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Ideal Containments under Flat Extensions and Interpolation on Linear Systems in P^2

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IDEAL CONTAINMENTS UNDER FLAT EXTENSIONS AND INTERPOLATION ON LINEAR SYSTEMS IN $\mathbb{P}^2$

by

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Fat points and their ideals have stimulated a lot of research but this dissertation concerns itself with aspects of only two of them, broadly categorized here as, the ideal containments and polynomial interpolation problems.

Ein-Lazarsfeld-Smith and Hochster-Huneke cumulatively showed that for all ideals I in $k[\mathbb{P}^n]$, $I^{(mn)} \subseteq I^m$ for all $m \in \mathbb{N}$. Over the projective plane, we obtain $I^{(4)} \subseteq I^2$. Huneke asked whether it was the case that $I^{(3)} \subseteq I^2$. Dumnicki, Szemberg and Tutaj-Gasinska show that if I is the saturated homogeneous radical ideal of the 12 points of the Hesse configuration, then $I^{(3)} \not\subseteq I^2$. Since then, additional examples have been found, but all of them, are the intersection loci of lines. Here we extend all the examples of $I^{(3)} \not\subseteq I^2$ to points that are not directly the intersection loci of lines but are the intersection loci of curves.

In the case of the interpolation problem, this dissertation makes the following contribution. Let k be an algebraically closed field of arbitrary characteristic. Let $q_1, \ldots, q_r$ be a set of not necessarily general points and let $p_1, \ldots, p_s$ be a set of general points in $\mathbb{P}^2$, $r + s \leq 8$. Let X be a blow up of the points with $e_1, \ldots, e_r$ and $E_1, \ldots, E_s$ the corresponding exceptional curves. Write $e = a_1e_1 + \cdots + a_re_r$ and $E = b_1E_1 + \cdots + b_sE_s$. For the two linear systems $[dL - e - E]$ and $[dL - e]$ with $[dL - e - E] \subseteq [dL - e]$, we give a condition sufficient to guarantee that

$h^0(X, dL - e - E) > \max\{0, h^0(X, dL - e) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$ and another condition necessary for $h^0(X, dL - e - E) > \max\{0, h^0(X, dL - e) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$. When $r = 7$, $s = 1$, $d = 3$, $a_j = 1$, $1 \leq j \leq 7$ and $b = 2$, we connect the discussion to quasi-elliptic fibrations and show that when $q_1 + \cdots + q_7$ is reduced, then $h^0(X, 3L - e_1 - \cdots - e_7 - 2E) > \max\{0, h^0(X, 3L - e_1 - \cdots - e_7) - 3\}$ if and only if $q_1 + \cdots + q_7$ is the union of the seven points of the Fano plane. Allowing infinitely near points, we obtain nonreduced subschemes $q_1 + \cdots + q_7$, consisting of essentially distinct points, that form part of the base loci of quasi-elliptic fibrations such that $h^0(X, 3L - e_1 - \cdots - e_7 - 2E) > \max\{0, h^0(X, 3L - e_1 - \cdots - e_7) - 3\}$.

DEDICATION

To my parents

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Chapter 1

Introduction

1.1 Fat Points in $\mathbb{P}^n$

Let k be an algebraically closed field of arbitrary characteristic. We work in $\mathbb{P}^n_k$. One can pick a point $p \in \mathbb{P}^n$ and using the correspondence of varieties and ideals in $k[\mathbb{P}^n]$, consider the maximal homogeneous ideal, $I(p) \subset k[\mathbb{P}^n]$, whose zero locus in $\mathbb{P}^n$ is exactly the point p which has multiplicity 1.

Expanding the notion of variety to the notion of scheme, in particular allowing for non-reduced varieties, we can consider the variety, mp, specified by the homogeneous ideal $I(p)^m$, $m \in \mathbb{N}$. Note that mp is supported at exactly p in $\mathbb{P}^n$ but we distinguish between mp and p by keeping in mind the different homogeneous coordinate rings $k[\mathbb{P}^n]/I(p)^m$ and $k[\mathbb{P}^n]/I(p)$ associated to each, respectively.

The point mp is called a fat point supported at p of multiplicity m. Just as we can consider a set of l distinct points $p_1 + \cdots + p_l \subseteq k[\mathbb{P}^n]$, $l \in \mathbb{N}$, each of multiplicity one, specified by the homogeneous ideal $I(p_1) \cap \cdots \cap I(p_l) \subseteq k[\mathbb{P}^n]$, where we have chosen the notation to emphasize the variety aspect of the set, we can also consider a set of l distinct fat points, $m_1 p_1 + \cdots + m_l p_l$, of varying multiplicities,

$m_1, \ldots, m_l$, in $\mathbb{P}^n$, supported at $p_1 + \cdots + p_l$, specified by the homogeneous ideal $I(p_1)^{m_1} \cap \cdots \cap I(p_l)^{m_l} \subset k[\mathbb{P}^n]$. In this dissertation, we consider an aspect each of two broad problems in the study of fat points: the ideal containments and polynomial interpolation problems. The two problems are not disparate but are in fact closely related in the sense that they are both concerned with properties of the ideal $I(p_1)^{m_1} \cap \cdots \cap I(p_l)^{m_l}$.

Start with a set of l distinct fat points $Z = m_1 p_1 + \cdots + m_l p_l$ in $\mathbb{P}^n_k$, k algebraically closed, of varying multiplicities, specified by the homogeneous ideal $I_Z = I(p_1)^{m_1} \cap \cdots \cap I(p_l)^{m_l} \subset k[\mathbb{P}^n]$. Let X be the surface obtained from the blow up of $\mathbb{P}^2$ at the points with L the total transform of a line in $\mathbb{P}^2$ to X and $E_1, \ldots, E_l$, the exceptional curves corresponding to the points $p_1, \ldots, p_l$, respectively. We can consider the divisor class $[dL - m_1 E_1 - \cdots - m_l E_l]$ on X, $d \in \mathbb{N}$, and its k-vector space of global sections $H^0(X, dL - m_1 E_1 - \cdots - m_l E_l)$. We can, also, consider the homogeneous component of degree d of the k-vector space of homogeneous polynomials I_Z specifying Z, $I_Z(d)$. Then it is well known that as k-vector spaces, $H^0(X, dL - m_1 E_1 - \cdots - m_l E_l) \cong I_Z(d)$ and hence $\dim_k I_Z(d) = \dim_k H^0(X, dL - m_1 E_1 - \cdots - m_l E_l)$.

The polynomial interpolation problem in $\mathbb{P}^2$ asks, for a set of points $p_1, \ldots, p_l$ with specified multiplicities $m_1, \ldots, m_l$, the number of homogeneous polynomials of degree d that vanish at each of the p_i with multiplicity at least m_i. When the base field is algebraically closed, it accepts as a reasonable answer, the number, $\dim_k I_Z(d) = \dim_k H^0(X, dL - m_1 E_1 - \cdots - m_l E_l)$. Let $I = I(p_1) \cap \cdots \cap I(p_l)$. The ideal containments problem, at least the aspect we consider here, seeks to understand when $I_Z \subseteq I^n$ when $m = m_1 = \cdots = m_l$. In the next two sections, we discuss the contexts of the ideal containments and polynomial interpolation problems in more detail by drawing on the literature. For each problem, we mention how our work fits in with previous work.

1.2 Ideal Containments

Let R be a commutative Noetherian domain. Let I be an ideal in R. We define the mth symbolic power of I to be the ideal

$$I^{(m)} = R \cap \bigcap_{P \in Ass_R(I)} I^m R_P \subseteq R_{(0)}.$$

In this dissertation we shall be interested in symbolic powers of homogeneous ideals of 0-dimensional subschemes in $\mathbb{P}^n$. In the case that the subscheme is reduced, the definition of the symbolic power takes a rather simple form by a theorem of Zariski and Nagata [17] and does not require passing to the localizations at various associated primes. Let $I \subseteq k[\mathbb{P}^n]$ be a homogeneous ideal of reduced points, $p_1, ..., p_l$, in $\mathbb{P}^n$ with k a field of any characteristic. Then $I = I(p_1) \cap \cdots \cap I(p_l)$ where $I(p_i) \subseteq k[\mathbb{P}^n]$ is the ideal generated by all forms vanishing at p_i, and the mth symbolic power of I is simply $I^{(m)} = I(p_1)^m \cap \cdots \cap I(p_l)^m$.

In [16], Ein, Lazarsfeld and Smith proved that if $I \subseteq k[\mathbb{P}^n]$ is the radical ideal of a 0-dimensional subscheme of $\mathbb{P}^n$, where k is an algebraically closed field of characteristic 0, then $I^{(mr)} \subseteq (I^{(r+1-n)})^m$ for all $m \in \mathbb{N}$ and $r \geq n$. Letting $r = n$, we get that $I^{(mn)} \subseteq I^m$ for all $m \in \mathbb{N}$. Hochster and Huneke in [31] extended this result to all ideals $I \subseteq k[\mathbb{P}^n]$ over any field k of arbitrary characteristic.

In [7] Bocci and Harbourne introduced a quantity $\rho(I)$, called the resurgence, associated to a nontrivial homogeneous ideal I in $k[\mathbb{P}^n]$, defined to be $\sup\{s/t : I^{(s)} \not\subseteq I^t\}$. It is seen immediately that if $\rho(I)$ exists, then for $s > \rho(I)t$, $I^{(s)} \subseteq I^t$. The results of [16, 31] guarantee that $\rho(I)$ exists since $I^{(mn)} \subseteq I^m$ implies that $\rho(I) \leq n$ for an ideal I in $k[\mathbb{P}^n]$. For an ideal I of points in $\mathbb{P}^2$, $I^{(mn)} \subseteq I^m$ gives $I^{(4)} \subseteq I^2$. According to [7] Huneke asked if $I^{(3)} \subseteq I^2$ for a homogeneous ideal I of points in $\mathbb{P}^2$. More generally Harbourne conjectured in [4] that if $I \subseteq k[\mathbb{P}^n]$ is a homogeneous ideal, then $I^{(rn-(n-1))} \subseteq I^r$ for all r. This led to the conjectures by Harbourne and Huneke in [27] for ideals I of points that $I^{(mn-n+1)} \subseteq \mathfrak{m}^{(m-1)(n-1)} I^m$ and $I^{(mn)} \subseteq \mathfrak{m}^{m(n-1)} I^m$ for $m \in \mathbb{N}$ where $\mathfrak{m}$ is the homogeneous maximal ideal of $k[\mathbb{P}^n]$.

The second conjecture remains open. Cooper, Embree, Ha and Hoefel give a counterexample in [12] to the first for $n = 2 = m$ for a homogeneous ideal $I \subseteq k[\mathbb{P}^2]$. The ideal I in this case is $I = (xy^2, yz^2, zx^2, xyz) = (x^2, y) \cap (y^2, z) \cap (z^2, x)$ whose zero locus in $\mathbb{P}^2$ is the 3 coordinate vertices of $\mathbb{P}^2$, $[0:0:1]$, $[0:1:0]$ and $[1:0:0]$ together with 3 infinitely near points, one at each of the vertices, for a total of 6 points. Clearly the monomial $x^2y^2z^2 \in (x^2, y)^3 \cap (y^2, z)^3 \cap (z^2, x)^3$ so $x^2y^2z^2$ is in $I^{(3)}$. Note $xyz \in I$ so $x^2y^2z^2 \in I^2$, but $x^2y^2z^2 \notin \mathfrak{m}I^2$.

Shortly thereafter a counterexample to the containment $I^{(3)} \not\subseteq I^2$ was given by Dumnicki, Szemberg and Tutaj-Gasinska in [15]. In this case I is the ideal of the 12 points dual to the 12 lines of the Hesse configuration. The Hesse configuration consists of the 9 flex points of a smooth cubic and the 12 lines through pairs of flexes. Thus I defines 12 points lying on 9 lines. Each of the lines goes through 4 of the points, and each point has 3 of the lines going through it. Specifically I is the saturated radical homogeneous ideal $I = (x(y^3 - z^3), y(x^3 - z^3), z(x^3 -$

$y^3)) \subset \mathbb{C}[\mathbb{P}^2]$. Its zero locus is the 3 coordinate vertices of $\mathbb{P}^2$ together with the 9 intersection points of any 2 of the forms $x^3 - y^3$, $x^3 - z^3$ and $y^3 - z^3$. The form $F = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$ defining the 9 lines belongs to $I^{(3)}$ since for each point in the configuration, 3 of the lines in the zero locus of F pass through the point, but $F \notin I^2$ and hence $I^{(3)} \not\subseteq I^2$. (Of course this also means that $I^{(3)} \not\subseteq \mathfrak{m}I^2$.) More generally, $I = (x(y^n - z^n), y(x^n - z^n), z(x^n - y^n))$ defines a configuration of $n^2 + 3$ points called a Fermat configuration [2]. For $n \geq 3$, we again have $I^{(3)} \not\subseteq I^2$ [28, 38] over any field of characteristic not 2 or 3 containing n distinct nth roots of 1.

Subsequent counterexamples to $I^{(3)} \subseteq I^2$ were given in [6], [3], [28], [14] and [38] including related counterexamples to $I^{(nr-n+1)} \subseteq I^r$ for ideals of points in $\mathbb{P}^n$ in positive characteristic given in [28]. All of the counterexamples to $I^{(3)} \subseteq I^2$ are ideals of points where the points are singular points of multiplicity at least 3 of a configuration of lines. By considering flat morphisms $\mathbb{P}^n \to \mathbb{P}^n$, we obtain many new counterexamples to $I^{(rn-n+1)} \subseteq I^r$, taking I to be the ideal of the fibers over the points of previously known counterexamples.

The idea for this comes from [18]. Suppose Δ is a matroid on $[s] = \{1, ..., s\}$ of dimension $s - 1 - c$ and and let $f_1, ..., f_s \in R = k[y_0, ..., y_n]$ be homogeneous polynomials that form an R-regular sequence, $n \geq c$. Suppose now that $\varphi : S = k[y_1, ..., y_s] \to R$ is a k-algebra map defined by $y_i \to f_i$. Then [18] shows that if $I_\Delta \subseteq S$ is the ideal of the matroid and m and r are positive integers, then $I_\Delta^{(m)} \subseteq I_\Delta^r$ if and only if $\varphi_*(I_\Delta)^{(m)} \subseteq \varphi_*(I_\Delta)^r$ where $\varphi_*(I_\Delta)$ denotes the ideal generated by $\varphi(I_\Delta)$ in R. Of course a natural question is whether $I^{(m)} \subseteq I^r$ if and only if $\varphi_*(I)^{(m)} \subseteq \varphi_*(I)^r$ for any saturated homogeneous ideal. This dissertation answers this question in the affirmative for ideals I of points in $\mathbb{P}^n$, relying on the ideas in [18].

1.3 Interpolation on Linear Systems

We work in the projective plane $\mathbb{P}^2_k$ over an algebraically closed field k of arbitrary characteristic. Let L denote the class of a line in $\mathbb{P}^2$. Then the Picard group of $\mathbb{P}^2$ is $\mathrm{Pic}(\mathbb{P}^2) = \{[dL] : d \in \mathbb{Z}\}$ where $[dL]$ is the linear equivalence class of curves defined by all homogeneous polynomials of degree d when $d > 0$. Given a set of distinct points $p_1, \ldots, p_n$ in $\mathbb{P}^2$ with assigned multiplicities $m_1, \ldots, m_n$, it is a classical question to ask how many polynomials, $P = P(x_0, x_1, x_2)$ in $R = k[x_0, x_1, x_2]$ of degree $d > 0$ are there such that P vanishes at each of the p_i to order at least m_i? Let $V = V(dL - m_1p_1 - \cdots - m_np_n)$ denote the vector space of all homogeneous polynomials of degree d vanishing to order at least m_i at the point p_i, $1 \leq i \leq n$, and let R_d be the k-vector space of all homogeneous polynomials of degree d. Notice that V is a subspace of R_d and moreover that V corresponds to the complete linear system of curves of degree d passing through p_i with multiplicity at least m_i, $\mathcal{L} = [dL - m_1p_1 - \cdots - m_np_n]$, while R_d corresponds to $[dL]$. A possible answer to the above question now is $\dim_k V$.

Note that, in char $k = 0$, if P vanishes at the point p_i to order at least m_i, then all the derivatives of P to order $m_i - 1$ must vanish at p_i. There are $\binom{m_i+1}{2} = \frac{m_i(m_i+1)}{2}$ such derivatives so that the requirement that P vanish at p_i to order at least m_i imposes $\frac{m_i(m_i+1)}{2}$ conditions on R_d. Since $\dim_k R_d = \binom{d+2}{2} = \frac{(d+2)(d+1)}{2}$, we have that $\dim_k V$ is at least $\frac{(d+2)(d+1)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}$. Also since $\dim_k V \geq 0$, we obtain the lower bound $\dim_k V \geq \max\{0, \frac{(d+2)(d+1)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}\}$. In the literature,

$\frac{(d+2)(d+1)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}$ is referred to us the virtual dimension of V, denoted $\mathcal{V}$, and $\max\{0, \frac{(d+2)(d+1)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}\}$ is referred to as the expected dimension of V, $\mathcal{E}$. So we have that $\mathcal{E} = \max\{0, \mathcal{V}\}$ and $\dim_k V \geq \mathcal{E}$. The numerics of finding a lower bound for $\dim_k V$ are the same in char $k = p > 0$ since a homogeneous polynomial F vanishes at points $p_1, \ldots, p_n$ to order at least $m_1, \ldots, m_n$ exactly when $F \in I_{p_1}^{m_1} \cap I_{p_2}^{m_2} \cap \cdots \cap I_{p_n}^{m_n}$ where I_{p_i} is the ideal of all homogeneous polynomials vanishing at p_i. But then $\dim_k(I_{p_1}^{m_1} \cap I_{p_2}^{m_2} \cap \cdots \cap I_{p_n}^{m_n})_d \geq \dim_k R_d - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}$. Everything said previously now follows. When $\dim_k V$ fails to achieve its lower bound, i.e. $\dim_k V > \mathcal{E}$, we say that $\mathcal{L}$ is a special linear system. For ease, we shall identify V and $\mathcal{L}$ with each other and also identify R_d with $[dL]$.

One might ask for instances when the lower bound is not achieved. Take $d = 1$ and consider three collinear points p_1, p_2 and p_3 in $\mathbb{P}^2$. For $\mathcal{L} = [L - p_1 - p_2 - p_3]$, we have that $\mathcal{V}(\mathcal{L}) = 3 - 3 = 0$ so that $\mathcal{E}(\mathcal{L}) = 0$ but $\dim_k \mathcal{L} = 1$. Hence $\dim_k \mathcal{L} > \mathcal{E}(\mathcal{L})$. In this case however, the points p_1, p_2 and p_3 are not general points since they are collinear. This leads one to ask whether there are any instances for which $\dim_k \mathcal{L} > \mathcal{E}(\mathcal{L})$ when the points $p_1, \ldots, p_n$ are general? Take $d = 2$ and consider two general points p_1 and p_2 each with assigned multiplicity 2 in $\mathbb{P}^2$. Let $\mathcal{L} = [2L - 2p_1 - 2p_2]$ be the linear system of conics vanishing with multiplicity at least 2 at each of p_1 and p_2. Note that $\mathcal{V}(\mathcal{L}) = 6 - 3 - 3 = 0$ so that $\mathcal{E}(\mathcal{L}) = 0$. Note, however, that $\dim_k \mathcal{L} > \mathcal{E}(\mathcal{L})$ since $\mathcal{L}$ contains the non-reduced conic that is the double line through the points p_1 and p_2. For another example, take $d = 4$ and consider the five general points $p_1, \ldots, p_5$ each with assigned multiplicity 2. Let $\mathcal{L} = [4L - 2p_1 - \cdots - 2p_5]$ be the linear system of curves of degree 4 in $\mathbb{P}^2$ vanishing with multiplicity at least 2 at each of the points. Then just as before one computes $\mathcal{V}(\mathcal{L}) = 0$ and hence $\mathcal{E}(\mathcal{L}) = 0$. Since 5 general points determine a conic, $\mathcal{L}$ contains the non-reduced curve of degree 4 that is double of the unique irreducible conic through the five points $p_1, \dots, p_5$. All the known examples for general points where $\dim_k \mathcal{L} > \mathcal{E}(\mathcal{L})$ are of this type i.e. they involve non-reduced curves in their fixed part. In fact Segre conjectured in 1961 [39] that given a linear system $\mathcal{L} = [dL - m_1 p_1 - \cdots - m_n p_n]$ in $\mathbb{P}^2$ such that $\dim_k \mathcal{L} > \mathcal{E}(\mathcal{L})$, the fixed part of $\mathcal{L}$ contains a non-reduced component.

In working with linear systems with assigned base points in $\mathbb{P}^2$, one finds that it is easier to blow up the points $p_1, \dots, p_n$ to obtain a birational projective morphism $\pi : X \to \mathbb{P}^2$ whose exceptional curves $E_1, \dots, E_n$ (see Definition 3.1.2) are contracted to $p_1, \dots, p_n$ respectively by the map π which is isomorphic away from the curves $E_1, \dots, E_n$. By so doing, one passes from cycles of the form $dL - m_1 p_1 - \cdots - m_n p_n$ to divisors $dL - m_1 E_1 - \cdots - m_n E_n$ where L is now understood to be the total transform to X of the original line L in $\mathbb{P}^2$. This allows one to take full advantage of the geometry of rational surfaces. Let $K_X = -3L + E_1 + \cdots + E_n$ denote the canonical divisor on X and $-K_X = 3L - E_1 - \cdots - E_n$ the anticanonical divisor. Given the complete linear system $\mathcal{L} = [dL - m_1 p_1 - \cdots - m_n p_n]$ on $\mathbb{P}^2$, its proper transform on X is the complete linear system $[dL - m_1 E_1 - \cdots - m_n E_n]$ of the divisor $dL - m_1 E_1 - \cdots - m_n E_n$. The virtual dimension of $[dL - m_1 E_1 - \cdots - m_n E_n]$ is defined to be $\mathcal{V}(dL - m_1 E_1 - \cdots - m_n E_n) = \frac{(dL - m_1 E_1 - \cdots - m_n E_n)^2 + (-K_X) \cdot (dL - m_1 E_1 - \cdots - m_n E_n)}{2} + 1$ and the expected dimension, $\mathcal{E}(dL - m_1 E_1 - \cdots - m_n E_n)$, just as before, is $\max\{0, \mathcal{V}(dL - m_1 E_1 - \cdots - m_n E_n)\}$. One easily checks that $\frac{(dL - m_1 E_1 - \cdots - m_n E_n)^2 + (-K_X) \cdot (dL - m_1 E_1 - \cdots - m_n E_n)}{2} + 1 = \frac{(d+2)(d+1)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}$ so that the linear systems $\mathcal{L}$ and $[dL - m_1 E_1 - \cdots - m_n E_n]$ on $\mathbb{P}^2$ and on X respectively have the same virtual and expected dimensions. Moreover $\dim_k \mathcal{L} = \dim_k [dL - m_1 E_1 - \cdots - m_n E_n]$ [29] so we abuse notation and refer to the linear system $[dL - m_1E_1 - \cdots - m_nE_n]$ as $\mathcal{L}$ and even go further to refer to its corresponding line bundle, $\mathcal{O}_X(\mathcal{L})$, also as $\mathcal{L}$.

Harbourne [22], Gimigliano [21] and Hirschowitz [30] all independently conjectured that if X is the blow up of n general points, $p_1, \ldots, p_n$, with assigned multiplicities $m_1, \ldots, m_n$, in $\mathbb{P}^2$, then the linear system $\mathcal{L} = |dL - m_1E_1 - \cdots - m_nE_n|$ is special if and only if there is an exceptional curve C such that nC, $n \geq 2$, sits in the fixed part of $\mathcal{L}$. Segre's conjecture is equivalent to the conjectures of Harbourne, Gimigliano and Hirschowitz if one considers the blowup of the points in Segre's statement or one considers the proper transform of the linear system to $\mathbb{P}^2$ of the Harbourne-Gimigliano-Hirschowitz statement. For two slightly different arguments showing the equivalence, see Ciliberto and Miranda [10] and Harbourne [25]. This allows us to refer to both conjectures by the acronym: the SHGH conjecture (Segre-Harbourne-Gimigliano-Hirschowitz). While there has been a substantial amount of evidence accumulated in favor of the truth of the conjecture, it still remains open. According to [25], Nagata's work in [36] proves the SHGH conjecture for $n \leq 9$ general points. The work of Harbourne in [24] also essentially proves the SHGH conjecture for $n \leq 9$ general points but a more explicit argument appears in [11].

In order to gain more insight into the SHGH conjecture and the problem of polynomial interpolation more broadly, Cook, Harbourne, Migliore and Nagel [33] took a slightly differently viewpoint. Instead of investigating when n general points $p_1, \ldots, p_n$ with assigned multiplicities $m_1, \ldots, m_n$ failed to impose independent conditions on the linear system $|dL|$ in $\mathbb{P}^2$, they considered when a single general point p with assigned multiplicity m failed to impose independent conditions on the linear system $|(m+1)L - q_1 - \cdots - q_n|$ where the points $q_1, \ldots, q_n$ are reduced points of $\mathbb{P}^2$ that are not necessarily general. Put $Z = q_1 + \cdots + q_n$, then Z is a reduced 0-dimensional subscheme. Let I_Z denote the ideal of forms in $k[x_0, x_1, x_2]$ that contain Z in their vanishing locus and let I_{Z+mp} be those that in addition vanish at p to multiplicity at least m. Finally let $I_{Z+mp}(m+1)$ be the forms of degree $m+1$ in I_{Z+mp}. Note that $I_{Z+mp}(m+1)$ is exactly what forms remain in $|(m+1)L - q_1 - \cdots - q_n|$ after mp has been imposed, i.e $I_{Z+mp}(m+1) = |(m+1)L - q_1 - \cdots - q_n - mp|$ and $I_Z(m+1) = |(m+1)L - q_1 - \cdots - q_n|$. To give the criterion in [33] for when mp fails to impose independent conditions on $I_Z(m+1)$, we recall three quantities from [33]. The multiplicity index is $m_Z := \min\{j \geq 0 : \dim_k I_{Z+jp}(j+1) > 0\}$, the Hilbert index is $t_Z := \min\{j \geq 0 : \dim_k I_Z(j+1) - \binom{j+1}{2} > 0\}$ and the speciality index is $\mu_Z = \min\{j \geq 0 : \dim_k I_{Z+jp}(j+1) - (\binom{j+3}{2} - |Z| - \binom{j+1}{2}) = 0\}$. Then it turns out by Theorem 2.16 in [33] that for some m, $\dim_k I_{Z+mp}(m+1) > \max\{0, \dim_k I_Z(m+1) - \binom{m+1}{2}\}$ if and only if $m_Z < t_Z$ and $m_Z \leq m < \mu_Z$. No classification in any degree of all such examples is known. Therefore, here, we make an effort in that direction by finding, in degree 3, all Z such that Z admits an unexpected curve (see Definition 3.2.3) by connecting the discussion to quasi-elliptic fibrations.

Moreover in this dissertation, we take points $q_1, \ldots, q_r$, $0 \leq r < 8$, not necessarily general, with assigned multiplicities $a_1, \ldots, a_r$, in $\mathbb{P}^2$ and consider the linear system $|dL - a_1q_1 - \cdots - a_rq_r| \subseteq |dL|$. We then impose general points $p_1, \ldots, p_s$, $1 < s \leq 8$, $r + s \leq 8$, with assigned multiplicities $b_1, \ldots, b_s$ and investigate when $\dim_k |dL - a_1q_1 - \cdots - a_rq_r - b_1p_1 - \cdots - b_sp_s| > \max\{\dim_k |dL - a_1q_1 - \cdots - a_rq_r| - \sum_{i=1}^{s} \binom{b_i+1}{2}, 0\}$. As usual, it is easier to work entirely with divisors, so we blow up the points $q_1, \ldots, q_r, p_1, \ldots, p_s$ to obtain a surface $\pi : X \to \mathbb{P}^2$. There are exceptional curves $e_1, \ldots, e_r, E_1, \ldots, E_s$ on X such that $\pi(e_j) = q_j$, $1 \leq j \leq r$, and $\pi(E_i) = p_i$, $1 \leq i \leq s$. Put $e = a_1 e_1 + \cdots + a_r e_r$ and $E = b_1 E_1 + \cdots b_s E_s$. If L also denotes the proper transform of a line in $\mathbb{P}^2$ to X, then $|dL - e - E|$ and $|dL - e|$ are the proper transforms of $|dL - a_1 q_1 - \cdots - a_r q_r - b_1 p_1 - \cdots - b_s p_s|$ and $|dL - a_1 q_1 - \cdots - a_r q_r|$ respectively. Our investigation now reduces to finding a necessary and sufficient condition for when $\dim_k |dL - e - E| > \max\{\dim_k |dL - e| - \sum_{i=1}^{s} \binom{b_i+1}{2}, 0\}$. Given the linear system $|dL - e|$, we can write $dL - e = F + N$ where F is a nef divisor on X and N is either trivial or a sum of smooth rational curves of negative self-intersection by Theorem 1.18 in [4]. We have that $\dim_k |dL - e| = \dim_k |F|$ and $\dim_k |dL - e - E| = \dim_k |F - E|$. The main result is Theorem 3.2.21 which states that if there is a smooth rational curve C such that either $C^2 = -1$ and $(F - E) \cdot C \leq -2$ or the char $k = 2$, $C^2 = -2$, $r + s = 8$, the class of $C = 3L - E_1 - \cdots - E_7 - 2E_8$ and $(F - E) \cdot C \leq -2$ then $\dim_k |F - E| > \dim_k |F| - \sum_{i=1}^{s} \binom{b_i+1}{2}$.

Chapter 2

Ideal Containments under Flat Extensions

Throughout this chapter, let $R = S = k[y_0, ..., y_n]$ and let $\{f_0, ..., f_n\} \subseteq R$ be an R-regular sequence of homogeneous elements of R of the same degree. Let $\varphi : S \to R$ be the k-algebra map given by $y_i \mapsto f_i$. For an ideal $I \subseteq S$, let $\varphi_*(I) \subseteq R$ denote the ideal generated by $\varphi(I)$.

2.1 Flat Extensions and Ideal Containments

Lemma 2.1.1. *Let $\varphi : S \to R$ be as above. Then R is a free graded S-module, hence R is faithfully flat as an S-module.*

Proof. It suffices to show that R is free over S since free modules are faithfully flat modules. Note that φ is injective since $\{f_0, ..., f_n\}$ is a regular sequence. It follows that $S \cong k[f_0, ..., f_n] \subseteq R$. So we identify S with $k[f_0, ..., f_n]$ and show that R is free over $k[f_0, ..., f_n]$. Since $\{f_0, ..., f_n\}$ is a maximal homogeneous R-regular sequence, it is a homogeneous system of parameters (sop). The reason is that every regular sequence is part of an sop and because R is Cohen-Macaulay (CM), every sop is a regular sequence (depth$R = \dim R$) and so if $\{f_0, ..., f_n\}$ is a maximal regular sequence, then it is an sop. Since $R = k[\mathbb{P}^n]$ is a positively graded affine k-algebra, the fact that $\{f_0, ..., f_n\}$ is a homogeneous sop is equivalent to R being a finite S-module by [9, Theorem 1.5.17]. Since both R and S are CM, depth$R = \dim R = n+1 = \dim S =$ depth$S$. By the Auslander-Buchsbaum formula [17, Exercise 19.8] [37, Theorem 15.3], $\mathrm{pd}_S R +$ depth $R =$ depth S. It follows that $\mathrm{pd}_S R = 0$. So looking at the minimal free resolution of R as an S-module, we see that R is a free S-module. Therefore R is a faithfully flat S-module. □

Lemma 2.1.2. *Let $I \subseteq S$ be a homogeneous saturated ideal defining a 0-dimensional subscheme of $\mathbb{P}^n$. Then $\varphi_*(I) \subseteq R$ also defines a 0-dimensional subscheme of $\mathbb{P}^n$.*

Proof. We start by showing that $R/\varphi_*(I)$ has the same Krull dimension as S/I. By the graded Auslander-Buchsbaum formula, $\mathrm{pd}_S(R/\varphi_*(I)) + \mathrm{depth}(R/\varphi_*(I))$ $= \mathrm{depth}(S) = \mathrm{pd}_S(S/I) + \mathrm{depth}(S/I)$. By 3.1 in [18], S/I and $R/\varphi_*(I)$ have the same graded Betti numbers so $\mathrm{pd}_S(S/I) = \mathrm{pd}_S(R/\varphi_*(I))$. Therefore $\mathrm{depth}(S/I) = \mathrm{depth}(R/\varphi_*(I))$. By 3.1 in [18] again, S/I is Cohen-Macaulay (CM) if and only if $R/\varphi_*(I)$ is CM. Since I defines an ideal of points and is saturated, we have that S/I is CM. It follows that $R/\varphi_*(I)$ is CM. For CM modules, the depth is the dimension so that $\dim S/I = \dim R/\varphi_*(I)$. Now since S/I and $R/\varphi_*(I)$ are both CM, $\mathrm{Ass}(R/\varphi_*(I))$ and $\mathrm{Ass}(S/I)$ are both unmixed with their elements having height $\mathrm{ht}(\varphi_*(I))$ and $\mathrm{ht}(I)$ respectively. But $\mathrm{ht}(\varphi_*(I)) = \mathrm{ht}(I)$ since $\dim S/I = \dim R/\varphi_*(I)$. It follows that the elements of $\mathrm{Ass}(R/\varphi_*(I))$ are all ideals of points. It follows that $\varphi_*(I)$ defines a 0-dimensional subscheme of $\mathbb{P}^n$. □

Lemma 2.1.3. *Let $I \subseteq S$ be a saturated homogeneous ideal such that the zero locus of I in $\mathbb{P}^n$ is 0-dimensional. Let $\varphi : S \to R$ be as above. Then $\varphi_*(I^{(m)}) = \varphi_*(I)^{(m)}$.*

Proof. By Lemma 2, $\varphi_*(I)$ is the defining ideal of a 0-dimensional subscheme so that $(\varphi_*(I))^{(m)} = \mathrm{Sat}((\varphi_*(I))^m)$ where $\mathrm{Sat}((\varphi_*(I))^m)$ denotes the saturation of the ideal $(\varphi_*(I))^m$. An ideal and its saturation have the same graded homogeneous components for high enough degree so that for $t \gg 0$, $((\varphi_*(I))^{(m)})_t = ((\varphi_*(I))^m)_t$.

Using again that the symbolic power of an ideal of a 0-dimensional subscheme in $\mathbb{P}^n$ is the saturation of the ordinary power, $I^{(m)} = \mathrm{Sat}(I^m)$, we have that $(I^{(m)})_t = (I^m)_t$ for $t \gg 0$. Therefore $(\varphi_*(I^{(m)}))_t = (I^{(m)} \otimes_S R)_t = (I^m \otimes_S R)_t = (\varphi_*(I^m))_t$ for $t \gg 0$. Since φ is a ring map, $\varphi_*(I^m) = (\varphi_*(I))^m$. This gives that $(\varphi_*(I^{(m)}))_t = ((\varphi_*(I))^m)_t$ for $t \gg 0$.

The last two paragraphs imply that $((\varphi_*(I))^{(m)})_t = \varphi_*(I^{(m)})_t$ for $t \gg 0$. Recall that $(\varphi_*(I))^{(m)}$ is saturated since it is the saturation of $(\varphi_*(I))^m$ and $\varphi_*(I^{(m)})$ is saturated by Lemma 3.1 in [18]. Two saturated graded homogeneous ideals that agree in degree t for $t \gg 0$, agree in all degrees. Hence $(\varphi_*(I))^{(m)} = \varphi_*(I^{(m)})$. □

Theorem 2.1.4. *Let $I \subseteq S$ be a saturated homogeneous ideal such that $V(I) \subseteq \mathbb{P}^n$ is a 0-dimensional subscheme. Let $\varphi : S \to R$ be given by $y_i \to f_i$, $0 \leq i \leq n$, where $\{f_0, ..., f_n\}$ is an R-regular sequence of homogeneous elements of R of the same degree. Let $\varphi_*(I)$ denote the ideal in R generated by $\varphi(I)$. Then $I^{(m)} \subseteq I^r$ if and only if $(\varphi_*(I))^{(m)} \subseteq (\varphi_*(I))^r$.*

Proof. ($\Longrightarrow$) Suppose that $I^{(m)} \subseteq I^r$. Then $\varphi(I^{(m)}) \subseteq \varphi(I^r)$ and so $\varphi_*(I^{(m)}) \subseteq \varphi_*(I^r)$. Since φ is a homomorphism, $\varphi(I^r) = (\varphi(I))^r$. Note that $\varphi(I^r)$ generates $\varphi_*(I^r)$ in R and $(\varphi(I))^r$ generates $(\varphi_*(I))^r$ in R. It follows that $\varphi_*(I^r) = (\varphi_*(I))^r$ since they have the same generating set. Now applying Lemma 3 we have that $(\varphi_*(I))^{(m)} = \varphi_*(I^{(m)}) \subseteq \varphi_*(I^r) = \varphi_*(I)^r$ concluding the forward direction.

($\Longleftarrow$) Suppose now that for some homogeneous ideals I and J of S, $I \not\subseteq J$

but $\varphi_*(I) \subseteq \varphi_*(J)$. Then there is a homogeneous element $f \in I \backslash J$ such that $\varphi(f) \in \varphi_*(J)$. We may assume with no loss in generality that $I = (f)$. We have the sequence

$$0 \to I \cap J \to I \oplus J \to I + J \to 0$$

with the first map given by $g \mapsto (g, -g)$ and the second map given by $(h, r) \mapsto h + r$. It is clear that the sequence is exact. Since φ is faithfully flat, we get an exact sequence

$$0 \to \varphi_*(I \cap J) \to \varphi_*(I) \oplus \varphi_*(J) \to \varphi_*(I + J) \to 0.$$

Since $\varphi_*(I) \subseteq \varphi_*(J)$, $\varphi_*(I + J) = \varphi_*(J)$. Then the map $\varphi_*(I) \oplus \varphi_*(J) \to \varphi_*(J)$ has kernel $\varphi_*(I)$. It follows that $\varphi_*(I \cap J) = \varphi_*(I)$. This is impossible since the generators of $\varphi_*(I \cap J)$ are the images of the generators of $I \cap J$ and thus have degree greater than degree f and hence greater than degree of $\varphi(f)$ which generates $\varphi_*(I) = I \otimes_S R \neq 0$.

So it is the case that $\varphi(f) \notin \varphi_*(J)$. Hence $\varphi_*(I) \not\subseteq \varphi_*(J)$. Therefore if $I^{(m)} \not\subseteq I^r$, then by Lemma 3, $(\varphi_*(I))^{(m)} = \varphi_*(I^{(m)}) \not\subseteq (\varphi_*(I))^r$. Hence $(\varphi_*(I))^{(m)} \subseteq (\varphi_*(I))^r$ if and only if $I^{(m)} \subseteq I^r$. $\square$

2.2 New Counterexamples to the Containment $I^{(rn-n+1)} \subseteq I^r \subseteq k[\mathbb{P}^n]$

Using the above result, we obtain many new counterexamples to the containment $I^{(3)} \subseteq I^2$ of ideals in $k[\mathbb{P}^2]$ and more generally counterexamples to the containment

$$I^{(nr-n+1)} \subseteq I^r \qquad (\star)$$

in $\mathbb{P}^n$. In particular if $I \subseteq k[\mathbb{P}^n]$ gives a counterexample to $(\star)$, then $\varphi_*(I)$ is a counterexample for any choice of homogeneous regular sequence $\{f_0, ..., f_n\}$ of elements of the same degree. We illustrate this below with a few examples.

Example 2.2.1. In this example, we work over $\mathbb{C}$. In [15], the Fermat configuration, for $n = 3$, was considered and its ideal $I = (x(y^3 - z^3), y(x^3 - z^3), z(x^3 - y^3)) \subseteq \mathbb{C}[x, y, z]$ was found to be a counterexample to the containment $I^{(3)} \subseteq I^2$. Recall the configuration consists of the 3 coordinate vertices and the 9 intersection points of $y^3 - z^3$ and $x^3 - z^3$. The ideal I is radical and all of the points in the configuration are reduced points. Now let $\varphi : \mathbb{C}[\mathbb{P}^2] \to \mathbb{C}[\mathbb{P}^2]$ by $x \to f = x^2 + y^2$, $y \to g = y^2 + z^2$ and $z \to h = x^2 + z^2$. One easily checks that $\{x^2 + y^2, y^2 + z^2, x^2 + z^2\}$ is a $\mathbb{C}[\mathbb{P}^2]$ - regular sequence. Then φ induces a map of schemes $\varphi^\# : \mathbb{P}^2 \to \mathbb{P}^2$ which is faithfully flat. Consider the scheme-theoretic fibers of $\varphi^\#$ over the Fermat configuration and call it the fibered Fermat configuration. Note that the fibered Fermat configuration is 0-dimensional. Since $\varphi^\#$ has degree 4, the fibers consist of 48 points of $\mathbb{P}^2$ where we count with multiplicity. The fibered Fermat configuration gives rise to the radical ideal $\varphi_*(I) = (f(g^3 - h^3), g(f^3 - h^3), h(f^3 - g^3)) \subseteq \mathbb{C}[\mathbb{P}^2]$ and by analyzing the ideal we see that the configuration consists of 4 multiplicity 1 points over each of the 3 coordinate vertices, given by $f = 0 = g$, $f = 0 = h$ and $g = 0 = h$. The remaining 36 points, each of multiplicity 1, in the configuration are the zero locus of $f^3 - h^3$ and $f^3 - g^3$. Since $I^{(3)} \not\subseteq I^2$, we have by Theorem 3 that $\varphi_*(I)^{(3)} \not\subseteq \varphi_*(I)^2$.

Example 2.2.2. We give another example of a fibered Fermat configuration whose ideal also gives a counterexample to the containment $I^{(3)} \subseteq I^2$. The difference here is that 36 of the points in the configuration have multiplicity 1 while the remaining points each have multiplicity 4. So there are still 48 points counting with multiplicity. Let $\varphi : \mathbb{C}[\mathbb{P}^2] \to \mathbb{C}[\mathbb{P}^2]$ by $x \to f = x^2$, $y \to g = y^2$ and $z \to h = z^2$. This faithfully flat ring map induces a morphism of schemes $\varphi^\# : \mathbb{P}^2 \to \mathbb{P}^2$ that is also flat. The fibers of $\varphi^\#$ over the Fermat configuration gives the fibered Fermat configuration that consists of the 36 points, each of multiplicity 1, of intersection of the degree 6 forms $f^3 - g^3$ and $g^3 - h^3$. The configuration has 3 more points each of multiplicity 4 over the 3 coordinate points. They are the zero loci of $f = 0 = g$, $f = 0 = h$ and $g = 0 = h$. So the fibered Fermat configuration here has points that are not all reduced. By Theorem 3, its nonradical ideal $\varphi_*(I)$ is a counterexample to the containment $\varphi_*(I)^{(3)} \subseteq \varphi_*(I)^2$.

Example 2.2.3. Similarly for the Fermat configurations considered in [28] for $n \geq 3$, we can construct new configurations of points, that may or may not be reduced in $\mathbb{P}^2$, that are the fibers of a morphism of schemes $\varphi^\# : \mathbb{P}^2 \to \mathbb{P}^2$. The morphism $\varphi^\#$ is induced by the ring map $\varphi : \mathbb{C}[\mathbb{P}^2] \to \mathbb{C}[\mathbb{P}^2]$ given by $x \to f$, $y \to g$ and $z \to h$ where $\{f, g, h\}$ is a homogeneous $\mathbb{C}[\mathbb{P}^2]$-regular sequence of the same degree. The Fermat configuration gives rise to a radical ideal $I = (x(y^j - z^j), y(x^j - z^j), z(x^j - y^j)) \subseteq \mathbb{C}[\mathbb{P}^2]$, $j \geq 3$, and for a choice of $\{f, g, h\}$, the fibered Fermat configuration gives rise to an ideal $\varphi_*(I) = (f(g^j - h^j), g(f^j - h^j), h(f^j - g^j))$, $j \geq 3$, not necessarily radical, that is also a counterexample to $\varphi_*(I)^{(3)} \subseteq \varphi_*(I)^2$. Here the Fermat configuration consists of the reduced j^2 points of intersection of $y^j - z^j$ and $x^j - y^j$ together with the 3 coordinate vertices for a total of $j^2 + 3$ points. If the degree of the homogeneous elements in $\{f, g, h\}$ is d, then the fibered configuration consists of the $d^2 j^2$ points of intersection of $g^j - h^j$ and $f^j - h^j$ together with the $3d^2$ fiber points over the three coordinate vertices that are the solutions of the three equations $f = 0 = g$, $f = 0 = h$ and $g = 0 = h$, counted with multiplicity. Again the points in the fibered configuration may or may not be reduced.

Example 2.2.4. Now we consider an example given in [6] that is inspired by the example of the Fermat configuration. Let $k = \mathbb{Z}/3\mathbb{Z}$ and let K be an algebraically closed field containing k. Note that $\mathbb{P}^2_K$ has 13 k-points and 13 k-lines such that each line contains 4 of the points and each point is incident to 4 of the lines. The forms $xy(x^2 - y^2)$, $xz(x^2 - z^2)$ and $yz(y^2 - z^2)$ vanish at all 13 points of $\mathbb{P}^2_k$ but the form $x(x^2 - y^2)(x^2 - z^2)$ does not vanish at the point $[1 : 0 : 0]$. One checks easily that the ideal $I = (xy(x^2 - y^2), xz(x^2 - z^2), yz(y^2 - z^2), x(x^2 - y^2)(x^2 - z^2)) \subseteq k[\mathbb{P}^2_K]$ is radical and its zero locus is the 13 k-points of $\mathbb{P}^2_K$. Then $F = x(x-z)(x+z)(x^2 - y^2)((x-z)^2 - y^2)((x+z)^2 - y^2)$ defines 9 lines meeting at 12 points with each point incident to 3 of the lines. It is not hard to see that $F \in I^{(3)}$ but $F \notin I^2$. So the reduced configuration that comes from $\mathbb{P}^2_k$ with the point $[1 : 0 : 0]$ removed together with all its incident lines gives rise to an ideal that is a counterexample to the containment $I^{(3)} \subseteq I^2$. Let $\varphi : k[\mathbb{P}^2_K] \to k[\mathbb{P}^2_K]$ be the ring map $x \to f = x^2$, $y \to g = y^2$ and $z \to h = z^2$. Applying the degree 4 morphism of schemes $\varphi^\# : \mathbb{P}^2_K \to \mathbb{P}^2_K$, induced by φ, and taking its fibers over the k-points, we get a configuration of 48 points. For each point in the original configuration, we get 4 points in the fibered configuration. The points in this new configuration are not all reduced. For instance over the point $[0 : 0 : 1]$, the fiber of $\varphi^\#$ is a point of multiplicity 4 in $\mathbb{P}^2_K$ given by the vanishing of y^2 and x^2. The ideal of the fibered configuration as schemes is the ideal $\varphi_*(I) = (fg(f^2 - g^2), fh(f^2 - h^2), gh(g^2 - h^2), f(f^2 - g^2)(f^2 - h^2))$. This ideal is not radical and since $\{f, g, h\} \subset \mathbb{P}^2_K$ is a regular sequence, we have by Theorem 3 that $\varphi_*(I)^{(3)} \not\subseteq \varphi_*(I)^2$. If instead we take $f = x^2 + y^2$, $g = y^2 + z^2$ and $h = x^2 + z^2$ in the above example, then the fibered configuration we obtain is a reduced configuration and the ideal $\varphi_*(I)$ is a radical ideal satisfying $\varphi_*(I)^{(3)} \not\subseteq \varphi_*(I)^2$.

Example 2.2.5. Variations of the above example are considered in $\mathbb{P}^n$ for various n in [28], giving counterexamples for the more general conjecture $I^{(nr-n+1)} \subseteq I^r$. We can apply our result to these to obtain new counterexamples to the more general containment.

Chapter 3

Interpolation on Linear Systems in $\mathbb{P}^2$

3.1 Interpolation with one or more General Points

We first give a definition for what it means for a set of points, $\{p_1, \ldots, p_n\}$ specified on an algebraic surface, X, to be in general position.

Definition 3.1.1. *A statement is true for n general points $p_1, \ldots, p_n$ on a surface X if the statement holds for any tuple $(q_1, \ldots, q_n) \in U \subseteq X^n$ such that U is nonempty and open.*

Definition 3.1.2. *Let X be a rational surface. An exceptional curve on X is a reduced, irreducible, non-singular curve with genus 0 and self-intersection -1. A nodal curve on X is a reduced, irreducible, nonsingular curve with genus 0 and self-intersection -2. In particular, by adjunction, a prime divisor C with $C^2 = C \cdot K_X = -1$ or $C^2 = -2$, $C \cdot K_X = 0$ is an exceptional or nodal curve respectively.*

Let $q_1, \ldots, q_r$ be points in $\mathbb{P}^2$, not necessarily general, and let $p_1, \ldots, p_s$ be general points in $\mathbb{P}^2$. Suppose that $r + s \leq 8$. Blow up the q_j, $1 \leq j \leq r$ and the p_i, $1 \leq i \leq s$ to obtain a surface X birational to $\mathbb{P}^2$. We have the exceptional curves e_j corresponding to the points q_j, $0 \leq j \leq r$ and the exceptional curves E_i corresponding to the points p_i, $1 \leq i \leq s$. Let L be the total transform of a line in $\mathbb{P}^2$ on X. Then the divisor class group on X is a free abelian group generated by $\{L, e_1, \ldots, e_r, E_1, \ldots, E_s\}$ with intersection form $L^2 = 1$, $e_1^2 = \cdots = e_r^2 = E_1^2 = \cdots = E_s^2 = -1$, $L \cdot e_j = L \cdot E_i = e_j \cdot E_i = 0$, $e_u \cdot e_v = 0$, $u \neq v$, $E_l \cdot E_k = 0$, $l \neq k$, $1 \leq j, u, v \leq r$ and $1 \leq i, l, k \leq s$.

For integers $a_j \geq 0$, $1 \leq j \leq r$ and $b_i \geq 0$, $1 \leq i \leq s$, put $e = \sum_{j=1}^{r} a_j e_j$ and $E = \sum_{i=1}^{s} b_i E_i$. For $t \geq 1$, consider the sub-linear system $|tL - e|$ of the linear system $|tL|$ on X. To the divisor classes $|tL - e|$ and $|tL - e - E|$ on X, we have the corresponding invertible sheaves $\mathcal{O}_X(tL - e)$ and $\mathcal{O}_X(tL - e - E)$ respectively such that the divisors corresponding to their global sections are the divisors in the classes $|tL - e|$ and $|tL - e - E|$. Then $\dim_k |tL - e| = \dim_k \Gamma(X, \mathcal{O}_X(tL - e))$ and similarly $\dim_k |tL - e - E| = \dim_k \Gamma(X, \mathcal{O}_X(tL - e - E))$ where Γ is the global sections functor. We have the isomorphisms $H^0(X, \mathcal{O}_X(tL - e)) \cong \Gamma(X, \mathcal{O}_X(tL - e))$ and $H^0(X, \mathcal{O}_X(tL - e - E)) \cong \Gamma(X, \mathcal{O}_X(tL - e - E))$ where $H^0(-)$ denotes the zeroth Cech cohomology groups of the invertible sheaves. In general, given an i-th Cech cohomology group, $H^i(-)$ of an invertible sheaf on X, we shall denote its dimension by $h^i(-)$. So, suppressing the sheaf notation $\mathcal{O}_X$, the dimensions of the linear systems $|tL - e|$ and $|tL - e - E|$ may be denoted as $h^0(X, tL - e)$ and $h^0(X, tL - e - E)$ respectively. We recall some definitions.

Definition 3.1.3. *Let X be a rational surface. A divisor D is effective if D is a nonnegative sum of prime divisors. We say that a divisor class D is effective if it contains an effective divisor. A divisor H is nef if for every effective divisor D on X, $H \cdot D \geq 0$. A divisor class is nef if it contains a nef divisor. Given a divisor on X, a complete linear system is the set of all effective divisors on X linearly equivalent to the divisor.*

Note that every divisor class can be written uniquely in terms of $L, e_1, \ldots, e_r, E_1, \ldots,$

E_s which allows us to regard $dL - \sum_{j=1}^{r} a_j e_j - \sum_{i=1}^{s} b_i E_i$ as either representing a divisor or a divisor class. Now consider $L - E_1$ first as a divisor. It is not effective since it is not a nonnegative sum of prime divisors. However, $L - E_1$ is an effective divisor class since it contains an effective divisor, namely, the proper transform of a line through the point p_1. We shall use complete linear system interchangeably with divisor class. Let $\mathcal{L}$ be a linear system on X containing an effective divisor. Then M is a fixed component of $\mathcal{L}$ if M is a prime divisor such that for every effective divisor N in $\mathcal{L}$, $N - M$ is an effective divisor. M is the fixed part of $\mathcal{L}$ if M is the sum of all the fixed components of $\mathcal{L}$.

In this dissertation, we shall be concerned with when *E is special on $|tL - e|$*.

Definition 3.1.4. *We say that E is special on $|tL - e|$ if $h^0(X, \mathcal{O}_X(tL - e - E)) > \max\{0, h^0(X, \mathcal{O}_X(tL - e)) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$. If there is a curve C such that $C \in H^0(X, \mathcal{O}_X(tL - e - E))$ but $h^0(X, \mathcal{O}_X(tL - e)) - \sum_{i=1}^{s} \binom{b_i+1}{2} \leq 0$, then C is called an unexpected curve.*

Readers familiar with the problem of polynomial interpolation on general points in $\mathbb{P}^2$ will know that the SHGH conjecture gives a criterion for when $h^0(X, \mathcal{O}_X(tL - e - E)) > \max\{0, h^0(X, \mathcal{O}_X(tL - e)) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$ for all s when $e = 0$. The SHGH conjecture is known to be true when $s \leq 9$ and that is what the next theorem states.

Theorem 3.1.5. *Let $s \leq 9$ and $r = 0$ so that we have $p_1, \ldots, p_s$ general points in $\mathbb{P}^2$ with assigned multiplicities $b_1, \ldots, b_s$. Blow up the points $p_1, \ldots, p_s$ to obtain the surface X as above. As usual, we say that E is special on $|tL|$ if $h^0(X, \mathcal{O}_X(tL - E)) > \max\{0, h^0(X, \mathcal{O}_X(tL)) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$. Then E is special on $|tL|$ or equivalently $|tL - E|$ is special if and only if tL - E is effective and there is an exceptional curve C with $(tL - E) \cdot C < -1$.*

This dissertation will subsume Theorem 3.1.5 for up to 8 points by considering not just the complete linear system $|tL|$ but also the linear systems of the form $|tL - e|$ where e is as above. A significant difference, however, is that while $|tL|$ is simultaneously nef, effective, has $h^1(X, tL) = 0$ and $|tL|$ is fixed component free, none of these need be true for $tL - e$, but in our situation it is helpful to know that $tL - e$ is effective, has $h^1(X, tL) = 0$ and $|tL - e|$ is fixed component free if $|tL - e|$ is nef. The next two lemmas are steps in that direction.

Lemma 3.1.6. *Let X be a smooth projective rational surface with $K_X^2 > 0$, then for a nef divisor F on X, $h^1(X, F) = 0$.*

Proof. Lemma II.5 in [23] $\square$

Lemma 3.1.7. *Let X be a blow up of at most $r + s \leq 8$ points in $\mathbb{P}^2$. Suppose that F is a nef divisor on X. Then F is, up to linear equivalence, effective and fixed component free with $h^1(X, \mathcal{O}_X(F)) = h^2(X, \mathcal{O}_X(F)) = 0$.*

Proof. This is essentially Theorem III.1 in [23]. The only difference here is the fact that F is fixed component free but that is also implicit in Theorem III.1 of [24]. We make everything explicit here. Since X is the blowup of points in $\mathbb{P}^2$, X is a smooth rational surface. The canonical divisor has the form $K_X = -3L + e_1 + \cdots + e_r + E_1 + \cdots + E_s$ so that $(K_X)^2 = 9 - (r + s) \geq 1$ since $r + s \leq 8$. Now $K_X^2 > 0$ implies $-K_X$ is effective (so $-K_X \cdot F \geq 0$) and, by the Hodge Index Theorem, that the intersection form on $K_X^\perp$ is negative definite (so $F \cdot (-K_X) > 0$).

Now by the previous lemma, $h^1(X, \mathcal{O}_X(F)) = 0$. Note that $h^2(X, \mathcal{O}_X(F)) = h^0(X, \mathcal{O}_X(K_X - F))$ by duality and $(K_X - F) \cdot F = K_X \cdot F - F^2$. From above,

$K_X \cdot F < 0$ and $F^2 \geq 0$ since F is nef. Therefore $(K_X - F) \cdot F < 0$. Since F is nef, $K_X - F$ is not effective so that $h^2(X, \mathcal{O}_X(F)) = 0$. This implies that $h^0(X, \mathcal{O}_X(F)) = \frac{F^2 + (-K_X) \cdot F}{2} + 1 \geq 2$, by the Riemann-Roch Theorem, and so F is effective.

If $F \cdot (-K_X) \geq 2$, then part (1) of Theorem 3.2.12 gives that F is base point free and hence fixed component free. Suppose now that $F \cdot (-K_X) = 1$. Suppose further, for the sake of contradiction, that F has a fixed component and write $F = H + N$ where H is the free part of F and N is its fixed part. Then by part (2) of Theorem 3.2.12, $H \in K_X^{\perp}$. But this implies that $H^2 < 0$ if H is not trivial. If H is trivial, then $H^2 = 0$. Since H is free, however, $H^2 \geq 0$. So it must be that $H^2 = 0$ and H is trivial. Therefore $F = N$. Hence $h^0(X, F) = h^0(X, N) = 1$. This contradicts the assertion above that $h^0(X, \mathcal{O}_X(F)) \geq 2$. Therefore F has no fixed components. □

Lemma 3.1.8. *Let X be a rational surface that is the blow up of $\mathbb{P}^2$ at up to 8 points of $\mathbb{P}^2$. Suppose that $tL - e$ is effective. We can, up to linear equivalence, write $tL - e = F + N$, where F is nef, effective and fixed component free and N is a sum of curves of negative self-intersection with $h^0(X, \mathcal{O}_X(N)) = 1$.*

Proof. Since $tL - e$ is effective, we can write $tL - e = F + N$ where F is fixed component free and N is the divisorial base locus of $tL - e$. Clearly F is effective and since F is fixed component free, $F \cdot D \geq 0$ for every effective divisor D on X so that F is nef. Since N is fixed, $h^0(X, \mathcal{O}_X(N)) = 1$. Write $N = n_1 N_1 + \cdots + n_t N_t$ where the N_i, $1 \leq i \leq t$ are the irreducible components of N. If $N_i^2 \geq 0$, then N is nef while fixed contradicting Lemma 3.1.7. Therefore $N_i^2 < 0$. □

Lemmas 3.1.6 and 3.1.8 allow us in Theorem 3.2.21 to come up with a characterization for when $h^0(X, \mathcal{O}_X(F-E)) > \max\{0, h^0(X, \mathcal{O}_X(F)) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$ that is analogous to the characterization for when $h^0(X, \mathcal{O}_X(tL-E)) > \max\{0, h^0(X, \mathcal{O}_X(tL)) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$ in the SHGH conjecture.

Lemma 3.1.9. *Let X be a blow up of $\mathbb{P}^2$ in a finite number of points $p_1, \ldots, p_k$. Suppose $\mathcal{L}$ is an effective linear system on X and let E be a smooth rational curve on X satisfying $E^2 = E \cdot K_X = -1$. If $\mathcal{L} \cdot E = -n$, then the divisorial base locus of $\mathcal{L}$ contains nE. Moreover $h^1(X, \mathcal{L}) \geq \binom{n}{2}$. Similarly let D be a smooth rational curve on X satisfying $D^2 = -2$ and $D \cdot K_X = 0$ with $\mathcal{L} \cdot D = -n$ where $n \geq 2$. Then $\lceil \frac{n}{2} \rceil D$ is in the divisorial base locus of $\mathcal{L}$. Moreover $h^1(X, \mathcal{L}) \geq \frac{n^2-1}{4} > 0$.*

Proof. By Bezout's theorem, $\mathcal{L} \cdot E \geq 0$ except possibly when $\mathcal{L}$ and E have a component in common. Since $\mathcal{L} \cdot E < 0$ and E is irreducible, we have that E sits in the base locus of $\mathcal{L}$. Now note that $(\mathcal{L} - (n-1)E) \cdot E = \mathcal{L} \cdot E - (n-1)E \cdot E = -n + (n-1) = -1$ and $(\mathcal{L} - nE) \cdot E = \mathcal{L} \cdot E + n = -n + n = 0$ so that nE sits in the base locus of $\mathcal{L}$. Now we have that $h^0(X, \mathcal{L}) = h^0(X, \mathcal{L} - nE)$ and hence by Riemann - Roch,

$$h^1(X, \mathcal{L}) + \frac{\mathcal{L}^2 + (-K_X) \cdot \mathcal{L}}{2} + 1 = h^1(X, \mathcal{L} - nE) + \frac{(\mathcal{L} - nE)^2 + (-K_X) \cdot (\mathcal{L} - nE)}{2} + 1.$$

This implies the following:

$$
\begin{aligned}
h^1(X,\mathcal{L}) &\geq \frac{(\mathcal{L}-nE)^2+(-K_X)\cdot(\mathcal{L}-nE)}{2} - \frac{\mathcal{L}^2+(-K_X)\cdot\mathcal{L}}{2} \\
&= \frac{n^2E^2-2n\mathcal{L}\cdot E+nK_X\cdot E}{2} \\
&\geq \frac{-n^2+2n^2-n}{2} \\
&= \frac{n(n-1)}{2} \\
&= \binom{n}{2}.
\end{aligned}
$$

Just as for E, $\mathcal{L}\cdot D<0$ implies that D sits in the base locus of $\mathcal{L}$. If n is even, note that $(\mathcal{L}-(\lceil\frac{n}{2}\rceil-1)D)\cdot D=-n+2\lceil\frac{n}{2}\rceil-2=-2$ while $(\mathcal{L}-\lceil\frac{n}{2}\rceil D)\cdot D=-n+2\lceil\frac{n}{2}\rceil=0$ and hence $\lceil\frac{n}{2}\rceil D$ is in the base locus of $\mathcal{L}$. When n is odd, a similar argument shows that $(\lceil\frac{n}{2}\rceil)D$ is in the base locus of $\mathcal{L}$. We show now that $h^1(X,\mathcal{L})\neq 0$. By the theorem of Riemann-Roch,

$$
\begin{aligned}
h^1(X,\mathcal{L}) &\geq \frac{(\mathcal{L}-\lceil\frac{n}{2}\rceil D)^2+(\mathcal{L}-\lceil\frac{n}{2}\rceil D)\cdot(-K_X)}{2} - \frac{\mathcal{L}^2+(\mathcal{L})\cdot(-K_X)}{2} \\
&= \frac{-2\lceil\frac{n}{2}\rceil^2+2n\lceil\frac{n}{2}\rceil}{2} \\
&= \lceil\frac{n}{2}\rceil(n-\lceil\frac{n}{2}\rceil)
\end{aligned}
$$

When n is even we get that $h^1(X,\mathcal{L})\geq\frac{n^2}{4}$ and when n is odd, we have that $h^1(X,\mathcal{L})\geq\frac{n^2-1}{4}$. Note that in both instances since $n\geq 2$, $h^1(X,\mathcal{L})>0$. $\square$

3.2 Interpolation with one General Point

We now consider a special case of the interpolation problem that was discussed at length in [33], namely when $s = 1$, $r = 7$ and there is an unexpected singular cubic curve. An example was given in [33] that showed that this situation occurs when the characteristic of the base field is 2. Let Z be the seven points of the Fano plane. One can show that the linear system of cubics containing Z in its base locus has dimension 3 so that there should be no singular cubic containing Z in its base locus with a general point of multiplicity 2. But again, one can check that $F = \alpha^2 xy(x+y) + \beta^2 xz(x+z) + \gamma^2 yz(y+z)$ is a cuspidal cubic vanishing at each point of Z with a cusp at $[\alpha : \beta : \gamma] \in \mathbb{P}^2$. We show that this is the only time that this situation can occur when the points are distinct, i.e., any reduced 0-dimensional subscheme admitting an unexpected cubic is the Fano plane.

Definition 3.2.1. *Let k be an algebraically closed field. Consider a collection of r points, $\{p_1, \ldots, p_r\}$, in $\mathbb{P}^2$ with $r \geq 1$. We say that $\{p_1, \ldots, p_r\}$ impose independent conditions on the homogeneous polynomials of degree d in $k[\mathbb{P}^2]$ if the codimension of the vector space of the homogeneous polynomials of degree d that vanish at all the points $\{p_1, \ldots, p_r\}$ is min $\{r, \binom{d+2}{2}\}$.*

Remark 3.2.2. *Given r points with $r \leq \binom{d+2}{2} - 1$, there is a nonzero homogeneous polynomial of degree d vanishing at each of the r points.*

We now reinterpret Definition 3.1.4 in the current situation. This reinterpretation is due to [33]. Note that Definition 3.1.4 and Definition 3.2.3 are not at variance. Both definitions capture the same idea in different situations, namely, if one finds a curve in excess of what there ought to be, then the curve is unexpected.

Definition 3.2.3. *Suppose that $Z = q_1 + \cdots + q_r$ is a reduced 0-dimensional subscheme in $\mathbb{P}^2$. Let $\mathcal{I}_Z$ be the sheaf of ideals of Z in $\mathbb{P}^2$. Let $p \notin Z$ be a general point in $\mathbb{P}^2$. We write mp for the fat point of multiplicity m supported at p and let $\mathcal{I}_{Z+mp}$ be the sheaf of ideals of the subscheme $Z + mp$. We say that Z admits an unexpected curve of degree $m+1$ if $h^0(\mathbb{P}^2, \mathcal{I}_{Z+mp}(m+1)) > max\ \{0, h^0(\mathbb{P}^2, \mathcal{I}_Z(m+1)) - \binom{m+1}{2}\}$.*

We now give a characterization, Theorem 2.16 in [33], for when a reduced 0-dimensional subscheme supported at only finitely many points admits an unexpected curve.

Theorem 3.2.4. *Let Z be a reduced 0-dimensional finite subscheme in $\mathbb{P}^2$. Then Z admits an unexpected curve if and only if $m_Z < t_Z$. In this case, $t_Z \leq \mu_Z$. Moreover Z admits an unexpected curve of degree $j+1$ if and only if $m_Z \leq j < \mu_Z$. Further, $h^0(\mathbb{P}^2, I_{Z+m_Z p}(m_Z+1)) = 1$.*

To prove that the 7 points of the Fano plane is the only Z with an unexpected curve of degree 3, we need some lemmas.

Lemma 3.2.5. *Suppose that $\{p_1, \ldots, p_7\}$ are points in $\mathbb{P}^2$ and assume that there is a line L that contains exactly 4 of the points. Suppose that p_8 is a double point of a cubic curve which vanishes at $\{p_1, \ldots, p_7\}$. Then p_8 is not a general point.*

Proof. Fix the 7 points $\{p_1, \ldots, p_7\}$ in $\mathbb{P}^2$ with exactly 4 of them, say $p_1, \ldots, p_4$, contained by a line L. Let C be a cubic curve going through $\{p_1, \ldots, p_7\}$. Then $C \cdot L \geq 4$ and so by Bézout's theorem, L is a component of C. So $C = L + Q$ where Q is a conic. We check two cases.

Suppose that Q is irreducible going through p_5, p_6, p_7. Then the choice of p_8 is one of the points in $L \cap Q$ but $L \cap Q \subseteq L$. Hence p_8 is always in the closed set L

and hence not general.

Suppose now that Q is reducible and let $Q = L_1 + L_2$, a union of lines. One of the lines, say L_1, contains at least two of the points p_5, p_6, p_7. It follows that $L + L_1$, contains at least 6 of the points $p_1, \ldots, p_7$. Therefore the double point of the cubic is contained in $L + L_1$. Hence the choice of p_8 is confined to a proper closed locus, hence p_8 is not general. $\square$

Lemma 3.2.6. *Suppose that Z is a reduced 0-dimensional subscheme in $\mathbb{P}^2$ such that Z admits an unexpected curve of degree 3. Then Z imposes exactly 7 independent conditions on forms of degree 3 and hence $|Z| \geq 7$.*

Proof. Note that if Z admits an unexpected curve of degree 3, then the least degree of an unexpected curve that Z admits is 3. This is because if Z admits an unexpected curve of degree $j+1 = 1$, then $Z + jp = Z$ and since $\binom{1}{2} = 0$, we have $h^0(\mathbb{P}^2, \mathcal{I}_{Z+jp}(j+1)) > h^0(\mathbb{P}^2, \mathcal{I}_Z(j+1)) - \binom{j+1}{2}$ which is impossible. Similarly if Z admits an unexpected curve of degree $j+1 = 2$, then since p is a general point, we have that $h^0(\mathbb{P}^2, \mathcal{I}_{Z+jp}(j+1)) = h^0(\mathbb{P}^2, \mathcal{I}_{Z+p}(2)) > h^0(\mathbb{P}^2, \mathcal{I}_Z(2)) - 1 = h^0(\mathbb{P}^2, \mathcal{I}_Z(j+1)) - \binom{j+1}{2}$ which is again impossible. Hence the least degree of an unexpected curve that Z admits is 3.

This implies that $m_Z + 1 = 3$ and hence $m_Z = 2$. By Theorem 3.2.4, $m_Z < t_Z$ when Z admits an unexpected curve so that $t_Z > 2$. It follows from the definition of t_Z that $h^0(\mathbb{P}^2, \mathcal{I}_Z(3)) - \binom{3}{2} \leq 0$ so that $h^0(\mathbb{P}^2, \mathcal{I}_Z(3)) \leq 3$. This implies that the points in Z impose at least 7 independent conditions on $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$. Therefore $|Z| \geq 7$.

Clearly $h^0(\mathbb{P}^2, \mathcal{I}_Z(3)) > 0$ so that Z does not impose 10 or more independent conditions on the space of cubics. If $h^0(\mathbb{P}^2, \mathcal{I}_Z(3)) = 1$, then the singular loci of the curves in $\mathcal{I}_Z(3)$ is a proper closed subset and Z does not admit an unexpected cubic. So Z does not impose 9 independent conditions.

Say that Z imposes 8 independent conditions on the space of cubics. Then $h^0(\mathbb{P}^2, \mathcal{I}_Z(3)) = 2$ and the space of cubics containing Z in its base locus with singular points forming an open set, U, in $\mathbb{P}^2$ is a pencil. Pick a point $p \in U$. Then there is a unique curve $C \in \mathcal{I}_Z(3)$ passing through p. Now since there is a curve in $\mathcal{I}_Z(3)$ with a singularity at p, it must be that C has a singularity at p. Since U is open, for a point p' in a neighborhood of p, there is a unique $C' \in \mathcal{I}_Z(3)$ with a singularity at p'. Since some of the points in the neighborhood of p lie on C, it must be that C has a multiple component in a neighborhood of p.

Since C is a cubic curve with a multiple component, C is comprised of two lines, one of which is multiple with the multiple line containing p. Since p is a general point however, the multiple line can contain at most one other point of Z. Therefore the non-multiple line of C contains at least 6 of the points of Z. But the space of cubics through Z where at least 6 of the points of Z are collinear must have dimension at least 5, i.e., $h^0(\mathbb{P}^2, \mathcal{I}_Z(3)) \geq 5$. Therefore Z cannot impose 8 independent conditions on the space of cubics and in fact must impose exactly 7 independent conditions. $\square$

Lemma 3.2.7. *Let Z be a reduced 0-dimensional subscheme in $\mathbb{P}^2$ such that Z admits an irreducible unexpected curve of degree 3. Then there is a subscheme $Z' \subset Z$ with $|Z'| = 7$ such that Z' admits an unexpected curve of degree 3.*

Proof. By the previous lemma, if Z admits an unexpected curve, C, of degree 3 then Z imposes at least 7 independent conditions and $|Z| \geq 7$. Let $Z' \subseteq Z$ be 7 points that impose independent conditions on the space of all cubics, $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ on $\mathbb{P}^2$.

Recall that Z admitting an unexpected curve means that $h^0(\mathbb{P}^2, \mathcal{I}_{Z+2p}(3)) > h^0(\mathbb{P}^2, \mathcal{I}_Z(3)) - 3$ for a general point p. Since Z imposes at least 7 independent conditions on cubics, $h^0(\mathbb{P}^2, \mathcal{I}_Z(3)) \leq 3$ and since $Z' \subseteq Z$ imposes exactly 7 independent conditions, $h^0(\mathbb{P}^2, \mathcal{I}_{Z'}(3)) = 3$. Since $h^0(\mathbb{P}^2, \mathcal{I}_{Z+2p}(3)) = 1$ by Theorem 3.2.4, there is a unique cubic containing Z with a double point at p and hence it contains Z' with a double point at p. Hence for a general point p, $h^0(\mathbb{P}^2, \mathcal{I}_{Z'+2p}(3)) \geq h^0(\mathbb{P}^2, \mathcal{I}_{Z+2p}(3)) = 1$. It follows that $h^0(\mathbb{P}^2, \mathcal{I}_{Z'+2p}(3)) > h^0(\mathbb{P}^2, \mathcal{I}_{Z'}(3)) - 3$ and so Z' admits an unexpected curve of degree 3. $\square$

We shall need the following lemma for Lemma 3.2.9 below.

Lemma 3.2.8. *Suppose that Z is a 0-dimensional subscheme of reduced points in $\mathbb{P}^2$ and let $p \in \mathbb{P}^2$ be a general point. Let C be a curve containing Z with multiplicity $\deg C - 1$ at the general point p. Then C is reduced. If C is not irreducible, then C is the union of lines through p and a curve C' such that C' is reduced and irreducible and the multiplicity of C' at p is $\deg C' - 1$ with C' being smooth away from p.*

Now put $Z' = Z \cap C'$ and $Z'' = Z - Z'$. Then Z' has multiplicity index, $m_{Z'}$, satisfying $m_{Z'} + |Z''| = m_Z$. Moreover every component of $C - C'$ passes through the general point p and exactly one other point of Z'' and hence $\deg C' = \deg C - |Z''| = (m_Z + 1) - |Z''| = (m_{Z'} + |Z''| + 1) - |Z''| = m_{Z'} + 1$.

Proof. Lemma 5.1 in [33]. $\square$

Lemma 3.2.9. *Suppose that Z' is a reduced 0-dimensional subscheme admitting an unexpected cubic, C, with $|Z'| = 7$. Then we have the following:*

1. *C is reduced and irreducible and hence no more than 3 points of Z' lie on a line and Z' is not contained in a conic.*
2. *The general cubic through Z' is reduced and irreducible.*
3. *Every cubic through Z' is singular.*

Proof. By Lemma 3.2.8 above, we know that C is reduced. If C is not irreducible, then Lemma 3.2.8 gives that C has a component C' that is an unexpected curve for a subscheme Y of Z'. Clearly C' cannot be a line since it would have to have a singularity of multiplicity 0 at a general point. So C' is a conic that is an unexpected curve for a subscheme $Y \subset Z'$ where $|Y| = 6$ by Lemma 3.2.8. Y however determines a unique conic so that the general point p is confined to a proper closed subset which is impossible. So C has to be irreducible. Since we have that C is irreducible, no more than 3 points of Z' are lie on a line and Z' is not contained in a conic.

We now argue that the general cubic through Z' is irreducible. Note that if the general cubic is reducible, then it consists of 3 lines or an irreducible conic and a line. Let's consider the space of cubics vanishing at Z' consisting of three lines. Note that two of the lines must each contain at least 2 of the points. Let those two lines be L_1 and L_2. Then L_1 and L_2 each contain at least 2 of the points and at most 3 of the points of Z' and L_3 contains at least one of the points of Z'. Hence there are finitely many choices for L_1 and L_2 and one projective dimension worth of choices for L_3. This is finitely many one dimensional families in a three dimensional vector space (hence 2 dimensional projectively). Therefore the general cubic cannot consist of three lines.

For the space of cubics, $Q + L$, vanishing at Z' consisting of an irreducible conic, Q, and a line, L, note that L contains at most 3 of the points and Q contains at least 4 of the points. Since Q is irreducible, no 3 or more of the points are collinear. So if Q has 4 points, then we have a pencil of conics and L is fixed. If Q has 5 points, then we have a unique conic and again L is fixed by 2 points. If Q has 6 points, then it is unique and L moves in pencil. Across all of these scenarios, we have finitely many one dimensional families of cubics $Q + L$ in a three dimensional space of cubics vanishing at Z'. So again, the general cubic cannot consist of an irreducible conic and a line. We conclude that the general cubic must be irreducible.

We now argue that every cubic through Z' is singular. By Lemma 3.2.6, we know that Z' imposes exactly 7 conditions so that $h^0(\mathbb{P}^2, \mathcal{I}_{Z'}(3)) = 3$. Pick a basis $\{F, G, H\}$ of $H^0(\mathbb{P}^2, \mathcal{I}_{Z'}(3))$. For a point $(A, B, C) \in \mathbb{P}^2$, we have the cubic $AF + BG + CH \in H^0(\mathbb{P}^2, \mathcal{I}_{Z'}(3))$. Now consider all the points $((A, B, C), (a, b, c)) \in \mathbb{P}^2 \times \mathbb{P}^2$ such that (a, b, c) is a singular point of the cubic $AF + BG + CH$. Since Z' admits an unexpected cubic, there are indeed such points. Let V be the closure of all such points in $\mathbb{P}^2 \times \mathbb{P}^2$.

Let $\pi_2 : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ be projection onto the second component. Let U be the open set of all the general points in the second component. Then $\pi_2^{-1}(U) \subset V$ is an open set and each general point $p \in U$ has a single point in its pre-image by π_2. Let W be the closure of $\pi_2^{-1}(U)$ in V. Then W is the component of V that is carried to the general points by π_2 and it is 2-dimensional. Let π_1 be projection onto the first component. If $\pi_1(W)$ is a point, then there is a single cubic containing Z' that has a singularity at every general point which is impossible. If $\pi_1(W)$ is contained in a curve of $\mathbb{P}^2$, then that means that every cubic with a singularity at a general point has a one dimensional singular locus. But this contradicts the fact that every general point has a unique cubic with a singularity at that general point. Since π_1 is a closed map and $\pi_1(W)$ is not contained in a curve, $\pi_1(W) = \mathbb{P}^2$. Therefore every cubic through Z', $AF + BG + CH$, has a singularity. □

Lemma 3.2.10. *Let $Z' = \{p_1, \ldots, p_7\}$ be a reduced 0-dimensional subscheme such that $|Z'| = 7$ and Z' admits an unexpected cubic. Blow up Z' to obtain a surface $X \to \mathbb{P}^2$ with exceptional curves $e_1, \ldots, e_7$ and let L be the total transform of a line in $\mathbb{P}^2$ on X. Then the anticanonical divisor class $-K_X$ is numerically effective.*

Proof. By Lemma 3.2.9, if Z' admits an unexpected cubic and C is the general cubic in $\mathcal{I}_{Z'}(3)$, then C is irreducible. To see that $-K_X$ is nef, we need only show that the proper transform of C when we blow up Z' belongs to the class of $-K_X$, $3L - e_1 - \cdots - e_7$. For this, it is enough to show that C has multiplicity 1 at every point of Z'. Let $\tilde{C}$ and C' be irreducible cubics in $\mathcal{I}_{Z'}(3)$ and suppose that both have a singularity at a point, say p_1, of Z'. Then $\tilde{C} \cdot C' = 4 + 6 = 10$ contrary to Bézout's theorem. So if there is an irreducible cubic in $\mathcal{I}_{Z'}(3)$ with a singularity at a point of Z', then it is unique. Therefore there at most 7 such irreducible cubics. Hence the general cubic C in $\mathcal{I}_{Z'}(3)$ cannot have a singularity at one of the points of Z'. $-K_X$ now has positive self-intersection and contains an irreducible section and so is nef. □

Corollary 3.2.11. *Let Z be a reduced 0-dimensional subscheme that admits an unexpected cubic. Then $|Z| = 7$.*

Proof. Note that $|Z| \geq 7$ by Lemma 3.2.6. By Lemma 3.2.7, Z has a subscheme, $Z' = p_1 + \cdots + p_7$, such that $|Z'| = 7$ and Z' admits an unexpected cubic. Blow up the points of Z to obtain a surface X with exceptional curves $e_1, \ldots, e_{|Z|}$ and L the total transform of a line. Consider the class $3L - e_1 - \cdots - e_7$ and note that by Lemma 3.2.10, it is nef. Furthermore, it is base point free by Theorem 3.2.12. Hence if $|Z| > 7$, then $h^0(X, 3L - e_1 - \cdots - e_{|Z|}) < h^0(X, 3L - e_1 - \cdots - e_7)$. But $h^0(X, 3L - e_1 - \cdots - e_7) = 3$ by Lemma 3.2.6 so that $h^0(X, 3L - e_1 - \cdots - e_{|Z|}) < 3$. This means that Z imposes more than 7 conditions on the space of cubics in $\mathbb{P}^2$ and hence cannot admit an unexpected curve by Lemma 3.2.6. Now it follows that $Z = Z'$. $\square$

The following result forms part of the main result, Theorem III.1, in [24].

Theorem 3.2.12. *Let X be a smooth, projective, anticanonical rational surface. Let $\mathcal{F}$ be a numerically effective divisor class on X. Write $\mathcal{F} = \mathcal{H} + \mathcal{N}$ where $\mathcal{H}$ is the class of the free part of $\mathcal{F}$ and $\mathcal{N}$ is the class of the fixed part of $\mathcal{F}$. Let $-\mathcal{K}_X$ denote the class of the effective anticanonical class on X and let D be a nonzero section of $-\mathcal{K}_X$.*

1. *Suppose that $(-\mathcal{K}_X) \cdot \mathcal{F} \geq 2$, then $h^1(X, \mathcal{F}) = 0$ and $\mathcal{F}$ is base point free so fixed component free ($\mathcal{N} = 0$).*

2. *Suppose that $(-\mathcal{K}_X) \cdot \mathcal{F} = 1$, then $h^1(X, \mathcal{F}) = 0$ and if $\mathcal{F}$ has no fixed components, then it has a unique base point on D. The class $\mathcal{F}$ has a fixed component if and only if $\mathcal{H} = r\mathcal{C}$ where $\mathcal{C} \in \mathcal{K}_X^{\perp}$ and $\mathcal{N} = \mathcal{N}_1 + \cdots + \mathcal{N}_t$ with $\mathcal{N}_i$ a smooth rational curve for every i, $\mathcal{N}_i^2 = -2$ and $\mathcal{N}_i \cdot \mathcal{N}_{i+1} = 1$ for $i < t$, $\mathcal{N}_t^2 = -1$, $\mathcal{N}_i \cdot \mathcal{N}_j = 0$ for $j > i + 1$, $C \cdot \mathcal{N}_1 = 1$, and $C \cdot \mathcal{N}_i = 0$ for $i > 1$ and finally $r = h^1(X, \mathcal{H})$ with $r > 1$ only if $C^2 = 0$.*

We will soon need the following theorem due to Bertini, [1], which guarantees the existence of a section that is reduced and irreducible and has no singular points outside the base locus of its linear system $\mathcal{F}$ in characteristic 0, when $\mathcal{F}$ is fixed component free and is not composed with a pencil.

Theorem 3.2.13 (Bertini)**.** *Let $V \subseteq \mathbb{P}^n_k$ be an algebraic variety where k is a field of characteristic 0. Suppose that $\mathcal{F}$ is a linear system on V that is fixed component free. If $\mathcal{F}$ is not composed with a pencil, then the sections of $\mathcal{F}$ that are reduced and irreducible comprise a dense open set in the parameter space $\mathbb{P}(\mathcal{F})^\vee$. Also the sections of $\mathcal{F}$ that have no singular points outside the base locus of $\mathcal{F}$ also form a dense open set in the parameter space $\mathbb{P}(\mathcal{F})^\vee$.*

We now introduce the notions of elliptic and quasi-elliptic fibrations and collect some facts about the Picard groups on elliptic and quasi-elliptic curves.

Definition 3.2.14. *Let X be a smooth projective surface and let B be a smooth curve. Let $\phi : X \to B$ be a surjective morphism such that the general fiber of ϕ is a curve of arithmetic genus 1. Then ϕ is a genus 1 fibration. If the general fiber is smooth, then it is an elliptic curve and the fibration is called an elliptic fibration. If the general fiber is not smooth, then the fiber is called a quasi-elliptic fibration.*

Lemma 3.2.15. *If the characteristic of the base field is not 2 or 3, then any genus 1 fibration is an elliptic fibration. In characteristics 2 and 3, the general fiber of a genus 1 fibration may be quasi-elliptic, in which case the general fiber is a cuspidal rational curve.*

Proof. Propositions 1.1 & 1.2 of [35]. □

Lemma 3.2.16. *Suppose that C is an elliptic curve in $\mathbb{P}^2$ over an algebraically closed field of characteristic p. Let $[m] : Pic(C) \to Pic(C)$, $m \in \mathbb{Z}$, $m \geq 2$, be the map given by $[m](x) = mx$ for $x \in Pic(C)$.*

1. *If $p \nmid m$, then $|ker([m])| = m^2$ and $ker([m]) \cong \mathbb{Z}_m \times \mathbb{Z}_m$.*

2. *If $ker([p])$ is not trivial, then for all $t > 0$, $ker([p^t]) \cong \mathbb{Z}_{p^t}$.*

3. *Let $m = p^t n$ such that $p \nmid n$ and $t > 0$.*

   a) *If $ker([p]) \neq \{O\}$, then $ker([m]) = ker([p^t]) \times ker([n]) \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_n \times \mathbb{Z}_n \cong \mathbb{Z}_m \times \mathbb{Z}_n$.*

   b) *If $ker([p]) = \{O\}$, then $ker([m]) = ker([p^t]) \times ker([n]) \cong \{O\} \times \mathbb{Z}_n \times \mathbb{Z}_n \cong \mathbb{Z}_n \times \mathbb{Z}_n$*

In particular, the number of m-torsion points on elliptic curves are always finite.

Proof. See Lemma III.8 in [5] for parts 1 and 2. For part 3, use the fact that p^t and n are coprime and hence $\ker([p^t n]) = \ker([p^t]) \times \ker([n])$. □

Lemma 3.2.17. *Suppose that C is a quasi-elliptic curve over an algebraically closed field, k, of characteristic p, prime. Then $Pic(C) \cong \mathbb{G}_a$ where $\mathbb{G}_a$ is the additive group of k. In particular, every non-trivial point of C is a p-torsion point under the isomorphism of $Pic(C)$ with the smooth points of C.*

Proof. See Proposition 5.2 in chapter 1, section 5 of [32]. □

We will need the semicontinuity principle to show that unexpected cubics occur only in characteristic 2. Before we state it, we define what it means for points $p_1, \dots, p_n$ in $\mathbb{P}^2$ to be essentially distinct.

Definition 3.2.18. *We say that points $p_1, \dots, p_n$ are essentially distinct if there is a sequence of blowings-up $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = \mathbb{P}^2$ such that p_i is a point on X_{i-1} and is the center of the blowing-up $X_i \to X_{i-1}$.*

Lemma 3.2.19. ***Semicontinuity Principle*** *Suppose that m, $m_1, \dots, m_r$ are nonnegative integers and let $p_1, \dots, p_r$ be general points in $\mathbb{P}^2$. Blow up the points to obtain a surface X and let $E_1, \dots, E_r$ be the corresponding exceptional curves with L the proper transform of a line in $\mathbb{P}^2$. Now let $p_1', \dots, p_r'$ be essentially distinct points of $\mathbb{P}^2$ and blow them up to obtain a surface X' with corresponding exceptional curves $E_1', \dots, E_r'$ and proper transform of a line L'. If $mL - m_1E_1 - \cdots - m_rE_r$ is an effective class, then $mL' - m_1E_1' - \cdots - m_rE_r'$ is an effective class.*

Proof. See Theorem I.1.6 in [26]. □

Theorem 3.2.20. *Let k be an algebraically closed field. Suppose that Z is a reduced finite subscheme of points in $\mathbb{P}^2_k$ that admits an unexpected curve of degree 3. Then the characteristic of the base field is 2.*

Proof. We know by Corollary 3.2.11 that $|Z| = 7$. [33] shows that if Z is the Fano plane of 7 points, then Z admits an unexpected curve of degree 3. It remains to show that if char $k \neq 2$, then no reduced 0-dimensional subscheme can admit an unexpected cubic.

Let Z admit an unexpected cubic. Blow up the points of Z to obtain a surface $X \to \mathbb{P}^2$. Note that $h^2(X, -K_X) = h^0(X, 2K_X)$ by duality and since the total transform of a line in $\mathbb{P}^2$ to X, L, is nef and $2K_X \cdot L = -6 < 0$, $2K_X$ is not effective. Hence $h^2(X, -K_X) = h^0(X, 2K_X) = 0$. Now by Riemann-Roch, $h^0(X, -K_X) = (-K_X)^2 + 1 + h^1(X, -K_X) = 3 + h^1(X, -K_X)$. Since $h^1(X, -K_X) \geq 0$, $-K_X$ is an effective divisor class. Suppose now that the characteristic of the base field is 0. By Lemma 3.2.10, $-K_X$ is numerically effective. Now note that $(-K_X)^2 = (3L - E_1 - \cdots - E_7)^2 = 9 - 7 = 2 > 0$ and hence taking $-K_X = F$ in Theorem 3.2.12, we have that $-K_X$ is base point free and hence fixed component free.

Note that the linear system of divisors of $-K_X$, $H^0(X, -K_X)$, has dimension 3 since it is just the linear system of cubics containing Z but Z imposes exactly 7 conditions on the space of cubics in $\mathbb{P}^2$ which has dimension 10. Another way to see this is that by Theorem 3.2.12, $h^1(X, -K_X) = 0$ and hence by the Riemann-Roch theorem, $h^0(X, -K_X) = 3$. Hence $-K_X$ is not a pencil. Moreover the general section of $-K_X$ is not composed with a pencil. To see this, note that Theorem 3.2.12 gives that $-K_X$ has no base points. So we get a map $\phi_{|-K_X|} : X \to \mathbb{P}^2$ where $2 = h^0(X, -K_X) - 1$. If $-K_X$ is composed with a pencil, then $\phi_{|-K_X|}(X)$ would be a curve. But that implies that the fibers of the morphism are disjoint and hence have self-intersection 0. But the sections of $-K_X$ are unions of fibers and so this would imply that $-K_X^2 = 0$. But this is false since $-K_X^2 = 2$. So $-K_X$ is indeed not composed with a pencil. In characteristic 0, theorem 3.2.13 now gives that the general section of $-K_X$ is smooth outside of its base points. But since $-K_X$ has no base points, we have that its general section is smooth.

Since $h^0(X, -K_X) = 3$, we can parameterize the sections of $H^0(X, -K_X)$ by $\mathbb{P}^2$. We do this by picking a basis $\{F, G, H\} \subset H^0(X, -K_X)$ and projectivizing $H^0(X, -K_X)$ by $AF + BG + CH \to (A, B, C) \in \mathbb{P}^2$. Then the non-general sections of $H^0(X, -K_X)$ correspond to some curve in $\mathbb{P}^2$ so that they just have projective dimension one. Since Z admits an unexpected cubic, for every general point p in $\mathbb{P}^2$, there is a non-general cubic containing Z with a singular point at p. Since we just have a one-dimensional (projective) amount of such cubics, each one must have a one-dimensional singular locus since the general points are two dimensional. Therefore such a cubic C has a linear component that is non-reduced. Since the non-reduced linear component contains a general point, it can contain at most one other point of Z. This implies that 6 points of Z are collinear. But this is impossible since the general section of $H^0(X, -K_X)$ is irreducible. Combining the last two sections gives that when the characteristic of the base field is 0, then no 0-dimensional subscheme can admit an unexpected cubic.

Suppose now that the characteristic is positive. By Lemma 3.2.9, we can assume that the general cubic containing Z is irreducible. Moreover for every general point, we obtain a pencil of cubics such that each cubic in the pencil is singular. Only one of the cubics has its singular point at p. Blow up the base points of the pencil to obtain a genus 1 fibration. Then the general fiber of the fibration is singular, in fact a cuspidal cubic by Lemma 3.2.15, but when char $k > 3$, then the general fiber is smooth, again by Lemma 3.2.15, so that char $k = 3$ or 2.

Say that char $k = 3$. The general fiber of the fibration has a single singular point of multiplicity 2. Let C be the image in $\mathbb{P}^2$ of one of the general fibers. Let C' be its proper transform to the surface X that is the blow up of $\mathbb{P}^2$ at the points of Z. For each point p of C', the Semicontinuity Principle, Lemma 3.2.19, gives that there is $C_p \in |-K_X|$ such that C_p is singular at the point p. Hence $C_p \cdot C' = 2$. Since the various C_p are linearly equivalent, if C_q is the curve for a point q of C', then $2p$ is linearly equivalent to $2q$ on C'. Therefore $p - q$ is a 2-torsion point. This means that for general points p and q on C', $p - q$ has 2-torsion. However, a cuspidal cubic in characteristic 3 has finitely many 2-torsion points by Lemma 3.2.17. This means that there can be only finitely many points p of C' such that a cubic through Z' has a singularity at p. Therefore Z' does not have an unexpected curve of degree 3 if char $k = 3$.

□

Theorem 3.2.21. *Let $q_1, \dots, q_r$ be points in $\mathbb{P}^2$ that are not necessarily general. Let $p_1, \dots, p_s$ be general points such that $r + s \leq 8$. Blow up the points to obtain a surface X. Let $a_1, \dots, a_r, b_1, \dots, b_s$ be nonnegative integers and put $e = \sum_{j=1}^{r} a_j e_j$ and $E = \sum_{i=1}^{s} b_i E_i$ where e_j is the blow up of the points q_j and E_i is the blow up of the points p_i. For some $t > 0$, consider the divisors $tL - e$ and $tL - e - E$ with $h^0(X, tL - e) > h^0(X, tL - e - E) > 0$. Write $tL - e = F + N$ where F is free and hence nef and N is the divisorial base locus of $tL - e$.*

1. *If there is an exceptional curve C such that $(F - E) \cdot C \leq -2$, then E is special on $tL - e$, i.e. $h^0(X, tL - e - E) > \max\{0, h^0(X, tL - e) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$.*
2. *Similarly, if there is a nodal curve D such that $(F - E) \cdot D \leq -2$, then E is special on $tL - e$, char $k = 2$ and $D = 3L - e_1 - \cdots - e_7 - 2E_8$.*

Proof. Write $tL - e = F + N$ where F is the free and hence nef part of $tL - e$ and N is the divisorial base locus of $tL - e$. Then since N is in the fixed part of $tL - e$, $h^0(X, tL - e) = h^0(X, F)$ and $h^0(X, F - E) = h^0(X, tL - e - E)$.

Since nef divisors are effective for 8 or fewer points, F is a nontrivial nef divisor, and $E \cdot L = 0$, $(F - E) \cdot L > 0$ and we have by duality that $h^2(X, F - E) = 0$. By the theorem of Riemann-Roch, $h^0(X, F - E) = \frac{(F-E)^2 + (-K_X) \cdot (F-E)}{2} + 1 + h^1(X, F - E)$. Since F is nef, $h^1(X, F) = h^2(X, F) = 0$ and so we can rewrite $h^0(X, F - E)$ to get $h^0(X, F - E) = h^0(X, F) - \sum_{i=1}^{s} \binom{b_i+1}{2} + h^1(X, F - E)$. Suppose now that there is an exceptional curve C with $(C)^2 = C \cdot K_X = -1$ such that $(F - E) \cdot C = -n$, $n \geq 2$ or a nodal curve D such that $D^2 = -2$, $D \cdot K_X = 0$ with $(F - E) \cdot D = -m$, $m \geq 2$. Then by Lemma 3.1.9, the divisorial base locus of $F - E$ contains nC or contains $\lceil \frac{m}{2} \rceil D$. In the first instance $h^1(X, F-E) > \binom{n}{2}$ and in the second scenario, $h^1(X, F-E) > \frac{m^2-1}{2}$. In both instances $h^1(X, F-E) > 0$ since $n \geq 2$ and also $m \geq 2$. Hence $h^0(X, F-E) > h^0(X, F) - \sum_{i=1}^{s} \binom{b_i+1}{2}$.

From the first paragraph of this argument, $h^0(X, tL-e-E) = h^0(X, F-E)$ and $h^0(X, F) - \sum_{i=1}^{s} \binom{b_i+1}{2} = h^0(X, tL-e) - \sum_{i=1}^{s} \binom{b_i+1}{2}$. From the second paragraph, we have that $h^0(X, F-E) > h^0(X, F) - \sum_{i=1}^{s} \binom{b_i+1}{2}$. Putting it together gives that $h^0(X, tL-e-E) > h^0(X, tL-e) - \sum_{i=1}^{s} \binom{b_i+1}{2}$. Hence E is special on $tL-e$.

Since $r+s \leq 8$, if D is a nodal curve on X, then D is the proper transform of a line through 3 of the points, or the proper transform of a conic through 6 of the points or the proper transform of a cubic through all 8 points with a singularity at one of the points. If D is the transform of a line or a conic, then $D \cdot E = 0$ since otherwise the points blown up to obtain E will not be general. So D must have the class $3L - e_1 - \cdots - e_7 - 2E$ where the $e_1, \ldots, e_7$ are the blow up of the points $q_1, \ldots, q_7$ and E is the blow up of the lone general point p. Let $Z = q_1 + \cdots + q_7$ be a 0-dimensional subscheme. Since p is a general point and there is a cubic containing Z in its vanishing locus with a singularity at p, Z admits an unexpected cubic. By Theorem 3.2.20, the characteristic of the base field is 2. $\square$

We shall need the following lemma from [19] for the proof of Theorem 3.2.23 and so we make note of it.

Lemma 3.2.22. *Let $p_1, \ldots, p_t$, $t \leq 8$, be points in $\mathbb{P}^2$. Blow them up to obtain a surface X and let $E_1, \ldots, E_t$ be the divisors corresponding to the blown up points and let L be the total transform of a general line in $\mathbb{P}^2$. Define $\mathcal{B}_t$, $\mathcal{L}_t$, $\mathcal{Q}_t$, $\mathcal{C}_t$ and $\mathcal{M}_8$ to be the following finite families:*

- $\mathcal{B}_t = \{E_1, \dots, E_t\}$;
- $\mathcal{L}_t = \{L - E_{i_1} - \cdots - E_{i_j} | 2 \leq j \leq t\}$;
- $\mathcal{Q}_t = \{2L - E_{i_1} - \cdots - E_{i_j} | 5 \leq j \leq t\}$;
- $\mathcal{C}_t = \{3L - 2E_{i_1} - E_{i_2} - \cdots - E_{i_j} | 7 \leq j \leq 8, j \leq t\}$; *and*
- $\mathcal{M}_8 = \{4L - 2E_{i_1} - 2E_{i_2} - 2E_{i_3} - E_{i_4} - \cdots - E_{i_8}, 5L - 2E_{i_1} - \cdots - 2E_{i_6} - E_{i_7} - E_{i_8},$
  $6L - 3E_{i_1} - 2E_{i_2} - \cdots - 2E_{i_8}\}$.

Let $Neg(X)$ be a set comprised of the classes of curves of negative self-intersection on X. Then $Neg(X) \subseteq \mathcal{B}_t \cup \mathcal{L}_t \cup \mathcal{Q}_t \cup \mathcal{C}_t \cup \mathcal{M}_8$.

Proof. See Proposition 4.1 in [19]. □

Theorem 3.2.23. *Let $q_1, \dots, q_r$ be points in $\mathbb{P}^2$, not necessarily general, and let $p_1, \dots, p_s$ be general points such that $r + s \leq 8$. Blow up the points to obtain a surface X and for some integers $a_1, \dots, a_r, b_1, \dots, b_s$, let $e = a_1 e_1 + \cdots + a_r e_r$ and $E = b_1 E_1 + \cdots + b_s E_s$ where the e_j are the blow up of the q_j and the E_i are the blow up of the p_i. For an integer $t > 0$, consider the divisors $tL - e$ and $tL - e - E$. Write $tL - e = F + N$ where F is free and hence nef and N is the divisorial base locus of $tL - e$. If $h^0(X, tL - e - E) > \max\{0, h^0(X, tL - e) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$, then either $r = 7$, $s = 1$ with a nodal curve D satisfying $D \cdot E > 1$ with D in the divisorial base locus of $F - E$ or there is an exceptional curve C with $C \cdot E > 0$ such that mC is in the divisorial base locus of $F - E$ with $m > 1$.*

Proof. We can write $tL - e = F + N$ where F is the free and hence nef part of $tL - e$ and N is the divisorial base locus of $tL - e$. Then $h^0(X, tL - e) = h^0(X, F)$ and $h^0(X, tL - e - E) = h^0(X, F - E)$. By Riemann-Roch, $h^0(X, F - E) = \frac{(F-E)^2 + (-K_X) \cdot (F-E)}{2} + 1 + h^1(X, F - E) - h^2(X, F - E)$. If we assume that $F - E$ is effective, then $h^2(X, F-E) = 0$ and we compute $\frac{(F-E)^2+(-K_X)\cdot(F-E)}{2} + 1$ to obtain $\frac{(F-E)^2+(-K_X)\cdot(F-E)}{2} + 1 = h^0(X, F) - \sum_{i=1}^{s} \binom{b_i+1}{2}$ ($h^1(X, F) = 0$ since F is nef and $h^2(X, F) = 0$ since F is effective.) Hence $h^0(X, F-E) = h^0(X, F) - \sum_{i=1}^{s} \binom{b_i+1}{2} + h^1(X, F-E)$. Then $h^0(X, F-E) = h^0(X, tL-e-E) > \max\{0, h^0(X, tL-e) - \sum_{i=1}^{s} \binom{b_i+1}{2}\} = \max\{0, h^0(X, F) - \sum_{i=1}^{s} \binom{b_i+1}{2}\}$ if and only if $h^0(X, F-E) > 0$ and $h^1(X, F-E) > 0$.

Note that the divisorial base locus of $F-E$ is not empty since $F-E$ is not nef. $F-E$ is not nef because $h^1(X, F-E) > 0$ on a blow up of 8 or fewer points. Suppose that there is no nodal curve D in the divisorial base locus of $F-E$ with $D \cdot E > 1$ and that there is no non-reduced exceptional curve C in the divisorial base locus of $F-E$ with $C \cdot E > 0$. Write $F - E = H + C_1 + \cdots + C_m + N_1 + \cdots + N_n$ where H is the free and hence the nef part of $F-E$ and $C_1 + \cdots + C_m + N_1 + \cdots + N_n$ is the divisorial base locus of $F-E$ with $C_i \cdot E > 0$ for all i and $N_j \cdot E = 0$ for all j. Since we are arguing by contradiction and we show below that the C_i are exceptional, we assume that the C_i are reduced. Because the C_i are reduced, they are all distinct but the N_j need not all be distinct. Moreover $0 \leq F \cdot N_j = (F-E) \cdot N_j$.

The claim now is that $C_i^2 = -1$ for all i. Fix an i and consider C_i. Note that all the curves in $C_1, \ldots, C_m, N_1, \ldots, N_n$ have negative self-intersection since otherwise they would be free. By Lemma 3.2.22, $C_i \in \mathcal{B}_t \cup \mathcal{L}_t \cup \mathcal{Q}_t \cup \mathcal{C}_t \cup \mathcal{M}_8$. We write $\mathcal{E}$ for an exceptional curve in $\{e_1, \ldots, e_r, E_1, \ldots, E_s\}$. Clearly the class of C_i is not $L - \mathcal{E}_{i_1} - \cdots - \mathcal{E}_{i_j}$, $3 \leq j \leq r$, since $C_i \cdot E > 0$ would imply that one of the general points is collinear with two other points which is not possible. Similarly the class of C_i is not $2L - \mathcal{E}_{i_1} - \cdots - \mathcal{E}_{i_j}$, $6 \leq j \leq r$, because combined with $C_i \cdot E > 0$, that would also imply that one of the general points is conconic with 6 or more of the other points which is not possible.

Finally the class of C_i is not $3L - 2\mathcal{E}_{i_1} - \mathcal{E}_{i_2} - \cdots - \mathcal{E}_{i_8}$; otherwise, if it were, and $\mathcal{E}_{i_1}$ is the preimage of a general point then D would be C_i which is not possible by assumption. Say that $\mathcal{E}_{i_1}$ is not the preimage of a general point and rather say that $\mathcal{E}_{i_2}$ is the preimage of a general point, p_{i_2}. Recalling that C_i is irreducible, $3L - 2\mathcal{E}_{i_1} - \mathcal{E}_{i_3} - \cdots - \mathcal{E}_{i_8}$ is the class of an irreducible section and since $(3L - 2\mathcal{E}_{i_1} - \mathcal{E}_{i_3} - \cdots - \mathcal{E}_{i_8})^2 = -1$, we have that $h^0(X, 3L - 2\mathcal{E}_{i_1} - \mathcal{E}_{i_3} - \cdots - \mathcal{E}_{i_8}) = 1$. It follows that p_{i_2} is confined to a closed locus which contradicts the assumption that it is general. Hence the class of C_i is not $3L - 2\mathcal{E}_{i_1} - \mathcal{E}_{i_2} - \cdots - \mathcal{E}_{i_8}$.

It follows now that C_i is an exceptional curve. Now for two exceptional curves C_i and C_j, if $C_i \cdot C_j > 0$, then $C_i + C_j$ moves and hence cannot be in the base locus. Hence $C_i \cdot C_j = 0$ for all i and j. Some of the C_i might meet the N_j or H. Let C be the sum of all those C_i and let C' be the sum of the remaining C_i. Note that $C + N_1 + \cdots + N_n + H$ is nef and to see that we check its intersection with its various components. For C_i appearing in C, $C_i \cdot C = -1$ but for some N_j, $C_i \cdot N_j > 0$ and $C_i \cdot H \geq 0$ so that $C_i \cdot (C + N_1 + \cdots + N_n + H) \geq 0$. For the N_j, we have from above that $N_j \cdot (F - E) \geq 0$ and since $F - E = C' + C + N_1 + \cdots + N_n + H$ and $N_j \cdot C' = 0$, we have that $N_j \cdot (C + N_1 + \cdots + N_n + H) \geq 0$. Finally, $H \cdot (C + N_1 + \cdots + N_n + H) \geq 0$ since H is free. By Lemma 3.1.7, $h^1(X, C + N_1 + \cdots + N_n + H) = 0$.

We now show that we can add the curves in C' onto $C + N_1 + \cdots + N_n + H$ one at a time while maintaining the vanishing of h^1. Let C_i be one of the curves in C. Take $C + N_1 + \cdots + N_n + H + C_i$ and consider the sequence of sheaves $0 \to \mathcal{O}_X(C + N_1 + \cdots + N_n + H) \to \mathcal{O}_X(C + N_1 + \cdots + N_n + H + C_i) \to \mathcal{O}_{C_i}(C + N_1 + \cdots + N_n + H + C_i) \to 0$. Since C_i is exceptional and $C_i \cdot (C + N_1 + \cdots + N_n + H + C_i) = C_i \cdot (C + N_1 + \cdots + N_n + H) + C_i \cdot C_i = 0 + -1 = -1$, the sequence devolves to $0 \to \mathcal{O}_X(C + N_1 + \cdots + N_n + H) \to \mathcal{O}_X(C + N_1 + \cdots + N_n + H + C_i) \to \mathcal{O}_{C_i}(-1) \to 0$. We take cohomology to get $0 \to H^0(X, \mathcal{O}_X(C + N_1 + \cdots + N_n + H) \to H^0(X, \mathcal{O}_X(C + N_1 + \cdots + N_n + H + C_i) \to H^0(C_i, \mathcal{O}_{C_i}(-1)) \to H^1(X, \mathcal{O}_X(C + N_1 + \cdots + N_n + H) \to H^1(X, \mathcal{O}_X(C + N_1 + \cdots + N_n + H + C_i) \to H^1(C_i, \mathcal{O}_{C_i}(-1)) \to \cdots$. Now by duality on C_i, we have that $h^1(C_i, \mathcal{O}_{C_i}(-1)) = h^0(C_i, \mathcal{O}_{C_i}(-2 + 1)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. The sequence $0 \to H^1(X, C + N_1 + \cdots + N_n + H + C_i) \to 0$ now gives that $h^1(X, C + N_1 + \cdots + N_n + H + C_i) = 0$. Iterating this argument gives that $h^1(X, C + N_1 + \cdots + N_n + H + C') = 0$. $\square$

3.3 Quasi-Elliptic Fibrations

We now make some basic observations about quasi-elliptic fibrations over an algebraically closed field k. In the next section, we shall make a connection between unexpected cubics and quasi-elliptic fibrations in order to understand better the sorts of 0-dimensional subschemes that admit unexpected cubics. We begin with the definitions of elliptic and quasi-elliptic surfaces.

Definition 3.3.1. *Let X be a smooth projective surface. We say X is an elliptic surface if there is a surjective map $\pi : X \to C$ where C is a smooth projective curve such that the general fiber of π is a smooth curve of genus one. We say that X is a quasi-elliptic surface if the general fiber of π is a reduced irreducible curve of genus one that is singular. In the first case, the map π is called an elliptic fibration and in the latter, a quasi-elliptic fibration.*

Note that since π is surjective and C is a curve, π is flat and the arithmetic genus of all the fibers is constant. So every fiber has genus one when the general fiber has genus one.

Proposition 3.3.2. *Suppose that X is a quasi-elliptic surface. Then the base field has characteristic* 2 *or* 3*. Moreover the general fiber of X has only one ordinary cusp.*

Proof. See Propositions 1.1 and 1.2 in [35] □

Definition 3.3.3. *A relatively minimal elliptic or quasi-elliptic surface is an elliptic or quasi-elliptic surface whose fibers do not contain exceptional curves.*

Note that an elliptic or quasi-elliptic surface may be relatively minimal without being minimal as an algebraic surface. It may not have exceptional curves in its fibers (vertical exceptional curves) but it may very well have exceptional curves that are not contained in any of its fibers (horizontal exceptional curves).

Definition 3.3.4. *Suppose that X is a smooth complete rational surface with an elliptic or quasi-elliptic fibration $\pi : X \to C$ such that X is relatively minimal, then X is called a rational elliptic or quasi-elliptic surface.*

Definition 3.3.5. *Let X be an elliptic or quasi-elliptic surface with fibration $\pi : X \to C$. We say that X is Jacobian when the fibration π admits a section. I.e the fibers of X are the anticanonical curves $H^0(X, -K_X)$.*

Note that when X admits a Jacobian elliptic or quasi-elliptic fibration, then that fibration is the unique Jacobian fibration on X and moreover its base curve is $\mathbb{P}^1$ obtained from the projectivization of $H^0(X, -K_X)$. The next proposition states that if X is the blow up of the 9 base points of a pencil of cubics such that the fibration provided on X by $H^0(X, -K_X)$ is quasi-elliptic, then X is relatively minimal.

Proposition 3.3.6. *Let X be the blow up of $\mathbb{P}^2$ at the 9 base points of a pencil of cubics such that X possesses a quasi-elliptic fibration whose fibers are the sections of $H^0(X, -K_X)$. Then X is a rational Jacobian minimal quasi-elliptic surface. Moreover every component of a reducible fiber is a rational curve of self-intersection -2.*

Proof. The only thing to demonstrate here is that the fibers of the fibration on X do not contain exceptional curves. Let C be an exceptional curve on X. By the adjunction formula $-2 = C^2 + C \cdot K_X$ so that $C \cdot (-K_X) = 1$. Let $f : X \to \mathbb{P}^1$ be the morphism giving the fibration on X. Let s and s_0 be two distinct points on $\mathbb{P}^1$ such that C is a component of $f^*(s)$. Note that $f^*(s)$ and $f^*(s_0)$ are linearly equivalent so that $f^*(s)|_C$ and $f^*(s_0)|_C$ are also linearly equivalent. Since $f^*(s)|_C$ is trivial, $f^*(s_0)|_C$ is also trivial. Hence $C \cdot f^*(s_0) = 0$. Since every section of $H^0(X, -K_X)$ is linearly equivalent to $f^*(s_0)$, this implies that $C \cdot (-K_X) = 0$ which is impossible. Hence no fiber of the fibration contains an exceptional curve.

Now let $X_s = m_1C_1 + \cdots + m_nC_n$, $m_i \geq 0$, be a reducible fiber. Then from the forerunning paragraph, $C_i \cdot K_X = 0$ and so $C_i \in K_X^\perp$. Since $K_X^\perp$ is negative semi-definite and the only elements $D \in K_X^\perp$ of self-intersection $D^2 = 0$ are multiples of K_X (Lemma II.4 in [24]), we have that $C_i^2 < 0$. Now by adjunction $2g - 2 = C_i^2$. Since $g \geq 0$, we have that $g = 0$ and $C_i^2 = -2$. $\square$

Proposition 3.3.7. *Let X be a quasi-elliptic surface obtained from the blow up of $\mathbb{P}^2$ at the 9 base points of a pencil of cubics. Then the fibers of X are all connected. Say that $X_s = m_1C_1 + \cdots + m_nC_n$ is a fiber. By connected, we mean that we cannot write $\{C_1, \ldots, C_m\}$ as a union of two disjoint sets $\{C_{i_1}, \ldots, C_{i_k}\}$ and $\{C_{j_1}, \ldots, C_{j_l}\}$ such that $C_i \cdot C_j = 0$ for all i's and j's.*

Proof. The fibers on X are simply the sections of $H^0(X, -K_X)$ so it suffices to show that for $C \in H^0(X, -K_X)$, $h^0(C, \mathcal{O}_C) = 1$. Consider the sequence $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$. For a $D \in H^0(X, -K_X)$, tensor the foregoing sequence with $\mathcal{O}_X(D)$ to get $0 \to \mathcal{O}_X(D - C) \to \mathcal{O}_X(D) \to \mathcal{O}_C(D) \to 0$. Since C and D are linearly equivalent, $\mathcal{O}_C(D)$ and $\mathcal{O}_X(D - C)$ are both trivial. We obtain the sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_C \to 0$ and take cohomology to get $0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(D)) \to H^0(C, \mathcal{O}_C) \to H^1(X, \mathcal{O}_X) \to \cdots$. Now $h^1(X, \mathcal{O}_X) = 0$, $h^0(X, \mathcal{O}_X) = 1$ and $h^0(X, \mathcal{O}_X(D)) = 2$ so that $h^0(C, \mathcal{O}_C) = 1$ and C is connected. $\square$

Now we know that given a quasi-elliptic fibration, X, obtained from the blow up of the base points of a pencil of cubics, a reducible fiber $X_s = m_1C_1 + \cdots + m_nC_n$, $m_i \geq 0$, on X has none of the C_i exceptional and in fact all the C_i are rational with self-intersection -2 (Proposition 3.3.6). Moreover X_s is connected (Proposition 3.3.7). The only question now is about the configuration of the C_i in X_s i.e., what is the possible structure of a reducible fiber?

To that end, note that from the components of X_s, we get the $\mathbb{Z}$-module, $M = \mathbb{Z}C_1 + \cdots + \mathbb{Z}C_n \subseteq K_X^{\perp}$ which has connected basis since X_s is connected. M also has a bilinear form inherited from the bilinear form on $K_X^{\perp}$ such that $C_i \cdot C_j \geq 0$ whenever $i \neq j$ since the C_i are all irreducible curves. By the negative semi-definiteness of $K_X^{\perp}$ and the fact that $X_s^2 = K_X^2 = 0$, the only elements of M with self-intersection 0 are the multiples of $X_s = m_1C_1 + \cdots + m_nC_n$ and for a $D = t_1C_1 + \cdots + t_nC_n$ that is not a multiple, $D^2 < 0$. All such modules M have been classified in [8] and we display the classification in Figure 4.1.

By way of example of how to interpret the figures, consider E_7 in Figure 4.1.

Label its nodes from left to right as $C_1, \ldots, C_7$ and label the node attached to C_4 as C_8. From E_7, we have the $\mathbb{Z}$-module $M = \mathbb{Z}C_1 + \cdots + \mathbb{Z}C_8$ with connected basis $\{C_1, \ldots, C_8\}$. According to the diagram for E_7, C_1 is attached to C_2 which is in turn attached to C_3 and so on. The first element X_s in M satisfying $X_s^2 = 0$ is $X_s = C_1 + 2C_2 + 3C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 2C_8$ and any other element $D \in M$ with $D^2 = 0$ is a multiple of X_s. All other elements have negative self-intersection. In particular the C_i have self-intersection -2. Hence the nodes represent -2-curves that are components of the fiber $X_s = C_1 + 2C_2 + 3C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 2C_8$, the number attached to the node is the multiplicity of that -2-curve in the fiber and the connections or lack thereof between the nodes indicate $C_i \cdot C_j$, $i, j \in \{1, \ldots, 8\}$. See Figure 4.2 for the analogue to Figure 4.1 that emphasizes the fiber aspect for the case of characteristic 2.

A quasi-elliptic surface X might have several reducible fibers so to see exactly how many and what fibers occur, we take advantage of the fact that $\mathbb{P}^2$ blown up at 9 points has Euler characteristic 12 and use that to count fibers. To proceed, we define the topological Euler characteristic of a variety X and recall a few propositions. Let X be a variety and pick an embedding of X into projective space. Let Ω be the cotangent bundle on X. Define the Hodge numbers $h^{i,j}$ by $h^{i,j} := h^j(X, \Omega^i) = \dim_k H^j(X, \Omega^i)$ for all $i, j \in \mathbb{Z}_{\geq 0}$. Define the kth Betti number, b_k, of Ω to be $\sum_{i+j=k} h^{i,j}$.

Definition 3.3.8. *The topological Euler characteristic of a variety $X \subseteq \mathbb{P}^n$ is defined to be $e(X) := \sum_{k \geq 0} (-1)^k b_k$.*

Note that because of various vanishing theorems, the sum indeed converges. Moreover by Hodge Theory, $e(X)$ is independent of embedding. On a curve

C of genus g, only the Hodge numbers $h^{0,0}$, $h^{0,1}$, $h^{1,0}$ and $h^{1,1}$ do not vanish automatically and so the only nonzero Betti numbers are $b_2 = h^{1,1}$, $b_1 = h^{1,0} + h^{0,1}$ and $b_0 = h^{0,0}$. Simple computations give that $h^{0,0} = h^{1,1} = 1$ and $h^{1,0} = h^{0,1} = g$ and hence $b_2 = b_0 = 1$ and $b_1 = 2g$. We now note that the topological Euler characteristic of C is $e(C) = b_2 - b_1 + b_0 = 2 - 2g$. Similarly for a point p in $\mathbb{P}^n$ considered as a variety, the only nonzero Betti number is $b_0 = h^{0,0}$ which has value $h^0(p, \Omega^0) = h^0(p, \mathcal{O}_p) = 1$. Hence $e(p) = 1$. The Euler characteristic has several nice properties but one that we use below in Proposition 3.3.10 is its inclusion-exclusion property. Namely, let X be the union of two proper closed subvarieties X_1 and X_2. Then $e(X) = e(X_1) + e(X_2) - e(X_1 \cap X_2)$.

Proposition 3.3.9 (Euler characteristic)**.** *Let $\phi : X \to B$ be a proper morphism from a smooth surface X to a smooth curve B. So ϕ is a fibration. Then the Euler characteristic of the surface is defined to be $e(X) = e(B)e(F) + \sum_{b \in B}(e(F_b) - e(F))$ where F_b is the fiber over a point $b \in B$ and F is the general nondegenerate fiber. In particular, if ϕ is a quasi-elliptic fibration, then $e(X) = 4 + \sum_{b \in B}(e(F_b) - 2)$.*

Proof. See page 137 in [34] for the general statement. If ϕ is quasi-elliptic, then $B \cong \mathbb{P}^1$ and F is a cuspidal cubic and hence $e(B) = e(\mathbb{P}^1) = e(F) = 2$. □

Note that the sum in the above proposition is actually finite since if F_b is nondegenerate, then $e(F_b) - e(F)$ is trivial. So in computing the Euler characteristic of a quasi-elliptic fibration, we need only concern ourselves with degenerate fibers.

Proposition 3.3.10. *Let $\pi : X \to C$ be a quasi-elliptic fibration and let F be some degenerate fiber. Suppose that F has n components. Then $e(F) = n + 1$.*

Proof. Say that F has n components, then F is a union of n rational curves. Hence each component F_i of F has $e(F_i) = 2$. From Figure 4.2, F has $n - 1$ points at which the intersection of its components are supported. Then by the inclusion-exclusion formula of the Euler characteristic, $e(F) = \sum_{i=1}^{n} e(F_i) - \sum_{F_i \cap F_j > 0} e(F_i \cap F_j) = 2n - (n-1) = n+1$ where we use the fact that $e(p) = 1$ for a point p. □

By way of example and also to explain the notation in the next theorem, we mention very briefly what it means for a quasi-elliptic fibration to have reducible fiber type $A_1^{\oplus 8}$ or reducible fiber type $D_6 \oplus A_1^{\oplus 2}$. A fibration of reducible fiber type $A_1^{\oplus 8}$ simply means that the fibration has 8 reducible fibers, mutually disjoint, each of type of A_1. A fiber of type A_1 consists of two rational curves meeting with multiplicity 2 at a point. Now a fibration of reducible fiber type $D_6 \oplus A_1^{\oplus 2}$ possesses three reducible fibers, two of which are of type A_1 and the third is of type D_6. A degenerate fiber of type D_6 consists of 7 connected rational curves with multiplicities and intersections as shown in Figure 4.2.

Theorem 3.3.11. *Suppose $p = 2$. Let $\phi : X \to \mathbb{P}^1$ be a Jacobian rational quasi-elliptic surface. Then the collection of reducible fibers of the fibration can be represented by one of the following extended Dynkin diagrams where a node of the diagram represents a rational curve of self-intersection -2 on X.*

$$A_1^{\oplus 8}, \quad A_1^{\oplus 4} \oplus D_4, \quad D_4^{\oplus 2}, \quad D_6 \oplus A_1^{\oplus 2}, \quad A_1 \oplus E_7, \quad E_8, \quad D_8$$

Proof. See Theorem 5.6.3 in [13]. The key point is that if $\phi : X \to \mathbb{P}^1$ is a quasi-elliptic fibration, then $e(X) = 4 + \sum_{b \in \mathbb{P}^1}(e(F_b) - 2) = 12$. For each of the fibers enumerated in Figure 4.1, their smooth loci can be given a group structure. The candidates for a quasi-elliptic fibration are those whose groups are annihilated by the characteristic $p = 2$ since the group structure of the irreducible fiber of Kodaira type II (the cuspidal cubic), k, is annihilated by $p = 2$. We go through the list of the Kodaira fiber types to determine them.

The reducible fibers with intersection graph A_1 are those of Kodaira fiber types I_2 (two rational curves meeting in two distinct points) and III (two rational curves meeting with tangency). The group structure on the smooth locus of I_2 is $\mathbb{Z}/2 \times k^*$ and that on the smooth locus of III is $\mathbb{Z} \times k$. Clearly $\mathbb{Z} \times k$ is annihilated by $p = 2$ but $\mathbb{Z}/2 \times k^*$ is not. Therefore the candidate for a reducible fiber with intersection graph A_1 is III, two rational curves meeting with tangency, and not I_2, two rational curves meeting in two distinct points.

The reducible fibers with intersection graph A_2 are the Kodaira fibers I_3 (a triangle of three rational curves) and IV (three rational curves meeting in a point). But I_3 has group structure $\mathbb{Z}/3 \times k^*$ and IV has group structure $\mathbb{Z}/3 \times k$. Clearly neither of these are annihilated by the characteristic so that none of the reducible fibers of a quasi-elliptic fibration can be a triangle of three rational curves (I_3) or three rational curves meeting in a point (IV). If the fiber has intersection graph A_n, $n \geq 3$, then it is the Kodaira fiber I_{n+1} (an $(n+1)$-cycle of rational curves) which has group structure $\mathbb{Z}/(n+1) \times k^*$ which is not annihilated by $p = 2$. So a reducible fiber cannot be an $(n+1)$-cycle of rational curves.

If the reducible fiber has intersection graph E_7 or E_8, then it is the Kodaira fiber III^* or II^* respectively (see E_7 and E_8 in Figure 4.2). III^* has the group $\mathbb{Z}/2 \times k$ and II^* has the group k both of which groups are annihilated by the characteristic. Therefore there are possibly reducible fibers with intersection graph E_7 or E_8. Now Kodaira fiber IV^* (see E_6 in Figure 4.1 and replace the nodes with lines such that two lines intersect for every two nodes with an edge) is the only fiber with intersection graph E_6. It has the group $\mathbb{Z}/3 \times k$ on its smooth locus and hence cannot be realized in characteristic 2.

Finally if a reducible fiber has intersection graph D_{4+n}, $0 \leq n \leq 8$, then it is the Kodaira fiber I_n^*, $0 \leq n \leq 8$, (see D_4, D_6 and D_8, for instance, in Figure 4.2) and hence has a group structure of $(\mathbb{Z}/2)^2 \times k$, n even, or $\mathbb{Z}/4 \times k$, n odd, on its smooth locus. Therefore when n is odd, there can be no reducible fibers with intersection graph D_{4+n}. A reducible fiber can have intersection graph D_{4+n} only when n is even.

From here, one can check that the only combinations of the fibers that satisfy the numerical condition of the Euler characteristic are those shown in the statement of the Theorem. Now one can give examples that realize each of the possibilities. □

Definition 3.3.12. *Let X be a smooth rational Jacobian quasi-elliptic surface and let F be a reducible fiber on X. The weight of F, $w(F)$, is the number of components of F of multiplicity* 1.

Theorem 3.3.13. *Let X be a smooth rational Jacobian quasi-elliptic surface. Suppose that $F_1, \ldots, F_n$ are the reducible fibers on X. Let $w(F_1), \ldots, w(F_n)$ denote the weights of the fibers. Let $C_1, \ldots, C_k$ be all the smooth irreducible curves of self-intersection -1 on X. Then $k = \sqrt{w(F_1) \cdots w(F_n)}$.*

In [13], not only are the configurations of the reducible fibers of quasi-elliptic fibrations in $\mathbb{P}^2$ enumerated, examples of fibrations that achieve the configurations are also provided. From these examples of fibrations, it is easy to blow up the base points of the fibration to obtain a surface $X \to \mathbb{P}^2$ together with the configuration of -2 and -1-curves on X. From there, for such an X, it is possible to determine all blow downs of X in order to obtain configurations of 9 base points that admit a quasi-elliptic fibration whose configuration of reducible fibers corresponds to the configuration of -2-curves on X.

3.4 Unexpected Curves and Quasi-Elliptic Fibrations

In this section, we shall situate our discussion of 0-dimensional subschemes Z that admit unexpected cubics within the context of quasi-elliptic fibrations. This point of view will allow us to see, almost immediately, that if Z is a reduced 0-dimensional subscheme that admits an unexpected cubic, then Z, in fact, contains the Fano plane. We go further to observe that if Z is not reduced, then Z forms part of the locus of base points of a quasi-elliptic fibration. We present instances of such occurrences.

Lemma 3.4.1. *Let X be a smooth rational surface with anticanonical class $-K_X = 3L - E_1 - \cdots - E_n$, $n \leq 8$. Then $-K_X$ is effective. In particular, $h^0(X, -K_X) = 9 - n + 1 + h^1(X, -K_X)$.*

Proof. By Serre duality, $h^2(X, -K_X) = h^0(X, 2K_X)$. Since $2K_X \cdot L = -6 < 0$, $2K_X$ is not effective. So $h^0(X, 2K_X) = 0$ and hence $h^2(X, -K_X) = 0$. By Riemann-Roch, $h^0(X, -K_X) = \frac{(-K_X)^2 + (-K_X)(-K_X)}{2} + 1 + h^1(X, -K_X) = 9 - n + 1 + h^1(X, -K_X)$. Since $h^1(X, -K_X) \geq 0$ and $n \leq 8$, $h^0(X, -K_X) \geq 1$. In particular $-K_X$ is effective. □

Lemma 3.4.2. *Let X be a smooth rational surface that is the blow up of $\mathbb{P}^2$ at the points $p_1, \ldots, p_n$, $n \leq 8$. Suppose that no more than 4 of the points lie on a line and no more than 6 of the points lie on a conic. Then $-K_X = 3L - E_1 - \cdots - E_n$ is nef.*

Proof. If $-K_X$ is not nef, then there is an effective and irreducible divisor D such that $(-K_X) \cdot D < 0$. This implies that for some irreducible component C of $-K_X$, $C \cdot D < 0$. By Bézout's theorem, $C = D$ and $D^2 < 0$. Since $-K_X = 3L - E_1 - \cdots - E_n$, $D \cdot L = 1$ or $D \cdot L = 2$. Note that $D \cdot L \neq 3$ since if it were, then D would be the proper transform of a cubic through all of the points with a singularity of multiplicity 2 at one of the points. Such a cubic does not meet $-K_X$ negatively however. Therefore D is the proper transform of a line through 2 or 3 of the points or the proper transform of a conic through 5 or 6 of the points. But this is impossible since $(-K_X) \cdot (L - E_i - E_j) = 1$, $(-K_X) \cdot (L - E_i - E_j - E_k) = 0$, $(-K_X) \cdot (2L - E_{i_1} - \cdots - E_{i_5}) = 1$ and $(-K_X) \cdot (2L - E_{i_1} - \cdots - E_{i_6}) = 0$. This contradicts our initial choice of D. So $-K_X$ is nef. $\square$

Lemma 3.4.3. *Let X be a smooth projective rational surface with $K_X^2 > 0$. Let F be a nef divisor on X. Then $h^2(X, F) = 0$ and similarly $h^1(X, F) = 0$.*

Proof. See Theorem III.1 in [23]. $\square$

Corollary 3.4.4. *Let X be a smooth rational surface that is the blow up of $\mathbb{P}^2$ at exactly 7 points with anticanonical class $-K_X = 3L - E_1 - \cdots - E_7$. Suppose that no more than 3 of the points lie on a line and no more than 6 of the points lie on a conic. Then $h^0(X, -K_X) = 3$.*

Proof. By Lemma 3.4.2, $-K_X$ is a nef class. By Lemma 3.4.3, $h^1(X, -K_X) = 0$. Since $n = 7$, we get from Lemma 3.4.1 that $h^0(X, -K_X) = (9 - 7) + 1 = 3$. $\square$

Lemma 3.4.5. *Consider the set $S = \{p_1, \ldots, p_7\} \subseteq \mathbb{P}^2$ over a field of characteristic 2. If there are 7 lines, $L_1, \ldots, L_7$, each containing exactly 3 of the points of S such that any pair of the lines meet at one of the points of S, then the seven lines have the Fano plane configuration.*

Proof. We may take $L_1 = \{p_1, p_2, p_3\}$. Now L_2 meets L_1 at one point and it does so at one of the points of S so we might as well take that point to be p_1. Then $L_2 = \{p_1, p_4, p_5\}$. The points p_6 and p_7 determine a line, call it L_3, which must meet both L_1 and L_2 at one of the points of S. L_3 cannot meet L_1 and L_2 at different points since otherwise it would contain more than 3 of the points of S. So $L_3 = \{p_1, p_6, p_7\}$.

Note that p_2 and p_4 determine a line, say L_4, and that line must meet L_3. Since L_4 already meets L_1 and L_2, it cannot meet L_3 at p_1. Nothing we have said so far distinguishes p_6 from p_7 so we may assume that L_4 meets L_3 at p_7. Hence $L_4 = \{p_2, p_4, p_7\}$. Similarly p_2 and p_5 determine a line L_5 which already meets L_1 at p_2, L_2 at p_5 and L_4 at p_2. It could, potentially, meet L_3 at either 6 or 7 but meeting L_3 at 7 would force it to meet L_4 twice so it must meet L_3 at 6. It follows that $L_5 = \{p_2, p_5, p_6\}$.

The points p_3 and p_4 give a line L_6. From the two forerunning paragraphs, L_6 already meets L_1, L_2 and L_4 at the points p_3, p_4 and p_4 again respectively. It has to meet L_3 and L_5 at points of S and the only choice is at the unique intersection of L_3 and L_5 if L_6 is to contain only 4 points of S. Hence $L_6 = \{p_3, p_4, p_6\}$. Finally p_3 and p_5 give a line L_7 which already meets the lines L_1, L_2, L_5 and L_6 at the points p_3 and p_5. The two remaining lines are L_3 and L_4. Therefore L_7 must contain p_7 to give that $L_7 = \{p_3, p_5, p_7\}$.

One can check that every point is now collinear with every other point via one of the lines $L_1, \ldots, L_7$ and so there are no more lines to be had. One can also check that $L_1, \ldots, L_7$ have the Fano plane configuration. $\square$

Lemma 3.4.6 (Four Point Lemma)**.** *Let k be an algebraically closed field and let $\{p_1, p_2, p_3, p_4\}$ and $\{q_1, q_2, q_3, q_4\}$ be two sets of general points in $\mathbb{P}^2_k$. There exists a unique projective map Φ taking p_i to q_i, $1 \leq i \leq 4$.*

Proof. See Lemma 11.2 in [20] and the discussion immediately after it. $\square$

Lemma 3.4.7. *Let k be an algebraically closed field of characteristic 2. If C is a cuspidal cubic in $\mathbb{P}^2_k$, then C is projectively equivalent to the normal form $x^3 + y^2z = 0$.*

Proof. Let C be a cuspidal cubic in $\mathbb{P}^2_k$. C is a linear combination of the monomials x^3, y^3, z^3, x^2y, x^2z, y^2x, y^2z, z^2x, z^2y, xyz and so for coefficients in k, $C = ax^3 + by^3 + cz^3 + dx^2y + ex^2z + fy^2x + gy^2z + hz^2x + iz^2y + jxyz$. Denote the cusp of C by p. By Lemma 3.4.6, we may suppose that $p = [0 : 0 : 1]$ and that the (repeated) tangent to C at p is $x = 0$. By Lemma 3.4.6 again, we may suppose that C passes through the point $q = [1 : 0 : 0]$ with tangent $z = 0$.

Since C contains p, the coefficient of z^3 must vanish. Since C has a singularity at p, we take derivatives to see that the coefficients of xz^2 and z^2y also vanish. Now $C = ax^3 + by^3 + dx^2y + ex^2z + fy^2x + gy^2z + jxyz$. Since $x^2 = 0$ is a reduced conic tangent to C at p such that each of its components touches C to order 3 at p, we have that the coefficients of y^2z and xyz must vanish. Hence $C = ax^3 + by^3 + dx^2y + ex^2z + fy^2x$.

C also contains the point q so that the coefficient of x^3 vanishes. Now C has a tangent at $z = 0$ which might either have contact order 2 if it is an ordinary tangent or contact order 3 if it is a flex. In the case that it has contact order 2, the coefficient of x^2y vanishes and we have that $C = ex^2z + fxy^2 + by^3$. In the case that the tangent has contact order 3, the coefficient of y^2x vanishes and we have that $C = ex^2z + by^3$.

Consider $C = ex^2z + fxy^2 + by^3$. If e is 0, then C reduces so e can't be 0. Similarly, b cannot be 0. If f is 0, then we are in the case that $C = ex^2z + by^3$ so we may assume $f \neq 0$. Now by scaling, we may take e, f and b to be 1. If $C = ex^2z + by^3$, then e and b cannot be trivial since otherwise C will reduce. Again, by scaling, we may take e and b to be 1. Note that $x^2z + y^3$ is a cubic with a cusp at $[0:0:1]$ and tangent $x = 0$ together with a flex at $[1:0:0]$ and tangent $z = 0$. Similarly $x^2z + xy^2 + y^3$ is a cubic with a cusp at $[0:0:1]$, a tangent $x = 0$ and a flex point $[1:1:0]$ with a flex tangent $x + y + z = 0$. The projective change of coordinates $x \to x$, $y \to x + y$ and $z \to y + z$ carries the cuspidal cubic $x^2z + xy^2 + y^3$ to the cuspidal cubic $x^2z + y^3$. $\square$

Theorem 3.4.8. *Let $Z \subseteq \mathbb{P}^2_k$, k algebraically closed, be a reduced 0-dimensional subscheme, that admits an unexpected cubic. Then the char $k = 2$ and Z is contained in the base points of some quasi-elliptic fibration. Moreover Z is the finite projective plane of order 2, i.e. the Fano plane.*

Proof. Let Z be as in the statement above. Then by Theorem 3.2.20, char $k = 2$, and by Corollary 3.2.11, $|Z| = 7$. Blow up the points of Z to obtain a surface X'. Then by Corollary 3.4.4, $h^0(\mathbb{P}^2, \mathcal{I}_{Z'}(3)) = h^0(X', -K_{X'}) = 3$. By lemma 3.2.9, the general section of $H^0(X', -K_{X'})$ is irreducible. Let D and D' be two such irreducible sections. Then we obtain a pencil $\mathcal{P} = \{mD + nD' | [m:n] \in \mathbb{P}^1\}$. Pick a $C \in H^0(X', -K_{X'})$ with a cusp at $p \in X'$. Then there is a $\tilde{D} \in \mathcal{P}$ such that $\tilde{D}$ contains p and is smooth at p.

Now C and $\tilde{D}$ form a pencil whose only base points are $p_8 = p$ and an infinitely near point p_9. Blow up the two base points to obtain a surface X fibered by genus 1 curves, the proper transforms to X of the curves of the pencil with basis C and $\tilde{D}$. By Lemma 3.2.9, every cubic through Z is singular and the general cubic is reduced and irreducible. It follows that the general section of $H^0(X, -K_X)$ is ultimately the proper transform of a cuspidal cubic. Hence the fibration is quasi-elliptic. Hence the 9 points that were blown up to obtain X are the base points of a quasi-elliptic fibration in $\mathbb{P}^2$. In particular Z consists of 7 of the 9 base points of a quasi-elliptic fibration.

If Z is reduced, then all of the points of Z are distinct in $\mathbb{P}^2$. We want to show that Z is the Fano plane by investigating the reducible fibers of X. Note that p_9 is infinitely near p_8 and p_8 is not infinitely near any other point since it is general. Theorem 3.3.11 gives us the numbers and arrangements of the possible reducible fibers on X, namely, $A_1^{\oplus 8}$, $A_1^{\oplus 4} \oplus D_4$, $A_1^{\oplus 2} \oplus D_6$, $A_1 \oplus E_7$, $D_4^{\oplus 2}$, E_8 and D_8. Each of the fibers maps to a cubic in $\mathbb{P}^2$ after we contract 9 exceptional curves in such a way that the resulting surface is $\mathbb{P}^2$. Since a cubic in $\mathbb{P}^2$ can have at most three components, if a fiber has more than four components, then some of its components must blow down to points in $\mathbb{P}^2$.

The fiber types with more than 4 components are D_4, D_6, E_7, E_8 or D_8. The point p_8 can never be infinitely near another point since it is general. So if we want 7 distinct points and the general double point p_8 from each of the fiber types mentioned at the beginning of the paragraph, then only one component of these fibers can blow down to a point and in fact, it must blow down to p_8 with p_9 infinitely near. Then there are at least three components left and since the fiber must map to a cubic, we end up with lines. But these lines came from -2-curves and hence p_8 would be collinear with two other points (neither of them p_9) and hence p_8 would not be general. Thus X cannot have any fiber types with more than four components.

So if Z is reduced, then the only possible reducible fiber type of X is $A_1^{\oplus 8}$. Since each of the fiber types here has two components, none of the components of any of the A_1 has to necessarily blow down to points. Since we want a general double point p_8 with p_9 infinitely near, blow down an exceptional curve onto one of the components of one of the A_1 and contract the resulting exceptional curve again to a point, call the point p_8. Then that A_1 blows down to a cuspidal cubic with a cusp at p_8 and p_9 infinitely near. There are seven A_1's left. Blow them down without creating any more infinitely near points. The only way one gets that is when one blows down each of the A_1's to a reducible cubic consisting of a line through 3 of the distinct points and a conic through 4 of the distinct points passing through p_8 with a tangent direction p_9.

By Lemma 3.4.7, $\mathbb{P}^2$ has a unique cuspidal cubic up to choice of coordinates. So now in $\mathbb{P}^2$, we want to blow up 7 points on this cuspidal cubic and get 7 lines each of which goes through 3 of the 7 points. These lines are members of disjoint fibers, so they can't meet except at the 7 points we're blowing up, and each has to go through 3 of the 7 points. By Lemma 3.4.5, the only way to achieve this is when the 7 points have the Fano plane configuration. Therefore up to coordinates, Z is the Fano plane. $\square$

3.5 Examples of Subschemes Admitting Unexpected Curves

Now we've seen that given a reduced 0-dimensional subscheme Z such that Z admits an unexpected cubic, $|Z| = 7$, and Z is contained in the intersection locus of some quasi-elliptic fibration. We now find other instances of 7 points that admit unexpected cubics other than the Fano plane. Since we show above that the Fano plane is the only instance in which all the points are reduced, the instances we find will all have non-reduced points (i.e., some of the points will be infinitely near). We work out a few examples in what follows. We do the first in some detail and for the remainder, we provide enough information for the interested reader to work out.

Definition 3.5.1. *Let $Z = \{p_1, \dots, p_r\}$ be essentially distinct (see Definition 3.2.18) and thus possibly infinitely near points in $\mathbb{P}^2$. If p_j is infinitely near p_i, then $j > i$ in our labeling. Let $p \notin Z$ be a general point in $\mathbb{P}^2$. Let $t > 1$ be an integer and tp be a fat point of multiplicity t supported at p. Blow up the points $p_1, \dots, p_r$ and p, in order, to obtain a surface X with L the total transform of a line from $\mathbb{P}^2$ to X and $E_1, \dots, E_r, E_p$, the exceptional curves corresponding to the blow up. Then Z admits an unexpected curve of degree $t+1$ if $h^0(X, (t+1)L - E_1 - \cdots - E_r - tE_p) > \max\{0, h^0(X, (t+1)L - E_1 - \cdots - E_r) - \binom{t+1}{2}\}$.*

So the definitions of unexpected curves carry over to schemes $Z = p_1 + \cdots + p_r$ where the points p_i are merely essentially distinct.

Lemma 3.5.2. *Let X be a smooth rational surface that is elliptic or quasi-elliptic. Let C be the generic fiber of X and let N be a component of a reducible fiber. Then the canonical homomorphism $\Phi : Pic(X) \to Pic(C)$ satisfies $\Phi(N) = 0$.*

Proof. Apply the Pic functor to the natural inclusion $C \subseteq X$ to get the map $\Phi : \mathrm{Pic}(X) \to \mathrm{Pic}(C)$ which is just restriction. We describe the restriction in a little more detail. Call a divisor on X that is a fiber of the fibration or a component of a fiber, a vertical fiber. If a divisor is not a fiber or a component of a fiber, refer to it as a horizontal fiber. Then for every divisor D on X, we write $D = D^v + D^h$ where D^v is the component of D that is vertical and D^h is the component of D that is horizontal. Then D^h and C have a nontrivial intersection supported at a finite number of points on C with degree $D^h \cdot C$. Since D^v is vertical, $D^v \cdot C = 0$ and in fact $D^v \cap C$ is trivial on C. The intersection, $D^h \cap C$, is in fact a divisor, $D|_C$, known as the restriction of D to C. Define $\Phi(D) = D|_C$. Now note that $D|_C$ is the trivial divisor on C if and only if D is a vertical divisor. Hence if N is a component of a fiber, then $\Phi(N) = 0$ in $\mathrm{Pic}^0(C)$. □

Example 3.5.3. E₈ Consider the quasi-elliptic surface of type E_8 with unique blow down indicated by the labeling of the Dynkin diagram in Figure 4.6. In $\mathbb{P}^2$, the intersection locus of the fibration consists of a point of $\mathbb{P}^2$ and 8 points infinitely near that one that are the intersection of a cuspidal cubic with its flex tangent taking with multiplicity 3. Let Z' consist of the first 7 of these 9 infinitely near points. For a general point p in $\mathbb{P}^2$, we'll find an unexpected cubic containing Z' in its vanishing locus and having a cusp at p. To this end, blow down the exceptional curve E_9 onto the nodal curve it attaches to (see Figure 4.6) and contract the resulting exceptional curve E_8 to obtain a surface X' with -2-curves $S = \{E_1 - E_2, E_2 - E_3, E_3 - E_4, L - E_1 - E_2 - E_3, E_4 - E_5, E_5 - E_6, E_6 - E_7\}$.

Pick a general point p_8' on X'. Blow up the point p_8' on X' to obtain the surface X with exceptional curves $E_1, \dots, E_7, E_8'$. Consider the class $D = 3L - E_1 -$

$\cdots - E_7 - 2E_8'$. We show that $h^0(X, \mathcal{O}_X(D)) > 0$, i.e. D is effective. Recall that since Z' formed part of the base locus of the quasi-elliptic fibration that we started with so there is a fiber of that fibration containing Z' and p_8' in its vanishing locus. Let C denote the proper transform onto X of such a cuspidal cubic and note that C is a section of the anticanonical class $-K_X = 3L - E_1 - \cdots - E_7 - E_8'$. Let $\Phi : \mathrm{Pic}(X) \to \mathrm{Pic}(C)$ be the canonical restriction map. Since C is a cuspidal cubic over an algebraically closed field of characteristic 2, $\mathrm{Pic}(C)$ has pure 2-torsion. Hence $\Phi(2K^\perp) = O$ in $\mathrm{Pic}(C)$ where $K^\perp$ is the subgroup of divisors in $\mathrm{Pic}(X)$ meeting the canonical divisor in O. Since C was a fiber of a fibration and all the -2-curves in S are components of another fiber of the fibration, we have that $\Phi(\langle S \rangle) = O$, by Lemma 3.5.2, where $\langle S \rangle$ is the subgroup generated by S in $\mathrm{Pic}(X)$. Therefore $\Phi(\langle S \rangle + 2K^\perp) = O$.

Now note that $3L - E_1 - \cdots - E_7 - 2E_8' = 3(L - E_1 - E_2 - E_3) + 2(E_1 - E_2) + 4(E_2 - E_3) + 6(E_3 - E_4) + 5(E_4 - E_5) + 4(E_5 - E_6) + 3(E_6 - E_7) + 2(E_7 - E_8)$ so that $3L - E_1 - \cdots - E_7 - 2E_8' \in \langle S \rangle + 2K^\perp$ so that $\Phi(3L - E_1 - \cdots - E_7 - 2E_8') = O$. Consider the sequence of sheaves $0 \to \mathcal{O}_X(D - C) \to \mathcal{O}_X(D) \to \mathcal{O}_C \to 0$ and take cohomology to obtain $0 \to H^0(X, \mathcal{O}_X(D - C)) \to H^0(X, \mathcal{O}_X(D)) \to H^0(C, \mathcal{O}_C(D)) \to H^1(X, \mathcal{O}_X(D - C)) \to H^1(X, \mathcal{O}_X(D)) \to H^1(C, \mathcal{O}_C(D)) \to \cdots$. Since $h^0(X, \mathcal{O}_X(D - C)) = h^0(X, \mathcal{O}_X(-E_8')) = 0$ and $h^2(X, \mathcal{O}_X(D - C)) = h^0(X, \mathcal{O}_X(-D)) = 0$ by duality, we have by Riemann-Roch that $h^1(X, \mathcal{O}_X(D - C)) = 0$. Therefore $H^0(X, \mathcal{O}_X(D))$ surjects onto $H^0(C, \mathcal{O}_C(D))$. Since $\mathcal{O}_C(D)$ is the line bundle of $\Phi(D) = O$ which is trivial, $\mathcal{O}_C(D)$ is the trivial bundle which implies $h^0(C, \mathcal{O}_C(D)) > 0$ and hence $h^0(X, \mathcal{O}_X(D)) > 0$. Therefore D is effective. By Corollary 3.4.4, $h^0(X, 3L - E_1 - \cdots - E_7) = 3$ and hence D is an unexpected cubic.

Example 3.5.4. $\mathbf{D_6 \oplus A_1^{\oplus 2}}$

1. Blow down E_9 onto $E_8 - E_9$ and then blow down E_8 in the first diagram of $D_6 \oplus A_1^{\oplus 2}$. Then the set of -2-curves on the resulting surface is $S = \{E_4 - E_5, E_5 - E_6, E_6 - E_7, E_1 - E_3, L - E_2 - E_4 - E_5, 2L - E_1 - E_3 - E_4 - E_5 - E_6 - E_7, L - E_1 - E_3 - E_2\}$. Now for a general point p'_8 with blow up E'_8, take $3L - E_1 - \cdots - E_7 - 2E'_8 = (E_1 - E_3) + (L - E_1 - E_2 - E_3) + (2L - E_1 - E_3 - E_4 - E_5 - E_6 - E_7) + 2(E_3 - E'_8)$. Just as in the previous example, one checks that $3L - E_1 - \cdots - E_7 - 2E'_8$ is effective and that $h^0(X, 3L - E_1 - \cdots - E_7) = 3$.

2. Blow down E_7 onto $E_6 - E_7$ and then blow down E_6 in the first diagram of $D_6 \oplus A_1^{\oplus 2}$. Then $S = \{L - E_2 - E_4 - E_5, L - E_1 - E_2 - E_3, E_4 - E_5, L - E_1 - E_4 - E_8, E_1 - E_3, E_8 - E_9, L - E_2 - E_8 - E_9\}$ and $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_8 - E_9 - 2E'_6 = (L - E_1 - E_2 - E_3) + (L - E_2 - E_8 - E_9) + (L - E_2 - E_4 - E_5) + 2(E_2 - E'_6)$.

3. Blow down E_9 onto $E_8 - E_9$ and then blow down E_8 in the second diagram of $D_6 \oplus A_1^{\oplus 2}$. Then $S = \{L - E_3 - E_4 - E_7, E_3 - E_4, E_2 - E_5, E_5 - E_6, L - E_1 - E_2 - E_7, 2L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6, E_1 - E_2\}$ and $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E'_8 = (L - E_1 - E_2 - E_7) + (2L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6) + (E_1 - E_2) + 2(E_2 - E'_8)$.

4. In the second diagram of $D_6 \oplus A_1^{\oplus 2}$, blow down E_6 onto the nodal curve $E_5 - E_6$ and then blow down E_5. Then $S = \{E_1 - E_2, L - E_1 - E_2 - E_7, E_7 -$

$E_8, E_8 - E_9, L - E_3 - E_4 - E_7, E_3 - E_4, L - E_7 - E_8 - E_9\}$. Then $3L - E_1 - E_2 - E_3 - E_4 - E_7 - E_8 - E_9 - 2E_5' = (L - E_7 - E_8 - E_9) + (L - E_3 - E_4 - E_7) + (L - E_1 - E_2 - E_7) + 2(E_7 - E_5')$.

Example 3.5.5. $\mathbf{D_4 \oplus A_1^{\oplus 4}}$

1. In the first diagram of $D_4 \oplus A_1^{\oplus 4}$, blow down E_9 onto $E_8 - E_9$ and then blow down E_8. Then $S = \{E_6 - E_7, L - E_5 - E_6 - E_7, 2L - E_1 - E_2 - E_3 - E_4 - E_6 - E_7, E_1 - E_2, E_3 - E_4, L - E_1 - E_2 - E_5, L - E_3 - E_4 - E_5\}$ and $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = (2L - E_1 - E_2 - E_3 - E_4 - E_6 - E_7) + (L - E_5 - E_6 - E_7) + (E_6 - E_7) + 2(E_7 - E_8')$.

2. In the first diagram of $D_4 \oplus A_1^{\oplus 4}$, blow down E_2 onto $E_1 - E_2$ and then blow down E_1. Then $S = \{E_8 - E_9, E_7 - E_8, E_6 - E_7, L - E_5 - E_6 - E_7, E_3 - E_4, 2L - E_3 - E_4 - E_6 - E_7 - E_8 - E_9, L - E_3 - E_4 - E_5\}$ and $3L - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - E_9 - 2E_2' = (L - E_5 - E_6 - E_7) + (2L - E_3 - E_4 - E_6 - E_7 - E_8 - E_9) + (E_6 - E_7) + 2(E_7 - E_2')$.

3. In the second diagram of $D_4 \oplus A_1^{\oplus 4}$, blow down E_3 onto $E_2 - E_3$ and then blow down E_2. Now $S = \{E_6 - E_7, E_8 - E_9, L - E_4 - E_6 - E_8, E_4 - E_5, L - E_1 - E_8 - E_9, L - E_1 - E_4 - E_5, L - E_1 - E_6 - E_7\}$ and $3L - E_1 - E_2 - E_5 - E_6 - E_7 - E_8 - E_9 - 2E_2' = 2(E_1 - E_2') + (L - E_1 - E_6 - E_7) + (L - E_1 - E_4 - E_5) + (L - E_1 - E_8 - E_9)$.

4. In the second diagram of $D_4 \oplus A_1^{\oplus 4}$, blow down E_9 onto $E_8 - E_9$ and then blow down E_8. Then $S = \{E_6 - E_7, E_4 - E_5, L - E_1 - E_2 - E_3, 2L - E_2 -$

$E_3 - E_4 - E_5 - E_6 - E_7, L - E_1 - E_4 - E_5, L - E_1 - E_6 - E_7, E_2 - E_3\}$ and $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = 2(E_1 - E_8') + (L - E_1 - E_2 - E_3) + (L - E_1 - E_4 - E_5) + (L - E_1 - E_6 - E_7)$.

5. In the third diagram of $D_4 \oplus A_1^{\oplus 4}$, blow down E_9 onto $E_8 - E_9$ and blow down E_8. Then $S = \{L - E_5 - E_6 - E_7, L - E_3 - E_4 - E_7, L - E_1 - E_2 - E_7, E_1 - E_2, E_3 - E_4, E_5 - E_6, 2L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6\}$ and $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = 2(E_6 - E_8') + (L - E_5 - E_6 - E_7) + (E_5 - E_6) + (2L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6)$.

6. In the third diagram of $D_4 \oplus A_1^{\oplus 4}$, blow down E_2 onto $E_1 - E_2$ and then blow down E_1. Then $S = \{L - E_5 - E_6 - E_7, E_8 - E_9, E_7 - E_8, L - E_3 - E_4 - E_7, E_3 - E_4, E_5 - E_6, L - E_7 - E_8 - E_9\}$ and $3L - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - E_9 - 2E_1' = 2(E_7 - E_1') + (L - E_5 - E_6 - E_7) + (L - E_3 - E_4 - E_7) + (L - E_7 - E_8 - E_9)$.

Example 3.5.6. $\mathbf{D_4^{\oplus 2}}$

1. Blow down E_9 onto $E_8 - E_9$ and then blow down E_8 in the first diagram of $D_4^{\oplus 2}$. Then $S = \{E_6 - E_7, E_4 - E_5, L - E_1 - E_2 - E_3, L - E_1 - E_4 - E_5, E_1 - E_2, L - E_1 - E_6 - E_7, E_2 - E_3\}$ and $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = 2(E_1 - E_8') + (L - E_1 - E_2 - E_3) + (L - E_1 - E_4 - E_5) + (L - E_1 - E_6 - E_7)$.

2. Blow down E_3 onto $E_2 - E_3$ in the first diagram of $D_4^{\oplus 2}$ and then blow down E_2. Then $S = \{E_6 - E_7, E_4 - E_5, L - E_4 - E_6 - E_8, E_8 - E_9, L - E_1 - E_8 -$

$E_9, L - E_1 - E_6 - E_7, L - E_1 - E_4 - E_5\}$ and $3L - E_1 - E_4 - E_5 - E_6 - E_7 - E_8 - E_9 - 2E_2' = 2(E_1 - E_2') + (L - E_1 - E_8 - E_9) + (L - E_1 - E_6 - E_7) + (L - E_1 - E_4 - E_5)$.

3. In the second diagram of $D_4^{\oplus 2}$, blow down E_9 onto $E_8 - E_9$ and then blow down E_8. Then $S = \{E_6 - E_7, L - E_1 - E_6 - E_7, 2L - E_2 - E_3 - E_4 - E_5 - E_6 - E_7, E_2 - E_3, E_4 - E_5, E_3 - E_4, L - E_1 - E_2 - E_3\}$. Now $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = 2(E_7 - E_8') + (L - E_1 - E_6 - E_7) + (E_6 - E_7) + (2L - E_2 - E_3 - E_4 - E_5 - E_6 - E_7)$.

Example 3.5.7. $\mathbf{E_7 \oplus A_1}$

1. In the first diagram of $E_7 \oplus A_1$, blow down E_3 onto $E_2 - E_3$ and then blow down E_3. We find that $S = \{L - E_1 - E_4 - E_5, E_5 - E_6, E_4 - E_5, E_6 - E_7, E_7 - E_8, E_8 - E_9, 2L - E_4 - E_5 - E_6 - E_7 - E_8 - E_9\}$ and that $3L - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - E_9 - 2E_2' = (2L - E_4 - E_5 - E_6 - E_7 - E_8 - E_9) + (L - E_1 - E_4 - E_5) + (E_4 - E_5) + 2(E_5 - E_2')$.

2. Blow down E_9 onto $E_8 - E_9$ in the first diagram of $E_8 - E_9$ and then blow down E_8. We now get that $S = \{E_6 - E_7, E_5 - E_6, E_4 - E_5, L - E_1 - E_4 - E_5, E_1 - E_2, E_2 - E_3, L - E_1 - E_2 - E_3\}$ and $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = 2(E_7 - E_8') + 3(L - E_1 - E_4 - E_5) + 4(E_5 - E_6) + 3(E_6 - E_7) + 2(E_4 - E_5) + 2(E_1 - E_2) + (E_2 - E_3)$.

3. Blow down E_9 onto $E_8 - E_9$ in the second diagram of $E_7 \oplus A_1$ and then blow down E_8. We see that $S = \{E_6 - E_7, E_5 - E_6, L - E_3 - E_4 - E_5, E_4 - E_5, E_3 -$

$E_4, L - E_1 - E_2 - E_3, E_1 - E_2\}$ and finally that $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = 2(E_7 - E_8') + 2(L - E_3 - E_4 - E_5) + (L - E_1 - E_2 - E_3) + 2(E_3 - E_4) + 3(E_4 - E_5) + 4(E_5 - E_6) + 3(E_6 - E_7)$

4. Consider the second diagram of $E_7 \oplus A_1$. Blow down E_2 onto $E_1 - E_2$ and then blow down the resulting exceptional curve E_1. Then $S = \{E_8 - E_9, E_7 - E_8, E_6 - E_7, L - E_3 - E_4 - E_5, E_3 - E_4, E_4 - E_5, E_5 - E_6\}$. We compute that $2(E_9 - E_1') + 3(L - E_3 - E_4 - E_5) + 6(E_5 - E_6) + 5(E_6 - E_7) + 4(E_7 - E_8) + 3(E_8 - E_9) + 4(E_4 - E_5) + 2(E_3 - E_4) = 3L - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - E_9 - 2E_1'$.

Example 3.5.8. $\mathbf{D_8}$

1. In the first diagram of D_8, blow down E_9 onto $E_8 - E_9$ and then blow down E_8. Then $S = \{E_6 - E_7, E_5 - E_6, E_4 - E_5, E_3 - E_4, E_2 - E_3, L - E_1 - E_2 - E_3, 2L - E_2 - E_3 - E_4 - E_5 - E_6 - E_7\}$ with $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = (L - E_1 - E_2 - E_3) + (2L - E_2 - E_3 - E_4 - E_5 - E_6 - E_7) + (E_2 - E_3) + 2(E_3 - E_8')$.

2. In the second diagram of D_8, blow down E_9 onto $E_8 - E_9$ and then blow down E_8. Then $S = \{E_1 - E_7, L - E_1 - E_2 - E_7, E_2 - E_3, E_3 - E_4, E_4 - E_5, E_5 - E_6, L - E_2 - E_3 - E_4\}$. Finally we compute that $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = 2(L - E_1 - E_2 - E_7) + (L - E_2 - E_3 - E_4) + (E_5 - E_6) + 2(E_4 - E_5) + 2(E_3 - E_4) + 2(E_2 - E_3) + (E_1 - E_7) + 2(E_7 - E_8')$.

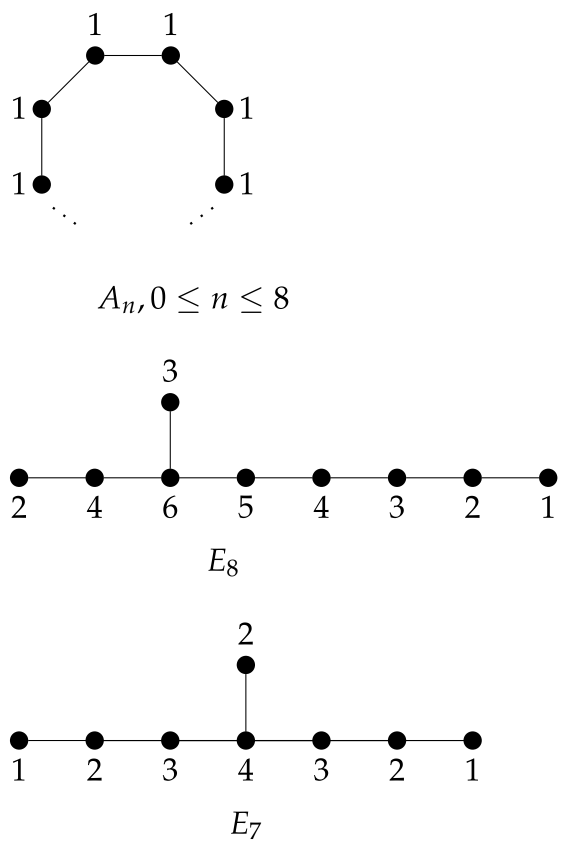
Example 3.5.9. $\mathbf{A_1^{\oplus 8}}$

1. Blow down E_8 onto $E_8 - E_9$ in the first diagram of $A_1^{\oplus 8}$ and then blow down E_8. Then $S = \{E_6 - E_7, E_4 - E_5, E_2 - E_3, 2L - E_2 - E_3 - E_4 - E_5 - E_6 - E_7, L - E_1 - E_6 - E_7, L - E_1 - E_4 - E_5, L - E_1 - E_2 - E_3\}$ and we compute that $(L - E_1 - E_6 - E_7) + (L - E_1 - E_4 - E_5) + (L - E_1 - E_2 - E_3) + 2(E_1 - E_8') = 3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8'$.

2. In the second diagram of $A_1^{\oplus 8}$, blow E_9 onto $E_8 - E_9$ and then blow down E_8. Then $S = \{L - E_1 - E_2 - E_3, L - E_1 - E_4 - E_5, L - E_1 - E_6 - E_7, L - E_2 - E_5 - E_6, L - E_2 - E_4 - E_7, L - E_3 - E_4 - E_6, L - E_3 - E_5 - E_7\}$. Now note that $3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - 2E_8' = (L - E_1 - E_2 - E_3) + (L - E_1 - E_4 - E_5) + (L - E_1 - E_6 - E_7) + 2(E_1 - E_8')$.

Chapter 4

Figures



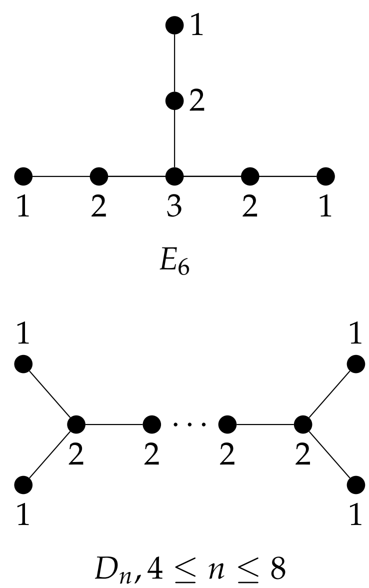
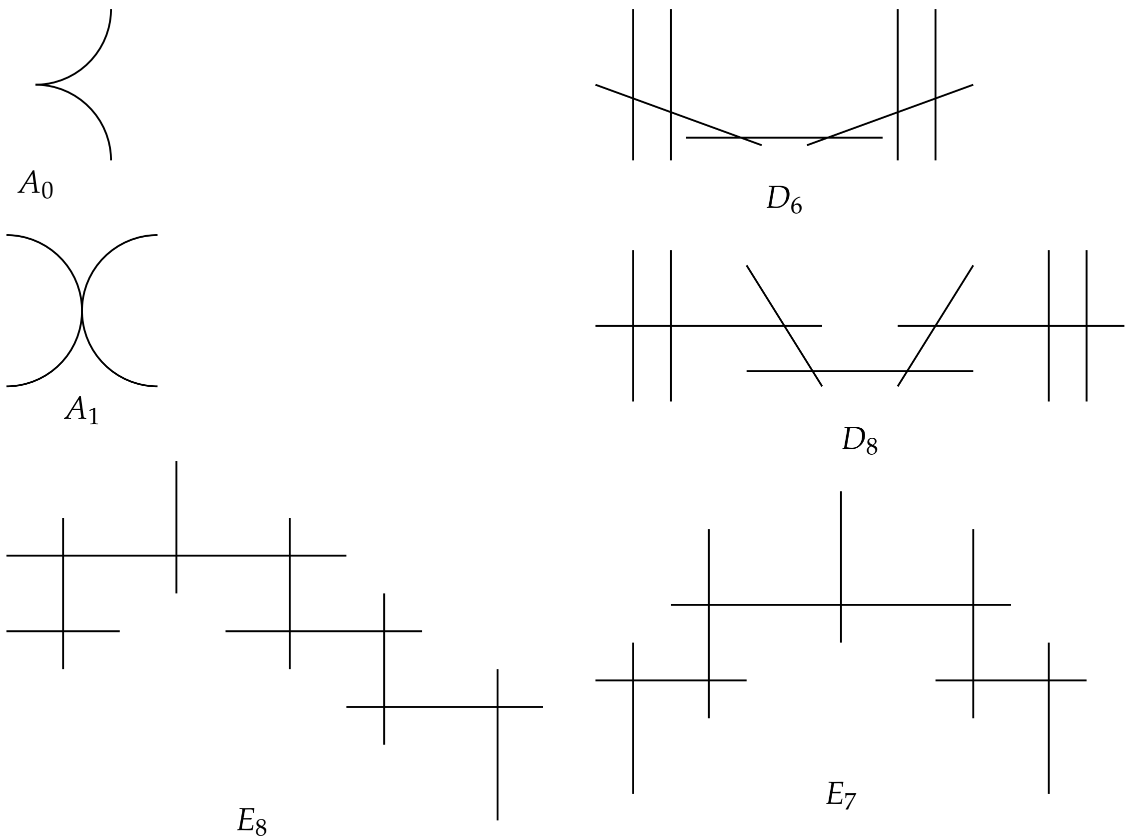


Figure 4.1: $\mathbb{Z}$-modules with connected basis $\{C_1, \ldots, C_m\}$ that give the possible reducible fibers of a rational Jacobian minimal quasi-elliptic fibration. D_n and A_n have $n+1$ vertices.



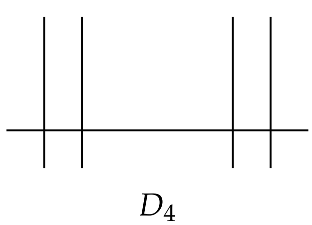
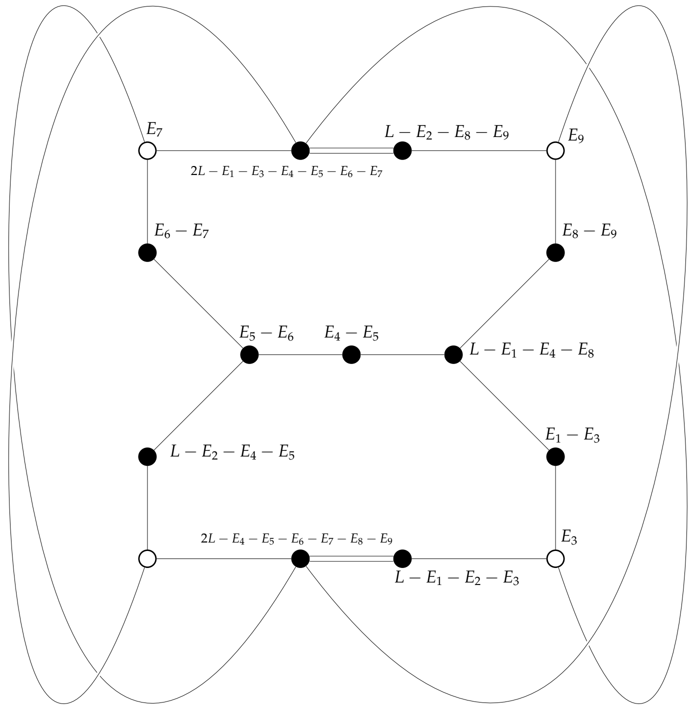


Figure 4.2: The various configurations of -2-curves that form a singular fiber type in a Jacobian quasi-elliptic fibration over a field of characteristic 2



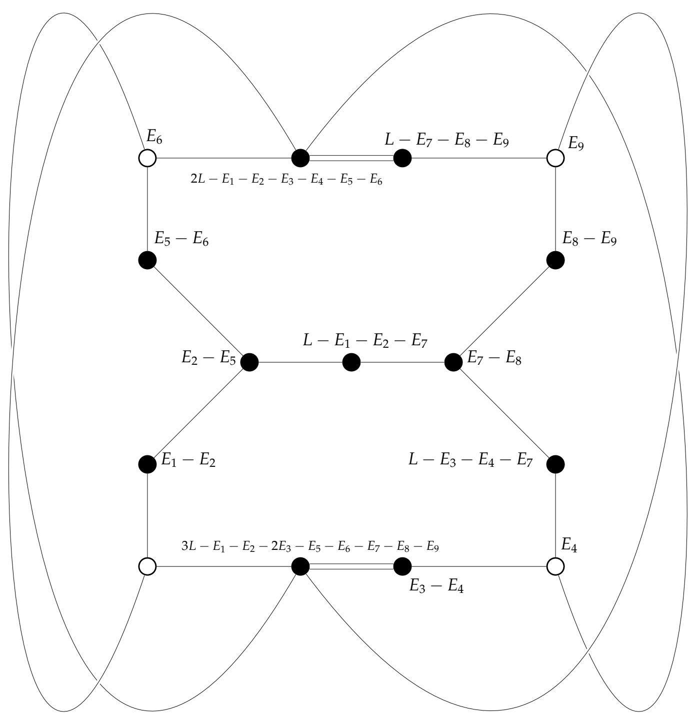


Figure 4.3: The blowings down of $D_6 \oplus A_1^{\oplus 2}$

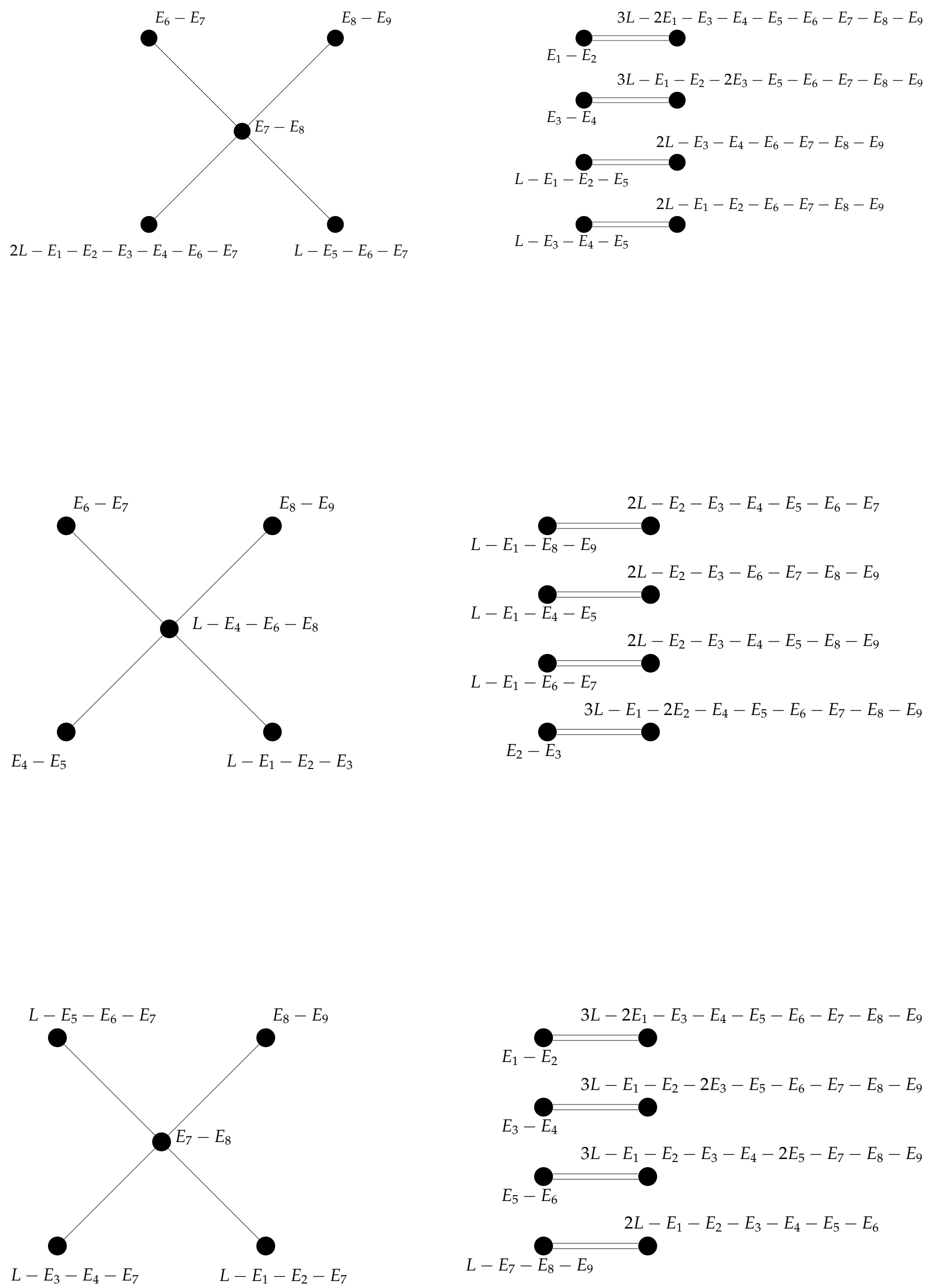


Figure 4.4: The blowings down of $D_4 \oplus A_1^{\oplus 4}$

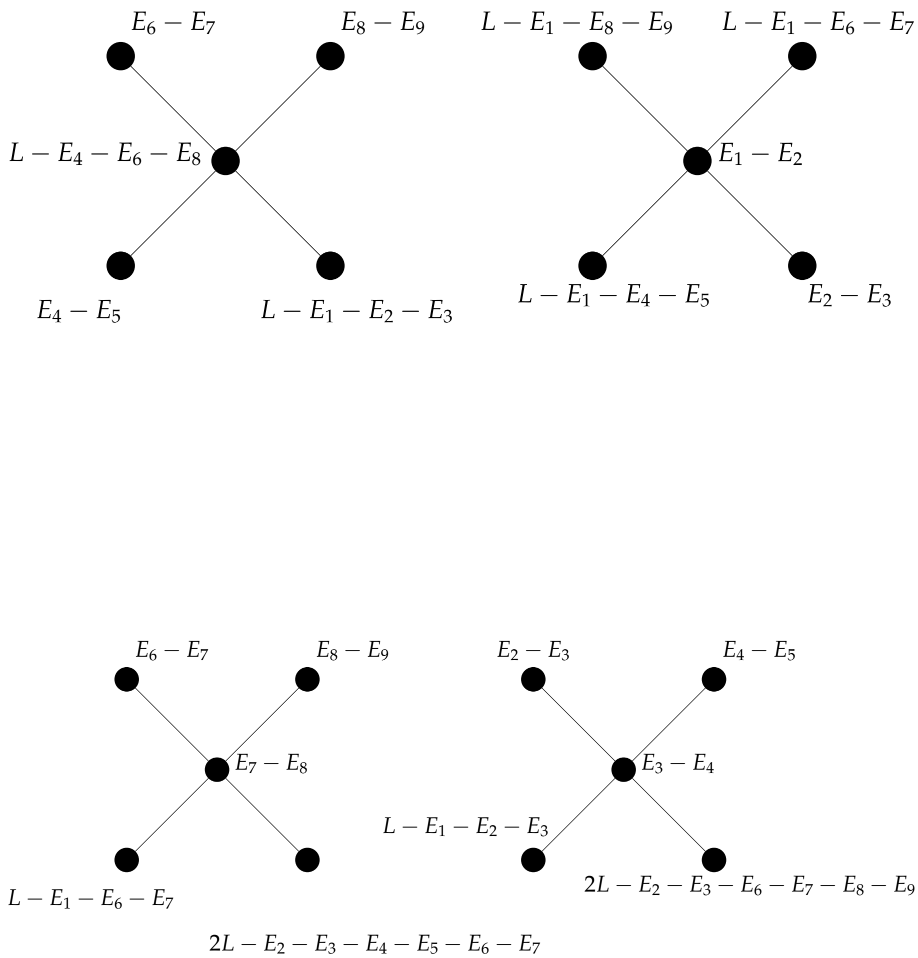


Figure 4.5: The blowings down of $D_4^{\oplus 2}$

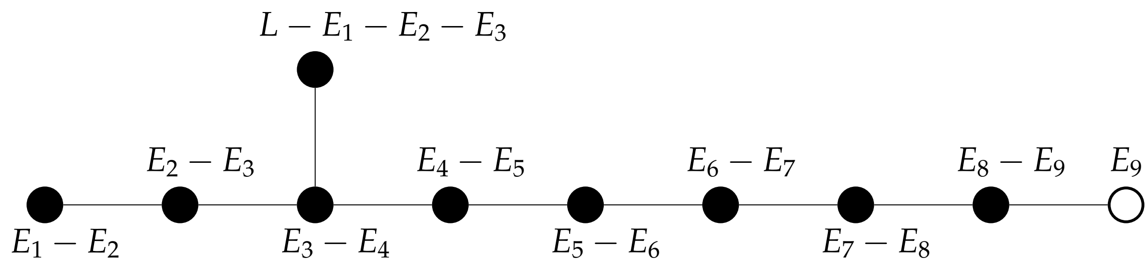


Figure 4.6: The unique blowing down of E_8

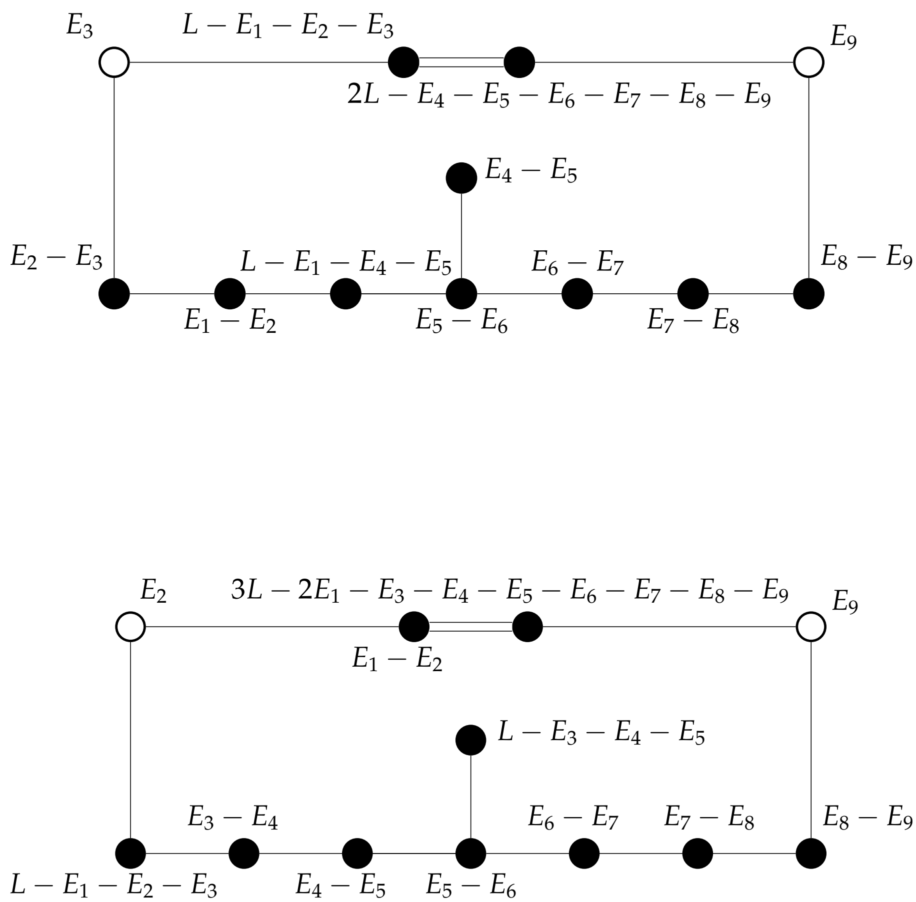


Figure 4.7: The blowings down of $E_7 \oplus A_1$

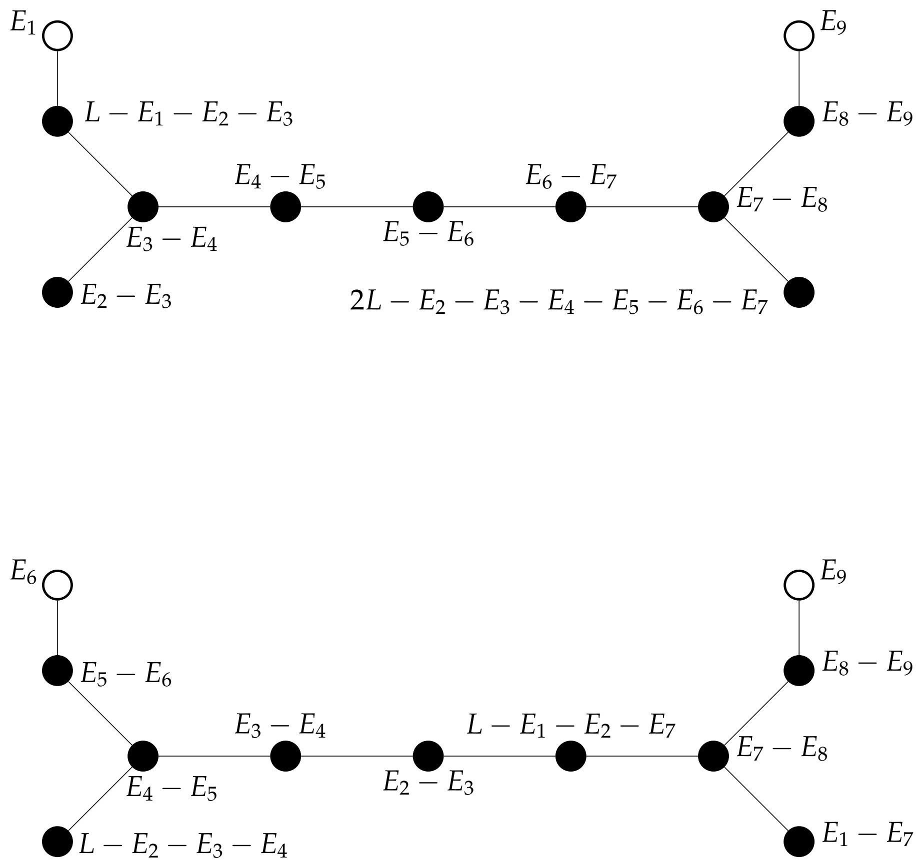


Figure 4.8: The blowings down of D_8

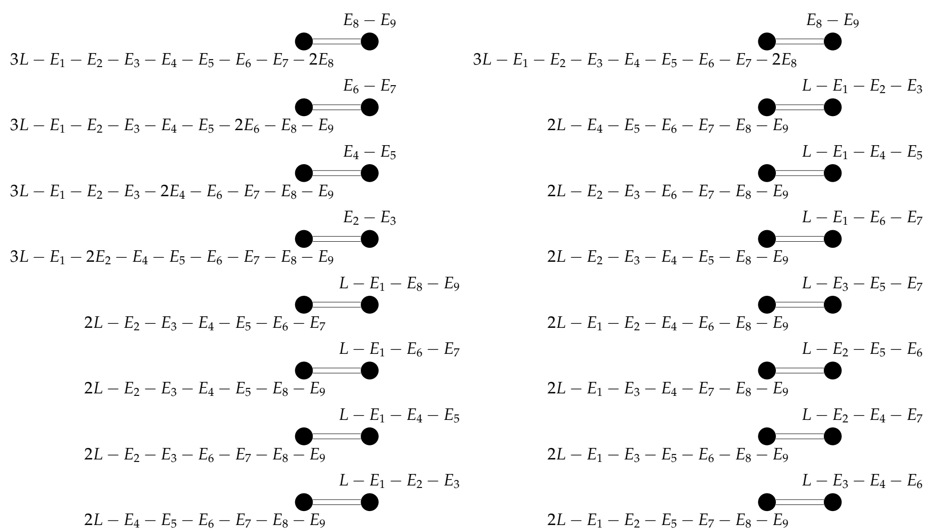


Figure 4.9: The blowings down of $A_1^{\oplus 8}$

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