2000

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Safe-Side Requirements in Life Insurance: A Corporate Perspective

Annamaria Olivieri* and Ermanno Pitacco†

Abstract†

Safe-side requirements concern the assumptions used to calculate premiums in relation to a set of more realistic assumptions. Roughly, safe-side requirements express the capability of premiums to generate positive margins. In a strictly actuarial framework, safe-side requirements are given in terms of some notion of expected profit, calling for assumptions that let such profit be nonnegative. An expected profit of zero, however, is not a realistic aim for the insurer.

We investigate the notion of conservative assumptions by adopting an unconventional approach. Our focus is the management of the financial resources coming both from premiums and from shareholders' capital. This leads to a general structure that includes as particular cases the results obtainable in a strictly actuarial environment.

Key words and phrases: technical basis, expected profit, portfolio fund, shareholders' capital, opportunity cost of capital, discounted cash flow

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†This research was partially funded by the Italian MURST, within the project "Modelli attuariali analitici e numerici per l'innovazione nelle assicurazioni di persone." A preliminary version was presented to the XVI Euro, held in Brussels in July 1998.

The authors are grateful to the anonymous referees for several useful remarks that have helped in the preparation of this paper.
1 Introduction

Though the concept of safe-side requirements has always been an integral part of actuarial science, it is only recently that it has become a core topic in Europe. This is especially true in Italy, as a consequence of the European Third Directive concerning life insurance regulation (Directive EEC, No. 96 of 1992). Because tariffs are no longer subject to approval, other tools must be used to monitor the stability of insurance companies.

Expressions such as “prudent,” “prudent valuation,” and so on are widely used in current legislation in order to define freedom in choosing premium ratings. These terms are often used vaguely or dubiously.

In this paper we have tried to outline a general structure that can help in understanding what a safe-side requirement is and to what it refers. We have adopted two approaches: (i) a purely actuarial approach, referring to some classical results of actuarial mathematics such as the notion of expected profit and the contribution formula of Homans (1863) (see also Haberman and Sibbett 1995, pages 287–297) as well as to some ideas developed in the framework of multistate models (see Hoem (1988) and Olivieri (1999)); and (ii) a corporate approach introducing the notion of shareholders’ capital, which is not usually considered in traditional actuarial mathematics. The corporate approach has proved to be more general than the purely actuarial one. The results obtained in the actuarial framework are special cases of the corporate approach. These approaches lead to a unitary formal structure.

Our basic set of assumptions are:

(a) Only life insurance policies issued on a single life are considered. Policies involving more than one life or health insurance policies are disregarded;

(b) As we are considering only net premiums, expenses and expense loadings are also disregarded;

(c) Benefits and premiums are specified at policy issue and remain unchanged throughout the insured period (hence, financial adjustments of benefits and/or premiums are not permitted);


2 The legislation is the Italian D.Lgs. 174/95 (published on the Gazzetta Ufficiale della Repubblica Italiana, n. 56, 18/5/1995), which introduces European legislation in Italy.
(d) Premiums are assumed to be paid at the start of each year and death benefits are paid at the end of the year of death (thus we adopt a time-discrete approach); and

(e) Though the underlying processes may vary, we consider expected values only (some remarks on the possibility of considering higher moments or, in general, probability distributions are made in Section 6).

The model is simple, but it makes the interpretation of the results easier. Moreover, comparisons with the traditional actuarial model are immediate.

Section 2 introduces the notion of prudence; this notion is referred to as a "first order basis" in comparison to a given set of more realistic assumptions that are referred to as a "second order basis." A formal structure is introduced under which various safe-side requirements can be classified. In Section 3 various definitions of safe-side requirements are given in a strictly actuarial framework. Section 4 adopts a corporate approach; some comparisons between the two approaches are then made. Section 5 discusses some numerical examples.

2 Technical Bases and Prudence

2.1 Some Preliminary Aspects

One of the objectives of actuarial valuations is to assess the adequacy of premiums and to forecast future payments by the insurer and the insured. This objective requires the choice of a convenient set of basic assumptions on which to base the forecast. In life insurance, such basic assumptions include demographic assumptions (e.g., mortality, morbidity, and lapse rates) and financial assumptions (e.g., interest and inflation rates). Throughout this paper, the set of assumptions used to derive net premiums will be called the technical basis.

When life insurance premiums are calculated according to the equivalence principle, the expected profit for the insurer is zero if a realistic technical basis is used. Hence, it is necessary to:

(i) Use a realistic technical basis and adopt premium principles other than the equivalence principle in order to include an explicit safety loading into premiums; or to

\[ \text{The equivalence principle states that the actuarial present value of premiums is equal to the actuarial present value of benefits.} \]
(ii) Adopt the equivalence principle and use a conservative technical basis in order to include an implicit safety loading into premiums.

In both cases, the safety loading leads to a positive expected profit. Among European life insurance companies, choice (ii) is commonly made.

The first order technical basis (briefly $TB_1$) is the set of conservative assumptions (i.e., assumptions favorable to the insurer) used in choice (ii). This set is relatively easy to define. It includes the valuation rate of interest (usually constant) as well as a mortality table.

The second order technical basis (briefly $TB_2$) is the set of realistic assumptions. The concept of second order basis is more complex. $TB_2$ must give a realistic description of the scenario facing the insurance company and the insured person. Thus, it should include assumptions about policyholder behavior, investment performance, and company behavior and assumptions about macro- and micro-economic forces.

It is likely that in the early days of actuarial practice the two concepts emerged simultaneously. Given the contractual relevance of $TB_1$, $TB_2$ has usually been expressed as a simple shift of $TB_1$. Hence, $TB_1$ and $TB_2$ were usually assumed to have the same structure; for example, for insurance policies $TB_1$ includes a lower interest rate and higher mortality than that included in $TB_2$, while for annuities $TB_1$ includes a lower interest rate and lower mortality than in $TB_2$.

In these days of easy access to high speed computers, it is important to adopt a more flexible structure for $TB_2$. For example, the financial assumption may include a (deterministic) term structure of interest rates or a convenient stochastic model; as to the demographical aspect, a projected table can represent the future trend of mortality. (A stochastic model can express the uncertainty of the projection.)

For the sake of simplicity and for obtaining results that can be compared to the traditional model, we will adopt a conventional structure for $TB_2$. On the other hand, a deterministic term structure of interest rates as well as a (deterministic) projected mortality table would not add significance to the considerations discussed below.

2.2 Formal Aspects

As we have stated in Section 2.1, safe-side requirements will be referred to as $TB_1$ in relation to a given realistic basis $TB_2$. We need a yardstick to assess whether $TB_1$ is on the safe side with respect to $TB_2$. From a formal viewpoint, this yardstick is represented by a vector-valued mapping of the two technical bases. Given the dynamic nature
of life insurance contracts and the length of these contracts, the vector could quantify safe-side requirements imposed on any single year. In other cases, the elements of the vector will represent the different components of the contract that can generate safety loadings and thus contribute to the expected profit. We will consider only deterministic mappings obtained as expected values of random variables.

If $\Gamma$ denotes the particular insurance policy, we will deal with functions $\Phi$ that map $(TB_1, TB_2, \Gamma)$ to quantities (typically expected profits) that are used to assess prudence. These quantities are specified in the proposed safe-side requirements. In particular, $\Gamma$ allows us to specify benefits and to determine premiums and reserves. If $\Phi$ is a vector-valued function, its elements will be denoted by $\phi_1, \phi_2, \ldots$.

As we have stated in Section 2.1, we adopt a constant rate of interest $i$ and a given set of mortality rates $q_y, y = 0, 1, \ldots$ for $TB_1$ and $i^*$ and $q^*_y, y = 0, 1, \ldots$ for $TB_2$. Thus we have

$$TB_1 = (i, \{q_y\})$$
$$TB_2 = (i^*, \{q^*_y\}).$$

As usual, we put $p_y = 1 - q_y$ and $p^*_y = 1 - q^*_y$. Traditional actuarial notation is used whenever possible.

We will focus on insurance policies with the following characteristics:

- $x$ is the issue age, and $\omega$ is the limiting age of the mortality table;
- Term $n$ years, where $n = 1, 2, \ldots, \omega - x$;
- The death benefit, paid at the end of the $t$th policy year of death, is $C_t$, where $t = 1, 2, \ldots, n$;
- Sum $S$ is paid in case of survival to age $x + n$ (when $n$ is finite);
- The premiums are paid at the start of each policy year (if the insured is then alive). The premium paid at time $k$ (i.e., age $x + k$) is $P_k, k = 0, 1, \ldots, n - 1$. The cases with single premiums and premiums payable for at most $m$ years (with $m \leq n$) are included; annuities are also included, by letting $P_k < 0$ (thus paid by the insurer).

Such a general insurance structure includes many practical policies such as term, whole life, endowment, pure endowment insurances, and immediate and deferred annuities.
We assume that premiums are calculated according to the equivalence principle (obviously, with $TB_1$). The net premium reserve at time $t$, $V_t$, is defined as

$$V_t = \sum_{h=0}^{n-t-1} C_{t+h+1} v^{h+1} h p_{x+t} q_{x+h+t} + S v^{n-t} n-t p_{x+t} - \sum_{h=0}^{n-t-1} P_{t+h} v^h h p_{x+t}$$

(1)

where $v = 1/(1+i)$. The boundary conditions are $V_0 = 0$ and $V_n = S$ (when $n$ is finite). $V_t$ satisfies the recurrence equation

$$(V_t + P_t) (1+i) = (C_{t+1} - V_{t+1}) q_{x+t} + V_{t+1}.$$  

(2)

3 Safe-Side Requirements in an Actuarial Framework

3.1 Profits and Second Order Reserves

From the recurrence equation (2), we get an expression for $u^*_{t+1}$, the annual profit at the end of the $t+1$st policy year, which is obtained evaluating the assets and liabilities using the realistic basis $TB_2$. We have

$$u^*_{t+1} = (V_t + P_t) (1+i^*) - (C_{t+1} - V_{t+1}) q^*_{x+t} - V_{t+1}$$

(3)

from which we obtain the contribution formula of Homans (1863)

$$u^*_{t+1} = (V_t + P_{t+1}) (i^* - i) + (C_{t+1} - V_{t+1}) (q_{x+t} - q^*_{x+t})$$

(4)

where the financial and demographic components of profits are:

$$f u^*_{t+1} = (V_t + P_{t+1}) (i^* - i) \quad \text{Financial Component} \quad (5)$$

$$d u^*_{t+1} = (C_{t+1} - V_{t+1}) (q_{x+t} - q^*_{x+t}) \quad \text{Demographic Component}. \quad (6)$$

The total future expected profit at time 0 is $u^*$ where
After substituting equation (3) into equation (7), and some algebraic manipulations, we obtain the following expression for total profit:

\[ u^* = \sum_{h=0}^{n-1} p_{h+1} X^* (1 + i^*)^{-h}. \]  

(8)

As before, total profit can be split into the financial and demographic components:

\[ f u^* = \sum_{h=0}^{n-1} f u_{h+1} X^* (1 + i^*)^{-h} \]  

(9)

\[ d u^* = \sum_{h=0}^{n-1} d u_{h+1} X^* (1 + i^*)^{-h}. \]  

(10)

More generally, we can define the expected future profit after time \( t \), i.e., in the interval \( [t, n] \), as

\[ u^*(t, n) = \sum_{h=0}^{n-t-1} u_{t+h+1} X^* (1 + i^*)^{-h}. \]  

Specifically, \( u^* = u^*(0, n) \). Also this expected profit can be split into its financial and demographic components; moreover, a result similar to equation (8) holds for \( u^*(t, n) \).

Finally, we define \( V_t^* \) as the second order reserve calculated using the realistic assumptions of \( TB_2 \)

\[ V_t^* = \sum_{h=0}^{n-t-1} C_{t+h+1} X^* q_{X+t+1} (1 + i^*)^{-h} + S_{n-t} X^* (1 + i^*)^{-(n-t)} \]  

(11)
3.2 Safe-Side Requirement (SSR) Definitions

Let us now turn to various safe-side requirements, which are referred to as $TB_1$ in relation to $TB_2$. We will then analyze the links between the various definitions; we will verify that some safe-side requirements (SSR) imply others.

Definition 1 (Naive SSR):

We say that $TB_1$ is on the safe side (with respect to $TB_2$) if and only if

- 1. Benefits are payable in case of survival only,
  2. $i \leq i^*$, and
  3. $q_{x+h-1} \leq q_{x+h-1}^*$ for $h = 1, 2, \ldots, n$;
  or

- 1. Benefits are payable in case of death only,
  2. $i \leq i^*$, and
  3. $q_{x+h-1} \geq q_{x+h-1}^*$ for $h = 1, 2, \ldots, n$.

In this case prudence is directly measured on the single elements of the two technical bases. The corresponding mapping $\Phi$ is a vector with $n + 1$ elements, given by

$$
\phi_h = q_{x+h-1} - q_{x+h-1}^* \quad \text{for} \quad h = 1, 2, \ldots, n
$$

$$
\phi_{n+1} = i^* - i.
$$

The safe-side requirement can be easily stated in terms of the mapping $\Phi$. Such a definition can be applied only to a restricted number of cases; for example, it cannot be used for policies that contain both survival and death benefits.

Definition 2 (Financial/Demographic Annual Profits SSR):

Let us consider the mapping $\Phi = [\phi_1, \phi_2, \ldots, \phi_{2n}]$ where

$$
\phi_{2t-1} = f_u^* \quad \text{for} \quad t = 1, 2, \ldots, n \quad (12)
$$

$$
\phi_{2t} = d_u^* \quad \text{for} \quad t = 1, 2, \ldots, n. \quad (13)
$$

We say that $TB_1$ is on the safe side if and only if $\phi_h \geq 0$ for $h = 1, 2, \ldots, 2n$. 

In this case, as in the following ones, the measurement of prudence relies on expected present values, which consist in equations (12) and (13) of the components of expected annual profit. From the definition of \( f \) \( u^*_t \) and \( d u^*_t \) (see equations (5) and (6)) conditions on the elements of \( TB_1 \) can be derived. The condition concerning the demographical assumption is based on the sum at risk \( (C_{t+1} - V_{t+1}) \); hence, it is more widely applicable than simply requiring that \( q_{x+h-1}^* \leq q_{x+h-1}^* \) or \( q_{x+h-1}^* \geq q_{x+h-1}^* \) for \( h = 1, 2, \ldots, n \).

**Definition 3 (Annual Profits SSR):**

Let \( \Phi = [\phi_1, \phi_2, \ldots, \phi_n] \), where

\[
\phi_t = u^*_t, \quad t = 1, 2, \ldots, n.
\]

We say that \( TB_1 \) is on the safe side if and only if \( \phi_t \geq 0 \) for \( t = 1, 2, \ldots, n \).

**Definition 4 (Total Profit SSR):**

Let

\[
\Phi = u^*.
\]

We say that \( TB_1 \) is on the safe side if and only if \( \Phi \geq 0 \).

**Definition 5 (Financial/Demographic Total Profit SSR):**

Let \( \Phi = [\phi_1, \phi_2] \), where

\[
\phi_1 = f u^*, \quad \phi_2 = d u^*.
\]

We say that \( TB_1 \) is on the safe side if and only if \( \phi_1 \geq 0, \phi_2 \geq 0 \).

**Definition 6 (Residual Profits SSR):**

Let \( \Phi = [\phi_1, \phi_2, \ldots, \phi_n] \), where

\[
\phi_t = u^*(t-1, n) \quad t = 1, 2, \ldots, n.
\]

We say that \( TB_1 \) is on the safe side if and only if \( \phi_t \geq 0 \) for \( t = 1, 2, \ldots, n \).

**Definition 7 (Reserves SSR):**

Let \( \Phi = [\phi_1, \phi_2, \ldots, \phi_n] \) where

\[
\phi_t = V_{t-1} - V_{t-1}^* \quad t = 1, 2, \ldots, n
\]
We say that $TB_1$ is on the safe side if and only if $\phi_t \geq 0$ for $t = 1, 2, \ldots, n$.

This requirement means that in each year what is set aside to meet future net liabilities ($V_t$) must be at least equal to the realistic value of future net liabilities themselves ($V_t^*$).

The quantity $V_{t-1} - V_{t-1}^*$ is called the "Loewy increment." (See Hoem (1988) and Loewy (1917).) From equations (1) and (11), after some manipulations we get

$$V_{t-1} - V_{t-1}^* = u^*(t, n)$$

(14)

which becomes, in particular, $V_0 - V_0^* = u^*(0, n)$, i.e.,

$$V_0^* = -u^*.$$  

(15)

Relations among the previous definitions can be easily found. Each definition involves a different degree of strictness. For example, a $TB_1$ which complies with Definition 2 also complies with Definition 3, the latter being less strict than the former. Let us adopt the following notation:

- $(Di)$ denotes Definition $i$, e.g., $(D3)$ refers to Definition 3; and
- $(Di) \Rightarrow (Dj)$ means that a first order technical basis $TB_1$ that is on the safe side according to $(Di)$ is also on the safe side according to $(Dj)$.

It can be easily verified that

$$(D1) \Rightarrow (D2) \Rightarrow (D3) \Rightarrow (D6) \Leftrightarrow (D7) \Rightarrow (D4)$$

and

$$(D2) \Rightarrow (D5) \Rightarrow (D4).$$

Note that a severe requirement that involves a higher premium could produce high profits per policy. High premiums may reduce the demand of the insurance policy, however, and profits for the entire portfolio.
The discussion in this section allows to verify the link between the first two safe-side requirements common in actuarial practice (i.e., \(D1\) and \(D2\), which are severe) and the seventh safe-side requirement (i.e., \(D7\), which has been recently proposed in actuarial literature, see Hoem (1988)). Moreover, the definitions above described give a general picture that allows to understand the meaning of the expression “prudent actuarial valuation.”

Definitions \((D1)\) through \((D7)\), as well as equations (14) and (15), show that the notion of prudence is linked to some notion of implied expected profit. Having used only expected values, it is impossible to consider explicitly measures of demographic or financial riskiness. As a result, the (positive) lower bound for profit can be chosen arbitrarily. It would be interesting, considering definition \((D4)\), to require conditions such as \(\Phi \geq u'\) where \(u'\) represents a minimum value for expected profit (or safety loading in terms of single premium) to be fixed in relation to the riskiness of the insurance contract.

We will briefly comment on an explicit consideration of risk in Section 6. In Section 4 we will introduce a structure that, although based on expected values only, is more general than the one just described. This structure allows us to single out positive lower bounds for profit.

4 Safe-Side Requirements and the Cost of Capital: Toward a Corporate Approach

4.1 Portfolio Fund, Discounted Cash Flow (DCF)

The safe-side requirements analyzed in Section 3 are minimal requirements in a corporate perspective. It is not enough for the policy to generate profits, but such profits must be enough to pay some minimum return to the invested capital. Because shareholders' capital is allocated at a portfolio level, in this section we refer not to a policy only, but to a portfolio of life insurance.

For simplicity, however, we consider a cohort of homogeneous policies, issued at the same time, identical in terms of insurance policy, age at entry, term, benefits, and premiums. Under such assumptions, \(S\) will be the total amount paid in case of survival of all policies at maturity. The amount actually paid at time \(n\) is a random variable that depends on the random number of survivors. At time 0, its expected value is given by \(S \times n p_x^*\) (according to \(TB2\)); similarly, \(V_t \times t p_x^*\) represents the (expected) portfolio reserve at time \(t\). Similar relations hold for the other quantities.
For the sake of brevity, we adopt the following notation

\[ \hat{V}_t = V_t \times t p^*_x \]
\[ \hat{u}_{t+1}^* = u^*_{t+1} \times t p^*_x. \]

For \( t = 0, 1, \ldots, n \), let \( Z_t \) denote the expected portfolio fund accumulated at time \( t \) (according to the information available at time 0) and let \( K_t \) \((t = 0, 1, \ldots, n)\) be the shareholders' capital flow withdrawn from \((K_t > 0)\) or paid to \((K_t < 0)\) the portfolio fund. The sign of \( K_t \) is determined from the point of view of shareholders. The behavior of the portfolio fund can then be described by

\[ Z_{t+1} = Z_t (1 + i^*) + P_t (1 + i^*) t p^*_x - C_{t+1} t p^*_x q^*_x + S - K_{t+1} \] (16)

for \( t = 0, 1, \ldots, n - 1 \), where we assume \( Z_0 = -K_0 \) with \( K_0 \leq 0 \). As \( K_t \) is assumed to be deterministic, no mortality factor is needed.

The analysis of cash flows is also considered in recent actuarial models, for example in profit testing techniques (see Goford (1985)). In that framework, however, shareholders' flows are not included—the main aim is the assessment of technical profit.

We will consider the sequence \( \{K_t\} \) as given flows. In concrete terms, flows depend both on corporate strategies and insurance regulation. In any case, we assume

\[ K_n = Z_{n-1} (1 + i^*) + P_n (1 + i^*) n p^*_x - C_n n-1 p^*_x q^*_x + S - n p^*_x \] (17)

so that \( Z_n = 0 \). Moreover, we suppose that flows \( K_t \) are chosen so that \( Z_t \geq \hat{V}_t \) for \( t = 1, 2, \ldots, n - 1 \). We define "free portfolio fund" as the (nonnegative) quantity \( Z_t - \hat{V}_t \), which consists of the financial resources in excess of the expected reserve. \( Z_t - \hat{V}_t \) represents the shareholders' capital globally linked to the portfolio at time \( t \).

Let \( G(\rho) \) denote the discounted cash flow (DCF) at time 0 for shareholders, calculated with a rate \( \rho \), i.e.,

\[ G(\rho) = \sum_{t=0}^{n} K_t (1 + \rho)^{-t}. \] (18)

The rate \( \rho \) represents the yield required from shareholders on the capital invested in the portfolio (i.e., the opportunity cost of capital). We
assume that \( p \geq i^* \). The DCF can be split into a sequence of periodic contributions. (This notion has been proposed, for a financial operation in general, by Peccati (1989).)

Let \( g_{t+1}(\rho) \) be the contribution at time \( t + 1 \) to the DCF \( G(\rho) \), evaluated at time 0. The splitting of DCF is based on the notion of outstanding capital at each time \( t \), i.e., the capital invested at that time, which, as seen above, is given by the free portfolio fund \( Z_t - \hat{V}_t \). The annual contribution to DCF can then be defined by amending the annual shareholders' flow \( K_{t+1} \) with the variation in the free portfolio fund. Hence, for \( t = 0, 1, \ldots, n - 1 \), we have

\[
\frac{-(Z_t - \hat{V}_t)}{(1 + \rho)^t} + \frac{(K_{t+1} + Z_{t+1} - \hat{V}_{t+1})}{(1 + \rho)^{t+1}}.
\]  

The structure of equation (19) is coherent with that of annual profit as defined in conventional life insurance mathematics. In the latter quantity, however, only debt capital (i.e., the reserve) is taken into consideration (see equation (3)). It can be easily verified that

\[
G(\rho) = \sum_{t=0}^{n-1} g_{t+1}(\rho).
\]  

Subtracting equation (3) (previously multiplied by \( t \rho^* \)) from (16) we obtain

\[
Z_{t+1} - \hat{V}_{t+1} = (Z_t - \hat{V}_t) (1 + i^*) - K_{t+1} + \hat{u}^*_{t+1}.
\]  

Solving for \( Z_t - \hat{V}_t \), with initial condition \( Z_0 - \hat{V}_0 = -K_0 \), we obtain

\[
Z_t - \hat{V}_t = -\sum_{h=0}^{t} K_h (1 + i^*)^{t-h} + \sum_{h=1}^{t} \hat{u}^*_h (1 + i^*)^{t-h}.
\]  

Substituting equation (21) into equation (19) we get

\[
g_{t+1}(\rho) = [\hat{u}^*_t - (Z_t - \hat{V}_t) (\rho - i^*)] (1 + \rho)^{-(t+1)}
\]  

for \( t = 0, 1, \ldots, n - 1 \). Equation (23) shows that the periodical contribution to DCF is equal to the annual profit amended by the loss incurred by investing the free portfolio fund at rate \( i^* \) instead of the required \( \rho \).
From equation (18), noting that $K_n$ is equal to the accumulated value of the insurance profit plus the accumulated value of shareholders' capital flows (as can be checked by substituting equation (22) into equation (17)), we obtain the following expression for DCF:

$$
G(\rho) = (1 + \rho)^{n-n} \left[ u^* \times (1 + i^*)^n \right.
$$

$$
+ \sum_{t=0}^{n} K_t \times (\frac{(1 + \rho)^{n-t} - (1 + i^*)^{n-t}}{1 + \rho}) \bigg].
$$

Expression (24) can be easily interpreted in terms of the loss originating from the difference between $\rho$ and $i^*$. Obviously, equation (24) could be obtained also by discounting back to time 0 the annual losses (i.e., by substituting (23) and (22) into (20)).

Equation (24) allows us to separate two components of DCF:

- The technical component,

$$
(1 + \rho)^{-n} \times u^* \times (1 + i^*)^n,
$$

which stems from the technical management of the insurance portfolio, and

- The capital component,

$$
(1 + \rho)^{-n} \sum_{t=0}^{n} K_t \times ((1 + \rho)^{n-t} - (1 + i^*)^{n-t}),
$$

which stems from the management of shareholders' capital flows.

4.2 More Safe-Side Requirements

The notions of DCF, splitting the DCF into annual contributions and then splitting the annual contributions themselves, suggest other safe-side requirements. We will consider mappings of the form $\Phi = \Phi(TB_1, TB_2, \Gamma, \rho)$.

**Definition 8 (DCF Annual Contributions SSR):**

Let $\Phi = [\phi_1(\rho), \phi_2(\rho), \ldots, \phi_n(\rho)]$ where
\[ \phi_t(\rho) = g_t(\rho) \quad t = 1, 2, \ldots, n, \]

\( \rho \) is the opportunity cost of capital, and \( g(\rho) \) is defined in equation (23). For a given \( \rho \), we say that \( TB_1 \) is on the safe side if and only if \( \phi_t(\rho) \geq 0 \) for \( t = 1, 2, \ldots, n \), i.e., if and only if

\[ \hat{u}_t^* \geq (Z_{t-1} - \hat{V}_{t-1})(\rho - i^*) \quad t = 1, 2, \ldots, n. \] (25)

Definition 8 (D8) leads to some interesting observations, especially when compared with (D3). Considering the nonnegativity of the free portfolio fund, the lower bound for \( \hat{u}_t^* \) depends on the difference between the rates \( \rho \) and \( i^* \) (which can reasonably be assumed to be nonnegative). Note in particular that:

(i) If we require a yield of \( \rho \) on the free portfolio fund that is higher than \( i^* \), equation (25) expresses a more severe condition than (D3), as a positive lower bound for the expected annual profit is imposed (unless the free portfolio fund is equal to zero). In addition, the entity of the lower bound depends on the value of the free portfolio fund; hence, it depends on the strategy concerning the choice of \( \{K_t\} \). (We will comment on some particular cases later.);

(ii) If \( \rho = i^* \), we have (D3);

(iii) The (overall) riskiness of the insurance business can be introduced into the safe-side requirement by properly choosing the rate \( \rho \) (\( \rho > i^* \)), which may include a risk premium.

Definition 9 (DCF SSR):

Let \( \Phi = G(\rho) \), where \( \rho \) is the opportunity cost of capital, and \( G(\rho) \) is defined in equation (18). For a given \( \rho \), we say that \( TB_1 \) is on the safe side if and only if \( G(\rho) \geq 0 \). According to equation (24), \( G(\rho) \geq 0 \) is achieved when

\[ u^* \geq (1 + i^*)^{-n} \sum_{t=0}^{n} -K_t ((1 + \rho)^{n-t} - (1 + i^*)^{n-t}). \] (26)
For certain choices of the sequence \( \{K_t\} \), the capital component of DCF can be positive, in which case equation (26) gives a negative bound for total profit.\(^4\) Equation (26) must then be modified as follows

\[
u^* \geq \max \left\{ 0, (1 + i^*)^{-n} \sum_{t=0}^{n} -K_t ((1 + \rho)^{n-t} - (1 + i^*)^{n-t}) \right\}.
\]

(27)

According to equation (27), we say that a \( TB_1 \) is on the safe side if and only if \( G(\rho) \geq 0 \) and \( u^* \geq 0 \).

Equation (27) can now be compared to (D4). The lower bound that is now required for total profit \( \nu^* \) depends both on the spread \( \rho - i^* \) as well as on the sequence of shareholders' capital flows \( \{K_t\} \). If \( \rho = i^* \) and/or the capital component of DCF is positive, we find \( u^* \geq 0 \), i.e., the requirement of (D4).

It is interesting to examine some cases related to particular choices of \( \{K_t\} \).

(i) Consider an initial investment followed by one withdrawal only at maturity, i.e., \( K_0 < 0, K_1 = K_2 = \cdots = K_{n-1} = 0 \). From equation (27) we find

\[
u^* (1 + i^*)^n \geq -K_0 [(1 + \rho)^n - (1 + i^*)^n].
\]

(28)

Equation (28) points out the need for the total profit to compensate the shareholders for missed returns occurring when the initial investment of shareholders' equity incurs a return of \( i^* \) instead of the required \( \rho \). In terms of annual profits, equations (25) and (22) lead to

\[
\hat{u}_{t+1}^* \geq -K_0 (1 + i^*)^t + \sum_{h=1}^{t} \hat{u}_h^* (1 + i^*)^{t-h} (\rho - i^*),
\]

(29)

which shows that the expected annual profit must cover the missed yield, equal to \( \rho - i^* \), obtained on shareholders' equity accumulated at the beginning of the year. As shown in equation (29),

\(^4\)This is unacceptable from the point of view of looking for premiums to meet the benefits. Selling some insurance policies at a loss, however, might be profitable for an entire portfolio in the long run if other policies generate cash flows that can be invested profitably elsewhere. Throughout the rest of this paper we will disregard this opportunity, as it is not allowed by insurance regulations in many countries.
shareholders' capital (which is equal to the free portfolio fund) at
time $t$ comes from the accumulated value of the initial investment
and previous (undistributed) annual profits.

(ii) Consider the case where $K_0 < 0$ and $K_h = \hat{u}_h^*$ for $h = 1, 2, \ldots, n-1$. In general terms, equations (25) and (22) lead to

$$\hat{u}_{t+1}^* \geq \left[ -K_0 (1 + i^*)^t - \sum_{h=1}^{t} K_h (1 + i^*)^{t-h} \right] (\rho - i^*).$$

(30)

When $K_h > 0$ for $h = 1, 2, \ldots, t$, equation (30) shows the decrease in the lower bound due to shareholders' capital withdrawals. When $K_h = \hat{u}_h^*$, $h = 1, 2, \ldots, t$, equation (30) becomes

$$\hat{u}_{t+1}^* \geq \left[ -K_0 (1 + i^*)^t \right] (\rho - i^*).$$

(31)

The loss is incurred only on the accumulated value of the initial investment. Multiplying both sides of equation (31) by $(1 + i^*)^{-(t+1)}$ and summing with respect to $t$, we obtain

$$u^* \geq -n K_0 (\rho - i^*) (1 + i^*)^{-1}.$$

(32)

The lower bound for total profit provided by equation (32) is more severe than that coming from equation (27), as equation (32) is obtained discounting the annual lower bounds with a higher factor. It can be easily verified that the lower bound implied by equation (28) is greater than that implied by equation (32) (and, therefore, than that implied by equation (27)). When $K_h = \hat{u}_h^*$, $h = 1, 2, \ldots, n - 1$, it is possible to reinvest annual profits at the rate $\rho$, hence weakening the bound on total profit.

(iii) As shown by the above mentioned examples, the notion of prudence depends on the strategy of shareholders' equity (as well as on the yield $\rho$). It is interesting to consider an objective strategy. To this aim, let us define the sequence $\{M_t\}$, $t = 0, 1, \ldots, n - 1$, where $M_t$ is the minimum solvency margin that must be assigned
(according to insurance regulation) to the insurance portfolio at
time \( t \). Suppose the shareholders set their capital flows such that

\[ Z_t - \hat{V}_t = M_t \quad \text{for} \quad t = 0, 1, \ldots, n - 1. \]

The consequent bounds on profit can be interpreted on one hand
as those implied by the opinion on the riskiness of the insurance
business expressed by current legislation (through \( \{ M_t \} \)), and on
the other hand by shareholders (through \( \rho \)).

5 Numerical Examples

We consider two types of policies: (i) a 15-year endowment insur­
ance with face value of 1,000 monetary units that is issued to an Italian
male age 50; and (ii) a 15-year deferred whole life annuity with annual
benefits of 100 monetary units that is issued to an Italian male age 50.
In both policies premiums are level and paid for 15 years, the second
order rate of interest is \( i^* = 0.06 \), and the opportunity cost of capital
is \( \rho = 0.08 \).

5.1 Endowment Insurance

For the endowment insurance, the second order level of mortal­
ity is derived from the Italian Table SIM1992 (which is referred to the
Italian male population, observed in 1992). Denoting by \( q_x^{SIM1992} \)
the rate of mortality calculated according to Table SIM1992, we assume
\( q_x^* = 0.70 q_x^{SIM1992} \).

In Table 1 the traditional approach is adopted; thus prudence is
analyzed only according to profits. Three different technical bases are
used as examples.

- In the first example (Columns (2) to (4) in Table 1), the \( TB_1 \) (\( i =
0.03, \ q = 1.2 \ q^* \)) complies with all safe-side requirements.

- In the second example (Columns (5) to (7) in Table 1), the \( TB_1 \)
(\( i = 0.03, \ q = 0.8 \ q^* \)) is used. This example does not satisfy
(D2). The financial profit in each year, however, is enough to allow
positive annual profits. Thus, (D3) and (D4) as well as (D6) and
(D7), are satisfied.
In the third example (Columns (8) to (10) in Table 1), a higher technical rate of interest has been chosen. Total profit is dramatically reduced as compared to the first example.

In Tables 2 to 4 the corporate approach is implemented. In each table, a different strategy of shareholders' capital flows has been adopted. In Table 2, we have an initial investment \( K_0 < 0 \) and a final withdrawal \( K_1 = K_2 = \cdots = K_{n-1} = 0, K_n > 0 \). The amount of the initial investment has been chosen according to a reasonable solvency fund to be assigned to the portfolio. Both of the technical bases, \( (i = 0.03, q = 1.2 q^*) \) and \( (i = 0.03, q = 0.8 q^*) \), satisfy (D8) and (D9) of prudence.

In Tables 3 and 4 withdrawals of shareholders' capital are permitted also at times \( t = 1, 2, \ldots, n-1 \). The requirements on annual and total profits are relaxed; note, however, that in Table 4 the \( \text{TB}_1 \) \( (i = 0.03, q = 0.8 q^*) \) cannot be adopted as the free portfolio fund becomes negative.

### 5.2 Deferred Annuity

For the deferred annuity, mortality rates are taken from a projected table, which is obtained from Table SIM1992 using an exponential projection model; it reflects the future expected (decreasing) trend of mortality. We point out that the limiting age of the mortality table is \( \omega = 109 \).

In Table 5 the traditional approach is adopted. It is difficult to cover financial losses with mortality profits (and mortality losses with financial profits) throughout the whole insurance period. In Table 6 the strategy \( K_0 < 0, K_1 = K_2 = \cdots = K_{n-1} = 0 \) is examined (also in this case, \( K_0 \) has been chosen according to a reasonable solvency fund to be assigned to the portfolio). Because of the length of the insurance contract, such strategy is unsatisfactory when (D8) is assumed. In Table 7 withdrawals of shareholders' capital at time \( t = 1, 2, \ldots \) are considered. (Their amount has been chosen according to the behavior of annual profits and to the interests required on the initial investment.) Because of the length of the contract, when a sequence \( \{K_t\} \) is given, there is not much freedom in the choice of \( \text{TB}_1 \).

Similar results can be obtained when expenses and other loadings are considered.
### Table 1
Endowment Insurance: Traditional Approach

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<th>$i = 3%$, $q_x = 1.2\ q_x^*$, and $P = 55.572$</th>
<th>$i = 3%$, $q_x = 0.8\ q_x^*$, and $P = 54.441$</th>
<th>$i = 4%$, $q_x = 1.2\ q_x^*$, and $P = 51.479$</th>
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$u^*$ 120.174  $u^*$ 108.885  $u^*$ 79.299
Table 2
Endowment Insurance: Corporate Approach

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<th>$L(\hat{u}^*)$</th>
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$K_n$ = 335.935, $G(p) = 85.901$, $G^T(p) = 90.791$, $G^C(p) = -4.890$, $L(u^*) = 6.473$

$K_n$ = 308.881, $G(p) = 77.372$, $G^T(p) = 82.262$, $G^C(p) = -4.890$, $L(u^*) = 6.473$

Notes: $L(\hat{u}^*)$ denotes the lower bound for $\hat{u}^*$; $L(u^*)$ denotes the lower bound for $u^*$; $G^T(p)$ denotes the technical component of DCF; and $G^C(p)$ denotes the capital component of DCF.
Table 3
Endowment Insurance: Corporate Approach

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Notes: $L(\hat{u}_t^*)$ denotes the lower bound for $\hat{u}_t^*$; $L(u^*)$ denotes the lower bound for $u^*$; $G^T(\rho)$ denotes the technical component of DCF; and $G^C(\rho)$ denotes the capital component of DCF.
### Table 4

#### Endowment Insurance: Corporate Approach

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$K_n$ 43.518  $K_n$ 16.464

$G(\rho)$ 97.152  $G(\rho)$ 88.623

$G^T(\rho)$ 90.791  $G^T(\rho)$ 82.262

$G^C(\rho)$ 6.361  $G^C(\rho)$ 6.361

$L(u^*)$ 0.000  $L(u^*)$ 0.000

Notes: $L(\hat{u}^*_t)$ denotes the lower bound for $\hat{u}_t^*$; $L(u^*)$ denotes the lower bound for $u^*$; $G^T(\rho)$ denotes the technical component of DCF; and $G^C(\rho)$ denotes the capital component of DCF.
Table 5
Deferred Annuity: Traditional Approach

\[ i = 3\%, \, q = 0.8\, q^*, \text{ and } P = 73.689 \]

\[ i = 3\%, \, q = q = 1.1\, q^* \text{ and } P = 64.979 \]

\[ i = 4\%, \, q = 0.8\, q^* \text{ and } P = 64.979 \]

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\[ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \]
Table 5 (continued)
Deferred Annuity: Traditional Approach

\[
\begin{array}{ccccccccc}
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 & \hat{u}_t^* & \hat{f}_t^* & \hat{d}_t^* & \hat{u}_t^* & \hat{f}_t^* & \hat{d}_t^* & \hat{u}_t^* & \hat{f}_t^* & \hat{d}_t^* \\
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46 & 2.317 & 0.807 & 1.510 & -0.011 & 0.588 & -0.599 & 1.999 & 0.521 & 1.478 \\
47 & 1.777 & 0.577 & 1.201 & -0.060 & 0.414 & -0.474 & 1.550 & 0.373 & 1.177 \\
48 & 1.332 & 0.400 & 0.932 & -0.084 & 0.283 & -0.367 & 1.174 & 0.259 & 0.915 \\
49 & 0.967 & 0.268 & 0.699 & -0.088 & 0.186 & -0.274 & 0.861 & 0.174 & 0.688 \\
50 & 0.680 & 0.173 & 0.508 & -0.081 & 0.118 & -0.199 & 0.612 & 0.112 & 0.500 \\
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53 & 0.184 & 0.035 & 0.149 & -0.036 & 0.022 & -0.058 & 0.170 & 0.023 & 0.147 \\
54 & 0.104 & 0.018 & 0.085 & -0.022 & 0.011 & -0.033 & 0.097 & 0.012 & 0.085 \\
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\hline
u^* & 326.378 & u^* & 239.886 & u^* & 206.533 \\
\hline
\end{array}
\]
Table 6
Deferred Annuity: Corporate Approach

\[ i = 3\% \text{ and } q = 1.2q^* \]

\[ i = 3\% \text{ and } q = 0.8q^* \]

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Table 6 (continued)

Deferred Annuity: Corporate Approach

\[ i = 3\% \text{ and } q = 1.2 \, q^* \]

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\[ K_n \] 12024.263 \hspace{1cm} 8294.630

\[ G(\rho) \] 68.250 \hspace{1cm} 28.470

\[ G^T(\rho) \] 108.334 \hspace{1cm} 68.554

\[ G^C(\rho) \] -40.084 \hspace{1cm} -40.084

\[ L(u^*) \] 120.762 \hspace{1cm} 120.762
Table 7
Deferred Annuity: Corporate Approach

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### Table 7 (Continued)

**Deferred Annuity: Corporate Approach**

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| $K_n$ | 58.797 | $K_n$ | -3670.836 |
| $G(\rho)$ | 230.785 | $G(\rho)$ | 191.005 |
| $G^T(\rho)$ | 108.334 | $G^T(\rho)$ | 68.554 |
| $G_C(\rho)$ | 122.451 | $G_C(\rho)$ | 122.451 |
| $L(u^*)$ | 0.000 | $L(u^*)$ | 0.000 |
6 Some Final Remarks

This paper first considers a purely actuarial approach to prudence and then shows how it is possible to introduce risk measures in a structure based on expected values only. Risk is introduced through the cost of capital and the amount of shareholders' capital (i.e., the free portfolio fund) linked to the insurance portfolio.

Safe-side requirements can also be formulated in a stochastic framework by considering the distribution of the random profits. Below is a possible definition:

**Definition 10 (A Possible Stochastic Safe-Side Requirement):**

Let $R$ denote the present value at time 0 of the future random profits, and let

$$ \Pr[R \leq r] = F_R(r \mid TB_1, TB_2). $$

We can say that a $TB_1$ is on the safe side if and only if

$$ F_R(0 \mid TB_1, TB_2) \leq p $$

where $p$ is a given bound.

Similar (and more strict) definitions can be given considering annual random profits or their components. Several analytical results can be used in this approach; among the more recent contributions, we mention Hesselager and Norberg (1996), which deals with multistate models.

An individual approach (i.e., based on a single contract) has the disadvantage that in order to limit the safety loading, low levels of $p$ must be chosen. A collective approach based on the entire portfolio may help in quoting competitive premiums; however, forecasts on the future size and composition of the portfolio are required, thus leading to a further element of uncertainty in the choice of the first order basis.

Finally, we must emphasize that an approach based on expected values only has the advantage that the results, relative to a whole portfolio, are linear in respect of those relative to single cohorts. On the other hand, a stochastic approach to prudence may lead to a more comprehensive classification of the notion of safe-side requirements.
References


