Spring 4-14-2016

Rigidity of the Frobenius, Matlis Reflexivity, and Minimal Flat Resolutions

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RIGIDITY OF THE FROBENIUS, MATLIS REFLEXIVITY, AND MINIMAL FLAT RESOLUTIONS

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Thomas Marley

Lincoln, Nebraska

May, 2016

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University of Nebraska, 2016

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Let R be a commutative, Noetherian ring of characteristic $p > 0$. Denote by $f : R \to R$ the Frobenius endomorphism, and let $R^{(e)}$ denote the ring R viewed as an R-module via f^e. Following on classical results of Peskine, Szpiro, and Herzog, Marley and Webb use flat, cotorsion module theory to show that if R has finite Krull dimension, then an R-module M has finite flat dimension if and only if $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for all $i > 0$ and infinitely many $e > 0$. Using methods involving the derived category, we show that one only needs vanishing for $\dim R + 1$ consecutive values of i for infinitely many values of e to conclude that M has finite flat dimension. We also study a general notion of Matlis duality and prove a change of rings result for Matlis reflexive modules. Finally, we determine some properties of minimal flat resolutions and prove that if the Frobenius map is finite, then tensoring with $R^{(e)}$ preserves minimal flat resolutions. We also demonstrate a version of the New Intersection Theorem for complexes of flat, cotorsion modules.

ACKNOWLEDGMENTS

First, I wish to acknowledge Jesus Christ, the Savior of the world, through whom all things, including mathematics, were made and without whom nothing would exist.

Second, I give my thanks to Tom Marley, without whose constant guidance and patient consideration I would never have been able to conduct this research and so enter into an academic career.

Third, I must also thank those with whom I have had mathematical discussions, in particular Brittney Falahola, Peder Thompson, and Mark Webb. I hope the conversations were as fruitful for them as they were for me. Also, I am indebted to Alexandra Seceleanu, Mark Walker, Mohammad Rammaha, and Andrew Suyker for serving on my Ph.D. committee.

Fourth, to the many professors and graduate students in UNL's Math Department who have prepared me so well to be a teacher of mathematics: Thank you for helping me to find my professional calling. To Nathan Wakefield, in particular, I owe a great deal of thanks for his many hours of mentorship and assistance.

Fifth, I may have never studied mathematics were it not for my high school math teacher, Rita Ahmann. I hope to inspire my students as she once inspired me. Also, to my undergraduate professors who so strongly encouraged me in my academic pursuits I give heartfelt thanks.

Sixth, to my friends and colleagues in the Mathematics Department, whose companionship made graduate school much more enjoyable and who have bought me more beer than I have bought for them: I hope I wasn't a drag on you, and I will settle the debt to you someday.

Seventh, I would also like to thank, in a special way, my friends from St. Teresa's Parish in Lincoln, NE. Only in Heaven will I know the graces that their prayers have gained for me.

Eighth, to my parents. Mom and Dad, thank you for allowing me to pursue my interests and God-given talents, even if that means I won't be close to home. Wherever I may go, my heart is never far from the farm.

Finally, to my wife and children. Though I would much rather be Papa at home than at work, the joy you have given me motivates me to work hard so that our time at home is more special. Thank you all for the sacrifices you have made for me.

DEDICATION

Dedicata Beate Mariae Virgini, Mater Dei.

Ora pro nobis.

GRANT INFORMATION

The author was partially supported by U.S. Department of Education grant P00A120068 (GAANN).

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Chapter 1

Introduction

Let R be a commutative Noetherian ring of characteristic $p > 0$. Since for any $r \in R$, $pr = 0$, the Binomial Theorem implies that the map $f : R \to R$ defined by $r \mapsto r^p$ is a ring homomorphism. We refer to this map as the *Frobenius endomorphism*. Often, it is useful to iterate this map: $f^{(e)}(r) = r^{p^e}$. For $e \geq 1$, we denote by $R^{(e)}$ the ring R considered as an R-module via the action from f^e. That is, if $r \in R$ and $s \in R^{(e)}$, $r \cdot s = r^{p^e} s$.

This seemingly simple map has been very influential in advances in the study of homological algebra. One of the first examples of its use was in [Kun69], where Kunz proved that R is a regular local ring if and only if $R^{(e)}$ is a flat R-module for some (equivalently, all) $e \geq 1$. A few years later, Peskine and Szpiro demonstrated the following theorem.

Theorem 1.1. *[PS73, Thèorém 1.7] Let R be a Noetherian ring of characteristic p and M a finitely generated R-module. If $\operatorname{pd}_R M < \infty$, then $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for all $i, e > 0$.*

Following on this result, Herzog was able to prove the following converse to Peskine and Szpiro's result:[1]

Theorem 1.2. *[Her74, Satz 3.1] Let R be a Noetherian ring of characteristic p and M a finitely generated R-module. Then, if $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for all $i > 0$ and infinitely many $e > 0$, it follows that $\operatorname{pd}_R M < \infty$.*

This theorem makes it natural for one to wonder how many values of i and e for which $\operatorname{Tor}_i^R(R^{(e)}, M)$ vanishes are necessary to guarantee that M has finite projective dimension. In [KL98], Koh and Lee showed the following:

Theorem 1.3. *[KL98, Proposition 2.6] Let R be a local ring of characteristic p and M a finitely generated R-module. If $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for $\operatorname{depth} R + 1$ consecutive values of i and some $e \gg 0$, then $\operatorname{pd}_R M < \infty$.*

Notice that all of the preceding results require M to be finitely generated. Recently, Marley and Webb [MW16] showed the following result where M is not assumed to be finitely generated:

Theorem 1.4. *[MW16, 3.5(a) and 4.2] Let R be a Noetherian ring of characteristic p and M an R-module. Then, the following hold:*

(a) If $\operatorname{fd}_R M < \infty$, then $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for all $i, e > 0$.

(b) If R has finite Krull dimension and $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for all $i > 0$ and infinitely many $e > 0$, then $\operatorname{pd}_R M < \infty$.

The proof of Theorem 1.4 uses flat, cotorsion theory studied extensively by Enochs [Eno84] and Enochs and Xu [EX97]. However, the use of flat, cotorsion theory does not seem to help establish a result analagous to Theorem 1.3. Instead, using complexes and the derived category, we are able to prove the following result in Chapter 2.

[1] In Herzog's original result, he assumed that R has finite Krull dimension. However, in light of [BM67, Lemma 4.5], this assumption is not necessary.

Theorem 1.5. *Let $(R, \mathfrak{m})$ be a local ring of characteristic p, and let M be an R-module. If there exist $\dim R + 1$ consecutive values of i such that $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for infinitely many $e > 0$, then $\operatorname{pd}_R M < \infty$. Moreover, if R is Cohen-Macaulay, one only needs vanishing for $\dim R + 1$ consecutive values of i for a single value e larger than the multiplicity of R for the result to hold.*

In Chapter 3, we study Matlis reflexive modules. For a commutative, Noetherian ring R, we let E (or E_R) denote the R-module $\bigoplus_{\mathfrak{m}\in\Omega} E_R(R/\mathfrak{m})$, where Ω is the set of maximal ideals of R, and $E_R(-)$ denotes the injective hull. It can be shown follows that E is a *minimal injective cogenerator* for R; i.e., E is injective, and it is the "smallest" R-module such that for each R-module M and each nonzero $x \in M$, there exists $f \in \operatorname{Hom}_R(M, E)$ such that $f(x) \neq 0$. Set $(-)^\vee = \operatorname{Hom}_R(-, E)$. Then, as E is a cogenerator, the natural evaluation map $M \to M^{\vee\vee}$ is injective for any R-module M. If the natural evaluation map $M \to M^{\vee\vee}$ is an isomorphism, we say that M is a *(Matlis) reflexive* R-module. The reference to Matlis in this definition should not come as a surprise. Indeed, recall the following theorems:

Theorem 1.6 ([Mat58], local setting; [Ooi76], semilocal setting)**.** *Suppose R is a semilocal ring complete with respect to its Jacobson radical.*

(a) If M is a finitely generated R-module, then $M^\vee$ is an Artinian R-module. Moreover, the natural map $M \to M^{\vee\vee}$ is an isomorphism.

(b) If N is an Artinian R-module, then $N^\vee$ is a finitely generated R-module. Moreover, the natural map $N \to N^{\vee\vee}$ is an isomorphism.

In particular, modules over complete semilocal rings that are either finitely generated or Artinian are reflexive.

The definition for (generalized) Matlis reflexivity first appeared in [BEGR00], whose main result is the following theorem.

Theorem 1.7. *[BEGR00, Theorem 12] Let R be a commutative Noetherian ring. If M is an R-module, M is reflexive if and only if there exists a finitely generated submodule $S \subseteq M$ such that M/S is Artinian and $R/\operatorname{Ann}_R M$ is a complete semilocal ring.*

This result has been used often in the literature; however, the proof given in [BEGR00] relies on a change of rings property of Matlis reflexive modules which is false in general. In Chapter 3, we patch the proof of Theorem 1.7, and we prove the following change of rings result for Matlis reflexive modules:

Theorem 1.8. *[DM15] Let R be a Noetherian ring, S a multiplicatively closed subset of R, and M an R_S-module.*

(a) If M is reflexive as an R-module, then M is reflexive as an R_S-module.

(b) If $S = R \setminus (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r)$, where each $\mathfrak{p}_i$ is a maximal ideal or a nonminimal prime ideal, then the converse to (a) holds.

Finally, in Chapter 4, we study flat, cotorsion module theory with applications to rings of characteristic $p > 0$. The theory of flat, cotorsion modules is in many ways "dual" to the theory of injective modules. Recall the following classical result.

Proposition 1.9. *[Ish65, Theorems 1.4, 1.5] Let E and E' be injective R-modules and F a flat module. Then, $\operatorname{Hom}_R(E, E')$ is a flat R-module and $\operatorname{Hom}_R(F, E')$ is an injective R-module.*

Let R be a Noetherian ring of finite Krull dimension, and let M be an R-module such that $\operatorname{fd}_R M < \infty$. One of the properties of minimal flat resolutions we establish is dual to a property of minimal injective resolutions: *Let* $\mathbf{F}$ *be a flat, cotorsion resolution of* M*. Then,* $\mathbf{F}$ *is minimal if and only if the maps in the complex* $k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, \mathbf{F})$ *are zero for all* $\mathfrak{p} \in \operatorname{Spec} R$*.* An application of this result is the following: *If the Frobenius map is finite and* $\mathbf{F}$ *is a minimal flat resolution of* M*, then* $R^{(e)} \otimes_R \mathbf{F}$ *is a minimal flat resolution of* $R^{(e)} \otimes_R M$*.*

Another focus of Chapter 4 is to prove a version of the New Intersection Theorem for complexes of flat, cotorsion modules. Recall the New Intersection Theorem by Roberts:

Theorem 1.10. *[Rob76, Intersection Theorem] Let* R *be a ring of characteristic* $p > 0$*. If*

$$\mathbf{F} : 0 \to F_r \to F_{r-1} \to \ldots \to F_0 \to 0$$

is a complex of finitely generated free modules such that $H_i(\mathbf{F})$ *is of finite length for all* i *and if* $r < \dim R$*, then* $\mathbf{F}$ *is exact.*

Our result is the following:

Theorem 1.11. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of characteristic* $p > 0$ *which is the homomorphic image of a Gorenstein ring, and let*

$$\mathbf{G} : 0 \to G_r \xrightarrow{\partial_r} \ldots \to G_1 \xrightarrow{\partial_1} G_0 \to 0$$

be a complex of flat, cotorsion R*-modules such that* $H_i(\mathbf{G})$ *is cotorsion and* $\operatorname{Supp}_R H_i(\mathbf{G}) \subseteq \{\mathfrak{m}\}$ *for all* i*. Suppose* $\mathbf{G}$ *is not exact, set* $j := \inf\{i : H_i(\mathbf{G}) \neq 0\}$*, and suppose* $\mathfrak{m}H_j(\mathbf{G}) \neq H_j(\mathbf{G})$*. Then,* $r \geq \dim R$*.*

The proof of this result relies on flat, cotorsion theory developed by Enochs and Xu. However, it should be noted that the previous theorem follows from a more general result by Foxby ([Fox79, Lemma 4.2]), who uses the derived category in his argument.

Chapter 2

Rigidity of the Frobenius

In this chapter, we prove the following rigidity property of the Frobenius.

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian ring. Let M be a complex of R-modules such that $s := \sup M$ is finite. Suppose for $d+1$ consecutive values of $i \geq s$ and infinitely many integers $e > 0$ that $\operatorname{Tor}_i^R(M, R^{(e)}) = 0$. Then, $\operatorname{fd}_R M < \infty$. Moreover, if R is Cohen-Macaulay, one only needs vanishing for $d+1$ consecutive values of $i \geq s$ for a single value e larger than the multiplicity of R for the result to hold.*

This result follows the classical work of [PS73] and [Her74] as well as more recent work of [KL98] and [MW16]. In order to establish our result, we make use of the derived category and certain results involving the Koszul complex with respect to a system of parameters.

2.1 Results with the Koszul Complex

For background on the derived category, see Appendix B.

Let $(R, \mathfrak{m}, k)$ be a local ring, and let $\mathbf{x} = \{x_1, \ldots, x_j\}$ be a sequence of elements from R. We denote by $K^R(\mathbf{x})$ the Koszul complex with respect to $\mathbf{x}$. If M is a complex of R-modules, then we denote $K^R(\mathbf{x}; M) := K^R(\mathbf{x}) \otimes_R M$.

Our work in this section will follow [AHIY12], where the authors use the Koszul complex on a sequence of generators for $\mathfrak{m}$. For our purposes, we need to use the Koszul complex with respect to a system of parameters of R. Our preliminary results, therefore, establish similar results from [AHIY12] for Koszul complexes with respect to systems of parameters.

We first need the following result of Eagon and Fraser [EF68]. Since the proof is not long, we include it for completeness.

Lemma 2.1.1. *Let R be a local ring and $\mathbf{x} = \{x_1, \ldots, x_d\}$ be a system of parameters for R. Set $I = (\mathbf{x})$ and $K = K^R(\mathbf{x})$. Then, there is an integer s such that the complex*

$$C^i : 0 \to I^{i-d} K_d \to I^{i-d+1} K_{d-1} \to \ldots \to I^i K_0 \to 0$$

is exact for all $i \geq s$.

Proof. Choose $w_1, \ldots, w_d \in K_1$ such that $\partial(w_j) = x_j$ for all $1 \leq j \leq d$. For all $1 \leq n \leq d$ and i sufficiently large, we have

$$Z_n(C^i) = Z_n(K) \cap I^{i-n} K_n = I\left(Z_n(K) \cap I^{i-n-1} K_n\right)$$

by the Artin-Rees Lemma. Increasing i, we may assume that the above equality holds for all n. Now, let $z \in Z_n(C^i)$. It follows that $z = \sum_{j=1}^{d} x_j v_j$, where each $v_j \in I^{i-n-1} K_n \cap Z_n(K)$. Now, we have that $z = \partial(w)$, where $w = \sum_{j=1}^{d} w_j v_j$, which finishes the proof. $\square$

Before stating the next proposition, we need the following definition.

Definition 2.1.2. Let $(R, \mathfrak{m})$ be a local ring and M a complex of R-modules. We define the *Loewy length* of M to be

$$\ell\ell_R M = \inf\{i \in \mathbb{N} : \mathfrak{m}^i M = 0\}.$$

The *homotopical Loewy length* (cf. [AIM06]) of M is defined to be

$$\ell\ell_{\mathbf{D}(R)} M = \inf\{\ell\ell_R V : M \simeq V \text{ in } \mathbf{D}(R)\}.$$

Proposition 2.1.3 (cf. [AHIY12], Proposition 4.1)**.** *Let M be an R-complex and $\mathbf{x} = \{x_1, \ldots, x_d\}$ a system of parameters. Then,*

$$\ell\ell_{\mathbf{D}(R)} K^R(\mathbf{x}; M) \leq \ell\ell_{\mathbf{D}(R)} K^R(\mathbf{x}) < \infty.$$

Proof. Let C^i be the complex in Lemma 2.1.1, and let s be an integer such that C^i is exact for all $i \geq s$. Then, the natural map $K^R(\mathbf{x}) \to K^R(\mathbf{x})/C^s$ is a quasi-isomorphism. Suppose that $\mathfrak{m}^l \subseteq (\mathbf{x})$. Since $(\mathbf{x})^s \left(K^R(\mathbf{x})/C^s\right) = 0$, we see that

$$\ell\ell_{\mathbf{D}(R)} K^R(\mathbf{x}) \leq sl < \infty.$$

Set $c = \ell\ell_{\mathbf{D}(R)} K^R(\mathbf{x})$, and let $\gamma : K^R(\mathbf{x}) \simeq V$, where $\mathfrak{m}^c V = 0$. Let $\epsilon : F \to M$ be a semi-free resolution. Then, we have

$$K^R(\mathbf{x}) \otimes_R M \xleftarrow[\simeq]{1\otimes\epsilon} K^R(\mathbf{x}) \otimes_R F \xrightarrow[\simeq]{\gamma\otimes 1} V \otimes_R F ,$$

and so $K^R(\mathbf{x}; M) \simeq V \otimes_R F$ in $\mathbf{D}(R)$. As $\mathfrak{m}^c(V \otimes_R F) = \mathfrak{m}^c V \otimes_R F = 0$, the claim follows. ☐

Now that we have established these preliminaries, the proof of the following result is the same as in [AHIY12].

Proposition 2.1.4 (cf. [AHIY12], Proposition 4.3)**.** *Let $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a homomorphism of local rings. For $\mathbf{x}$ a system of parameters of S, set $K^S = K^S(\mathbf{x})$, $c = \ell\ell_{\mathbf{D}(S)} K^S$, and assume that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}^c$. Then, for any complex L in $\mathbf{D}(R)$, there exists an isomorphism of graded k-vector spaces*

$$\mathrm{Tor}_*^R(L, K^S) \cong \mathrm{Tor}_*^R(L, k) \otimes_k H_*(K^S).$$

Proof. Let V be a complex of S-modules such that $K^S \simeq V$ in $\mathbf{D}(S)$ and $\mathfrak{n}^c V = 0$. It follows that $\mathfrak{m} \cdot V = 0$, so V is a complex of k-vector spaces. In particular, $H_i(V)$ is a k-vector space for each i. Thus, the canonical surjection $Z_i(V) \to H_i(V)$ has a splitting σ_i. Now, composing the injection $Z_i(V) \to V_i$ with σ_i, we get a morphism $H(V) \to V$ which is a quasi-isomorphism. Hence, $V \simeq H(V) \simeq H(K^S)$ in $\mathbf{D}(R)$, we see that $H(K^S)$ is a complex of k-vector spaces.

Therefore, we have the following isomorphisms in $\mathbf{D}(R)$:

$$L \otimes_R^{\mathbf{L}} K^S \simeq L \otimes_R^{\mathbf{L}} H(K^S) \simeq (L \otimes_R^{\mathrm{L}} k) \otimes_k^{\mathbf{L}} H(K^S).$$

The result follows by taking homology and applying the Künneth isomorphisms (e.g., [Avr98, Proposition 1.3.4]). ☐

We finish this section with two more preliminary results which will help us in the next section.

Lemma 2.1.5. *Let* $\mathbf{x} = \{x_1, \ldots, x_r\}$ *be a sequence of elements from* R. *Let* M *be a complex of* R*-modules such that* $H_i(M) = 0$ *for all* $i = t, \ldots, t+j$, *where* $j \geq r$. *Then,* $H_i(K^R(\mathbf{x}; M)) = 0$ *for all* $i = t+r, \ldots, t+j$.

Proof. We induce on r. For $r = 1$, there exists a short exact sequence of complexes

$$0 \to M \to K^R(x_1; M) \to M[-1] \to 0$$

([BH93, 1.6.12]). The corresponding long exact sequence gives

$$H_i(M) \to H_i(K^R(x_1; M)) \to H_{i-1}(M).$$

So, $H_i(K^R(x_1; M)) = 0$ for $i = t+1, \ldots, t+j$ since both $H_i(M)$ and $H_{i-1}(M)$ are zero for these values of i.

Suppose the lemma is true for $r-1$ elements. Let $K' = K^R(x_1, \ldots, x_{r-1})$ and $M' = K' \otimes_R M$. By the induction hypothesis, $H_i(M') = 0$ for $i = t+r-1, \ldots, t+j$. Now, $K^R(\mathbf{x}; M) \cong K^R(x_r; M')$. The result now follows by the $r = 1$ case. $\square$

Proposition 2.1.6. *Let* $\varphi : R \to S$ *be a homomorphism of local rings, and let* $\mathbf{x} = \{x_1, \ldots, x_r\}$ *be a sequence of elements in* S. *Let* M *be a complex of* R*-modules, and let* $l \geq r$. *If* $\operatorname{Tor}_i^R(S, M) = 0$ *for* $i = t, \ldots, t+l$, *then* $\operatorname{Tor}_i^R(K^S(\mathbf{x}), M) = 0$ *for* $i = t+r, \ldots, t+l$.

Proof. Note that $\operatorname{Tor}_i^R(K^S(\mathbf{x}), M) = H_i(K^S(\mathbf{x}) \otimes_R F) \cong H_i(K^S(\mathbf{x}) \otimes_S (S \otimes_R F))$, where F is a semi-free resolution of M. Observe that $H_i(S \otimes_R F) = \operatorname{Tor}_i^R(S, M) = 0$ for all $i = t, \ldots, t+l$. The result now follows from Lemma 2.1.5 applied to S. $\square$

2.2 Rigidity of the Frobenius Map

Definition 2.2.1. For a local ring $(R, \mathfrak{m})$, we define

$$c(R) := \inf\{\ell\ell_{\mathbf{D}(R)} K^R(\mathbf{x}) : \mathbf{x} \text{ a system of parameters of } R\}.$$

Our results concern local rings of characteristic p. Given such a ring R, we let $R^{(e)}$ denote the ring R viewed as an R-module via the eth iteration of the Frobenius map.

Proposition 2.2.2. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional local ring of characteristic $p > 0$ and M an R-complex. Let e be an integer such that $p^e \geq c(R)$. Suppose that $\operatorname{Tor}_i^R(M, R^{(e)}) = 0$ for $i = t, t+1, \ldots, \ell$ for some $\ell \geq t + d$. Then, $\operatorname{Tor}_i^R(M, k) = 0$ for $i = t + d, \ldots, \ell$.*

Proof. Set $S = R^{(e)}$, and let $\mathbf{x}$ be a system of parameters for S such that $c(R) = \ell\ell_{\mathbf{D}(S)} K^S(\mathbf{x})$. Set $K^S = K^S(\mathbf{x})$, and note by Proposition 2.1.6 that $\operatorname{Tor}_i^R(M, K^S) = 0$ for $i = t + d, \ldots, \ell$. Finally, applying Proposition 2.1.4 and noting that $H_0(K^S) \neq 0$, we see that $\operatorname{Tor}_i^R(M, k) = 0$ for $i = t + d, \ldots, \ell$. $\square$

Theorem 2.2.3. *[CIM] Let R be a d-dimensional Noetherian ring and M an R-complex such that $s = \sup M < \infty$. Suppose there exists $t \geq s + d$ such that $\operatorname{Tor}_t^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Then, $\operatorname{fd}_R M < \infty$.*

As promised in the introduction to this chapter, we now prove the following extension of [MW16, Theorem 4.2].

Theorem 2.2.4. *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian ring. Let M be a complex of R-modules such that $s := \sup M$ is finite. Suppose that there exists $t \geq s$ such that for infinitely many integers $e > 0$, $\operatorname{Tor}_i^R(M, R^{(e)}) = 0$ for $i = t, t+1, \ldots, t+d$. Then, $\operatorname{fd}_R M < \infty$.*

Proof. By Theorem 2.2.3, we need only show that $\operatorname{Tor}_{t+d}^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. For $\mathfrak{p} \in \operatorname{Spec} R$, since localization commutes with the Frobenius, it follows that $\operatorname{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}^{(e)}) = 0$ for $i = t, t+1, \dots, t+d$. But $\dim R_\mathfrak{p} \leq d$, so by Proposition 2.2.2, it follows that $\operatorname{Tor}_{t+d}^{R_\mathfrak{p}}(M_\mathfrak{p}, k(\mathfrak{p})) = 0$, which finishes the proof. □

Question 2.2.5. Following Theorem 1.3, does there exist a single value of e for which the vanishing of $\operatorname{Tor}_i^R(M, R^{(e)})$ for $\dim R + 1$ values of i implies that $\operatorname{fd}_R M < \infty$?

The difficulty of answering the above question is that we need $p^e \geq c(R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ in order to apply Proposition 2.2.2 to reach our desired conclusion. In the general setting where R is a local ring, it is not clear how to bound $\{c(R_\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} R\}$. We do, however, have such a result for Cohen-Macaulay rings. First, recall the following definition:

Definition 2.2.6. Let R be a commutative ring and $\mathfrak{a}$ be an ideal of R. An ideal $\mathfrak{b}$ is called a *reduction* of $\mathfrak{a}$ if $\mathfrak{b} \subseteq \mathfrak{a}$ and for some $r > 0$, $\mathfrak{a}^{r+1} = \mathfrak{b}\mathfrak{a}^r$.

Lemma 2.2.7. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring such that $R/\mathfrak{m}$ is infinite. Then, $c(R) \leq e(R)$, where $e(R)$ is the multiplicity of R.*

Proof. Since $R/\mathfrak{m}$ is infinite, we can choose $\mathbf{x}$ to be a system of parameters for R which forms a minimal reduction of $\mathfrak{m}$ ([Mat89, Theorem 14.14]). Now, since R is Cohen-Macaulay, we have

$$e(R) = \lambda_R(R/(\mathbf{x})) \geq \ell\ell_R(R/(\mathbf{x})) = \ell\ell_{\mathbf{D}(R)} K^R(\mathbf{x}) \geq c(R).$$

□

Theorem 2.2.8. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and M an R-complex such that $s = \sup M < \infty$. Suppose for some e with $p^e \geq e(R)$ we have $\operatorname{Tor}_t^R(M, R^{(e)}) = 0$ for $d+1$ consecutive values of $t \geq s$. Then,* $\operatorname{fd}_R M < \infty$.

Proof. By passing to faithful flat extensions, we may assume that R is complete and that $R/\mathfrak{m}$ is infinite, which also implies that $k(\mathfrak{p})$ is infinite for all $\mathfrak{p} \in \operatorname{Spec} R$. By [Lec64, Remarks following Theorem 1], it follows that $e(R) \geq e(R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. So, $p^e \geq c(R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ by Lemma 2.2.7. By Proposition 2.2.2, it follows that $\operatorname{Tor}_{t+d}^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$, which implies that $\operatorname{fd}_R M < \infty$ by Theorem 2.2.3. $\square$

Chapter 3

Matlis Reflexivity

Throughout this chapter R will denote a commutative Noetherian ring, and E will denote the R-module $\bigoplus_{\mathfrak{m}\in\Omega} E_R(R/\mathfrak{m})$, where Ω is the set of maximal ideals of R. Recall from the introduction that an R-module M is said to be (Matlis) reflexive if the natural map $M \to M^{\vee\vee}$ is an isomorphism.

The definition for (generalized) Matlis reflexivity first appeared in [BEGR00], whose main results are the following theorems.

Theorem 3.1. *[BEGR00, Theorems 9 and 12] Let R be a commutative Noetherian ring.*

(a) If M is a finitely generated R-module, M is reflexive if and only if $R/\operatorname{Ann}_R(M)$ is a complete semilocal ring.

(b) If M is an R-module, M is reflexive if and only if there exists a finitely generated submodule $S \subseteq M$ such that M/S is Artinian and $R/\operatorname{Ann}_R M$ is a complete semilocal ring.

While the former statement follows directly from the latter statement in the previous theorem, the proof of the latter relies on the former, so we keep the results separate.

To prove the forward direction of Theorem 3.1(a), the authors assert the following "change of rings" principal for Matlis reflexivity ([BEGR00, Lemma 2]):

(*) *Let S be a multiplicatively closed subset of R and suppose M is an R_S-module. Then, M is reflexive as an R-module if and only if M is reflexive as an R_S-module.*

However, the proof of (*) given in [BEGR00] is incorrect, and the "if" part is false in general. Since (*) is used in the forward implication of Theorem 3.1(a), we will patch the proof of this result in Section 3.1. Then, we will demonstrate the error in the proof of (*) as well as a counterexample to the "if" part in Section 3.2. Next, in Section 3.3, we will demonstrate some results about products of rings which will be useful for us in Section 3.4, where we conclude by exploring for which multiplicatively closed subsets the statement (*) is true. Indeed, we prove the following:

Theorem 3.2. *[DM15] Let R be a Noetherian ring, S a multiplicatively closed subset of R, and M an R_S-module.*

(a) If M is reflexive as an R-module, then M is reflexive as an R_S-module.

(b) If $S = R \setminus (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r)$, where each $\mathfrak{p}_i$ is a maximal ideal or a nonminimal prime ideal, then the converse to (a) holds.

3.1 Fixing Theorem 3.1

The following two lemmata will be useful in the sequel:

Lemma 3.1.1. *[BEGR00, Lemma 1] Let M be an R-module and I some ideal of R such that $IM = 0$. Then, M is reflexive as an R-module if and only if M is reflexive as an $R/\operatorname{Ann}_R M$-module.*

Proof. By adjunction, we have $\mathrm{Hom}_R(M, E) = \mathrm{Hom}_{R/I}(M, E_{R/I})$ since $\mathrm{Hom}_R(R/I, E_R) = E_{R/I}$. The result follows readily. $\square$

Lemma 3.1.2. *[BEGR00, Lemma 5] Let $0 \to M \to N \to L \to 0$ be a short exact sequence of R-modules. Then, N is a reflexive R-module if and only if M and L are reflexive R-modules.*

Proof. The result follows from applying the Snake Lemma to the following commutative diagram, where the vertical maps are given by the natural evaluation maps, which are injective:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M^{\vee\vee} \arrow[r] & N^{\vee\vee} \arrow[r] & L^{\vee\vee} \arrow[r] & 0
\end{tikzcd}$$

$\square$

Let M be a finitely generated R-module. We will now show that for M to be a reflexive R-module, it suffices to show that $M \cong M^{\vee\vee}$ via any isomorphism. The proof we give below is the same as the proof of [Chr00, Proposition 1.1.9], though in that result, the author uses the duality functor $(-)^* = \mathrm{Hom}_R(-, R)$.

For any R-module N, let $i_N : N \to N^{\vee\vee}$ be the canonical map given by evaluation. A direct computation demonstrates that the canonical map $i_{N^\vee} : N^\vee \to (N^\vee)^{\vee\vee}$ is split injective with splitting given by the dual map $(i_N)^\vee : (N^\vee)^{\vee\vee} \to N^\vee$. We now have

Proposition 3.1.3. *Let M be a finitely generated module such that $M \cong M^{\vee\vee}$. Then, the natural map $i_M : M \to M^{\vee\vee}$ is an isomorphism.*

Proof. Let $M \xrightarrow{f} M^{\vee\vee}$ be the given isomorphism, and consider the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_M"] & M^{\vee\vee} \arrow[d, "i_{M^{\vee\vee}}"] \\ M^{\vee\vee} \arrow[r, "f^{\vee\vee}"] & (M^{\vee\vee})^{\vee\vee}. \end{tikzcd}$$

Note that the map $i_{M^{\vee\vee}}$ is split injective (set $N = M^{\vee}$ in the preceding remark). Let $s_{M^{\vee\vee}}$ be its splitting. It now follows that i_M is split injective with splitting $s_M := f^{\vee\vee} s_{M^{\vee\vee}} f^{-1}$. Now, $s_M i_M = \mathrm{id}_M$, and so s_M is a surjective homomorphism. Thus, $f s_M$ is a surjective endomorphism of the (finitely generated) module $M^{\vee\vee}$. But surjective endomorphisms of finitely generated modules are isomorphisms ([Mat89, Theorem 2.4]), so $f s_M$ is an isomorphism. Thus, s_M is an isomorphism as f is, and since s_M splits i_M, it follows that i_M is an isomorphism. $\square$

For a semilocal Noetherian ring R, let $J(R)$ denote its Jacobson radical. We let $\widehat{R}^{J(R)}$, or $\widehat{R}$, denote the $J(R)$-adic completion of the ring R, and when we say that a module is complete over a semilocal ring R, we mean that it is complete with respect to $J(R)$.

Lemma 3.1.4. *Let R be a semilocal ring and M a finitely generated R-module. Then, M is complete with respect to $J(R)$ if and only if M is reflexive as an R-module.*

Proof. Set $E = \bigoplus_{\mathfrak{m}\in\Omega} E_R(R/\mathfrak{m})$. First, note that $\mathrm{Hom}_R(E,E) \cong \prod_{\mathfrak{m}\in\Omega} \widehat{R_{\mathfrak{m}}}$. Indeed, for $\mathfrak{m} \neq \mathfrak{n}$, $\mathrm{Hom}_R(E_R(R/\mathfrak{m}), E_R(R/\mathfrak{n})) = \mathrm{Hom}_{R_\mathfrak{n}}(E_R(R/\mathfrak{m})_\mathfrak{n}, E_R(R/\mathfrak{n})) = 0$ by adjunction. It follows that

$$\mathrm{Hom}_R(E,E) = \prod_{\mathfrak{m}\in\Omega} \mathrm{Hom}_R(E_R(R/\mathfrak{m}), E_R(R/\mathfrak{m})) \cong \prod_{\mathfrak{m}\in\Omega} \widehat{R_{\mathfrak{m}}}.$$

Moreover, using the Chinese Remainder Theorem, we have the following ring isomorphisms:

$$\begin{aligned}
\widehat{R} &\cong \varprojlim R/J(R)^n \\
&\cong \varprojlim \prod_{\mathfrak{m}\in\Omega} R/\mathfrak{m}^n \\
&\cong \prod_{\mathfrak{m}\in\Omega} \varprojlim R/\mathfrak{m}^n \\
&\cong \prod_{\mathfrak{m}\in\Omega} \varprojlim (R_{\mathfrak{m}})/(\mathfrak{m}^n R_{\mathfrak{m}}) \\
&\cong \prod_{\mathfrak{m}\in\Omega} \widehat{R_{\mathfrak{m}}}.
\end{aligned}$$

As M is finitely generated, applying [Ish65, Lemma 1.6], it follows that $\mathrm{Hom}_R(\mathrm{Hom}_R(M,E),E) \cong M \otimes_R \mathrm{Hom}_R(E,E) \cong \prod_{\mathfrak{m}\in\Omega} \widehat{M_{\mathfrak{m}}}^{\mathfrak{m}} \cong \widehat{M}^{J(R)}$, so the result follows by Proposition 3.1.3. □

The following lemma is the main component in fixing Theorem 3.1(a).

Lemma 3.1.5. *Let R be a semilocal ring. Suppose there exists a finitely generated, complete (equivalently, reflexive) R-module M. Set $I = \mathrm{Ann}_R M$. Then, R/I is complete with respect to the $J(R/I)$-adic topology.*

Proof. By Lemma 3.1.1, M is a reflexive as an R-module if and only if it is reflexive as an R/I-module. So we may assume in addition that M is a faithful R-module.

Say $M = Rx_1 + \ldots + Rx_n$. Then there is an injective map from R to M^n by sending r to $(rx_1, \ldots, rx_n)$. Tensoring with $\widehat{R}$, the $J(R)$-adic completion of R, we have an injection $\widehat{R} \hookrightarrow M^n$. As M^n is a Noetherian R-module, this says that $\widehat{R}$ is a finitely generated R-module.

Consider the short exact sequence $0 \to R \to \widehat{R} \to \widehat{R}/R \to 0$. Tensoring with $R/J(R)$, as $R/J(R) \cong \widehat{R/J(R)}$, we see that $(R/J(R)) \otimes_R (\widehat{R}/R) = 0$. As $\widehat{R}/R$ is finitely generated, it follows that $\widehat{R}/R = 0$ by Nakayama's Lemma. $\square$

Remark 3.1.6. Let M be an R-module, and let N be an $R_{\mathfrak{p}}$-module for some $\mathfrak{p} \in \operatorname{Spec} R$. Then, $\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$. Indeed, note

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, N_{\mathfrak{p}})) \cong \operatorname{Hom}_R(M, N).$$

In particular, if $\mathfrak{p} \notin \operatorname{Supp}_R M$, it follows that $\operatorname{Hom}_R(M, N) = 0$.

Lemma 3.1.7. *Let R be a commutative ring and M be an R-module. Let $\{A_i\}_{i\in\Lambda}$ be a collection of R-modules, set $A = \bigoplus_{i\in\Lambda} A_i$, and suppose* $\operatorname{Hom}_R(M, A_i) = 0$ *for all $i \neq j$. Then,* $\operatorname{Hom}_R(M, A) = \operatorname{Hom}_R(M, A_j)$.

Proof. The split exact sequence $0 \to \oplus_{i\neq j} A_i \to \oplus A_i \to A_j \to 0$ yields the (split) exact sequence

$$0 \to \operatorname{Hom}_R\left(M, \bigoplus_{i\neq j} A_i\right) \to \operatorname{Hom}_R\left(M, \bigoplus_{i\in\Lambda} A_i\right) \to \operatorname{Hom}_R(M, A_j) \to 0.$$

So it suffices to show that $\operatorname{Hom}_R\left(M, \bigoplus_{i\neq j} A_i\right) = 0$. But as $0 \to \bigoplus_{i\neq j} A_i \to \prod_{i\neq j} A_i$ is exact, we have that

$$0 \to \operatorname{Hom}_R\left(M, \bigoplus_{i\neq j} A_i\right) \to \operatorname{Hom}_R\left(M, \prod_{i\neq j} A_i\right),$$

which, as $\operatorname{Hom}_R\left(M, \prod_{i\neq j} A_i\right) = \prod_{i\neq j} \operatorname{Hom}_R(M, A_i) = 0$, gives the result. $\square$

Proposition 3.1.8. *[BEGR00, cf. Proposition 8] Let M be a reflexive R-module and suppose that* $\operatorname{Hom}_R(M, E_R(R/\mathfrak{m})) = 0$ *for some maximal ideal $\mathfrak{m}$. Then,* $\operatorname{Hom}_R(M^\vee, E_R(R/\mathfrak{m})) = 0$.

Proof. Suppose that $\operatorname{Hom}_R(M^\vee, E_R(R/\mathfrak{m})) \neq 0$. Note that the surjection $E_R \to E_R(R/\mathfrak{m})$ induces a surjection $\operatorname{Hom}_R(M^\vee, E_R) \to \operatorname{Hom}_R(M^\vee, E_R(R/\mathfrak{m}))$. Thus, taking the composition

$$M \overset{\cong}{\hookrightarrow} M^{\vee\vee} = \operatorname{Hom}_R(M^\vee, E_R) \twoheadrightarrow \operatorname{Hom}_R(M^\vee, E_R(R/\mathfrak{m})),$$

we see that $\operatorname{Hom}_R(M, \operatorname{Hom}_R(M^\vee, E_R(R/\mathfrak{m}))) \neq 0$. However, by adjunction, this implies that $\operatorname{Hom}_R(M^\vee, \operatorname{Hom}_R(M, E_R(R/\mathfrak{m})) \neq 0$, which is a contradiction as $\operatorname{Hom}_R(M, E_R(R/\mathfrak{m})) = 0$. □

We are now ready to patch the proof of the following result.

Theorem 3.1(a). *Let M be a finitely generated R-module. If M is a reflexive R-module, then $R/\operatorname{Ann}_R M$ is a complete semilocal ring.*

Proof. Suppose M is reflexive. Then, we have that $M^\vee = \bigoplus_{\mathfrak{m}\in\Omega} \operatorname{Hom}_R(M, E_R(R/\mathfrak{m}))$ is also reflexive. By [BEGR00, Lemma 6], it follows that this direct sum cannot be infinite. Therefore, there is a set $\{\mathfrak{n}_1, \ldots, \mathfrak{n}_t\} \subseteq \Omega$ such that $\operatorname{Hom}_R(M, E_R(R/\mathfrak{m})) = 0$ for $\mathfrak{m} \notin \{\mathfrak{n}_1, \ldots, \mathfrak{n}_t\}$ and $\operatorname{Hom}_R(M, E_R(R/\mathfrak{n}_i)) \neq 0$. Thus, $M^\vee = \bigoplus_{i=1}^t M_i$, where $M_i = \operatorname{Hom}_R(M, E_R(R/\mathfrak{n}_i))$. As $M_i \subseteq M^\vee$, which is reflexive, it follows that M_i is reflexive for each i by Lemma 3.1.2. Further, note that $\operatorname{Supp}_R M_i = \{\mathfrak{n}_i\}$ as M is finitely generated and $\operatorname{Supp}_R E_R(R/\mathfrak{n}_i) = \{\mathfrak{n}_i\}$. For $\mathfrak{m} \neq \mathfrak{n}_i$, note that

$$\begin{aligned} \operatorname{Hom}_R(M_i, E_R(R/\mathfrak{m})) &= \operatorname{Hom}_R(M_i, \operatorname{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m}, E_R(R/\mathfrak{m}))) \\ &= \operatorname{Hom}_{R_\mathfrak{m}}((M_i)_\mathfrak{m}, E_R(R/\mathfrak{m})) \\ &= 0. \end{aligned}$$

Set $I_i = \operatorname{Ann}_R M_i$. We claim that $\operatorname{Hom}_R(R/I_i, E_R(R/\mathfrak{m})) = 0$ for $\mathfrak{m} \neq \mathfrak{n}_i$. In particular, this will show that $I_i \not\subseteq \mathfrak{m}$ for all $\mathfrak{m} \neq \mathfrak{n}_i$. It will then follow that $(R/I_i, \mathfrak{n}_i/I_i)$

is a local ring. To prove the claim, as M is reflexive and by Remark 3.1.6 and Lemma 3.1.7, we have that

$$\begin{aligned} M &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(M,E),E) \\ &\cong \bigoplus_{i=1}^{t} \operatorname{Hom}_R(\operatorname{Hom}_R(M,E_R(R/\mathfrak{n}_i)),E_R(R/\mathfrak{n}_i)) \\ &= \bigoplus_{i=1}^{t} \operatorname{Hom}_R(M_i,E_R(R/\mathfrak{n}_i)). \end{aligned}$$

Thus, as M is a finitely generated R-module, $M_i^\vee = \operatorname{Hom}_R(M_i, E_R(R/\mathfrak{n}_i))$ is a finitely generated R-module, too. Also, $I_i = \operatorname{Ann}_R M_i = \operatorname{Ann}_R M_i^\vee$. Therefore, there is an injection $0 \to R/I_i \to (M_i^\vee)^l$, for some $l \in \mathbb{N}$. Applying $\operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$, $\mathfrak{m} \neq \mathfrak{n}_i$, it follows that

$$\operatorname{Hom}_R((M_i^\vee)^l, E_R(R/\mathfrak{m})) \to \operatorname{Hom}_R(R/I_i, E_R(R/\mathfrak{m})) \to 0$$

is exact. But the leftmost module is 0 by Proposition 3.1.8, so $\operatorname{Hom}_R(R/I_i, E_R(R/\mathfrak{m})) = 0$, too, which finishes the proof of the claim.

Now, note $I = \operatorname{Ann}_R M = \operatorname{Ann}_R(M^\vee) = \bigcap_{i=1}^t I_i$, and so $R/I = R/I_1 \times \ldots \times R/I_t$ by the Chinese Remainder Theorem. As each R/I_i is a local ring, R/I is a semilocal ring. Finally, as M is a finitely generated, reflexive, faithful R/I-module, Lemma 3.1.5 applies to show that R/I is complete with respect to $J(R/I)$, which finishes the proof. $\square$

3.2 Examples and Counterexamples

Let R be a Noetherian ring and $S \subseteq R$ some multiplicatively closed subset of R. In [BEGR00], the authors assert that $\operatorname{Hom}_R(R_S, E_R) = E_{R_S}$. In this section, we will show that this is not necessarily the case.

The following result is a slight generalization of [MS95, Lemma 4.1], and the proof is very similar.

Proposition 3.2.1. *Let R be a Noetherian ring, and let $\mathfrak{q} \nsubseteq \mathfrak{p}$ be distinct prime ideals. Then,*

$$\operatorname{Ass}_R \operatorname{Hom}_R(R_\mathfrak{p}, E_R(R/\mathfrak{q})) = \{\mathfrak{l} \in \operatorname{Spec} R : \mathfrak{l} \subseteq \mathfrak{p} \cap \mathfrak{q}\}.$$

Proof. Let $\mathfrak{l} \in \operatorname{Ass}_R \operatorname{Hom}_R(R_\mathfrak{p}, E_R(R/\mathfrak{q}))$. Then, $\operatorname{Hom}_R(R/\mathfrak{l}, \operatorname{Hom}_R(R_\mathfrak{p}, E_R(R/\mathfrak{q}))) \neq 0$. From this statement, we conclude two things via adjunction:

i) $\operatorname{Hom}_R(R/\mathfrak{l} \otimes_R R_\mathfrak{p}, E_R(R/\mathfrak{q})) \neq 0$; and

ii) $\operatorname{Hom}_R(R_\mathfrak{p}, \operatorname{Hom}_R(R/\mathfrak{l}, E_R(R/\mathfrak{q})) \neq 0$.

Now, i) shows that $R/\mathfrak{l} \otimes_R R_\mathfrak{p} \neq 0$, so, in particular, $\mathfrak{l}R_\mathfrak{p} \neq R_\mathfrak{p}$, whence it follows that $\mathfrak{l} \cap (R \setminus \mathfrak{p}) = \emptyset$, and so $\mathfrak{l} \subseteq \mathfrak{p}$. The utility of ii) lies in the fact that

$$\operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{l}, E_R(R/\mathfrak{q})) = V(\mathfrak{l}) \cap \{\mathfrak{q}\}$$

as $R/\mathfrak{l}$ is finitely generated. Since $\operatorname{Hom}_R(R/\mathfrak{l}, E_R(R/\mathfrak{q})) \neq 0$, it follows that $\mathfrak{l} \subseteq \mathfrak{q}$. Thus, $\mathfrak{l} \subseteq \mathfrak{p} \cap \mathfrak{q}$.

Conversely, let $\mathfrak{l} \subseteq \mathfrak{p} \cap \mathfrak{q}$. First, note that

$$\operatorname{Hom}_R(R_\mathfrak{p}/\mathfrak{l}R_\mathfrak{p}, E_R(R/\mathfrak{q})) \cong \operatorname{Hom}_{R/\mathfrak{l}}(R_\mathfrak{p}/\mathfrak{l}R_\mathfrak{p}, \operatorname{Hom}_R(R/\mathfrak{l}, E_R(R/\mathfrak{q})))$$

via adjunction. As $\mathfrak{l} \subseteq \mathfrak{q}$, $R/\mathfrak{q}$ is an $R/\mathfrak{l}$-module, $\operatorname{Hom}_R(R/\mathfrak{l}, E_R(R/\mathfrak{q})) \cong E_{R/\mathfrak{l}}(R/\mathfrak{q})$. Further, $\operatorname{Hom}_R(R_\mathfrak{p}/\mathfrak{l}R_\mathfrak{p}, E_R(R/\mathfrak{q}))$ is isomorphic to a submodule of $\operatorname{Hom}_R(R_\mathfrak{p}, E_R(R/\mathfrak{q}))$. Combining these facts, we have

$$\operatorname{Ass}_R \operatorname{Hom}_{R/\mathfrak{l}}(R_\mathfrak{p}/\mathfrak{l}R_\mathfrak{p}, E_{R/\mathfrak{l}}(R/\mathfrak{q})) \subseteq \operatorname{Ass}_R \operatorname{Hom}_R(R_\mathfrak{p}, E_R(R/\mathfrak{q})).$$

As $R/\mathfrak{l}$ is a domain and $\mathfrak{l} \subseteq \mathfrak{p} \cap \mathfrak{q}$, we may reduce to the case where R is a domain and strive to show that $0 \in \operatorname{Ass}_R(R_\mathfrak{p}, E_R(R/\mathfrak{q}))$.

To this end, let $x \in \mathfrak{q} \setminus \mathfrak{p}$, and let $\mathfrak{r} \subseteq \mathfrak{q}$ be a minimal prime over xR, so $\operatorname{ht} \mathfrak{r} = 1$ by Krull's Principal Ideal Theorem. We claim that $R_\mathfrak{p} + R_\mathfrak{r}$ is a subring of Q, the quotient field of R. It suffices to show that $R_\mathfrak{p} + R_\mathfrak{r}$ is multiplicatively closed. Let $s \notin \mathfrak{p}$ and $t \notin \mathfrak{r}$. Then, it follows that $(s,t)R \nsubseteq \mathfrak{p} \cup \mathfrak{r}$ by prime avoidance, and so there exist $a, b \in R$, such that $u = as + bt \in (R \setminus \mathfrak{p}) \cap (R \setminus \mathfrak{r})$. Then,

$$\frac{1}{s} \cdot \frac{1}{t} = \frac{1}{st} = \frac{u}{ust} = \frac{a}{tu} + \frac{b}{su},$$

which is in $R_\mathfrak{p} + R_\mathfrak{r}$. This proves the claim. If $R_\mathfrak{p} + R_\mathfrak{r}$ is a proper subring of Q, i.e., if it is not a field, then there exists some nonzero prime ideal N of $R_\mathfrak{p} + R_\mathfrak{r}$. Note that $0 \neq N \cap R$ as N is nonzero and R is a domain; further $N \cap R$ is a prime ideal of R. Also, $N \cap R \subseteq \mathfrak{r}$, which tells us that $N \cap R = \mathfrak{r}$ as $\operatorname{ht} \mathfrak{r} = 1$. Similarly, $N \cap R \subseteq \mathfrak{p}$, and so $\mathfrak{r} \subseteq \mathfrak{p}$, which contradicts the fact that $x \notin \mathfrak{p}$. Therefore, $R_\mathfrak{p} + R_\mathfrak{r} = Q$, and so

$$\frac{R_\mathfrak{p}}{R_\mathfrak{p} \cap R_\mathfrak{r}} \cong \frac{Q}{R_\mathfrak{r}} \neq 0.$$

Further, note that $Q/R_\mathfrak{r}$ is an $R_\mathfrak{q}$-module. Indeed, $Q/R_\mathfrak{r} \otimes_R R_\mathfrak{q} = Q_\mathfrak{q}/(R_\mathfrak{r})_\mathfrak{q} = Q/R_\mathfrak{r}$ as $\mathfrak{r} \subseteq \mathfrak{q}$. Thus, by Matlis Duality in the local ring $(R_\mathfrak{q}, \mathfrak{q}R_\mathfrak{q})$, we have that $\operatorname{Hom}_R(Q/R_\mathfrak{r}, E_R(R/\mathfrak{q})) \cong \operatorname{Hom}_{R_\mathfrak{q}}(Q/R_\mathfrak{r}, E_R(R/\mathfrak{q})) \neq 0$.

Let $f \in \operatorname{Hom}_R(Q/R_{\mathfrak{r}}, E_R(R/\mathfrak{q}))$, and suppose that $af = 0$ for some $a \in R$. Then,

$$0 = af(Q/R_{\mathfrak{r}}) = f(a \cdot Q/R_{\mathfrak{r}}) = f(Q/R_{\mathfrak{r}}),$$

and so $f = 0$. Thus, for any non-zero element of $\operatorname{Hom}_R(Q/R_{\mathfrak{r}}, E_R(R/\mathfrak{q}))$, we have that $\operatorname{Ann}_R f = 0$. Therefore,

$$0 \in \operatorname{Ass}_R\left(\frac{R_{\mathfrak{p}}}{R_{\mathfrak{p}} \cap R_{\mathfrak{r}}}, E_R(R/\mathfrak{q})\right) \subseteq \operatorname{Ass}_R \operatorname{Hom}_R(R_{\mathfrak{p}}, E_R(R/\mathfrak{q})),$$

which completes the proof. $\square$

This result leads to two different examples.

Example 3.2.2. Let $(R, \mathfrak{m})$ be a local ring of dimension at least two and $\mathfrak{p}$ any prime which is neither maximal nor minimal. By Proposition 3.2.1, $\operatorname{Ass}_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, E_R(R/\mathfrak{m})) = \operatorname{Spec} R_{\mathfrak{p}}$, and so $\operatorname{Hom}_R(R_{\mathfrak{p}}, E_R(R/\mathfrak{m})) \ncong E_{R_{\mathfrak{p}}}$.

Example 3.2.3. Let R be a Noetherian domain which is not local, and let $\mathfrak{m} \neq \mathfrak{n}$ be distinct maximal ideals of R. Then, $(0) \in \operatorname{Ass}_R \operatorname{Hom}_R(R_{\mathfrak{m}}, E_R(R/\mathfrak{n}))$, which is a submodule of $\operatorname{Hom}_R(R_{\mathfrak{m}}, E_R)$. Hence, $\operatorname{Hom}_R(R_{\mathfrak{m}}, E_R) \ncong E_{R_{\mathfrak{m}}}$.

Melkersson and Schenzel provide another example:

Example 3.2.4. [MS95, p. 127] Let R be a local Noetherian domain such that the completion of R has a nonminimal prime contracting to (0) in R. If $Q = R_{(0)}$ is the field of fractions in R, then it follows that $\operatorname{Hom}_R(Q, E_R)$ is not an Artinian Q-vector space, so $\operatorname{Hom}_R(Q, E_R) \ncong E_Q$.

We conclude this section showing that the converse to part (a) of Theorem 3.2 is not true in general. If R is a domain and Q is its field of fractions, it follows readily that Q is reflexive as a $Q = R_{(0)}$-module. However, as the following result shows, Q is rarely reflexive as an R-module.

Proposition 3.2.5. *Let R be a Noetherian domain and Q its field of fractions. Then, Q is reflexive as an R-module if and only if R is a complete local domain of dimension at most one.*

Proof. Suppose that R is a one-dimensional complete local domain with maximal ideal $\mathfrak{m}$, and let $E = E_R(R/\mathfrak{m})$. Then, by [Sch15, Theorem 2.5], $\operatorname{Hom}_R(Q, E) \cong Q$. But the evaluation map of the Matlis double dual is always injective, so $Q \to \operatorname{Hom}_R(\operatorname{Hom}_R(Q, E), E)$ is an isomorphism.

Conversely, suppose that Q is reflexive as an R-module. By Theorem 3.1(b), it follows that R is a complete semilocal domain, hence local. (Indeed, from Proposition 3.1.4, $R = \prod_{\mathfrak{m}\in\Omega} \widehat{R_{\mathfrak{m}}}$, so if there is more than one maximal ideal, R will not be a domain.) Thus, we need only show that $\dim R \leq 1$. Theorem 3.1(b) gives us that there exists a finitely generated submodule $N \subseteq Q$ such that Q/N is Artinian. Moreover, $\operatorname{Ann}_R(N) = 0$, so $\dim R = \dim N$. Applying local cohomology to the short exact sequence $0 \to N \to Q \to Q/N \to 0$, and noting that $H^i_{\mathfrak{m}}(Q/N) = 0$ for all $i \geq 1$ and $H^i_{\mathfrak{m}}(Q) = 0$ for all $i \geq 1$, we see that $H^i_{\mathfrak{m}}(N) = 0$ for all $i \geq 2$. Thus, $\dim N \leq 1$, which finishes the proof. $\square$

3.3 Products of Rings

In this section, we establish some results about products of rings which will help us in the sequel.

Lemma 3.3.1. *Let $R = R_1 \times \ldots \times R_n$, where R_i is a commutative ring for each i. Consider R-modules $M = M_1 \times \ldots \times M_n$ and $N = N_1 \times \ldots \times N_n$, where $M_i = R_iM$ and $N_i = R_iN$ for each i. Then,* $\operatorname{Hom}_R(M_i, N_j) = 0$ *for $i \neq j$. In particular,*

$$\operatorname{Hom}_R(M, N) = \operatorname{Hom}_{R_1}(M_1, N_1) \times \ldots \times \operatorname{Hom}_{R_n}(M_n, N_n).$$

Proof. If $i \neq j$, note that $\operatorname{Hom}_R(R/R_i, N_j) \cong (0 :_{N_j} R_i) = N_j$. Thus, we have

$$\begin{aligned}
\operatorname{Hom}_R(M_i, N_j) &\cong \operatorname{Hom}_R(M_i, \operatorname{Hom}_R(R/R_i, N_j)) \\
&\cong \operatorname{Hom}_R(M_i \otimes_R R/R_i, N_j) \\
&\cong \operatorname{Hom}_R(M_i/M_i, N_j) \\
&= 0.
\end{aligned}$$

If $j = i$, we have that $\operatorname{Hom}_R(M_i, N_i) \cong \operatorname{Hom}_{R_i}(M_i, N_i)$ as $R_lM_i = 0$ for all $l \neq i$, and the result follows. □

The next result demonstrates that showing reflexivity for a product of rings reduces to showing reflexivity on each factor.

Lemma 3.3.2. *Let $R = R_1 \times \ldots \times R_n$, where each R_i is a Noetherian ring, and let Ω_i denote the set of maximal ideals of R_i. Let M be an R-module, and write $M = M_1 \times \ldots \times M_n$, where $M_i = R_iM$. Then, M is reflexive as an R-module if and only if each M_i is reflexive as an R_i-module.*

Proof. For each $1 \leq i \leq n$, set $E_i = \bigoplus_{\mathfrak{m} \in \Omega_i} E_{R_i}(R_i/\mathfrak{m})$, the minimal injective cogenerator of R_i. If $\mathfrak{m}_i \in \Omega_i$, note that $\widetilde{\mathfrak{m}_i} = R_1 \times \ldots \times \mathfrak{m}_i \times \ldots \times R_n$ is a maximal ideal of R and that each maximal ideal of R arises in this way. Further, note that

$$E_R(R/\widetilde{\mathfrak{m}_i}) = E_{R_i}(R_i/\mathfrak{m}_i)$$

for each i. Indeed, $R_{\widetilde{\mathfrak{m}_i}} \cong (R_i)_{\mathfrak{m}_i}$, so

$$E_R(R/\widetilde{\mathfrak{m}_i}) \cong E_{R_{\widetilde{\mathfrak{m}_i}}} \cong E_{(R_i)_{\mathfrak{m}_i}}(R_i/\mathfrak{m}_i) \cong E_{R_i}(R_i/\mathfrak{m}_i).$$

Hence, it follows that the minimal injective cogenerator of R is $E_R = E_1 \oplus \ldots \oplus E_n$, where $E_i = R_i E_R$.

By Lemma 3.3.1, note that

$$\operatorname{Hom}_R(M, E) = \bigoplus_{i=1}^{n} \operatorname{Hom}_{R_i}(M_i, E_i).$$

Further, as $\operatorname{Hom}_{R_i}(M_i, E_i)$ is an R_i-module, it follows that

$$\operatorname{Hom}_R(\operatorname{Hom}_{R_i}(M_i, E_i), E) = \operatorname{Hom}_{R_i}(\operatorname{Hom}_{R_i}(M_i, E_i), E_i)$$

for each i. Therefore,

$$\operatorname{Hom}_R(\operatorname{Hom}_R(M, E), E) = \bigoplus_{i=1}^{n} \operatorname{Hom}_{R_i}(\operatorname{Hom}_{R_i}(M_i, E_i), E_i).$$

Now, the natural maps $M_i \to \operatorname{Hom}_{R_i}(\operatorname{Hom}_{R_i}(M_i, E_i), E_i)$ are isomorphisms for each i if and only if the natural map $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, E_R), E_R)$ is an isomorphism, which finishes the proof. $\square$

We end this section with a result on localizing products of rings.

Lemma 3.3.3. *Let $R = R_1 \times \ldots \times R_n$ where each R_i is a commutative ring, and let $S \subset R$ be a multiplicatively closed subset of R. Let $f_i : R \to R_i$ be the canonical projection map, and set $S_i = f_i(S)$. Then, S_i is multiplicatively closed subset of R_i, and*

$$R_S = (R_1)_{S_1} \times \ldots \times (R_n)_{S_n}$$

Proof. Define $\varphi : R_S \to (R_1)_{S_1} \times \ldots \times (R_n)_{S_n}$ by $\varphi(\frac{a}{s}) = (\frac{f_i(a)}{f_i(s)})$. We claim that φ is an isomorphism of rings. To show that φ is well-defined, we will show that the map $\widetilde{f}_i : R_S \to (R_i)_{S_i}$ given by $\frac{a}{s} \mapsto \frac{f_i(a)}{f_i(s)}$ is well-defined for each i. If $\frac{a}{s} = \frac{b}{t} \in R_S$, note that $at = bs$, so $f_i(at) = f_i(bs)$, or $f_i(a)f_i(t) = f_i(b)f_i(s)$ as f_i is a ring homomorphism. But this means that $\frac{f_i(a)}{f_i(s)} = \frac{f_i(b)}{f_i(t)}$ in $(R_i)_{S_i}$, which proves the claim. That φ is a homomorphism follows from the fact that each f_i is a ring homomorphism.

To show injectivity, suppose that $\varphi(\frac{a}{s}) = 0$. Then, for each i, $\frac{f_i(a)}{f_i(s)} = 0$, so there exists $t_i \in S$ such that $0 = f_i(t_i)f_i(a) = f_i(t_ia)$ for all i. Taking $t = t_1 \ldots t_n \in S$, we have that $f_i(ta) = 0$ for all i. As $R = R_1 \times \ldots \times R_n$, it follows that $ta = 0$, whence $\frac{a}{s} = 0$.

Finally, to show surjectivity of φ, let $\left(\frac{r_i}{s_i}\right) \in (R_1)_{S_1} \times \cdots \times (R_n)_{S_n}$. Choose $t_i \in S$ such that $f_i(t_i) = s_i$ for each i. As $R = R_1 \times \cdots \times R_n$, there exists $a \in R$ such that $f_i(a) = s_1 \cdots r_i \cdots s_n$ for each i. It follows that

$$\varphi\left(\frac{r}{t_1 \ldots t_n}\right) = \left(\frac{r_i}{s_i}\right),$$

which completes the proof. $\square$

3.4 Change of Rings Results

We are now prepared to prove Theorem 3.2(a).

Theorem 3.4.1. *Let R be a Noetherian ring, S a multiplicatively closed subset of R, and M an R_S-module. If M is a reflexive R-module, then M is a reflexive R_S-module.*

Proof. To show that M is R_S-reflexive, note that M is reflexive as an R_S-module if and only if M is reflexive as an $R_S/\operatorname{Ann}_{R_S} M_S$-module by Lemma 3.1.1. So, we may assume without loss of generality that $\operatorname{Ann}_{R_S} M = 0$, which implies that $\operatorname{Ann}_R M = 0$. In particular, it follows from Theorem 3.1 that R is a complete semilocal ring. Thus, the proof of Lemma 3.1.4 shows that $R = R_1 \times \ldots \times R_k$, where each R_i is a complete local ring and $\operatorname{Ann}_{R_i} M = 0$. By Lemma 3.3.3, it follows that $R_S = (R_1)_{S_1} \times \ldots \times (R_k)_{S_k}$, where S_i is the image of S under the canonical projection $R \to R_i$. To show that M is R_S-reflexive, by Lemma 3.3.2, it suffices to show that $R_i M$ is a reflexive $(R_i)_{S_i}$-module for each i. Thus, we may reduce to the case that $(R, \mathfrak{m})$ is a complete local ring and $\operatorname{Ann}_R M = 0$.

Since M is reflexive as an R-module, it follows from Theorem 3.1 that there exists a short exact sequence

$$0 \to N \to M \to X \to 0$$

such that N is finitely generated and X is Artinian. Now, if $S \cap \mathfrak{m} = \emptyset$, $R_S = R$, and there is nothing to show. Otherwise, $X_S = 0$, and it follows that $M \cong N_S$, a finitely generated R_S-module. If we can show that R_S is Artinian (hence, semilocal and complete), it will follow from Theorem 1.6 that M is a reflexive R_S-module.

To this end, note that $\operatorname{Ann}_R N_S = \operatorname{Ann}_R M = 0$, so $\operatorname{Ann}_R N = 0$, and $\dim R = \dim N$. Since M is an R_S-module and $\mathfrak{m} \cap S \neq \emptyset$, we see that $H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{m}R_S}(M) = 0$ for all i. Further, X is Artinian, so $H^i_{\mathfrak{m}}(X) = 0$ for $i \geq 1$. Thus, by looking at the long exact sequence on local cohomology, we conclude that $H^i_{\mathfrak{m}}(N) = 0$ for all $i \geq 2$, which means that $\dim R = \dim N \leq 1$. So $\dim R_S = 0$, and R_S is Artinian. $\square$

Now, to show part (b) of Theorem 3.2, we need the following result credited to F. Schmidt on Henselian local rings found in [BKKN67]. Recall that a Henselian ring is a local ring $(R, \mathfrak{m}, k)$ satisfying the following: *Let $F(x) \in R[x]$ be a monic polynomial and $\overline{F}(x) \in k[x]$, the polynomial obtained from reducing the coefficients in F modulo $\mathfrak{m}$. If there are monic, relatively prime polynomials $g, h \in k[x]$ such that $\overline{F}(x) = g(x)h(x)$, then there exist monic polynomials $G, H \in R[x]$ such that $F(x) = G(x)H(x)$, and $\overline{G} = g$ and $\overline{H} = h$.*

Proposition 3.4.2. *[BKKN67, Satz 2.3.11] Let $(R, \mathfrak{m})$ be a local Henselian domain which is not a field, and let F be the field of fractions of R. Suppose that V is a discrete valuation ring with field of fractions F. Then, $R \subseteq V$.*

Proof. We will first show that $\mathfrak{m} \subseteq V$. Let k be the residue field of R and $a \in \mathfrak{m}$. Let n be any positive integer which does not divide the characteristic of k. We claim that the polynomial $P(x) = x^n - (1 + a)$ has a root b in R. Indeed, $\overline{P}(x) = x^n - 1 = (x - 1)^l g(x) \in k[x]$, where $l < n$ and $g(1) \neq 0$. Because R is Henselian and $\overline{P}(x)$ has linear factors, so does $P(x)$, which proves the claim. Let v be the valuation on F associated to V. Then, it follows that $nv(b) = v(1 + a)$. If $v(a) < 0$, it follows that $v(b) \leq -1$, and so $v(1 + a) \leq -n$. But n can be arbitrarily large, so we reach a contradiction. Thus, $v(a) \geq 0$, and $a \in V$.

Now, let $c \in R$, and choose some nonzero $d \in \mathfrak{m}$. (Here, we are using that R is not a field.) If $v(c) < 0$, it follows that $v(c^\ell d) < 0$ for ℓ sufficiently large. But $c^\ell d \in \mathfrak{m}$, which is a contradiction. Thus, $v(c) \geq 0$, and $c \in V$, which finishes the proof. $\square$

Before we proceed, we need the following result:

Theorem 3.4.3. *[HS06, Theorem 6.3.3] Let R be a Noetherian domain with field of fractions F, and let $0 \neq \mathfrak{p} \in \operatorname{Spec} R$. Then, there exists a discrete valuation ring V of F such that $\mathfrak{m}_V \cap R = \mathfrak{p}$.*

For a Noetherian ring R, let $\operatorname{Min} R$ and $\operatorname{Max} R$ denote the set of minimal and maximal primes of R, respectively. Let $\mathrm{T}(R) = (\operatorname{Spec} R \setminus \operatorname{Min} R) \cup \operatorname{Max} R$.

Lemma 3.4.4. *Let R be a Noetherian ring and $\mathfrak{p} \in \mathrm{T}(R)$. If $R_{\mathfrak{p}}$ is Henselian then the natural map $\varphi : R \to R_{\mathfrak{p}}$ is surjective; i.e., $R/\ker\varphi \cong R_{\mathfrak{p}}$.*

Proof. By replacing R with $R/\ker\varphi$, we may assume that φ is injective. Thus, $\mathfrak{p}$ contains every minimal prime of R. Indeed, let $x \in \mathfrak{q}$ for some minimal prime $\mathfrak{q}$ so that $rx = 0$ for some $r \in R$. If $x \notin \mathfrak{p}$, it follows that $\varphi(x) = 0$.

To prove the lemma, it suffices to show that every element of the form $\frac{1}{u}$ in $R_{\mathfrak{p}}$ is also in R. In other words, if $u \in R \setminus \mathfrak{p}$, we need to show that u is a unit in R. To do so, we need only show that the image of u in $R/\mathfrak{q}$ is a unit for every $\mathfrak{q} \in \operatorname{Min} R$. Indeed, if $(u) + \mathfrak{q} = R$ for all minimal primes $\mathfrak{q}$, it follows that $(u) + \left(\bigcap_{\mathfrak{q} \in \operatorname{Min} R} \mathfrak{q}\right) = R$, and so $ru + j = 1$ for some $r, j \in R$ with j nilpotent. But then $ru = 1 - j$, which is easily seen to be a unit, and so u is a unit.

Thus, we can assume that R is a domain. (Note that $(R/\mathfrak{q})_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}$ is still Henselian.) If $R_{\mathfrak{p}}$ is a field, then $\mathfrak{p} \in \mathrm{T}(R)$, $\mathfrak{p}$ is both minimal and maximal in $\operatorname{Spec} R$, and it follows that R is a field. Since $u \notin \mathfrak{p} = (0)$, it follows that u is a unit in R. Now, suppose that $R_{\mathfrak{p}}$ is not a field, and suppose u is not a unit in R. Then $u \in \mathfrak{n}$ for some maximal ideal $\mathfrak{n}$ of R. By Theorem 3.4.3, there exists a discrete valuation ring V with the same field of fractions as R such that $\mathfrak{m}_V \cap R = \mathfrak{n}$. As $R_{\mathfrak{p}}$ is Henselian, $R_{\mathfrak{p}} \subseteq V$ by Proposition 3.4.2. But as $u \notin \mathfrak{p}$, u is a unit in $R_{\mathfrak{p}}$, hence in V, contradicting $u \in \mathfrak{n} \subseteq \mathfrak{m}_V$. Thus, u is a unit in R and $R = R_{\mathfrak{p}}$. $\square$

Proposition 3.4.5. *Let R be a Noetherian ring and $S = R \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r)$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in \mathrm{T}(R)$. Suppose R_S is complete with respect to its Jacobson radical. Then, the natural map $\varphi : R \to R_S$ is surjective.*

Proof. First, we may assume that $\mathfrak{p}_j \nsubseteq \bigcup_{i \neq j} \mathfrak{p}_i$ for all j. Also, by passing to the ring $R/\ker\varphi$, we may assume φ is injective. Moreover, we note that if $\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_t}$ are the ideals in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ containing $\ker\varphi$, it is easily seen that $(R/\ker\varphi)_S = (R/\ker\varphi)_T$ where $T = R \setminus (\mathfrak{p}_{i_1} \cup \cdots \cup \mathfrak{p}_{i_t})$. Hence, we may assume each $\mathfrak{p}_i$ contains $\ker\varphi$.

Now, as R_S is semilocal and complete, the map $\psi : R_S \to R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_r}$ given by $\psi(u) = (\frac{u}{1}, \ldots, \frac{u}{1})$ is an isomorphism. For each i, let $\rho_i : R \to R_{\mathfrak{p}_i}$ be the natural map. Since $R \to R_S$ is an injection, $\cap_i \ker\rho_i = (0)$. It suffices to prove that u is a unit in R for every $u \in S$. As $R_{\mathfrak{p}_i}$ is complete, hence Henselian, we have that ρ_i is surjective for each i by Lemma 3.4.4. Thus, u is a unit in $R/\ker\rho_i$ for every i; i.e., $(u) + \ker\rho_i = R$ for $i = 1, \ldots, r$. Then $(u) = (u) + (\cap_i \ker\rho_i) = R$. Hence, u is a unit in R, and we are done. □

We now prove part (b) of the Theorem 3.2:

Theorem 3.4.6. *Let R be a Noetherian ring, $S = R \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r)$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in \mathrm{T}(R)$ and M an R_S-module. If M is a reflexive R_S-module, then M is reflexive as an R-module.*

Proof. We may assume $M \neq 0$. Let $S = R \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r)$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in \mathrm{T}(R)$. Let $I = \operatorname{Ann}_R M$, whence $I_S = \operatorname{Ann}_{R_S} M$. As in the proof of Proposition 3.4.5, we may assume each $\mathfrak{p}_i$ contains I.

Now, to show that M is reflexive as a R-module, we may assume by Lemma 3.1.1 that $\operatorname{Ann}_R M = 0$. Hence, as $(\operatorname{Ann}_R M)_S = \operatorname{Ann}_{R_S} M$, it follows that $\operatorname{Ann}_{R_S} M = 0$.

Note that this implies the natural map $R \to R_S$ is injective. As M is R_S-reflexive, R_S is complete with respect to its Jacobson radical by Theorem 3.1. By Proposition 3.4.5, we have that $R \cong R_S$, and hence, M is R-reflexive. $\square$

Question 3.4.7. Does Theorem 3.2(b) hold in the case $S = \{x^n : n \geq 0\}$ for some $x \in R$?

Chapter 4

Flat, Cotorsion Theory over Noetherian Rings

In this chapter, we make use of flat, cotorsion module theory and minimal flat resolutions ([Xu96], [Eno84], [EJ11]). After a summary of relevant portions of this theory, we will demonstrate two properties of minimal flat resolutions which resemble properties satisfied by minimal free and minimal injective resolutions. As an application of these properties, we show that in a certain case, the Frobenius functor preserves minimal flat resolutions. This result elaborates more on Marley and Webb's study in [MW16]. Finally, we study a flat, cotorsion version of the New Intersection Theorem as proved by Roberts [Rob76].

4.1 Background on Flat, Cotorsion Theory

Throughout this section, let R be a Noetherian ring.

Definition 4.1.1. An R-module M is called *cotorsion* if for every flat R-module F we have $\operatorname{Ext}^1_R(F, M) = 0$ (equivalently, $\operatorname{Ext}^i_R(F, M) = 0$ for all $i \geq 1$).

It is a well-known result of Matlis [Mat58] that injective modules over Noetherian rings have a decomposition into a direct sum of injective hulls $E_R(R/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} R$. While no such decomposition is known for flat modules, Enochs [Eno84] showed that flat, cotorsion modules do have a decomposition which is analogous to Matlis's decomposition for injective modules. The following is the main result of [Eno84].

Theorem 4.1.2. *An R-module F is flat and cotorsion if and only if $F \cong \prod_{\mathfrak{p} \in \operatorname{Spec} R} T_{\mathfrak{p}}$, where $T_{\mathfrak{p}}$ is the completion of a free $R_{\mathfrak{p}}$-module. Furthermore, the modules $T_{\mathfrak{p}}$ appearing in such a decomposition are uniquely determined (up to isomorphism) by F.*

Recall that the decomposition of injective modules allows us to study the structure of injective resolutions and Bass numbers. It turns out that we are able to do the same for flat, cotorsion resolutions. The dual notion of injective envelopes is the following:

Definition 4.1.3. Let M be an R-module. An R-homomorphism $\varphi : F \to M$ is called a *flat cover* of M if

(a) F is flat;

(b) for every map $\psi : G \to M$ with G flat, there exists a homomorphism $g : G \to F$ such that $\varphi g = \psi$; and

(c) if $h : F \to F$ satisfies $\varphi h = \varphi$, then h is an isomorphism.

φ is called a *flat precover* if φ satisfies (a) and (b) but not necessarily (c).

Flat covers were shown to exist for all modules and all rings in [BEBE01], though the result had been previously shown for commutative, Noetherian rings of finite Krull dimension [Xu95]. It follows that flat covers are always surjective, that flat covers of flat modules are isomorphisms, and that flat covers are unique up to isomorphism. Thus, we will occasionally abuse language by referring to F as the flat cover of M.

By [Eno84, Lemma 2.2 and Corollary], the kernel of a flat cover is cotorsion and the flat cover of a cotorsion module is also cotorsion.

Definition 4.1.4. [EX97] Let M be an R-module. A *minimal flat resolution* of M is a complex of flat modules

$$\ldots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \ldots \to F_0 \to 0$$

such that $H_i(\mathbf{F}) = 0$ for $i > 0$, $H_0(\mathbf{F}) = M$, and $F_i \to \operatorname{coker} \varphi_{i+1}$ is a flat cover.

Any two minimal flat resolutions of M are chain isomorphic, and if $\operatorname{fd}_R M = n < \infty$, then the length of a minimal flat resolution of M is n (see [Web15, Lemmata 2.2.3 and 2.2.4]).

Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Then, the minimal free resolution of M, call it $(\mathbf{F}, \partial)$, satisfies $\operatorname{im} \partial_{i+1} \subseteq \mathfrak{m} F_i$. Enochs and Xu show that the same property holds for minimal flat resolutions:

Lemma 4.1.5. *[EX97, Proof of Theorem 2.2] Let $(R, \mathfrak{m})$ be a local ring, and let $\mathbf{F} \xrightarrow{\partial} M$ be a minimal flat resolution of a cotorsion R-module M. Then, $\operatorname{im} \partial_{i+1} \subseteq \mathfrak{m} F_i$ for all $i \geq 0$.*

As stated above, the flat cover of a cotorsion module is cotorsion, and so, by Theorem 4.1.2, such a flat cover would have a decomposition. Moreover, since the kernel of a flat cover is cotorsion, we can make the following definition, using terminology from [MW16].

Definition 4.1.6. Let M be an R-module and $\mathbf{F}$ a minimal flat resolution of M. Then, for $i > 0$ and $\mathfrak{p} \in \operatorname{Spec} R$, define the *Enochs-Xu numbers* of M to be $\pi_i(\mathfrak{p}, M) = \pi(\mathfrak{p}, F_i)$.

If M is cotorsion in the previous definition, then F_0 is also cotorsion, and we may define $\pi_0(\mathfrak{p}, M) = \pi(\mathfrak{p}, F_0)$. If M is not cotorsion, we set $\pi_0(\mathfrak{p}, M) = \pi(\mathfrak{p}, C_R(F_0))$, where $C_R(F_0)$ is the cotorsion envelope of F_0 (see [Web15, Definition 2.2.8]).

Due to the duality between flat modules and injective modules, Enochs and Xu [EX97] referred to these invariants as "invariants dual to the Bass numbers" or "dual Bass numbers." The connection actually goes deeper. Recall for a finitely generated R-module M that the Bass numbers of M are given by $\dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$. Enochs and Xu prove the following:

Theorem 4.1.7. *[EX97, Theorem 2.2] Let M be a cotorsion R-module. Then, for all $i \geq 0$,*

$$\pi_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, M)).$$

Consequently, the Enochs-Xu numbers $\pi_i(\mathfrak{p}, M)$ do not depend on the choice of the minimal flat resolution $\mathbf{F}$ of M.

4.2 Minimal Flat Resolutions

Lemma 4.2.1 (cf. [MW16], Lemma 2.5(b))**.** *Let R be a Noetherian ring, and let M be an R-module. Suppose M has a cotorsion resolution*

$$\ldots \to G_i \xrightarrow{\partial_i} G_{i-1} \to \ldots \to G_1 \xrightarrow{\partial_1} G_0 \to M \to 0;$$

i.e., this sequence is exact and each G_j is cotorsion. Then, M is cotorsion if either of the following hold:

(a) R has finite Krull dimension.

(b) $G_n = 0$ for some $n \geq 0$.

Proof. If condition (a) holds, this result is [MW16, Lemma 2.5(b)]. Suppose (b) holds so that $G_j = 0$ for all $j \geq n$. Let F be a flat R-module; we want to show that $\operatorname{Ext}^1_R(F, M) = 0$. Set $C_i = \operatorname{coker} \partial_i$ for each $i \geq 1$. Thus, we have a short exact sequence

$$0 \to C_{i+1} \to G_{i-1} \to C_i \to 0$$

for each $i \geq 1$. As G_i is cotorsion for all i, we have isomorphisms $\operatorname{Ext}^j_R(F, C_i) \cong \operatorname{Ext}^{j+1}_R(F, C_{i+1})$ for all $j \geq 1$ and all $i \geq 1$. In particular, setting $j = i$ and noting that $C_1 \cong M$, we find that

$$\operatorname{Ext}^1_R(F, M) \cong \operatorname{Ext}^i_R(F, C_i)$$

for all $i \geq 1$. In particular, setting $i = n+1$, $\operatorname{Ext}^1_R(F, M) \cong \operatorname{Ext}^{n+1}_R(F, C_{n+1})$. Finally, as $G_n = 0$, $C_{n+1} = 0$, and so $\operatorname{Ext}^1_R(F, M) = 0$. □

Remark 4.2.2. Let R be Noetherian and M an R-module. Suppose that $\mathbf{F} \xrightarrow{\partial} M$ is a flat cotorsion resolution of M. If either condition (a) or (b) of Lemma 4.2.1 holds, then it follows that M and $\ker \partial_i$ are cotorsion for all i. Indeed, $\cdots \to F_{i+2} \to F_{i+1} \to \ker \partial_i \to 0$ is a flat, cotorsion resolution of $\ker \partial_i$.

In this section, we will demonstrate two different properties of minimal flat resolutions. The first property can be considered the analogue for flat, cotorsion resolutions of the following classical result: *Let $(R, \mathfrak{m})$ be a local ring; if $\mathbf{F} \to M$ is a free resolution of a finitely generated R-module M, then there exists a minimal free resolution of M that is a direct summand of $\mathbf{F}$.*

We begin with a lemma.

Lemma 4.2.3. *Let R be a commutative, Noetherian ring, and let $\partial : G \to M$ be a surjection of a flat R-module G onto an R-module M. Suppose that $K := \ker \partial$ is cotorsion. Let N be a direct summand of M, and let $\theta : N \to M$ and $\psi : M \to N$ be maps so that $\psi\theta = \mathrm{id}_N$. If F is a flat cover of N, F is a direct summand of G.*

More explicitly, if $\varphi : F \to N$ is a flat cover, there exist maps $g : G \to F$ and $f : F \to G$ making the diagram

$$\begin{tikzcd} G \arrow[r, "\partial"] \arrow[d, shift left, "g"] & M \arrow[r] \arrow[d, shift left, "\psi"] & 0 \\ F \arrow[r, "\varphi"] \arrow[u, shift left, "f"] & N \arrow[r] \arrow[u, shift left, "\theta"] & 0 \end{tikzcd}$$

commute with $gf = \mathrm{id}_F$.

Proof. First, as φ is a flat cover of N, we have a map $g : G \to F$ making the following diagram commute:

$$\begin{tikzcd} & G \arrow[dl, dashed, "g"'] \arrow[d, "\psi\partial"] & \\ F \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd} .$$

Now, applying $\mathrm{Hom}_R(F, -)$ to $0 \to K \to G \to M \to 0$, since K is cotorsion, we have a short exact sequence

$$0 \to \mathrm{Hom}_R(F, K) \to \mathrm{Hom}_R(F, G) \xrightarrow{\alpha \mapsto \partial\alpha} \mathrm{Hom}_R(F, M) \to 0,$$

so there exists some map $j : F \to G$ such that $\partial j = \theta\varphi$. Note for $gj : F \to F$ that $\varphi gj = \psi\partial j = \psi\theta\varphi = \varphi$, so gj is an isomorphism as $\varphi : F \to N$ is a flat cover.

Let $h = gj : F \to F$ so that $\varphi h = \varphi$; equivalently, $\varphi h^{-1} = \varphi$. Define $f := jh^{-1}$, and note that

$$\begin{tikzcd} G \arrow[r, "\partial"] \arrow[d, shift left, "g"] & M \arrow[r] \arrow[d, shift left, "\psi"] & 0 \\ F \arrow[r, "\varphi"] \arrow[u, shift left, "f"] & N \arrow[r] \arrow[u, shift left, "\theta"] & 0 \end{tikzcd}$$

commutes as $\partial f = \partial jh^{-1} = \theta\varphi h^{-1} = \theta\varphi$. As $gf = \mathrm{id}_F$, the lemma is proved. $\square$

Remark 4.2.4. With the same notation as in the previous result, we can construct maps $\gamma : \ker\partial \to \ker\varphi$ and $\epsilon : \ker\varphi \to \ker\partial$ such that $\gamma\epsilon = \mathrm{id}_{\ker\varphi}$. Indeed, if we take the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker\partial \arrow[r, "\beta"] \arrow[d, shift left, dashed, "\gamma"] & G \arrow[r, "\partial"] \arrow[d, shift left, "g"] & M \arrow[r] \arrow[d, shift left, "\psi"] & 0 \\ 0 \arrow[r] & \ker\varphi \arrow[r, "\alpha"] \arrow[u, shift left, dashed, "\epsilon"] & F \arrow[r, "\varphi"] \arrow[u, shift left, "f"] & N \arrow[r] \arrow[u, shift left, "\theta"] & 0 \end{tikzcd}$$

the maps $\gamma : \ker\partial \to \ker\varphi$ and $\epsilon : \ker\varphi \to \ker\partial$ defined by $\gamma(x) = \alpha^{-1}(g(\beta(x)))$ and $\epsilon(y) = \beta^{-1}(f(\alpha(y)))$ are well defined as α and β are injective. As $\gamma\epsilon = \mathrm{id}_{\ker\varphi}$, the claim follows.

Proposition 4.2.5. *Let R be a Noetherian ring, and let $\mathbf{G} \xrightarrow{\partial} M$ be a flat, cotorsion resolution of an R-module M. Suppose that condition (a) or (b) of Lemma 4.2.1 holds. Then, any minimal flat resolution of M is a direct summand of $\mathbf{G}$.*

Proof. Since any two minimal flat resolutions of M are chain isomorphic, it suffices to prove that there exists a minimal flat resolution of M that is a direct summand of $\mathbf{G}$.

First, recall that $K_i = \ker \partial_i$ is cotorsion for all $i \geq 0$ by Remark 4.2.2. Let $F_0 \xrightarrow{\varphi_0} M$ be a flat cover, and let $L_0 = \ker \varphi_0$. From Lemma 4.2.3 and Remark 4.2.4, it follows that there exist maps

$$\begin{tikzcd}
0 \arrow[r] & K_0 \arrow[r] \arrow[d, shift left, "\gamma_0"] & G_0 \arrow[r, "\partial_0"] \arrow[d, shift left, "g_0"] & M \arrow[r] \arrow[d, "="] & 0 , \\
0 \arrow[r] & L_0 \arrow[r] \arrow[u, shift left, "\epsilon_0"] & F_0 \arrow[r, "\varphi_0"] \arrow[u, shift left, "f_0"] & M \arrow[r] & 0
\end{tikzcd}$$

where all of the squares commute, $\gamma_0 \epsilon_0 = \mathrm{id}_{L_0}$, and $g_0 f_0 = \mathrm{id}_{F_0}$.

For each $i > 0$, let $\widetilde{\partial_i}$ denote the (surjective) map $G_i \to K_{i-1}$. For the inductive step, assume that for all $i < n$ we have modules F_i, maps $F_i \xrightarrow{\varphi_i} F_{i-1}$, where F_i is a flat cover of $L_{i-1} := \operatorname{im} \varphi_i = \ker \varphi_{i-1}$, and maps as in the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_i \arrow[r] \arrow[d, shift left, "\gamma_i"] & G_i \arrow[r, "\widetilde{\partial_i}"] \arrow[d, shift left, "g_i"] & K_{i-1} \arrow[r] \arrow[d, shift left, "\gamma_{i-1}"] & 0 \\
0 \arrow[r] & L_i \arrow[r] \arrow[u, shift left, "\epsilon_i"] & F_i \arrow[r, "\varphi_i"] \arrow[u, shift left, "f_i"] & L_{i-1} \arrow[r] \arrow[u, shift left, "\epsilon_{i-1}"] & 0
\end{tikzcd}$$

such that $\gamma_i \epsilon_i = \mathrm{id}_{L_i}$ and $g_i f_i = \mathrm{id}_{F_i}$.

Let $F_n \xrightarrow{\varphi_n} L_{n-1}$ be a flat cover, and set $L_n = \ker \varphi_n$. Then we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] & G_n \arrow[r, "\widetilde{\partial_n}"] & K_{n-1} \arrow[r] \arrow[d, shift left, "\gamma_{n-1}"] & 0 \\
0 \arrow[r] & L_n \arrow[r] & F_n \arrow[r, "\varphi_n"] & L_{n-1} \arrow[r] \arrow[u, shift left, "\epsilon_{n-1}"] & 0
\end{tikzcd}$$

Applying Lemma 4.2.3, we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] \arrow[d, shift left, "\gamma_n"] & G_n \arrow[r] \arrow[d, shift left, "g_n"] & K_{n-1} \arrow[r] \arrow[d, shift left, "\gamma_{n-1}"] & 0 \\
0 \arrow[r] & L_n \arrow[r] \arrow[u, shift left, "\epsilon_n"] & F_n \arrow[r] \arrow[u, shift left, "f_n"] & L_{n-1} \arrow[r] \arrow[u, shift left, "\epsilon_{n-1}"] & 0
\end{tikzcd}$$

where all of the squares commute, $g_n f_n = \mathrm{id}_{F_n}$ and $\gamma_n \epsilon_n = \mathrm{id}_{L_n}$.

By induction, we construct a flat resolution

$$\mathbf{F} : \ldots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \ldots \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

and chain maps $\mathbf{g} : \mathbf{G} \to \mathbf{F}$ and $\mathbf{f} : \mathbf{F} \to \mathbf{G}$ over id_M such that $\mathbf{gf} := \mathrm{id}_{\mathbf{F}}$. To finish the proof, it remains to observe that we chose each F_i to be a flat cover of $\operatorname{im} \varphi_i$, so that $\mathbf{F}$ is minimal. □

The next result follows immediately.

Corollary 4.2.6. *Let R be a Noetherian ring of finite Krull dimension, and let $\mathbf{G}$ be a flat cotorsion resolution of an R-module M as in the previous result. Then, $\pi_i(\mathfrak{p}, M) \leq \pi(\mathfrak{p}, G_i)$ for all i.*

Now, we demonstrate another property of minimal flat resolutions, which is dual, in a sense, to the following classical result: *Let M be a finitely generated R-module, and $M \to \mathbf{I}$ an injective resolution. Then, $\mathbf{I}$ is a minimal injective resolution if and only if for all $\mathfrak{p} \in \operatorname{Spec} R$, the maps of the complex $\operatorname{Hom}_R(k(\mathfrak{p}), \mathbf{I}_\mathfrak{p})$ are zero, where $k(\mathfrak{p}) = (R/\mathfrak{p})_\mathfrak{p}$.*

Proposition 4.2.7. *Let R be a Noetherian commutative ring, M a cotorsion R-module, and F a flat, cotorsion module with $F \xrightarrow{\varphi} M \to 0$ exact. Set $K = \ker \varphi$. Then, the following are equivalent:*

a) $F \xrightarrow{\varphi} M$ *is a flat cover;*

b) K *is cotorsion, and* $\operatorname{Hom}_R(R_{\mathfrak{p}}, F) \to \operatorname{Hom}_R(R_{\mathfrak{p}}, M)$ *is a flat cover for all* $\mathfrak{p} \in \operatorname{Spec} R$.

c) K *is cotorsion, and* $k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, F) \to k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, M)$ *is an isomorphism for all* $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. The implication $a) \Rightarrow b)$ is [EX97, Theorem 2.7].
$b) \Rightarrow c)$: Note that $\operatorname{Hom}_R(R_{\mathfrak{p}}, M)$ is a cotorsion $R_{\mathfrak{p}}$-module and $\operatorname{Hom}_R(R_{\mathfrak{p}}, F)$ is a flat $R_{\mathfrak{p}}$-module by [MW16, Lemma 2.4]. Hence, it suffices to show this implication for $(R, \mathfrak{m}, k)$ local and $\mathfrak{p} = \mathfrak{m}$.

As $F \to M \to 0$ is exact and K is cotorsion, we have an exact sequence

$$k \otimes_R K \xrightarrow{\psi} k \otimes_R F \to k \otimes_R M \to 0.$$

We claim that ψ is the zero map. Indeed, let G be a flat, cotorsion R-module that is a flat cover of K making

$$\begin{tikzcd} G \arrow[rr] \arrow[dr, two heads] & & F \\ & K \arrow[ur, hook] & \end{tikzcd}$$

commute. Note that such a G exists since K is cotorsion. We have a diagram

$$k \otimes_R G \xrightarrow{\widetilde{\psi}} k \otimes_R F \xrightarrow{f} k \otimes_R M \to 0.$$

Now, by Lemma 4.1.5, $\widetilde{\psi}$ is the zero map, which means that f is an isomorphism.
$c) \Rightarrow a)$: As K is cotorsion, it follows that $F \xrightarrow{\varphi} M$ is a flat precover. Suppose that $F \xrightarrow{\varphi} M$ is not a flat cover. Then, by [Eno84, Lemma 1.1] there exists $L \subseteq K$ that

is a direct summand of F. In particular, L is flat and cotorsion, so $\widehat{R_{\mathfrak{p}}} \subseteq L$ for some $\mathfrak{p} \in \operatorname{Spec} R$.

We claim that the induced map $\tau : k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, F) \to k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, M)$ is not an isomorphism. Indeed, we have an exact sequence

$$\begin{tikzcd} k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, K) \arrow[r, "\sigma"] & k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, F) \arrow[r, "\tau"] & k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, M) \arrow[r] & 0 . \\ k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, L) \arrow[u, hook] \arrow[ur, dashed] & & & \end{tikzcd}$$

Note, $0 \neq k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, L)$, which injects into $k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, F)$ as tensor products preserve direct summands. Therefore, $\operatorname{im} \sigma \neq 0$, so τ is not an isomorphism. □

Theorem 4.2.8. *Let R be a Noetherian ring, M an R-module, and $\mathbf{F} \xrightarrow{\partial} M$ a flat cotorsion resolution of M. Suppose that condition (a) or (b) of Lemma 4.2.1 holds. Then, $\mathbf{F}$ is minimal if and only if the maps in the complex $k(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, \mathbf{F})$ are zero for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. Consider the exact sequence $F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\alpha_i} C_{i+1} \to 0$, where $C_{i+1} = \operatorname{coker} \partial_{i+1}$. Note that C_{i+1} is cotorsion for all i by Lemma 4.2.1. For each $\mathfrak{p} \in \operatorname{Spec} R$, and all $i \geq 0$, ∂_i and α_i induce maps

$$\operatorname{Hom}_R(R_{\mathfrak{p}}, F_{i+1}) \xrightarrow{\overline{\partial}_{i+1,\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, F_i) \xrightarrow{\overline{\alpha}_{i,\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, C_{i+1}) \to 0,$$

which induce maps

$$k(\mathfrak{p}) \otimes \operatorname{Hom}_R(R_{\mathfrak{p}}, F_{i+1}) \xrightarrow{\widetilde{\partial}_{i+1,\mathfrak{p}}} k(\mathfrak{p}) \otimes \operatorname{Hom}_R(R_{\mathfrak{p}}, F_i) \xrightarrow{\widetilde{\alpha}_{i,\mathfrak{p}}} k(\mathfrak{p}) \otimes \operatorname{Hom}_R(R_{\mathfrak{p}}, C_{i+1}) \to 0.$$

Note that $\mathbf{F}$ is minimal if and only if $F_i \xrightarrow{\alpha_i} C_{i+1}$ is a flat cover for all $i \geq 0$, if and only if $\widetilde{\alpha}_{i,\mathfrak{p}}$ is an isomorphism for all $i \geq 0$, for all $\mathfrak{p} \in \operatorname{Spec} R$ (by Proposition 4.2.7), if and only if $\widetilde{\partial}_{i+1,\mathfrak{p}}$ is the zero map for all $i \geq 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. □

In [Tho16, Section 9.3], Thompson shows that the property just established for minimal flat resolutions is equivalent to the notion of a minimal complex defined in [AM02].

4.3 An Application in Characteristic p

Suppose that R is a Noetherian ring of prime characteristic, and let $f : R \to R$ denote the Frobenius map. For some $e \geq 1$, let $R^{(e)}$ denote the ring R viewed as an R-module via f^e; i.e., $rs := r^{p^e}s$ for $r \in R$ and $s \in R^{(e)}$. We say that the Frobenius map is *finite* if $R^{(e)}$ is finitely generated as an R-module.

Remark 4.3.1. Let $e \geq 1$. If the Frobenius map on R is finite and F is a flat, cotorsion R-module, we claim that $R^{(e)} \otimes_R F$ is a flat, cotorsion $R^{(e)}$-module. Indeed, $R^{(e)} \otimes_R F$ is clearly a flat $R^{(e)}$-module. That tensoring with $R^{(e)}$ commutes with direct products and completions follows since $R^{(e)}$ is finitely generated. Further, if $F = \prod_{\mathfrak{p}\in \operatorname{Spec} R} \widehat{R_{\mathfrak{p}}^{X_{\mathfrak{p}}}}$, we have the following isomorphisms of R-modules:

$$\begin{aligned} R^{(e)} \otimes_R F &= R^{(e)} \otimes_R \left(\prod_{\mathfrak{p}\in \operatorname{Spec} R} \widehat{R_{\mathfrak{p}}^{X_{\mathfrak{p}}}}\right) \\ &\cong \prod_{\mathfrak{p}\in \operatorname{Spec} R} \left(R^{(e)} \otimes_R \widehat{R_{\mathfrak{p}}^{X_{\mathfrak{p}}}}\right) \\ &\cong \prod_{\mathfrak{p}\in \operatorname{Spec} R} \widehat{R^{(e)} \otimes_R R_{\mathfrak{p}}^{X_{\mathfrak{p}}}} \\ &\cong \prod_{\mathfrak{p}\in \operatorname{Spec} R} \widehat{(R_{\mathfrak{p}}^{(e)})^{X_{\mathfrak{p}}}}. \end{aligned}$$

The last isomorphism holds since localization commutes with the Frobenius (e.g., [Mar14, Proposition 2.1]). Hence, if the Frobenius map is finite, we have that tensoring with $R^{(e)}$ preserves the Enochs-Xu numbers; i.e., $\pi(\mathfrak{p}, F) = \pi(\mathfrak{p}_e, R^{(e)} \otimes_R F)$ for all $\mathfrak{p} \in \operatorname{Spec} R$, where $\mathfrak{p}_e \in \operatorname{Spec} R^{(e)}$ such that $\mathfrak{p}_e \cap R = \mathfrak{p}$.

The following lemma will be useful for us.

Lemma 4.3.2 (Lemma 2.6, [MW16])**.** *Let R be a Noetherian ring of finite Krull dimension. Let M, T, and F be R-modules such that M is finitely generated, T is flat, and F is flat and cotorsion. Then, the natural map $\psi : M \otimes_R \operatorname{Hom}_R(T, F) \to \operatorname{Hom}_R(T, F \otimes_R M)$ is an isomorphism.*

Theorem 4.3.3. *Let R be a finite-dimensional Noetherian ring of prime characteristic, M an R-module, and $e \geq 1$. Suppose that $\operatorname{fd}_R M < \infty$, and let $\mathbf{F} \to M$ be a minimal flat resolution of M. If the Frobenius map is finite, then $R^{(e)} \otimes_R \mathbf{F}$ is a minimal flat $R^{(e)}$-resolution of $R^{(e)} \otimes_R M$.*

Proof. By [MW16, Theorem 3.5 (a)] and Remark 4.3.1, it follows that $R^{(e)} \otimes_R \mathbf{F}$ is a flat, cotorsion resolution of $R^{(e)} \otimes_R M$. Thus, we need only show that $R^{(e)} \otimes_R \mathbf{F}$ is minimal. Let $S = R^{(e)}$, $\mathfrak{q} \in \operatorname{Spec} S$, and $k_S(\mathfrak{q}) = (S/\mathfrak{q})_{\mathfrak{q}}$. We claim that the complex $k_S(\mathfrak{q}) \otimes_S \operatorname{Hom}_S(S_{\mathfrak{q}}, S \otimes_R \mathbf{F})$ has zero differential. The result will then follow from Theorem 4.2.8.

Let $\mathfrak{p} = f^{-1}(\mathfrak{q})$, and set $k_R(\mathfrak{p}) = (R/\mathfrak{p})_{\mathfrak{p}}$. Observe for all $i \geq 0$:

$$\begin{aligned}
S \otimes_R k_R(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, F_i) &\cong (S_{\mathfrak{q}}/\mathfrak{q}^{[p]}S_{\mathfrak{q}}) \otimes_S S \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, F_i) \\
\text{(Lemma 4.3.2)} \quad &\cong (S_{\mathfrak{q}}/\mathfrak{q}^{[p]}S_{\mathfrak{q}}) \otimes_S \operatorname{Hom}_R(R_{\mathfrak{p}}, S \otimes_R F_i) \\
&\cong (S_{\mathfrak{q}}/\mathfrak{q}^{[p]}S_{\mathfrak{q}}) \otimes_S \operatorname{Hom}_R(R_{\mathfrak{p}}, \operatorname{Hom}_S(S, S \otimes_R F_i)) \\
&\cong (S_{\mathfrak{q}}/\mathfrak{q}^{[p]}S_{\mathfrak{q}}) \otimes_S \operatorname{Hom}_S(S_{\mathfrak{q}}, S \otimes_R F_i).
\end{aligned}$$

Now, we have that the induced map

$$k_R(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, F_{i+1}) \to k_R(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, F_i)$$

is the zero map for all $i \geq 0$. Tensoring with S and using the isomorphism above, we have

$$(S_{\mathfrak{q}}/\mathfrak{q}^{[p]}S_{\mathfrak{q}}) \otimes_S \operatorname{Hom}_S(S_{\mathfrak{q}}, S \otimes_R F_{i+1})) \to (S_{\mathfrak{q}}/\mathfrak{q}^{[p]}S_{\mathfrak{q}}) \otimes_S \operatorname{Hom}_S(S_{\mathfrak{q}}, S \otimes_R F_i)$$

is the zero map for all $i \geq 0$. Applying $S/\mathfrak{q} \otimes_S -$, we have

$$k_S(\mathfrak{q}) \otimes_S \operatorname{Hom}_S(S_{\mathfrak{q}}, S \otimes_R F_{i+1}) \to k_S(\mathfrak{q}) \otimes_S \operatorname{Hom}_S(S_{\mathfrak{q}}, S \otimes_R F_i)$$

is the zero map for all $i \geq 0$, which finishes the proof. $\square$

We conclude by noting the following consequence of Theorem 4.3.3 and Remark 4.3.1.

Corollary 4.3.4. *Let R be a finite-dimensional Noetherian ring of prime characteristic, M a cotorsion R-module, $\mathfrak{p} \in \operatorname{Spec} R$, and $e \geq 1$. If the Frobenius map is finite, then $\pi_i(\mathfrak{p}, M) = \pi_i(\mathfrak{q}, R^{(e)} \otimes_R M)$, where $\mathfrak{q} \in \operatorname{Spec} R^{(e)}$ and $\mathfrak{q} \cap R = \mathfrak{p}$.*

4.4 The New Intersection Theorem

In this section, we will study the annihilator of the eth iterations of the Frobenius on an R-module M. As an application of this fact, we will demonstrate a result resembling the New Intersection Theorem (see [Rob76]). First, a few preliminary results.

Lemma 4.4.1. *Let $(R, \mathfrak{m})$ be a local ring, I be an ideal of R, and M be an R-module. Then, there is an isomorphism $(M/IM)^{\vee\vee} \cong M^{\vee\vee}/IM^{\vee\vee}$, where $(-)^{\vee} = \operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$. In particular, if $M \neq IM$, then $M^{\vee\vee} \neq IM^{\vee\vee}$.*

Proof. Let $I = (x_1, \dots, x_n)$. We have an exact sequence

$$R^n \xrightarrow{(x_1,\dots,x_n)} R \to R/I \to 0.$$

Tensoring with M, the sequence

$$M^n \xrightarrow{(x_1,\dots,x_n)} M \to M/IM \to 0$$

remains exact. Applying $(-)^{\vee\vee}$, we have

$$(M^{\vee\vee})^n \xrightarrow{(x_1,\dots,x_n)} M^{\vee\vee} \to (M/IM)^{\vee\vee} \to 0,$$

is also exact, which finishes the proof. $\square$

Lemma 4.4.2. *[Mar14, 3.7, 3.8] Let $\varphi : R \to S$ be a homomorphism of local rings such that S is finitely generated as an R-module. Let E_R and E_S be the injective hulls of the residue fields of R and S, respectively. Then,*

(a) $\operatorname{Hom}_R(S, E_R) \cong E_S$, *and*

(b) for any R-module M, $\operatorname{Hom}_S(S \otimes_R M, E_S) \cong \operatorname{Hom}_R(S, \operatorname{Hom}_R(M, E_R))$ *as S-modules.*

Proposition 4.4.3. *Let $\varphi : R \to S$ be a homomorphism of local rings such that S is finitely generated as an R-module. Let E_R and E_S be the injective hulls of the residue fields of R and S, respectively. Then, for any R-module M, we have*

$$S \otimes_R \operatorname{Hom}_R(\operatorname{Hom}_R(M, E_R), E_R) \cong \operatorname{Hom}_S(\operatorname{Hom}_S(S \otimes_R M, E_S), E_S).$$

Proof. By Lemma 4.4.2 and hom-tensor adjunction, we have

$$\begin{aligned}
\operatorname{Hom}_S(\operatorname{Hom}_S(S \otimes_R M, E_S), E_S) &\cong \operatorname{Hom}_S(\operatorname{Hom}_S(S \otimes_R M, E_S), \operatorname{Hom}_R(S, E_R)) \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_S(S \otimes_R M, E_S), E_R) \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_R(S, \operatorname{Hom}_R(M, E_R)), E_R) \\
&\cong S \otimes_R \operatorname{Hom}_R(\operatorname{Hom}_R(M, E_R), E_R);
\end{aligned}$$

the last isomorphism follows from [Ish65, Lemma 1.6] □

Let R be a Noetherian local ring of characteristic $p > 0$. For what follows, we will use another notion of the Frobenius functor. If $f : R \to R$ denotes the Frobenius homomorphism of R, let R^{f^e} be the ring R viewed as an R-R-bimodule via the action $r \cdot s = rs$ and $s \cdot r = sr^{p^e}$ for $r \in R$, $s \in R^f$. The functor $F_R^e(-) = R^{f^e} \otimes_R -$, which takes R-modules to R-modules, is called the *Frobenius functor*. Let M be an R-module. For $r \in R$, $s \in R^{f^e}$, and $x \in M$, we have $r \cdot (s \otimes x) = rs \otimes x$ and $s \otimes rx = sr^{p^e} \otimes x$. If R^f is finite as a right R-module, we say that R is *f-finite*.

In Proposition 4.4.3, if we take $R = S$ and let φ be the Frobenius endomorphism f, we obtain the following:

Corollary 4.4.4. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and M an R-module. Suppose R is f-finite. Set $(-)^\vee = \operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$. Then, $F_R^e(M^{\vee\vee}) \cong F_R^e(M)^{\vee\vee}$.*

Remark 4.4.5. (a) From Remark 4.3.1, if R is f-finite and F is a flat, cotorsion R-module, then $F_R(F) \cong F$, so the Frobenius functor preserves flat, cotorsion R-modules.

(b) Moreover, Theorem 4.3.3 shows that when R is f-finite and $\mathbf{F} \to M$ is a finite minimal flat resolution of a cotorsion R-module M, then $F_R(\mathbf{F})$ is a minimal flat resolution of $F_R(M)$.

Proposition 4.4.6. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of characteristic $p > 0$, and let M be an R-module such that $\mathfrak{m}M \neq M$. Then, $\bigcap_e \operatorname{Ann}_R(F_R^e(M)) = 0$.*

Proof. We first make a few reductions. Note that we can assume without loss of generality that R is complete as

$$\operatorname{Ann}_R(F_R^e(M))\widehat{R} \subseteq \operatorname{Ann}_{\widehat{R}}(F_{\widehat{R}}^e(\widehat{R} \otimes_R M)).$$

Further, we reduce to the case that the Frobenius map is finite. Indeed, there exists a faithfully flat ring extension $R \to S$ where the Frobenius map on S is finite. (This is standard; see [Mar14, 3.11].) As S is an R-algebra, it follows from [Mar14, Proposition 2.1] that $F_S^e(S \otimes_R M) \cong S \otimes_R F_R^e(M)$ for all R-modules M. Hence, for each $e \geq 0$, we have that

$$\operatorname{Ann}_S F_S^e(S \otimes_R M) = \operatorname{Ann}_S S \otimes_R F_R^e(M) \supseteq \operatorname{Ann}_R F_R^e(M)S$$

as $R \to S$ is a faithfully flat extension. Hence, if $\bigcap_e \operatorname{Ann}_S F_S^e(S \otimes_R M) = 0$, it follows that $\bigcap_e \operatorname{Ann}_R F_R^e(M) = 0$.

Finally, following Corollary 4.4.4, we have $\operatorname{Ann}_R(F_R(M^{\vee\vee})) = \operatorname{Ann}_R(F_R(M)^{\vee\vee}) = \operatorname{Ann}_R(F_R(M))$. Moreover, Lemma 4.4.1 guarantees that $M^{\vee\vee} \neq \mathfrak{m}M^{\vee\vee}$. Since $M^{\vee\vee}$ is cotorsion ([Eno84, Lemma 2.1]), we may assume that M is cotorsion.

Now, let $\mathbf{F} : \ldots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$ be a minimal flat resolution of M. From Lemma 4.1.5, it follows $\varphi_1(F_1) \subseteq \mathfrak{m}F_0$. As $\mathfrak{m}M \neq M$, it follows that $F_0 \otimes_R k \cong M \otimes_R k \neq 0$, so, in particular, $\pi(\mathfrak{m}, F_0) \neq 0$. Recalling that $F_R^e(F_i) = R^{f^e} \otimes_R F_i$, note that we have $R^{f^e} \otimes F_1 \xrightarrow{1\otimes\varphi_1} R^{f^e} \otimes F_0$. Let $u \otimes r \in F_R^e(F_1)$; observe

$$(1 \otimes \varphi_1)(u \otimes r) = u \otimes \varphi_1(r) = u \otimes \sum a_i v_i,$$

where $v_i \in F_0$ and $a_i \in \mathfrak{m}$. But $u \otimes \sum a_i v_i = \sum u a_i^{p^e} \otimes v_i = \sum a_i^{p^e}(u \otimes v_i)$ which is in $\mathfrak{m}^{[p^e]} F_R^e(F_0)$. Hence, $\operatorname{im}(1 \otimes \varphi_1) \subseteq \mathfrak{m}^{[p^e]} F_R(F_0)$.

Now, as tensoring with R^{f^e} is right exact and R is f-finite, it follows that

$$\begin{aligned} F_R^e(M) &\cong \frac{F_R^e(F_0)}{\operatorname{im}(F_R^e(F_1) \to F_R^e(F_0))} \\ &\twoheadrightarrow \frac{F_R^e(F_0)}{\mathfrak{m}^{[p^e]} F_R^e(F_0)} \\ &\cong \frac{F_0}{\mathfrak{m}^{[p^e]} F_0}. \end{aligned}$$

As F_0 is flat and cotorsion, we have $F_0 = \prod T(\mathfrak{q})$, where the product is over $\operatorname{Spec} R$, and $T(\mathfrak{q})$ is the completion of a free R_q-module. Moreover, $\mathfrak{m}^{[p^e]} \prod T(\mathfrak{q}) = \prod \mathfrak{m}^{[p^e]} T(\mathfrak{q})$, and so

$$\frac{F_0}{\mathfrak{m}^{[p^e]} F_0} \cong \prod \frac{T(\mathfrak{q})}{\mathfrak{m}^{[p^e]} T(\mathfrak{q})} = \frac{T(\mathfrak{m})}{\mathfrak{m}^{[p^e]} T(\mathfrak{m})} \oplus \prod_{\mathfrak{q} \subsetneq \mathfrak{m}} \frac{T(\mathfrak{q})}{\mathfrak{m}^{[p^e]} T(\mathfrak{q})}.$$

However, for all $\mathfrak{q} \subsetneq \mathfrak{m}$, we have $\mathfrak{m}^{[p^e]} \not\subseteq \mathfrak{q}$, so it follows that $\mathfrak{m}^{[p^e]} T(\mathfrak{q}) = T(\mathfrak{q})$. Thus, we have

$$\frac{F_0}{\mathfrak{m}^{[p^e]} F_0} = \frac{\widehat{A}}{\mathfrak{m}^{[p^e]} \widehat{A}},$$

where $A = \oplus_\Lambda R$ is a free R-module. Notice that $\Lambda \neq \emptyset$ as $\pi(\mathfrak{m}, F_0) > 0$. By [MW16, Lemma 4.1 (c)] and as $\mathfrak{m}^{[p^e]}$ is $\mathfrak{m}$-primary, it follows that

$$\frac{\widehat{A}}{\mathfrak{m}^{[p^e]}\widehat{A}} = \widehat{\frac{A}{\mathfrak{m}^{[p^e]}A}} = \frac{A}{\mathfrak{m}^{[p^e]}A}.$$

Thus, we have that

$$F_R^e(M) \twoheadrightarrow \bigoplus_\Lambda \frac{R}{\mathfrak{m}^{[p^e]}} \twoheadrightarrow \frac{R}{\mathfrak{m}^{[p^e]}}.$$

Now, if $x \in \bigcap_e \operatorname{Ann}_R F_R^e(M)$, it follows that $xR \subseteq \mathfrak{m}^{[p^e]}$ for all $e \geq 0$, so $x \in \bigcap_e \mathfrak{m}^{[p^e]} = 0$. ∎

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension n, and let $\mathfrak{a}_i = \operatorname{Ann}_R H_\mathfrak{m}^i(R)$ for all $i \geq 0$. Note that $\mathfrak{a}_i = R$ for all $i > n$. Define $\mathfrak{b}_i = \mathfrak{a}_0 \ldots \mathfrak{a}_i$ for all $i \geq 0$.

The following result generalizes [Rob76, Theorem 1] and [BH93, Theorem 8.1.2], and it follows the proof of the latter with the necessary modifications.

Proposition 4.4.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and*

$$\mathbf{G} : 0 \to G_m \to \ldots \to G_0 \to 0$$

be a complex of flat R-modules such that $\operatorname{Supp}_R(H_i(\mathbf{G})) \subseteq \{\mathfrak{m}\}$ for each $i = 0, \ldots, m$. Then, $\mathfrak{b}_{m-i}$ annihilates $H_i(\mathbf{G})$ for $i = 0, \ldots, m$.

Proof. Let $x_1, \ldots, x_n$ be a system of parameters for R, and let $\mathbf{C}$ be the Čech complex viewed as a homological sequence; that is

$$\mathbf{C} : 0 \to C_n \to \ldots \to C_1 \to C_0 \to 0$$

where

$$C_j = \bigoplus_{1 \le i_1 < \ldots < i_{n-j} \le n} R_{x_{i_1} \ldots x_{i_{n-j}}}.$$

Note that $H^i_m(R) \cong H_{n-i}(\mathbf{C})$.

Consider the first quadrant bicomplex $\mathbf{C} \otimes \mathbf{G}$:

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \leftarrow & C_0 \otimes_R G_q & \leftarrow & C_1 \otimes_R G_q & \leftarrow & \ldots & \leftarrow & C_p \otimes_R G_q & \leftarrow & \ldots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \leftarrow & C_0 \otimes_R G_1 & \leftarrow & C_1 \otimes_R G_1 & \leftarrow & \ldots & \leftarrow & C_p \otimes_R G_1 & \leftarrow & \ldots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \leftarrow & C_0 \otimes_R G_0 & \leftarrow & C_1 \otimes_R G_0 & \leftarrow & \ldots & \leftarrow & C_p \otimes_R G_0 & \leftarrow & \ldots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & \ldots & & 0 & & \ldots
\end{array}$$

For each p, let $H_q(C_p \otimes_R G)$ denote the qth homology of $C_p \otimes \mathbf{G}$, i.e., the qth homology of the pth column of the above complex. We consider the spectral sequence $E^1_{pq} = H_q(C_p \otimes \mathbf{G})$. By [Rot79, 11.18], this spectral sequence converges to the homology of the total complex $\mathbf{T}$ of $C \otimes \mathbf{G}$. Further, as C_p is flat, we have $H_q(C_p \otimes \mathbf{G}) \cong C_p \otimes H_q(\mathbf{G})$. As $\operatorname{Supp}_R H_q(\mathbf{G}) \subseteq \{\mathfrak{m}\}$, each element of $H_q(\mathbf{G})$ is annihilated by a power of $\mathfrak{m}$. Therefore,

$$E^\infty_{pq} = E^1_{pq} = \begin{cases} 0 & p < n \\ H_q(\mathbf{G}) & p = n \end{cases}.$$

In particular, the spectral sequence collapses. Thus, by [Wei94, 5.2.7], it follows that $H_{i+n}(\mathbf{T})$ is the only non-zero element of E^1_{pq} for which $p+q=i+n$, which shows that $H_{i+n}(\mathbf{T}) \cong H_i(\mathbf{G})$.

Now, we fix p and let E^1_{pq} be the homology of the rows of the above bicomplex; i.e., $E^1_{pq} = H_q(\mathbf{C} \otimes G_p)$. But $H_q(\mathbf{C} \otimes G_p) \cong H^{n-q}_{\mathfrak{m}}(G_p) \cong H^{n-q}_{\mathfrak{m}}(R) \otimes_R G_p$ as G_p is flat. Thus, it follows that $\mathfrak{a}_{n-q} E^1_{pq} = 0$ for $0 \leq q \leq n$. As each E^r_{pq} is a subquotient of E^1_{pq}, it follows that $\mathfrak{a}_{n-q} E^r_{pq}$ for all $r \geq 1$, whence $\mathfrak{a}_{n-q} E^\infty_{pq} = 0$. By [Rot79, Theorem 11.19], this spectral sequence again converges to the homology of the total complex $\mathbf{T}$. Thus, for each $0 \leq i \leq m$, there exists a filtration

$$0 = J_s H_{i+n}(\mathbf{T}) \subseteq J_{s+1} H_{i+n}(\mathbf{T}) \subseteq \ldots \subseteq J_t H_{i+n}(\mathbf{T}) = H_{i+n}(\mathbf{T}),$$

where

$$\frac{J_j H_{i+n}(\mathbf{T})}{J_{j-1} H_{i+n}(\mathbf{T})} \cong E^\infty_{j,i+n-j}$$

for each $s+1 \leq j \leq i+n$. But $E^\infty_{j,i+n-j} = 0$ if $j > m$ or $i+n-j > n$, so the only non-zero part of our filtration is when $i \leq j \leq m$:

$$0 = J_{i-1} H_{i+n}(\mathbf{T}) \subseteq J_i H_{i+n}(\mathbf{T}) \subseteq \ldots \subseteq J_m H_{i+n}(\mathbf{T}) = H_{i+n}(\mathbf{T}).$$

Recalling that $\mathfrak{a}_{n-(i+n-j)} E^\infty_{j,i+n-j} = 0$ for each $i \leq j \leq m$ and by the above filtration, we have that

$$\mathfrak{a}_{n-(i+n-i)} \mathfrak{a}_{n-(i+n-(i+1))} \cdots \mathfrak{a}_{n-(i+n-m)} = \mathfrak{a}_0 \mathfrak{a}_1 \ldots \mathfrak{a}_{m-i} = \mathfrak{b}_{m-i}$$

annihilates $H_{i+n}(\mathbf{T})$. But the first part of the argument shows that $H_{i+n}(\mathbf{T}) \cong H_i(\mathbf{G})$, which proves the claim. $\square$

Let $\mathfrak{a}_i$ and $\mathfrak{b}_i$ be defined as above. When R is the homomorphic image of a Gorenstein ring (e.g., a complete local ring), it follows that $\dim R/\mathfrak{a}_i \leq i$ by [BH93, Theorem 8.1.1(b)], whence $\dim R/\mathfrak{b}_i \leq i$ for each $i \geq 0$. Note that this does not hold if R is not the homomorphic image of a Gorenstein ring (see [BH93, Remark 8.1.5]).

The following is a version of the New Intersection Theorem for complexes of flat, cotorsion modules (cf. [Rob76]).

Theorem 4.4.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$ which is the homomorphic image of a Gorenstein ring, and let*

$$\mathbf{G} : 0 \to G_r \xrightarrow{\partial_r} \ldots \to G_1 \xrightarrow{\partial_1} G_0 \to 0$$

be a complex of flat, cotorsion R-modules such that $H_i(\mathbf{G})$ is cotorsion and $\operatorname{Supp}_R H_i(\mathbf{G}) \subseteq \{\mathfrak{m}\}$ for all i. Suppose $\mathbf{G}$ is not exact, set $j := \inf\{i : H_i(\mathbf{G}) \neq 0\}$, and suppose $\mathfrak{m}H_j(\mathbf{G}) \neq H_j(\mathbf{G})$. Then, $r \geq \dim R$.

Proof. First, we claim that $\operatorname{im} \partial_i$ and $\ker \partial_i$ are cotorsion for all $0 \leq i < r$. Indeed, $\ker \partial_r = H_r(\mathbf{G})$ is cotorsion. Let F be a flat R-module. Applying $\operatorname{Hom}_R(F, -)$ to the short exact sequence

$$0 \to \ker \partial_r \to G_r \to \operatorname{im} \partial_r \to 0$$

gives that $\operatorname{im} \partial_r$ is cotorsion as both G_r and $\ker \partial_r$ are cotorsion. Showing that $\operatorname{im} \partial_i$ and $\ker \partial_i$ are cotorsion for all $0 \leq i \leq r$ follows from induction and the short exact sequences

$$0 \to \operatorname{im} \partial_{i+1} \to \ker \partial_i \to H_i(\mathbf{G}) \to 0 \quad \text{and} \quad 0 \to \ker \partial_i \to G_i \to \operatorname{im} \partial_i \to 0.$$

We now show that we may assume that $H_0(\mathbf{G}) \neq 0$. For if $H_0(\mathbf{G}) = 0$, we have a short exact sequence

$$0 \to K \to G_1 \xrightarrow{\partial_1} G_0 \to 0,$$

where $K = \ker \partial_1$. As K is cotorsion, we have a short exact sequence

$$0 \to \operatorname{Hom}_R(G_0, K) \to \operatorname{Hom}_R(G_0, G_1) \xrightarrow{h \mapsto \partial_1 h} \operatorname{Hom}_R(G_0, G_0) \to 0,$$

so there exists $g : G_0 \to G_1$ such that $\partial_1 g = \mathrm{id}_{G_0}$. In particular, ∂_1 splits. Thus, by splitting off irrelevant portions of $\mathbf{G}$, we may assume that $\mathfrak{m}H_0(\mathbf{G}) \neq H_0(\mathbf{G})$.

Let $M := H_0(\mathbf{G})$. Tensoring $\mathbf{G}$ with the eth power of the Frobenius map gives us another complex $F_R^e(\mathbf{G})$ with components $F_R^e(\mathbf{G})_i = F_R^e(G_i)$, all of which are flat for each $e \geq 0$. As the tensor product is a right exact functor, it follows that $H_0(F_R^e(\mathbf{G})) \cong F_R^e(M)$. Also, note that $\operatorname{Supp}_R H_i(F_R^e(\mathbf{G})) \subseteq \{\mathfrak{m}\}$. Indeed, it suffices to show that $F_R^e(\mathbf{G})_x$ is exact for all $x \in \mathfrak{m}$. To that end, note that the Frobenius commutes with localization, and so $F_R^e(\mathbf{G})_x \cong F_R^e(\mathbf{G}_x)$. Now, $\mathbf{G}_x$ is exact as $\operatorname{Supp}_R H_i(\mathbf{G}) \subseteq \{\mathfrak{m}\}$, so by [MW16, Corollary 3.5], we have that $F_R^e(\mathbf{G}_x)$ is exact, which proves the claim. Thus, by Proposition 4.4.7, it follows that $0 = \mathfrak{b}_r H_0(F_R^e(\mathbf{G})) = \mathfrak{b}_r F_R^e(M)$ for all $e \geq 0$. By Proposition 4.4.6, $\mathfrak{b}_r = 0$. Finally, as R is the homomorphic image of a Gorenstein ring, we have

$$\dim R = \dim R/\mathfrak{b}_r \leq r,$$

which proves the theorem.

□

Remark 4.4.9. It may be of interest to note that we could also have reduced to the case where $\partial_1(G_1) \subseteq \mathfrak{m}G_0$ in the previous proof. Indeed, say that $G_1 \xrightarrow{\partial_1} G_0 \xrightarrow{\partial_0} M \to 0$ is an exact sequence. From Proposition 4.2.5, there are maps $g_i : G_i \to F_i$ and $f_i : F_i \to G_i$ for $i = 0, 1$ such that we can rewrite $G_1 = F_1 \oplus \ker g_1$ and $G_0 = F_0 \oplus \ker g_0$, where

$F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$ is part of a minimal flat resolution of M. It follows that

$$\partial_1 = \begin{pmatrix} \varphi_1 & 0 \\ 0 & h \end{pmatrix}.$$

But note that $h : \ker g_1 \to \ker g_0$ is surjective. Indeed, if $x \in \ker g_0$, it follows that $\partial_0(x) = 0$, so $x \in \operatorname{im} \partial_1$, whence it must be that $x \in \operatorname{im} h$. Thus, as we saw above, since $\ker \partial_1$ is cotorsion, it follows that $0 \to \ker \partial_1 \to G_1 \to G_0 \to 0$ splits. Therefore, we may write $G_1 = F_1 \oplus \ker g_0 \oplus \ker g_1$, and we note

$$\partial_1 = \begin{pmatrix} \varphi_1 & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So splitting off $\ker g_0$, we may assume that the tail end of our complex is

$$F_1 \oplus \ker g_0 \xrightarrow{\left(\begin{smallmatrix} \varphi_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} F_0 \to M \to 0.$$

Since $F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$ is a minimal flat resolution, it follows from Lemma 4.1.5 that $\varphi_1(F_1) \subseteq \mathfrak{m}F_0$. Thus, $\operatorname{im} \partial_1 \subseteq \mathfrak{m}G_0$.

The following proposition gives different proof of the previous result in the case when $\mathbf{G}$ is concentrated in degree 0.

Proposition 4.4.10. *Let R be a Noetherian local ring. Let F be a flat, cotorsion R-module with $\operatorname{Supp}_R F = \{\mathfrak{m}\}$. Then, R is an Artinian ring.*

Proof. As F is flat and cotorsion, it follows that $F \cong \prod_{\mathfrak{p} \in \operatorname{Spec} R} T(\mathfrak{p})$. We claim that $T(\mathfrak{m}) \neq 0$ and $T(\mathfrak{p}) = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$. If $T(\mathfrak{m}) = 0$, as $F \neq 0$, there exists some $\mathfrak{p} \in \operatorname{Spec} R$ such that $T(\mathfrak{p}) \neq 0$. Note that we can write F as $T(\mathfrak{p}) \oplus \prod_{\mathfrak{q} \neq \mathfrak{p}} T(\mathfrak{q})$ for some $\mathfrak{p} \in \operatorname{Spec} R$. By assumption, $F_{\mathfrak{p}} = 0$, but $T(\mathfrak{p})_{\mathfrak{p}} \neq 0$, which is a contradiction. This shows that $T(\mathfrak{m}) \neq 0$ and that $T(\mathfrak{p}) = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$.

Hence, we have that $F \cong \widehat{R^{(X)}}$ and $\operatorname{Supp}_R F = \{\mathfrak{m}\}$. Therefore, as $R \hookrightarrow \widehat{R} \hookrightarrow \widehat{R^{(X)}}$, we have that $\operatorname{Supp}_R(R) = \{\mathfrak{m}\}$, which finishes the proof. □

Remark 4.4.11. (i) Using the derived category, Foxby proves a stronger version of Theorem 4.4.8 (see [Fox79, Lemma 4.2]).

(ii) As a consequence of Foxby's result, it follows that if there exists a flat R-module F such that $\operatorname{Supp}_R F = \{\mathfrak{m}\}$, then R must be Artinian. (Note that this is Proposition 4.4.10 without the cotorsion assumption on F). We outline the proof of this result in Appendix A.

Appendix A

An Improvement of Proposition 4.4.10

Let $(R, \mathfrak{m})$ be a local, Noetherian ring. Recall that in Section 4.4, we showed that if there exists a flat, cotorsion R-module F with $\operatorname{Supp}_R F = \{\mathfrak{m}\}$, then R is Artinian. The point of this appendix is to prove the same result without the assumption that F be cotorsion. Our proof is a specialization of the arguments in [Fox79], where the arguments are given in the derived category.

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and let M be an R-module. We define $\operatorname{depth} M = \inf\{i : \operatorname{Ext}^i_R(k, M) \neq 0\}$.

Lemma A.1. *Let M be an R-module. Then,* $\sup\{i : H^i_{\mathfrak{m}}(M) \neq 0\} \leq \dim M$.

Proof. Recall that M is a direct limit of finitely generated modules. Since direct limits (being exact) commute with local cohomology, and since the result holds for finitely generated modules, the result follows. □

Lemma A.2. *Let M be an R-module. Then,* $\operatorname{depth} M = \inf\{i : H^i_{\mathfrak{m}}(M) \neq 0\}$.

Proof. Let $0 \to M \to \mathbf{I}$ be a minimal injective resolution of M. Recall that $\Gamma_m(I^i) = E^{\mu_i}$, where $E = E_R(R/\mathfrak{m})$ and μ_i is the ith Bass number of M with respect to $\mathfrak{m}$. In particular, if $t = \operatorname{depth} M$, then

$$\Gamma_{\mathfrak{m}}(\mathbf{I}) : 0 \to \ldots \to 0 \to E^{\mu_t} \to E^{\mu_{t+1}} \to \ldots,$$

so it follows that $t \leq \inf\{i : H^i_{\mathfrak{m}}(M) \neq 0\}$.

Further, we have that $\operatorname{Hom}_R(k, \Gamma_{\mathfrak{m}}(\mathbf{I})) \cong \operatorname{Hom}_R(k, \mathbf{I})$, so by identifying $\operatorname{Hom}_R(k, \Gamma_{\mathfrak{m}}(I^i))$ as the socle of $\Gamma_{\mathfrak{m}}(I^i)$, we view $\operatorname{Hom}_R(k, \mathbf{I})$ as a subcomplex of $\Gamma_{\mathfrak{m}}(\mathbf{I})$. Finally, recall from [Rob80, Proposition 2.5] (the same proof works for M not finitely generated) that the maps of the complex $\operatorname{Hom}_R(k, \mathbf{I})$ are all zero since $\mathbf{I}$ is minimal. Hence, $0 \neq \operatorname{Ext}^t_R(k, M) = \operatorname{soc} E^{\mu_t} \subseteq H^t_{\mathfrak{m}}(M)$, which proves the claim. □

The following is now immediate.

Corollary A.3. *For any R-module M,* $\operatorname{depth} M \leq \dim M$*.*

For an R-module M, we define the *small support* of M to be

$$\operatorname{supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R : \operatorname{Tor}^R_i(k(\mathfrak{p}), M) \neq 0, \text{ for some } i\}.$$

Remark A.4. Note for $\mathfrak{p} \in \operatorname{Spec} R$ that $\operatorname{supp}_R M \subseteq \operatorname{Supp}_R M$ for all R-modules M as $\operatorname{Tor}^R_i(k(\mathfrak{p}), M) \cong \operatorname{Tor}^{R_{\mathfrak{p}}}_i(k(\mathfrak{p}), M_{\mathfrak{p}})$. Further, if F is a flat R-module and $\mathfrak{p} \in \operatorname{supp}_R F$, then $k(\mathfrak{p}) \otimes_R F \neq 0$. Moreover, if $\operatorname{Supp}_R F = \{\mathfrak{m}\}$, then $\operatorname{supp}_R F = \{\mathfrak{m}\}$, and we get that $F \neq \mathfrak{m}F$.

Lemma A.5. *Let M be an R-module, and let F be a flat R-module. Then,* $\operatorname{supp}_R(M \otimes_R F) = \operatorname{supp}_R M \cap \operatorname{supp}_R F$*.*

Proof. Note that $\operatorname{Tor}_i^R(k(\mathfrak{p}), M\otimes_R F) \cong \operatorname{Tor}_i^R(k(\mathfrak{p}), M)\otimes_R F$. So if $\mathfrak{p} \in \operatorname{supp}_R(M\otimes_R F)$, $\mathfrak{p} \in \operatorname{supp}_R M$. Further, $k(\mathfrak{p})\otimes_R(\mathbf{G}\otimes_R F)$ is homologically non-trivial, where $\mathbf{G}$ is a flat resolution of M. In particular, $k(\mathfrak{p}) \otimes_R F \neq 0$. Conversely, if $\mathfrak{p} \in \operatorname{supp}_R M \cap \operatorname{supp}_R F$, then $\mathfrak{p}_\mathfrak{p} \in \operatorname{supp}_{R_\mathfrak{p}} M_\mathfrak{p} \cap \operatorname{supp}_{R_\mathfrak{p}} F_\mathfrak{p}$, so we may assume that $\mathfrak{p} = \mathfrak{m}$. In this case, $\operatorname{Tor}_i^R(k, M)$ is a non-zero vector space for some i, and so $\operatorname{Tor}_i^R(k, M) \otimes_R F \neq 0$. The result now follows. $\square$

Corollary A.6. *Let M be an R-module, and let F be a flat R-module such that $\mathfrak{m} \in \operatorname{supp}_R M \cap \operatorname{supp}_R F$. Then,*

$$\operatorname{depth}_R M \leq \dim(M \otimes_R F).$$

Proof. Note from Lemma A.5 and Corollary A.3 that $t := \operatorname{depth}(M\otimes_R F) \leq \dim(M\otimes_R F)$. But $0 \neq H^t_\mathfrak{m}(M \otimes_R F) \cong H^t_\mathfrak{m}(M) \otimes F$, so $\operatorname{depth} M \leq t$. $\square$

We now prove a generalization of Proposition 4.4.10.

Proposition A.7. *Let $(R, \mathfrak{m})$ be a local ring and F a flat module such that* $\operatorname{Supp}_R F = \{\mathfrak{m}\}$. *Then,* $\dim R = 0$.

Proof. Note that every element of F is annihilated by a power of $\mathfrak{m}$. Hence, $H^0_\mathfrak{m}(F) = F$. By Remark A.4, we have $F/\mathfrak{m}F \neq 0$. We claim that $\dim R = 0$. If not, there exists a prime $\mathfrak{p} \neq \mathfrak{m}$ such that $\dim R/\mathfrak{p} \geq 1$. Hence, $\operatorname{depth} R/\mathfrak{p} \geq 1$.

Now, as F is flat, $\operatorname{depth} R/\mathfrak{p} \leq \operatorname{depth} R/\mathfrak{p} \otimes_R F$. Since every element of F is annihilated by a power of $\mathfrak{m}$, $H^0_\mathfrak{m}(R/\mathfrak{p}\otimes F) = R/\mathfrak{p}\otimes F \neq 0$. Therefore, $\operatorname{depth} R/\mathfrak{p}\otimes F = 0$ by Lemma A.2. But $\operatorname{depth} R/\mathfrak{p} \geq 1$, a contradiction. Hence, $\dim R = 0$. $\square$

If $\mathfrak{m} \in \operatorname{supp}_R F$ and R possesses a big Cohen-Macaulay module, we are able to say more.

Proposition A.8 (See [Fox79], Lemma 4.2)**.** *Suppose R possesses a big Cohen-Macaulay module, and F is a flat R-module with $\mathfrak{m} \in \operatorname{supp}_R F$. Then,* $\dim R \leq \dim F$.

Proof. Let C be a big Cohen-Macaulay module for R. Then, Corollary A.6 gives

$$\dim R = \operatorname{depth} C \leq \dim(C \otimes_R F) \leq \dim F,$$

as $\operatorname{Supp}_R(C \otimes_R F) \subseteq \operatorname{Supp}_R F$. $\square$

Appendix B

Background on the Derived Category

Our reference for this section is [Chr00, Appendix A].

Let R be a commutative Noetherian ring. Let (X, ∂) be a complex of R-modules; i.e.,

$$(X, \partial) : \ldots \to X_{i+1} \xrightarrow{\partial_{i+1}} X_i \xrightarrow{\partial_i} X_{i-1} \to \ldots$$

where $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$ for all i. We can think of an R-module M as an R-complex by setting $M_0 = M$ and $M_i = 0$ for all other i. We set $\sup X = \sup\{i : H_i(X) \neq 0\}$ and define $\inf X$ similarly. A complex is called *non-trivial* if at least one of its homology modules is non-zero. Let (Y, ϵ) be another complex of R-modules. A *morphism* of complexes $\varphi : X \to Y$ is a family of $(\varphi_i)_{i\in\mathbb{Z}}$ of R-linear maps $\varphi_i : X_i \to Y_i$ making the following square commute for each i:

$$\begin{tikzcd} X_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "\varphi_{i+1}"] & X_i \arrow[d, "\varphi_i"] \\ Y_{i+1} \arrow[r, "\epsilon_{i+1}"] & Y_i \end{tikzcd}$$

A morphism of complexes $\varphi : X \to Y$ induces a homomorphism on homology $H(\varphi) : H(X) \to H(Y)$. When the induced homomorphism on homology is an isomorphism for all i, φ is called a *quasi-isomorphism*, and we write $X \xrightarrow{\simeq} Y$. Note that this is not an equivalence relation.

Let X, Y, Z be R-complexes, and let $\alpha : X \to Y$ and $\beta : Y \to Z$ be morphisms. Then,

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$$

is a *short exact sequence* of R-complexes if it is exact in each degree. Moreover, a short exact sequence like the one above induces a long exact sequence on homology:

$$\ldots \to H_{i+1}(Z) \xrightarrow{\partial_{i+1}} H_i(X) \xrightarrow{\alpha_i} H_i(Y) \xrightarrow{\beta_i} H_i(Z) \to \ldots$$

If X and Y are complexes of R-modules, we can form complexes $X \otimes_R Y$ and $\operatorname{Hom}_R(X, Y)$ in the standard way (see [Chr00, A.2.1, A.2.4]). Moreover, many standard morphisms which hold for R-modules can be extended to R-complexes (see [Chr00, A.2]).

Let X be an R-complex such that $\inf X > -\infty$; i.e., X is a R-complex that is homologically bounded below. A *semi-flat (resp. free; projective) resolution* of X is a bounded below complex F of flat (resp. free; projective) R-modules such that $F \xrightarrow{\simeq} X$. It is a classical result that homologically bounded below R-complexes have semi-flat (free; projective) resolutions, F, such that $F_\ell = 0$ for $\ell < \inf X$ (see [Chr00, A.3.2]).

Dually, let Y be an R-complex such that $\sup Y < \infty$. A *semi-injective* resolution of Y is a bounded above complex, I, of injective R-modules such that $Y \xrightarrow{\simeq} I$. It follows that all such complexes Y have a semi-injective resolution I such that $I_\ell = 0$ for all $\ell > \sup Y$.

We denote the derived category of R by $\mathbf{D}(R)$. The objects of $\mathbf{D}(R)$ are complexes of R-modules, and if $\varphi : X \to Y$ is a morphism of complexes, then φ induces a morphism φ in $\mathbf{D}(R)$. Moreover, quasi-isomorphisms of R-complexes are isomorphisms in $\mathbf{D}(R)$.

For complexes X and Y of R-modules such that $\inf X > -\infty$ and/or $\inf Y > -\infty$, we define $X \otimes_R^{\mathbf{L}} Y$ to be the isomorphism class in the category of R-complexes represented by $F \otimes_R Y$ and $X \otimes_R F'$, where F and F' are semi-flat resolutions of X and Y, respectively. It can be shown that $X \otimes_R^{\mathbf{L}} Y$ is well-defined and does not depend on the choice of F, F' ([Chr00, A.4.1]). Moreover, if X or Y is a bounded below complex of flat modules, then $X \otimes_R^{\mathbf{L}} Y = X \otimes_R Y$. We denote $\operatorname{Tor}_i^R(X, Y) := H_i(X \otimes_R^{\mathbf{L}} Y)$. Observe that if X and Y are concentrated in degree 0, then this is the standard interpretation of $\operatorname{Tor}_i^R(X, Y)$.

Now, suppose that X and Y are R-complexes with $\inf X > -\infty$ and/or $\sup Y < \infty$. We define $\mathbf{R}\operatorname{Hom}_R(X, Y)$ to be the isomorphism class in the category of R-complexes represented by $\operatorname{Hom}_R(P, Y)$ and $\operatorname{Hom}_R(X, I)$, where P is a semi-projective resolution of X and I is a semi-injective resolution of Y. Again, this definition is independent of the choice of P and I. We denote $\operatorname{Ext}_R^i(X, Y) = H_{-i}(\mathbf{R}\operatorname{Hom}_R(X, Y))$, and observe that if X and Y are concentrated in degree 0, then this is the standard definition of $\operatorname{Ext}_R^i(X, Y)$.

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