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JOINS AND COVERS IN INVERSE SEMIGROUPS AND TIGHT C^*-ALGEBRAS

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Abstract

We show Exel's tight representation of an inverse semigroup can be described in terms of joins and covers in the natural partial order. Using this, we show that the C^*-algebra of a finitely aligned category of paths, developed by Spielberg, is the tight C^*-algebra of a natural inverse semigroup. This includes as a special case finitely aligned higher-rank graphs: that is, for such a higher-rank graph Λ, the tight C^*-algebra of the inverse semigroup associated to Λ is the same as the C^*-algebra of Λ.

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1. Introduction

C^*-algebras generated by partial isometries include many important families specified by generators and relations, such as graph algebras, higher-rank graph algebras [7, 11], Kellendonk's C^*-algebras of tilings [8, 9] and C^*-algebras of quasilattice-ordered groups [18]. There is a close connection between partial isometries on a Hilbert space, where $A \in B(\mathcal{H})$ is a partial isometry if and only if $AA^*A = A$, and inverse semigroups, where each element s satisfies $ss^*s = s$ and $s^*ss^* = s^*$ for a *unique* element s^*. But there is no equivalent of addition in an inverse semigroup and this issue prevents a straightforward translation of the partial isometries' relations to inverse semigroups. For instance, while there is a natural inverse semigroup associated to a graph (first developed in [1] and rediscovered independently in each of [13, 19]), the C^*-algebra of this inverse semigroup is not the C^*-algebra of the graph, but rather the Toeplitz C^*-algebra.

Exel has developed a crucial notion, tightness, for Hilbert-space representations of inverse semigroups [3] and several authors have shown that the tight C^*-algebra, that is, the one universal for tight representations, is the 'right' one for various families. That is, the tight C^*-algebra of a graph inverse semigroup is the graph algebra, and similarly for C^*-algebras of a large family of semigroupoids (which includes singly-aligned higher-rank graph algebras) [3], for Kellendonk's tiling C^*-algebras [6], and for some Cuntz–Li-type crossed products relating to number fields [22].

We give an alternative approach to tightness, using the natural partial order of an inverse semigroup and finite joins in that partial order as a way to formulate finite sums in an inverse semigroup. For inverse semigroups of operators, such joins are always available: all possible joins, finite or not, exist as operators and can be added to the codomain of a representation; see Proposition 1.1. General inverse semigroups, such as the natural inverse semigroups for graphs and other combinatorial objects, can have (finite) sets of elements that 'should have' a join but do not. Following Lenz [16], these finite sets that should have a join can be identified as covers. It is then natural to study representations of inverse semigroups as operators that send covers to joins of the operators. One of our main results is that for Hilbert-space representations of an inverse semigroup, cover-to-join and tight are equivalent, Corollary 2.3. Thus, the tight C^*-algebra of an inverse semigroup S is universal for the cover-to-join Hilbert-space representations of S, Corollary 2.5.

In fact, this is a special case of a more general result for homomorphisms between inverse semigroups. For $\theta : S \to T$, tightness is defined if T has a Boolean algebra of idempotents, while cover-to-join is defined if T is finitely complete, that is, all possible finite joins exist, and finitely distributive, that is, $a \vee B = \vee\{ab : b \in B\}$ for each element a and finite subset B and similarly for multiplication on the right by a. Then, for T satisfying both conditions, a homomorphism $\theta : S \to T$ is tight if and only if it is cover-to-join, Theorem 2.2.

To demonstrate the usefulness of this description of tight C^*-algebras, we apply these methods to a very general construction of a C^*-algebra from a combinatorial object, a category of paths, introduced by Spielberg [20]. Spielberg's construction generalises both higher-rank graphs [11] and positive cones in discrete ordered groups, including Nica's quasilattice ordered groups [18]. A category of paths is a left and right cancellative category with the property that morphisms do not have inverses. For many of his results, Spielberg imposes an additional condition, that the category of paths be finitely aligned; see Section 3, following Proposition 3.4 for the definition. This condition is motivated by a similar condition for higher-rank graphs.

We observe that there is a natural inverse semigroup associated to a category of paths, Proposition 3.1, and that finitely aligned categories of paths can be characterised in terms of the associated inverse semigroup; see Proposition 3.6. In the finitely aligned case, Spielberg's C^*-algebra is the tight C^*-algebra of this inverse semigroup, Theorem 3.7 and Corollary 3.8. This result is new for finitely aligned higher-rank graph algebras, extending the result that singly-aligned higher-rank graphs are tight C^*-algebras in [3].

This paper builds on extensive work by Exel [3, 4], Lenz [16], and Lawson [14, 15], who have laid the foundational work for C^*-algebras of inverse semigroups and, of course, Spielberg's interesting categories of paths construction [20]. Further, we would like to thank Spielberg for helpful comments about this work.

1.1. Background. For consistency with adjoints of operators, we use s^* for the inverse of an element s in an inverse semigroup. Given an inverse semigroup S and $A \subseteq S$, $E(A)$ denotes the idempotents of A.

By an *inverse semigroup of operators* we mean a collection of partial isometries in some C^*-algebra so that, using the usual multiplication and adjoint, the collection is an inverse semigroup.

Let $\mathcal{P}$ be a semilattice of projections in a C^*-algebra $\mathcal{A}$, that is, $\mathcal{P}$ is closed under products. Define $\mathrm{PI}(\mathcal{P})$ to be the set of all partial isometries X in $\mathcal{A}$, that is, elements satisfying $X = XX^*X$ and $X^* = X^*XX^*$, such that: (1) $X^*X, XX^* \in \mathcal{P}$; and (2) $X^*\mathcal{P}X \subseteq \mathcal{P}$ and $X\mathcal{P}X^* \subseteq \mathcal{P}$. From [2, Proposition 1], $\mathrm{PI}(\mathcal{P})$ is an inverse semigroup of operators with semilattice of idempotents $\mathcal{P}$, so that for any inverse semigroup of operators $S \subset \mathcal{A}$ with $E(S) \subseteq \mathcal{P}$, we have $S \subseteq \mathrm{PI}(\mathcal{P})$.

Proposition 1.1. *If S is an inverse semigroup of operators, then S is contained in an infinitely distributive inverse semigroup of operators.*

Proof. We may assume $S \subset B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. By [12, Proposition 1.4.20], it suffices to verify the result for the lattice of idempotents $E(S)$ contained in the projections of $B(\mathcal{H})$. Since the projections in $E(S)$ all commute, it is easy to verify finite distributivity using $P \vee Q = P + Q - PQ$. Infinite joins are strong-operator-topology limits of an increasing net of finite joins, so using the SOT-continuity of multiplication on bounded sets gives distributivity for infinite joins; see [17, Section 3.2] for the details of these arguments. So adding all infinite joins to $E(S)$ gives a semilattice of projections, $\mathcal{P}$, and $\mathrm{PI}(\mathcal{P})$ is an inverse semigroup of operators which is infinitely distributive and contains S. $\square$

2. Tight and cover-to-join

The key point of this section is that, with the right restriction on the codomain of an inverse semigroup homomorphism, tight and cover-to-join are the same.

In [4, Definitions 2.6 and 6.1], tightness is defined first for a homomorphism on the semilattice of idempotents and then extended to one on an inverse semigroup. Let S be an inverse semigroup. First, given finite sets X and Y in $E(S)$, possibly empty, define

$$E^{X,Y} := \{e \in E : e \leq x \text{ for all } x \in X, ey = 0 \text{ for all } y \in Y\}$$

and, for a set $F \subseteq E(S)$, we call a finite set $Z \subseteq F$ a *cover* of F if, for all $f \in F$, there is $z \in Z$ so that there is a nonzero element below both z and f. Since zf is the meet of z and f, this is equivalent to requiring $zf \neq 0$.

For a homomorphism $\theta : E \to B$ from a semilattice to a Boolean algebra B, θ is defined to be *tight* if for all finite covers Z of $E^{X,Y}$,

$$\bigvee \theta(Z) = \Big(\bigwedge \theta(X)\Big) \wedge \Big(\bigwedge \neg\theta(Y)\Big). \tag{2.1}$$

A homomorphism $\theta : S \to T$ between inverse semigroups S and T with $E(T)$ a Boolean algebra is then defined to be *tight* if $\theta|_{E(S)}$ is tight.

We also need Lenz's relation from [16, Definition 5.1], although we adopt the arrow notation of [15, page 3]. For an element a and a finite set B, we say $a \longrightarrow B$ if, for each $x \leq a$, there is some $b \in B$ so that there is a nonzero element below both x and b. For finite sets A and B, we write $A \longrightarrow B$ if, for each $a \in A$, $a \longrightarrow B$. Lastly, $A \longleftrightarrow B$ if $A \longrightarrow B$ and $B \longrightarrow A$. Lemma 2.2 of [15] gives the key properties of this relation; for example, it is transitive, and if $a \longrightarrow b$ then $a^* \longrightarrow b^*$ and if, in addition, $bc \neq 0$, then $ac \longrightarrow bc$.

Crucially, for a finite set C and an element a with $C \subseteq a^{\downarrow}$ (where $a^{\downarrow} := \{x \in S : x \leq a\}$), call C a *cover* of a if $a \longrightarrow C$. To distinguish between these two different uses of cover, notice that Exel's definition is always applied to sets denoted $E^{X,Y}$ while this second definition is applied to single elements. Of course, for $E(S)$, C being a cover of a is equivalent to C being a cover of $E^{\{a\},\varnothing}$. A homomorphism $\theta : S \to T$ between inverse semigroups is called a *cover-to-join* map if for each cover C of an element $s \in S$, $\vee\theta(C)$ exists and equals $\theta(s)$.

Observe that, in a distributive inverse semigroup, if some finite set F satisfies $\vee F = a$, then F is a cover for a. To see this, observe that $F \subseteq a^{\downarrow}$ and for any $b \leq a$ with $b \neq 0$, $b = b \wedge a = \vee\{b \wedge f : f \in F\}$ and so $b \wedge f_0 \neq 0$ for some $f_0 \in F$.

The following result is a natural extension to general inverse semigroups of [15, Lemma 3.14(1)]; since this result appears only in the first version of that paper on the arXiv, we give a proof here.

Proposition 2.1. *Let S be an inverse semigroup and let $\theta : S \to T$ be a homomorphism to a finitely complete distributive inverse semigroup. Then θ is cover-to-join if and only if the restriction of θ to $E(S)$ is cover-to-join.*

Proof. The forward direction is clear. Suppose that $\theta|_{E(S)}$ is cover-to-join and let C be a cover of a in S. Define $d(x) = x^*x$. First, we claim that $\{d(c) : c \in C\}$ is a cover of $d(a)$. Let $0 \neq e \leq d(a)$. Putting $a' = ae$, we have $d(a') = e$ and $0 \neq a' \leq a$. Thus there exists some $c \in C$ and some x so that $0 \neq x \leq a', c$. Therefore, $0 \neq d(x) \leq d(a), d(c)$, proving the claim.

Since $\theta|_{E(S)}$ is cover-to-join, the claim implies

$$\theta(d(a)) = \bigvee_{c \in C} \theta(d(c)).$$

Multiplying this equation by $\theta(a)$ gives $\theta(a) = \bigvee \theta(c)$. □

We can now prove the main result of this section.

Theorem 2.2. *Let S be an inverse semigroup with zero and T a finitely complete distributive inverse semigroup so that $E(T)$ is a Boolean algebra.*

Then a homomorphism $\theta : S \to T$ is tight if and only if it is a cover-to-join map.

Proof. Suppose that $\theta|_{E(S)}$ is tight. By Proposition 2.1, it suffices to establish that $\theta|_{E(S)}$ is cover-to-join. Assume $Z \subset E(S)$ is a cover for $x \in E(S)$. Then Z is a cover for $E^{\{x\},\varnothing}$

and so by tightness

$$\bigvee \theta(Z) = \theta(x).$$

Thus, $\theta|_{E(S)}$ (and hence, θ) is a cover-to-join map.

Conversely, suppose that θ is a cover-to-join map and $Z \subset E(S)$ is a cover for $E^{X,Y}$. Then $Z \cup Y$ is a cover for $\wedge X$ and so

$$\Big(\bigvee \theta(Z)\Big) \vee \Big(\bigvee \theta(Y)\Big) = \theta(\wedge X).$$

Before taking the meet of both sides with $\neg \bigvee \theta(Y) = \bigwedge \neg\theta(Y)$, notice that $z \wedge y = 0$ for all $z \in Z$ and $y \in Y$ so $\theta(z) \le \neg\theta(y)$. It follows that $\bigvee \theta(Z) \le \bigwedge \neg\theta(Y)$ and so when we take the meet, we obtain (2.1). □

By Proposition 1.1, inverse semigroups of operators can always be enlarged to be finitely complete, distributive, and have a Boolean algebra of idempotents. Thus, we have the following corollary.

COROLLARY 2.3. *Let S be an inverse semigroup and $\theta : S \to B(\mathcal{H})$ a representation of S on Hilbert space. Then θ is tight if and only if it is cover-to-join.*

In general, the conditions on T in Theorem 2.2 are essential. For example, for the inverse semigroup $\mathcal{S}$ of [4, Section 7], the identity map $i : \mathcal{S} \to \mathcal{S}$ is tight in the sense of Exel but is not a cover-to-join map. In contrast to Proposition 2.1, note that $i|_{E(\mathcal{S})}$ is a cover-to-join map, even though i is not. What this inverse semigroup $\mathcal{S}$ shows is that defining a homomorphism $\theta : S \to T$ to be tight when $\theta|_{E(S)}$ satisfies (2.1) can create complications if T is not distributive.

We should mention a result similar to Theorem 2.2 in a different context. Exel, in [5], defines tightness of representations of semigroupoids, a generalisation of categories. He shows, in [5, Proposition 7.4], that, under mild conditions, if the semigroupoid is a category, then a representation is tight if and only if it satisfies an appropriate version of cover-to-join for a semigroupoid.

The following result, a restatement of [3, Theorem 13.3], gives the universal property of the tight C^*-algebra of an inverse semigroup.

THEOREM 2.4 (Exel). *For a countable inverse semigroup S, there is a C^*-algebra $C^*_{\text{tight}}(S)$, and a canonical tight representation $i : S \to C^*_{\text{tight}}(S)$, so that for each tight representation $\theta : S \to \mathcal{B}(\mathcal{H})$, there is a unique representation $\overline{\theta} : C^*_{\text{tight}}(S) \to \mathcal{B}(\mathcal{H})$ so that $\theta = \overline{\theta} \circ i$.*

That is, the following diagram commutes:

$$\begin{tikzcd}
S \arrow[d, "i"] \arrow[dr, dotted, "\overline{\theta}"] & \\
C^*_{\text{tight}}(S) \arrow[r, "\theta"] & \mathcal{B}(\mathcal{H})
\end{tikzcd}$$

Thus, there is a one-to-one correspondence between tight representations of S and representations of $C^*_{\text{tight}}(S)$. The next result then follows from Corollary 2.3.

Corollary 2.5. *For a countable inverse semigroup S, $C^*_{\text{tight}}(S)$ is universal for the cover-to-join Hilbert-space representations of S. That is, there is a one-to-one correspondence between representations of $C^*_{\text{tight}}(S)$ and cover-to-join representations of S.*

3. Categories of paths

To show that the connections developed in the previous section can be usefully applied to tight C^*-algebras, we look at the C^*-algebra of a finitely aligned category of paths and show that this is a tight C^*-algebra for a natural inverse semigroup. Before giving our results, we need to outline Spielberg's work. In [20], Spielberg introduces the notion of a category of paths, a generalisation of the path category of a directed graph that includes the categories of higher-rank graphs as well as many other examples. He builds a C^*-algebra for a finitely aligned category of paths and shows that, when the category is a higher-rank graph (necessarily finitely aligned), this C^*-algebra is isomorphic to the C^*-algebra of the higher-rank graph. Our goal in this section is to show that the C^*-algebra of a category of paths is the tight C^*-algebra of a natural inverse semigroup, the inverse semigroup of zigzag maps.

Briefly, a category of paths, say Λ, is a small category with left and right cancellation and no inverses. That is, a category whose class of objects is a set and for morphisms α, β, and γ, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$ and $\alpha\gamma = \beta\gamma$ implies $\alpha = \beta$. Finally, if $\alpha\beta$ equals the identity map on $s(\beta)$, then each of α and β is this identity map. We routinely identify an object (or vertex) with the identity morphism on the object and view a morphism, α, as a map from its source, $s(\alpha)$, to its range, $r(\alpha)$.

For $\alpha \in \Lambda$, there is a right shift map, also denoted α, from $s(\alpha)\Lambda$ to $\alpha\Lambda$ via $\beta \mapsto \alpha\beta$. Also, there is a left shift map, σ^α, from $\alpha\Lambda$ to $s(\alpha)\Lambda$ via $\alpha\beta \mapsto \beta$. Both left shift maps and right shift maps are one-to-one, by the cancellation properties of Λ, and, as functions on subsets of Λ, the shift maps α and σ^α are inverses of each other.

Define a *zigzag* as a $2n$-tuple of elements $\zeta = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$, where $r(\alpha_i) = r(\beta_i)$ and $s(\alpha_{i+1}) = s(\beta_i)$, and the associated zigzag map as the composition of shift maps $\phi_\zeta = \sigma^{\alpha_1}\beta_1 \cdots \sigma^{\alpha_n}\beta_n$. Note that ϕ_ζ is one-to-one, being a composition of one-to-one maps. The reverse of ζ is $\overline{\zeta} = (\beta_n, \alpha_n, \ldots, \beta_1, \alpha_1)$. Finally, we use $A(\zeta) \subset \Lambda$ for the domain of $\zeta \in \mathcal{Z}$. Note that left and right shift maps embed in $\mathcal{Z}$: the left-shift map σ^α as $(\alpha, r(\alpha))$ and the right shift β as $(r(\beta), \beta)$.

We use $\mathcal{ZM}$ for the set of zigzag maps and call $\phi_{\overline{\zeta}}$ the reverse of ϕ_ζ. We can identify $A(\zeta)$ with the zigzag map of $\overline{\zeta}\zeta$, the right identity of ζ in $\mathcal{ZM}$. Using composition as the operation, $\mathcal{ZM}$ becomes an inverse semigroup. Precisely, for maps $f : A \to B$ and $g : C \to D$, we define $f \circ g$ to have domain $g^{-1}(B \cap C)$ and range $f(B \cap C)$. The symmetric inverse monoid on Λ is the set of all one-to-one maps between subsets of Λ, with this composition operation and f^{-1} the inverse of f. The symmetric inverse monoid on a set is well known to be an inverse semigroup.

Proposition 3.1. *For a category of paths Λ, $\mathcal{ZM}$ forms an inverse semigroup with the reverse as the inverse.*

Proof. Observe that $\phi_{\overline{\zeta}}$ is the inverse function of ϕ_ζ, so the reverse on $\mathcal{ZM}$ corresponds to the usual inverse in the symmetric inverse monoid on Λ. Since the collection of zigzag maps is a subset of the symmetric inverse monoid which is closed under compositions and inverses, it is an inverse subsemigroup, and hence an inverse semigroup itself. □

For convenience, we will use ζ instead of ϕ_ζ, writing $\zeta \in \mathcal{ZM}$ to clarify that we are working with the zigzag map instead of the zigzag itself.

From the category of paths Λ, Spielberg first builds a groupoid G and then defines the boundary of the category, $\partial\Lambda$, which can be identified with a subset of the unit space. Then the C^*-algebra of the category, $C^*(\Lambda)$, is defined as $C^*(G|_{\partial\Lambda})$, where $G|_{\partial\Lambda}$ is the reduction of G to $\partial\Lambda$. We refer the reader to [20] for this construction and concentrate here on characterisations of $C^*(G|_{\partial\Lambda})$ in terms of representations of $\mathcal{ZM}$.

First, [20, Theorem 6.1] characterises the representations of $C^*(G)$. Essentially, they are in one-to-one correspondence with families of Hilbert space operators $\{T_\zeta : \zeta \in \mathcal{ZM}\}$, where $\mathcal{ZM}$ is the inverse semigroup of zigzag maps, satisfying:

(1) $T_\zeta T_\eta = T_{\zeta\eta}$;
(2) $T_\zeta^* = T_{\overline{\zeta}}$;
(3) $T_\zeta^* T_\zeta = \bigvee_{i=1}^n T_{\eta_i}^* T_{\eta_i}$ if $A(\zeta) = \bigcup_{i=1}^n A(\eta_i)$.

This is a slight modification of the statement of [20, Theorem 6.1], which indexes the partial isometries by the zigzags but imposes a fourth condition, that $T_\zeta = T_\zeta^* T_\zeta$ if $\phi_\zeta = \mathrm{id}_{A(\zeta)}$. But this condition is equivalent to requiring that if two zigzags ζ and η have the same zigzag maps, then $T_\zeta = T_\eta$. Indeed, if $\phi_\zeta = \phi_\eta$, then $\overline{\zeta}\eta = \mathrm{id}_{A(\zeta)}$ and so the extra condition implies $T_\zeta^* T_\eta = T_\eta^* T_\zeta T_\zeta^* T_\eta$. Multiplying on the left by T_ζ and observing that T_η and T_ζ are partial isometries with the same initial and final projections by condition (3), it follows that $T_\eta = T_\zeta$. The other direction is immediate from $\mathrm{id}_{A(\zeta)} = \mathrm{id}_{A(\zeta)}^{-1}\mathrm{id}_{A(\zeta)}$.

Clearly, such a family $\{T_\zeta : \zeta \in \mathcal{ZM}\}$ satisfying the first two conditions is equivalent to an inverse semigroup homomorphism, $T : \mathcal{ZM} \to B(\mathcal{H})$ given by $T(\zeta) = T_\zeta$. To describe the third condition in terms of this representation, we need to develop another concept: a homomorphism of inverse semigroups, θ, is *finitely join-preserving* if, whenever $a = \vee C$ for a finite set C and an element a, then $\theta(a) = \vee\theta(C)$. Analogous to Proposition 2.1, we have the following result.

Proposition 3.2. *Let S and T be inverse semigroups and $\theta : S \to T$ be a homomorphism. Then θ is finitely join-preserving if and only if the restriction of θ to $E(S)$ is finitely join-preserving.*

Proof. Because of the parallels to Proposition 2.1, we give only an outline. For the nontrivial direction, let $a = \vee C$ and define $d(x) = x^* x$. It follows from

[12, Proposition 1.4.17] that $d(a) = \vee d(C)$, and so by the assumption on θ, $\theta(d(a)) = \vee\theta(d(C))$. Multiplying by a and using [12, Proposition 1.4.18], we have $\theta(a) = \vee\theta(C)$. □

Applying this proposition to the representation T of $\mathcal{ZM}$, we have the following corollary.

Corollary 3.3. *Let $\mathcal{ZM}$ be the inverse semigroup of zigzag maps associated to a category of paths. An inverse semigroup representation $T : \mathcal{ZM} \to B(\mathcal{H})$ satisfies condition (3) if and only if the representation is finitely join-preserving.*

Since $a = \vee C$ implies that C is a cover of a, we immediately have the following observation.

Proposition 3.4. *For a homomorphism θ between inverse semigroups, if θ is cover-to-join, then it is finitely join-preserving.*

Join-preserving representations properly include cover-to-join representations. A trivial example is the identity map on a three-element chain of idempotents, say $\{0, s, 1\}$. Then s is a cover for 1 and since $s \neq 1$, the identity map is not cover-to-join, even though it is join-preserving.

Spielberg calls a category of paths *finitely aligned* if for every pair of elements α and β, there is a finite subset G of Λ so that

$$\alpha\Lambda \cap \beta\Lambda = \bigcup_{\epsilon\in G} \epsilon\Lambda.$$

Zigzag maps in the finitely aligned case admit a simple form: every zigzag map is a finite union of maps of the form $\gamma\sigma^\delta$, with $\gamma, \delta \in \Lambda$ [20, Lemma 3.3]. In fact, it follows from the fourth paragraph of the proof of [20, Theorem 6.3] that this form is canonical, that is, if $\bigcup_i \gamma_i\sigma^{\delta_i} = \bigcup_j \alpha_j\sigma^{\beta_j}$ then both finite unions have the same number of elements and there is a permutation π so that $\gamma_i = \alpha_{\pi(i)}$ and $\delta_i = \beta_{\pi(i)}$. We thank Jack Spielberg for clarifying our initially ambiguous discussion of this point.

We call maps of the form $\gamma\sigma^\delta$ elementary zigzag maps.

Remark 3.5. For two elementary zigzags maps, $a = \alpha\sigma^\beta$ and $b = \gamma\sigma^\delta$, observe that $a \leq b$ if and only if

$$\alpha = \gamma\epsilon \quad \text{and} \quad \beta = \delta\epsilon,$$

for some $\epsilon \in s(\alpha)\Lambda$. The reverse direction is immediate and the forward direction follows from observing that $a \leq b$ implies $\beta\Lambda \subseteq \delta\Lambda$, which implies that $\beta = \delta\beta'$ for some β' in Λ. Further, $\alpha = \alpha\sigma^\beta(\beta) = \gamma\sigma^\delta(\beta) = \gamma\beta'$.

Steinberg [21] defined a poset to be a *weak semilattice* if the intersection of principal downsets is finitely generated as a downset. Lenz [16] and Lawson [15] used the stronger condition that the intersection be singly generated. Both conditions hold for all E-unitary and 0-E-unitary inverse semigroups. It is worth noting that the weak semilattice condition appears, without a name, in [10] as a necessary and sufficient condition for a cocycle to have closed range—see Proposition 3.9 and the final paragraph of the paper.

Proposition 3.6. *Let Λ be a category of paths and $\mathcal{ZM}$ the associated inverse semigroup of zigzag maps. Then Λ is finitely aligned if and only if $\mathcal{ZM}$ is a weak semilattice.*

Proof. Viewing Λ as a subset of $\mathcal{ZM}$, it is immediate that if $\mathcal{ZM}$ is a weak semilattice then Λ is finitely aligned.

Conversely, suppose that Λ is finitely aligned. Let $x = \gamma_1\sigma^{\delta_1}$ and $y = \gamma_2\sigma^{\delta_2}$. Since Λ is finitely aligned, there is a finite set G_δ such that

$$\delta_1\Lambda \cap \delta_2\Lambda = \bigcup_{\nu\in G_\delta} \nu\Lambda.$$

Let G'_δ be the subset of those $\nu \in G_\delta$ for which there is an elementary zigzag map $\kappa\sigma^\lambda \in x^\downarrow \cap y^\downarrow$ with $\lambda \in \nu\Lambda$.

If an elementary zigzag map $\kappa\sigma^\lambda$ is below both x and y, then by Remark 3.5 $\lambda \in \delta_1\Lambda \cap \delta_2\Lambda$ and so $\lambda \in \nu\Lambda$ for some $\nu \in G_\delta$. Since $\kappa\sigma^\lambda$ is below both x and y, it follows that $x(\lambda) = y(\lambda)$. Writing $\lambda = \nu\lambda'$ for a path λ' with $s(\nu) = r(\lambda')$, we have that $x(\lambda) = x(\nu)\lambda'$. Thus $s(x(\nu)) = s(\nu)$ and, further,

$$\kappa = \kappa\sigma^\lambda(\lambda) = x(\lambda) = x(\nu)\lambda'.$$

Thus, $\kappa\sigma^\lambda$ is below $x(\nu)\sigma^\nu$.

We claim that a finite set of generators for $x^\downarrow \cap y^\downarrow$ is $\{x(\nu)\sigma^\nu : \nu \in G'_\delta\}$. By the previous paragraph, each element of $x^\downarrow \cap y^\downarrow$ is below an element of this set. On the other hand, we know that $x(\nu) = y(\nu)$ for each $\nu \in G'_\delta$ and so, using Remark 3.5 again, $x(\nu)\sigma^\nu$ will be below both x and y.

Next, if we allow x and y to be arbitrary zigzag maps, then each map is a finite join of elementary zigzags. By applying the above reasoning to each possible pair of elementary zigzags making up x and y and then taking a union, we have a finite generating set for the order ideal $x^\downarrow$ and $y^\downarrow$. □

Given a finite union $\bigcup_i \gamma_i\sigma^{\delta_i}$, a finitely join-preserving representation of $\mathcal{ZM}$ must map this element to

$$\bigvee_i T_{\gamma_i}T^*_{\delta_i}.$$

So a finitely join-preserving representation of $\mathcal{ZM}$, in the finitely aligned case, is determined by the image of elements of Λ. Spielberg shows [20, Theorem 6.3] that conditions (1)–(3) above have a simpler equivalent form that involves T_α only for those $\alpha \in \Lambda$:

(1′) $T^*_\alpha T_\alpha = T_{s(\alpha)}$;
(2′) $T_\alpha T_\beta = T_{\alpha\beta}$ if $s(\alpha) = r(\beta)$;
(3′) $T_\alpha T^*_\alpha T_\beta T^*_\beta = \bigvee_{\gamma\in\alpha\vee\beta} T_\gamma T_\gamma*$.

Here $\alpha \vee \beta$ denotes the minimal common extensions of α and β, that is, the generators of the downset $\alpha\Lambda \cap \beta\Lambda$.

Thus, representations of $\mathcal{ZM}$ for a finitely aligned countable category of paths Λ are determined by the image of Λ itself. Spielberg characterises, in [20, Theorem 8.2], the representations of $C^*(\Lambda)$ for Λ a finitely aligned countable category of paths such that the associated groupoid is amenable, in terms of this image, that is, $\{T_\alpha : \alpha \in \Lambda\}$. There are four conditions, the three conditions (1′)–(3′) and a fourth condition, which we will now outline.

Call F an *exhaustive set* for a vertex $v \in \Lambda$ if $s(\beta) = v$ for all $\beta \in F$ and for every α with $s(\alpha) = v$, there is some $\beta \in F$ such that $\alpha\Lambda \cap \beta\Lambda$ is nonempty. If F is a finite exhaustive set for v, then it follows that $\{\beta\sigma^\beta : \beta \in F\}$ is a cover for v in the inverse semigroup $\mathcal{ZM}$.

It is convenient to use $FE(v)$ for the collection of finite exhaustive sets at v. With this in hand, the fourth condition is:

(4′) $T_v = \bigvee_{\beta \in F} T_\beta T_\beta^*$ if $F \in FE(v)$.

Theorem 3.7. *Let Λ be a countable finitely aligned category of paths and $\mathcal{ZM}$ be the inverse semigroup of zigzag maps. There is a one-to-one correspondence between families of operators $\{T_\alpha : \alpha \in \Lambda\}$ satisfying conditions (1′)–(4′), and cover-to-join representations of $\mathcal{ZM}$. In particular, if the groupoid of Λ is amenable, then representations of $C^*(\Lambda)$ correspond to cover-to-join representations of $\mathcal{ZM}$.*

Proof. Suppose that π is a cover-to-join representation of $\mathcal{ZM}$ and consider the family $\{T_\alpha\}$ of operators where $T_\alpha = \pi(\sigma^{r(\alpha)}\alpha)$. Conditions (1) and (2) follow from the fact that π is a $*$-homomorphism. We observed after the definition of cover-to-join that such a map is necessarily finitely join-preserving and so by Corollary 3.3, conditions (1)–(3) hold. By Spielberg's [20, Theorem 6.3], this is equivalent to conditions (1′)–(3′). Finally, suppose that v is a vertex in Λ and $F \in FE(v)$. As noted above, $\{\beta\sigma^\beta : \beta \in F\}$ is a cover for v in $\mathcal{ZM}$ and so condition (4′) follows from the fact that π is cover-to-join.

Conversely, suppose that $\{T_\alpha : \alpha \in \Lambda\}$ is a family of operators satisfying conditions (1′)–(4′). We will show that the induced representation π of $\mathcal{ZM}$ is cover-to-join.

Let $\{z_1, z_2, \ldots, z_n\}$ be a cover for x in $\mathcal{ZM}$. Since every zigzag map is a union of maps of the form $\gamma\sigma^\delta$, we may assume (possibly enlarging the cover) that $z_i = \gamma_i\sigma^{\delta_i}$ for each i.

Initially we will suppose that $x = \gamma\sigma^\delta$. For each i, since $z_i \le x$, by Remark 3.5, there is β_i in $s(\delta)\Lambda$ such that $\delta_i = \delta\beta_i$ and $\gamma_i = \gamma\beta_i$.

Let $v = s(\delta)$. We claim that $\{\beta_1, \ldots, \beta_n\} \in FE(v)$. To see this, let $\alpha \in v\Lambda$ and consider the zigzag map $\gamma\alpha\sigma^{\delta\alpha} \le \gamma\sigma^\delta$. Since $\{z_1, z_2, \ldots, z_n\}$ cover x, there exists i and a nonzero zigzag z such that $z \le z_i$ and $z \le \gamma\alpha\sigma^{\delta\alpha}$. In particular, there is some $\beta \in \Lambda$

common to the domains of z_i and $\delta\alpha\sigma^{\delta\alpha}$. So

$$\beta \in \delta\alpha\Lambda \cap \delta\beta_i\Lambda.$$

Thus $\sigma^\delta(\beta) \in \alpha\Lambda \cap \beta_i\Lambda$, that is, $\alpha\Lambda \cap \beta_i\Lambda$ is not empty. This shows that $\{\beta_1, \dots, \beta_n\}$ is exhaustive at v. By condition (4′),

$$T_v = \bigvee_{i=1}^{n} T_{\beta_i} T_{\beta_i}^*.$$

Multiplying on the left by T_γ and on the right by T_δ^* and using $\delta_i = \delta\beta_i$ and $\gamma_i = \gamma\beta_i$,

$$T_\gamma T_\delta^* = \bigvee_{i=1}^{n} T_{\gamma_i} T_{\delta_i}^*,$$

which proves that $\pi(x) = \bigvee \pi(z_i)$.

Next, suppose that $x = \bigcup_{j=1}^{m} \alpha_j \sigma^{\beta_j}$ and let $e_j = (\alpha_j\sigma^{\beta_j})^* \alpha_j\sigma^{\beta_j}$. Again, let $\{z_i = \gamma_i\sigma^{\delta_i} \colon i = 1, \dots, n\}$ be a cover for x. Then $\{z_i e_j : i = 1, \dots, n\}$ covers the elementary zigzag map $xe_j = \alpha_j\sigma^{\beta_j}$. By the previous two paragraphs,

$$\pi(xe_j) = \bigvee_{i=1}^{n} \pi(z_i e_j) = \bigvee_{i=1}^{n} \pi(z_i)\pi(e_j).$$

Since π is join-preserving (by Corollary 3.3) we have

$$\begin{aligned}
\pi(x) &= \bigvee_{j=1}^{m} \pi(xe_j) \\
&= \bigvee_{j=1}^{m} \bigvee_{i=1}^{n} \pi(z_i)\pi(e_j) \\
&= \bigvee_{i=1}^{n} \pi(z_i)\pi(x^*x) = \bigvee_{i=1}^{n} \pi(z_i),
\end{aligned}$$

where the last equality holds since $z_i \le x$. Thus π is cover-to-join. □

By the universal property of tight C*-algebras, Corollary 2.5, we have the following result.

Corollary 3.8. *For Λ a countable finitely aligned category of paths with $\mathcal{ZM}$ its inverse semigroup of zigzag maps, if the groupoid of Λ is amenable, then $C^*_{\text{tight}}(\mathcal{ZM})$ is isomorphic to $C^*(\Lambda)$.*

Finitely aligned categories of paths include finitely aligned higher-rank graphs, so this shows that their C*-algebras are tight. The best previous result for higher-rank graphs along these lines was for singly-aligned higher rank graphs (or, more generally, for higher-rank graphs with the little pull-back property) by Exel in [3, Section 20], as a consequence of a result for a family of semigroupoids. Exel also mentions there that his methods should work for all finitely aligned higher-rank graphs, so Corollary 3.8 is expected. Nonetheless, we believe that this approach is sufficiently different to be interesting.

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