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Abstract†

A simple stochastic model of an insurer's underwriting and related investment operations is used to determine the optimal amounts of written premiums for one period for the insurer's book of business. The written premium for each class is determined by the solution of a constrained optimization problem. The insurer's objective function is the expected profit on a book of business over the period. The insurer has a safety constraint where a certain portion of capital and surplus can be depleted with a small probability. This paper provides an explicit solution for optimum expected profit and corresponding written premiums by classes. Due to the closed form nature of the solution for expected profit, insights are given as to how the optimum expected profit depends upon the model's parameters.

Key words and phrases: *written premiums, profits, risk, capital, surplus, objective function*

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1 Introduction

In a simple deterministic world with only one class of business, an underwriter would be concerned only with the profitability of this class of business. If the class is perceived as profitable, then the underwriter will commit as much available capacity as possible to writing that class of business. Considering even a single class of business in a real world (i.e., where risk exists), however, one has to balance profitability with risk.

Insurers typically have several classes of business, so insurance management considers business expediency as well as diversification. As these classes may be interdependent, management is interested in a balanced book of business consisting of different classes of business. Thus the management's concern with the overall profitability of a book of business must be balanced against the assumption of an acceptable level of risk. This approach to maximizing the profitability and controlling the amount of assumed risk is consistent with modern investment portfolio theory.

Markowitz (1952) introduced an important portfolio selection criterion in the field of finance. The Markowitz approach seeks to maximize the return on an investment portfolio by requiring the portfolio's standard deviation to be within an acceptable range. In this paper, we use a different portfolio criterion for selecting the optimal amounts of written premiums for an insurer book of business.

We define an insurer as any risk-bearing entity. An insurer may be a primary insurance company, a reinsurer, or a group of insurance companies. The insurer writes various classes of business. What constitutes a class of business depends upon the context of a given situation. For a primary insurer, the class of business may be fire, allied lines, private passenger auto liability, and other annual statement lines. For a reinsurer, the class of business may be property pro-rata, property excess, property cat covers, casualty pro-rata, casualty excess, and other suitable classes. For a group of insurers, the classes of business may be different profit centers.

The time horizon for which the optimal premiums are to be determined is short (for example, one year or less). There are certain reasons for this short time frame. First, for a fixed model it is unlikely that the functional relationship among variables would stay the same over a long time period. Second, due to changes in environment (economic, legal, regulatory, and technological), it would not be possible for a single model to represent the behavior of an insurer's operation over an extended time period.

Brubaker (1979) considered the same constrained optimization problem. His solution for optimal premium levels is based on a numeric iterative procedure that could be (but not easily) extended to situations where there are many classes of business. This paper improves Brubaker's approach by providing an explicit closed form solution to the constrained optimization problem. This solution provides insights into relationships between expected profits and input model variables. Other examples of constrained optimization problems have appeared in Ang and Lai (1987) and Meyers (1991).

For each class of business considered, the losses and expenses have a depleting effect upon the insurer's capital and surplus. Written premium and investment income attributable to a class of business increase the level of capital and surplus. The insurer's capital and surplus serve as safety margins against unexpected adverse underwriting and investment results. In addition, capital and surplus are needed to support future growth. It is assumed that management does not want the total capital and surplus to be depleted by more than a specified amount within a given time period. Losses, expenses, and investment income are considered to be random variables. Written premiums, portion of capital and surplus at risk, and the probability of depletion are viewed as deterministic decision variables to be controlled by management. The stochastic model introduced below considers the underwriting and investment income supporting the insurance operation of an insurer in a simple fashion.

2 The Model, Objective Function, and Constraint

Consider an insurance company with m classes of business. The total random profit, Π, over the single period of interest is

$$\begin{aligned}\Pi &= \sum_{i=1}^{m}(w_i + I_i - L_i - E_i)\\ &= \sum_{i=1}^{m} w_i R_i\end{aligned}$$

where, for the ith class, L_i, I_i, and E_i denote the losses, investment income, and the expenses, respectively, and w_i is the written premium and R_i is the profit per unit of written premium.

Let us define the following vectors:

$$\mathbf{r} = (r_1, r_2, \ldots, r_m)^T \quad \text{where } r_i = E[R_i]$$
$$\mathbf{R} = (R_1, R_2, \ldots, R_m)^T$$

and

$$\mathbf{w} = (w_1, w_2, \ldots, w_m)^T.$$

The total profit for this period can be written in vector form as

$$\Pi = \sum_{i=1}^{m} w_i R_i = \mathbf{w}^T \mathbf{R}.$$

The total expected profit, π, is

$$\pi = E[\mathbf{w}^T \mathbf{R}] = \mathbf{w}^T \mathbf{r} \tag{1}$$

and the variance of the total profit is

$$\text{Var}[\mathbf{w}^T \mathbf{R}] = \mathbf{w}^T \mathbf{V} \mathbf{w} \tag{2}$$

where $\mathbf{V}$ is the variance-covariance matrix for $\mathbf{R}$. Note that $\mathbf{V}$ is an $m \times m$ symmetric matrix:

$$\mathbf{V} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{pmatrix}$$

where

$$\sigma_{ij} = \text{Cov}[R_i, R_j] = E[(R_i - r_i)(R_j - r_j)]$$
$$\sigma_{ii} = \text{Var}[R_i].$$

We seek to maximize the expected profit, $\mathbf{w}^T \mathbf{r}$, subject to restrictions on the use of capital and surplus as specified below. Let C denote the insurer's total capital and surplus at the beginning of the period under review.

We assume that during this period, management is not willing to have more than a specific portion, k, of capital and surplus be depleted with a small probability, α. In other words, the probability that the total loss and the total expense arising from all classes of business may exceed the total written premium, the total investment income attributable to its underwriting operation during the period, and a portion of capital and surplus, kC, is at most α. We can write this statement as

$$\Pr[\sum_{i=1}^{m}(L_i + E_i) > \sum_{i=1}^{m}(w_i + I_i) + kC] = \Pr[kC + \mathbf{w}^T\mathbf{R} \le 0] \le \alpha. \qquad (3)$$

Assuming that $\mathbf{R}$ has a multivariate normal distribution, equation (3) reduces to

$$\Pr[\mathbf{w}^T\mathbf{R} \le -kC] = \Pr\left[Z \le \frac{-\mathbf{w}^T\mathbf{r} - kC}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}\right] \le \alpha. \qquad (4)$$

where

$$Z = \frac{\mathbf{w}^T\mathbf{R} - \mathbf{w}^T\mathbf{r}}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}$$

has a standard normal distribution. If we replace the inequality $(\le \alpha)$ in equation (4) by equality then we have

$$\Pr\left[Z \le -\frac{\mathbf{w}^T\mathbf{r} + kC}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}\right] = \alpha$$

which, by the symmetry of the normal distribution, implies that

$$\Pr\left[Z > \frac{\mathbf{w}^T\mathbf{r} + kC}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}\right] = \alpha. \qquad (5)$$

It then follows that

$$z_\alpha = \frac{\mathbf{w}^T\mathbf{r} + kC}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}$$

or

$$z_\alpha\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}} = \mathbf{w}^T\mathbf{r} + kC \qquad (6)$$

where z_α is the $100(1-\alpha)$ percentile of standard normal random variable, i.e.,

$$\Pr[Z > z_\alpha] = \alpha.$$

Equation (6) defines the safety constraint condition, due to limitations on the use of capital and surplus in our optimization problem. Now we proceed with the solution of the optimization problem.

3 The Optimization Problem

Our optimization problem is the determination of the vector of written premiums so that our objective function, total expected profit, is maximized subject to constraint as specified by equation (6). That is,

$$\text{Maximize}\mathbf{w}^T\mathbf{r}$$

subject to the constraint

$$z_\alpha\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}} = \mathbf{w}^T\mathbf{r} + kC.$$

Before proceeding with the derivation of the optimal solution for $\mathbf{w}$, the vector of written premiums, the following steps will facilitate the derivation of our results.

First, we note some special properties of the variance-covariance matrix $\mathbf{V}$. Second, we introduce vectors $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{w}}$ based on linear transformations of the original vectors $\mathbf{r}$ and $\mathbf{w}$. These transformed vectors facilitate solution of the optimization problem and make interpretation of our results easier. Third, we reformulate the original constrained optimization problem in terms of the transformed vectors $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{w}}$. The optimization problem is solved in terms of the transformed vectors. Finally, the solution of the constrained optimization problem is restated in terms of original vectors $\mathbf{r}$ and $\mathbf{w}$.

First, it is well-known that any variance-covariance matrix such as $\mathbf{V}$ is a nonnegative definite matrix; see, for example, Schott (1997, Chapter 1, page 23). Using Cholesky decomposition,[1] of $\mathbf{V}$ there exists an upper-triangular matrix $\mathbf{B}$ such that

[1]The Cholesky decomposition of a matrix is a well known algorithm for expressing a square matrix as a product of a lower-triangular matrix and an upper-triangular matrix; see, for example, Burden and Faires (1997, Chapter 6.6, page 410).

$$\mathbf{V} = \mathbf{B}^T\mathbf{B}. \tag{7}$$

If we assume further that $\mathbf{V}$ is nonsingular, i.e., $\mathbf{V}$ is a positive definite matrix, then $\mathbf{B}$ is also nonsingular.

Second, we introduce $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{w}}$ as follows:

$$\tilde{\mathbf{w}} = \mathbf{B}\mathbf{w} \tag{8}$$

$$\tilde{\mathbf{r}} = (\mathbf{B}^T)^{-1}\mathbf{r}. \tag{9}$$

The objective function in terms of transformed vectors is

$$\mathbf{w}^T\mathbf{r} = (\mathbf{B}^{-1}\tilde{\mathbf{w}})^T(\mathbf{B}^T\tilde{\mathbf{r}}) = \tilde{\mathbf{w}}^T\tilde{\mathbf{r}}.$$

Also, note that

$$\begin{aligned}\mathbf{w}^T\mathbf{V}\mathbf{w} &= (\mathbf{B}^{-1}\tilde{\mathbf{w}})^T\mathbf{B}^T\mathbf{B}(\mathbf{B}^{-1}\tilde{\mathbf{w}})\\ &= \tilde{\mathbf{w}}^T\tilde{\mathbf{w}}.\end{aligned}$$

Equation (6) can now be rewritten as

$$z_\alpha\sqrt{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}} = \tilde{\mathbf{w}}^T\tilde{\mathbf{r}} + kC. \tag{10}$$

To summarize, the optimization problem in terms of transformed vectors is to maximize $\tilde{\mathbf{w}}^T\tilde{\mathbf{r}}$ subject to the constraint of equation (10).

4 The Solution

The traditional approach to solving this optimization problem is by first defining a Lagrangian function $\mathcal{L}$:

$$\mathcal{L} = \tilde{\mathbf{w}}^T\tilde{\mathbf{r}} - \lambda[z_\alpha\sqrt{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}} - \tilde{\mathbf{w}}^T\tilde{\mathbf{r}} - kC]. \tag{11}$$

Next we determine the partial derivatives of H with respect to $\tilde{\mathbf{w}}$ and λ.

$$\frac{\partial\mathcal{L}}{\partial\tilde{w}_i} = \tilde{r}_i - \lambda[\frac{z_\alpha\tilde{w}_i}{\sqrt{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}}} - \tilde{r}_i] \tag{12}$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(z_\alpha \sqrt{\tilde{\mathbf{w}}^T \tilde{\mathbf{w}}} - \tilde{\mathbf{w}}^T \tilde{\mathbf{r}} - kC). \tag{13}$$

Setting these partial derivatives to zero yields

$$\tilde{r}_i - \lambda[\frac{z_\alpha \tilde{w}_i}{\sqrt{\tilde{\mathbf{w}}^T \tilde{\mathbf{w}}}} - \tilde{r}_i] = 0 \tag{14}$$

and

$$z_\alpha \sqrt{\tilde{\mathbf{w}}^T \tilde{\mathbf{w}}} - \tilde{\mathbf{w}}^T \tilde{\mathbf{r}} - kC = 0,$$

which is simply the constraint equation (10). Equation (14) implies

$$\tilde{w}_i = a\tilde{r}_i \tag{15}$$

where

$$a = \frac{(1+\lambda)\sqrt{\tilde{\mathbf{w}}^T \tilde{\mathbf{w}}}}{\lambda z_\alpha}. \tag{16}$$

Note that the a is a scalar and does not depend upon a particular i, though it does depend on the unknown premiums. Equation (15) also can be written in vector form as

$$\tilde{\mathbf{w}} = a\tilde{\mathbf{r}}. \tag{17}$$

Substituting equation (17) into the constraint equation (10) gives

$$a z_\alpha \sqrt{\tilde{\mathbf{r}}^T \tilde{\mathbf{r}}} - a\tilde{\mathbf{r}}^T \tilde{\mathbf{r}} - kC = 0. \tag{18}$$

Solving equation (18) for a, we have

$$a = \frac{kC}{z_\alpha \sqrt{\tilde{\mathbf{r}}^T \tilde{\mathbf{r}}} - \tilde{\mathbf{r}}^T \tilde{\mathbf{r}}} \tag{19}$$

which yields the vector of the optimum premium as

$$\tilde{\mathbf{w}}^* = a\tilde{\mathbf{r}} = \left[\frac{kC}{z_\alpha \sqrt{\tilde{\mathbf{r}}^T \tilde{\mathbf{r}}} - \tilde{\mathbf{r}}^T \tilde{\mathbf{r}}}\right] \tilde{\mathbf{r}}. \tag{20}$$

The maximum expected profit, $\pi^{\max}$, is

$$\pi^{\max} = \frac{kC}{\frac{z_\alpha}{\sqrt{\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}} - 1}. \tag{21}$$

We now rewrite our results in terms of the original vectors $\mathbf{r}$ and $\mathbf{w}$ using the equations (7), (8), and (9):

$$a = \frac{kC}{z_\alpha\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} - \mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} \tag{22}$$

and the vector of the optimum premium is

$$\mathbf{w}^* = \left[\frac{kC}{z_\alpha\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} - \mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}\right]\mathbf{V}^{-1}\mathbf{r} \tag{23}$$

and

$$\begin{aligned}\pi^{\max} &= \left[\frac{kC}{z_\alpha\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} - \mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}\right]\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r} \\ &= \frac{kC}{\frac{z_\alpha}{\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}} - 1}. \end{aligned} \tag{24}$$

Note for the maximum expected profit to be positive, we must ensure that

$$\left(\frac{z_\alpha}{\sqrt{\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}} - 1\right) = \left(\frac{z_\alpha}{\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}} - 1\right) > 0. \tag{25}$$

This imposes restrictions on the selection of α and of the portfolio of risks assumed in terms of the vector of expected profits $\mathbf{r}$ and the matrix of variance-covariance $\mathbf{V}$.

Next we must prove that $\pi^{\max}$ is indeed the maximum premium. To this end, we must prove that the Hessian matrix, $\mathbf{H}$, of second order partial derivatives is negative semidefinite. Specifically let

$$g_i = \frac{\partial}{\partial \tilde{w}_i}[z_\alpha\sqrt{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}} - \tilde{\mathbf{w}}^T\tilde{\mathbf{r}} - kC] \tag{26}$$

$$h_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \tilde{w}_i \partial \tilde{w}_j} \quad \text{for } i, j = 1, 2, \ldots, m, \text{ and} \tag{27}$$

$$\mathbf{H} = \{h_{ij}\} \tag{28}$$

where these derivatives are evaluated at $\tilde{\mathbf{w}}^*$ (given in equation (20)). Following Varian (1992, Chapter 27), to prove that $\pi^{\max}$ is a maximum we must prove that

$$\mathbf{y}^T\mathbf{H}\mathbf{y} \leq 0 \tag{29}$$

for any vector $\mathbf{y}^T = (y_1, \ldots, y_m)$ satisfying

$$\mathbf{g}^T\mathbf{y} = 0 \tag{30}$$

where $\mathbf{g}^T = (g_1, \ldots, g_m)$.

When evaluated at $\tilde{\mathbf{w}}^*$, it is easily seen that

$$g_i = \frac{z_\alpha \tilde{w}_i^*}{\theta^{1/2}} - \tilde{r}_i \tag{31}$$

$$h_{ij} = \begin{cases} \lambda \dfrac{z_\alpha \tilde{w}_i^* \tilde{w}_j^*}{\theta^{3/2}} & \text{if } i \neq j, \text{ and} \\ -\lambda \dfrac{z_\alpha}{\theta^{3/2}}(\theta + (\tilde{w}_i^*)^2) & \text{if } i = j \end{cases} \tag{32}$$

where $\theta = \tilde{\mathbf{w}}^{*T}\tilde{\mathbf{w}}^*$. In matrix terms,

$$\mathbf{g} = \left(\frac{a z_\alpha}{\theta^{1/2}} - 1\right)\tilde{\mathbf{r}} \tag{33}$$

after using equation (20) and

$$\begin{aligned} \mathbf{H} &= -\lambda \frac{z_\alpha}{\theta^{3/2}}(\theta\mathbf{I} + \tilde{\mathbf{w}}^*\tilde{\mathbf{w}}^{*T}) \\ &= -\lambda \frac{z_\alpha}{\theta^{3/2}}(\theta\mathbf{I} + a^2\tilde{\mathbf{r}}\tilde{\mathbf{r}}^T) \end{aligned} \tag{34}$$

where $\mathbf{I}$ is the identity matrix. Substituting equation (33) into equation (30) implies that the constraint reduces to

$$\tilde{\mathbf{r}}^T\mathbf{y} = \mathbf{y}^T\tilde{\mathbf{r}} = 0. \tag{35}$$

Hence for $\mathbf{y}$ satisfying equation (35),

$$\begin{aligned} \mathbf{y}^T\mathbf{H}\mathbf{y} &= -\lambda \frac{z_\alpha}{\theta^{3/2}}(\theta\mathbf{y}^T\mathbf{I}\mathbf{y} + a^2\mathbf{y}^T\tilde{\mathbf{r}}\tilde{\mathbf{r}}^T\mathbf{y}) \\ &= -\lambda \frac{z_\alpha}{\theta^{1/2}}\mathbf{y}^T\mathbf{y}. \end{aligned} \tag{36}$$

Thus **H** is negative semidefinite if and only if $\lambda \geq 0$ (assuming, of course, that $\alpha > 0.5$, i.e., $z_\alpha > 0$).

To prove that $\lambda \geq 0$ when evaluated at $\mathbf{w}^*$, we substitute equation (20) into equation (16) to give

$$a = \frac{(1+\lambda)\sqrt{a^2\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}}{\lambda z_\alpha}$$

which reduces to

$$1 = \frac{(1+\lambda)\sqrt{\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}}{\lambda z_\alpha}$$

which implies

$$\lambda = \frac{1}{\dfrac{z_\alpha}{\sqrt{\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}} - 1}. \tag{37}$$

But under the requirement that equation (25) holds, $\lambda \geq 0$, and the proof is completed.[2]

Let us summarize our results with regard to the optimum premium vector and maximum expected profit as a proposition.

Proposition 1. *Based on the following three premises:*

1. *The vector of profits per unit of written premiums,* **R**, *has a multivariate normal distribution with mean vector* **r** *and variance-covariance matrix* **V**;
2. *The insurer's constraint for the time period under consideration is the probability that the insurer's total losses exceed kC dollars of its capital and surplus is at most α; and*
3. **r**, **V**, *and α are such that*

$$\left(\frac{z_\alpha}{\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}} - 1\right) > 0.$$

The vector of optimum written premium ($\mathbf{w}^*$) *is given by*

$$\mathbf{w}^* = \left[\frac{kC}{z_\alpha\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} - \mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}\right]\mathbf{V}^{-1}\mathbf{r}$$

[2]The author thanks the editor for proving that $\pi^{\max}$ is maximum.

and the maximum expected profit, $\pi^{\max}$, for the book of business is

$$\pi^{\max} = \frac{kC}{\dfrac{z_\alpha}{\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}} - 1}.$$

Some implications of Proposition 1 are:

- As the portion of capital and surplus exposed to the risk of depletion, k, increases, then expected profit increases; and
- As α increases, then the expected profit increases.

Finally, expressions for the vector of written premiums and expected profit can be simplified in the case of uncorrelated classes of business because the variance-covariance matrix becomes a diagonal matrix with diagonal elements corresponding to the variances of the respective classes.

We note that

$$\begin{aligned}\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r} &= \sum_{i=1}^{m} \frac{r_i^2}{\sigma_{ii}} \\ &= \sum_{i=1}^{m} \frac{1}{\mathrm{CV}_i^2}\end{aligned}$$

where CV_i is the coefficient of variation for the ith class.

Thus, in the case of uncorrelated classes of business, the class with the smallest coefficient of variation contributes most to total expected profit. [See equation (24).] Furthermore,

$$\mathbf{V}^{-1}\mathbf{r} = \begin{pmatrix} \dfrac{r_1}{\sigma_{11}} \\ \vdots \\ \dfrac{r_m}{\sigma_{mm}} \end{pmatrix}.$$

It can be noted from equation (23) that the optimum amount of written premium, w_i, is inversely proportional to the variances σ_{ii}, in the case of uncorrelated classes of business.

5 Numerical Examples

5.1 Example 1

Let us consider the example stated as Case 7 by Brubaker (1979). Brubaker provides an iterative solution for optimal premiums. The setting is as follows:

- $k = 0.5$, $C = 300$ million, $\alpha = 0.001$, $z_\alpha = 3.1$;
- Three classes of business ($m = 3$);
- Expected profit per unit of premium: $r_i = 0.05$, for $i = 1, 2, 3$;
- Variance of each class: $\sigma_{ii} = (0.075)^2$, for $i = 1, 2, 3$;
- Correlation coefficients among classes: $\rho_{12} = -0.5$ and $\rho_{13} = \rho_{23} = 0$.

Using our notation,

$$\mathbf{r} = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.05 \end{pmatrix}$$

and

$$\mathbf{V} = (0.075)^2 \begin{pmatrix} 1.0 & -0.5 & 0.0 \\ -0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}.$$

As

$$\mathbf{V}^{-1} = \begin{pmatrix} 237.037 & 118.519 & 0.0 \\ 118.519 & 237.037 & 0.0 \\ 0.0 & 0.0 & 177.778 \end{pmatrix}$$

and $\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r} = 2.222$, the optimum vector of premium is

$$\mathbf{w}^* = \begin{pmatrix} 1,111.6 \\ 1,111.6 \\ 555.8 \end{pmatrix}$$

and the maximum expected profit is $\pi^{\max} = 138.9$.

Note the upper-triangular matrix B is

$$\mathbf{B} = \begin{pmatrix} 0.07500 & -0.03750 & 0 \\ 0 & 0.06495 & 0 \\ 0 & 0 & 0.07500 \end{pmatrix}$$

and

$$\tilde{\mathbf{r}} = \mathbf{B}^{-1}\mathbf{r} = \begin{pmatrix} 0.667 \\ 1.155 \\ 0.667 \end{pmatrix}.$$

As the expected profit per unit of premium and the standard deviation for each class are the same, it is tempting to infer that the three classes contribute equally to total profitability. Considering the correlation structure among the three classes, $\mathbf{V}$, and replacing $\mathbf{r}$ by $\tilde{\mathbf{r}}$ that summarizes the information about $(\mathbf{r}, \mathbf{V})$, we note that the components of the vector $\tilde{\mathbf{r}}$ are not all equal. Thus, the three classes impact the insurer's total profitability in unequal ways.

Matrix operations and the Cholesky decompositions can be done using readily available software such as SAS, S-Plus, Maple, or Mathematica.[3]

5.2 Example 2

Let us consider an example of an insurer with four classes of business ($m = 4$) and with $C = 300$ million. The insurer faces the following expected profit per unit of premium vector and variance-covariance matrix:

$$\mathbf{r} = \begin{pmatrix} 0.05 \\ 0.06 \\ 0.07 \\ 0.08 \end{pmatrix}$$

[3]SAS is a registered trademark of: SAS Institute Inc., Cary, NC 27512-8000, USA; S-Plus is a registered trademark of: MathSoft, 101 Main Street, Cambridge, MA 02142, USA; MAPLE is a registered trademark of: Waterloo Maple Software, 450 Phillip Street, Waterloo ON N2L 5J2, CANADA; and Mathematica is a registered trademark of: Wolfram Research, Inc., 100 Trade Center Drive, Champaign IL 61820-7237, USA.

and

$$\mathbf{V} = (0.075)^2 \begin{pmatrix} 1.0 & -0.4 & -0.5 & -0.6 \\ -0.4 & 2.0 & -0.5 & 0.3 \\ -0.5 & -0.5 & 3.0 & 0.1 \\ -0.6 & 0.3 & 0.1, & 4.0 \end{pmatrix}$$

We will calculate the optimum premium and profits for various levels of α and k.

The following are easily derived:

$$\mathbf{V}^{-1} = \begin{pmatrix} 243.359 & 56.237 & 48.897 & 31.064 \\ 56.234 & 106.979 & 27.212 & -0.268 \\ 48.8971 & 27.212 & 71.827 & 3.498 \\ 31.064 & -0.268 & 3.498 & 49.037 \end{pmatrix}$$

$$\mathbf{B}^T = \begin{pmatrix} 0.075 & 0 & 0 & 0 \\ -0.03 & 0.1017 & 0 & 0 \\ -0.0375 & -0.0387 & 0.1182 & 0 \\ -0.045 & 0.0033 & -0.0084 & 0.1428 \end{pmatrix}$$

and

$$\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r} = 2.853.$$

Tables 1 through 5 display the results of our calculations.

Table 1
Optimum Premiums and Profits: Case k = 1.00

		Optimum Premiums $\mathbf{w}^*$				
α	z_α	w_1^*	w_2^*	w_3^*	w_4^*	Profit
0.001	3.090	2719.45	1409.03	1189.88	723.27	361.67
0.005	2.576	4295.32	2225.54	1879.38	1142.39	571.25
0.010	2.326	5981.06	3098.97	2616.97	1590.73	795.44
0.025	1.960	14058.64	7284.22	6151.26	3739.05	1869.70

Table 2
Optimum Premiums and Profits: Case $k = 0.75$

α	z_α	Optimum Premiums $\mathbf{w}^*$				Profit
		w_1^*	w_2^*	w_3^*	w_4^*	
0.001	3.090	2039.59	1056.78	892.41	542.45	271.25
0.005	2.576	3221.49	1669.15	1409.54	856.79	428.43
0.010	2.326	4485.80	2324.23	1962.73	1193.05	596.58
0.025	1.960	10543.98	5463.16	4613.44	2804.29	1402.27

Table 3
Optimum Premiums and Profits: Case $k = 0.50$

α	z_α	Optimum Premiums $\mathbf{w}^*$				Profit
		w_1^*	w_2^*	w_3^*	w_4^*	
0.001	3.090	1359.73	704.52	594.94	361.63	180.84
0.005	2.576	2147.66	1112.77	939.69	571.19	285.62
0.010	2.326	2990.53	1549.49	1308.49	795.37	397.72
0.025	1.960	7029.32	3642.11	3075.63	1869.53	934.85

In our model, the maximum expected profit and the vector of optimum premiums are functions of k, C, α (or z_α), $\mathbf{r}$, and $\mathbf{V}$. In Tables 1 to 5, C, $\mathbf{r}$, and $\mathbf{V}$ are held constant while k and α are varied. We shall now explain the effect of changes in k and α on the maximum expected profit.

First, keeping α constant, as k increases we are exposing a larger portion of our capital and surplus to a fixed level of risk, α, in order to support our insurance operation. Thus we see higher profit levels as shown in Tables 1 to 5. This is consistent with the idea that greater risk bearing should be compensated by higher profit (return) levels.

Similarly, keeping k constant, as α increases we are exposing a fixed portion of our capital and surplus (k) to a larger level of risk, α, in order to support our insurance operation. Thus we see higher profit levels as shown in Tables 1 to 5. Again, this is consistent with the idea that greater risk bearing should be compensated by higher profit (return) levels. To illustrate this in the extreme case: as $\alpha \to 0$ (i.e., the level of risk decrease to zero), $z_\alpha \to \infty$ and both the optimum premium vector (equation (23)) and the maximum expected profit (equation (24))

Table 4
Optimum Premiums and Profits: Case $k = 0.25$

		Optimum Premiums $\mathbf{w}^*$				
α	z_α	w_1^*	w_2^*	w_3^*	w_4^*	Profit
0.001	3.090	679.86	352.26	297.47	180.82	90.42
0.005	2.576	1073.83	556.38	469.85	285.60	142.81
0.010	2.326	1495.27	774.74	654.24	397.68	198.86
0.025	1.960	3514.66	1821.05	1537.81	934.76	467.42

Table 5
Optimum Premiums and Profits: Case $k = 0.10$

		Optimum Premiums $\mathbf{w}^*$				
α	z_α	w_1^*	w_2^*	w_3^*	w_4^*	Profit
0.001	3.090	271.95	140.90	118.99	72.33	36.17
0.005	2.576	429.53	222.55	187.94	114.24	57.12
0.010	2.326	598.11	309.90	261.70	159.07	79.54
0.025	1.960	1405.86	728.42	615.13,	373.91	186.97

decrease to zero. In other words, the best way to avoid all risk is to write no business altogether.

Note that these observations are easily derived mathematically because, in equation (24), $\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r} = 2.853$, so $\pi^{\max}$ increases as k or α increases. Recall that as α increases, z_α decreases.

6 Conclusion

We have introduced a simple model to represent underwriting and related investment income for a class of business of an insurer. An important decision for the management of insurance companies is the determination of written premiums for respective classes of business. A rational solution to this problem requires balancing profitability against risk to the insurer. This paper explains risk-taking in terms of the extent to which management is willing to lose a portion of its capital and surplus during a short time horizon.

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