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Comment on “Wave-Breaking Limits for Relativistic Electrostatic Waves in a One-Dimensional Warm Plasma” [Phys. Plasmas 13, 123102 (2006)]

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Comment on "Wave-breaking limits for relativistic electrostatic waves in a one-dimensional warm plasma" [Phys. Plasmas 13, 123102 (2006)]

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In this Comment, several incorrect and misleading claims made by Trines and Norreys (TN) in Ref. 1 are addressed.

(1) TN claim that the warm-plasma model used by Schroeder, Esarey, and Shadwick (SES), Ref. 2, to calculate the maximum attainable amplitude of a traveling electron plasma wave, is unsuitable because it does not satisfy "Taub's fundamental identity" (which is simply the Schwarz inequality for moments of the phase-space distribution). This claim is false. The Taub inequality [Eq. (4.4) or (4.5) of Ref. 3] is a statement about distribution functions (without dynamical content) and restricts the possible form of an equation of state connecting the pressure, density, and internal energy. The warm-plasma model is an asymptotic treatment of the Vlasov-Maxwell equation and, as a result, *predicts* a particular relationship between these "state variables." This relationship must automatically satisfy Taub's inequality.

Explicitly, the Taub inequality[3] is $wT^{\mu}_{\mu}\geqslant n_p^2$. Here, $T^{\mu\nu}=\int d\Omega f p^{\mu}p^{\nu}$ is the energy-momentum stress tensor [with $p^{\nu}=(\gamma,\gamma\beta)$ the particle four-momentum, $d\Omega$ the invariant momentum space volume, and f the phase-space density], and $w=U_{\mu}T^{\mu\nu}U_{\nu}$ is the energy density, with $U^{\mu}=J^{\mu}/(J^{\nu}J_{\nu})^{1/2}$ the fluid momentum defined by Eckart,[4] where $J^{\mu}=\int d\Omega f p^{\mu}$ is the four-current, and $n_p=(J^{\nu}J_{\nu})^{1/2}$ is the proper density. The invariant density is defined as $h=\int d\Omega f=T^{\mu}_{\mu}$. Normalizing the current by the invariant density, $u^{\mu}=J^{\mu}/h$, the stress tensor can be expressed as $T^{\mu\nu}=\int d\Omega f p^{\mu}p^{\nu}=hu^{\mu}u^{\nu}+\Theta^{\mu\nu}$, where $\Theta^{\mu\nu}=\int d\Omega f(p^{\mu}-u^{\mu})(p^{\nu}-u^{\nu})$ is the second-order centered momentum moment. The energy density can be re-written in terms of centered moments,

$$w=n_p^2/h+(h/n_p)^2[R^{\alpha\nu}_{\alpha\nu}-h\epsilon^4]/4, \tag{1}$$

where $\epsilon^2=-\Theta^{\mu}_{\mu}/h=(n_p/h)^2-1=u^{\mu}u_{\mu}-1$ is the invariant measure of thermal spread and $R^{\alpha\beta\mu\nu}=\int d\Omega f(p^{\alpha}-u^{\alpha})(p^{\beta}-u^{\beta})(p^{\mu}-u^{\mu})(p^{\nu}-u^{\nu})$ is the fourth-order centered moment. The above equations are the result of definitions (re-writing in terms of centered moments) and are *exact*.

Using Eq. (1), the Taub inequality $wT^{\mu}_{\mu}\geqslant n_p^2$ can be expressed as $[R^{\alpha\nu}_{\alpha\nu}-h\epsilon^4]\geqslant 0$. Applying the Schwarz inequality yields $h^2\epsilon^4=(\Theta^{\mu}_{\mu})^2\leqslant hR^{\alpha\nu}_{\alpha\nu}$. Therefore, the Taub inequality is *always* satisfied within the warm-plasma model (to all orders in ϵ).

In the warm-plasma approximation we take ϵ to be a small parameter (which is valid for nonrelativistic plasma temperatures) and expand to order $\mathcal{O}(\epsilon^2)$. The energy density in the warm-plasma approximation is $w=n_p^2/h$, and the Taub inequality is *always* satisfied $(wT^{\mu}_{\mu}=n_p^2)$ within the warm-plasma approximation. And, as proved above, calculating and including the evolution of higher-order centered moments will also satisfy the Taub inequality (to all orders in ϵ).

The value of Taub's inequality is that is provides constraints for the case where an equation of state must be assumed to close the equations of motion. This is not the case for the warm-plasma model (which derives moments of the distribution), and thus the Taub inequality provides no additional information.

[Note to reader: we have read the Response by TN.[5] The error made by TN is inconsistently expanding quantities only to order $\mathcal{O}(\epsilon^2)$ and then incorrectly making conclusions about terms of order $\mathcal{O}(\epsilon^4)$.]

(2) TN state that the heat flow can be expressed in first-order and second-order centered moments. This statement is false for a collisionless plasma. In general, without viscosity (i.e., the collisionless case), the energy-momentum stress tensor can be written as[4,6] $T^{\mu\nu}=wU^{\mu}U^{\nu}+\Delta^{\mu}_{\alpha}T^{\alpha\beta}\Delta^{\nu}_{\beta}+q^{\mu}U^{\nu}+q^{\nu}U^{\mu}$, where $\Delta^{\mu\nu}=g^{\mu\nu}-U^{\mu}U^{\nu}$ is the projection tensor and $q^{\mu}=U_{\nu}T^{\nu\alpha}\Delta^{\mu}_{\alpha}$ is the heat flow (difference of the energy flow and flow of enthalpy). Evaluating the heat flow for a collisionless plasma yields

$$q^{\mu}=(h/n_p)^2U^{\mu}[h\epsilon^4-R^{\alpha\nu}_{\alpha\nu}]/4-(h/n_p)Q^{\mu\alpha}_{\alpha}/2, \tag{2}$$

where $Q^{\alpha\mu\nu}=\int d\Omega f(p^{\alpha}-u^{\alpha})(p^{\mu}-u^{\mu})(p^{\nu}-u^{\nu})$ is the third-order centered momentum moment. The above expression for the heat flow is *exact* (the heat flow expressed in terms of the centered moments). The above expression is also relativistically covariant, correctly describing relativistic fluid dynamics. In the warm-plasma model, $Q^{\alpha\mu\nu}\sim h\epsilon^3$ and $R^{\alpha\beta\mu\nu}\sim h\epsilon^4$. The heat flow is proportional to the third-order (and higher) centered moments, $q^{\mu}\sim h\epsilon^3$. Although the heat flow may superficially appear as a second-order centered moment, for the case of a fluid without viscosity (i.e., *collisionless*), the only contribution to the heat flow is proportional to the third-order and higher centered moments.

TN also claim that the asymptotic approach inherent in the warm-plasma approximation may lead to "possibly even violation of the first and second laws of thermodynamics." This implication is also false. The warm-plasma model satisfies energy-momentum conservation and conserves

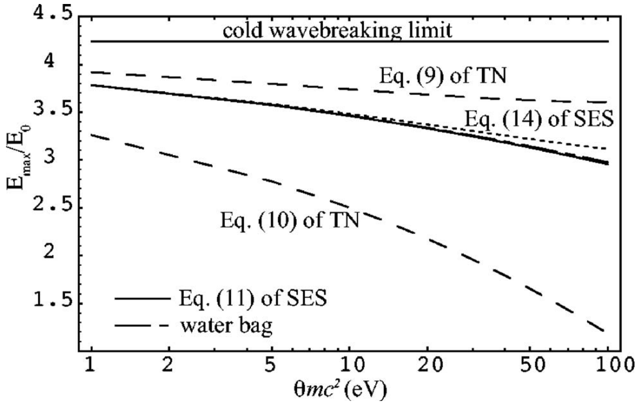

FIG. 1. Wave-breaking field amplitude E_{max}/E_0 versus initial plasma temperature θmc^2 (for $\gamma_\varphi=10$ and $\gamma_\perp=1$): analytic solution assuming warm-plasma approximation [Eq. (11) of SES, Ref. 2] (solid curve), wave-breaking field assuming a water-bag distribution (dashed-dotted curve), lowest-order corrections to the cold wave-breaking field [Eq. (14) of SES, Ref. 2] (dotted curve), and the "almost identical" poor estimations [Eqs. (9) and (10) of TN, Ref. 1] (dashed curves). The cold wave-breaking field result is also shown (uppermost curve).

entropy.[7] It should be noted that the warm-plasma approximation is an asymptotic approximation that assumes the contributions to the bulk fields from the higher-order moments are small in comparison to the lower-order moments (not that the higher-order moments are identically zero).

(3) TN state that SES, in Ref. 2, "erroneously" claim the evolution of the second moment is "a representation of an adiabatic process." This statement by TN is false. Nowhere in Ref. 2 is it stated or implied that the warm approximation relies on an adiabatic assumption. The evolution of the second moment of the phase space distribution derived in Ref. 2 is based on asymptotics (closure is obtained assuming a warm plasma). Assumption of adiabaticity was not used in Ref. 2. Conservation of entropy in the warm-plasma model has precisely the same origin as it does in the unapproximated Vlasov equation; namely, the absence of collisions.

(4) In commenting on Eqs. (9) and (10) of Ref. 1, TN state, "Note that both results are almost identical to the result obtained by Schroeder *et al.*, for this regime." This characterization is false and misleading. The result derived by SES, Ref. 2, is the correct lowest-order thermal corrections to the cold wave-breaking field (expansion of the full analytic warm wave-breaking result calculated in Ref. 2). Equations (9) and (10) of TN are very poor predictions of the wave-breaking field (owing to the crude boundary conditions assumed by TN in Ref. 1).

To illustrate this point, plotted in Fig. 1 versus initial plasma temperature θmc^2 is the full analytic solution of the warm wave-breaking limit (solid curve) [Eq. (11) of Ref. 2], and the lowest-order corrections to the cold wave-breaking limit [Eq. (14) of Ref. 2] (dotted curve). It is straightforward to evaluate the wave-breaking field assuming a potential from a water-bag distribution [Eq. (5) of Ref. 1] without making the subsequent approximations of TN. Doing so yields the dashed-dotted curve shown in Fig. 1. As Fig. 1 shows, the full analytic solution assuming the warm-plasma approximation [Eq. (11) of Ref. 2] and the solution assuming a water-bag distribution (a result not contained in Ref. 1) yield curves that are nearly indistinguishable on the scale of Fig. 1. These results do differ, and the difference is do to the specifics of the water-bag distribution and is consistent with the asymptotic warm-plasma approximation. (Using a different initial phase-space distribution would yield a third curve whose difference from the water-bag would also be consistent with the warm-plasma approximation.) Figure 1 shows that the result derived by SES [Eq. (14) of Ref. 2], by expanding the full analytic solution in Ref. 2, provides an accurate description of the wave-breaking amplitude and is the correct lowest-order thermal corrections to the cold wave-breaking field. As discussed in Ref. 2, Eq. (14) of SES is valid in the limit $\theta \ll \gamma_\perp^2/\gamma_\varphi^2 \ll 1$; i.e., for relativistic phase velocities (typical of plasma waves ponderomotively driven by short-pulse lasers in underdense plasma). Using the full analytic expression derived by SES [Eq. (11) of Ref. 2] yields, in the limit $\gamma_\varphi^2\theta \ll 1$,

$$\frac{E_{max}^2}{E_0^2} \simeq 2\gamma_\perp(\gamma_\varphi - 1) - 2\gamma_\varphi\left[\frac{4}{3}\beta_\varphi(3\beta_\varphi^2\gamma_\varphi^2\gamma_\perp^2\theta)^{1/4} - (3\beta_\varphi^2\gamma_\varphi^2\theta)^{1/2}\right], \tag{3}$$

which is also valid for nonrelativistic phase velocities. With $\beta_\varphi \simeq 1$, Eq. (3) reduces to Eq. (14) of SES. Here, E_{max} is the wave-breaking field, $E_0 = mc\omega_p/e$, γ_φ is the Lorentz factor of the wave phase velocity, $\gamma_\perp$ is the Lorentz factor of the transverse quiver motion in a laser field, and θ is the initial plasma temperature normalized to mc^2. As discussed in Ref. 2, the full analytic expression for the wave-breaking field for arbitrary phase velocity, i.e., Eq. (11) of Ref. 2 [as well as Eq. (3) above] reduces to the result of Coffey[8] in the limit $\theta \ll \beta_\varphi^2 \ll 1$. Note that in Eq. (3) we have kept the first two lowest-order corrections to the cold result. Higher-order corrections can be derived by simply expanding the full analytical result of SES [Eq. (11) of Ref. 2].

The results of TN [Eqs. (9) and (10) of Ref. 1], rather than being "almost identical," are particularly weak bounds. They are not the correct lowest order corrections to the cold result. Moreover, as illustrated in Fig. 1, the cold wave-breaking field yields a more accurate result than using Eq. (10) of TN for $\gamma_\varphi^2\theta < 1$. The parameter regime plotted in Fig. 1 is the regime relevant for typical laser-plasma accelerator experiments (past and present).

(Note to reader: we have read the Response by TN,[5] and all the published equations are accurately plotted in Fig. 1.)

(5) TN incorrectly claim that no trapping can occur for field amplitudes below the wave-breaking limit. TN also claim that trapping is not possible for plasma wave phase velocities with $\beta_\varphi = 1$. Both these statements are, in general, false. TN use these erroneous claims to attempt to justify the singularities of the water-bag model (TN state in Ref. 1, "for $v_\varphi = 1$, the separatrix is located at $v = 1$ at the phase of maximum compression; i.e., it is simply out of reach and no particles will be pushed across, no matter how large the pressure. Thus, in the limit $\gamma_\varphi \to \infty$ there should be no upper bound induced by wave breaking for the wave amplitude at

all."). Particle trapping can occur below the wave-breaking limit and for $\beta_\varphi=1$.

The quasi-static Hamiltonian for single-particle motion is $H=(1+u^2)^{1/2}-\beta_\varphi u-\phi=\text{const}$, where u is the electron momentum normalized to mc, β_φ is the plasma wave phase velocity normalized to c, and ϕ is the potential of the excited plasma wave in a plasma of arbitrary temperature. From this Hamiltonian, the initial momentum required for an electron to be on a trapped orbit in a one-dimensional plasma wave is[9]

$$u_t=\gamma_\varphi\beta_\varphi(1-\gamma_\varphi\phi_{\min})-\gamma_\varphi[(1-\gamma_\varphi\phi_{\min})^2-1]^{1/2}, \tag{4}$$

where u_t is the initial electron momentum required for trapping (normalized to mc) and $\phi_{\min}$ is the minimum potential of the of the excited plasma wave (in a plasma of arbitrary temperature). Trapping will occur for all electrons with initial momentum $\geqslant u_t$. Significant trapping will occur below the warm wave-breaking limit for a Maxwellian distribution. Only if the electron distribution is artificially terminated (e.g., by choosing an unphysical distribution, such as a water-bag) will no electrons be trapped before the wave-breaking field is reached. For a physical distribution (e.g., a Maxwellian), the tails of the distribution will be trapped below the wave-breaking field, and the fraction of trapped electrons can be calculated from Eq. (4) with the warm-plasma wave potential.

In the limit $\beta_\varphi\to 1$, Eq. (4) reduces to $u_t=(\phi_{\min}-1/\phi_{\min})/2$. The minimum potential is bounded for any thermal plasma $-1<\phi_{\min}<0$, and therefore trapping (of electrons with initial momentum $\geqslant u_t$) can always occur in a thermal plasma (even in the limit $\beta_\varphi\to 1$).

It is well known that the characteristic trajectories of the warm-plasma equations do not correspond to individual particle orbits (unlike the cold case, where there is a one-to-one correspondence between characteristic trajectories of the fluid equations and particle orbits). Nonetheless, as has been shown through detailed kinetic comparisons,[10] for the range of temperatures considered in Ref. 9, the bulk fields are very close to the those of the cold plasma $E_{\text{warm}}-E_{\text{cold}}\sim\theta E_0$ (with $\theta\sim 10^{-5}$).[7] The single particle orbits are completely determined by the bulk fields (i.e., ϕ in the single-particle Hamiltonian). As a result, examining test-particles moving in the cold fields is a completely natural approximation, and using the potential $\phi_{\min}$ derived from the cold fluid equations in Eq. (4) is an excellent approximation for determining the single particle orbits in plasma waves below the wave-breaking limit.[9]

(6) TN claim that the warm-fluid model breaks down before the wave-breaking field is reached. This claim is false. In the work of SES,[2] the wave-breaking field is defined (via fluid theory) as the maximum amplitude of a periodic electron plasma wave. This maximum amplitude was solved (asymptotically) in Ref. 2 assuming a warm plasma. The warm-plasma approximation remains valid at the wave-breaking limit and no singularities or divergences appear, i.e., the warm-fluid model does not break down at the wave-breaking limit. Naturally, the absence of a traveling-wave solution does not imply any "breakdown" of the fluid model. Breakdown of the fluid model can only be meaningfully defined as the appearance of singularities or violation of the assumptions leading to the fluid equations. In the results of SES, neither of these phenomena are manifest.

The calculation in SES is a carefully ordered asymptotic expansion of Vlasov-Maxwell dynamics. In asymptotic theory, simply assuming a small parameter does not guarantee that the asymptotic solutions will remain consistent with the initial assumption. The warm-plasma approximation is not self-fulfilling ("using an approximation to justify itself" as erroneously suggested by TN). Nothing in the theory forces the momentum spread to remain small. The fact that the plasma temperature remains small through wave-breaking is a prediction of the theory. In this case, as is consistent with the theory of asymptotics, it is not a tautology to take this as evidence that the warm-plasma approximation is valid; i.e., the warm-plasma approximation is internally self-consistent. Including additional terms in the asymptotics (i.e., those pertaining to third and higher moments), however, would give more accurate results.

The warm-plasma model will be an excellent approximation provided the temperature is nonrelativistic. This is the case for short-pulse laser-plasma experiments. If the initial temperature is relativistic, then the plasma wave evolution and wave-breaking limit will be *distribution dependent* and a strong function of the higher-order moments of the distribution, which will be determined from the specific form of the distribution. In this relativistic temperature regime, choosing an unphysical distribution (e.g., the water-bag used by TN) is problematic. The unbounded solutions (that predict singularities in the plasma density) derived from the unphysical water-bag distribution do not indicate the physical response of a plasma, but rather the breakdown of the collisionless plasma assumption. Physical observables are finite, and singularities produced by a model do not have physical meaning and cannot be "applied judiciously" as claimed by TN.

(Note to reader: we have read the Response of TN[5] and we stand by all the points made above in our Comment).

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