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BRIDGE SPECTRA OF CABLES OF 2-BRIDGE KNOTS

by

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We compute the bridge spectra of cables of 2-bridge knots. We also give some results about bridge spectra and distance of Montesinos knots.

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DEDICATION

For Liana.

Table of Contents

Chapter 1

Introduction

Knot theory is the study of knotted curves in space. Informally, one should imagine a rope which is tangled up and then has the ends connected to form one continuous loop. A major question in the field is, given a knot, can we untangle it? This is, in general, a very hard question to answer for a specific knot. So, to make some progress, we use a type of tool to help us, called knot invariants. An invariant is a function that gives a value that does not change even if you change how you are looking at the knot. Often these invariants have properties that we can exploit to answer important questions.

One such question has to do with connecting two knots to make a single knot. If we take two knots which are not tangled together, break each apart and connect them together to form a single knot, can we get the knots to untie to a simple circle instead of being knotted? In 1954, Horst Schubert defined an invariant called the bridge number. This is defined as the smallest number of local maxima in any way you look at a knot. He proved a very important property of this invariant: that it must go up when connecting two non-trivial knots. Using the fact that only the unknot, or the knot without any "knotting" in it, has a bridge number of one, this implies that he answered the question we posed at the beginning of this paragraph in the negative.

The bridge spectrum of a knot is a generalization of Schubert's bridge number and we explore some of its properties here. In this dissertation, we compute the bridge spectra of a variety of classes of knots. Until now, the classes of knots for which the bridge spectrum was known was relatively short: iterated torus knots (which include torus knots), 2-bridge knots, high distance knots, and partial results for twisted torus knots. In this dissertation, we add to this list all cables of 2-bridge knots and a class of generalized Montesinos knots, including all pretzel knots, satisfying a condition on their tangled regions.

The bridge spectrum of a knot K is a strictly decreasing list of nonnegative integers first defined by Doll [4] and Morimoto and Sakuma [14]. This list is obtained from (g, b) -splittings of a knot in S^3. A (g, b)*-splitting* of a knot K is a Heegaard splitting of $S^3 = V_1 \cup_\Sigma V_2$ such that the genus of V_i is g, Σ intersects K transversely, and $V_i \cap K$ is a collection of b trivial arcs for $i = 1, 2$. The *genus g bridge number* $b_g(K)$ is the minimum number b for which a (g, b)-splitting exists. The *bridge spectrum* for K, which we denote $\mathbf{b}(K)$, is the list

$$(b_0(K), b_1(K), b_2(K), \ldots).$$

Note that $b_0(K)$ is the classical bridge number, $b(K)$, first defined by Schubert [19]. The fact that bridge spectrum is strictly decreasing comes from the process called *meridional stabilization* where, given a (g, b)-splitting, one can create a $(g+1, b-1)$-splitting. Thus, every bridge spectrum is bounded above componentwise by the following sequence, which is called a *stair-step* spectrum,

$$\left(b_0\left(K\right), b_0\left(K\right) - 1, b_0\left(K\right) - 2, \ldots, 2, 1, 0\right).$$

An invariant of a bridge surface of a knot is the distance, where we are considering

distance in terms of the curve complex of the bridge surface (see page 10 and Definition 2.8.1 for relevant terms). Tomova showed in [21], with results of Bachman and Schleimer from [1], the following theorem, as stated by Zupan in [24]:

Theorem 1.0.1. *[21] Suppose K is a knot in S^3 with a $(0, b)$-bridge sphere Σ of sufficiently high distance (with respect to b). Then any (g', b')-bridge surface Σ' satisfying $b' = b_{g'}(K)$ is the result of meridional stabilizations performed on Σ. Thus*

$$\mathbf{b}(K) = (b_0(K), b_0(K) - 1, b_0(K) - 2, \ldots, 2, 1, 0).$$

A natural question to ask is the following:

Question 1.0.2. *Given a knot, K, if the bridge spectrum of K is stair-step, does K necessarily have a $(0, b)$-bridge sphere that is of high distance?*

We answer this question in the negative in Section 4.2 by showing the following two results:

Corollary 4.2.2. *Given a pretzel knot $K_n = K_n(p_1, \ldots, p_n)$ with $\gcd(p_1, \ldots, p_n) \neq 1$, then the primitive bridge spectrum is stair-step, i.e. $\hat{\mathbf{b}}(K_n) = (n, n-1, \ldots, 2, 1, 0)$ and $\mathbf{b}(K_n) = (n, n-1, \ldots, 3, 2, 0)$.*

Proposition 4.2.4. *If $K_n(p_1, \ldots, p_n)$ is a pretzel knot with $n \geq 4$, and P a genus zero bridge surface for K_n, then $d(P, K_n) = 1$.*

By these two results, we see that stair-step bridge spectrum does not imply high distance.

Theorem 1.0.1 tells us that a generic knot has stair-step bridge spectrum. There is a *gap* at index g in the bridge spectrum if $b_g(K) < b_{g-1}(K) - 1$. Most work regarding bridge spectra focuses on finding bridge spectra with gaps, see [24], [3]. This includes

Theorem 5.2.1 and is the topic of most of the conjectures in Chapter 6. The rest of this section is devoted to the statement Theorem 5.2.1.

Lustig and Moriah [11] define generalized Montesinos knots, which include Montesinsos knots and pretzel knots. Generalized Montesinos knots, see Defintion 2.9.2, are defined with a set of pairs of integers, $\{(\beta_{i,j}, \alpha_{i,j})\}_{i,j}$; when $\gcd\{\alpha_{i,j}\} \neq 1$, we show that these generalized Montesinos knots have the property $t(K) + 1 = b(K) = b_0(K)$, where $t(K)$ is the tunnel number, see section 2.7. For a (g, b)-splitting, with $b > 0$, we have $t(K) \leq g + b - 1$, which comes to us from Morimoto, Sakuma, and Yokota [15]. From these two facts, we can quickly conclude that generalized Montesinos knots with $\gcd\{\alpha_{i,j}\} \neq 1$ have stair-step bridge spectra. We also show that pretzel knots, which are generalized Montesinos knots, have distance one, and so they do not satisfy the hypothesis of Tomova's theorem.

We are interested in the behavior of the bridge spectrum under cabling, a special case of taking a satellite of a knot. In the same paper that Schubert [19] defined bridge number, he proved the following, which we will make use of here. Schultens gives a modern proof of this result in [20]. See Section 2.5.

Theorem 1.0.3. *[19],[20] Let K be a satellite knot with companion J and pattern of index n. Then $b_0(K) \geq n \cdot b_0(J)$.*

In Chapter 5 we show:

Theorem 5.2.1. *Let $K_{p/q}$ be a non-torus 2-bridge knot and $T_{m,n}$ an (m, n)-torus knot. If $K := \mathsf{cable}(T_{m,n}, K_{p/q})$ is a cable of $K_{p/q}$ by $T_{m,n}$, then the bridge spectrum of K is $\mathbf{b}(K) = (2m, m, 0)$.*

In Chapter 2 we give the relevant background and notation needed for the the proofs of Chapter 4 and Theorem 5.2.1. In Chapter 3 we give some necessary results and lemmas for the proof of the Theorem 5.2.1. Chapter 4 has proofs of Corollary 4.2.2

and Proposition 4.2.4 and details about a class of knots which are stair-step but not of high distance. In Chapter 5 we prove Theorem 5.2.1. Finally, in Chapter 6 we give some conjectures based on these results.

Chapter 2

Notation and Background

2.1 Preliminaries

We assume a general knowledge of the basics of knot theory and 3-manifolds. See [6] and [18] for more detail on any topics in this section. A *knot* K is an embedded copy of S^1 in S^3. More precisely, a knot K is the isotopy class of images of embeddings of $S^1 \hookrightarrow S^3$. A *link* is multiple copies of S^1 simultaneously embedded in S^3. Two knots are *equivalent* if there is an ambient isotopy taking one knot to the other. Let $N(\cdot)$ and $\eta(\cdot)$ denote closed and open regular neighborhoods. The space $E(K) := S^3 - \eta(K)$ is called the *exterior* of the knot K and $\partial E(K)$ is a torus. We will use $|A|$ to denote the number of connected components of A.

Let M be a 3-manifold, $S \subset M$ an embedded surface and $J \subset M$ an embedded 1-manifold. We will use $M(J)$ and S_J to denote $M - \eta(J)$ and $S - \eta(J)$, respectively. We use the notation (M, J) to denote the pair of M and J. An embedded 1-manifold $\gamma \subset S$ is *essential* if $S \setminus \eta(\gamma)$ does not have a disk component in S. A submanifold $M' \subset M$ is *proper* if $\partial M' \subset \partial M$. A *compressing disk* D for S is a embedded disk in M with $D \cap S = \partial D$ but ∂D does not bound a disk in S. A *boundary compressing*

disk, or *∂-compressing disk*, Δ for S is an embedded disk in M such that $\Delta \cap S = \gamma$ is an arc with $\gamma \subset \partial\Delta$, γ is essential in S and $\partial\Delta - \gamma$, another arc, is a subset of ∂M. The surface S is *incompressible* if there does not exist a compressing disk D for S; S is *∂-incompressible* if there does not exist a ∂-compressing disk Δ for S. The surface S is said to be *essential* if S is incompressible, ∂-incompressible and not isotopic into ∂M.

2.2 2-bridge knots and rational tangles

One way to create a 2-bridge knot is to start with a rational tangle.

Definition 2.2.1. *Given a rational number $p/q \in \mathbb{Q}$, write*

$$\frac{p}{q} = r + \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_k}}}}$$

such that $r \in \mathbb{Z}$, $b_i \in \mathbb{Z} \setminus \{0\}$, then $r + [b_1, b_2, \ldots, b_k]$ denotes a partial fraction decomposition *of p/q.*

It should be noted that partial fraction decompositions are not unique in general. But if we assume the b_i all have the same sign and k is odd, then the partial fraction decomposition is unique, see [10]. Given a partial fraction decomposition $\frac{p}{q} = r + [b_1, b_2, \ldots, b_k]$, one can produce a diagram that is formed from a vertical 3 braid with a_1 half-twists between the left two strands, below that, add a_2 half-twists between the right two strands. Then alternate between the left and right two strands

in this way for each consecutive a_i. Finally, add a fourth strand on the left, cap and connect the new left two strands at the top and connect the right two strands at the bottom. The diagram we have created is called a *4-plat rational tangle*. See Figure 2.1.

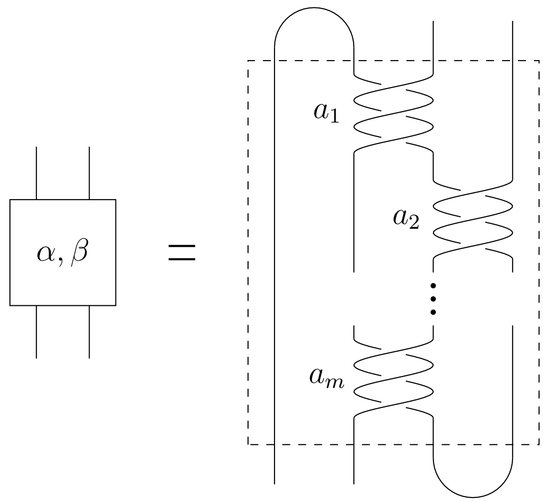

Figure 2.1: A rational tangle in the 4-plat form.

The "pillowcase" is a term used by Hatcher and Thurston in [7]. Technically, one should define it as $I^2 \sqcup_{\partial I^2} I^2$, where one copy of I^2 is the "front" of the pillowcase and the other is the "back." A *2-bridge tangle* is given by a fraction $p/q \in \mathbb{Q}$ by placing lines of slope p/q on the front of the pillowcase and connecting them with lines of slope $-p/q$ on the back of the pillowcase. See Figure 2.2. For a more detailed treatment of this section, see [7].

These two different diagrams yield isotopic tangles if and only if they represent the same rational number. We can create a *2-bridge knot* by taking a 4-plat rational tangle or the 2-bridge tangle and connecting the top two strands together and the bottom two strands together.

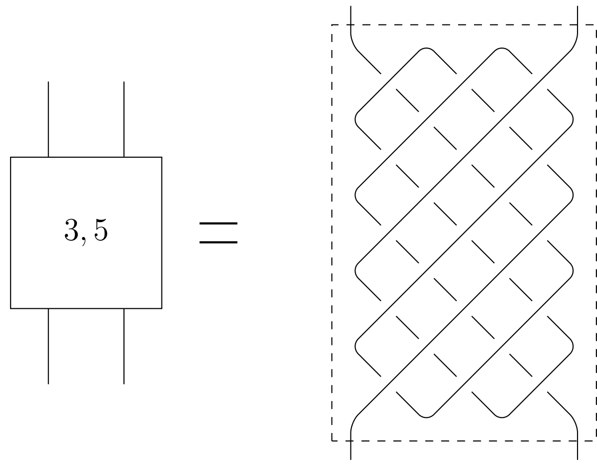

Figure 2.2: A 2-bridge rational tangle of $\frac{3}{5}$.

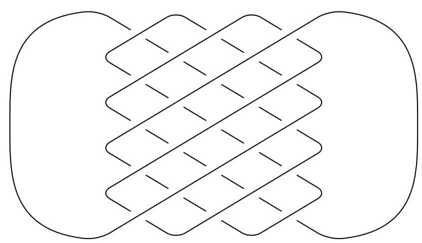

Figure 2.3: The 2-bridge knot corresponding to $\frac{3}{5}$.

2.3 Bridge Spectrum

Given a 3-manifold M with boundary, a *trivial arc* is a properly embedded arc α, see page 6, that cobounds a disk with an arc $\beta \subset \partial M$; i.e., $\alpha \cap \beta = \partial\alpha = \partial\beta$ and there is an embedded disk $D \subset M$ such that $\partial D = \alpha \cup \beta$. We call the disk D a *bridge disk*. A *bridge splitting* of a knot K in M is a decomposition of (M, K) into $(V_1, A_1) \cup_\Sigma (V_2, A_2)$, where each V_i is a handlebody with boundary Σ and $A_i \subset V_i$ is a collection of trivial arcs for $i = 1, 2$. One should note that when we exclude the knot K, a bridge splitting is a *Heegaard splitting of M*. For all g and b, a *(g, b)-splitting* of (M, K) is a bridge splitting with $g(V_i) = g$ and $|A_i| = b$ for $i = 1, 2$. The surface Σ is

called a *bridge surface.* Throughout this dissertation, we will be focusing on bridge splittings of (S^3, K).

Definition 2.3.1. *The* genus g bridge number *of a knot K in S^3, $b_g(K)$, is the minimum b such that a (g, b)-splitting of K exists. We also require that for $b_g(K)$ to be zero, the knot must be able to be isotoped into a genus g Heegaard surface.*

Definition 2.3.2. *The* bridge spectrum *of a knot K in S^3, $\mathbf{b}(K)$, is the list of genus g bridge numbers:*

$$(b_0(K), b_1(K), b_2(K), \ldots).$$

As mentioned earlier, the genus zero bridge number is the classical bridge number, except in the case of the unknot. A simple closed curve in the boundary of a handlebody is called *primitive* if it transversely intersects the boundary of a properly embedded essential disk of the handlebody in a single point. Some define $b_g(K) = 0$ when K can be isotoped into a genus g Heegaard surface and is primitive. Here is another invariant, $\hat{b}_g(K)$ with the following added requirement.

Definition 2.3.3. *The* (primitive) genus g bridge number *of a knot K in S^3, $\hat{b}_g(K)$, is the minimum b such that a (g, b)-splitting of K exists. We also require that for $b_g(K)$ to be zero, the knot must be able to be isotoped into a genus g Heegaard surface and is primitive.*

Definition 2.3.4. *The* (primitive) bridge spectrum *of a knot K in S^3, $\hat{\mathbf{b}}(K)$ is the list of genus g bridge numbers:*

$$(\hat{b}_0(K), \hat{b}_1(K), \hat{b}_2(K), \ldots).$$

In the next section we will see that the process of meridional stabilization forces the bridge spectrum and primitive bridge spectrum to be strictly decreasing, see Proposition 2.4.1. Thus, when $b_g(K) = 0$, we can discard the bridge number for higher genus than g. The only potential difference between these two spectra is the last non-zero term in the sequence. We make the relation between bridge spectrum and primitive bridge spectrjm rigorous for future use.

Proposition 2.3.5. *For a knot K, if $b_g(K) \neq 0$, then $b_g(K) = \hat{b}_g(K)$. If $b_g(K) = 0$, then $\hat{b}_g(K) \in \{0, 1\}$.*

Proof. If $b_g(K) \geq 1$, then by definition, $\hat{b}_g(K) = b_g(K)$. If $b_g(K) = 0$, then K embeds in a genus g Heegaard surface, Σ. If K is primitive on one side of Σ, then $\hat{b}_g(K) = 0$. If K is not primitive on either side of Σ, then through the process of elementary stabilization, we can create a $(g, 1)$-splitting. Hence, $\hat{b}_g(K) \leq 1$, completing the proof. □

For example, consider the following well known proposition:

Proposition 2.3.6. *Given a non-trivial torus knot $K = T_{p,q}$ in S^3, we have $\mathbf{b}(K) = (\min\{p, q\}, 0)$, while $\hat{\mathbf{b}}(K) = (\min\{p, q\}, 1, 0)$.*

Proof. Schubert proved that $b_0(K) = \hat{b}_0(K) = \min\{p, q\}$, which is greater than one for nontrivial knots. For $b_1(K)$, by definition, a torus knot embeds on a genus one surface. Hence, $b_1(K) = 0$. For $\hat{b}_1(K)$, we see that a nontrivial torus knot cannot embed on a torus and intersect an essential disk of a genus one torus only once. By definition of a torus knot, it must intersect the meridian disk p times and the longitudinal disk q times. Hence, $\hat{b}_1(K) \geq 1$, but we can easily see that we can make a $(1, 1)$-splitting, thus $\hat{b}_1(K) = 1$. And for $\hat{b}_2(K)$, we can easily embed a torus knot on a genus two surface with one handle having only a single arc of the knot. □

So they are distinct invariants. But there are knots for which they coincide. The proof above also gives us part of the proof of the following proposition. The other direction follows directly from the definition of genus one bridge number.

Proposition 2.3.7. *A knot K, in S^3, is a torus knot if and only if $b_1(K) = 0$.*

Thus, any non-torus knot has $b_1(K) \geq 1$, which gives us the following, along with the fact that bridge spectrum are strictly decreasing sequences.

Proposition 2.3.8. *Let K be a 2-bridge knot that is not a torus knot in S^3; then $\hat{\mathbf{b}}(K) = \mathbf{b}(K) = (2, 1, 0)$.*

2.4 Operations on bridge splittings and multiple bridge splittings

This section is a summary of tools that we will need in the proof of our theorem. Many come from work of numerous people but directly these can also be found in [24]. There are three main ways to operate on a bridge surface on a knot in S^3 to obtain a new bridge splitting: stabilization, perturbation, and meridional stabilization.

The genus of a surface can be increased by adding a handle which does not interact with the knot through a process called *elementary stabilization*. A properly embedded arc α, see page 6, is said to be *boundary parallel* in a 3-manifold M if it isotopic rel boundary into ∂M. Let $(S^3, K) = (V_1, A_1) \cup_\Sigma (V_2, A_2)$ and let α be a boundary parallel arc in V_1 such that $\alpha \cap A_1 = \emptyset$. Then let $W_1 = V_1 - \eta(\alpha)$, $W_2 = V_2 \cup N(\alpha)$, and $\Sigma' = \partial W_1 = \partial W_2$. Then $(S^3, K) = (W_1, A_1) \cup_{\Sigma'} (W_2, A_2)$ is a new bridge splitting with the genus of Σ' one higher than Σ. We can also run this process in reverse. If D_i are compressing disks in (V_i, A_i) for $\Sigma - \eta(K)$, and $|D_1 \cap D_2| = 1$, then $\partial N(D_1 \cup D_2)$

is a 2-sphere which intersects Σ in a single curve. Then compression along this curve yields a new bridge surface Σ'' of lower genus, and Σ is said to be *stabilized*.

The number of trivial arcs in A_1 and A_2 can be increased by one each, through the process of *elementary perturbation*. Add in a canceling pair of trivial arcs in a sideways "S" shape, cutting through the surface; i.e. a strand which passed down through the surface transversally can be perturbed and a section of the arc can be brought up through the surface again, creating two new arcs. Again, for the reverse direction, if there are two bridge disks on either side of Σ, which intersect in a single point contained in J, one may construct an isotopy which cancels two arcs of A_1 and A_2, creating a new surface Σ'', and Σ is *perturbed*.

Bridge spectra are always strictly decreasing sequences. To see this, take any (g, b)-splitting of a knot. Then, by definition, we have a decomposition of (S^3, k) as $(V_1, A_1) \cup_\Sigma (V_2, A_2)$. Intuitively, take any trivial arc α in, say, A_1; then $N(\alpha) \subset V_1$ is a closed neighborhood of α in V_1. Take this neighborhood from V_1 and move it to V_2, which produces one higher genus handlebodies, and one less trivial arc. More carefully, let $W_1 = (V_1 - \eta(\alpha))$ and $W_2 = (V_2 \cup N(\alpha))$. Let $B_1 = A_1 - \{\alpha\}$, and $B_2 = A_2 \cup \alpha$. Then B_2 is A_2 with two arcs combined into a single arc by connecting them with α. Also, notice that $g(W_i) = g(V_i) + 1$. Thus, we have $(S^3, k) = (W_1, B_1) \cup_{\Sigma'} (W_2, B_2)$, which is a $(g + 1, b - 1)$-splitting. This process is called *meridional stabilization*. This process proves the following proposition.

Proposition 2.4.1. *If K is a knot in S^3, with $\hat{b}_g(K) \geq 1$, then $\hat{b}_{g+1}(K) \leq \hat{b}_g(K) - 1$. Similarly, if $b_g(K) \geq 1$, then $b_{g+1}(K) \leq b_g(K) - 1$.*

The next corollary is immediate from Proposition 2.4.1.

Corollary 2.4.2. *If K is a knot in S^3, then its bridge spectrum and primitive bridge spectrum are bounded above, component-wise, by $(b_0(K), b_0(K) - 1, b_0(K) - 2, \ldots)$.*

Equivalently, for every knot K, and every $g \leq b_0(K)$, there is a $(g, b_0(K) - g)$-splitting for K.

For a bridge surface Σ in (S^3, K), one can sometimes find two bridge disks on opposite sides of Σ which have two points of intersection in K. In this case, the component of J is isotopic into Σ and Σ is called *cancelable.*

A (g, b)-surface for a knot K is said to be *irreducible* if it is not stabilized, perturbed, meridionally stabilized, or cancelable. Thus, if $b_g(K) < b_{g-1}(K) - 1$, then a (g, b)-surface Σ satisfying $b = b_g(K)$ must be irreducible.

In this paragraph, we will use definitions on page 6. Let Σ be a bridge surface for (S^3, K) which yields the splitting $(S^3, K) = (V_1, A_1) \cup_\Sigma (V_1, A_1)$. Then Σ is called *weakly reducible* if there exist disjoint disks D_1 and D_2, that are either both compressing, both bridge, or one of each, such that $D_i \subset (V_i, A_i)$ for $i = 1, 2$, for Σ_J. If Σ is not weakly reducible, perturbed, or cancelable, then Σ is called *strongly irreducible.* By considering bridge disks as embedded in $M(J)$, one can see that perturbed and cancelable surfaces will be weakly reducible; hence, in $M(J)$, Σ is strongly irreducible if and only if it is not weakly reducible.

For a more details about the following comcepts, see [8]. Let F be a disjoint union of closed oriented surfaces. A *compression body* C is a handlebody or the 3-manifold obtained by attaching 1-handles to $F \times \{1\} \subset F \times I$. Then let $\partial_- C = F \times \{0\}$ and let $\partial_+ C = \partial C - \partial_- C$. An arc in a compression body is said to be *vertical* if it is isotopic to $\{x\} \times I$ for $x \in F$. Next, a *multiple bridge splitting* is the following: let (M, J) contain a collection $\mathscr{S} = \{\Sigma_0, S_1, \Sigma_1, \ldots, S_d, \Sigma_d\}$ of disjoint surfaces transverse to J, such that (M, J) cut along $\mathscr{S}$ is a collection of compression bodies containing trivial arcs $\{(C_0, \tau_0), (C'_0, \tau'_0), \ldots, (C_d, \tau_d), (C'_d, \tau'_d)\}$, where

- $(C_i, \tau_i) \cup_{\Sigma_i} (C'_i, \tau'_i)$ is a bridge splitting of a submanifold (M_i, J_i), where $M_i =$

$C_i \cup C'_i$ and $J_i = \tau_i \cup \tau'_i$, for each i,

- $\partial_- C_i = \partial_- C'_{i-1} = S_i$ for $1 \le i \le d$,
- $\partial M = \partial_- C_0 \cup \partial_- C'_d$, and
- $J = \bigcup_{i=1}^{d} (\tau_i \cup \tau'_i)$.

The surfaces Σ_i are called *thick* and the surfaces S_j are called *thin*. The thick surface Σ_i is *strongly irreducible* if it is strongly irreducible in the manifold $(C_i, \tau_i) \cup_{\Sigma_i} (C'_i, \tau'_i)$, and a multiple bridge splitting is called *strongly irreducible* if each thick surface is strongly irreducible and no compression body is trivial. A compression body is trivial if it is homeomorphic to $\Sigma_i \times I$ with τ_i only vertical arcs. This leads us to the following theorem of Hayashi and Shimokawa [8], which we present in the same way that Zupan does in [24]. Theorem 2.4.3 is the basis for one of the two major cases in the proof of Theorem 5.2.1.

Theorem 2.4.3. *[8],[24, Theorem 2.8] Let M be a 3-manifold containing a 1-manifold J. If (M, J) has a strongly irreducible multiple bridge splitting, then $\partial(M - \eta(J))$ and every thin surface are incompressible. On the other hand, if $\partial(M - \eta(J))$ is incompressible in $M - \eta(J)$ and Σ is a weakly reducible bridge splitting for (M, J), then (M, J) has a strongly irreducible multiple bridge splitting $\{\Sigma_0, S_1, \Sigma_1, \dots, S_d, \Sigma_d\}$ satisfying*

$$g(\Sigma) = \sum_{i=0}^{d} g(\Sigma_i) - \sum_{i=1}^{d} g(S_i).$$

2.5 Cable spaces and Cables

A cable space, loosely, is the solid torus with a torus knot taken out of its interior. For more information on cable spaces, see [5] and [24]. More precisely, let $T_{m,n}$ be

the torus knot on the standardly embedded torus in S^3 intersecting the meridian transversely in m points and intersecting the longitude transversely in n points. Let V be the solid torus $S^1 \times D^2$ with the torus knot pushed into the interior of V off the boundary. The *cable space* $C_{m,n}$ is $V - \eta(T_{m,n})$. Cable spaces are Seifert fibered. For more information on Seifert fibered spaces, see Waldhausen [22]. There one will find that essential surfaces, see page 6 in Seifert fibered spaces are either vertical or horizontal. A *vertical* surface in a Seifert fibered space made up of a union of fibers. A *horizontal* surface in a Seifert fibered space is transverse to any fiber it intersects. In a cable space $C_{p,q}$, the notation $\partial_+ C_{p,q}$ is used to denote the boundary of the solid torus V and $\partial_- C_{p,q}$ denotes the boundary of the (p, q)-torus knot.

The following is Zupan's Lemma 3.2 and 3.3.

Lemma 2.5.1. *[24, Lemma 3.2] Suppose $S \subset C_{p,q}$ is incompressible. If each component of $S \cap \partial_+ C_{p,q}$ has integral slope, then each component of $S \cap \partial_- C_{p,q}$ also has integral slope.*

Lemma 2.5.2. *[24, Lemma 3.3] Suppose $S \subset C_{p,q}$ is incompressible. Then $S \cap \partial_+ C_{p,q}$ is meridional if and only if $S \cap \partial_- C_{p,q}$ is meridional.*

A *cable* of a knot is defined in the following way. Given a knot K_0 in S^3, and a torus knot $T_{m,n}$, the cable $K := \mathsf{cable}(T_{m,n}, K_0)$ is the knot obtained by taking $S^3 \setminus \eta(K_0)$ and gluing in the solid torus V with $T_{m,n}$ pushed slightly into the interior of V. The space V is glued in so that the cable has the preferred framing, that is, the usual longitude of V is mapped to the the trivial element in $H_1(E(K_0))$. Then the knot K is called the (m, n)-*cable* of K_0.

2.6 Incompressible surfaces in 2-bridge knot complements

This section will mostly be devoted to Hatcher's and Thurston's result about essential surfaces, in the complement of 2-bridge knots. See [7] for more information.

We will devote this paragraph to describing how these surfaces $S_n(n_1, n_2, \ldots, n_{k-1})$ are defined. Given a continued fraction decomposition $r + [b_1, b_2, \ldots, b_k]$ of $\frac{p}{q}$, form the corresponding 4-plat rational tangle in S^3, see Definition 2.2, and create a link by connecting the top two strands together and bottom two strands together. Since we are only investigating knots in this dissertation, we will assume that q is odd in the reduced fraction $\frac{p}{q}$. Isotope the knot into a vertical square tower, see Figure 2.4. This knot has $k-1$ *inner horizontal plumbing squares*, one between each twisted region. For each inner plumbing square, there is a complementary *outer plumbing square* which exists in the same horizontal plane in S^3, thinking of S^3 as $(S^2 \times [0,1])/(S^2 \times \{0\}, S^2 \times \{1\})$, see Figure 2.5. The surface $S_n(n_1, n_2, \ldots, n_{k-1})$, where $n \geq 1$ and $0 \leq n_i \leq n$, consists of n parallel sheets running close to the vertical bands of this tower form of the knot. At the i-th plumbing square, n_i of the n sheets run into the inner plumbing square and the other $n - n_i$ sheets run into the outer plumbing square. See Figure 2.6 for a small example.

The branched surface $\Sigma[b_1, \ldots, b_k]$ is obtained by a single sheet running vertically and branching at each plumbing square into the inner and the outer plumbing square, see Figure 2.5. So for each continued fraction decomposition $r + [b_1, b_2, \ldots, b_k]$, the branched surface $\Sigma[b_1, \ldots, b_k]$ carries many not necessarily connected surfaces $S_n(n_1, n_2, \ldots, n_{k-1})$.

Hatcher and Thurston, [7], give the following classification of incompressible surfaces.

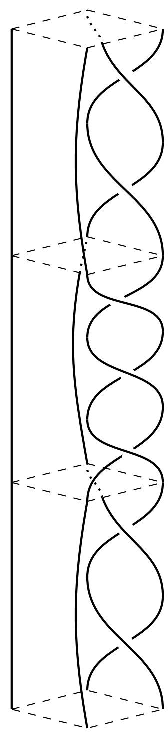

Figure 2.4: A 2-bridge knot in square tower form.

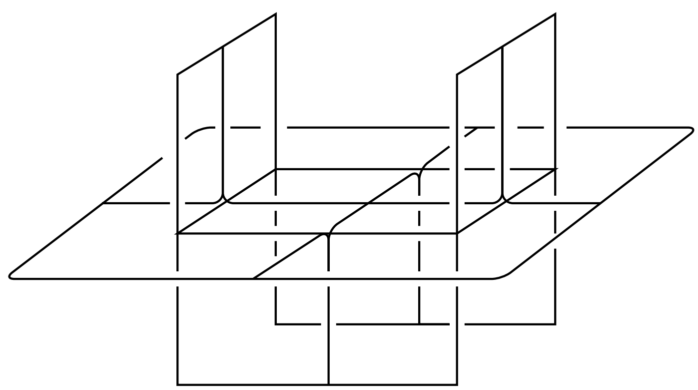

Figure 2.5: A plumbing square, with n vertical sheets, n_i sheets in the inner plumbing square, and $n - n_i$ sheets in the outer square.

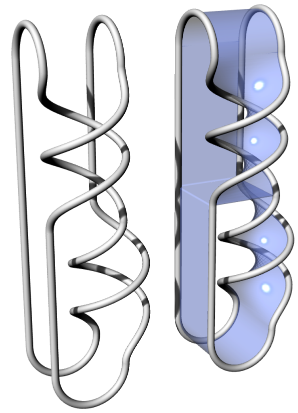

Figure 2.6: On the left, the figure 8 knot, and on the right the same knot with the surface $S_1(1)$.

Theorem 2.6.1. *[7, Theorem 1] Let $\frac{p}{q}$ be a rational number with continued fraction decomposition $r + [b_1, b_2, \ldots, b_k]$.*

1. *A closed incompressible surface in $S^3 - K_{p/q}$ is a torus isotopic to the boundary of a tubular neighborhood of $K_{p/q}$.*

2. *A non-closed incompressible, ∂-incompressible surface in $S^3 - K_{p/q}$ is isotopic to one of the surfaces $S_n(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$, for some continued fraction expansion $p/q = r + [b_1, \ldots, b_k]$ with $|b_i| \geq 2$ for each i.*

3. *The surface $S_n(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$ is incompressible and ∂-incompressible if and only if $|b_i| \geq 2$ for each i.*

4. *Surfaces $S_n(n_1, \ldots, n_{k-1})$ carried by distinct $\Sigma[b_1, \ldots, b_k]$'s with $|b_i| \geq 2$ for each i are not isotopic.*

5. *The relation of isotopy among the surfaces $S_n(n_1, \ldots, n_{k-1})$ carried by a given $\Sigma[b_1, \ldots, b_k]$ with $|b_i| \geq 2$ for each i is generated by:*

$S_n(n_1, \ldots, n_{i-1}, n_i, \ldots, n_{k-1})$ *is isotopic to* $S_n(n_1, \ldots, n_{i-1}+1, n_i+1, \ldots, n_{k-1})$ *if* $b_i = \pm 2$. *(When* $i = 1$ *this means* $S_n(n_1, n_2, \ldots, n_{k-1})$ *is isotopic to* $S_n(n_1 + 1, n_2 \ldots, n_{k-1})$*, and similarly when* $i = k$*.)*

Remark 2.6.2. *The main points of this theorem that we will use are* (1) *and* (2)*, which gives us that the only closed incompressible surfaces in the complement of a 2-bridge knot* $K_{p/q}$ *are isotopic to the boundary of* $E(K_{p/q})$ *and that every incompressible surface with boundary is isotopic to some* $S_n(n_1, \ldots, n_k)$*.*

2.7 Tunnel Number

Recall $E(K)$ is the exterior of a knot. A family of mutually disjoint properly embedded arcs Γ in the exterior $E(K)$ of a knot K is said to be an *unknotting tunnel system* if $E(K) - \eta(\Gamma)$ is homeomorphic to a handlebody.

Definition 2.7.1. *The* tunnel number *of a knot* K*,* $t(K)$*, is the minimum number of arcs in an unknotting tunnel system, over all unknotting tunnel systems for* K*.*

Morimoto gives an equivalent definition in [13], which we will use here. For a knot K in S^3, there is a Heegaard splitting (V_1, V_2) of S^3 such that a handle of V_1 contains K as a core of V_1.

Definition 2.7.2. *The minimum genus of* V_1 *minus one, over all Heegaard splittings satisfying the above fact, is the tunnel number,* $t(K)$*.*

2.8 Distance

For a more thorough discussion on distance, see [21]. Given a compact, orientable, properly embedded surface S in a 3-manifold M, the 1-skeleton of the *curve complex*,

$\mathcal{C}(S)$, is the graph whose vertices correspond to isotopy classes of essential simple closed curves in S such that two vertices are connected if the corresponding isotopy classes have disjoint representatives. For two subsets A and B of $\mathcal{C}(S)$, the *distance* between them, $d(A, B)$ is defined to be the length of the shortest path from an element of A to an element of B.

For any subset $X \subset S^3$, let X_K be $E(K) \cap X$.

Definition 2.8.1. *[21]*

Suppose M is a closed, orientable irreducible 3-manifold containing a knot K and suppose P_K is a bridge surface for K splitting M into handlebodies V and W. Let $\mathcal{V}$ (resp $\mathcal{W}$) be the set of all essential simple closed curves on P_K that bound disks in V_K (resp. W_K). Then the distance $d(P, K) := d(\mathcal{V}, \mathcal{W})$ *measured in $C(P_K)$.*

2.9 Generalized Montesinos Knots

J. Montesinos defined the class of knots and links that now bear his name in 1973 in [12]. As stated earlier on page 7, given a rational number $\frac{\beta}{\alpha} \in \mathbb{Q}$, there is a unique continued fraction decomposition $\frac{\beta}{\alpha} = [a_1, a_2, \ldots, a_m]$ where $a_i \neq 0$ for all $i = 1, \ldots, m$ and m is odd. We also recall that for each rational number, there is an associated rational tangle, see page 8 and Figure 2.2.

Definition 2.9.1. *A* Montesinos knot or link $M(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_n}{\alpha_n}|e)$ *is the knot in Figure 2.7 where each β_i, α_i for $i = 1, \ldots, n$ represents a rational tangle given by $\frac{\beta_i}{\alpha_i}$, and e represents the number of positive half-twists. If e is negative, we have negative half-twists instead.*

Lustig and Moriah in [11] defined the class of knots which we describe in the rest of this section. They based their definition off of Boileau and Zieschang [2], who prove

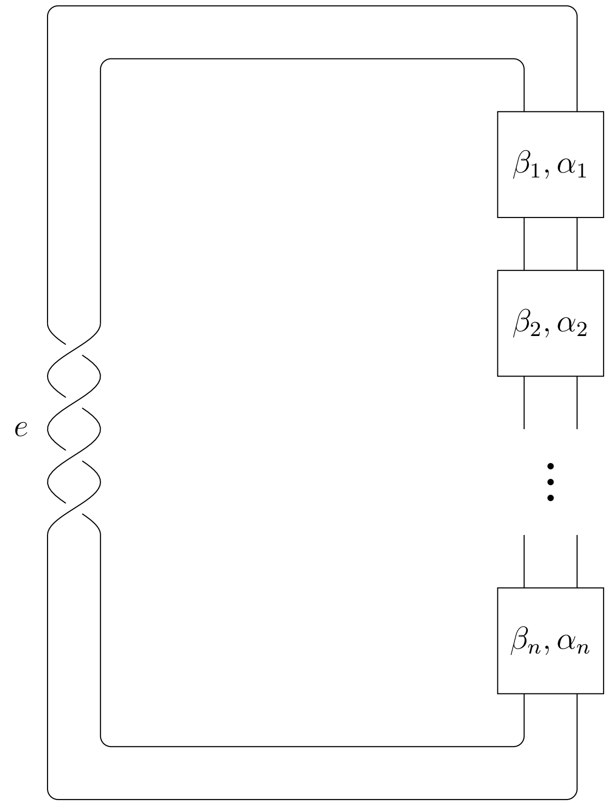

Figure 2.7: The Montesinos link $M(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_n}{\alpha_n}|e)$

that any Montesinos knot $M(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_n}{\alpha_n}|e)$, which does not have integer tangles, i.e., $\alpha_i \neq 1$ for all i, has bridge number n. Consider Figure 2.8. Each $\alpha_{i,j}, \beta_{i,j}$ is the 4-plat diagram from the rational tangle defined by $\alpha_{i,j}/\beta_{i,j}$. An n-braid is n disjoint arcs in I^3 with initial points in $I \times \{\frac{1}{2}\} \times \{1\}$, end points on $I \times \{\frac{1}{2}\} \times \{0\}$, and the arcs strictly decreasing in the third component of I^3. A double of a braid is obtained by duplicating each arc in an ϵ-neighborhood of the original, possibly with twisting of an arc and its duplicate, so it becomes a $2n$-braid. In a generalized Montesinos knot, each B_j is a double of an n-braid. For a more in depth description, see [2] and [11]. They exhibit a number diagrams which show that every Montesinos Knot and in particular, every pretzel knot, is a Generalized Montesinos Knot. One should

note that for a pretzel knot $K_n = (p_1, \ldots, p_n)$, the corresponding rational tangles are $p_k = \alpha_{i,j}/\beta_{i,j}$.

Definition 2.9.2. *A* generalized Montesinos knot or link*,*
$K = M\left(\{(\beta_{i,j}, \alpha_{i,j})\}_{i=1,j=1}^{i=\ell,j=m}, \{B_i\}_{j=1}^{j=\ell-1}\right)$ *is the knot in Figure 2.8 where each* $\beta_{i,j}, \alpha_{i,j}$ *for* $i = 1, \ldots, m$ *and* $j = 1, \ldots, \ell$ *represents a rational tangle given by* $\frac{\beta_{i,j}}{\alpha_{i,j}}$*, and* B_i *represents a* $2n$*-braid, which is obtained by doubling an* n*-braid.*

The main result from Lustig and Moriah's paper that we use below is the following. Recall that $rk(G)$, the rank of the group G, is the minimum number of generators; the notation $t(K)$ is the tunnel number, see Definition 2.7.1; and $b(K)$ is the bridge number, which we are usually denoted as $b_0(K)$, the genus zero bridge number, see Definition 2.3.1.

Theorem 2.9.3. *[11, Theorem 0.1] Let* K *be a generalized Montesinos knot/link as in Figure 2.8 below, with* $2n$*-plats. Let* $\alpha = gcd(\alpha_{i,j} : i = 1, \ldots, \ell; j = 1, \ldots, m)$*. If* $\alpha \neq 1$ *then* $rk(\pi_1(S^3 - K)) = t(K) + 1 = b(K) = n$*.*

2.10 Results on bridge surfaces

In [24], Zupan introduced the bridge spectrum and in the same paper produced the results which appear in this section.

Let M be a 3-manifold with boundary, let P be a subsurface of ∂M, and let A be a properly embedded surface in M. Then a *P-∂-compressing disk* for A is a ∂-compressing disk Δ for A with the added condition that $\Delta \cap \partial M \subset P$. Also, A is *P-essential* if A is incompressible and there does not exist a P-∂-compressing disk for A in M. Similarly, A is *P-strongly irreducible* if A is separating and admits either compressing or P-∂-compressing disks on either side but admits no pair of disjoint

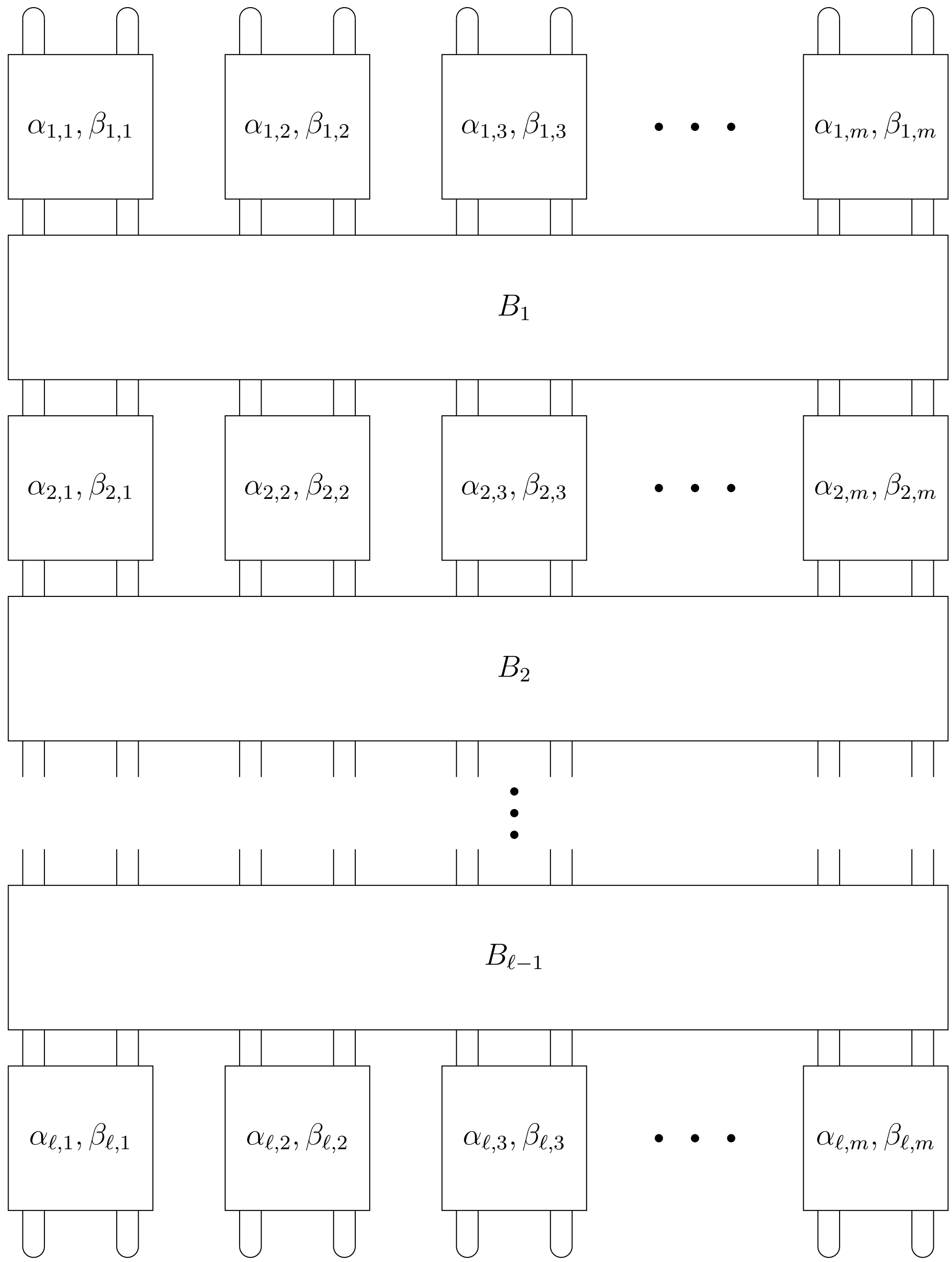

Figure 2.8: A generalized Monetesinos knot

disks on opposite sides. Two surfaces A and B are *almost transverse* if A is transverse to B except for a single saddle tangency. The following lemmas of Zupan will be used in Chapters 3 to 5 below. For the following lemma, recall for a 3-manifold M and an embedded 1-manifold $J \subset M$, $M(J)$ is $M - \eta(J)$ and for an embedded surface $\Sigma \subset M$, Σ_J is $\Sigma - \eta(J)$.

Lemma 2.10.1. *[24, Lemma 5.2] Let M be a compact 3-manifold and J a properly embedded 1-manifold, with $Q := \partial N(J)$ in $M(J)$. Suppose Σ is a strongly irreducible bridge splitting surface for (M, J), and let $S \subset M(J)$ be a collection of properly embedded essential surfaces such that for each component c of the boundary of each element of S, either $c \subset Q$ or $c \subset \partial M$. Then one of the following must hold:*

1. *After isotopy, Σ_J is transverse to each element of S and each component of $\Sigma_J \setminus \eta(S)$ is Q-essential in $M(J) \setminus \eta(S)$.*
2. *After isotopy, Σ_J is transverse to S, one component of $\Sigma_J \setminus \eta(S)$ is Q-strongly irreducible and all other components are Q-essential in $M(J) \setminus \eta(S)$.*
3. *After isotopy, Σ_J is almost transverse to S and each component of $\Sigma_J \setminus \eta(S)$ is Q-essential in $M(J) \setminus \eta(S)$.*

Lemma 2.10.2. *[24, Lemma 6.1] Let J be a knot in a 3-manifold M and let $K = \mathsf{cable}(T_{m,n}, J)$ be a (m, n)-cable of J. If $\Sigma \subset M$ is a Heegaard surface such that $J \subset \Sigma$ and if there is a compressing disk D for Σ such that $|D \cap J| = 1$, then there exists an embedding of K in M such that $K \subset \Sigma$.*

This lemma tells us that for $K_{p/q}$ a 2-bridge knot, $K = \mathsf{cable}(T_{m,n}, K_{p/q})$ satisfies $b_2(K) = 0$, since every $K_{p/q}$ can be isotoped into a genus 2 Heegaard surface with each handle having a single arc of the knot traversing it.

Chapter 3

Results about surfaces

In this Chapter we introduce and prove some technical results needed for the proof of Theorem 5.2.1.

Lemma 3.1. *The surface $S_n(n_1, \dots, n_k)$ has Euler characteristic $-n(k-1)$.*

Proof. Consider the single sheeted surface $S_1(n_1, \dots, n_{k-1})$, see 2.6. The vertical bands deformation retract to an edge and the plumbing squares deform to a vertex. As our knot lives in S^3, the outer squares can be deformed to a point at infinity. See Figure 3.1. Thus, we get a graph with $k-1$ vertices and $2(k-1)$ edges. The Euler characteristic of this graph and surface is $-(k-1)$. Given an n-sheeted surface, $S_n(n_1, \dots, n_{k-1})$, there are n copies of this graph, not taking into account how these graphs are connected. But, this is only a change in adjacency, not in the number of vertices and edges, so it has no effect on the Euler characteristic. Thus, $\chi\left(S_n(n_1, \dots, n_{k-1})\right) = -n(k-1)$. $\square$

From this lemma, together with Theorem 2.6.1 part (2), we see that all non-closed incompressible surfaces in the exterior of any non-torus 2-bridge knot have negative Euler characteristic. This is because k represents the number of steps in the partial fraction decomposition, which in turn corresponds to the number of twist regions in

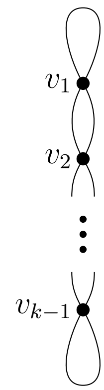

Figure 3.1: The deformation retract of $S_1(n_1, \dots, n_{k-1})$.

the vertical diagram of the 2-bridge knot, see Figure 2.6. So, for a 2-bridge knot $K_{p/q}$, k is zero if and only if $K_{p/q}$ is the unknot and k is one if and only if $K_{p/q}$ is a torus knot. This proves the follow lemma.

Lemma 3.2. *The surfaces $S_n(n_1, \dots, n_k)$ of a non-torus 2-bridge knot have negative Euler characteristic.*

Another important property of 2-bridge knots is that they are meridionally small.

Definition 3.3. *A knot $K \subset S^3$ is* meridionally small *if $E(K)$ contains no essential surface S with ∂S consisting of meridian curves of $N(K)$.*

Lemma 3.4. *Every 2 bridge knot is meridionally small.*

Proof. We notice from Remark 2.6.2 that all essential surfaces in $E(K)$ either are disjoint from $\partial E(K)$ or intersect $\partial E(K)$ with integer slope. In either case, they do not intersect the boundary in meridian curves. □

Zupan, in [23], proves the following theorem.

Theorem 3.0.3. *[Theorem 6.6], [23] For a knot J in S^3, the following are equivalent:*

1. *J is meridionally small.*

2. *Every cable of J is meridionally small.*

3. *There exists a cable of J that is meridionally small.*

The following is a modification of Lemma 3.5 from [24].

Lemma 3.5. *Let* $K = \mathsf{cable}(T_{m,n}, K_{p/q})$ *be a cable of a 2-bridge knot. Suppose* $S \subset E(K)$ *is an essential surface. If* S *is not isotopic to the boundary torus of* $K_{p/q}$, *then* $S \cap \partial E(K)$ *is nonempty and has integral slope.*

Proof. Let T denote the boundary torus of $E(K_{p/q})$. Recall that $C_{m,n}$ is a cable space, see page 16. After isotopy, we may assume $|S \cap T|$ is minimal.
Claim: Each component of $S \cap C_{m,n}$ is incompressible in $C_{m,n}$ and each component of $S \cap E(K_{p,q})$ is incompressible in $E(K_{p,q})$.

To prove this claim, let D be a compressing disk for $S \cap C_{m,n}$ in $C_{m,n}$. Then ∂D bounds a disk $D' \subset S$ by the incompressibility of S, where $D' \cap T \neq \emptyset$. By the irreducibility of $E(K)$, there is an isotopy of S pushing D' onto D which reduces $|S \cap T|$, yielding a contradiction. The argument is similar for $S \cap E(K_{p,q})$ being incompressible in $E(K_{p,q})$, proving the claim.

If $S \cap T = \emptyset$, then $S \subset C_{m,n}$ or $S \subset E(K_{p,q})$. If $S \subset C_{m,n}$, then by our claim, each component of $S \cap C_{m,n}$ is incompressible in $C_{m,n}$, and thus S is closed and vertical and the only such surfaces are boundary parallel by [22]. Similarly, if $S \subset E(K_{p,q})$, then by our claim, each component of $S \cap E(K_{p,q})$ is incompressible in $E(K_{p,q})$ and then by Remark 2.6.2, S is closed and thus boundary parallel. Thus, S is isotopic to T.

If $S \cap T \neq \emptyset$, then $S \cap E(K_{p,q})$ is one of the surfaces classified by Theorem 2.6.1 and so $S \cap E(K_{p,q})$ has integer slope. The surface T is also $\partial_+ C_{m,n}$. Then we apply

Lemma 2.5.1 which tells us that S has integer slope in $\partial_- C_{m,n}$, thus $S \cap \partial E(K)$ has integer slope. $\square$

Chapter 4

Bridge spectra of Montesinos and pretzel knots

Before determining the bridge spectra in Section 4.2, we begin Section 4.1 with further results on the relationship between tunnel number and bridge spectrum.

4.1 Tunnel number and bridge spectrum

The next proposition appears in [15] without proof; we provide one here for completeness. Also, recall a knot is primitive in a (g, b)-splitting if it transversely intersects the boundary of a properly embedded essential disk of the handlebody in a single point. See Section 2.3.

Proposition 4.1.1. *If K is a knot with a (g, b)-splitting satisfying either $b > 0$, or $b = 0$ and K is primitive, then $t(K) \leq g + b - 1$.*

Proof. Recall Definition 2.7.2, the tunnel number of a knot K is the minimum genus minus one over all Heegaard surfaces with that contain K as its core. Suppose first $b > 0$; we can meridionally stabilize each of the b arcs, see Section 13. This takes the

genus g surface and adds in b more handles, which gives us a $g+b$ genus surface. This is precisely the situation in Definition 2.7.2. Hence, the new Heegaard surface is of genus $g+b$ with K as the core. Thus the tunnel number is at most $g+b-1$.

If $b=0$ and the knot is primitive, then by definition, there is an essential disk for the Heegaard surface which intersects the knot exactly once. This essential disk must be a meridian disk and thus, the knot must have a tunnel number that is at most $g-1$. □

This is the driving force behind the following proposition:

Proposition 4.1.2. *If K is a knot K with $t(K)+1=b(K)=b_0(K)$, then K has the stair-step primitive bridge spectrum, $\hat{\mathbf{b}}(K)=(\hat{b}_0(K),\hat{b}_0(K)-1,\ldots,2,1,0)$.*

Proof. By Proposition 4.1.1, for any (g,b)-splitting, $t(K)\leq g+b-1$, and so $b_0(K)-1=t(K)\leq g+b_g(K)-1$ for all g, and so $b_g(K)\geq b_0(K)-g$. Corollary 2.4.2 states that for every knot K, and every g, there is a $(g,b_0(K)-g)$-splitting for K. Since $b_g(K)\leq b_0(K)-g$, we get that $b_g(K)=b_0(K)-g$ for all g. □

4.2 Bridge spectra of Montesinos knots and pretzel knots

Theorem 4.2.1. *A generalized Montesinos knot or link,*
$K=M\left(\{(\beta_{i,j},\alpha_{i,j})\}_{i=1,j=1}^{i=\ell,j=m},\{B_i\}_{j=1}^{j=\ell-1}\right)$ *with* $\alpha=\gcd\{\alpha_{i,j}\}\neq 1$ *has the stair-step primitive bridge spectrum,* $\hat{\mathbf{b}}(K)=(\hat{b}_0(K),\hat{b}_0(K)-1,\ldots,2,1,0)$.

Proof. This is immediate from Proposition 4.1.2 and Theorem 2.9.3. □

Notice that Theorem 4.2.1 implies that any pretzel knot $K_n=K_n(p_1,\ldots,p_n)$ also has the stair-step primitive bridge spectrum if $\alpha=\gcd(p_1,\ldots,p_n)\neq 1$. It should be

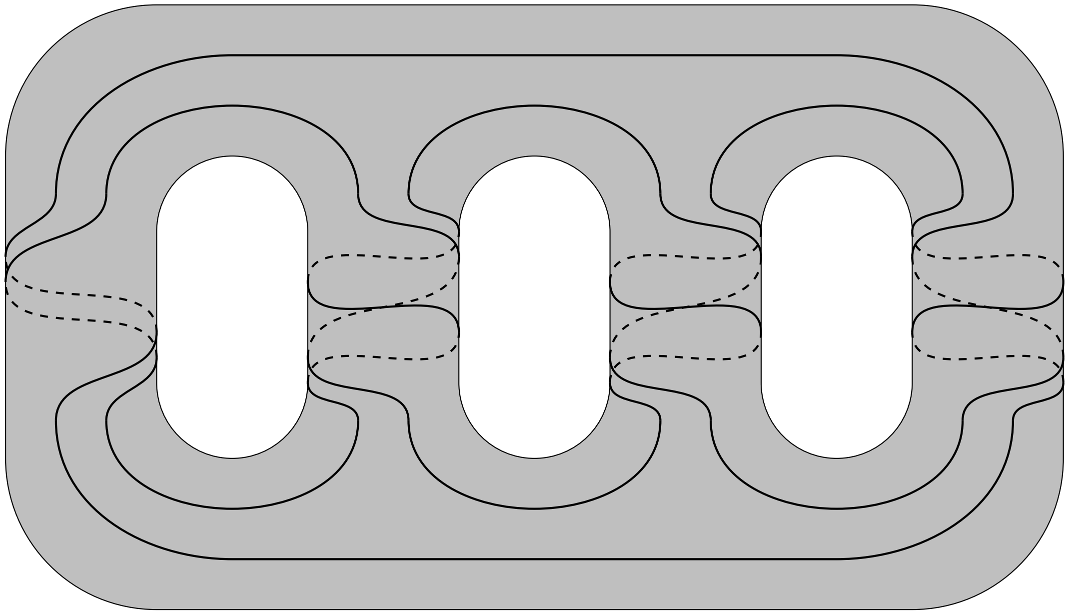

Figure 4.1: A pretzel knot $K_4(2,-3,-3,3)$ embedding on a genus three surface.

noted that pretzel knots K_n with $\alpha \neq 1$, via proposition 2.3.5, while having stair-step primitive bridge spectra, have the bridge spectrum $\mathbf{b}(K_n) = (n, n-1, \ldots, 3, 2, 0)$. This is because any pretzel knot embeds into a genus $n-1$ surface. See, for example, Figure 4.2. These facts prove the following corollary.

Corollary 4.2.2. *Given a $K_n = K_n(p_1, \ldots, p_n)$ pretzel knot with* $\gcd(p_1, \ldots, p_n) \neq 1$, *the primitive bridge spectrum is stair-step, i.e. $\hat{\mathbf{b}}(K_n) = (n, n-1, \ldots, 2, 1, 0)$, and $\mathbf{b}(K_n) = (n, n-1, \ldots, 3, 2, 0)$.*

One should ask what happens when $\alpha = 1$, to which we do not have a complete answer here. Morimoto, Sakuma, and Yokota in [15] show that there are Montesinos knots K which have $\alpha = 1$ and $t(K) + 2 = b(K)$. Here is their theorem as stated by Hirasawa and Murasugi in [9]:

Theorem 4.2.3. *[9]*

The Montesinos knot $K = M(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_r}{\alpha_r}|e)$ *has tunnel number one if and only if one of the following conditions is satisfied.*

1. $r = 2$.

2. $r = 3, \beta_2/\alpha_2 \equiv \beta_3/\alpha_3 \equiv \pm\frac{1}{3}$ *in* $\mathbb{Q}/\mathbb{Z}$*, and* $e + \sum_{i=1}^{3} \beta_i/\alpha_i = \pm 1/(3\alpha_1)$.

3. $r = 3, \alpha_2$ *and* α_3 *are odd, and* $\alpha_1 = 2$.

Any pretzel link $K_3(p_1, p_2, 2)$ is a knot if and only if p_1 and p_2 are odd. Therefore, by Theorem 4.2.3, every pretzel knot of this form has tunnel number one but bridge number three. Thus, we cannot use Proposition 4.1.2 to compute the bridge spectrum. It is unknown if there is a pretzel knot or Montesinos knot which has bridge spectrum $(3, 1, 0)$.

Another point of interest is that any n-pretzel knot K_n with $n \geq 4$ is a distance one knot.

Proposition 4.2.4. *If* $K_n(p_1, \ldots, p_n)$ *is a pretzel knot with* $n \geq 4$*, and* P *is a genus zero bridge surface for* K_n*, then* $d(P, K_n) = 1$.

Proof. Let K_n be a pretzel knot with $n \geq 4$. Arrange K_n in S^3 so that the rational tangles are all intersecting the plane in S^3 where $z = 0$. Let P be the plane where $z = 0$. Choose one disk between the first two tangles on one side of P and another on the other side between the third and fourth tangle, as in Figure 4.2. These two disks are bridge disks for the knot. If we thicken these disks by taking an ϵ neighborhood of each, and then consider the boundary of these neighborhoods, we obtain disks which bound essential simple closed curves in P. Since they are disjoint curves, the vertices they correspond to in the curve complex are adjacent. Hence, $d(P, K_n) \leq 1$. If the distance is zero, we would have disks on opposite sides of P which bound essential

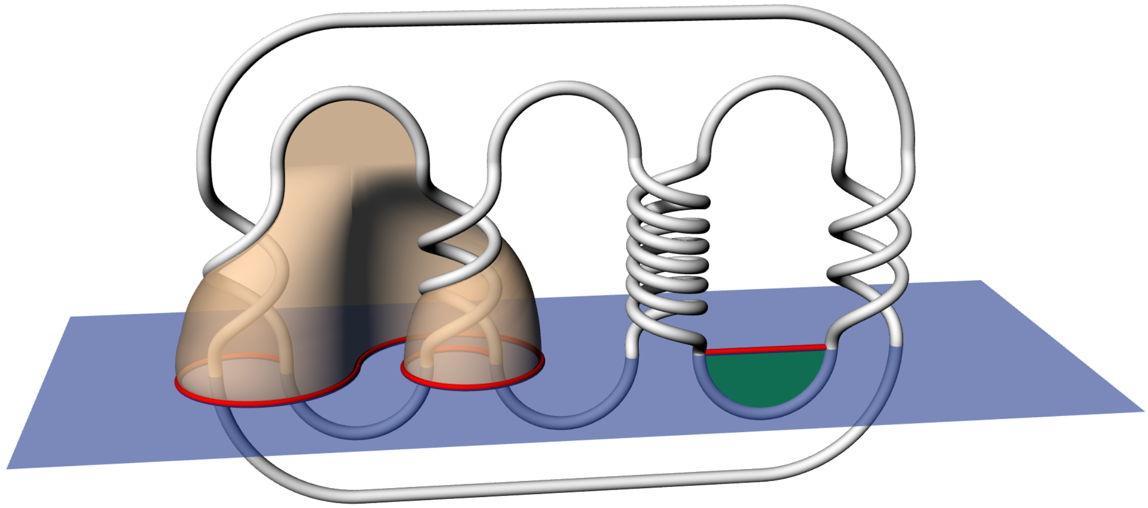

Figure 4.2: A four branched pretzel knot with disjoint bridge disks on opposite sides of the bridge surface.

simple closed curves that are isotopic to each other, so we can isotope our disks to have the same boundary, and form a sphere. This implies that our knot has at least two components, one on each side of the sphere, contradicting our assumption that K_n is a knot. Hence, $d(P, K_n) = 1$. □

This gives us that any pretzel knot $K_n = K_n(p_1, \ldots, p_n)$ with $\gcd\{p_1, \ldots, p_n\} \neq 1$ and $n \geq 4$ has a stair-step bridge spectrum and is not high distance. Thus, one could not find the bridge spectrum of K_n with Theorem 1.0.1. This proves the next corollary.

Corollary 4.2.5. *There exists knots with distance 1 and stair-step bridge spectrum.*

Thus, stair-step does not imply high distance. As an example, $K_n(3p_1, 3p_2, \ldots, 3p_n)$ for $n \geq 4$ for any pretzel knot $K_n(p_1, \ldots, p_n)$ has $\mathbf{b}(K_n) = (n, n-1, \ldots, 1, 0)$ and distance one.

Chapter 5

Bridge spectra of cables of 2-bridge knots

In this Chapter we prove Theorem 5.2.1. To do this, we compute the genus zero, one and two bridge numbers of cables of 2-bridge knots. The majority of the work here is establishing a lower bound for the genus one bridge number, which is done is Section 5.2.

5.1 Genus zero and genus two bridge numbers

From Schubert's and Schulten's result on bridge number of satellite knots, Theorem 1.0.3, we have the following.

Lemma 5.1.1. *Let $K_{p/q}$ be a 2-bridge knot, $T_{m,n}$ a torus knot, and $K = \mathsf{cable}(T_{m,n}, K_{p/q})$, then $b(K) = b_0(K) = 2m$.*

Proof. From Theorem 1.0.3, we have $b_0(K) \geq 2m$. To prove the upper bound, consider Figure 2.6. If we were to cable $K_{p/q}$, we would produce a knot with $2m$ maxima, which proves the upper bound. $\square$

A similar lemma is required for the genus two bridge number. We hope to relay some intuition by noticing that any cable of a two bridge knot embeds on a genus two surface.

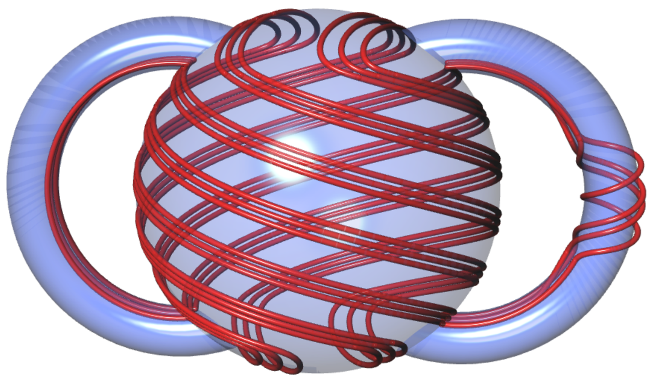

Figure 5.1: A cable of a 2-bridge knot embedded on a genus two surface

Lemma 5.1.2. *Let K be a cable of a non-torus 2-bridge knot by $T_{m,n}$. Then $b_2(K) = 0$.*

Proof. Recall Proposition 2.3.8, which tells us that 2-bridge knots embed on a genus two surface. From the Figure 2.3 we see that this embedding can have a single arc of the knot on a handle, i.e. there is a meridian disk which intersects the knot once. Thus, the remark following Lemma 2.10.2 tells us that the cable can embed in a genus two surface. See Figure 5.1. □

5.2 Genus one bridge number

Theorem 5.2.1. *Let $K_{p/q}$ be a non-torus 2-bridge knot and $T_{m,n}$ an (m,n)-torus knot. If $K := \mathsf{cable}(T_{m,n}, K_{p/q})$ is a cable of $K_{p/q}$ by $T_{m,n}$, then the bridge spectrum of K is $\mathbf{b}(K) = (2m, m, 0)$.*

Proof. The previous two lemmas give us all but $b_1(K)$. We observe from Figure 5.1 that eliminating either handle will produce an embedding with m trivial arcs on either side of the torus. Hence, $b_1(K) \leq m$. For the rest of the proof, we need to show that $b_1(K) \geq m$. In Section 2.4, we note that every bridge surface is either strongly irreducible or weakly reducible. This dichotomy gives us two main cases for the proof of the theorem.

Case 1:. Suppose that Σ is a strongly irreducible genus 1 bridge surface for K. Let J, also denote the knot K, let Σ be the bridge surface, and let S denote $T := \partial E(K_{p/q})$. With these choices, apply Lemma 2.10.1 and get one of three situations, giving us the following subcases.

Subcase A: After isotopy, Σ_K is transverse to T and each component of $\Sigma_K \setminus \eta(T)$ is $\partial N(K)$-essential in $E(K) \setminus \eta(T)$.

The exterior $E(K)$ is split along T into $E(K_{p/q})$ and $C_{m,n}$ and Σ_K is $\partial N(K)$-essential in both. The cable space $C_{m,n}$ can be decomposed into $(T^2 \times I) \cup V$, where $V = D^2 \times S^1$, a solid torus. By assumption, the bridge surface Σ is transverse to K.
Claim: The surface $\Sigma \cap V$ is a collection of meridian disks.
To prove the claim, notice that $\Sigma \cap V$ is also essential in V and cannot be a ∂-parallel annulus or ∂-parallel disk, as these surfaces are disjoint from K or are $\partial N(K)$-∂-compressible. Thus, each component of $\Sigma \cap V$ is a meridian disk.

By Lemma 2.5.2 we know that $\Sigma \cap T$ has meridian slope also. This is also the boundary slope of $\Sigma \cap E(K_{p/q})$. But by Theorem 2.6.1, no essential surface has a boundary slope of $1/0$, giving us a contradiction and completing this subcase.

Subcase B: After isotopy, Σ_K is transverse to T, one component of $\Sigma_K \setminus \eta(T)$ is $\partial N(K)$-

strongly irreducible and all other components are $\partial N(K)$-essential in $E(K) \setminus \eta(T)$. Note that $\Sigma_K \setminus \eta(T) = [\Sigma_K \cap E(K_{p/q})] \cup [\Sigma_K \cap C_{m,n}]$.

First assume that $\Sigma_K \cap C_{m,n}$ is $\partial N(K)$-strongly irreducible. This means that $\Sigma_K \cap E(K_{p/q})$ is $\partial N(K)$-essential. Again by Theorem 2.6.1, we know that $\Sigma_K \cap E(K_{p/q})$ is isotopic to an essential surface $S_n = S_n(n_1, n_2, \ldots, n_{k-1})$ and has negative Euler characteristic by Lemma 3.2. The bridge surface Σ is a torus by assumption, and thus must have Euler characteristic zero. We can only increase the Euler characteristic of S_n by gluing on disks. Assume that the number of components of $\Sigma \cap T$ is minimal. It must be that $\Sigma \cap T$ is a collection of essential, simple closed curves in T. These curves must be the boundaries of disks in order for us to increase the Euler characteristic, but the only disks that have essential curves in T as their boundary are meridian disks, which have boundary slope $1/0$. This gives a contradiction of Theorem 2.6.1.

Now, assume that $\Sigma_K \cap E(K_{p/q})$ is $\partial N(K)$-strongly irreducible and $\Sigma_K \cap C_{m,n}$ is $\partial N(K)$-essential. This implies that $\Sigma_K \cap C_{m,n}$ is a collection of meridian disks, which means that $|\Sigma \cap K| = m \cdot |\Sigma \cap \partial E(K_{p/q})| = m \cdot b'$, where $b' := |\Sigma \cap \partial E(K_{p/q})|$ and $\Sigma \cap \partial E(K_{p/q})$ is a $(1, b')$-bridge surface for $K_{p/q}$. Since $b_1(K_{p/q}) = 1$, then $b' \geq 1$ therefore $b_1(K) \geq m$, completing this subcase.

Subcase C: After isotopy, Σ_K is almost transverse to T, and each component of $\Sigma_K \setminus \eta(T)$ is $\partial N(K)$-essential in $M(J) \setminus \eta(T)$. Any saddlepoint tangency in a genus one torus will produce a figure 8 shaped intersection, c. When cut along an open neighborhood of c, the torus splits into two disjoint components, one an annulus and the other a disk or a single disk. In either case, the disk will be inessential, which gives a contradiction to $\Sigma_K \setminus \eta(T)$ being $\partial N(K)$-essential.

Case 2:. Suppose that Σ is weakly reducible. We apply Theorem 2.4.3, which gives us

that there exists a multiple bridge splitting $\{\Sigma_0, S_1, \Sigma_1, \ldots, S_d, \Sigma_d\}$ of (S^3, K), that is strongly irreducible and

$$g(\Sigma) = \sum_{i=0}^{d} g(\Sigma_i) - \sum_{i=1}^{d} g(S_i). \qquad (\star)$$

Since Σ is a genus one bridge surface, this sum must equal one. By Theorem 3.0.3, K is meridionally small since K is the cable of a meridionally small knot, see Definition 3.3. Hence $E(K)$ contains no essential meridional surfaces, and so $S_i \cap K = \emptyset$ for all $i = 1, \ldots, d$. Thus, by Lemma 3.5, we have that each S_i is isotopic to T. If we have a multiple bridge splitting $\{\Sigma_0', S_1', \Sigma_1', S_2', \Sigma_2'\}$ with S_1' and S_2' isotopic to T, then $\Sigma_0' \sqcup_{S_1'} \Sigma_1'$ or $\Sigma_1' \sqcup_{S_2'} \Sigma_2'$ would be isotopic to Σ_0' or Σ_2'. Thus, we can assume without loss of generality that $d = 1$ and our multiple bridge splitting is $\{\Sigma_0, S_1, \Sigma_1\}$. This is a strongly irreducible multiple bridge splitting and one of Σ_0 or Σ_1 is contained in the cable space $C_{m,n}$ and the other is contained in the exterior of the 2-bridge knot, $E(K_{p/q})$. Let $\Sigma_0 \subset C_{m.n}$ and $\Sigma_1 \subset E(K_{p/q})$. As K is not contained in $E(K_{p/q})$, Σ_1 must be a Heegaard surface. Since the tunnel number of a 2-bridge knot is 1, by Proposition 4.1.1, then $g(\Sigma_1) = g(E(K_{p/q})) = 2$. This is because the genus of a 3-manifold is the minimum genus of a Heegaard splitting. We also have $g(S_1) = 1$, and thus by the equation $(\star)$ above,

$$1 = 2 + g(\Sigma_0) - 1.$$

This tells us that $g(\Sigma_0) = 0$, but Σ_0 needs to be a bridge surface for $C_{m,n}$ and thus, a Heegaard surface, which gives a contradiction as the Heegaard genus of $C_{m,n}$ is greater than zero, since $C_{m,n}$ is not S^3, the only 3-manifold with a genus zero Heegaard splitting. This completes the final case of the proof.

□

Chapter 6

Generalizations and Conjectures

Montesinos knots are n rational tangles connected as in Figure 2.7. The 2-bridge knots are the special case in which $n = 1$. A nontorus 2-bridge knot has primitive bridge spectrum $(2, 1, 0)$. This generalizes to some Montesinos knots, as we see from Theorem 4.2.1. A special class of Montesinos knots are n-pretzel knots, having $\beta_i = 1$ for all i; these also then have stair-step primitive bridge spectra if $\gcd\{\alpha_i\} \neq 1$. But these knots embed on a genus $n-1$ torus, see Figure 4.1. Hence, Corollary 4.2.2 states that their bridge spectrum is $(n, n-1, \ldots, 3, 2, 0)$.

Question 6.1. *Are pretzel knots exactly the class of Montesinos knots which have the property that* $\hat{\mathbf{b}}(K) \neq \mathbf{b}(K)$*?*

So, as with the Theorem 5.2.1, we would like to know what happens to bridge spectra under the operation of cabling. In general, we do not know. But examples of cabling that fails to produce an m-stair-step bridge spectrum, coming from cables of pretzel knots.

Conjecture 6.2. *Let* $P(p_1, p_2, \ldots, p_j)$ *be a pretzel knot and* $T_{m,n}$ *an* (m, n)*-torus knot. If* $K = \mathsf{cable}(T_{m,n}, P(p_1, p_2, \ldots, p_j)$ *is a* $T_{m,n}$ *cable of the* $P(p_1, p_2, \ldots, p_j)$*, then the*

bridge spectrum of K *is* $\mathbf{b}(K) = (mj, m(j-1), \ldots, 3m, 2m, \min\{m, |2\cdot\sum_{i=1}^{j} p_i - n|\}, 0)$.

We conjecture that the degeneration we see in the last nonzero bridge number in Conjecture 6.2 is analogous to the degeneration we see in iterated torus knots, see [24]. Where as when we move to 2-bridge knots, we have no degeneration, which in this analogy, perhaps corresponds to Montesinos knots which are not pretzel. Thus, this leads us to Conjecture 6.3, which stems from question 3.5 in [17].

A note on the direction of the possible proof for Conjecture 6.2: It seems that methods similar to the ones used in this dissertation in the proof of the Theorem 5.2.1 might work for $n = 3$, but not for $n \geq 4$. This is a consequence of Oertel's work in [16] where he shows that Montesinos knots with four or more branches have closed incompressible surfaces in their exterior. This stands in sharp contrast to the situation for 2-bridge knots, which played a key role in our proof of Theorem 5.2.1.

Conjecture 6.3. *The bridge spectrum of an* (m, n)*-cable of a Montesinos knot* M_0 *is* m*-stair-step if and only if* M_0 *is not a pretzel knot.*

There is less direct evidence in support of this conjecture at this time. In a similar vein, though, is the following conjecture. We noticed that cabling is mostly well behaved in the cases we have studied. There exists some degeneration at the end of the bridge spectrum, but we do not believe there can be degeneration elsewhere.

Conjecture 6.4. *Let* J *be a knot with bridge spectrum* $\mathbf{b}(J) = (b_0(J), \ldots, b_g(J), 0)$ *and a cable* $K = \mathsf{cable}(T_{m,n}, J)$*. Then*

$$\mathbf{b}(K) = (m \cdot b_0(J), m \cdot b_1(J), \ldots, m \cdot b_{g-1}(J), m \cdot b_g(J), b_{g+1}(K), 0)\,.$$

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