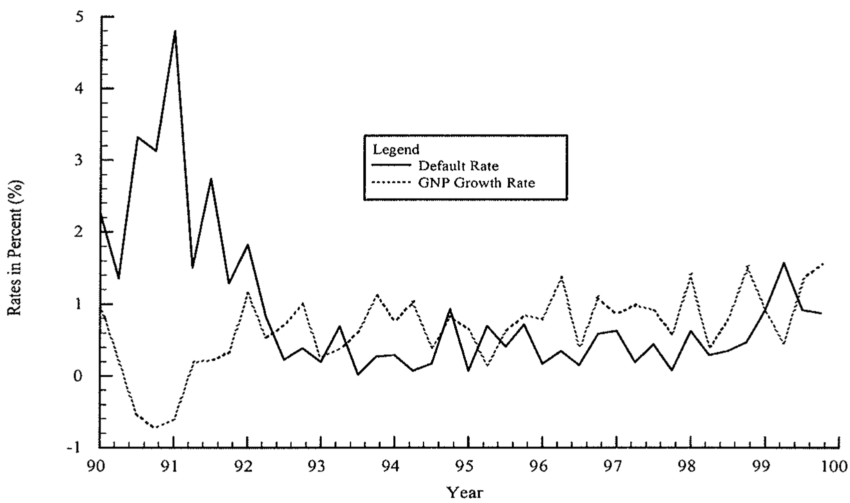
$$\begin{aligned}\sqrt{\widehat{QD}_t} &= 1.156 + .279\sqrt{QD}_{t-1} - 0.846\, QG_{t-1} \\ &\quad + \varepsilon_{D,t} - 0.732\, \varepsilon_{G,t-1} \qquad (10)\\ \widehat{QG}_t &= 1.197 - 0.624\sqrt{QD}_{t-1} + \varepsilon_{G,t} \qquad (11)\end{aligned}$$

with

$$\hat{\Sigma} = \begin{pmatrix} 0.082 & -0.045 \\ -0.045 & 0.204 \end{pmatrix}. \qquad (12)$$

The final model shows a strong first-order contemporary lead-lag relationship between the quarterly high-yield default level and the quarterly growth rate.

Figure 8
Quarterly High-Yield Default Rates and GNP Growth Rates, 1990–1999



Data Sources: Liesner (1989), ftp://ftp.bls.gov/pub/special.requests/cpi/cpiai.txt and http://www.bea.doc.gov/bea/ARTICLES/NATIONAL/NIPA/1998/0898nip3.pdf

6 Closing Comments

The aggregate bond default rates were below average in 1998 and 1999. Based on the fitted equations (3) to (6), the forecast of the default rates for 2000, 2001, and 2002 are 0.341%, 0.417%, and 0.562% respectively.

Based on the equations (7) to (9), the model's forecast for quarterly high-yield default rates for 2000 are 0.896%, 0.928%, 0.899%, and 0.908% respectively. These figures imply an annual forecast of 3.6% high-yield default rate in 2000, which is somewhat more pessimistic than the forecast of 2.8% produced by Altman et al. (2000).

In this paper we have illustrated multiple time series modeling techniques through the analysis of U.S. corporate bond default data. This method has the advantage of being simple to use. The iterative cycles of tentative specification, estimation, and diagnostic checking parallel those of the well-known Box-Jenkins (1976) method. The methodology has been implemented by some time series computer packages, such as SCA (Liu and Hudak, 1994). Vector time series models might be useful to other actuarial applications, say, stochastic asset modeling (Wilkie, 1995), pension simulation (Knox, 1993), and solvency assessment (Hardy, 1993). Research in some of these topics is in process.

There are many books and research papers related to other aspects of default risk or credit risk. Interested readers may refer to Duffie and Huang (1996), Duffie and Singleton (1998), Jarrow (1998), Altman (1999), and Jarrow and Turnbull (2000).

References

Altman, E.I. "Market Dynamics and Investment Performance of Distressed Securities." In *Workouts and Turnarounds II: Global Restructuring for the Next Century*, edited by Cushman, R., Napoli, D. and Sigoloff, S. Homewood, Ill.: Business One Irwin, 1999.

Altman, E.I. and Kishore, V.M. "Defaults and Returns on High Yield Bonds." In *The Handbook of Corporate Debt Instruments.* (ed. F.J. Fabozzi). New Hope, PA: Frank J. Associates, 1998: pp. 255-272.

Altman, E.I., Cooke, D. and Kishore, V.M. "Defaults and Returns on High Yield Bonds: Analysis Through 1998 and Default Outlook for 1999-2001." *Working Paper Series S-99-10.* New York University Salomon Center, 1999.

Altman, E.I., Hukkawala, N. and Kishore, V.M. "Defaults and Returns on High Yield Bonds: Analysis Through 1999 and Default Outlook for 2000-2002." *Working Paper Series S-00-10.* New York University Salomon Center, 2000.

Bartlett, M.S. "Further Aspects of the Theory of Multiple Regression." *Proceedings of Cambridge Philosophical Society* 34 (1938): 33-40.

Box, G.E.P. and Cox, D.R. "An Analysis of Transformations." *Journal of the Royal Statistical Society* B26 (1964): 211-252.

Box, G.E.P and Jenkins, G.M. *Time Series Analysis: Forecasting and Control.* San Francisco, CA: Holden-Day, 1976.

Caouette, J.B., Altman, E.I. and Narayanan, P. *Managing Credit Risk: The Next Great Financial Challenge.* New York, N.Y.: John Wiley & Sons, 1998.

Chan, W.S. "Outlier Analysis of Annual Retail Price Inflation: A Cross-Country Study." *Journal of Actuarial Practice* 6 (1998): 149-172.

De Jong, P. and Penzer, J. "Diagnosing Shocks in Time Series." *Journal of the American Statistical Association* 93 (1998): 796–806.

Dickey, D.A. and Fuller, W.A. "Likelihood Ratio Statistics for Autoregressive Time Series with a Unit Root." *Econometrica* 49 (1981): 1057–1072.

Duffie, D. and Huang, M. "Swap Rates and Credit Quality." *Journal of Finance* 51 (1996): 921–950.

Duffie, D. and Singleton, K. "Simulating Correlated Defaults." *Working Paper.* Graduate School of Business, Stanford University, 1998.

Engle, R.F. and Granger, C.W.J. "Co-integration and Error Correction: Representation, Estimation and Testing." *Econometrica*, 55 (1987): 251–276.

Engle, R.F. and Granger, C.W.J. *Long-Run Economic Relationships: Readings in Cointegration.* Oxford, England: Oxford University Press, 1991.

Frees, E., Kung, Y.C., Rosenberg, M., Young, V. and Lai, S.W. "Forecasting Social Security Assumptions." *North American Actuarial Journal* 1 (1997): 49–82.

Hardy, M.R. "Stochastic Simulation in Life Office Solvency Assessment." *Journal of the Institute of Actuaries* 120 (1993): 131–151.

Jarrow, R. "Current Advances in the Modelling of Credit Risk." *Derivative: Tax, Regulation, Finance* May/June (1998): 1-12.

Jarrow, R. "The Intersection of Market and Credit Risk." *Journal of Banking and Finance* 12 (2000): 271–299.

Knox, D.M. "A Critique of Defined Contribution Plans Using a Simulation Approach." *Journal of Actuarial Practice* 1 (1993): 49–68.

Liesner, T. *One Hundred Years of Economic Statistics.* New York, N.Y.: Facts on File, 1989.

Liu, L.M. and Hudak, G.B. *Forecasting and Time Series Analysis Using the SCA Statistical System.* Chicago, Ill.: Scientific Computing Associates, 1994.

Lütkepohl, H. *Introduction to Multiple Time Series Analysis.* Berlin, Germany: Springer-Verlag, 1993.

MacKinnon, J.G. "Critical Values of Cointegration Tests." In *Long-Run Economic Relationships: Readings in Cointegration,* Engle, R.F. and Granger, C.W.J. (Editors). Oxford, England: Oxford University Press, 1991.

McLeod, A.I. and Li, W.K. "Diagnostic Checking ARMA Time Series Models using Squared-Residual Autocorrelations." *Journal of Time Series Analysis* 1983: 269–273.

Rao, B.B. *Cointegration for the Applied Economist.* New York, N.Y.: St. Martin's Press, 1994.

Reinsel, G.C. *Elements of Multivariate Time Series Analysis.* New York, N.Y.: Springer-Verlag, 1997.

Tiao, G.C. and Box, G.E.P. "Modeling Multiple Time Series with Applications." *Journal of the American Statistical Association* 76 (1981): 802–816.

Tiao, G.C. and Tsay, R.S. "Multiple Time Series Modeling and Extended Sample Cross-Correlations." *Journal of Business and Economic Statistics* 1 (1983): 43–56.

Vanderhoof, I.T., Albert, F., Tenenbein, A. and Verni, R. "The Risk of Asset Default — Report of the Society of Actuaries C-1 Risk Task Force of the Committee on Valuation and Related Areas." *Transactions of the Society of Actuaries* 46 (1989):547–591.

Wilkie, A.D. "More on a Stochastic Investment Model for Actuarial Use." *British Actuarial Journal* 1 (1995): 777–964.

Zurcher, M.L. "C-1 Bond Risk Analysis." *ARCH* 3 (1993): 627–639.