Independent Claim Report Lags and Bias in Forecasts Using Age-to-Age Factor Methodology

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Abstract

This paper finds that when report lags are assumed to be independent, the age-to-age factor method produces biased estimates when applied to claim count development data. Two distributions are considered as models for the ultimate number of claims for an accident period: (i) a Poisson distribution, and (ii) a negative binomial distribution. In the Poisson case, the assumption of independent report lags implies the independence of the total number of claims reported in any two periods. In the negative binomial case, however, assuming that report lags are independent does not imply that increments are independent, and a somewhat different argument is required. Finally, it is proved that weighted average forecasts exhibit a smaller bias than do straight average estimates.

Key words and phrases: *loss development, Poisson, negative binomial, report lag*

1 Introduction

Stanard (1985) observes an apparent bias in forecasts of ultimate claims when commonly used reserving methods are applied to simulated data. His approach is to specify a stochastic model of the emergence of claims over time and use it to generate data to be used as input

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to various reserving methods. One of the methods he selects is the familiar age-to-age factor method—he finds that the method produces overstated forecasts of ultimate claims in certain cases.

Stanard's simulation model assumes that the report lag of each claim is independent. This assumption has been presented in other work, particularly that of Weissner (1978, 1981). As I will prove, however, when report lags are assumed to be independent, the age-to-age factor method is biased when applied to claim count development data. Two models are considered for the ultimate number of claims in an accident period: (i) the Poisson distribution and (ii) the negative binomial distribution.

In the Poisson model, the assumption of independent report lags implies the independence of the total number of claims reported in any two periods and provides an example of an emergence process with independent increments. A general argument may be made to show that the age-to-age factor methodology gives biased results when the underlying process is known to have independent development increments. In the negative binomial model, which is the model specified by Stanard, assuming that report lags are independent does not imply that increments are independent, and a somewhat different argument is required.

The arguments presented here will use Jensen's inequality. Stanard notes in Appendix A of his paper that the observed bias is likely due to the fact that the expected value of the ratio of two non-constant random variables is not necessarily equal to the ratio of their expected values, i.e., in general

$$\frac{E[X]}{E[Y]} \neq E[\frac{X}{Y}].$$

Jensen's inequality may be used to show that, under certain conditions,

$$E\left[\frac{X}{Y}\right] > \frac{E[X]}{E[Y]}.$$

These ratios will arise as the usual claims development or age-to-age factors.

Finally, I will prove that weighted average forecasts exhibit a smaller bias than straight average estimates.

2 Preliminaries

2.1 Notation and Assumptions

For simplicity, the claims activity is divided into n consecutive and disjoint time periods of equal length, such as weeks, months, quarters, years, etc. For $i, j = 1, 2, \ldots$, let $X_{i,j}$ denote the number of incidents occurring in period i that are reported as claims in period $i + j - 1$ (i.e., with lag $j - 1$). The incremental development triangle at the end of the nth period is displayed in Table 1.

Table 1
Number of Accident Period i Claims Reported with Lag $j - 1$

	j						
i	1	2	...	n-$i+1$	...	$n-1$	n
1	$X_{1,1}$	$X_{1,2}$	...	$X_{1,n-i+1}$	...	$X_{1,n-1}$	$X_{1,n}$
2	$X_{2,1}$	$X_{2,2}$	...	$X_{2,n-i+1}$	...	$X_{2,n-1}$	
$\vdots$	$\vdots$	$\vdots$	...	...			
i	$X_{i,1}$	$\vdots$	$\vdots$	$X_{i,n-i+1}$			
$\vdots$	$\vdots$	$\vdots$	...				
$n-1$	$X_{n-1,1}$	$X_{n-1,2}$					
n	$X_{n,1}$						

These data are more commonly summarized as a cumulative development triangle (as shown in Table 2), where

$$S_{i,j} = \sum_{k=1}^{j} X_{i,k}.$$

The assumptions, however, will be stated in terms of the $X_{i,j}$s.

The basic problem for data given in this format is to estimate the total number of claims arising from each accident period from the number reported through the end of period n and from the claim reporting pattern. It is sufficient for our purposes to consider only the problem of forecasting the next reporting increment.

then it follows that

$$\Omega_{i,n-i+2} > E[X_{i,n-i+2} \mid S_{i,n-i+1} > 0] \equiv E[X_{i,n-i+2}].$$

Proof: Observe that, due to the independence of accident periods,

$$\begin{aligned}
\Omega_{i,n-i+2} &= E\left[S_{i,n-i+1} \frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0,\ S_{i,n-i+1} > 0 \right] \\
&= E\left[S_{i,n-i+1} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0,\ S_{i,n-i+1} > 0 \right] \\
&\quad \times E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0,\ S_{i,n-i+1} > 0 \right] \\
&= E\left[S_{i,n-i+1} \mid S_{i,n-i+1} > 0 \right] \\
&\quad \times E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0 \right]. \qquad (4)
\end{aligned}$$

Because of the independence of increments, it is also true that

$$\begin{aligned}
&E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0 \right] \\
&\qquad = (i-1)E[X_{k,n-i+2}]\, E\left[\frac{1}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0 \right]. \qquad (5)
\end{aligned}$$

Using Jensen's inequality, with $g(\mathbf{x}) = x_1 + \cdots + x_{i-1}$, one deduces that

$$\begin{aligned}
&E\left[\frac{1}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0 \right] \\
&\qquad > \frac{1}{E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0 \right]}
\end{aligned}$$

and this inequality may be strengthened by noting that

$$E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]$$
$$\leq E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \mid S_{k,n-i+1} > 0, k = 1, 2, \dots, i-1\right]$$
$$= (i-1)E\left[S_{i,n-i+1} \mid S_{i,n-i+1} > 0\right].$$

Thus

$$E\left[\frac{1}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]$$
$$> \frac{1}{(i-1)E\left[S_{i,n-i+1} \mid S_{i,n-i+1} > 0\right]}. \tag{6}$$

Substituting equations (5) and (6) into (4) completes the proof. □

Readers will observe at this point that Theorem 1 is a statement of fact regarding ratios of independent random variables; it does not rely on the specific distribution of the underlying process. This should not be surprising because the age-to-age factor methodology also does not rely on the specific distribution of the underlying process. Intuition is the main guide in the construction of forecasts relying on identical distributions by lag. The conclusion is not that the age-to-age factor method is biased absolutely, but that it is not compatible with a claims process assumed to have independent increments.

3.2 Independent Increments from Independent Claims Lags: The Poisson Case

I will now prove that when the report lags are independent and the distribution of ultimate accident period claims is Poisson with constant mean λ then assumption (2) holds. The proof relies on two well-known properties of Poisson processes: the number of claims reported with lag $j-1$ is Poisson with mean λp_j,[1] where p_j is the probability that a claim from accident period i is reported in period $i+j-1$. In addition, the number of claims reported with lag $j-1$ and with lag $k-1$ are also independent.[2] Formally, this may be stated as:

[1] See, for example, Karlin and Taylor (1994, Chapter 5, Theorem 5.2, page 243).

In either case, the mean of M is,

$$E[M] = \frac{\alpha}{\beta}. \tag{9}$$

Proposition 5. *When N (the distribution of ultimate claims) is negative binomial with parameters $\alpha > 0$ and $\beta > 0$ and the report lags are independent, then N_j (the number of claims reported with lag $j - 1$) is negative binomial with parameters α and β_j where $\beta_j = \beta / p_j$.*

Proof: Again, $[N_j = k \mid N = n]$ has a binomial distribution when the report lags are independent and $n \geq k$.

$$\begin{aligned}\Pr[N_j = k] &= \sum_{n=0}^{\infty} \Pr[N_j = k \mid N = n] \Pr[N = n] \\ &= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p_j^k (1 - p_j)^{n-k} \frac{\Gamma(\alpha + n)}{n!\Gamma(\alpha)} \left(\frac{\beta}{1+\beta}\right)^{\alpha} \left(\frac{1}{1+\beta}\right)^{n}.\end{aligned}$$

Changing the summation variable to $r = n - k$ produces

$$\begin{aligned}\Pr[N_j = k] &= \frac{(p_j)^k}{k!\Gamma(\alpha)} \left(\frac{\beta}{1+\beta}\right)^{\alpha} \left(\frac{1}{1+\beta}\right)^{k} \sum_{r=0}^{\infty} \frac{\Gamma(\alpha + k + r)}{r!} \left(\frac{1 - p_j}{1+\beta}\right)^{r} \\ &= \frac{\Gamma(\alpha + k)}{k!\Gamma(\alpha)} \left(\frac{\beta}{1+\beta}\right)^{\alpha} \left(\frac{p_j}{1+\beta}\right)^{k} \left(1 - \frac{1 - p_j}{1+\beta}\right)^{-(\alpha+k)} \\ &\quad \times \sum_{r=0}^{\infty} \frac{\Gamma(\alpha + k + r)}{r!\Gamma(\alpha + k)} \left(\frac{1 - p_j}{1+\beta}\right)^{r} \left(1 - \frac{1 - p_j}{1+\beta}\right)^{\alpha+k} \\ &= \frac{\Gamma(\alpha + k)}{k!\Gamma(\alpha)} \left(\frac{\beta}{1+\beta}\right)^{\alpha} \left(\frac{p_j}{1+\beta}\right)^{k} \left(1 - \frac{1 - p_j}{1+\beta}\right)^{-(\alpha+k)} \\ &= \frac{\Gamma(\alpha + k)}{k!\Gamma(\alpha)} \left(\frac{\beta}{p_j+\beta}\right)^{\alpha} \left(\frac{p_j}{p_j+\beta}\right)^{k}\end{aligned}$$

and the proposition is proved. □

Proposition 6. *When the distribution of ultimate claims, N, is negative binomial with parameters $\alpha > 0$ and $\beta > 0$ and the report lags are independent, $[N_j \mid N_k = s]$ has a negative binomial with parameters $\alpha_{jk} = \alpha + s$ and $\beta_{jk} = (\beta + p_k)/p_j$ provided $p_j > 0$. In particular,*

$$E[N_j \mid N_k] = \frac{(\alpha + N_k)p_j}{\beta + p_k}.$$

Proof: Again, $[N_j = k \mid N = n]$ is binomial when the report lags are independent and $n \geq k$.

$$\begin{aligned}\Pr[N_j = r \mid N_k = s] &= \frac{\Pr[N_j = r, N_k = s]}{\Pr[N_k = s]} \\ &= \frac{\sum_{n=0}^{\infty} \Pr[N_j = r, N_k = s \mid N = n]\Pr[N = n]}{\sum_{n=0}^{\infty} \Pr[N_k = s \mid N = n]\Pr[N = n]}.\end{aligned}$$

Again, $[N_j = r, N_k = s \mid N = n]$ is multinomial, so the numerator may be rewritten as

$$\begin{aligned}&\sum_{n=0}^{\infty} \Pr[N_j = r, N_k = s \mid N = n]\Pr[N = n] \\ &\quad = \sum_{n=r+s}^{\infty} \left[\frac{n!}{r!s!(n-r-s)!} p_j^r p_k^s (1 - p_j - p_k)^{n-r-s} \right. \\ &\qquad \left. \times \frac{\Gamma(\alpha + n)}{n!\Gamma(\alpha)} \left(\frac{\beta}{1+\beta}\right)^{\alpha} \left(\frac{1}{1+\beta}\right)^{n}\right]\end{aligned}$$

and the denominator may be rewritten as

$$\begin{aligned}&\sum_{n=0}^{\infty} \Pr[N_k = s \mid N = n]\Pr[N = n] \\ &\quad = \sum_{n=s}^{\infty} \frac{n!}{s!(n-s)!} p_k^s (1 - p_k)^{n-s} \frac{\Gamma(\alpha + n)}{n!\Gamma(\alpha)} \left(\frac{\beta}{1+\beta}\right)^{\alpha} \left(\frac{1}{1+\beta}\right)^{n}.\end{aligned}$$

The numerator and denominator can be summed separately and reduced to give

$$\Pr[N_j = r \mid N_k = s] = \frac{\Gamma(\alpha_{jk} + r)}{r!\Gamma(\alpha_{jk})} \left(\frac{\beta_{jk}}{1+\beta_{jk}}\right)^{\alpha} \left(\frac{1}{1+\beta_{jk}}\right)^{r}$$

thus proving the proposition. □

One implication of Proposition 3 is that the increments are no longer independent. The following fact is also required:

Proposition 7. *When the distribution of ultimate claims, N, is negative binomial with parameters $\alpha > 0$ and $\beta > 0$ and the report lags are independent, then for $p_j > 0$ and $p_k > 0$,*

$$E[N_j \mid N_k > 0] = \frac{\alpha p_j}{\beta}\left(\frac{1-\left(\frac{\beta}{\beta+p_k}\right)^{\alpha+1}}{1-\left(\frac{\beta}{\beta+p_k}\right)^{\alpha}}\right). \qquad (10)$$

Proof: Proceeding in a now familiar fashion but using the convenient, alternative form of the negative binomial probabilities, one sees that:

$$\Pr\left[N_j = r \mid N_k > 0\right] = \frac{\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}\Pr\left[N_j = r,\, N_k = s \mid N = n\right]\Pr\left[N = n\right]}{\Pr\left[N_k > 0\right]}.$$

The denominator on the right side of this equation is

$$\Pr\left[N_k > 0\right] = 1 - \left(\frac{\beta}{\beta+p_k}\right)^{\alpha}.$$

The numerator on the right side of this equation is

$$
\begin{aligned}
&\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}\Pr\left[N_j = r, N_k = s \mid N = n\right]\Pr\left[N = n\right] \\
&= \sum_{s=1}^{\infty}\sum_{n=r+s}^{\infty}\frac{n!}{r!s!\left(n-r-s\right)!}p_j^r p_k^s\left(1-p_j-p_k\right)^{n-r-s} \\
&\quad\times\frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)}\int_0^{\infty}\frac{e^{-\lambda}\lambda^n}{n!}\lambda^{\alpha-1}e^{-\beta\lambda}d\lambda \\
&= \frac{\beta^{\alpha}p_j^r}{\Gamma\left(\alpha\right)r!}\int_0^{\infty}\left\{\sum_{s=1}^{\infty}\frac{\left(\lambda p_k\right)^s}{s!}\sum_{n=0}^{\infty}\frac{\left[\left(1-p_j-p_k\right)\lambda\right]^n}{n!}\right\} \\
&\quad\times\lambda^{\alpha+r-1}e^{-(\beta+1)\lambda}d\lambda \\
&= \frac{\beta^{\alpha}p_j^r}{\Gamma\left(\alpha\right)r!}\left\{\int_0^{\infty}\lambda^{\alpha+r-1}e^{-(\beta+p_j)\lambda}d\lambda\right. \\
&\quad\left.-\int_0^{\infty}\lambda^{\alpha+r-1}e^{-(\beta+p_j+p_k)\lambda}d\lambda\right\} \\
&= \frac{\Gamma\left(\alpha+r\right)}{\Gamma\left(\alpha\right)r!}\beta^{\alpha}p_j^r\left\{\frac{1}{\left(\beta+p_j\right)^{\alpha+r}}-\frac{1}{\left(\beta+p_j+p_k\right)^{\alpha+r}}\right\}.
\end{aligned}
$$

It follows that

$$
\Pr\left[N_j = r \mid N_k > 0\right] = \frac{\dfrac{\Gamma\left(\alpha+r\right)}{\Gamma\left(\alpha\right)r!}p_j^r\left\{\dfrac{1}{\left(\beta+p_j\right)^{\alpha+r}}-\dfrac{1}{\left(\beta+p_j+p_k\right)^{\alpha+r}}\right\}}{\dfrac{1}{\beta^{\alpha}}-\dfrac{1}{\left(\beta+p_k\right)^{\alpha}}}.
$$

It is straightforward to sum this expression to obtain the result:

$$
\begin{aligned}
E\left[N_j \mid N_k > 0\right] &= \sum_{r=1}^{\infty} r \Pr\left[N_j = r \mid N_k > 0\right] \\
&= \sum_{r=1}^{\infty} \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)(r-1)!} p_j^r \\
&\quad \times \frac{\left\{ \dfrac{1}{\left(\beta + p_j\right)^{\alpha+r}} - \dfrac{1}{\left(\beta + p_j + p_k\right)^{\alpha+r}} \right\}}{\dfrac{1}{\beta^{\alpha}} - \dfrac{1}{(\beta + p_k)^{\alpha}}} \\
&= \frac{\alpha p_j}{\dfrac{1}{\beta^{\alpha}} - \dfrac{1}{(\beta + p_k)^{\alpha}}} \sum_{r=0}^{\infty} \frac{\Gamma(\alpha + 1 + r)}{\Gamma(\alpha + 1) r!} p_j^r \\
&\quad \times \left\{ \frac{1}{\left(\beta + p_j\right)^{\alpha+1+r}} - \frac{1}{\left(\beta + p_j + p_k\right)^{\alpha+1+r}} \right\}
\end{aligned}
$$

which is a difference of two negative binomial forms. This may be simplified as

$$
\begin{aligned}
E\left[N_j \mid N_k > 0\right] &= \alpha p_j \left\{ \frac{\dfrac{1}{\beta^{\alpha+1}} - \dfrac{1}{(\beta + p_k)^{\alpha+1}}}{\dfrac{1}{\beta^{\alpha}} - \dfrac{1}{(\beta + p_k)^{\alpha}}} \right\} \\
&= \frac{\alpha p_j}{\beta} \left\{ \frac{1 - \left(\dfrac{\beta}{\beta + p_k}\right)^{\alpha+1}}{1 - \left(\dfrac{\beta}{\beta + p_k}\right)^{\alpha}} \right\}
\end{aligned}
$$

and the proposition is proved. □

The key task may now be addressed, that is a theorem without the restrictions of Assumption 2.

Theorem 2. *When Assumption 1 holds and the distribution of ultimate claims is negative binomial, the expected value of the weighted average forecast is always greater than the expected value of the actual change, i.e.,*

$$\Omega_{i,n-i+2} > E[X_{i,n-i+2} \mid S_{i,n-i+1} > 0]$$

where $\Omega_{i,n-i+2}$ *is defined in equation (3).*

Proof: As in the proof of Theorem 1,

$$\Omega_{i,n-i+2} = E\left[S_{i,n-i+1} \mid S_{i,n-i+1} > 0\right] \times E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]$$

due to the independence of accident periods. In the proof of Theorem 1, it was possible to separate the expectation operator containing the ratio. As has been shown, however, independence of increments does not hold here—some other mechanism must be employed. To this end, one fixes the $S_{k,n-i+1}$ and computes the expectation in successive steps. But

$$E\left[\frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]$$
$$= E\left[\frac{E[\sum_{k=1}^{i-1} X_{k,n-i+2} \mid S_{1,n-i+1},\ldots,S_{i-1,n-i+1}]}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]$$
$$= E\left[\frac{\sum_{k=1}^{i-1} E[X_{k,n-i+2} \mid S_{k,n-i+1}]}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right].$$

Proposition 3 implies that this expression is equal to

$$\frac{p_{n-i+2}}{\beta + \pi_{n-i+1}} E\left[\frac{\sum_{k=1}^{i-1}(S_{k,n-i+1} + \alpha)}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]$$
$$= \frac{p_{n-i+2}}{\beta + \pi_{n-i+1}}\left(1 + (i-1)\alpha E\left[\frac{1}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1} > 0\right]\right)$$

where $\pi_k = p_1 + \ldots p_k$ for $k = 1, 2, \ldots$. Jensen's inequality implies that in turn

$$\frac{p_{n-i+2}}{\beta+\pi_{n-i+1}}\left(1+(i-1)\alpha E\left[\frac{1}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid \sum_{k=1}^{i-1} S_{k,n-i+1}>0\right]\right)$$
$$>\frac{p_{n-i+2}}{\beta+\pi_{n-i+1}}\left(1+\frac{(i-1)\alpha}{E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \mid \sum_{k=1}^{i-1} S_{k,n-i+1}>0\right]}\right)$$
$$\geq\frac{p_{n-i+2}}{\beta+\pi_{n-i+1}}\left(1+\frac{\alpha}{E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \mid \sum_{k=1}^{i-1} S_{k,n-i+1}>0\right]}\right).$$

From Proposition 2, one can see that

$$E\left[\sum_{k=1}^{i-1} S_{k,n-i+1} \mid \sum_{k=1}^{i-1} S_{k,n-i+1}>0\right] \leq (i-1)E\left[S_{k,n-i+1} \mid S_{k,n-i+1}>0\right]$$
$$=\frac{(i-1)\alpha\pi_{n-i+1}}{\beta\left(1-\left(\frac{\beta}{\beta+\pi_{n-i+1}}\right)^{\alpha}\right)}.$$

Substituting these results into the expression for $\Omega_{i,n-i+2}$ yields

$$\Omega_{i,n-i+2}>\frac{\alpha p_{n-i+2}}{\beta(\beta+\pi_{n-i+1})}\left(\frac{(\beta+\pi_{n-i+1})^{\alpha+1}-\beta^{\alpha+1}}{(\beta+\pi_{n-i+1})^{\alpha}-\beta^{\alpha}}\right)$$
$$=E\left[X_{k,n-i+2} \mid S_{k,n-i+1}>0\right]$$

from equation (10) of Proposition 4. The theorem is thus proved. □

5 Average Factors: Straight versus Weighted

There has been much discussion in actuarial circles regarding the merits of weighted average development factors as opposed to straight (unweighted) average development factors. In this section, it will be demonstrated that the straight average estimator,

$$\bar{X}_{i,n-i+2}=\frac{1}{i-1}\sum_{k=1}^{i-1} S_{i,n-i+1}\frac{X_{k,n-i+2}}{S_{k,n-i+1}},$$

cannot reduce or eliminate the bias seen in the weighted average estimator. For brevity, attention is restricted to the case of independent increments.

Before stating and proving the final theorem, one more proposition is needed:

Proposition 8. *For any set of positive m numbers $\{y_1, y_2, \ldots, y_m\}$,*

$$\sum_{k=1}^{m} \frac{1}{y_k} \geq \frac{m^2}{\sum_{k=1}^{m} y_k}.$$

Proof: Let Y denote the discrete random variable such that

$$\Pr[Y = y_k] = \frac{1}{m}, \qquad \text{for } k = 1, \ldots, m.$$

Applying Jensen's inequality yields

$$E\left[\frac{1}{Y}\right] = \frac{1}{m}\sum_{k=1}^{m} \frac{1}{y_k} \geq \frac{1}{E[Y]} = \frac{m}{\sum_{k=1}^{m} y_k}$$

and the proposition is proved. □

Theorem 3. *When both expectations are defined, the expected value of the straight average prediction is greater than the expected value of the weighted average prediction. That is,*

$$E[\bar{X}_{i,n-i+2} \mid S_{k,n-i+1} > 0,\ k = 1, \ldots, i] \geq E[\hat{X}_{i,n-i+2} \mid S_{k,n-i+1} > 0,\ k = 1, \ldots, i]. \quad (11)$$

Proof: Again making use of the independence and symmetry between the periods, proving equation (11) is equivalent to proving the following inequalities:

$$\frac{1}{(i-1)} \sum_{k=1}^{i-1} E\left[S_{i,n-i+1} \frac{X_{k,n-i+2}}{S_{k,n-i+1}} \mid S_{k,n-i+1} > 0\right]$$
$$\geq E\left[S_{i,n-i+1} \frac{\sum_{k=1}^{i-1} X_{k,n-i+2}}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid S_{1,n-i+1} > 0, \ldots, S_{i-1,n-i+1} > 0\right]$$

or

$$E\left[\sum_{k=1}^{i-1}\left[\frac{1}{S_{k,n-i+1}} \mid S_{k,n-i+1} > 0\right]\right]$$
$$\geq (i-1)^2 E\left[\frac{1}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid S_{1,n-i+1} > 0, \ldots, S_{i-1,n-i+1} > 0\right]$$

or

$$E\left[\sum_{k=1}^{i-1}\frac{1}{S_{k,n-i+1}} - \frac{(i-1)^2}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \mid S_{1,n-i+1} > 0, \ldots, S_{i-1,n-i+1} > 0\right] \geq 0.$$

But as $S_{k,n-i+1} > 0$, $k = 1, \ldots, i-1$,

$$\sum_{k=1}^{i-1}\frac{1}{S_{k,n-i+1}} - \frac{(i-1)^2}{\sum_{k=1}^{i-1} S_{k,n-i+1}} \geq 0$$

from Proposition 5 and, therefore, is the expectation of this quantity. This proves the theorem. □

6 Beyond Claim Counts: Possible Extensions of the Results

To this point, only claim count development data have been considered. As actuaries are concerned also with the development of the amounts of paid and reported claims, it is natural to ask whether these results can be generalized to include the analysis of claim amounts.

The general case based on the assumption of independent increments may be adapted to include specific examples of paid claim development. Medical malpractice indemnity payments exhibit special behavior to which the work presented above may be applied. Such a claim typically is closed either with no payment or with a single payment of a final award or settlement. By defining the variable $X_{i,j}$ to be the total amount paid for claims closing in period j (not the number of claims reported in a given period), assumptions 1 and 2 of Section 3 will be met in this simplest of situations.

One requires that the closure lags of claims are independent and that, once determined, the amount of a claim cannot change in subsequent periods. In reality, there are subtle problems with this. One must tabulate amounts for incidents rather than claims: a medical incident may generate several claims whose closure lags are not only interdependent, they are the same. Settlements could be paid as periodic payments which means that the amounts are actually paid over many periods violating the independence assumption. In practice, annuities may be purchased at the time of closure to fund the payments and limit the payment to a single period.

Allocated loss adjustment expense (ALAE) payments for the same medical malpractice business do not exhibit these properties although they are often combined with indemnity for the purposes of development analysis. Payments of ALAE are made incrementally from the time of the claim report to the time of its closure. Partial payments related to the same claim or incident will appear in different periods. Hence, the increments $X_{i,j}$ cannot be expected to be independent when ALAE is included.

Although a bias argument similar to the negative binomial case might be constructed, the interdependence of the payment increments is more complicated than the claim count increments. An essential part of the negative binomial example is being able to specify the nature of the interdependence. For ALAE payments, it is not clear what the nature of this interdependence would be.

For reported loss amounts, a similar problem arises. In this case, the value of the claim may include not only numerous partial payments but also changing estimates of the unpaid portion of the claim. The case reserve estimates are included in order to stabilize the development and bring the initial value of the claim as close as possible to its final value. This introduces the possibility of negative increments and serves only to complicate their interdependence.

Extending the results to development of claim amounts is difficult. Perhaps the most promising approach would be to consider particular models presented by Stanard in his original paper where the claim amount structure is specified.

7 Closing Comments

It is not the purpose of this paper to advocate one set of assumptions regarding the independence of report lags over another. If one believes that expected development increments are directly proportional to the accumulated total claims at a given point in time, then one might conclude that methods based on independent increment assumptions produce understated results.

It is, however, apparent that Stanard's simulation test of the development method produces the correct observation. If one believes that individual report lags are independent, then the loss development methods will produce overstated results. One thing that the analytical work presented here does not show is the magnitude of the bias. Stanard's work produced measures of that in specific cases. The key point is that there is a fundamental incompatibility between loss development techniques and methods relying on independent report lags.

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