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Module categories for group algebras over commutative rings

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Module categories for group algebras over commutative rings

with an appendix by Greg Stevenson

by

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Abstract

We develop a suitable version of the stable module category of a finite group G over an arbitrary commutative ring k. The purpose of the construction is to produce a compactly generated triangulated category whose compact objects are the finitely presented kG-modules. The main idea is to form a localisation of the usual version of the stable module category with respect to the filtered colimits of weakly injective modules. There is also an analogous version of the homotopy category of weakly injective kG-modules and a recollement relating the stable category, the homotopy category, and the derived category of kG-modules.

Key Words: Group algebra, weakly projective module, weakly injective module, stable category, homotopy category, derived category.

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Contents

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1. Introduction

The purpose of this paper is to develop a version of the stable module category of a finite group G over an arbitrary commutative ring k. We start with the category $\mathsf{Mod}(kG)$ whose objects are all kG-modules and whose morphisms are the module homomorphisms. If k were a field, the stable module category $\mathsf{StMod}(kG)$ would be formed by taking the same objects, but with morphisms

$$\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N)$$

where $\mathrm{PHom}_{kG}(M,N)$ is the linear subspace consisting of those morphisms that factor through some projective kG-module. This is then a triangulated category that is compactly generated, with compact objects the finitely generated kG-modules; see [12]. The crucial property that makes this work is that the group algebra kG is self injective, so projectives modules are the same as the injective modules.

Over an arbitrary commutative ring k however, projective modules are no longer injective, so one needs to adjust this construction. Instead, we use the fact that weakly projective modules are weakly injective, where "weakly" means "relative to the trivial subgroup". Thus we define $\mathrm{PHom}_{kG}(M,N)$ to be the morphisms that factor through a weakly projective module, and construct $\mathsf{StMod}(kG)$ as above. This is again a triangulated category, and the compact objects in it are the finitely presented kG-modules; we write $\mathsf{stmod}(kG)$ for the full subcategory of modules isomorphic in $\mathsf{StMod}(kG)$ to a finitely presented module.

This is not quite good enough to make a decent stable module category, as there are several problems with this construction:

– The category $\mathsf{StMod}(kG)$ is usually not compactly generated by $\mathsf{stmod}(kG)$.

– The functor $k_{(|G|)}\otimes_k -\colon \mathsf{StMod}(kG) \to \mathsf{StMod}(k_{(|G|)}G)$ is not an equivalence of categories, where $k_{(|G|)}$ denotes k localised by inverting all integers coprime to $|G|$.

The reason for both of these is that filtered colimits of weakly projective modules are not necessarily weakly projective; see Example 3.3.

To remedy these deficiencies, we form the Verdier quotient of $\mathsf{StMod}(kG)$ with respect to the localising subcategory of *cw-injectives*, namely filtered colimits of weakly injective modules. The principal features of this quotient category, which we denote $\mathsf{StMod}^{\mathrm{cw}}(kG)$, are collected in the result below.

Theorem 1.1 *The following statements hold:*

(1) The category $\mathsf{StMod}^{\mathrm{cw}}(kG)$ *is a compactly generated triangulated category.*

(2) The natural functor $\mathsf{stmod}(kG) \to \mathsf{StMod}^{\mathrm{cw}}(kG)$ *identifies* $\mathsf{stmod}(kG)$ *with the full subcategory consisting of the compact objects.*

(3) The functor $k_{(|G|)} \otimes -\colon \mathsf{StMod}^{\mathrm{cw}}(kG) \to \mathsf{StMod}^{\mathrm{cw}}(k_{(|G|)}G)$ *is an equivalence of triangulated categories.*

(4) An object in $\mathsf{StMod}^{\mathrm{cw}}(kG)$ *is zero if and only if its image in* $\mathsf{StMod}^{\mathrm{cw}}(k_{(p)}G)$ *is zero for each prime p dividing $|G|$.*

These results are proved in Sections 5 and 6, building on properties of weakly injective modules and their filtered colimits, presented in the preceding sections.

The equivalence in (3) above is deduced from general results on $\mathsf{StMod}^{\mathrm{cw}}(kG)$ in Section 6, concerning base change along a homomorphism $k \to k'$ of commutative rings, and leading to the following local-global principle:

Theorem 1.2 *A kG-module M is cw-injective if and only if $M_{\mathfrak{p}}$ is a cw-injective $k_{\mathfrak{p}}G$-module for every maximal ideal $\mathfrak{p} \subseteq k$.*

In Section 8 we develop a version of the homotopy category of weakly injective kG-modules, namely, we introduce a Verdier quotient $\mathsf{K}_{\mathrm{pur}}(\mathsf{WInj}\,kG)$ of the homotopy category of weakly injective kG-modules that has the following properties.

Theorem 1.3 *The category* $\mathsf{K}_{\mathrm{pur}}(\mathsf{WInj}\,kG)$ *is a compactly generated triangulated category and the compact objects form a full triangulated category that is equivalent to the bounded derived category* $\mathsf{D}^b(\mathsf{mod}\,kG)$ *of finitely presented kG-modules. Moreover, there is a recollement*

$$\mathsf{StMod}^{\mathrm{cw}}(kG) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} \mathsf{K}_{\mathrm{pur}}(\mathsf{WInj}\,kG) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} \mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}\,kG)$$

where $\mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}\,kG)$ *denotes the pure derived category of kG-modules.*

When k is a field, the thick tensor ideals of $\mathsf{stmod}(kG)$, and the localising tensor ideals of $\mathsf{StMod}(kG)$, have been classified, in terms of appropriate subsets of the Zariski spectrum of the cohomology ring $H^*(G,k)$; see [4, 5]. One of our motivations for undertaking the research reported here is to investigate possible extensions of these results to a general commutative ring k. The tensor product of kG-modules again induces a tensor triangulated structure on the stable module category. However, in Section 7 we produce an infinite chain of thick tensor ideals of $\mathsf{stmod}(\mathbb{Z}G)$ for any group G with at least two elements, and explain why this implies that the classification in terms of cohomological support [4] does not carry over verbatim to this context.

The same example shows that the spectrum of the tensor triangulated category $\mathsf{stmod}(\mathbb{Z}G)$ in the sense of Balmer [1] is disconnected. Again, this is in contrast to the case when the ring of coefficients is a field. A detailed explanation of this phenomenon can be found in Appendix A.

2. Weakly projective and weakly injective modules

In this section we recall basic results on relative projectives and relative injectives for group algebras over commutative rings, obtained by Gaschütz [10] and D. G. Higman [13], building on the work of Maschke and Schur. See also [3, §3.6]. The novelty, if any, is a systematic use of Quillen's [26] notion of exact categories.

Throughout this work G is a finite group, k a commutative ring of coefficients and kG the group algebra. The category of kG-modules is denoted $\mathsf{Mod}(kG)$, and $\mathsf{mod}(kG)$ is its full subcategory consisting of all finitely presented modules.

Restriction to the trivial subgroup

The functor $\mathsf{Mod}(kG) \to \mathsf{Mod}(k)$ that restricts a kG-module to the trivial subgroup has a left adjoint $kG \otimes_k -$ and a right adjoint $\operatorname{Hom}_k(kG,-)$. These functors are isomorphic, since G is finite; we identify them, and write $M{\uparrow}^G$ for $kG \otimes_k M$; the G action is given by $h(g \otimes m) = hg \otimes m$. Adjunction yields, for each kG-module M, natural morphisms of kG-modules

$$\begin{aligned} &\iota_M \colon M \longrightarrow M{\uparrow}^G &&\text{and}\quad \pi_M \colon M{\uparrow}^G \longrightarrow M \quad\text{where}\\ &\iota_M(m) = \sum_{g\in G} g \otimes g^{-1}m &&\text{and}\quad \pi_M(g\otimes m) = gm. \end{aligned}$$

Note that ι_M is a monomorphism whilst π_M is an epimorphism.

Weakly projective and weakly injective modules

An *exact category* in the sense of Quillen [26] (see also Appendix A of Keller [16]) is an additive category with a class of short exact sequences satisfying certain axioms. For example, every additive category is an exact category with respect to the *split exact structure*.

We give $\mathsf{Mod}(kG)$ the structure of an exact category with respect to the k-split short exact sequences. Said otherwise, we consider for $\mathsf{Mod}(k)$ the split exact structure, and declare that a sequence of kG-modules is *k-split exact* if it is split exact when restricted to the trivial subgroup.

For any kG-module M, the following sequences of kG-module are k-split exact:

$$0 \longrightarrow M \xrightarrow{\iota_M} M{\uparrow}^G \longrightarrow \mathrm{Coker}\,\iota_M \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Ker}\,\pi_M \longrightarrow M{\uparrow}^G \xrightarrow{\pi_M} M \longrightarrow 0.$$

Indeed, each element of $M{\uparrow}^G$ can be written uniquely as $\sum_{g\in G} g \otimes_k m_g$. The map assigning such an element to m_1, where 1 is the identity of G, is a k-splitting of ι_M, whilst the map $M \to M{\uparrow}^G$ assigning m to $1 \otimes_k m$ is a k-splitting of π_M.

Definition 2.1 A kG-module P is said to be *weakly projective* if $\mathrm{Hom}_{kG}(P,-)$ preserves exactness of k-split exact sequences; equivalently, given a k-split exact sequence $0 \to L \to M \to N \to 0$ of kG-modules, any morphism $P \to N$ lifts to M.

In the same vein, a kG-module P is *weakly injective* if $\mathrm{Hom}_{kG}(-,P)$ preserves exactness of k-split exact sequences; equivalently, given a k-split exact sequence $0 \to L \to M \to N \to 0$ of kG-modules, any morphism $L \to P$ extends to M.

The trace map

We recall a characterisation of weakly projective and injective kG-modules in terms of the trace map.

Definition 2.2 If M and N are kG-modules, the *trace map*

$$\mathrm{Tr}_G \colon \mathrm{Hom}_k(M,N) \to \mathrm{Hom}_{kG}(M,N)$$

is defined by

$$\mathrm{Tr}_G(\theta)(m) = \sum_{g\in G} g(\theta(g^{-1}m)) \quad \text{for } m \in M.$$

A straightforward calculation justifies the following properties of the trace map.

Lemma 2.3 *Fix a homomorphism $\theta \in \mathrm{Hom}_k(M,N)$.*

(1) If $\alpha\colon M' \to M$ is a morphism of kG-modules, then $\mathrm{Tr}_G(\theta\alpha) = \mathrm{Tr}_G(\theta)\alpha$.

(2) If $\beta\colon N \to N'$ is a morphism of kG-modules, then $\mathrm{Tr}_G(\beta\theta) = \beta\,\mathrm{Tr}_G(\theta)$.

For any kG-module M, let $\theta_M \in \operatorname{End}_k(M{\uparrow}^G)$ be the map defined by

$$\theta_M(g \otimes m) = \begin{cases} 1 \otimes m & g = 1 \\ 0 & g \neq 1. \end{cases}$$

Lemma 2.4 *The map* $\operatorname{Tr}_G(\theta_M)$ *is the identity morphism on* $M{\uparrow}^G$.

Theorem 2.5 *Let* $\rho\colon M \to N$ *be a morphism of* kG*-modules. The following conditions are equivalent:*

(1) In $\mathsf{Mod}(kG)$*, the map* ρ *factors through the surjection* $\pi_N : N{\uparrow}^G \to N$.

(2) In $\mathsf{Mod}(kG)$*, the map* ρ *factors through the injection* $\iota_M : M \to M{\uparrow}^G$.

(3) There exists $\theta \in \operatorname{Hom}_k(M,N)$ *such that* $\operatorname{Tr}_G(\theta) = \rho$.

Proof: (1) $\Rightarrow$ (3): If there exists a homomorphism $\rho' : M \to N{\uparrow}^G$ of kG-modules with $\rho = \pi_N\rho'$ then $\rho = \operatorname{Tr}_G(\pi_N\theta_N\rho')$, by Lemmas 2.3 and 2.4.

(2) $\Rightarrow$ (3) follows as above.

(3) $\Rightarrow$ (1) and (2): Suppose there exists $\theta \in \operatorname{Hom}_k(M,N)$ with $\operatorname{Tr}_G(\theta) = \rho$. The map $\rho' \in \operatorname{Hom}_{kG}(M, N{\uparrow}^G)$ with $\rho'(m) = \sum_{g\in G} g \otimes \theta(g^{-1}m)$ satisfies $\pi_N\rho' = \rho$, and $\rho'' \in \operatorname{Hom}_{kG}(M{\uparrow}^G, N)$ with $\rho''(g \otimes m) = g\theta(m)$ satisfies $\rho''\iota_M = \rho$. $\square$

Criteria for weak projectivity and weak injectivity

The result below is due to D. G. Higman [13]. Part (5) is called *Higman's criterion* for weak projectivity.

Theorem 2.6 *The following conditions on a* kG*-module* P *are equivalent:*

(1) The module P *is weakly projective.*

(2) The natural morphism $\pi_P : P{\uparrow}^G \to P$ *is a split surjection.*

(3) The module P *is weakly injective.*

(4) The natural morphism $\iota_P : P \to P{\uparrow}^G$ *is a split injection.*

(5) There exists $\theta \in \operatorname{End}_k(P)$ *such that* $\operatorname{Tr}_G(\theta) = \mathrm{id}_P$.

(6) There exists a kG*-module* M *such that* P *is a direct summand of* $M{\uparrow}^G$.

In particular, $M{\uparrow}^G$ *is weakly projective and weakly injective for any* kG*-module* M.

Proof: We first prove that conditions (1), (2), and (5) are equivalent; analogous arguments settle the equivalence of (3), (4) and (5).

(1) $\Rightarrow$ (2): If P is weakly projective then the identity morphism of P lifts to a splitting of $P\!\uparrow^G \to P$.

(2) $\Rightarrow$ (5): If $\alpha\colon P \to P\!\uparrow^G$ is a splitting then by Lemma 2.3 we have

$$\mathrm{Tr}_G(\pi_P \theta_{P\uparrow^G} \alpha) = \pi_P \mathrm{Tr}_G(\theta_{P\uparrow^G})\alpha = \pi_P \alpha = \mathrm{id}_P\,.$$

(5) $\Rightarrow$ (1): Let $\theta \in \mathrm{End}_k(P)$ with $\mathrm{Tr}_G(\theta) = \mathrm{id}_P$. Given a k-split exact sequence $0 \to L \to M \to N \to 0$ and a morphism $\alpha\colon P \to N$ of kG-modules, choose a k-lift $\widetilde{\alpha}\colon P \to M$ of α; it follows from Lemma 2.3 that $\mathrm{Tr}_G(\widetilde{\alpha}\theta)$ is a kG-lift of α.

(4) $\Rightarrow$ (6): Take $M = P$.

(6) $\Rightarrow$ (5): A module of the form $M\!\uparrow^G$ satisfies (5) by Lemma 2.4. It remains to observe that this property descends to direct summands, by Lemma 2.3. $\square$

The theorem above shows that the exact category $\mathsf{Mod}(kG)$ is a *Frobenius category*. This means that there are enough projective objects and enough injective objects, and that both coincide; see Section I.2 of Happel [12] for details.

When k is a field, weak injectivity is equivalent to injectivity. The following corollary of Theorem 2.6 is thus a generalisation of Maschke's theorem.

Corollary 2.7 *Let k be a commutative ring and G a finite group. The following conditions are equivalent:*

(1) Each kG-module is weakly injective.

(2) The trivial kG-module k is weakly injective.

(3) The order of G is invertible in k.

Proof: Both non-trivial implications follow from Higman's criterion: (2) $\Rightarrow$ (3) because $\mathrm{Tr}_G(\theta) = |G|\theta$ for any $\theta \in \mathrm{End}_k(k) \cong k$, whilst (3) $\Rightarrow$ (1) because $\mathrm{Tr}_G(\frac{1}{|G|}\mathrm{id}_P) = \mathrm{id}_P$ for any kG-module P. $\square$

3. An example

In this section, we construct modules that are filtered colimits of weakly injective kG-modules but are not themselves weakly injective. In what follows, given an integer n, we write $k_{(n)}$ for the localisation of k that inverts all integers r such that $(r,n) = 1$. For any kG-module M, we denote $M_{(n)}$ the kG-module $k_{(n)} \otimes_k M$; this can also be described as a filtered colimit:

$$M_{(n)} = \mathrm{colim}_{(r,n)=1}(M \xrightarrow{r} M)\,.$$

Definition 3.1 Let k be a commutative ring, G a finite group, and n an integer. For any kG-module M define $\varGamma_n M$ by forming the following exact sequence

$$0 \longrightarrow M \longrightarrow M{\uparrow}^G \oplus M_{(n)} \longrightarrow \varGamma_n M \longrightarrow 0 \tag{3.1}$$

where the monomorphism sends $m \in M$ to $(\iota_M(m), m)$. This sequence is k-split exact, since ι_M is k-split.

Theorem 3.2 *If $|G|$ divides n, the kG-module $\varGamma_n M$ is a filtered colimit of weakly injective kG-modules.*

Proof: It is expedient to use the stable category, $\mathsf{StMod}(kG)$, defined in Section 5.

The kG-module $\varGamma_n M$ is given by the following pushout

$$\begin{tikzcd} M \arrow[r, "\iota_M"] \arrow[d] & M{\uparrow}^G \arrow[d] \\ M_{(n)} \arrow[r] & \varGamma_n M \end{tikzcd}$$

and this is a filtered colimit of pushouts of the form

$$\begin{tikzcd} M \arrow[r, "\iota_M"] \arrow[d, "r"] & M{\uparrow}^G \arrow[d] \\ M \arrow[r] & M[r] \end{tikzcd}$$

with $(r,n) = 1$. In particular, $\varGamma_n M$ is a filtered colimit of the $M[r]$. The hypothesis is that $|G|$ divides n, so one gets

$$\mathrm{Tr}_G\Big(\frac{n}{|G|}\mathrm{id}_M\Big) = \frac{n}{|G|}\mathrm{Tr}_G(\mathrm{id}_M) = \frac{n}{|G|}(|G|\mathrm{id}_M) = n\,\mathrm{id}_M\,.$$

Therefore the multiplication morphism $M \xrightarrow{n} M$ factors through a weakly injective module, by Theorem 2.5. For each integer r with $(r,n) = 1$, there exist integers a and b such that $na + rb = 1$, so the preceding computation implies that the morphism $M \xrightarrow{r} M$ is an isomorphism in the stable category $\mathsf{StMod}(kG)$, with inverse the multiplication by b. It follows that the morphism on the right in the diagram is also an isomorphism. Therefore each $M[r]$ is weakly injective. □

The kG-module $\varGamma_n M$ need not be itself weakly injective.

Example 3.3 Let G be a finite group and set $k = \mathbb{Z}$. Given $n > 1$, the $\mathbb{Z}G$-module $\varGamma_n\mathbb{Z}$ constructed above is weakly injective if and only if $|G| = 1$.

To prove this, first observe that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{(n)},\mathbb{Z}) = 0$ so $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}_{(n)},\mathbb{Z}) = 0$ as well. If $\Gamma_n\mathbb{Z}$ is weakly injective, then the sequence (3.1) splits and $\iota_{\mathbb{Z}}$ is a split monomorphism, since the second component of the inverse of $\mathbb{Z} \to \mathbb{Z}{\uparrow}^G \oplus \mathbb{Z}_{(n)}$ is zero. It follows that $\mathbb{Z}$ is weakly injective, by Theorem 2.6, and therefore $|G|$ is invertible in $\mathbb{Z}$, by Corollary 2.7. Thus $|G| = 1$. The other direction is clear.

4. Filtered colimits of weakly injective modules

Motivated by the example in the previous section we study filtered colimits of weakly injective modules.

Definition 4.1 We say that a kG-module is *cw-injective* if it is isomorphic in $\mathsf{Mod}(kG)$ to a filtered **c**olimit of **w**eakly injective modules.

Weakly injective modules are cw-injective, but not conversely; see Example 3.3. However, these notions coincide for finitely presented modules, as is proved in part (2) of the next result.

Lemma 4.2 *Let M be a kG-module.*

(1) The module M is cw-injective if and only if every morphism $N \to M$ with N finitely presented factors through a weakly injective module.

(2) If M is finitely presented and cw-injective, then it is weakly injective.

Proof: (1) If $M = \mathrm{colim}\, M_\alpha$, a filtered colimit of weakly injective modules M_α, then each morphism $N \to M$ with N finitely presented factors through one of the structure morphisms $M_\alpha \to M$.

As to the converse, one has that $M = \mathrm{colim}_{N \to M} N$, with the colimit taken over the category of morphisms $N \to M$ where N runs through all finitely presented kG-modules. Suppose that each morphism $N \to M$ factors through a weakly injective module. Thus each morphism $N \to M$ factors through $N{\uparrow}^G$, by Theorem 2.5. The module $N{\uparrow}^G$ is finitely presented and it follows that the morphisms $N \to M$ with N weakly injective form a cofinal subcategory. Thus M is a filtered colimit of weakly injective kG-modules.

(2) Let M be a finitely presented kG-module. If M is cw-injective, then (1) implies that the identity morphism id_M factors through a weakly injective module. Thus M is weakly injective, by Theorem 2.6. □

Next we present criteria for cw-injectivity in terms of the purity of certain canonical morphisms. For background on purity, see the books of Jensen and Lenzing [15] and Prest [24, 25].

Definition 4.3 An exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of kG-modules is said to be *pure exact* if for every right kG-module N, the sequence

$$0 \to N \otimes_{kG} M' \to N \otimes_{kG} M \to N \otimes_{kG} M'' \to 0$$

is exact. This is equivalent to the statement that for any finitely presented kG-module P, each morphism $P \to M''$ lifts to a morphism $P \to M$, and to the statement that the sequence is a filtered colimit of split exact sequences.

A morphism $M \to M''$ appearing in a pure exact sequence as above is called a *pure epimorphism*, whilst $M' \to M$ is called a *pure monomorphism*.

Criteria for cw-injectivity

The following tests for cw-injectivity are analogues of the criterion for weak injectivity formulated in Theorem 2.6.

Theorem 4.4 *The following properties of a kG-module M are equivalent:*

(1) The module M is cw-injective.

(2) The natural morphism $\pi_M : M{\uparrow}^G \to M$ is a pure epimorphism.

(3) The natural morphism $\iota_M : M \to M{\uparrow}^G$ is a pure monomorphism.

Proof: (1) $\Rightarrow$ (2) and (3): If M is a filtered colimit of weakly injective modules M_α, then $M{\uparrow}^G \to M$ is a filtered colimit of split epimorphisms $M_\alpha{\uparrow}^G \to M_\alpha$ and hence is a pure epimorphism. Similarly, $M \to M{\uparrow}^G$ is a filtered colimit of split monomorphisms $M_\alpha \to M_\alpha{\uparrow}^G$ and hence is a pure monomorphism.

(2) $\Rightarrow$ (1): When $M{\uparrow}^G \to M$ is a pure epimorphism any morphism $N \to M$ with N finitely presented lifts to $M{\uparrow}^G$, so Lemma 4.2 yields that M is cw-injective.

(3) $\Rightarrow$ (1): Let $N \to M$ be a morphism of kG-modules with N finitely presented. We obtain the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & N{\uparrow}^G \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & M{\uparrow}^G \arrow[r] & M' \arrow[r] & 0.
\end{tikzcd}$$

When $M \to M{\uparrow}^G$ is a pure monomorphism, $M{\uparrow}^G \to M'$ is a pure epimorphism, so the morphism $N' \to M'$ lifts to $M{\uparrow}^G$; note that N' is also finitely presented. Thus the morphism $N \to M$ extends to $N{\uparrow}^G$, so M is cw-injective, by Lemma 4.2. □

Corollary 4.5 *The class of cw-injective modules is closed under*

(1) filtered colimits,

(2) products, and

(3) pure submodules and pure quotient modules.

Proof: We use the characterisation from Lemma 4.2 to verify (1) and (2).

(1) Let $M = \operatorname{colim} M_\alpha$ be a filtered colimit of cw-injective modules. Each morphism $N \to M$ with N finitely presented factors through one of the structure morphisms $M_\alpha \to M$. The morphism $N \to M_\alpha$ then factors through a weakly injective module. Thus M is cw-injective.

(2) Let $M = \prod_\alpha M_\alpha$ be a product of cw-injective modules and fix a finitely presented module N. Each morphism $N \to M_\alpha$ factors through a weakly injective module and therefore through the natural morphism $N \to N{\uparrow}^G$. Thus each morphism $N \to M$ factors through $N{\uparrow}^G$, and it follows that M is cw-injective.

(3) Let $M \to N$ be a pure monomorphism with N cw-injective. An application of Theorem 4.4 yields that the composite $M \to N \to N{\uparrow}^G$ is a pure monomorphism; it factors through the natural monomorphism $M \to M{\uparrow}^G$, so the latter is pure as well. Another application of Theorem 4.4 shows that M is cw-injective.

The argument for a pure quotient module is analogous. □

5. Stable module categories

In this section we introduce the stable module category $\mathsf{StMod}(kG)$ and an appropriate variant which we denote by $\mathsf{StMod}^{\mathrm{cw}}(kG)$. We begin with a brief discussion of compactly generated triangulated categories.

Compactly generated triangulated categories

Let T be a triangulated category with suspension $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$. For a class C of objects we define its *perpendicular categories* to be the full subcategories

$$\mathsf{C}^\perp = \{Y \in \mathsf{T} \mid \operatorname{Hom}_{\mathsf{T}}(\Sigma^n X, Y) = 0 \text{ for all } X \in \mathsf{C}, n \in \mathbb{Z}\}$$
$$^\perp\mathsf{C} = \{X \in \mathsf{T} \mid \operatorname{Hom}_{\mathsf{T}}(X, \Sigma^n Y) = 0 \text{ for all } Y \in \mathsf{C}, n \in \mathbb{Z}\}.$$

We write $\mathsf{Thick}(\mathsf{C})$ and $\mathsf{Loc}(\mathsf{C})$ for the thick subcategory and the localising subcategory, respectively, of T generated by C.

Suppose that T has set-indexed coproducts. An object $X \in \mathsf{T}$ is *compact* if the representable functor $\mathrm{Hom}_{\mathsf{T}}(X,-)$ preserves set-indexed coproducts. The category T is *compactly generated* if there is a set C of compact objects such that T admits no proper localising subcategory containing C.

The following proposition provides a useful method to construct compactly generated triangulated categories and to identify its subcategory of compact objects. As usual, T^{c} denotes the full subcategory of compact objects in T.

Proposition 5.1 *Let* T *be a triangulated category with set-indexed coproducts,* C *a set of compact objects in* T*, and set* $\mathsf{V} = \mathsf{C}^{\perp}$*. Then the composite*

$$\mathsf{Loc}(\mathsf{C}) \rightarrowtail \mathsf{T} \twoheadrightarrow \mathsf{T}/\mathsf{V}$$

is an equivalence of triangulated categories. Moreover, the category T/V *is compactly generated and the equivalence identifies* $\mathsf{Thick}(\mathsf{C})$ *with* $(\mathsf{T}/\mathsf{V})^{\mathsf{c}}$.

Proof: Set $\mathsf{U} = \mathsf{Loc}(\mathsf{C})$. Observe that since C consists of compact objects, the Verdier quotient T/U does exist, in that the morphisms between a given pair of objects form a set; see [23, Proposition 9.1.19]. It follows that the Verdier quotient T/V exists, since $\mathsf{U}^{\perp} = \mathsf{V}$. Moreover, the composite $\mathsf{U} \rightarrowtail \mathsf{T} \twoheadrightarrow \mathsf{T}/\mathsf{V}$ is an equivalence, by [23, Theorem 9.1.16], with the warning that there $\mathsf{U}^{\perp}$ is denoted ${}^{\perp}\mathsf{U}$. The category U is compactly generated by construction. The description of the compacts follows from standard properties of compact objects; see Lemma 2.2 in [22]. □

Corollary 5.2 *If* $\mathsf{T}' \subseteq \mathsf{T}$ *is a localising subcategory containing* $\mathsf{Loc}(\mathsf{C})$*, then the natural functor*

$$\mathsf{T}'/(\mathsf{V} \cap \mathsf{T}') \longrightarrow \mathsf{T}/\mathsf{V}$$

is an equivalence of triangulated categories.

Proof: This follows from Lemma 5.3 below, since the functor induces an equivalence between the categories of compact objects. □

The following test for equivalence of triangulated categories is implicit in [17, §4.2]; for details see [5, Lemma 4.5].

Lemma 5.3 *Let* $F\colon \mathsf{S} \to \mathsf{T}$ *be an exact functor between compactly generated triangulated categories and suppose* F *preserves set-indexed coproducts. If* F *restricts to an equivalence* $\mathsf{S}^c \xrightarrow{\sim} \mathsf{T}^c$*, then* F *is an equivalence of categories.*

The stable module category

The following lemma describes the morphisms that are annihilated when one passes to the stable module category; it is an immediate consequence of Theorems 2.5 and 2.6.

Lemma 5.4 *For a morphism $M \to N$ of kG-modules, the following conditions are equivalent:*

(1) The morphism factors through some weakly projective module.

(2) The morphism factors through the natural epimorphism $N{\uparrow}^G \to N$.

(3) The morphism factors through some weakly injective module.

(4) The morphism factors through the natural monomorphism $M \to M{\uparrow}^G$.

The *stable module category* $\mathsf{StMod}(kG)$ has the same objects as $\mathsf{Mod}(kG)$, but the morphisms are given by

$$\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N)$$

where $\mathrm{PHom}_{kG}(M,N)$ is the k-submodule of $\mathrm{Hom}_{kG}(M,N)$ consisting of morphisms $M \to N$ satisfying the equivalent conditions of Lemma 5.4. As $\mathsf{Mod}(kG)$ is a Frobenius category, $\mathsf{StMod}(kG)$ is a triangulated category; see [12, Section I.2]. The exact triangles in it are induced by the k-split exact sequences of kG-modules.

Let $\mathsf{stmod}(kG)$ be the full subcategory of $\mathsf{StMod}(kG)$ consisting of objects isomorphic to finitely presented kG-modules. This is a thick subcategory consisting of compact objects. The result below is an immediate consequence of Lemma 4.2.

Lemma 5.5 *In $\mathsf{StMod}(kG)$, the perpendicular category $(\mathsf{stmod}\,kG)^{\perp}$ coincides with the localising subcategory whose objects are the cw-injective kG-modules.*

Set $\mathsf{V} = (\mathsf{stmod}\,kG)^{\perp}$ and modify the stable module category by defining

$$\mathsf{StMod}^{\mathrm{cw}}(kG) = \mathsf{StMod}(kG)/\mathsf{V}.$$

This Verdier quotient exists by Proposition 5.1, since $\mathsf{stmod}(kG)$ consists of compact objects. Given Lemma 5.5, the following result is a special case of Proposition 5.1.

Theorem 5.6 *The composite functor*

$$\mathsf{Loc}(\mathsf{stmod}\,kG) \rightarrowtail \mathsf{StMod}(kG) \twoheadrightarrow \mathsf{StMod}^{\mathrm{cw}}(kG)$$

is an equivalence of triangulated categories. The category $\mathsf{StMod}^{\mathrm{cw}}(kG)$ is compactly generated and the equivalence identifies $\mathsf{stmod}(kG)$ with the full subcategory consisting of all compacts objects of $\mathsf{StMod}^{\mathrm{cw}}(kG)$.

For later use, we record a remark about functors between exact categories.

Lemma 5.7 *Let $F \colon \mathsf{C} \to \mathsf{D}$ be a functor between exact categories. If F admits a right adjoint that is exact, then F preserves projectivity; if it admits a left adjoint that is exact, then F preserves injectivity.*

Proof: Suppose F has a right adjoint, say F', that is exact. For any projective object X of C, it follows from the isomorphism

$$\mathrm{Hom}_{\mathsf{D}}(F(X),-) \cong \mathrm{Hom}_{\mathsf{C}}(X,F'(-))$$

that $\mathrm{Hom}_{\mathsf{D}}(F(X),-)$ maps an epimorphism in D to a surjective map, that is to say, that $F(X)$ is a projective object of D.

The argument for the other case is exactly analogous. □

The tensor product

The category of kG-modules carries a tensor product; it is the tensor product over k with diagonal G-action. Note that this tensor product is exact in each variable with respect to the k-split exact structure.

Lemma 5.8 *Let M,N be kG-modules.*

(1) If M is weakly injective, then $M \otimes_k N$ is weakly injective.

(2) If M is cw-injective, then $M \otimes_k N$ is cw-injective.

Proof: The functor $- \otimes_k N$ has $\mathrm{Hom}_k(N,-)$ as a right adjoint that is exact with respect to the k-split exact structure. Thus, Lemma 5.7 implies that $- \otimes_k N$ preserves weak projectivity, which coincides with weak injectivity. The functor $- \otimes_k N$ preserves filtered colimits, and therefore cw-injectivity as well. □

It follows from the lemma that the tensor product on $\mathsf{Mod}(kG)$ passes to a tensor product on $\mathsf{StMod}(kG)$; this is compatible with the triangulated structure, so that $\mathsf{stmod}(kG)$ and $\mathsf{StMod}(kG)$ are tensor triangulated categories. The cw-injective kG-modules form a tensor ideal in $\mathsf{StMod}(kG)$ and hence $\mathsf{StMod}^{\mathrm{cw}}(kG)$ inherits a tensor triangulated structure.

6. Base change

In this section we prove a number of results that track the behaviour of the stable module category $\mathsf{StMod}^{\mathrm{cw}}(kG)$ under changes of the coefficient ring k. We remind the reader that $\mathsf{Mod}(kG)$ is the category of kG-modules with exact structure defined by the k-split short exact sequences.

Let $\phi\colon k \to k'$ be a homomorphism of commutative rings. Assigning the element $\sum \alpha_g g$ to $\sum \phi(\alpha_g) g$ is then a homomorphism of rings $kG \to k'G$. We consider the corresponding *restriction functor* $\mathsf{Mod}(k'G) \to \mathsf{Mod}(kG)$ and the *base change functor* $k' \otimes_k -\colon \mathsf{Mod}(kG) \to \mathsf{Mod}(k'G)$.

Exact structure

Base change and restriction are compatible with the exact structure for modules over kG and $k'G$ respectively.

Lemma 6.1 *Let $\phi\colon k \to k'$ be a ring homomorphism.*

(1) Restriction takes k'-split exact sequences of $k'G$-modules to k-split exact sequences of kG-modules.

(2) Base change takes k-split exact sequences of kG-modules to k'-split exact sequences of $k'G$-modules.

(3) Restriction and base change preserve weak injectivity and cw-injectivity.

Proof: Parts (1) and (2) are clear. As to (3), it follows from Lemma 5.7 that restriction and base change preserve weak injectivity, because the former is right adjoint to the latter, and they are both exact functors; one uses again the property that weak injectivity and weak projectivity coincide. Since both functors also preserve filtered colimits, they preserve cw-injectivity as well. $\square$

Localisation at $|G|$

Recall that for any kG-module M and integer n we write $M_{(n)}$ for the localisation of M inverting all integers coprime to n.

Theorem 6.2 *For any integer n that is divisible by $|G|$ the localisation functor*

$$(-)_{(n)}\colon \mathsf{StMod}^{\mathrm{cw}}(kG) \longrightarrow \mathsf{StMod}^{\mathrm{cw}}(k_{(n)}G)$$

is an equivalence of triangulated categories, and induces an equivalence

$$(-)_{(n)}\colon \mathsf{stmod}(kG) \longrightarrow \mathsf{stmod}(k_{(n)}G).$$

Proof: To begin with note that, for any n, localisation does induce an exact functor of triangulated categories, by Lemma 6.1. It suffices to prove that this is an equivalence on $\mathsf{StMod}^{\mathrm{cw}}(kG)$; the second equivalence follows by restriction to compact objects; see Theorem 5.6.

Consider the restriction functor

$$F\colon \mathsf{StMod}^{\mathrm{cw}}(k_{(n)}G) \longrightarrow \mathsf{StMod}^{\mathrm{cw}}(kG).$$

The composite $F(-)_{(n)}$ is the identity. As to the other composite, for each kG-module M, the morphism in Definition 3.1 induces an exact triangle

$$M \longrightarrow F(M_{(n)}) \longrightarrow \varGamma_n M \longrightarrow$$

in $\mathsf{StMod}(kG)$, as $M{\uparrow}^G$ is zero there. By Theorem 3.2, the kG-module $\varGamma_n M$ is cw-injective, so the first morphism in the triangle defines a natural isomorphism in $\mathsf{StMod}^{\mathrm{cw}}(kG)$ from the identity functor to $F((-)_{(n)})$. Thus $(-)_{(n)}$ is an equivalence of categories with inverse F. □

Remark 6.3 As Example 3.3 shows, $\varGamma_n M$ need not be weakly injective, so the localisation functor

$$(-)_{(n)} \colon \mathsf{StMod}(kG) \longrightarrow \mathsf{StMod}(k_{(n)}G)$$

is not necessarily an equivalence of categories.

Concerning Lemma 6.1, it is easy to construct examples of kG-modules that are weakly injective after base change along a homomorphism $k \to k'$, but are not themselves weakly injective. We now focus on certain classes of homomorphisms where such a phenomenon cannot occur.

Pure monomorphisms

The notion of a pure monomorphism has been recalled in Definition 4.3.

Lemma 6.4 *Let $\phi\colon k \to k'$ be a homomorphism of rings that is a pure monomorphism of k-modules, and let M be a kG-module. If $k' \otimes_k M$ is cw-injective as a kG-module, then so is M.*

Proof: The canonical morphism $\phi \otimes_k M \colon M \to k' \otimes_k M$ is a pure monomorphism of kG-modules, since for any right kG-module N we have

$$N \otimes_{kG} (\phi \otimes_k M) \cong \phi \otimes_k (N \otimes_{kG} M)$$

where the second is a monomorphism, by the hypothesis on ϕ. Thus the desired statement follows from the fact that a pure submodule of a cw-injective module is cw-injective, by Corollary 4.5. □

Proposition 6.5 *Let $\{k \to k_i\}_{i\in I}$ be homomorphisms of rings such that the induced map $X \to \prod_i (k_i \otimes_k X)$ is a monomorphism for any k-module X. If M is a kG-module with the property that $k_i \otimes_k M$ is cw-injective as kG-module for each $i \in I$, then M is cw-injective.*

Proof: We claim that for any kG-module M, the natural morphism

$$\eta_M \colon M \longrightarrow \prod_{i\in I} (k_i \otimes_k M)$$

is a pure monomorphism of kG-modules. Indeed, for any right kG-module N the morphism $\eta_{(N\otimes_{kG}M)}$ factors as

$$N \otimes_{kG} M \xrightarrow{N\otimes_{kG}\eta_M} N \otimes_{kG} \prod_{i\in I}(k_i \otimes_k M) \longrightarrow \prod_{i\in I}\big(k_i \otimes_k (N \otimes_{kG} M)\big)$$

where the morphism on the right is the canonical one. Since $\eta_{(N\otimes_{kG}M)}$ is a monomorphism, by hypothesis, so is $N \otimes_{kG} \eta_M$. This settles the claim.

Now if each $k_i \otimes_k M$ is cw-injective as a kG-module, then so is their product and hence also its pure submodule M, by Corollary 4.5. □

A local-global principle

From Proposition 6.5 we obtain the following local-to-global type results for detecting cw-injectivity. As usual, the localisation of a k-module M at a prime ideal $\mathfrak{p}$ of k is denoted $M_{\mathfrak{p}}$.

Corollary 6.6 *A kG-module M is cw-injective if and only if $M_{\mathfrak{p}}$ is a cw-injective $k_{\mathfrak{p}}G$-module for every maximal ideal $\mathfrak{p} \subseteq k$.*

Proof: If M is cw-injective, so is each $M_{\mathfrak{p}}$ by Lemma 6.1. The converse is a special case of Proposition 6.5, since the canonical map $X \to \prod_{\mathfrak{p}} X_{\mathfrak{p}}$ is a monomorphism for each k-module X; see [20, Theorem 4.6]. □

The next result adds to Theorem 6.2.

Corollary 6.7 *Fix an integer n divisible by $|G|$, and let M be a kG-module. The following conditions are equivalent:*

(1) The kG-module M is cw-injective.

(2) The $k_{(n)}G$-module $M_{(n)}$ is cw-injective.

(3) The $k_{(p)}G$-module $M_{(p)}$ is cw-injective for every prime p dividing n.

Proof: (1) $\Leftrightarrow$ (2): This follows from Theorem 6.2.

(2) $\Rightarrow$ (3): This is by Lemma 6.1, applied to the homomorphism $k_{(n)} \to k_{(p)}$.

(3) $\Rightarrow$ (2): We apply Proposition 6.5 to the family $k_{(n)} \to k_{(p)}$, as p ranges over the primes dividing n. To this end, let X be a $k_{(n)}$-module, and note that there is a natural isomorphism

$$k_{(p)} \otimes_{k_{(n)}} X \xrightarrow{\sim} \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(n)}} X$$

with X viewed as a $\mathbb{Z}_{(n)}$-module via restriction. Since the ideals (p) in $\mathbb{Z}_{(n)}$, where p divides n, is a complete list of maximal ideals of $\mathbb{Z}_{(n)}$, it follows from [20, Theorem 4.6] that the induced map $X \to \prod_p X_{(p)}$ is a monomorphism, as desired. □

Epimorphisms

Recall that a homomorphism $R \to S$ of rings is an *epimorphism* (in the category of rings) if and only if the canonical morphism $S \otimes_R M \to M$ is an isomorphism for each S-module M; equivalently, if the multiplication map $S \otimes_R S \to S$ is an isomorphism. This means that the restriction functor $\mathsf{Mod}(S) \to \mathsf{Mod}(R)$ is fully faithful. For example, each localisation $R \to R[\Sigma^{-1}]$ inverting the elements of a subset $\Sigma \subseteq R$ is an epimorphism.

Lemma 6.8 *If $\phi\colon k \to k'$ is a ring epimorphism, then so is the induced homomorphism $kG \to k'G$.*

Proof: If ϕ is an epimorphism, then the induced morphism $k' \otimes_k k' \to k'$ is an isomorphism. It follows that the induced morphism $k'G \otimes_{kG} k'G \to k'G$ is an isomorphism. Thus $kG \to k'G$ is a ring epimorphism. □

Proposition 6.9 *Let $k \to k'$ be a ring epimorphism and set*

$$\mathsf{U} = \{M \in \mathsf{StMod}^{\mathrm{cw}}(kG) \mid k' \otimes_k M = 0\}$$
$$\mathsf{V} = \{M \in \mathsf{StMod}^{\mathrm{cw}}(kG) \mid M \xrightarrow{\sim} k' \otimes_k M\}.$$

Then base change induces an equivalence of triangulated categories

$$\mathsf{StMod}^{\mathrm{cw}}(kG)/\mathsf{U} \xrightarrow{\sim} \mathsf{StMod}^{\mathrm{cw}}(k'G)$$

while restriction induces an equivalence of triangulated categories

$$\mathsf{StMod}^{\mathrm{cw}}(k'G) \xrightarrow{\sim} \mathsf{V}.$$

Proof: Restriction followed by base change is naturally isomorphic to the identity. By general principles, this implies that restriction is fully faithful, while base change is up to an equivalence a quotient functor; see Proposition 1.3 of Chap. I in [9]. □

Remark 6.10 Let $k \to k'$ be an epimorphism such that k' is finitely presented as a k-module, and set

$$\mathsf{C} = \{M \in \mathsf{stmod}(kG) \mid k' \otimes_k M = 0\}$$
$$\mathsf{D} = \{M \in \mathsf{stmod}(kG) \mid M \xrightarrow{\sim} k' \otimes_k M\}.$$

Then C and D form thick tensor ideals of $\mathsf{stmod}(kG)$ such that the inclusion of C admits a right adjoint while the inclusion of D admits a left adjoint. Moreover, $\mathsf{C}^{\perp} = \mathsf{D}$ and $\mathsf{C} = {}^{\perp}\mathsf{D}$. Thus the inclusions of C and D and their adjoints form a localisation sequence

$$\mathsf{C} \rightleftarrows \mathsf{stmod}(kG) \rightleftarrows \mathsf{D}.$$

In this situation the Balmer spectrum of $\mathsf{stmod}(kG)$ is disconnected, provided that $\mathsf{C} \neq 0 \neq \mathsf{D}$. Recall from [1] that for any essentially small tensor triangulated category T there is an associated spectral topological space $\operatorname{Spc}\mathsf{T}$ which provides a universal notion of tensor-compatible supports for objects of T. The above localisation sequence yields a decomposition

$$\operatorname{Spc}(\mathsf{stmod}\,kG) = \operatorname{Spc}\mathsf{C} \coprod \operatorname{Spc}\mathsf{D}$$

by Theorem A.5. We refer to Appendix A for further explanations.

In the next section we describe an example that illustrates the preceding remark.

7. Thick subcategories

Let G be a finite group and fix a prime $p \in \mathbb{Z}$.

For each positive integer n, the canonical homomorphism $\phi_n : \mathbb{Z} \to \mathbb{Z}/p^n$ is an epimorphism of rings, and $\mathbb{Z}/p^n$ is finitely presented over $\mathbb{Z}$. Restriction via ϕ_n identifies the $\mathbb{Z}/p^n G$-modules with all $\mathbb{Z}G$-modules that are annihilated by p^n.

Let D_n be the full subcategory of $\mathsf{stmod}(\mathbb{Z}G)$ whose objects are the $\mathbb{Z}G$-modules isomorphic to modules annihilated by p^n; this is a tensor ideal thick subcategory and restriction via ϕ_n identifies it with $\mathsf{stmod}(\mathbb{Z}/p^n G)$. Note that any kG-module M is in D_n when $M \otimes_{\mathbb{Z}} M$ is in D_n. This is easy to verify and means that D_n is radical in the sense of Balmer [1].

When p does not divide $|G|$, one has $\mathsf{D}_n = 0$ for each n, by Theorem 6.2.

Proposition 7.1 *When p divides $|G|$ one has $\mathsf{D}_n \neq \mathsf{D}_{n+1}$ for each $n \geq 1$, so these tensor ideal thick subcategories form a strictly increasing chain:*

$$\mathsf{D}_1 \subsetneq \mathsf{D}_2 \subsetneq \mathsf{D}_3 \subsetneq \ldots \subseteq \mathsf{stmod}(\mathbb{Z}G).$$

Moreover, $\mathbb{Z} \notin \mathsf{D}_n$ for any n, so $\bigcup_{n\geqslant 1} D_n$ is a proper thick subcategory of $\mathsf{stmod}(\mathbb{Z}G)$.

Proof: Fix an $n \geq 1$. Evidently $\mathbb{Z}/p^{n+1}$ is in D_{n+1}; we claim that it is not in D_n. Indeed, the inclusion $\mathsf{D}_n \to \mathsf{stmod}(\mathbb{Z}G)$ admits $\mathbb{Z}/p^n \otimes_{\mathbb{Z}} -$ as a left adjoint; see Remark 6.10. It thus suffices to show that the natural morphism $\varepsilon\colon \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$ is not an isomorphism in $\mathsf{stmod}(\mathbb{Z}G)$. Assume to the contrary that it admits an inverse, say a morphism $\sigma\colon \mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1}$. Observe that $\sigma(1) = ap$, for some integer a, since σ is to be $\mathbb{Z}$-linear.

In $\mathsf{stmod}(\mathbb{Z}G)$ the morphism $\varepsilon\sigma$ is the identity map of $\mathbb{Z}/p^n$, so that in $\mathsf{mod}(kG)$

the endomorphism $\mathrm{id}-\varepsilon\sigma$ of $\mathbb{Z}/p^n$ admits a factorisation

$$\begin{tikzcd} \mathbb{Z}/p^n \arrow[rr, "\mathrm{id}-\varepsilon\sigma"] \arrow[dr, "\kappa"'] & & \mathbb{Z}/p^n \\ & (\mathbb{Z}/p^n)G \arrow[ur, "\pi"'] & \end{tikzcd}$$

where π is the natural morphism; see Theorem 2.5. Write $\kappa(1) = \sum_{g\in G} g \otimes c_g$ with $c_g \in \mathbb{Z}/p^n$. Note that $c_g = c_1$ for each g, since κ is $\mathbb{Z}G$-linear. Thus the commutativity of the diagram above yields

$$1-ap = (\mathrm{id}-\varepsilon\sigma)(1) = \pi(\sum_{g\in G} g\otimes c_1) = |G|c_1 \quad \text{in } \mathbb{Z}/p^n.$$

This is a contradiction, for p divides $|G|$. Thus, $\mathsf{D}_n \neq \mathsf{D}_{n+1}$, as claimed.

It remains to verify that $\mathbb{Z}$ is not in D_n for any $n \geq 1$. As above, it suffices to verify that the natural morphism $\mathbb{Z} \to \mathbb{Z}/p^n$ is not an isomorphism in $\mathsf{stmod}(\mathbb{Z}G)$, but this is clear because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^n,\mathbb{Z}) = 0$. □

In order to connect the preceding computation to the problem of classifying thick subcategories of the stable module category, we recall some basics concerning relative cohomology; see [3, §3.9].

Relative cohomology

Let k be a commutative ring and G a finite group. A *relative projective resolution* of a kG-module M is a complex P of kG-modules with each P^i weakly projective, $P^i = 0$ when $i > 0$, and equipped with a morphism of complexes $\varepsilon\colon P \to M$ of kG-modules such that the augmented complex

$$\cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is k-split exact. Relative projective resolutions exist and are unique up to homotopy; they can be constructed starting with the morphism $\pi_M : M{\uparrow}^G \to M$. For any such P, and kG-module N, we set

$$\mathrm{Ext}^*_{G,1}(M,N) = H^*(\mathrm{Hom}_{kG}(P,N)). \tag{7.1}$$

These k-modules are independent of the choice of P. Note that

$$\mathrm{Ext}^*_{G,1}(M,N) \cong \mathrm{Ext}^*_{kG}(M,N)$$

when M is projective over k, since any kG-projective resolution of M is a relative projective resolution. In particular, $\mathrm{Ext}^*_{G,1}(k,k) \cong \mathrm{Ext}^*_{kG}(k,k)$.

As usual, $\mathrm{Ext}^*_{G,1}(M,M)$ is a graded k-algebra, and $\mathrm{Ext}^*_{G,1}(M,N)$ is a graded bimodule with $\mathrm{Ext}^*_{G,1}(N,N)$ acting on the left and $\mathrm{Ext}^*_{G,1}(M,M)$ on the right. The functor $-\otimes_k M$, with diagonal G-action, induces a homomorphism

$$\zeta_M\colon \mathrm{Ext}^*_{G,1}(k,k) \to \mathrm{Ext}^*_{G,1}(M,M)$$

of graded k-algebras, and the $\mathrm{Ext}^*_{G,1}(k,k)$ action on $\mathrm{Ext}^*_{G,1}(M,N)$ through ζ_M and through ζ_N coincide, up to the usual sign.

For a kG-module M we set

$$H^*(G,M) = \mathrm{Ext}^*_{G,1}(k,M).$$

The k-algebra $H^*(G,k)$ is graded-commutative and $H^*(G,M)$ is a graded module over it. Observe that for each kG-module N, there is a natural isomorphism of graded $H^*(G,k)$ modules:

$$\mathrm{Ext}^*_{G,1}(M,N) \cong H^*(G,\mathrm{Hom}_k(M,N)). \tag{7.2}$$

Indeed, let P be a kG-projective resolution of k, so that $P\otimes_k M$ with diagonal G-action is a relative projective resolution of M. The isomorphism above is induced by the adjunction isomorphism

$$\mathrm{Hom}_{kG}(P\otimes_k M,N) \cong \mathrm{Hom}_{kG}(P,\mathrm{Hom}_k(M,N)).$$

Supports

Let now k be a noetherian commutative ring and M a kG-module. The ring $H^*(G,k)$ is then finitely generated as a k-algebra, hence a noetherian ring, and the $H^*(G,k)$-module $H^*(G,M)$ is finitely generated whenever M is finitely generated; see Evens [8]. From (7.2) one then deduces that $\mathrm{Ext}^*_{G,1}(M,N)$ is finitely generated over $H^*(G,k)$ for any pair of finitely generated kG-modules M and N.

We write $\mathcal{V}_G$ for the homogeneous prime ideals of $H^*(G,k)$ not containing the ideal of elements of positive degree, $H^{\geqslant 1}(G,k)$, and for each finitely generated kG-module M, set

$$\mathcal{V}_G(M) = \{\mathfrak{p}\in\mathcal{V}_G \mid \mathrm{Ext}^*_{G,1}(M,M)_{\mathfrak{p}} \neq 0\}.$$

This is a closed subset of $\mathcal{V}_G$ that we call the *support of M*. It coincides with

$$\{\mathfrak{p}\in\mathcal{V}_G \mid \mathrm{Ext}^*_{G,1}(M,N)_{\mathfrak{p}} \neq 0 \text{ for some } N\in\mathsf{mod}\,kG\}$$

since the action of $H^*(G,k)$ on $\mathrm{Ext}^*_{G,1}(M,N)$ factors through that of $\mathrm{Ext}^*_{G,1}(M,M)$.

To formulate the next result, we recall that k is *regular* if it is noetherian and each finitely generated k-module has a finite projective resolution. For any k-module M, we write $\mathrm{Supp}_k M$ for the subset $\{\mathfrak{p}\in\mathrm{Spec}\,k \mid M_{\mathfrak{p}}\neq 0\}$ of $\mathrm{Spec}\,k$.

Proposition 7.2 *Let k be a regular ring. Given finitely generated k-modules M,N with trivial G-action, if $\operatorname{Supp}_k M \subseteq \operatorname{Supp}_k N$, then $\mathcal{V}_G(M) \subseteq \mathcal{V}_G(N)$.*

Proof: It is not hard to verify that

$$\operatorname{Supp}_k M = \operatorname{Supp}_k \operatorname{End}_k(M).$$

Indeed, as M is finitely generated localisation at a prime $\mathfrak{p}$ induces an isomorphism

$$\operatorname{End}_k(M)_\mathfrak{p} \cong \operatorname{End}_{k_\mathfrak{p}}(M_\mathfrak{p}).$$

Thus, after localisation, we have to verify $M = 0$ if and only if $\operatorname{End}_k(M) = 0$, and this is clear.

Consequently, if $\operatorname{Supp}_k M \subseteq \operatorname{Supp}_k N$, then $\operatorname{Supp}_k \operatorname{End}_k(M) \subseteq \operatorname{Supp}_k \operatorname{End}_k(N)$. Then Hopkins' theorem [11, Theorem 11] (see also [21, Theorem 1.5]) implies that $\operatorname{End}_k(M)$ is in the thick subcategory generated by $\operatorname{End}_k(N)$, in the derived category of k-modules, and hence in the derived category of kG-modules; this is where we need the trivial action. It follows from (7.2) that $\mathcal{V}_G(M) \subseteq \mathcal{V}_G(N)$. $\square$

Remark 7.3 When k is a field, assigning a subcategory C of $\mathsf{stmod}(kG)$ to its support, $\bigcup_{M \in \mathsf{C}} \mathcal{V}_G(M)$, induces a bijection between the tensor ideal thick subcategories and the specialisation closed subsets of $\mathcal{V}_G$. This result is due to Benson, Carlson, and Rickard [4, Theorem 3.4] for k algebraically closed, and proved in [5, Theorem 11.4] in general. It follows from Proposition 7.1 that such a classification result cannot hold when $k = \mathbb{Z}$: Pick a prime p dividing $|G|$; then $\mathbb{Z}/p^n$ is not in the tensor ideal thick subcategory of $\mathsf{stmod}(\mathbb{Z}G)$ generated by $\mathbb{Z}/p$ when $n > 1$. However, we claim that there are equalities

$$\mathcal{V}_G(\mathbb{Z}/p^n) = \mathcal{V}_G(\mathbb{Z}/p) \quad \text{for all } n \geq 1.$$

This follows from Proposition 7.2, since $\operatorname{Supp}_\mathbb{Z}(\mathbb{Z}/p^n) = \{(p)\}$ for each $n \geq 1$.

8. The homotopy category of weakly injective modules

In this section we study the homotopy category of weakly injective modules and a pure variant. As before, k will be a commutative ring and G a finite group. An important ingredient of this discussion is an additional exact structure[1] for the category of kG-modules.

Let $\mathsf{WInj}kG$ denote the full subcategory of weakly injective (equivalently, weakly projective) kG-modules, and $\mathsf{winj}kG$ is the full subcategory of finitely presented modules in $\mathsf{WInj}kG$.

[1] In what follows, this exact structure and the homotopy categories and derived categories based on it have an implicit dependence on k that is only partially reflected in the notation.

Exact structures

Recall that a sequence $0 \to M' \to M \to M'' \to 0$ in $\mathsf{Mod}(kG)$ is k-split exact if it is split exact when restricted to the trivial subgroup. The sequence is *k-pure exact* if it is pure exact when restricted to the trivial subgroup.

Lemma 8.1 *For an exact sequence $\eta\colon 0 \to M' \to M \to M'' \to 0$ of kG-modules, the following conditions are equivalent:*

(1) The sequence is k-pure exact.

(2) For every finitely presented weakly injective kG-module P, each morphism $P \to M''$ lifts to a morphism $P \to M$.

Proof: Recall from Definition 4.3 that η is a pure exact sequence of kG-modules if and only if for every finitely presented kG-module Q, each morphism $Q \to M''$ lifts to a morphism $Q \to M$. Next observe that each finitely presented weakly injective kG-module is a direct summand of $Q\uparrow^G = kG \otimes_k Q$ for some finitely presented k-module Q. Applying

$$\mathrm{Hom}_{kG}(kG \otimes_k Q, -) \cong \mathrm{Hom}_k(Q, -)$$

to the morphism $M \to M''$ yields the assertion. □

For any additive category C we write $\mathsf{K}(\mathsf{C})$ for the homotopy category of cochain complexes in C. As usual, $\mathsf{K}^b(\mathsf{C})$ is the full subcategory of bounded complexes.

For a specified exact structure on C, a complex $X = (X^n, d^n)$ is said to be *acyclic* if there are exact sequences

$$0 \to Z^n \xrightarrow{u^n} X^n \xrightarrow{v^n} Z^{n+1} \to 0$$

in C such that $d^n = u^{n+1}v^n$ for all $n \in \mathbb{Z}$. We say X is *acyclic in almost all degrees* if such exact sequences exist for almost all n.

The bounded derived category

We consider the category of finitely presented kG-modules with the k-split exact structure and define the corresponding *bounded derived category* as the Verdier quotient

$$\mathsf{D}^b(\mathsf{mod}\,kG) = \mathsf{K}^b(\mathsf{mod}\,kG)/\mathsf{K}^b_{\mathrm{ac}}(\mathsf{mod}\,kG),$$

where $\mathsf{K}^b_{\mathrm{ac}}(\mathsf{mod}\,kG)$ is the full subcategory of acyclic complexes, with respect to the k-split structure, in $\mathsf{K}^b(\mathsf{mod}\,kG)$. The natural functor

$$\mathsf{K}^{+,b}(\mathsf{winj}\,kG) \longrightarrow \mathsf{D}^b(\mathsf{mod}\,kG) \tag{8.1}$$

is an equivalence of triangulated categories; its inverse identifies a complex with its weakly injective resolution. Here $\mathsf{K}^{+,b}(\mathsf{winj}kG)$ is the full subcategory of $\mathsf{K}(\mathsf{winj}kG)$ of complexes X with $X^n = 0$ for $n \ll 0$ and X acyclic in almost all degrees.

Proposition 8.2 *The composite functor*

$$\mathsf{mod}(kG) \rightarrowtail \mathsf{D}^b(\mathsf{mod}kG) \twoheadrightarrow \mathsf{D}^b(\mathsf{mod}kG)/\mathsf{K}^b(\mathsf{winj}kG)$$

induces an equivalence of triangulated categories

$$\mathsf{stmod}(kG) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}kG)/\mathsf{K}^b(\mathsf{winj}kG).$$

Proof: Adapt the proof for group algebras over a field; see Theorem 2.1 in [27]. □

Given bounded complexes X, Y of finitely presented kG-modules and $n \in \mathbb{Z}$, set

$$\operatorname{Ext}^n_{G,1}(X,Y) = \operatorname{Hom}_{\mathsf{D}^b(\mathsf{mod}kG)}(X, Y[n]).$$

This notation is consistent with our previous definition (7.1).

The pure derived category

The *pure derived category* of $\mathsf{Mod}(kG)$ is by definition the following Verdier quotient

$$\mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}kG) = \mathsf{K}(\mathsf{Mod}kG)/\mathsf{K}_{k\text{-pac}}(\mathsf{Mod}kG),$$

where $\mathsf{K}_{k\text{-pac}}(\mathsf{Mod}kG)$ denotes the full subcategory consisting of complexes that are acyclic with respect to the k-pure exact structure.

Proposition 8.3 *The derived category* $\mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}kG)$ *is a compactly generated triangulated category. The inclusion* $\mathsf{winj}kG \to \mathsf{Mod}(kG)$ *induces an equivalence of triangulated categories between* $\mathsf{K}^b(\mathsf{winj}kG)$ *and the full subcategory of compact objects in* $\mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}kG)$.

Proof: Consider the triangulated category $\mathsf{K}(\mathsf{Mod}kG)$ and identify each kG-module with the corresponding complex concentrated in degree zero. Then each object in $\mathsf{winj}kG$ is compact in $\mathsf{K}(\mathsf{Mod}kG)$ and

$$(\mathsf{winj}kG)^{\perp} = \mathsf{K}_{k\text{-pac}}(\mathsf{Mod}kG).$$

Now apply Proposition 5.1. □

Lemma 8.4 *The composite* $\mathsf{K}(\mathsf{WInj}kG) \rightarrowtail \mathsf{K}(\mathsf{Mod}kG) \twoheadrightarrow \mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}kG)$ *induces an equivalence of triangulated categories*

$$\mathsf{K}(\mathsf{WInj}kG)/\mathsf{K}_{k\text{-pac}}(\mathsf{WInj}kG) \xrightarrow{\sim} \mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}kG).$$

Proof: This is an immediate consequence of Corollary 5.2. □

Remark 8.5 When k is a field, every exact sequence of k-modules is pure exact, so $\mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}\,kG)$ is the usual derived category of kG-modules. On the other extreme, if G is trivial, $\mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}\,kG)$ is the pure derived category of $\mathsf{Mod}(k)$ studied in [18].

The pure homotopy category

The *pure homotopy category* of $\mathsf{WInj}\,kG$ is by definition the following Verdier quotient

$$\mathsf{K}_{\mathrm{pur}}(\mathsf{WInj}\,kG) = \mathsf{K}(\mathsf{WInj}\,kG)/\mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}\,kG)$$

where $\mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}\,kG)$ denotes the full subcategory consisting of complexes that are acyclic with respect to the pure exact structure.

Proposition 8.6 *The homotopy category* $\mathsf{K}_{\mathrm{pur}}(\mathsf{WInj}\,kG)$ *is compactly generated and the natural functor* (8.1) *induces an equivalence of triangulated categories between the full subcategory of compact objects in* $\mathsf{K}_{\mathrm{pur}}(\mathsf{WInj}\,kG)$ *and* $\mathsf{D}^b(\mathsf{mod}\,kG)$.

Proof: Choose for each $M \in \mathsf{mod}(kG)$ a weakly injective resolution $\boldsymbol{i}\,M$. Arguing as in the proof of [19, Lemma 2.1], one can verify that the morphism $M \to \boldsymbol{i}\,M$ induces, for each $X \in \mathsf{K}(\mathsf{WInj}\,kG)$, an isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,kG)}(\boldsymbol{i}\,M, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,kG)}(M, X).$$

Thus each $\boldsymbol{i}\,M$ is compact in $\mathsf{K}(\mathsf{WInj}\,kG)$ and

$$\{\boldsymbol{i}\,M \mid M \in \mathsf{mod}(kG)\}^{\perp} = \mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}\,kG).$$

Now apply Proposition 5.1. □

Recollements

A *recollement* is by definition a diagram of exact functors

$$\begin{tikzcd} \mathsf{T}' \arrow[r, "I" description] & \mathsf{T} \arrow[l, shift right=3, "I_\rho"'] \arrow[l, shift left=3, "I_\lambda"] \arrow[r, "Q" description] & \mathsf{T}'' \arrow[l, shift right=3, "Q_\rho"'] \arrow[l, shift left=3, "Q_\lambda"] \end{tikzcd}$$

satisfying the following conditions, where S denotes the full subcategory consisting of objects $X \in \mathsf{T}$ satisfying $Q(X) = 0$.

(1) The functors (I_λ, I), (I, I_ρ), (Q_λ, Q), (Q, Q_ρ) form adjoint pairs.

(2) The functor I induces an equivalence $\mathsf{T}' \xrightarrow{\sim} \mathsf{S}$.

(3) The functor Q induces an equivalence $\mathsf{T}/\mathsf{S} \xrightarrow{\sim} \mathsf{T}''$.

Proposition 8.7 *Let* T *be a triangulated category with set-indexed products and coproducts. Fix a pair of essentially small thick subcategories* C *and* D *consisting of compact objects such that* $\mathsf{D} \subseteq \mathsf{C}$*. Then there is an induced recollement of compactly generated triangulated categories*

$$\begin{tikzcd} (\mathsf{D}^{\perp})/(\mathsf{C}^{\perp}) \arrow[r, "I" description] & \mathsf{T}/(\mathsf{C}^{\perp}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q" description] & \mathsf{T}/(\mathsf{D}^{\perp}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

Restricting the left adjoints of I *and* Q *to the full subcategories of compact objects yields up to equivalence and idempotent completion a sequence of exact functors:*

$$\mathsf{C}/\mathsf{D} \twoheadleftarrow \mathsf{C} \leftarrowtail \mathsf{D}\,.$$

Proof: We have a pair of localising subcategories $\mathsf{C}^{\perp} \subseteq \mathsf{D}^{\perp}$ which induce a quotient functor $\mathsf{T}/(\mathsf{C}^{\perp}) \twoheadrightarrow \mathsf{T}/(\mathsf{D}^{\perp})$ with kernel $(\mathsf{D}^{\perp})/(\mathsf{C}^{\perp})$; see Proposition 2.3.1 of Chap. II in [29]. The categories $\mathsf{C}^{\perp}$ and $\mathsf{D}^{\perp}$ form subcategories of T that are closed under set-indexed products and coproducts. Thus I and Q preserve set-indexed products and coproducts. It follows from Brown representability that I and Q both have left and right adjoints.

A left adjoint of a coproduct preserving functor preserves compactness. The description of the categories of compact objects follows from Proposition 5.1. $\square$

A recollement

We apply Proposition 8.7: Let $\mathsf{T} = \mathsf{K}(\mathsf{WInj}kG)$ and set

$$\mathsf{C} = \mathsf{K}^{+,b}(\mathsf{winj}kG) \quad \text{and} \quad \mathsf{D} = \mathsf{K}^{b}(\mathsf{winj}kG).$$

Note that

$$\mathsf{C}^{\perp} = \mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}kG) \quad \text{and} \quad \mathsf{D}^{\perp} = \mathsf{K}_{k\text{-pac}}(\mathsf{WInj}kG).$$

The inclusion $\mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}kG) \rightarrowtail \mathsf{K}_{k\text{-pac}}(\mathsf{WInj}kG)$ induces the following commutative diagram of exact functors:

$$\begin{tikzcd} \mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}kG) \arrow[r, equal] \arrow[d, tail] & \mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}kG) \arrow[d, tail] & \\ \mathsf{K}_{k\text{-pac}}(\mathsf{WInj}kG) \arrow[r, tail] \arrow[d, two heads] & \mathsf{K}(\mathsf{WInj}kG) \arrow[r, two heads] \arrow[d, two heads] & \mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}kG) \arrow[d, equal] \\ \mathsf{StMod}^{\mathrm{cw}}(kG) \arrow[r, tail, "I"] & \mathsf{K}_{\mathrm{pur}}(\mathsf{WInj}kG) \arrow[r, two heads, "Q"] & \mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}kG). \end{tikzcd}$$

The occurrence of the stable module category in this diagram is explained by the following lemma.

Lemma 8.8 *Taking a complex of kG-modules to its cycles in degree zero induces an equivalence of triangulated categories*

$$\mathsf{K}_{k\text{-pac}}(\mathsf{WInj}\,kG)/\mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}\,kG) \xrightarrow{\sim} \mathsf{StMod}^{\mathrm{cw}}(kG).$$

Proof: Denote by $\mathsf{K}_{\mathrm{ac}}(\mathsf{WInj}\,kG)$ the category of complexes that are acyclic with respect to the k-split exact structure. Taking cycles in degree zero induces an equivalence of triangulated categories

$$\mathsf{K}_{\mathrm{ac}}(\mathsf{WInj}\,kG) \xrightarrow{\sim} \mathsf{StMod}(kG).$$

This functor identifies $\mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}\,kG) \cap \mathsf{K}_{\mathrm{ac}}(\mathsf{WInj}\,kG)$ with the category of cw-injective kG-modules by Lemma 5.5, and induces therefore an equivalence

$$\mathsf{K}_{\mathrm{ac}}(\mathsf{WInj}\,kG)/(\mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}\,kG) \cap \mathsf{K}_{\mathrm{ac}}(\mathsf{WInj}\,kG)) \xrightarrow{\sim} \mathsf{StMod}^{\mathrm{cw}}(kG).$$

Now observe that

$$\mathsf{K}_{\mathrm{ac}}(\mathsf{WInj}\,kG) \subseteq \mathsf{K}_{k\text{-pac}}(\mathsf{WInj}\,kG)$$

and apply Corollary 5.2. □

The following result establishes the analogue of Proposition 6.1 in [6] for general rings of coefficients.

Theorem 8.9 *The functors I and Q induce a recollement*

$$\begin{tikzcd} \mathsf{StMod}^{\mathrm{cw}}(kG) \arrow[r, "I" description] & \mathsf{K}_{\mathrm{pur}}(\mathsf{WInj}\,kG) \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \arrow[r, "Q" description] & \mathsf{D}_{\mathrm{pur}}(\mathsf{Mod}\,kG) \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \end{tikzcd}.$$

Restricting the left adjoints of I and Q to the full subcategories of compact objects yields up to equivalence the following sequence

$$\mathsf{stmod}(kG) \twoheadleftarrow \mathsf{D}^b(\mathsf{mod}\,kG) \hookleftarrow \mathsf{K}^b(\mathsf{winj}\,kG).$$

Proof: We apply Proposition 8.7. The functor Q is a quotient functor by Lemma 8.4, and its kernel identifies with $\mathsf{K}_{k\text{-pac}}(\mathsf{WInj}\,kG)/\mathsf{K}_{\mathrm{pac}}(\mathsf{WInj}\,kG)$; see Proposition 2.3.1 of Chap. II in [29]. The description of the kernel as stable module category follows from Lemma 8.8. The description of the compact objects follows from Propositions 8.2, 8.3, and 8.6. □

Remark 8.10 The adjoints of I and Q in this recollement admit explicit descriptions. More precisely, $I_\lambda = \boldsymbol{t}k \otimes_k -$, $I_\rho = \mathrm{Hom}_k(\boldsymbol{t}k, -)$, $Q_\lambda = \boldsymbol{p}k \otimes_k -$, $Q_\rho = \mathrm{Hom}_k(\boldsymbol{p}k, -)$. Here, we write $\boldsymbol{p}k$ for a weakly projective resolution, and $\boldsymbol{t}k$ for a Tate resolution of the trivial kG-module k; see [6, §6].

A. Disconnecting spectra via localisation sequences

BY GREG STEVENSON

Let S be an essentially small tensor triangulated category, for instance the stable module category $\mathsf{stmod}(kG)$ of a group algebra kG. In [1] Balmer has associated to S a spectral topological space $\operatorname{Spc}\mathsf{S}$ which provides a universal notion of tensor-compatible supports for objects of S. We briefly recall the definition of $\operatorname{Spc}\mathsf{S}$. A proper tensor ideal (our tensor ideals are always assumed to be thick) P of S is *prime* if $s\otimes s' \in \mathsf{P}$ implies $s \in \mathsf{P}$ or $s' \in \mathsf{P}$ for all $s,s' \in \mathsf{S}$. The *spectrum* of S is then defined to be the set $\operatorname{Spc}\mathsf{S}$ of prime tensor ideals of S endowed with the *Zariski topology* which is given by the basis of closed subsets

$$\{\operatorname{supp}_{\mathsf{S}} s = \{\mathsf{P} \in \operatorname{Spc}\mathsf{S} \mid s \notin \mathsf{P}\} \mid s \in \mathsf{S}\}.$$

The subset $\operatorname{supp}_{\mathsf{S}} s$ is called the *support* of the object s. More generally one can define the support of any subcategory $\mathsf{C} \subseteq \mathsf{S}$ by

$$\operatorname{supp}_{\mathsf{S}} \mathsf{C} = \bigcup_{c\in \mathsf{C}} \operatorname{supp}_{\mathsf{S}} c.$$

The spectrum is contravariantly functorial with respect to strong monoidal exact functors [1, Proposition 3.6] i.e., given a strong monoidal exact functor $i_*\colon \mathsf{R} \to \mathsf{S}$ there is a spectral map $\operatorname{Spc}(i_*)\colon \operatorname{Spc}\mathsf{S} \to \operatorname{Spc}\mathsf{R}$.

We show that if a tensor triangulated category S admits a semiorthogonal decomposition given by a pair of tensor ideals then the spectrum of S is disconnected. In the remainder of this section, S is an essentially small tensor triangulated category, with unit object $\mathbf{1}$, and

$$\mathsf{R} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathsf{S} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathsf{T}$$

is a localisation sequence such that both R and T are tensor ideals of S, when viewed as subcategories via the embeddings i_* and j_*. We recall that this diagram being a *localisation sequence*, or *semiorthogonal decomposition*, means that i_* and j_* are fully faithful exact functors such that $i^!$ is right adjoint to i_* and j^* is left adjoint to j_*. Moreover, $(i_*\mathsf{R})^\perp = j_*\mathsf{T}$ and $i_*\mathsf{R} = {}^\perp(j_*\mathsf{T})$.

In general one must ask that both $i_*\mathsf{R}$ and $j_*\mathsf{T}$ are tensor ideals; $i_*\mathsf{R}$ being an ideal does not necessarily imply $j_*\mathsf{T}$ is an ideal and vice versa. However, there is a special case in which these implications hold and we feel it is worth making explicit. We say the tensor triangulated category S is *rigid* if the symmetric monoidal structure on S is closed, with internal hom $\mathcal{H}om(-,-)$ and, setting $s^\vee = \mathcal{H}om(s,\mathbf{1})$ for $s \in \mathsf{S}$, the natural morphism

$$s^\vee \otimes s' \to \mathcal{H}om(s,s')$$

is an isomorphism for all $s,s' \in \mathsf{S}$. For further details the interested reader can consult Section 1 of [7] in the general case and Appendix A of [14] for further details in the triangulated case.

Lemma A.1 *Let* S *be an essentially small rigid tensor triangulated category. If* R *is a tensor ideal of* S*, then so are the thick subcategories* $\mathsf{R}^\perp$ *and* ${}^\perp\mathsf{R}$.

Proof: It is clear from the adjunction between the tensor product and internal hom that $\mathsf{R}^\perp$ is closed under the functors $\mathcal{H}om(s,-)$ for $s \in \mathsf{S}$. As $\mathcal{H}om(s,-) \cong s^\vee \otimes (-)$ and the duality $(-)^\vee$ is a contravariant equivalence it is thus immediate that $\mathsf{R}^\perp$ is an ideal as claimed. The proof that ${}^\perp\mathsf{R}$ is an ideal is similarly trivial. □

We now return to the setting where S is only assumed to be tensor triangulated and commence the proof of the promised result on disconnectedness of the spectrum. Recall that an object $s \in \mathsf{S}$ is said to be $\otimes$-*idempotent* if $s \otimes s \cong s$.

Lemma A.2 *There are* $\otimes$*-idempotent objects* $\Gamma_{\mathsf{R}}\mathbf{1}$ *and* $L_{\mathsf{R}}\mathbf{1}$ *such that*

$$\Gamma_{\mathsf{R}}\mathbf{1} \otimes (-) \cong i_*i^! \quad \text{and} \quad L_{\mathsf{R}}\mathbf{1} \otimes (-) \cong j_*j^*.$$

In particular, $\Gamma_{\mathsf{R}}\mathbf{1} \otimes L_{\mathsf{R}}\mathbf{1} \cong 0$.

Proof: This is standard, see for instance [2, Theorem 2.13] □

Lemma A.3 *Given objects* $r \in \mathsf{R}$*,* $s \in \mathsf{S}$*, and* $t \in \mathsf{T}$ *there are isomorphisms*

$$s \otimes i_*r \cong i_*(i^!s \otimes r) \quad \text{and} \quad s \otimes j_*t \cong j_*(j^*s \otimes t).$$

Proof: The proof of the second isomorphism is almost exactly as in [28, Lemma 8.2] and the proof of the first is similar. □

Lemma A.4 *The map* $\mathrm{Spc}(j^*)\colon \mathrm{Spc}\,\mathsf{T} \to \mathrm{Spc}\,\mathsf{S}$ *gives a homeomorphism onto* $\mathrm{supp}_{\mathsf{S}}\, j_*\mathsf{T}$ *and* $\mathrm{Spc}(i^!)\colon \mathrm{Spc}\,\mathsf{R} \to \mathrm{Spc}\,\mathsf{S}$ *is a homeomorphism onto* $\mathrm{supp}_{\mathsf{S}}\, i_*\mathsf{R}$. *Furthermore, there are equalities*

$$\mathrm{supp}_{\mathsf{S}}\, i_*\mathsf{R} = \mathrm{supp}_{\mathsf{S}}\, \Gamma_{\mathsf{R}}\mathbf{1} \quad \text{and} \quad \mathrm{supp}_{\mathsf{S}}\, j_*\mathsf{T} = \mathrm{supp}_{\mathsf{S}}\, L_{\mathsf{R}}\mathbf{1},$$

so that both subsets are closed in $\mathrm{Spc}\,\mathsf{S}$.

Proof: The arguments for R and T are similar so we only prove the result for T. We already know from [1, Proposition 3.11] that the localisation j^* induces a homeomorphism

$$\operatorname{Spc}\mathsf{T} \xrightarrow{\sim} \{\mathsf{P} \in \operatorname{Spc}\mathsf{S} \mid \mathsf{R} \subseteq \mathsf{P}\},$$

and it remains to identify this subset with the support of $\mathsf{R}^{\perp} = j_*\mathsf{T}$.

First observe that since $\Gamma_{\mathsf{R}}\mathbf{1} \otimes L_{\mathsf{R}}\mathbf{1} \cong 0$ every prime tensor ideal must contain one of these idempotents. Furthermore, no proper ideal can contain both idempotents as they build $\mathbf{1}$ via the localisation triangle

$$\Gamma_{\mathsf{R}}\mathbf{1} \to \mathbf{1} \to L_{\mathsf{R}}\mathbf{1} \to \Sigma\Gamma_{\mathsf{R}}\mathbf{1}.$$

So for $\mathsf{P} \in \operatorname{Spc}\mathsf{S}$ we have $\mathsf{R} \subseteq \mathsf{P}$ if and only if $L_{\mathsf{R}}\mathbf{1} \notin \mathsf{P}$ if and only if $j_*\mathsf{T} \not\subseteq \mathsf{P}$. This completes the argument via the following trivial fact

$$\operatorname{supp}_{\mathsf{S}} j_*\mathsf{T} = \{\mathsf{P} \in \operatorname{Spc}\mathsf{S} \mid j_*\mathsf{T} \not\subseteq \mathsf{P}\}.$$

The final statement now follows as

$$\begin{aligned}\operatorname{supp}_{\mathsf{S}} L_{\mathsf{R}}\mathbf{1} &= \{\mathsf{P} \in \operatorname{Spc}\mathsf{S} \mid L_{\mathsf{R}}\mathbf{1} \notin \mathsf{P}\}\\ &= \{\mathsf{P} \in \operatorname{Spc}\mathsf{S} \mid \Gamma_{\mathsf{R}}\mathbf{1} \in \mathsf{P}\}\\ &= \{\mathsf{P} \in \operatorname{Spc}\mathsf{S} \mid \mathsf{R} \subseteq \mathsf{P}\}\\ &= \operatorname{Spc}(j^*)(\operatorname{Spc}\mathsf{T})\\ &= \operatorname{supp}_{\mathsf{S}} j_*\mathsf{T}.\end{aligned}$$

□

From this point onward let us be lazy and identify R and T with thick subcategories of S and view their spectra as subsets of $\operatorname{Spc}\mathsf{S}$.

Theorem A.5 *The subsets* $\operatorname{Spc}\mathsf{R} = \operatorname{supp}_{\mathsf{S}}\mathsf{R}$ *and* $\operatorname{Spc}\mathsf{T} = \operatorname{supp}_{\mathsf{S}}\mathsf{T}$ *of* $\operatorname{Spc}\mathsf{S}$ *are open and closed, and there is a decomposition*

$$\operatorname{Spc}\mathsf{S} = \operatorname{Spc}\mathsf{R} \coprod \operatorname{Spc}\mathsf{T}.$$

In particular, if neither R *nor* T *is the zero ideal the space* $\operatorname{Spc}\mathsf{S}$ *is disconnected.*

Proof: By [1, Proposition 3.11] and the last lemma respectively we can describe $\operatorname{Spc}\mathsf{R}$ as both

$$\{\mathsf{P} \in \operatorname{Spc}\mathsf{S} \mid \mathsf{T} \subseteq \mathsf{P}\} \quad \text{and} \quad \operatorname{supp}_{\mathsf{S}}\mathsf{R}.$$

It is clear from the first description that $\operatorname{Spc}\mathsf{R}$ is generalisation closed and from the second that $\operatorname{Spc}\mathsf{R}$ is specialisation closed. By the last lemma this subset is constructible (since it is the support of an object) and hence it is both closed and open. The argument for T is the same. The existence of the claimed decomposition is clear: by Lemma A.2 we have $\otimes$-orthogonal Rickard idempotents, precisely one of which must be contained in any non-zero proper prime ideal (this can also be used to argue that both subsets in question are open). □

For completeness we give an easier proof of a stronger result in the rigid case.

Proposition A.6 *Suppose* S *is as above but is furthermore assumed to be rigid. Then* S *is equivalent to* $\mathsf{R} \oplus \mathsf{T}$. *In particular the spectrum decomposes as in the above theorem.*

Proof: Using rigidity of S we have, for $i \in \mathbb{Z}$,

$$\mathrm{Hom}(L_{\mathsf{R}}\mathbf{1}, \Sigma^i \Gamma_{\mathsf{R}}\mathbf{1}) \cong \mathrm{Hom}(\Sigma^{-i}\mathbf{1}, (L_{\mathsf{R}}\mathbf{1})^{\vee} \otimes \Gamma_{\mathsf{R}}\mathbf{1}) \cong 0,$$

the final isomorphism since thick tensor ideals are closed under $(-)^{\vee}$ and $\mathsf{R} \cap \mathsf{T} = 0$. Hence the localisation triangle

$$\Gamma_{\mathsf{R}}\mathbf{1} \to \mathbf{1} \to L_{\mathsf{R}}\mathbf{1} \to \Sigma\Gamma_{\mathsf{R}}\mathbf{1}$$

splits yielding $\mathbf{1} \cong \Gamma_{\mathsf{R}}\mathbf{1} \oplus L_{\mathsf{R}}\mathbf{1}$. It is now clear that S splits. □

Corollary A.7 *If* S *has a decomposition, as in the assumptions of this section, given by non-zero tensor ideals and* $\mathbf{1}$ *is indecomposable then* S *is not rigid.*

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