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Mariarosaria Coppola
University of Naples Federico II, coppola@dms.unina.it

Emilia Di Lorenzo
University of Naples Federico II, diloremi@dmsna.dms.unina.it

Marilena Sibillo
University of Sassari., sibillo@ssmain.uniss.it

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Risk Sources in a Life Annuity Portfolio: Decomposition and Measurement Tools

Mariarosaria Coppola,* Emilia Di Lorenzo,† and Marilena Sibillo‡

Abstract§

The paper considers a model for a homogeneous portfolio of whole life annuities immediate. The aim is to study two risk factors: the investment risk and the insurance risk. A stochastic model of the rate of return is used to study these risk factors. Measures of the insurance risk and the investment risk for the entire portfolio are suggested. The problem of the longevity risk is presented, and its consequences with different projections of the mortality tables are analyzed. The model is applied to some concrete cases, and several illustrations show the importance of the two components of the riskiness in terms of the number of policies in the portfolio. Understanding these risks will allow insurance companies to control, to some extent, the overall risk of their annuity portfolios.

Key words and phrases: *Ornstein-Uhlenbeck process, investment risk, insurance risk, longevity risk, moments of insurance functions*

*Mariarosaria Coppola, Ph.D., is a researcher in the area of stochastic analysis of financial and insurance risk at the University of Naples "Federico II".

Dr. Coppola's address is: Dipartimento di Matematica e Statistica, Facolta' di Economia, Universita' degli Studi di Napoli "Federico II", via Cintia, Complesso Monte S.Angelo 80126 Napoli, ITALY. Internet address: *coppola@dms.unina.it*

†Emilia Di Lorenzo, Ph.D., is an associate professor of financial mathematics at the University of Naples "Federico II". She has been grant-holder at the Italian National Research Council and researcher at the University of Salerno.

Dr. Di Lorenzo's address is: Dipartimento di Matematica e Statistica, Facolta' di Economia Universita' degli Studi di Napoli "Federico II", via Cintia, Complesso Monte S.Angelo 80126 Napoli, ITALY. Internet address: *diloremi@dmsna.dms.unina.it*

‡Marilena Sibillo, Ph.D., is an associate professor of financial mathematics at the University of Sassari.

Dr. Sibillo's address is: Istituto Economico e Aziendale, Facolta' di Economia, Universita' degli Studi di Sassari, Via Rolando 1 07100 Sassari, ITALY. Internet address: *sibillo@ssmain.uniss.it*

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1 Introduction

Most of the problems faced by an insurer managing a portfolio of life insurance policies are based on the investment risk (due to interest rates) and insurance risk (due to mortality) and on their interactions. Because of the nature of these risks, most of the research has been done on the present value of a single policy within a framework whereby both interest rates and mortality are random. Recently the focus has shifted to similar problems concerning an entire portfolio of policies. Among the contributions in this area are Norberg (1993), Parker (1993), (1994a), (1994b), (1996), and (1997), and Frees (1998).

Norberg (1993) gave the first two moments of the present value of stochastic payment streams and applied them to a portfolio of temporary insurance contracts. Parker (1993) studied moments of the present value of future cash flows modeling the force of interest by (i) a white noise, (ii) a Wiener process, and (iii) an Ornstein-Uhlenbeck process. Parker found moments of the present value of a portfolio of benefits relating to life policies (1994a) and endowment insurance policies (1994b) by modeling the force of interest using a Vasicek model; see Vasicek (1977). Parker (1996) proposed two methods to obtain the limiting distribution of the present value of a portfolio of benefits. Parker (1997) provided an interesting paper on the interaction between investment and insurance risks for a portfolio of life insurance policies with random curtate future lifetimes. Using the Vasicek model for the rate of return Parker considered the variance as a measure of the riskiness of a portfolio and divided it into insurance and investment risks. Frees (1998) showed the utility of the coefficient of determination for quantifying the relative importance of each source of uncertainty where there are more than two sources of risks.

The aim of the paper is to study the risk of an annuity portfolio by dividing this risk into two components: an investment risk and an insurance risk. We offer some ways of controlling these by means of the variability measures of the expected value of the life annuities portfolio with respect to each of these two components.

In dealing with a portfolio of life insurance policies, it is well-known that the effect of accidental deviations of mortality can be reduced by using pooling techniques. But as pointed out in Marocco and Pitacco (1998) and Olivieri (1998), however, in the case of a portfolio of life annuities, a phenomenon not controllable by pooling techniques is the *longevity risk*, which is the systematic deviations of the actual number of deaths from the expected number of deaths due to the improvements in future mortality. The longevity risk produces actuarial losses in the case of a life annuity portfolio, while in the case of life insurance contracts it produces actuarial gains. For these reasons it seems particularly useful to include suitable projections of mortality improvements in the case of a life annuity portfolio.

In Section 2 we propose the random variables in a portfolio of homogeneous whole life annuities immediate and we obtain the first two moments of the present value of the portfolio and of the average cost per policy. Section 3 presents a description of the stochastic process used to model the instantaneous rate of return, while in Section 4 we consider the two sources of risk and their measures for the entire portfolio; the longevity risk is introduced also. In Section 5, the model is applied and several illustrations concerning the importance of the two components of the riskiness, as they relate to the number of policies in portfolio, are presented.

2 Portfolio of Life Annuities

Let us consider a portfolio of c homogeneous whole life annuity-immediate policies. These policies are assumed to have been issued to c lives each age x and pay an annual benefit of one unit payable at the end of each year to each of the survivors. For $i = 1, 2, \ldots, c$, let T_i be the random variable representing the curtate-future-lifetime of the ith life insured and let Z_i be the random variable representing the present value of the lifetime annuity benefits for the ith annuitant:

$$Z_i = \begin{cases} 0 & \text{if } T_i = 0; \\ \sum_{h=1}^{T_i} e^{-y(h)} & \text{if } T_i = 1, 2, \ldots, \end{cases} \tag{1}$$

where:

$$y(t) = \int_0^t \delta_s \, ds, \quad t > 0,$$

with δ_s being the random instantaneous rate of return at time s that is used for discounting the payments.

Moreover we suppose (see, for example, Bowers et al., 1987, Chapters 3 and 8, and Parker 1994a) that the following assumptions hold:

(i) For $i = 1, 2, \ldots, c$, the T_is are independent and identically distributed;

(ii) Given knowledge of $y(h)$ for $h = 1, 2, \ldots$, the Z_is are independent and identically distributed for $i = 1, 2, \ldots, c$; and

(iii) For $i = 1, 2, \ldots, c$, the T_is and δ_s are mutually independent.

The random Z_i variables are independent only when conditioning on the knowledge of the sequence of $y(h)$s for $h = 1, 2, \ldots$. In general they are not independent, as the same rates of return are used for discounting the payments.

For our valuations it is necessary to compute the first and the second moments of Z_i that are:

$$E[Z_i] = E[E[Z_i \mid T_i]] = \sum_{h=1}^{\infty} {}_hp_x E[e^{-y(h)}] \tag{2}$$

$$E[Z_i^2] = \sum_{h=1}^{\infty} {}_hp_x E[e^{-2y(h)}] + 2\sum_{h=2}^{\infty} {}_hp_x \sum_{r=1}^{h-1} E[e^{-y(r)}e^{-y(h)}]. \tag{3}$$

The proof of equation (3) is easily derived as follows:
Proof:

$$\begin{aligned} E[Z_i^2] &= E[E[Z_i^2 \mid \{y(h)\}_{h=1}^{\infty}]] \\ &= \sum_{h=1}^{\infty} E[(\sum_{k=1}^{h} e^{-y(k)})^2]\, {}_{h|1}q_x \\ &= \sum_{h=1}^{\infty} E[(\sum_{k=1}^{h} e^{-y(k)})^2]({}_hp_x - {}_{h+1}p_x) \\ &= E[e^{-2y(1)}]p_x + \sum_{h=1}^{\infty} \left\{ E[(\sum_{k=1}^{h+1} e^{-y(k)})^2] - E[(\sum_{k=1}^{h} e^{-y(k)})^2]\, {}_{h+1}p_x \right\} \\ &= E[e^{-2y(1)}p_x + \sum_{h=2}^{\infty} {}_hp_x(\sum_{r=1}^{h-1} 2e^{-y(r)}e^{-y(h)} + e^{-2y(h)})] \end{aligned}$$

and equation (3) holds. □

Let $Z(c)$ denote the total present value for the entire portfolio of c annuities, i.e.,

$$Z(c) = \sum_{i=1}^{c} Z_i . \tag{4}$$

The first two moments of $Z(c)$ are:

$$E[Z(c)] = c \sum_{h=1}^{\infty} {}_h p_x E[e^{-y(h)}] \tag{5}$$

$$\begin{aligned} E[Z(c)^2] &= E[\sum_{i=1}^{c} Z_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{c} Z_i Z_j] \\ &= \sum_{i=1}^{c} E[Z_i^2] + \sum_{\substack{i,j=1 \\ i \neq j}}^{c} E[Z_i Z_j]. \end{aligned} \tag{6}$$

Next we need an expression for $E[Z_i Z_j]$. But, by virtue of assumptions (i), (ii), and (iii) (Parker 1994a),

$$\begin{aligned} E[Z_i Z_j] &= E[E[Z_i Z_j \mid \{y(h)\}_{h=1}^{\infty}]] \\ &= E[E[Z_i \mid \{y(h)\}_{h=1}^{\infty}] E[Z_j \mid \{y(h)\}_{h=1}^{\infty}]] \\ &= E[E[Z_1 \mid \{y(h)\}_{h=1}^{\infty}] E[Z_2 \mid \{y(h)\}_{h=1}^{\infty}]] \\ &= E[Z_1 Z_2] \\ &= E[\sum_{h=1}^{T_1} e^{-y(h)} \sum_{k=1}^{T_2} e^{-y(k)}] \\ &= E[E[\sum_{h=1}^{T_1} e^{-y(h)} \sum_{k=1}^{T_2} e^{-y(k)} \mid \{y(r)\}_{r=1}^{\infty}]] \\ &= E[\sum_{h=1}^{\infty} {}_h p_x e^{-y(h)} \sum_{k=1}^{\infty} {}_k p_x e^{-y(k)}] \\ &= \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_h p_x \, {}_k p_x E[e^{-y(h)-y(k)}]. \end{aligned} \tag{7}$$

Therefore equation (6) can be written as:

$$
\begin{aligned}
E[Z(c)^2] &= cE[Z_i^2] \\
&\quad + \sum_{\substack{i,j=1 \\ i \neq j}}^{c} \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_hp_x {}_kp_x E[e^{-y(h)-y(k)}] \\
&= cE[Z_i^2] \\
&\quad + c(c-1) \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_hp_x {}_kp_x E[e^{-y(h)-y(k)}]. \qquad (8)
\end{aligned}
$$

Finally, from equations (5) and (8), we can obtain the variance of $Z(c)$.

For our analysis it will be useful to consider the average cost per policy, $Z(c)/c$, of the portfolio under consideration.

3 Stochastic Rate of Return

One of the problems facing insurance companies is the financial risk arising from fluctuations of their rate of return. To investigate this problem we follow Di Lorenzo, Sibillo, and Tessitore (1997) and model the instantaneous global rate of return ($Y(t)$) as a sum of two components: a deterministic component ($\delta(t)$) and a stochastic component ($X(t)$) that describes the deviations of the instantaneous global rate of return from its expected value, $\delta(t)$. This means that $Y(t)$ can be written as:

$$
Y(t) = \delta(t) + X(t). \qquad (9)
$$

We suppose that $\delta(t)$ is determined by forecasts based on the existing investments. In addition, $\{X(t), 0 \leq t < +\infty\}$ is an Ornstein-Uhlenbeck process, with parameters $\beta > 0$ and $\sigma > 0$ and initial value $X(0) = 0$. $X(t)$ is characterized by the following stochastic differential equation:

$$
dX(t) = -\beta X(t)dt + \sigma dW(t) \qquad (10)
$$

where $W(t)$ is a standard Wiener (Brownian motion) process.

It follows from equation (9) that the stochastic present value at time 0 of a payment of one monetary unit at time t is given by:

$$\begin{aligned} e^{-y(t)} &= e^{-\int_0^t Y(s)ds} \\ &= e^{-\int_0^t (\delta(s)+X(s))ds} \\ &= v(t)F(t) \end{aligned} \tag{11}$$

where

$$v(t) = e^{-\int_0^t \delta(s)ds} \tag{12}$$

and

$$F(t) = e^{-\int_0^t X(s)ds}. \tag{13}$$

Clearly $v(t)$ is the deterministic discounting factor and $F(t)$ is the stochastic discounting factor. $F(t)$ is log normally distributed with parameters $-E[\int_0^t X(s)ds]$, and $\mathrm{Var}[\int_0^t X(s)ds]$ and its rth moment about the origin is given by the formula

$$E[(F(t))^r] = \exp\{-rE[\int_0^t X(s)ds] + \frac{1}{2}r^2\mathrm{Var}[\int_0^t X(s)ds]\}. \tag{14}$$

Using the fact that $E[X(t)] = 0$ and letting:

$$\phi(t) = \mathrm{Var}[\int_0^t X(s)ds] \tag{15}$$

we obtain (Crow and Shimizu 1988):

$$E[F(t)] = e^{\frac{1}{2}\phi(t)} \tag{16}$$

and

$$\mathrm{Var}[F(t)] = e^{\phi(t)}[e^{\phi(t)} - 1]. \tag{17}$$

Finally, according to Di Lorenzo, Sibillo, and Tessitore (1997), the autocovariance function can be written as follows:

$$\mathrm{Cov}[F(h), F(k)] = e^{\frac{1}{2}(\phi(h)+\phi(k))}[e^{\Phi(h,k)} - 1] \tag{18}$$

where:

$$\Phi(h,k) = \mathrm{Cov}[\int_0^h X(s)ds, \int_0^k X(s)ds].$$

4 Measures of Sources of Uncertainty

As Frees (1998) points out, it is important to identify the factors affecting the total risk. To this end, we will consider mortality and stochastic interest as risk factors and make actuarial valuations using an instantaneous total rate of return (interest income plus capital gains and losses) represented by the stochastic process defined in equations (9) and (10). Moreover, we will take into account the mortality component, both relating to the riskiness caused by random mortality deviations, and to the riskiness caused by improvements in mortality trend.

After identifying the risk factors, we must study ways to manage them. The risk control tools are different depending on the risk components considered. For example,

- The risk due to random deviations of the numbers of deaths from their expected values can be controlled by means of pooling techniques and reinsurance;
- The investment risk can be controlled by various well-known financial risk management techniques such as immunization techniques and hedging strategies (Frees 1998); and
- The longevity risk (due to an improved mortality trend) can be controlled by using projected mortality tables that are constructed on the basis of forecasts of the future mortality trend (Marocco and Pitacco 1998 and Olivieri 1998).

In light of the above considerations, it is important to quantify the contribution of each risk factor to the total riskiness of the portfolio. It is for this purpose that we want to study the mortality and investment components of the life annuity portfolio considered in Section 2.

4.1 Insurance and Investment Risk Measures

For valuation purposes, it seems reasonable to adopt a simple measure of the two risk components affecting the portfolio. We adopt a well-known formula for the decomposition of the variance and apply it to the variance of the present value of the annuity portfolio.

First we observe that $\mathrm{Var}[Z(c)]$, the variance of the present value of the portfolio considered in our study, can be decomposed in two ways as follows (Parker 1997):

$$\mathrm{Var}[Z(c)] = E[\mathrm{Var}[Z(c) \mid \{T_i\}_{i=1}^{c}]] + \mathrm{Var}[E[Z(c) \mid \{T_i\}_{i=1}^{c}]] \qquad (19)$$

and

$$\begin{aligned}\mathrm{Var}[Z(c)] = E[\mathrm{Var}[Z(c) \mid \{y(k)\}_{k=1}^{\infty}]] \\ + \mathrm{Var}[E[Z(c) \mid \{y(k)\}_{k=1}^{\infty}]].\end{aligned} \tag{20}$$

In equation (19), $\mathrm{Var}[E[Z(c) \mid \{T_i\}_{i=1}^{c}]]$ provides a measure of the variability of $Z(c)$ caused by cash flows connected to random events (mortality, survival), after averaging out the effect of the stochastic discounting factors. Thus, we have the following definition:

Definition 1. *The insurance risk measure is* $\mathrm{Var}[E[Z(c) \mid \{T_i\}_{i=1}^{c}]]$.

Analogously, $E[\mathrm{Var}[Z(c) \mid \{T_i\}_{i=1}^{c}]]$ is an average over cash flows connected to random events of the variability in $Z(c)$ due to the stochastic rate of return, and it can be considered as an investment risk measure. In equation (20), however, $\mathrm{Var}[E[Z(c) \mid \{y(k)\}]]$ is a measure of the variability of $Z(c)$ due to the effect of the stochastic discounting factors as the effect of random events connected with mortality and survival have been averaged out, so it is a measure of the investment risk. Thus, we have the following definition:

Definition 2. *The investment risk measure is* $\mathrm{Var}[E[Z(c) \mid \{y(k)\}_{k=1}^{\infty}]]$.

We choose equation (20) for our valuations, because, as Parker (1997) explains, it allows us to clearly relate the risk components to the number of policies. We get:

$$\begin{aligned}\mathrm{Var}[E[Z(c) \mid \{y(k)\}_{k=1}^{\infty}]] &= \mathrm{Var}[E[\sum_{i=1}^{c} Z_i \mid \{y(k)\}_{k=1}^{\infty}]] \\ &= \mathrm{Var}[c \sum_{h=1}^{\infty} {}_hp_x e^{-y(h)}] \\ &= c^2 \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_hp_x {}_kp_x \mathrm{Cov}[e^{-y(h)}, e^{-y(k)}]\end{aligned} \tag{21}$$

also given by:

$$\begin{aligned}\mathrm{Var}[E[Z(c) \mid \{y(k)\}_{k=1}^{\infty}]] = c^2 \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_hp_x {}_kp_x E[e^{-y(h)-y(k)}] \\ - (c \sum_{h=1}^{\infty} {}_hp_x E[e^{-y(h)}])^2\end{aligned}$$

and

$$E[\mathrm{Var}[Z(c) \mid \{y(k)\}_{k=1}^{\infty}]] = E[\mathrm{Var}[\sum_{i=1}^{c} Z_i \mid \{y(k)\}_{k=1}^{\infty}]]$$
$$= E[c\mathrm{Var}[Z_i \mid \{y(k)\}_{k=1}^{\infty}]]$$
$$= cE[E[Z_i^2 \mid \{y(k)\}_{k=1}^{\infty}]]$$
$$- cE[(E[Z_i \mid \{y(k)\}_{k=1}^{\infty}])^2]. \quad (22)$$

With regard to the average cost per policy, $Z(c)/c$, we get:

$$\mathrm{Var}[E[\frac{Z(c)}{c} \mid \{y(k)\}_{k=1}^{\infty}]] = \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} {}_h p_x {}_k p_x \mathrm{Cov}(e^{-y(h)}, e^{-y(k)}) \quad (23)$$

and

$$E[\mathrm{Var}[\frac{Z(c)}{c} \mid \{y(k)\}_{k=1}^{\infty}]] = \frac{1}{c}(E[E[Z_i^2 \mid \{y(k)\}_{k=1}^{\infty}]]$$
$$- E[[E[Z_i \mid \{y(k)\}_{k=1}^{\infty}]]^2]). \quad (24)$$

4.2 The Longevity Risk

Together with the risk due to accidental deviations of death frequencies from their expected values, the improvements of mortality trends at adult ages have consequences on all life insurance contracts. As life annuities are contracts pertaining to survival benefits, the calculation of present values should be based on mortality tables with built-in mortality projections, because unexpected improvements in future mortality at the older ages could result in an underestimation of future costs and result in actuarial losses.

Definition 3. *The longevity risk is the systematic deviation of the actual number of deaths from their expected values across the older ages.*

By analyzing mortality trend in terms of survival functions, two aspects known as *rectangularization* and *expansion* emerge. *Rectangularization* refers to the higher concentration of deaths around the mode of the curve of deaths, lowering the risk for the insurer. *Expansion* refers to the random advancement of the mode of curve of deaths toward the ultimate life time (Olivieri and Pitacco 1999) and hence a higher risk for the insurer. Longevity risk is the result of rectangularization and expansion acting jointly (Marocco and Pitacco 1998). It can be mitigated by using projected mortality tables;, that is, tables constructed on the basis of a forecast of the future mortality trend (Pitacco 1998).

5 Numerical Illustrations

Let us consider a portfolio of c whole life annuities immediate as described in Section 2. We will quantify the insurance and investment risks on the basis of equations (21) to (24) and four different mortality tables.

Following Olivieri (1998), we assume that the basic distribution of future lifetimes can be represented by a Weibull distribution, i.e., the survival function from age 0 to age x, $s(x)$, is given by:

$$s(x) = e^{-(x/\alpha)^{\gamma}}, \quad x > 0,$$

where $\alpha > 0$ and $\gamma > 0$ are constant parameters. The projected survival function from age 0 to age x is also assumed to follow a Weibull distribution. The basic mortality table and the three projected tables with increasing survival probabilities are based on the parameters α and γ suggested by Olivieri (1998). These parameter values are given below.

Parameter Values

Survival Tables	α	γ
Basic	82.7	7.00
Pessimistic Projection	83.5	8.00
Realistic Projection	85.2	9.15
Optimistic Projection	87.0	10.45

The parameters β and σ of the force of interest process (equation (9)) used in our calculations are determined in a manner similar to Di Lorenzo, Sibillo, and Tessitore (1997). As the Ornstein-Uhlenbeck process, $X(t)$, (equation (9)) represents the deviations of the force of interest from its expected values, we use the differences between the actual observed rates and the corresponding forecasted rates. Then by means of the covariance equivalence principle (Pandit and Wu 1983 and Parker 1994), we can estimate β and σ from these differences.

Using data from Italian short-term (three months) bonds, regularly reported in *Statistical Bulletin*, we obtain $\delta = 0.09$, $\beta = 0.11$, and $\sigma = 0.005$.

Tables 1 and 2 show the mean, variance, investment risk component, and insurance risk component of the present value of a portfolio of c annuities issued at age 65. Table 1 is based on $c = 15$, while Table 2 is based on $c = 1000$.

Tables 3 and 4 show the mean, variance, investment risk component, and insurance risk component of the present value of the average cost per policy of a portfolio of c annuities issued at age 65. Table 3 is based on $c = 15$, while Table 4 is based on $c = 1000$.

Tables 5 and 6 show the mean, variance, investment risk component, and insurance risk component of the present value of a portfolio of c annuities issued at age 45. Table 5 is based on $c = 15$, while Table 6 is based on $c = 1000$.

Tables 7 and 8 show the mean, variance, investment risk component, and insurance risk component of the present value of the average cost per policy of a portfolio of c annuities issued at age 45. Table 7 is based on $c = 15$, while Table 8 is based on $c = 1000$.

Table 1
Present Value of Annuity Portfolio at Age 65 with c = 15

		Projections		
	Basic	Pessimistic	Realistic	Optimistic
$E[Z(c)]$	106.654	110.001	114.706	120.257
$\mathrm{Var}[Z(c)]$	199.384	196.662	196.012	197.376
$\mathrm{Var}[E[Z(c)\vert\{y\}]]$	94.698	102.631	114.973	131.174
$E[\mathrm{Var}[Z(c)\vert\{y\}]]$	104.686	94.031	81.039	66.202

Table 2
Present Value of Annuity Portfolio at Age 65 with c = 1000

		Projections		
	Basic	Pessimistic	Realistic	Optimistic
$E[Z(c)]$	7110.24	7333.41	7647.04	8017.12
$\mathrm{Var}[Z(c)]$	427861.00	462405	516394.00	587408.00
$\mathrm{Var}[E[Z(c)\vert\{y\}]]$	420882.00	456136.00	510992.00	582995.00
$E[\mathrm{Var}[Z(c)\vert\{y\}]]$	6979.00	6269.00	5402.00	4413.00

From Tables 1 and 2 we observe that the mean value of Z(c) increases with the projection; the global variance, for $c = 15$, decreases, except for the optimistic projection, while it always increases for $c = 1000$. Analyzing the two risk components we note that for both values of c the financial risk increases with the projection, while the insurance risk decreases.

Tables 3 and 4 show a similar behavior to Tables 1 and 2, respectively. The numerical results for the global variance are confirmed if we study it as function of the c:

Table 3
Present Value of Average Cost per Policy at Age 65 with $c = 15$

		Projections		
	Basic	Pessimistic	Realistic	Optimistic
$E[\frac{Z(c)}{c}]$	7.11024	7.33341	7.64704	8.01712
$\mathrm{Var}[\frac{Z(c)}{c}]$	0.88614	0.87404	0.87116	0.87725
$\mathrm{Var}[E[\frac{Z(c)}{c}\mid\{y\}]]$	0.42088	0.45613	0.51099	0.58299
$E[\mathrm{Var}[\frac{Z(c)}{c}\mid\{y\}]]$	0.46526	0.41791	0.36017	0.29426

Table 4
Present Value of Average Cost per Policy at Age 65 with $c = 1000$

		Projections		
	Basic	Pessimistic	Realistic	Optimistic
$E[\frac{Z(c)}{c}]$	7.11024	7.33341	7.64704	8.01712
$\mathrm{Var}[\frac{Z(c)}{c}]$	0.42786	0.46240	0.51639	0.58740
$\mathrm{Var}[E[\frac{Z(c)}{c}\mid\{y\}]]$	0.42088	0.45613	0.51099	0.58299
$E[\mathrm{Var}[\frac{Z(c)}{c}\mid\{y\}]]$	0.00698	0.00627	0.00540	0.00441

$$\begin{aligned}
\mathrm{Var}[\frac{Z(c)}{c}]_{\mathrm{pess}} &= -53.7790 + \frac{60.5039 + 54.2351(c-1)}{c} \\
&= 0.4561 + \frac{6.2688}{c} \\
\mathrm{Var}[\frac{Z(c)}{c}]_{\mathrm{real}} &= -58.4772 + \frac{64.3908 + 58.9882(c-1)}{c} \\
&= 0.5110 + \frac{5.4026}{c} \\
\mathrm{Var}[\frac{Z(c)}{c}]_{\mathrm{opt}} &= -64.2742 + \frac{69.2707 + 64.8572(c-1)}{c} \\
&= 0.5830 + \frac{4.4135}{c}.
\end{aligned}$$

So the variance related to the pessimistic projection is greater than the variance related to the realistic projection for $c < 16$; moreover, the variance related to the realistic projection is greater than the variance related to the optimistic projection for $c < 14$.

Table 5
Present Value of Annuity Portfolio at Age 45 with $c = 15$

		Projections		
	Basic	Pessimistic	Realistic	Optimistic
$E[Z(c)]$	143.506	145.974	148.913	151.390
$\mathrm{Var}[Z(c)]$	263.391	264.082	269.890	276.967
$\mathrm{Var}[E[Z(c)\mid\{y\}]]$	227.782	239.457	254.678	268.497
$E[\mathrm{Var}[Z(c)\mid\{y\}]]$	35.609	24.625	15.212	8.470

Table 6
Present Value of Annuity Portfolio at Age 45 with $c = 1000$

		Projections		
	Basic	Pessimistic	Realistic	Optimistic
$E[Z(c)]$	9567.07	9731.6	9927.55	10092.7
$\mathrm{Var}[Z(c)]$	1014670	1065890	1132920	1193880
$\mathrm{Var}[E[Z(c)\mid\{y\}]]$	1012360	1064250	1131900	1193320
$E[\mathrm{Var}[Z(c)\mid\{y\}]]$	2310	1640	1020	560

For all values of c, the financial risk increases and the insurance risk decreases when the projection increases. We observe that the decreasing behavior of the insurance risk is stronger when the number of policies is small. From a mathematical point of view, we can justify this behavior by means of equation (24) in which the dependence of $E[\mathrm{Var}[\frac{Z(c)}{c} \mid \{y(k)\}]]$ on c is evident.

For every fixed survival table, the global variance of $\frac{Z(c)}{c}$ decreases as c increases. In particular, the financial risk takes the same value (from equation (23) we see that $\mathrm{Var}[E[\frac{Z(c)}{c} \mid \{y(k)\}]]$ does not depend on c), while the insurance risk decreases to zero as c tends to infinity (see equation (24)).

We can repeat analogous considerations about Tables 5, 6, 7, and 8. Observe that for $x = 45$ the global variance always increases; in fact we have:

Table 7
Present Value of Average Cost per Policy at Age 65 with $c = 15$

		Projections		
	Basic	Pessimistic	Realistic	Optimistic
$E[\frac{Z(c)}{c}]$	9.56706	9.73160	9.92753	10.0926
$\mathrm{Var}[\frac{Z(c)}{c}]$	1.17062	1.17369	1.19951	1.23096
$\mathrm{Var}[E[\frac{Z(c)}{c}\mid\{y\}]]$	1.01236	1.06425	1.13190	1.19332
$E[\mathrm{Var}[\frac{Z(c)}{c}\mid\{y\}]]$	0.15826	0.10944	0.06761	0.03764

Table 8
Present Value of Average Cost per Policy at Age 65 with $c = 1000$

		Projections		
	Basic	Pessimistic	Realistic	Optimistic
$E[\frac{Z(c)}{c}]$	9.56706	9.73160	9.92753	10.0926
$\mathrm{Var}[\frac{Z(c)}{c}]$	1.01467	1.06589	1.13292	1.19388
$\mathrm{Var}[E[\frac{Z(c)}{c}\mid\{y\}]]$	1.01236	1.06425	1.13190	1.19332
$E[\mathrm{Var}[\frac{Z(c)}{c}\mid\{y\}]]$	0.00231	0.00164	0.00102	0.00056

$$\begin{aligned}
\mathrm{Var}[\frac{Z(c)}{c}]_{\text{pess}} &= -94.5385 + \frac{97.2269 + 95.599(c-1)}{c} \\
&= 1.0605 + \frac{1.6279}{c} \\
\mathrm{Var}[\frac{Z(c)}{c}]_{\text{real}} &= -98.3670 + \frac{100.497 + 99.495(c-1)}{c} \\
&= 1.1280 + \frac{1.0020}{c} \\
\mathrm{Var}[\frac{Z(c)}{c}]_{\text{opt}} &= -101.6260 + \frac{103.368 + 102.814(c-1)}{c} \\
&= 1.1880 + \frac{0.5460}{c}.
\end{aligned}$$

The variance related to the pessimistic projection is greater than the variance related to the realistic projection for $c < 10$; moreover, the variance related to the realistic projection is greater than the variance related to the optimistic projection for $c < 8$.

6 Summary and Concluding Remarks

We have analyzed and quantified two risk sources for a portfolio of life annuities: the investment risk and the insurance risk. This analysis was done in a framework in which both mortality and rates of returns are random.

The global rate of return is modeled as the sum of two components: a deterministic one, which considers the existing investments of the company, and a stochastic one, representing the deviations of the real rate of return from its anticipated values. The stochastic component is an Ornstein-Uhlenbeck process with a mean reversion level of zero.

We also consider the longevity risk, the risk due to the improvements in mortality trend. The effects of the mortality improvements are investigated using different projected mortality tables.

On the basis of the numerical examples presented, we may conclude that the insurance risk decreases when the projection increases. On the other hand, the financial risk increases when the projection increases, because the company could be exposed for a longer period to a risk of systematic nature. Moreover, the mean value of the present value of the cash flows connected to the portfolio increases when the projection increases, because the insurer could bear bigger costs.

In conclusion, the numerical results presented in Section 6 show how the use of projected mortality tables allows the insurer to front the risk of greater costs and how the exposure to the financial risk and to the insurance risk varies, depending on the longevity of the lives insured.

One area for future research is the development of the model presented in the paper, focusing on the effect of the randomness of the projections in the valuations concerning the considered portfolio. Such research can lead to the determination of the systematic risk component due to the type of randomness depicted by the survival functions used for constructing mortality tables.

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