5-2018

Resolutions of Finite Length Modules over Complete Intersections

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RESOLUTIONS OF FINITE LENGTH MODULES OVER COMPLETE INTERSECTIONS

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Mark E. Walker

Lincoln, Nebraska

May, 2018

RESOLUTIONS OF FINITE LENGTH MODULES OVER COMPLETE INTERSECTIONS

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University of Nebraska, 2018

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The structure of free resolutions of finite length modules over regular local rings has long been a topic of interest in commutative algebra. Conjectures by Buchsbaum-Eisenbud-Horrocks and Avramov-Buchweitz predict that in this setting the minimal free resolution of the residue field should give, in some sense, the smallest possible free resolution of a finite length module. Results of Tate and Shamash describing the minimal free resolution of the residue field over a local hypersurface ring, together with the theory of matrix factorizations developed by Eisenbud and Eisenbud-Peeva, suggest analogous lower bounds for the size of free resolutions of finite length modules of infinite projective dimension over such rings. In this dissertation we describe both positive and negative results pertaining to these lower bounds. By refining an argument of Charalambous, we show that the lower bounds hold in certain multigraded settings. We are also able to obtain results for finite free resolutions of multigraded modules, recovering results of Charalambous and Santoni. For the local case, however, we use a construction of Iyengar-Walker to provide examples showing that the lower bounds do not always hold. In order to accomplish this, we make use of the theory of higher matrix factorizations developed by Eisenbud-Peeva to investigate the structure of free resolutions over complete intersections of arbitrary codimension.

DEDICATION

To my wife, Hannah.

ACKNOWLEDGMENTS

I would like to begin by thanking my advisor, Mark Walker, for his guidance throughout this process. Without his suggestions, patience, support, and so on, this work would not have been possible. Along those lines, I thank my committee readers, Lucho Avramov and Tom Marley, for their comments and suggestions on this document.

I owe a great deal to the UNL commutative algebra community. In particular, I thank Chris Evans, Nick Packauskas, Josh Pollitz, Michael Brown, Eric Canton, Doug Dailey, Haydee Lindo, Kat Shultis, Peder Thompson, Mark Webb, and Andrew Windle. I was also fortunate enough to meet some remarkable friends and colleagues outside of my research area, including Corbin Groothuis, Meggan Hass, Kelsey Wells, Lara Ismert, Mitch Hamidi, Karina Uhing, and Christina Edholm.

My decision to go to graduate school was due in part to the professors I worked with as an undergraduate at Buena Vista University. Specifically, I would like to acknowledge Ben Donath, Lanny Grigsby, and Tim McDaniel.

Moving outside of mathematics, I thank Brian Christiansen, Conor Dondale, Kasey Kunzmann, and Brian Patterson for each offering me their unique perspectives on life and (occasionally) mathematics over the years.

Finally, I would not be where I am without the love and support of my family. So to my parents, Roy and Brenda, my sister, Mara, and my wife, Hannah, I say thank you.

GRANT INFORMATION

The author was partially supported by U.S. Department of Education grant P00A120068 (GAANN).

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Chapter 1

Introduction

Commutative algebra is the study of commutative rings and their modules. Modules over certain rings can be quite simple to describe. For example, if k is a field, then a finitely generated module over k is nothing more than a finite-dimensional vector space over k. As we work with more complicated rings, however, we find that their module theories become more complicated as well. This makes it necessary to develop tools and techniques that, in certain situations, allow us to reduce the problem of studying arbitrary modules over arbitrary rings to the problem of studying vector spaces over fields, which is much more familiar territory. One of the most fruitful methods for accomplishing this reduction is to study free resolutions of modules.

If $(R, \mathfrak{m}, k)$ is a local Noetherian ring, a general recurring principle is that the minimal free resolution of the residue field $R/\mathfrak{m} = k$ plays a crucial role in understanding both the structure of R and its module theory. For example, we have the following central result which initiated a rich interplay between the areas of commutative algebra and homological algebra:

Theorem 1.0.1 (Auslander-Buchsbaum-Serre). *For a local Noetherian ring* $(R, \mathfrak{m}, k)$*, the following are equivalent:*

1. *R is a regular local ring.*
2. *Every finitely generated R-module has a finite free resolution.*
3. *The R-module k has a finite free resolution.*

When R is a regular local ring of Krull dimension d, we are actually able to say exactly what a minimal free resolution of its residue field is. Namely, it is a Koszul complex on a minimal set of generators of its maximal ideal. This yields an equality

$$\beta_i^R(k) = \binom{d}{i}$$

where $\beta_i^R(k)$, the i^{th} Betti number of k, is the rank of the free module appearing in the i^{th} step of the minimal free resolution of k.

After understanding the minimal free resolution of the residue field, it is natural to consider resolutions of modules of finite length, which are essentially the modules that can be "built" from the residue field. More formally, if R is a commutative ring, then an R-module M has finite length if there exists a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

of R-modules such that $M_{i+1}/M_i \cong R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i \subseteq R$ for each $0 \leqslant i < n$. The following conjecture from [11] predicts that, step by step, the Koszul complex gives the smallest possible resolution among all resolutions of finite length modules:

Conjecture 1.0.2 (Buchsbaum-Eisenbud-Horrocks Conjecture)**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of Krull dimension d, and let $M \neq 0$ be a finite length R-module of finite projective dimension. Then for all i there is an inequality*

$$\beta_i^R(M) \geqslant \binom{d}{i}.$$

This conjecture, henceforth referred to as the BEH Conjecture, has received significant attention over the years, see for example Santoni [28] and Charalambous [15] for results in the multigraded case and Chang [14] and Burman [13] for results in the Loewy length two case. Also, in [6] it is noted that for local rings of dimension $d \leqslant 4$ the conjecture holds for elementary reasons.

Avramov-Buchweitz introduced a weaker form of the BEH conjecture in [6], which predicts that the total sum of the Betti numbers of a nonzero finite length module will be at least as large as the total sum of the Betti numbers coming from the Koszul complex of a system of parameters:

Conjecture 1.0.3 (Total Rank Conjecture)**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of Krull dimension d, and let $M \neq 0$ be a finite length R-module of finite projective dimension. Then there is an inequality*

$$\sum_{i=0}^{d} \beta_i^R(M) \geqslant 2^d.$$

Using the Evans-Griffith Syzygy Theorem (see [20]), Avramov-Buchweitz proved the conjecture when $d = 5$ under the assumption that R contains its residue field (see [6]). Due to recent work of André and Bhatt (see [1] and [9]) on the Direct Summand Conjecture, the assumption about the residue field is no longer needed. In [31], Walker proved the conjecture when char $k \neq 2$ with additional assumptions on R, including

the case when R is a regular local ring.

So far, these conjectures have all revolved around modules having finite free resolutions. Once we move out of the regular local case, however, we must contend with infinite free resolutions. The next simplest case comes from rings of the form $R = Q/(f)$, where $(Q, \mathfrak{n}, k)$ is a regular local ring, say of Krull dimension $d+1$, and $f \in \mathfrak{n}^2$ is a nonzero element. Shamash [29] and Tate [30] provide an explicit structure for the minimal free resolution of k as an R-module, and in particular demonstrate an equality

$$\beta_i^R(k) = 2^d$$

for all $i \geqslant d$. Eisenbud [18] went on to describe the asymptotic structure of arbitrary free resolutions over such rings: given any finitely generated R-module M, its minimal free resolution eventually becomes periodic of period 2 after at most $d+1$ steps. The periodic part of the resolution comes from a matrix factorization of the element f, and there is an equality

$$\beta_i^R(M) = \beta_{d+1}^R(M)$$

for all $i \geqslant d+1$ (see [18]). Based off of the same intuition as in the regular case, we might expect that any finite length R-module of infinite projective dimension will have a larger minimal free resolution, at least asymptotically, than the residue field. This leads to the following conjecture:

Conjecture 1.0.4 (Matrix Factorization Conjecture). *Let $(Q, \mathfrak{n}, k)$ be a regular local ring of dimension $d+1$, let $f \in \mathfrak{n}^2$ be Q-regular, and set $R = Q/(f)$. If M is a finite length R-module of infinite projective dimension, then there is an inequality*

$$\beta_n^R(M) \geqslant 2^d \qquad \textit{for all } n > d.$$

We call this the Matrix Factorization Conjecture (MFC) because the inequality is equivalent to saying that the matrices appearing in a matrix factorization giving the periodic part of the minimal free resolution of M are at least as big as those appearing in a matrix factorization giving the periodic part of the minimal free resolution of k. Although this conjecture does not appear explicitly in the literature, it is suggested both by the aforementioned results of Shamash and Tate and also results of Eisenbud-Peeva in [19] which relate the sizes of certain matrix factorizations to other invariants of modules over complete intersections, which we describe below.

Let $R = Q/(f_1, ..., f_c)$ be a complete intersection of codimension c. This means that $(Q, \mathfrak{n}, k)$ is a regular local ring and $f_1, ..., f_c \in \mathfrak{n}^2$ is a Q-regular sequence. If M is a finitely generated R-module of infinite projective dimension, results of and Avramov [2] and Gulliksen [22] show that the Betti numbers of M are eventually given by two polynomials, one for the even Betti numbers and one for the odd Betti numbers, each of degree j for some $0 \leqslant j \leqslant c-1$ and each with the same leading coefficient. The value $j+1$ is known as the complexity of M. Avramov-Buchweitz [5] give the following description of the leading coefficient of these polynomials:

Theorem 1.0.5. *([5], Theorem 7.3) With the notation as above, if the complexity of M is $d \geqslant 1$, there exist a positive integer β-$deg_R(M)$ and polynomials $p_{even}(t)$ and $p_{odd}(t) \in \mathbb{Q}[t]$ of degree $\leqslant d-2$ such that*

$$\beta_n^R(M) = \frac{\beta\text{-}deg_R(M)}{2^{d-1}(d-1)!} n^{d-1} + \begin{cases} p_{even}(n) & \text{for even } n \gg 0 \\ p_{odd}(n) & \text{for odd } n \gg 0. \end{cases}$$

The number β-$\deg_R(M)$ is the *Betti degree* of M. Avramov-Buchweitz [5] go on to introduce the following conjecture:

Conjecture 1.0.6 (Betti Degree Conjecture)**.** *With the notation as above, there is an inequality β-$deg_R(M) \geqslant 2^{d-1}$. Equivalently,*

$$\lim_{n\to\infty} \frac{\beta_n^R(M)}{n^{d-1}} \geqslant \frac{1}{(d-1)!}.$$

Eisenbud-Peeva give explicit formulas in [19] for certain Poincaré series over complete intersections that relate Betti degrees to the size of certain matrix factorizations. Their results, together with the Betti Degree Conjecture (BDC) and the motivation described earlier, lead one directly to the statement of the MFC. By recent work of Iyengar-Walker in [24], the BDC has been shown to be false in general.

In this work we explore the relationships between these various conjectures, paying special attention to the MFC. In Chapter 2 we study the graded analogue of the MFC. By refining an argument of Charalambous we show that the conjecture holds for multigraded modules over a certain class of multigraded hypersurface rings:

Theorem 1.0.7. *Let S be a G-graded finitely generated algebra over a field k, where G is an abelian group, that is also a hypersurface ring. Let $R = S[x]$ be the polynomial ring over S. If the graded MFC holds for all G-graded S-modules, then it also holds*

for all $G\times\mathbb{Z}$-graded R-modules. In particular, if S is multigraded and the graded MFC holds for all multigraded S-modules, then the graded MFC holds for all multigraded R-modules.

Additionally, we are able to recover results of Charalambous and Santoni pertaining to the graded analogue of the BEH Conjecture, and we give a new result pertaining to the graded analogue of the Total Rank Conjecture:

Theorem 1.0.8. *([28], Corollary 2.6) Let S be a G-graded finitely generated algebra over a field k, where G is an abelian group, and let $R = S[x]$ be the polynomial ring over S. If the graded BEH Conjecture holds for all G-graded S-modules, then it holds for all $G \times \mathbb{Z}$-graded R-modules. In particular, if S is multigraded and the graded BEH Conjecture holds for all multigraded S-modules, then the graded BEH Conjecture holds for all multigraded R-modules.*

Theorem 1.0.9. *Let S be a G-graded finitely generated algebra over a field k, where G is an abelian group, and let $R = S[x]$ be the polynomial ring over S. If the graded Total Rank Conjecture holds for all G-graded S-modules, then it holds for all $G \times \mathbb{Z}$-graded R-modules. In particular, if S is multigraded and the graded Total Rank Conjecture holds for all multigraded S-modules, then the graded Total Rank Conjecture holds for all multigraded R-modules.*

In Chapter 3 we return to the local setting. After recalling some basic definitions and constructions, we review the theory of higher matrix factorizations. Along the way we provide a new proof of the following essential result, originally due to Eisenbud-Peeva:

Theorem 1.0.10. *([19], Theorem 1.3.1) Let $(Q, \mathfrak{n}, k)$ be a regular local ring with infinite residue field k, let $f_1, ..., f_c \in \mathfrak{n}^2$ be a Q-regular sequence, and set $R =$*

$Q/(f_1, ..., f_c)$. If M is a finitely generated R-module, then for all $n \gg 0$ the module $Syz_n^R(M)$ is a higher matrix factorization module with respect to some generating set $f_1', ..., f_c'$ of $(f_1, ..., f_c)$.

Next, we revisit the Betti Degree Conjecture. Although it is known to be false in general (see [24]), we provide some specific cases where the conjecture does hold. In particular, we will show

Proposition 1.0.11. *Let $(R, \mathfrak{m}, k)$ be a complete intersection, and let M be a finitely generated R-module of infinite projective dimension whose Betti numbers are eventually polynomial. Then the Betti Degree Conjecture holds for M.*

Finally, we return our attention to the Matrix Factorization Conjecture. We begin by relating it to some of the other existing conjectures, and then we show that the conjecture holds in certain cases. However, using the Iyengar-Walker construction, we show

Theorem 1.0.12. *There exist hypersurfaces $R = Q/(f)$ with f an element of the cube of the maximal ideal of Q such that the MFC fails over R.*

Drawing inspiration from the theory of Knörrer periodicity, we produce additional counterexamples over "quadratic hypersurfaces" (those whose defining relations are in the square, but not the cube, of the maximal ideal) that are not immediately obtainable through the Iyengar-Walker construction.

Proposition 1.0.13. *If $R = Q/(f)$ is a local hypersurface ring for which the MFC fails, and the characteristic of the residue field of R is different from 2, then the MFC also fails for $R^{\#\#} = Q[[x, y]]/(f + xy)$.*

Having seen that the MFC fails in general, we end by asking what lower bounds are obtainable for the asymptotic size of minimal free resolutions of finite length modules over hypersurface rings. Using an elementary argument, we give a first step:

Proposition 1.0.14. *Let R be a local hypersurface ring of dimension d and let M be a finite length R-module of infinite projective dimension. Then there is an inequality*

$$\beta_n^R(M) \geqslant \frac{d}{2} \qquad \textit{for all } n > d.$$

Chapter 2

Resolutions of Finite Length Multigraded Modules

Given a polynomial ring $R = k[x_1, ..., x_d]$ with coefficients in a field k, we often exploit the fact that it has a graded structure. The standard way to grade R is to declare that each variable x_i has degree one, in which case we have a decomposition $R = \bigoplus_{n \geqslant 0} R_n$, where R_n is the k-vector space spanned by all monomials $x_1^{a_1} \cdots x_d^{a_d}$ satisfying $a_i \geqslant 0$ for each $1 \leqslant i \leqslant d$ and $a_1 + \cdots + a_d = n$. In particular, we say that R is a $\mathbb{Z}$-graded ring.

In this section we study properties of modules with a more refined graded structure. Instead of considering $\mathbb{Z}$-graded modules over $\mathbb{Z}$-graded rings, we will focus our attention on $\mathbb{Z}^n$-graded modules over $\mathbb{Z}^n$-graded rings. It turns out that this additional graded structure allows us to place additional restrictions on free resolutions of such modules that need not be present in more general settings. After developing the necessary preliminaries we will proceed to our main results, which pertain to modules of both finite and infinite projective dimension.

2.1 Preliminaries

We begin by collecting the necessary terminology for multigraded rings and modules.

Definition 2.1.1. *([10], Section 5.1) Let* $(G, +)$ *be an Abelian group. A* G*-graded ring is a ring* R *together with a decomposition* $R = \bigoplus_{a \in G} R_a$ *as an Abelian group such that* $R_a R_b \subseteq R_{a+b}$ *for all* $a, b \in G$*. A* G*-graded* R*-module is a module* M *with a decomposition* $M = \bigoplus_{a \in G} M_a$ *as an Abelian group such that* $R_a M_b \subseteq M_{a+b}$ *for all* $a, b \in G$*.*

If $G = \mathbb{Z}^n$ in the previous definition for some $n > 0$ we say that R is a *multigraded ring*. If M is a $\mathbb{Z}^n$-graded R-module, we say that M is a *mutigraded* R*-module*.

Example 2.1.2. *If* k *is a field, the polynomial ring* $R = k[x_1, ..., x_n]$ *is a* $\mathbb{Z}^n$*-graded ring with the degree of* x_i*, denoted* $|x_i|$*, given by* $(0, ..., 1, ..., 0)$ *where* 1 *is in the* i^{th} *component.*

Example 2.1.3. *The multigraded* R*-submodules of* $R = k[x_1, ..., x_n]$ *as above are precisely the collection of monomial ideals of* R*. Moreover, if* $I \subseteq R$ *is a monomial ideal, then* R/I *is a multigraded* R*-module.*

2.2 The Infinite Projective Dimension Case

2.2.1 Endomorphisms of Tor Modules

We will let k be a field, G an abelian group, S a G-graded finitely generated k-algebra of (Krull) dimension $d - 1$, and $R = S[x]$ the polynomial ring over S with $|x| = (e_G, 1) \in G \times \mathbb{Z}$ (e_G is the identity element of G). M will denote an R-module. As usual, we set $\beta_i^R(M) = \dim_k \operatorname{Tor}_i^R(M, k)$, and we will adopt the notation of [15] by setting

$$\gamma_i^R(M) = \operatorname{rank}_R \operatorname{Syz}_i^R(M).$$

Note that for each $i \geqslant 0$, $\gamma_i^R(M)$ is well-defined for all finite length R-modules M since $\operatorname{rank}_R(M) = 0$ ([10], Proposition 1.4.5). We begin with a series of results from Charalambous:

Lemma 2.2.1. *([15], Lemma 1) Any R-module M fits into a functorial short exact sequence of R-modules*

$$0 \longrightarrow R \otimes_S M \xrightarrow{\phi} R \otimes_S M \xrightarrow{\epsilon} M \longrightarrow 0$$

where $\phi(1 \otimes m) = x \cdot (1 \otimes m) - 1 \otimes x \cdot m$ and $\epsilon(r \otimes m) = rm$. Moreover, ϕ is equivalent to multiplication by x if and only if $x \cdot M = 0$.

Lemma 2.2.2. *([15], Lemma 2) There are isomorphisms $Tor_i^R(R \otimes_S M, N) \cong Tor_i^S(M, N)$ for $i \geqslant 0$.*

Suppose M is an R-module of finite length. Then we obtain an exact sequence of finite dimensional k-vector spaces

$$\operatorname{Tor}_i^R(R \otimes_S M, k) \xrightarrow{\phi_i} \operatorname{Tor}_i^R(R \otimes_S M, k) \xrightarrow{\epsilon_i} \operatorname{Tor}_i^R(M, k) \longrightarrow \operatorname{Tor}_{i-1}^R(R \otimes_S M, k).$$

From this we see that Image $\epsilon_i \cong$ Coker ϕ_i for all i. Moreover, as ϕ_i is an endomorphism of k-vector spaces, there is an equality

$$\dim_k \text{ Coker } \phi_i = \dim_k \text{ Ker } \phi_i$$

(see [15], Section 3). These facts lead to

Lemma 2.2.3. *([15], Lemma 4) If M is a finite length R-module, then for each $i \geqslant 0$*

$$dim_k\ Tor_i^R(M,k) = dim_k\ Coker\ \phi_i + dim_k\ Coker\ \phi_{i-1}.$$

From now on, we can (and will) assume that ϕ_i is an endomorphism on $\mathrm{Tor}_i^S(M,k)$, as in [15].

Lemma 2.2.4. *([15], Lemma 5) Let M be a finite length R-module. Then $\gamma_i^R(M) = dim_k\ Coker\ \phi_{i-1}$ for $i \geqslant 1$.*

2.2.2 Extending Results of Charalambous

Let $R = S[x]$ as before. In addition to viewing S as a subring of R, we can also view it as a quotient of R, since $S = R/(x)$. We begin by recalling a result of Nagata, and Shamash independently, that illustrates the utility of the latter interpretation. Although we will state it in terms of graded modules over graded rings, we remark that in [29] the result was stated in terms of modules over local rings (see also [26]).

Proposition 2.2.5. *([29], Corollary 1) For a finitely generated graded R-module M such that $x \cdot M = 0$, there is an equality of Poincaré series*

$$\mathcal{P}_M^S(t) = \frac{\mathcal{P}_M^R(t)}{(1+t)}.$$

In particular, for each i there is an equality

$$\beta_i^R(M) = \beta_i^S(M) + \beta_{i-1}^S(M).$$

From this, we have the following useful corollary:

Corollary 2.2.6. *Let M be a finitely generated S-module such that $x \cdot M = 0$. Then $pd_S(M) < \infty$ if and only if $pd_R(M) < \infty$.*

Let M be a graded ($G \times \mathbb{Z}$-graded) R-module of finite length. As was done in [15], we may decompose M as

$$M = M_0 \oplus \cdots \oplus M_j$$

where $M_i := \bigoplus_{g \in G} M_{(g,i)}$ is a G-graded S-module of finite length and $x \cdot M_i \subseteq M_{i+1}$ for each i. As we are interested in modules of infinite projective dimension, it will be important to understand the values $\mathrm{pd}_S(M_i)$.

Lemma 2.2.7. *If $pd_S(M_i) < \infty$ for all i, then $pd_R(M) < \infty$.*

Proof. We proceed by induction on j. If $j = 0$, then $M = M_0$, and so we may use Proposition 2.2.5. If $j > 0$, we can write a short exact sequence of R-modules

$$0 \to M_{\geqslant 1} \to M \to M_0 \to 0$$

(since $M_0 \cong M/M_{\geqslant 1}$, it is an R-module satisfying $x \cdot M_0 = 0$). By assumption, $\mathrm{pd}_S(M_i) < \infty$ for all i. Thus, we see that $\mathrm{pd}_R(M_{\geqslant 1}), \mathrm{pd}_R(M_0) < \infty$ by the inductive hypothesis. An inspection of the induced exact sequence of Tor modules now yields $\mathrm{pd}_R(M) < \infty$.

□

When we prove the main result of this section, we will ultimately want to reduce to the case where $\mathrm{pd}_S(M_0) = \infty$. Iterating the next result allows us to make this reduction.

Lemma 2.2.8. *Let M be an R-module as before. If $pd_S(M_0) < \infty$, then*

$$\gamma_i^R(M) \geqslant \gamma_i^R(M_{\geqslant 1}) \qquad \textit{for all } i \geqslant d,$$

with equality for $i > d$.

Proof. We begin with the graded version of a construction from ([8], Lemma 2.3). Let $F_\bullet \to M$ and $G_\bullet \to M_0$ be minimal graded R-free resolutions, and let $\varphi : F_\bullet \to G_\bullet$ be a lifting of the natural surjection of R-modules $M \to M_0$. Since F_0 maps onto M_0 by way of composition, φ_0 must be surjective by Nakayama's Lemma. We obtain a mapping cone

$$\mathrm{Cone}(\varphi) : \cdots \to F_1 \oplus G_2 \to F_0 \oplus G_1 \to G_0 \to 0$$

with the usual differential. Now set

$$T_\bullet = \cdots \to F_2 \oplus G_3 \to F_1 \oplus G_2 \to \mathrm{Ker}\begin{pmatrix}\varphi_0 & \partial_1^G\end{pmatrix} \to 0$$

with $T_0 = \mathrm{Ker}\begin{pmatrix}\varphi_0 & \partial_1^G\end{pmatrix}$.

We claim that $T_\bullet$ is a graded free resolution of $M_{\geqslant 1}$, following the argument of ([8], Lemma 2.3): First note that $\mathrm{Ker}\begin{pmatrix}\varphi_0 & \partial_1^G\end{pmatrix}$ is a free R-module, since $\begin{pmatrix}\varphi_0 & \partial_1^G\end{pmatrix}$ is a surjection from the free R-module $F_0 \oplus G_1$ onto the free R-module G_0. Also, it is clear that $H_i(T_\bullet) \cong H_{i+1}(\mathrm{Cone}(\varphi))$ for $i \geqslant 0$. The exact sequence

$$0 \to G_\bullet \to \mathrm{Cone}(\varphi) \to \Sigma F_\bullet \to 0$$

induces an exact sequence on homology:

$$\cdots \to H_i(G_\bullet) \to H_i(\mathrm{Cone}(\varphi)) \to H_{i-1}(F_\bullet) \to H_{i-1}(G_\bullet) \to \cdots$$

From this we see that $H_i(T_\bullet) = 0$ for $i \geqslant 1$ ($F_\bullet$ and $G_\bullet$ are acyclic). Another inspection of the exact sequence on homology yields $H_0(T_\bullet) = H_1(\mathrm{Cone}(\varphi)) \cong M_{\geqslant 1}$. Thus, $T_\bullet$ is a graded free resolution of $M_{\geqslant 1}$.

Since $\mathrm{pd}_S(M_0) < \infty$ and $x \cdot M_0 = 0$, we know that $\mathrm{pd}_R(M_0) < \infty$ by Corollary 2.2.6. Thus, $T_\bullet$ is minimal starting in homological degree d. It now follows that $\mathrm{Syz}_i^R(M) \cong \mathrm{Syz}_i^R(M_{\geqslant 1})$ for $i > d$. In particular, we get equalities

$$\gamma_i^R(M) = \gamma_i^R(M_{\geqslant 1}) \qquad \text{for } i > d.$$

As for $i = d$, we observe the exact sequence of Tor modules:

$$\mathrm{Tor}_{d+1}^R(M_0, k) \to \mathrm{Tor}_d^R(M_{\geqslant 1}, k) \to \mathrm{Tor}_d^R(M, k) \to \mathrm{Tor}_d^R(M_0, k) \to \mathrm{Tor}_{d-1}^R(M_{\geqslant 1}, k)$$

By assumption, $\mathrm{Tor}_{d+1}^R(M_0, k) = 0$, and so $\beta_d^R(M_{\geqslant 1}) \leqslant \beta_d^R(M)$. Therefore, we'll have

$$\begin{aligned} \gamma_d^R(M) &= \beta_d^R(M) - \gamma_{d+1}^R(M) \\ &\geqslant \beta_d^R(M_{\geqslant 1}) - \gamma_{d+1}^R(M) \\ &= \beta_d^R(M_{\geqslant 1}) - \gamma_{d+1}^R(M_{\geqslant 1}) \\ &= \gamma_d^R(M_{\geqslant 1}), \end{aligned}$$

which completes the proof.

$\square$

2.2.3 Verifying the Lower Bound

We now arrive at the main result of Section 2.2:

Theorem 2.2.9. *Let S be a G-graded finitely generated algebra over a field k, where G is an abelian group, that is also a hypersurface ring. Let $R = S[x]$ be the polynomial ring over S. If the graded MFC holds for all G-graded S-modules, then it also holds for all $G \times \mathbb{Z}$-graded R-modules. In particular, if S is multigraded and the graded MFC holds for all multigraded S-modules, then the graded MFC holds for all multigraded R-modules.*

Proof. Set $\dim S = d - 1$. Decompose $M = M_0 \oplus \cdots \oplus M_j$, with each M_p a finite length G-graded S-module and $x \cdot M_p \subseteq M_{p+1}$. Fixing $i > d$, we obtain a direct sum decomposition

$$\operatorname{Tor}_{i-1}^S(M, k) \cong \operatorname{Tor}_{i-1}^S(M_0, k) \oplus \cdots \oplus \operatorname{Tor}_{i-1}^S(M_j, k).$$

The argument from ([15], Theorem 1) carries over to show that $\operatorname{Tor}_{i-1}^S(M_0, k)$ is a direct summand of Coker ϕ_{i-1}, where

$$\phi_{i-1} : \operatorname{Tor}_{i-1}^S(M, k) \to \operatorname{Tor}_{i-1}^S(M, k)$$

is the map induced from multiplication by x. Combining Lemmas 2.2.7 and 2.2.8, we

may assume without loss of generality that $\mathrm{pd}_S(M_0) = \infty$. We now have

$$\begin{aligned} \gamma_i^R(M) &= \dim_k \text{ Coker } \phi_{i-1} \quad \text{(by Lemma 2.2.4)} \\ &\geqslant \dim_k \text{ Tor}_{i-1}^S(M_0, k) \\ &= \beta_{i-1}^S(M_0) \\ &\geqslant 2^{d-1}. \end{aligned}$$

The result now follows from the equality $\beta_i^R(M) = \gamma_i^R(M) + \gamma_{i+1}^R(M)$. $\square$

Example 2.2.10. *Let $S = k[x_0]/(x_0^n)$ for some $n > 1$. Then S is a zero-dimensional hypersurface ring, and so the graded MFC holds trivially for S. Theorem 2.2.9 implies that the graded MFC holds for multigraded modules over the ring*

$$R = S[x_1, ..., x_d] = k[x_0, x_1, ..., x_d]/(x_0^n).$$

Example 2.2.11. *The Theorem also applies to the graded hypersurface ring $R = k[x_0, x_1, ..., x_d]/(x_0^n x_1^m)$ for $n, m > 1$. Indeed, we can view $R = S[x_2, ..., x_d]$ where $S = k[x_0, x_1]/(x_0^n x_1^m)$. The graded analogue of Proposition 3.3.3 below shows that the graded MFC holds for S.*

2.3 The Finite Projective Dimension Case

We now return to the case of finite free resolutions, where we will see that our methods from the previous section can be applied to both the BEH Conjecture and the Total Rank Conjecture.

2.3.1 The Buchsbaum-Eisenbud-Horrocks Conjecture

We begin with the BEH Conjecture.

Theorem 2.3.1. *([28], Corollary 2.6) Let S be a G-graded finitely generated algebra over a field k, where G is an abelian group, and let $R = S[x]$ be the polynomial ring over S. If the graded BEH Conjecture holds for all G-graded S-modules, then it also holds for all $G \times \mathbb{Z}$-graded R-modules.*

Proof. Set $\dim S = d-1$. Decompose $M = M_0 \oplus \cdots \oplus M_j$ as before. Fixing $i \geqslant 0$, the argument from ([15], Theorem 1) again implies that $\mathrm{Tor}_i^S(M_0, k)$ is a direct summand of Coker ϕ_i, where

$$\phi_i : \mathrm{Tor}_i^S(M, k) \to \mathrm{Tor}_i^S(M, k)$$

is the map induced from multiplication by x. By assumption, $\mathrm{pd}_R(M) < \infty$, and so Lemma 2.2.3 now implies that $\mathrm{pd}_S(M_0) < \infty$. Thus, it follows that

$$\begin{aligned} \gamma_{i+1}^R(M) &= \dim_k \text{ Coker } \phi_i \quad \text{(by Lemma 2.2.4)} \\ &\geqslant \dim_k \mathrm{Tor}_i^S(M_0, k) \\ &\geqslant \tbinom{d-1}{i}. \end{aligned}$$

But then $\beta_i^R(M) = \gamma_i^R(M) + \gamma_{i+1}^R(M) \geqslant \binom{d-1}{i-1} + \binom{d-1}{i} = \binom{d}{i}$, which completes the proof. □

Example 2.3.2. *([15], Corollary 1; [28], Corollary 3.6) The graded BEH Conjecture holds for $\mathbb{Z}^d$-graded modules of finite length over $R = k[x_1, ..., x_d]$.*

Example 2.3.3. *([28], Corollary 3.6) Let S be a regular local ring (or a regular graded k-algebra) of dimension $d \leqslant 4$. As mentioned before, the (graded) BEH Conjecture holds for S. Theorem 2.3.1 implies that the graded BEH Conjecture also holds for $\mathbb{Z}^n$-graded modules of finite length over $R = S[x_1, ..., x_n]$.*

2.3.2 The Total Rank Conjecture

We now prove an analogous result for the Total Rank Conjecture.

Theorem 2.3.4. *Let S be a G-graded finitely generated algebra over a field k, where G is an abelian group, and let $R = S[x]$ be the polynomial ring over S. If the graded Total Rank Conjecture holds for all G-graded S-modules, then it also holds for all $G \times \mathbb{Z}$-graded R-modules.*

Proof. Set dim $S = d - 1$. Decompose $M = M_0 \oplus \cdots \oplus M_j$ as before. Fixing $i \geqslant 1$, $\mathrm{Tor}^S_{i-1}(M_0, k)$ is again a direct summand of Coker ϕ_{i-1}, where $\phi_{i-1} : \mathrm{Tor}^S_{i-1}(M, k) \to \mathrm{Tor}^S_{i-1}(M, k)$ is the map induced from multiplication by x (see [15], Theorem 1). By assumption, $\mathrm{pd}_R(M) < \infty$, and so Lemma 2.2.3 now implies that $\mathrm{pd}_S(M_0) < \infty$. It follows that

$$\begin{aligned}
\sum_{i=1}^{d} \gamma_i^R(M) &= \sum_{i=1}^{d} \dim_k \mathrm{Coker}(\phi_{i-1}) \quad \text{(by Lemma 2.2.4)} \\
&\geqslant \sum_{i=1}^{d} \dim_k \mathrm{Tor}^S_{i-1}(M_0, k) \\
&= \sum_{i=0}^{d-1} \beta_i^S(M_0) \\
&\geqslant 2^{d-1}.
\end{aligned}$$

Since $\sum_{i=0}^{d} \beta_i^R(M) = 2\left(\sum_{i=1}^{d} \gamma_i^R(M)\right)$, the desired result follows at once. $\square$

Example 2.3.5. *Let S be either a Noetherian local ring (or a graded k-algebra) of dimension $d \leqslant 5$. Then the (graded) Total Rank Conjecture holds for S (see [6]). Theorem 2.3.4 now implies that the graded Total Rank Conjecture holds for $\mathbb{Z}^n$-graded modules of finite length and finite projective dimension over $R = S[x_1, ..., x_n]$.*

Chapter 3

Resolutions over Local Complete Intersections

We now turn our attention to free resolutions over local complete intersections. First, we recall the theory of higher matrix factorizations from [19] which describes the asymptotic structure of minimal free resolutions over complete intersections of arbitrary codimension. We then proceed to the Betti Degree Conjecture from [5]. While it was shown in [24] to fail in general, we demonstrate certain classes of modules and complete intersections for which it holds. We also produce additional counterexamples to the conjecture, using [24] as a starting point. In the closing section we return to the Matrix Factorization Conjecture. After demonstrating certain situations where the conjecture holds, we use higher matrix factorizations together with the Iyengar-Walker construction to produce families of counterexamples to the conjecture.

3.1 Preliminaries

Throughout this section $(Q, \mathfrak{n}, k)$ will denote a Noetherian local ring, $f \in \mathfrak{n}$ will denote a Q-regular element, and $R = Q/(f)$ will be the quotient of Q by f. Our main goal will be to take homological information over the ring Q and translate it into homological information over the ring R.

As a first example of this translation, we recall a result of Shamash [29] which

takes a Q-free resolution of an R-module M and produces an R-free resolution of M. The statement is taken from [4].

Proposition 3.1.1. *([29], Lemma 2) If $R = Q/(f)$ with f a Q-regular element, M is an R-module, and $U_\bullet$ is a Q-free resolution of M, then there is a family of Q-linear endomorphisms $\boldsymbol{\sigma} = \{\sigma^{[j]} \in Hom_Q(U_\bullet, U_\bullet) \mid |\sigma^{[j]}| = 2j-1\}_{j\geqslant 0}$, such that*

$$\sigma^{[0]} = \partial^{U_\bullet}; \quad \sigma^{[0]}\sigma^{[1]} + \sigma^{[1]}\sigma^{[0]} = f \cdot id^{U_\bullet}; \quad \sum_{j=0}^{n} \sigma^{[n]}\sigma^{[n-j]} = 0.$$

If $\{x^{(i)} \mid |x^{(i)}| = 2i\}_{i\geqslant 0}$ is a linearly independent set over R, then

$$\cdots \longrightarrow \bigoplus_{i=0}^{n} Rx^{(i)} \otimes_Q U_{n-2i} \xrightarrow{\partial} \bigoplus_{i=0}^{n-1} Rx^{(i)} \otimes_Q U_{n-1-2i} \longrightarrow \cdots$$

is an R-free resolution $Sh(\boldsymbol{\sigma}, U_\bullet)$ of M with differential

$$\partial(x^{(i)} \otimes u) = \sum_{j=0}^{i} x^{(i-j)} \otimes \sigma^{[j]}(u).$$

The collection $\boldsymbol{\sigma}$ is referred to as a system of higher homotopies of f. We will call $\mathrm{Sh}(\boldsymbol{\sigma}, U_\bullet)$ the *Shamash resolution* of M over R, suppressing the system of higher homotopies $\boldsymbol{\sigma}$ and the Q-free resolution $U_\bullet$ unless specificity is required.

We remark that the Shamash resolution need not be minimal. For instance, taking $M = R$ from the Proposition we see that any Q-free resolution of R and any system of higher homotopies $\boldsymbol{\sigma}$ of f produces an infinite free resolution of R as an R-module. There are, however, situations in which the Shamash resolution is minimal. We record one such instance now.

Proposition 3.1.2. *([29], Theorem 1) Let f be a Q-regular element, let $R = Q/(f)$, and suppose M is an R-module such that $f \in \mathfrak{n}(0 :_Q M)$. If $U_\bullet$ is a minimal Q-free resolution of M and $\boldsymbol{\sigma}$ is any system of higher homotopies of f, then $Sh(\boldsymbol{\sigma}, U_\bullet)$ is a minimal R-free resolution of M.*

We now recall a construction that involves modding out Q by a regular sequence whose length may be bigger than one. Let $\underline{f} = f_1, ..., f_c$ be a Q-regular sequence, and set $R = Q/(f_1, ..., f_c)$. Given a complex $F_\bullet$ of finitely generated free R-modules, we construct a sequence of c degree -2 chain maps on $F_\bullet$ known as the Eisenbud operators.

Definition 3.1.3. *([18], Section 1) Let $(\widetilde{F_\bullet}, \widetilde{\partial^{F_\bullet}})$ be a lifting of $(F_\bullet, \partial^{F_\bullet})$ to Q. That is, for each $i \in \mathbb{N}$, choose a map $\widetilde{\partial_i^F} : \widetilde{F_i} \to \widetilde{F_{i-1}}$ of free Q-modules such that $\widetilde{\partial_i^{F_\bullet}} \otimes R = \partial_i^{F_\bullet}$. Since $\partial^{F_\bullet} \circ \partial^{F_\bullet} = 0$ and $\underline{f}$ is a regular sequence, it follows that*

$$\widetilde{\partial^{F_\bullet}} \circ \widetilde{\partial^{F_\bullet}} = \sum_{i=1}^{c} f_i \tilde{t}_i$$

for some degree -2 maps $\tilde{t}_i$. We then define the Eisenbud operators *associated to $f_1, ..., f_c$ to be the maps $t_i = \tilde{t}_i \otimes R$ for $1 \leqslant i \leqslant c$.*

In [18], Eisenbud shows that the t_i's are chain maps of degree -2, and that they are independent of the choice of liftings up to homotopy.

Suppose $R = Q/(f_1, ..., f_c)$ as above, and let $F_\bullet$ be the minimal R-free resolution of a finitely generated R-module M. In this case we will denote the Eisenbud operators associated to $f_1, ..., f_c$ by $t_1^M, ..., t_c^M$. If we need to be specific about the generating set $\underline{f} = f_1, ..., f_c$ we will write $t_1^M(\underline{f}), ..., t_c^M(\underline{f})$. When $c = 1$ we write $R = Q/(f)$ and denote the Eisenbud operator associated to f by t_f^M. With this notation, we record another situation when the Shamash resolution is minimal.

Proposition 3.1.4. *([7], Proposition 6.2) Let f be a Q-regular element, let $R = Q/(f)$, and suppose M is an R-module. If $U_\bullet$ is a minimal Q-free resolution of M and $\boldsymbol{\sigma}$ is a system of higher homotopies for f, then $Sh(\boldsymbol{\sigma}, U_\bullet)$ is minimal if and only if t_f^M is surjective.*

3.2 Asymptotic Properties of Resolutions

We now explore the asymptotic structure of minimal free resolutions over complete intersections. After beginning with the hypersurface case, we review the theory of higher matrix factorizations from [19], where we provide a new proof that sufficiently high syzygies over arbitrary complete intersections having an infinite residue field are higher matrix factorization modules. Next, we proceed to our study of the Betti Degree Conjecture. Although the conjecture fails in general (see [24]), we will demonstrate a special case when the conjecture still holds. Namely, we will show that the Betti Degree Conjecture holds for modules whose Betti numbers are eventually given by a single polynomial. Together with results from [3], this will imply that the Betti Degree Conjecture holds for all complete intersections of minimal multiplicity.

3.2.1 Matrix Factorizations

Let $(Q, \mathfrak{n}, k)$ be a regular local ring, let $f \in \mathfrak{n}^2$ be a Q-regular element, and set $R = Q/(f)$. We now recall the theory of matrix factorizations from [18] and its role in describing the module theory of R.

Definition 3.2.1. *([18], Section 5) A matrix factorization of f is an ordered pair of maps of finitely generated free Q-modules $(\varphi : F \to G, \psi : G \to F)$ such that $\varphi \circ \psi = f \cdot id_G$ and $\psi \circ \varphi = f \cdot id_F$.*

Notes:

(1) Given a matrix factorization $(\varphi : F \to G, \psi : G \to F)$ we will suppress the free modules and simply write (φ, ψ).

(2) Because f is Q-regular, one can deduce that φ and ψ are injective and that F and G have the same rank.

Definition 3.2.2. *([32], Definition 7.1) A morphism between matrix factorizations (φ_1, ψ_1) and (φ_2, ψ_2) is a pair of morphisms (α, β) making the following diagram commute:*

$$\begin{tikzcd} Q^{n_1} \arrow[r, "\psi_1"] \arrow[d, "\alpha"'] & Q^{n_1} \arrow[r, "\varphi_1"] \arrow[d, "\beta"'] & Q^{n_1} \arrow[d, "\alpha"'] \\ Q^{n_2} \arrow[r, "\psi_2"] & Q^{n_2} \arrow[r, "\varphi_2"] & Q^{n_2} \end{tikzcd}$$

The matrix factorizations above are said to be isomorphic *if α and β are isomorphisms.*

Let $\mathrm{mf}(Q, f)$ denote the category of matrix factorizations of f over Q. We say that the matrix factorizations $(1, f), (f, 1) \in \mathrm{mf}(Q, f)$ are *trivial* ([18], Section 6). Define $[\mathrm{mf}(Q, f)]$ to be the category with the same objects as $\mathrm{mf}(Q, f)$ but whose morphisms are classes of morphisms from $\mathrm{mf}(Q, f)$ modulo maps that factor through direct sums of trivial matrix factorizations. Now let $\underline{\mathrm{MCM}}(R)$ denote the category of maximal Cohen-Macaulay R-modules whose morphisms are classes of morphisms of MCM R-modules modulo maps that factor through a free R-module (for details see [32], Chapter 7). Then we have the following result:

Theorem 3.2.3. *([18], Corollary 6.3) There is an equivalence of categories*

$$cok : [mf(Q, f)] \to \underline{MCM}(R)$$

given on objects by $cok(\varphi,\psi) = coker\ \varphi$.

Defining a reduced matrix factorization (φ, ψ) to be one in which each entry of φ and ψ is in $\mathfrak{n} \subseteq Q$ ([18], Section 6), the above equivalence can also be interpreted as an equivalence between reduced matrix factorizations of f over Q and MCM R-modules having no non-trivial free summand.

The inverse of this equivalence can be described on objects as follows (see [32], Chapter 7): given a MCM R-module M with no nontrivial free summands, its minimal Q-free resolution has the form

$$0 \longrightarrow Q^n \xrightarrow{\varphi} Q^n \longrightarrow M \longrightarrow 0.$$

The fact that $\mathrm{pd}_Q(M) = 1$ follows from the Auslander-Buchsbaum formula, and the fact that the free Q-modules appearing in the resolution of M have the same rank follows from the fact that $f \cdot M = 0$. Moreover, since f annihilates M, there exists a homotopy $\psi : Q^n \to Q^n$ such that $\varphi \circ \psi = f \cdot \mathrm{id}_{Q^n}$. So M gets assigned to the matrix factorization (φ, ψ) in $[\mathrm{mf}(Q, f)]$, which is reduced because M has no non-trivial free summand.

We now describe an equivalence demonstrated by Buchweitz in [12] between $[\mathrm{mf}(Q, f)]$, $\underline{\mathrm{MCM}}(R)$, and the singularity category $D^b_{\mathrm{sing}}(R)$ of R. We begin with the definition of the derived category.

Definition 3.2.4. *([21], III.2.1) Let R be a ring and let $C(R)$ denote the category of chain complexes of R-modules. The derived category, denoted by $D(R)$, is a category together with a functor $\pi : C(R) \to D(R)$ such that*

1. *$\pi(f)$ is an isomorphism for any quasi-isomorphism f.*

2. *Any functor $\varphi : C(R) \to X$ transforming quasi-isomorphisms into isomorphisms factors uniquely through $D(R)$. That is, there exists a unique functor $\psi : D(R) \to X$ such that $\varphi = \psi \circ \pi$.*

Recall that a quasi-isomorphism of complexes is a morphism $f : A_\bullet \to B_\bullet$ of chain complexes such that the induced map on homology $H(f) : H(A_\bullet) \to H(B_\bullet)$ is an isomorphism. Next we recall the definition of the bounded derived category.

Definition 3.2.5. *([12], Section 0.2) The bounded derived category of R, denoted by $D^b(R)$, is the full subcategory of $D(R)$ whose objects have finitely generated total homology.*

Finally we arrive at the definition of the singularity category. Recall from [12] that a perfect complex is a complex $X_\bullet \in D^b(R)$ that is isomorphic in $D^b(R)$ to a bounded complex of finitely generated projective R-modules. The full subcategory $D^b_{\mathrm{perf}}(R)$ of perfect complexes can be shown to be a thick triangulated subcategory of $D^b(R)$, and thus it is possible to take the Verdier quotient of $D^b(R)$ by $D^b_{\mathrm{perf}}(R)$ (see [12]).

Definition 3.2.6. *([12], Definition 1.2.2) The singularity category of R, denoted by $D^b_{sing}(R)$, is the Verdier quotient category $D^b(R)/D^b_{perf}(R)$.*

Having gathered the necessary terminology, we close with the statement of the aforementioned equivalences.

Theorem 3.2.7. *([12], Theorem 4.4.1) Let $R = Q/(f)$ be a local hypersurface ring. Then there are equivalences of categories*

$$[mf(Q,f)] \xrightarrow{cok} \underline{MCM}(R) \xrightarrow{i} D^b_{sing}(R).$$

On objects, the functors are given by $cok(\varphi, \psi) = coker\ \varphi$ *and* $i(M) = M$ *(*M *is viewed as a complex concentrated in degree zero).*

3.2.2 Higher Matrix Factorizations

Our purpose in this section is to recall the theory of higher matrix factorizations and to give a new proof of a version of Theorem 1.3.1 from [19] for complete intersections. Instead of beginning in the natural way by defining a higher matrix factorization, we begin with an equivalent notion (at least in our setting).

Definition 3.2.8. *([19], Definition 6.1.1) Let* $(Q, \mathfrak{n}, k)$ *be a Cohen-Macaulay local ring, let* $f_1, ..., f_c \in \mathfrak{n}$ *be a regular sequence, and set* $R = Q/(f_1, ..., f_c)$*. We say that a finitely generated* R*-module* M *of finite projective dimension over* Q *is a* stable syzygy *(or* stable*) with respect to* $f_1, ..., f_c$ *if* $c = 0$ *and* $M = 0$*, or if* $c \geqslant 1$ *and:*

1. $M = Syz_2^R(L)$ *for some finitely generated* R*-module* L *such that* L *has finite projective dimension over* Q*,* L *is MCM without free summand, and the Eisenbud operator* t_c^L *is surjective.*

2. *The module* $\widetilde{M} = Syz_2^{\widetilde{R}}(L)$ *is a stable syzygy with respect to* $f_1, ..., f_{c-1}$*, where* $\widetilde{R} = Q/(f_1, ..., f_{c-1})$*.*

Notes:

(1) From the definition one sees that if Q is regular, then for $c = 1$ a finitely generated R-module M is a stable syzygy if and only if M is MCM without free summand (see [19], Proposition 6.1.6).

(2) If M is stable with respect to $f_1, ..., f_c$, then so are all of its syzygies (see [19], Proposition 6.1.5).

The benefit of working with stable syzygies is that their minimal R-free resolutions can be constructed inductively, beginning with a finite free resolution over the regular local ring Q. This is clear in the case of $c = 1$, but requires more justification in arbitrary codimension. By Proposition 3.1.4, if we can build a minimal $\widetilde{R}$-free resolution of a stable syzygy R-module M, then we can simply apply the Shamash construction to obtain its minimal R-free resolution. Assuming by induction that we understand the resolutions of stable syzygies in codimension $c-1$, with the base case being MCM modules without free summands over hypersurface rings, we need only find a way to navigate between the modules M, L, and $\widetilde{M}$ from the definition. This is the content of the following construction.

Construction: (The Box Complex: [19], Section 6.3) Let $R = Q/(f)$ with $(Q, \mathfrak{n}, k)$ local and $f \in \mathfrak{n}^2$ a Q-regular element, let L be a finitely generated R-module, and set $M = \operatorname{Syz}_2^R(L)$. Let

$$F_\bullet : \quad \cdots \longrightarrow F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow L \longrightarrow 0$$

denote a minimal Q-free resolution of L. Since $F_\bullet$ is a Q-free complex that resolves an R-module, there exists a system of higher homotopies $\boldsymbol{\sigma}$ on $F_\bullet$ with respect to f. Let $\theta := \sigma^{[1]}$ with components $\{\theta_i : F_i \to F_{i+1}\}_{i \geqslant 0}$, and let $\tau := \sigma^{[3]}$ with components $\{\tau_i : F_i \to F_{i+3}\}_{i \geqslant 0}$. Given the above information, one can show the following:

Proposition 3.2.9. *([19], Box Theorem 6.3.2) A Q-free resolution of $M = Syz_2^R(L)$*

is given by the complex

$$\begin{tikzcd} Box(F_\bullet): & \cdots \arrow[r] & F_4 \arrow[r, "\partial_4"] & F_3 \arrow[r, "\partial_3"] & F_2 \\ & & & \oplus & \oplus \\ & & & F_1 \arrow[r, "\partial_1"] \arrow[uur, "\theta_1"] & F_0 \end{tikzcd}$$

The resolution is minimal if and only if θ_1 *is minimal. In particular, it is minimal if* L *admits a surjective Eisenbud operator. Moreover, a homotopy for* f *on* $Box(F_\bullet)$ *is given by*

$$\begin{pmatrix} \theta_2 & \tau_0 \\ \partial_2 & \theta_0 \end{pmatrix}, (\theta_3\ \tau_1), \theta_4, \theta_5, \theta_6, \cdots \quad .$$

We now illustrate the construction by resolving a stable syzygy of complexity two in codimension two.

Example 3.2.10. *Suppose* $R = Q/(f_1, f_2)$ *is a complete intersection of codimension two and that* M *is a finitely generated* R*-module that is stable with respect to* f_1, f_2*. By definition,* $M = Syz_2^R(L)$ *where* L *is MCM without free summand, the minimal* R*-free resolution of* L *is the Shamash resolution, and* $\tilde{M} = Syz_2^{Q/(f_1)}(L)$ *is MCM without free summand. Then* $\tilde{M}$ *has a two-periodic minimal* $Q/(f_1)$*-free resolution:*

$$F_\bullet : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

and so L *has a minimal* $Q/(f_1)$*-free resolution of the form:*

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow 0$$

Applying the box complex construction yields a minimal $Q/(f_1)$-free resolution of M:

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & 0 \\
 & & & \oplus & \oplus & \\
 & & & G_1 \arrow[r] \arrow[uur] & G_0 &
\end{tikzcd}$$

Applying the Shamash construction yields the minimal R-free resolution of M (letting bars denote tensoring down to R):

$$\cdots \longrightarrow \overline{F_1}^n \oplus \overline{G_1} \longrightarrow \overline{F_0}^n \oplus \overline{G_0} \longrightarrow \cdots \longrightarrow \overline{F_1} \oplus \overline{G_1} \longrightarrow \overline{F_0} \oplus \overline{G_0} \longrightarrow 0.$$

Having seen how to build resolutions of stable syzygies, we gather the preliminaries necessary to prove that high syzygies of modules over complete intersections are always stable.

Lemma 3.2.11. *Let $R = Q/(f_1, ..., f_c)$ be a complete intersection of codimension c, let M be a finitely generated R-module, let $F_\bullet$ be the minimal R-free resolution of M, and suppose there is a sequence $\underline{f}' = \{f_1', ..., f_c'\}$ generating $(f_1, ..., f_c)$ such that $t_c^M(\underline{f}') : F_\bullet \to F_\bullet$ is surjective. Let $\{f_1'', ..., f_{c-1}''\}$ be another set of generators for $(f_1', ..., f_{c-1}')$ and set $f_c'' = f_c'$. Then $\underline{f}'' = \{f_1'', ..., f_c''\}$ generates $(f_1, ..., f_c)$ and $t_c^M(\underline{f}'') : F_\bullet \to F_\bullet$ is surjective.*

Proof. The change of rings property of the Eisenbud operators ([18], Proposition 1.7) implies that $t_c^M(\underline{f}')$ and $t_c^M(\underline{f}'')$ are homotopic. Thus, the surjectivity of $t_c^M(\underline{f}')$ is equivalent to the surjectivity of $t_c^M(\underline{f}'')$ since $F_\bullet$ is a minimal complex. $\square$

Construction: (The Iterated Box Complex) Let $R = Q/(f)$ with $(Q, \mathfrak{n}, k)$ local and $f \in \mathfrak{n}^2$ a Q-regular element, let L be a finitely generated R-module, and set

$M_n = \mathrm{Syz}_{2n}^R(L)$ for $n > 0$. Suppose

$$F_\bullet : \cdots \to F_3 \to F_2 \to F_1 \to F_0$$

is a Q-free resolution of L. We construct a Q-free resolution of M_n for each n as follows:

- $n = 1$: Let $\boldsymbol{\sigma}^{F_\bullet}$ be a system of higher homotopies on $F_\bullet$ for f and set $\mathrm{Box}_2(F_\bullet) = \mathrm{Box}(F_\bullet)$. Then as we have seen before, $\mathrm{Box}_2(F_\bullet)$ is a Q-free resolution of M_1.

- $n > 1$: Suppose $\mathrm{Box}_{2(n-1)}(F_\bullet)$ is a Q-free resolution of M_{n-1}. Let $\boldsymbol{\sigma}^{\mathrm{Box}_{2(n-1)}(F_\bullet)}$ be a system of higher homotopies on $\mathrm{Box}_{2(n-1)}(F_\bullet)$ for f. Then setting $\mathrm{Box}_{2n}(F_\bullet) = \mathrm{Box}\left(\mathrm{Box}_{2(n-1)}(F_\bullet)\right)$, the original arguments for the box complex carry through to show that $\mathrm{Box}_{2n}(F_\bullet)$ is a Q-free resolution of M_n.

Note: Suppose $F_\bullet$ is minimal and $t_f^L : F_\bullet \to F_\bullet$ is surjective. Then $\mathrm{Box}_{2n}(F_\bullet)$ is a minimal Q-free resolution of M_n for each $n > 0$ by Proposition 3.1.4.

Theorem 3.2.12. *([19], Theorem 1.3.1) Let $(Q, \mathfrak{n}, k)$ be a regular local ring with infinite residue field k, let $f_1, ..., f_c \in \mathfrak{n}^2$ be a Q-regular sequence, and set $R = Q/(f_1, ..., f_c)$. If N is a finitely generated R-module, then for all $i \gg 0$ the syzygy modules $Syz_i^R(N)$ are stable with respect to some regular sequence $f_1', ..., f_c'$ generating $(f_1, ..., f_c)$.*

Proof. We proceed by induction on c. When $c = 1$, being a stable syzygy is equivalent to being MCM without free summand, so this case follows from ([18], Theorem 6.1). Now suppose $c > 1$. After taking a generic choice of generators $\underline{f'} = \{f_1', ..., f_c'\}$ of $(f_1, ..., f_c)$ and passing to a sufficiently high syzygy we may assume N is MCM and $t_c^N(\underline{f'})$ is surjective ([18], Theorem 3.1). Let $\widetilde{R} = Q/(f_1', ..., f_{c-1}')$ and suppose $Y_\bullet$ is the

minimal $\widetilde{R}$-free resolution of N. By induction, there is a $j \gg 0$ such that $\mathrm{Syz}_{2j+2}^{\widetilde{R}}(N)$ is stable with respect to $f_1'', ..., f_{c-1}''$, some generic choice of generators of $(f_1', ..., f_{c-1}')$. Using Lemma 3.2.11, we may assume that $f_i' = f_i''$ for $1 \leqslant i \leqslant c-1$.

We claim that the module $M = \mathrm{Syz}_{2j+2}^{R}(N)$ is stable with respect to $f_1', ..., f_c'$. To see this, set $L = \mathrm{Syz}_{2j}^{R}(N)$ (so that $M = \mathrm{Syz}_2^R(L)$). First, note that L is MCM without free summand since N is. Also, $t_c^L(\underline{f}')$ is surjective since L is a syzygy of N and $t_c^N(\underline{f}')$ is surjective. Finally, we verify that $\mathrm{Syz}_2^{\widetilde{R}}(L)$ is stable with respect to $f_1', ..., f_{c-1}'$. Using the iterated box complex construction, we see that the minimal $\widetilde{R}$-free resolution of L is $\mathrm{Box}_{2j}(Y_\bullet)$:

$$\begin{tikzcd}
\mathrm{Box}_{2j}(Y_\bullet): & \cdots \arrow[r] & Y_{2j+2} \arrow[r] & Y_{2j+1} \arrow[r] & Y_{2j} \\
 & & & \oplus & \oplus \\
 & & & \bigoplus_{k=1}^{j-1} Y_{2k+1} \arrow[r] \arrow[uur] & \bigoplus_{k=1}^{j-1} Y_{2k}
\end{tikzcd}$$

But then $\mathrm{Syz}_2^{\widetilde{R}}(L) = \mathrm{Syz}_{2j+2}^{\widetilde{R}}(N)$, which is stable with respect to $f_1', ..., f_{c-1}'$ by assumption. Thus, we arrive at the desired result.

□

Finally, we arrive at the definition of a higher matrix factorization. In the case where the ring Q below is a regular local ring (or more generally a Gorenstein local ring) the notion of a stable syzygy is the same as the notion of a higher matrix factorization module (see [19], Theorem 7.2.1).

Definition 3.2.13. *([19], Definition 1.2.2) Let $f_1, ..., f_c \in Q$ be elements in a commutative ring, and set $R(p) = Q/(f_1, ..., f_p)$ for $1 \leqslant p \leqslant c$ with $R = R(c)$. A higher*

matrix factorization (d, h) *with respect to* $f_1, ..., f_c$ *is:*

- *A pair of free finitely generated* Q*-modules* A_0, A_1 *with filtrations*

$$0 \subseteq A_s(1) \subseteq \cdots \subseteq A_s(c) = A_s, \quad \textit{for } s = 0, 1$$

  such that each $A_s(p-1)$ *is a free summand of* $A_s(p)$*;*

- *A pair of maps* d, h *preserving filtrations,*

$$\bigoplus_{q=1}^{c} A_0(q) \xrightarrow{h} A_1 \xrightarrow{d} A_0$$

  where we regard $\bigoplus\limits_{q=1}^{c} A_0(q)$ *is filtered by the submodules* $\bigoplus_{q \leqslant p} A_0(q)$*;*

such that, writing

$$A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{d_p} A_0(p) \qquad 1 \leqslant p \leqslant c$$

for the induced maps, we have

1. $d_p h_p = f_p \cdot id_{A_0(p)}$ *mod* $(f_1, ..., f_{p-1}) A_0(p)$
2. $\pi_p h_p d_p \equiv f_p \cdot \pi_p$ *mod* $(f_1, ..., f_{p-1})(A_1(p)/A_1(p-1))$*, where* π_p *is the natural projection* $A_1(p) \to A_1(p)/A_1(p-1)$*.*

Finally, we define the module of the higher matrix factorization (d, h) *to be* $M = M(c)$*, where*

$$M(p) = Coker(R(p) \otimes d_p), \quad 1 \leqslant p \leqslant c,$$

and refer to M *as a higher matrix factorization module (HMF module).*

In the case when stable syzygies are equivalent to higher matrix factorization modules, the proof of ([19], Theorem 6.4.2) shows that the maps d_p and h_p are essentially liftings of various homotopy maps and differentials from resolutions of intermediate stable syzygies $M(p)$ over the intermediate quotient rings $R(p)$.

It turns out that one can give a more thorough description of the relationship between the modules $M(1), ..., M(c) = M$ (see [19] for more details). We start with a definition:

Definition 3.2.14. *([19], Definition 7.3.1) Let S be a Gorenstein local ring and let M be a finitely generated S-module. Choose an integer $q >$ depth $S -$ depth M. We set*

$$App_S(M) = Syz^S_{-q}(Syz^S_q(M))$$

and refer to it as the essential CM S-approximation of M.

Note that the definition is independent of q by construction. In particular, $\mathrm{App}_S(M)$ is an invariant of the module M. We now use this invariant to describe the aforementioned relationship between $M(1), ..., M(c) = M$:

Theorem 3.2.15. *([19], Theorem 7.4.1) Let Q be a Gorenstein local ring, and suppose $M = M(c)$ is the module over $R = R(c) = Q/(f_1, ..., f_c)$ of a stable minimal matrix factorization of a regular sequence $f_1, ..., f_c$.*

1. *For every $p = 1, ..., c$ we have:*

$$M(p) = App_{R(p)}(M).$$

2. *Suppose that $M = Syz_j^R(N)$ for some R-module N with $j >$ depth $Q -$ depth N. For every $p = 1, ..., c$ we have:*

$$M(p) = Syz_j^{R(p)}(N).$$

3.2.3 The Betti Degree Conjecture

We have seen in the introduction that the Betti numbers of every finitely generated module over a complete intersection are eventually determined by two polynomials of the same degree with the same leading coefficient (see Theorem 3.2.18 below). In this section we examine the leading coefficient more closely by exploring the Betti Degree Conjecture. We begin with a definition:

Definition 3.2.16. *([4], Section 4.2) Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let M be a finitely generated R-module. The complexity of M as an R-module, denoted $cx_R\ M$, is*

$$cx_R\ M = \inf\{d \in \mathbb{N} \mid \beta_n^R(M) \leqslant \beta n^{d-1} \text{ for some real number } \beta \text{ and for all } n \gg 0\}.$$

In the same way that the Auslander-Buchsbaum-Serre Theorem classifies regular local rings as those for which all modules have finite projective dimension, we have the following result of Gulliksen [23] which classifies complete intersections in terms of polynomial growth of Betti numbers:

Theorem 3.2.17. *([23], Theorem 2.3) For a Noetherian local ring $(R, \mathfrak{m}, k)$, the following are equivalent:*

a) R is a complete intersection.

b) $cx_R\ M < \infty$ for all finitely generated R-modules M.

c) $\mathrm{cx}_R\, k < \infty$.

We now recall a result stated in the introduction due to Avramov-Buchweitz:

Theorem 3.2.18. *([5], Theorem 7.3) Let $(R, \mathfrak{m}, k)$ be a local complete intersection, and let M be a finitely generated R-module of complexity $d \geqslant 1$. Then there exist a positive integer $\beta\text{-}deg_R(M)$ and polynomials $p_{even}(t)$ and $p_{odd}(t) \in \mathbb{Q}[t]$ of degree $\leqslant d-2$ such that*

$$\beta_n^R(M) = \frac{\beta\text{-}deg_R(M)}{2^{d-1}(d-1)!} n^{d-1} + \begin{cases} p_{even}(n) & \text{for even } n \gg 0 \\ p_{odd}(n) & \text{for odd } n \ll 0 \end{cases}$$

The Betti Degree Conjecture states that, in the setting above, there is an inequality $\beta\text{-deg}_R(M) \geqslant 2^{d-1}$. In [5], Avramov-Buchweitz remark that the conjecture holds for all modules M over all complete intersections R such that $\mathrm{cx}_R\, M \leqslant 2$. They also note that the conjecture holds for the residue field of a complete intersection of arbitrary codimension. In the main result of this section, we show that the Betti Degree Conjecture holds whenever the module in question has its Betti numbers governed by exactly one polynomial:

Proposition 3.2.19. *Let $(R, \mathfrak{m}, k)$ be a local complete intersection and let M be a finitely generated R-module of complexity $d \geqslant 1$. If the Betti numbers of M are eventually given by a single polynomial, i.e., $\beta_n^R(M) = p(n)$ for all $n \gg 0$ for some $p(t) \in \mathbb{Q}[t]$, then $2^{d-1} \mid \beta\text{-}deg_R(M)$. In particular, $\beta\text{-}deg_R(M) \geqslant 2^{d-1}$.*

Proof. Perhaps after passing to a high syzygy of M, we may assume without loss of generality that $\beta_n^R(M) = p(n)$ for $n \geqslant 0$. Then by Theorem 3.2.18 we may write

$$p(n) = \frac{\beta\text{-deg}_R(M)}{2^{d-1}(d-1)!} n^{d-1} + p_{d-2} n^{d-2} + \cdots .$$

In particular, we note that $p(t)$ takes on integer values for all $t \in \mathbb{N}$. Next, define a new function $\Delta_1(n) := p(n+1) - p(n)$. Then for each $1 < i \leqslant d-1$ set $\Delta_i(n) := \Delta_{i-1}(n+1) - \Delta_{i-1}(n)$. Note that each Δ_i is an integer valued function. We now claim that for each i the leading term of Δ_i is $\frac{\beta\text{-deg}_R(M)}{2^{d-1}(d-1-i)!}n^{d-1-i}$. To see this, we proceed by induction on i.

For $i = 1$, we have

$$\begin{aligned}
\Delta_1(n) &= p(n+1) - p(n) \\
&= \left(\tfrac{\beta\text{-deg}_R(M)}{2^{d-1}(d-1)!}(n+1)^{d-1} + \cdots\right) - \left(\tfrac{\beta\text{-deg}_R(M)}{2^{d-1}(d-1)!}n^{d-1} + \cdots\right) \\
&= \tfrac{\beta\text{-deg}_R(M)}{2^{d-1}(d-1)!}\tbinom{d-1}{1}n^{d-2} + \cdots \\
&= \tfrac{\beta\text{-deg}_R(M)}{2^{d-1}(d-1-1)!}n^{d-1-1} + \cdots
\end{aligned}$$

Now suppose the result holds for all $1 \leqslant j < i \leqslant d-1$. Then

$$\begin{aligned}
\Delta_i(n) &= \Delta_{i-1}(n+1) - \Delta_{i-1}(n) \\
&= \left(\tfrac{\beta\text{-deg}_R(M)}{2^{d-1}(d-i)!}(n+1)^{d-i} + \cdots\right) - \left(\tfrac{\beta\text{-deg}_R(M)}{2^{d-1}(d-i)!}n^{d-i} + \cdots\right) \\
&= \tfrac{\beta\text{-deg}_R(M)}{2^{d-1}(d-i)!}\tbinom{d-i}{1}n^{d-i-1} + \cdots \\
&= \tfrac{\beta\text{-deg}_R(M)}{2^{d-1}(d-1-i)!}n^{d-1-i} + \cdots\ ,
\end{aligned}$$

which gives the desired form. In particular, Δ_{d-1} is an integer-valued function satisfying

$$\Delta_{d-1}(n) = \frac{\beta\text{-deg}_R(M)}{2^{d-1}(d-1-(d-1))!}n^{d-1-(d-1)} = \frac{\beta\text{deg}_R(M)}{2^{d-1}}.$$

It follows that $2^{d-1} \mid \beta\text{-deg}_R(M)$, and so in particular $\beta\text{-deg}_R(M) \geqslant 2^{d-1}$. $\square$

3.2.4 Betti Numbers and HMF Modules

As we have seen, over a complete intersection $R = Q/(f_1, ..., f_c)$ of codimension c every finitely generated R-module M has a sufficiently high syzygy that is stable with respect to some generic choice of minimal generators of $(f_1, ..., f_c)$. Given the explicit construction of the minimal R-free resolutions of stable syzygy modules, we are able to describe the Poincaré series and the Betti degrees of such modules.

With the notation above, let M be a finitely generated R-module that is stable with respect to $f_1, ..., f_c$. Then M is a HMF module ([19], Theorem 7.2.1). Among many other things, this implies that M comes with two finitely generated free Q modules A_1, A_0, filtrations

$$0 \subseteq A_s(1) \subseteq \cdots \subseteq A_s(c) = A_s \quad \text{for } s = 0, 1$$

and direct sum decompositions $A_s(p) = A_s(p-1) \oplus B_s(p)$ for $s = 0, 1$ and $1 < p \leqslant c$ (we also set $A_s(1) = B_s(1)$ for $s = 0, 1$). It turns out that these free modules are key players in determining the Betti numbers of M, as the following result shows:

Theorem 3.2.20. *([19], Corollary 5.2.1) Keeping the notation from above, suppose M is a HMF module over R with respect to $f_1, ..., f_c$. Then the Poincaré series of M over R has the form*

$$P^R_M(t) = \sum_{p=1}^{c} \frac{1}{(1-t^2)^{c-p+1}} \left(\ rank_R(B_1(p)) \cdot t + rank_R(B_0(p))\ \right).$$

Moreover, the Betti numbers of M over R are given by the following two polynomials:

$$b^M_{even}(2t) = \sum_{p=1}^{c} \binom{c-p+t}{c-p} rank_R(B_0(p))$$

$$b_{odd}^{M}(2t+1) = \sum_{p=1}^{c} \binom{c-p+t}{c-p} rank_R(B_1(p)).$$

Note: In ([19], Corollary 5.2.3) it is shown that if

$$\gamma = \min_{1 \leqslant q \leqslant c} \{q \mid \mathrm{rank}_R(B_0(q)), \mathrm{rank}_R(B_1(q)) \neq 0\},$$

then $\mathrm{rank}_R(B_0(\gamma)) = \mathrm{rank}_R(B_1(\gamma))$ (this last equality reflects the fact that the "bottom layer" of a HMF always comes from a classical matrix factorization). So, if M is a HMF module over R with respect to $f_1, ..., f_c$ and we have

$$\gamma = \min_{1 \leqslant q \leqslant c} \{q \mid \mathrm{rank}_R(B_0(q)), \mathrm{rank}_R(B_1(q)) \neq 0\},$$

then ([19], Corollary 5.2.3) yields an equality

$$\beta\text{-}\mathrm{deg}_R(M) = \mathrm{rank}_R(B_0(\gamma)) = \mathrm{rank}_R(B_1(\gamma)).$$

3.3 The Matrix Factorization Conjecture

In our final section, we examine the Matrix Factorization Conjecture in the local setting. We first show that it implies the Total Rank Conjecture and the Betti Degree Conjecture for modules of maximal complexity over complete intersections. After listing some situations for which the Matrix Factorization Conjecture holds, we produce a family of counterexamples which build off of the Iyengar-Walker construction.

3.3.1 Relationship with other Conjectures

We begin by showing that the Matrix Factorization Conjecture implies the Total Rank Conjecture.

Proposition 3.3.1. *If the Matrix Factorization Conjecture is true, then the Total Rank Conjecture is true for regular local rings.*

Proof. Let $M \neq 0$ be a finite length Q-module, where $(Q, \mathfrak{n}, k)$ is a regular local ring of dimension d. Then a positive power of $\mathfrak{n}$ annihilates M, and so in particular there is some regular element $g \in \mathfrak{n}$ such that $g \cdot M = 0$. Setting $f = g^2$ we see that M is a finite length module over $R = Q/(f)$. Moreover, $f = g^2 \in \mathfrak{n}(0 :_Q M)$, and so the Shamash resolution is a minimal R-free resolution of M by Proposition 3.1.2. This yields an equality of Poincaré series

$$\mathcal{P}_M^R(t) = \frac{\mathcal{P}_M^Q(t)}{(1-t^2)}.$$

Now fix $j > d$. Then we have an equality

$$\beta_j^R(M) = \sum_{i \geqslant 0} \beta_{2i}^Q(M) = \sum_{i \geqslant 0} \beta_{2i+1}^Q(M).$$

As we are assuming the validity of the Matrix Factorization Conjecture, we know that $\beta_j^R(M) \geqslant 2^{d-1}$, and so

$$\sum_{i=0}^{d} \beta_i^Q(M) = \sum_{i \geqslant 0} \beta_{2i}^Q(M) + \sum_{i \geqslant 0} \beta_{2i+1}^Q(M) \geqslant 2^{d-1} + 2^{d-1} = 2^d.$$

$\square$

Proposition 3.3.2. *If the Matrix Factorization Conjecture is true, then the Betti Degree Conjecture is true for modules of maximal complexity.*

Proof. Let $(R, \mathfrak{m}, k)$ be a complete intersection of codimension c, say $R = Q/(f_1, ..., f_c)$, and let M be a finitely generated R-module of complexity c with minimal free resolution $F_\bullet$. We begin by making a series of reductions.

Since the Betti numbers of M remain unchanged after passing to a faithfully flat extension, we may assume that Q (and R) has an infinite residue field. Next, we may pass to a high syzygy and assume M is MCM over R. Now let $\underline{x}$ be a maximal R-regular sequence in $\mathfrak{m} \setminus \mathfrak{m}^2$. Then $\underline{x}$ is an M-regular sequence as well, and so $F_\bullet \otimes R/(\underline{x})$ is the minimal $R/(\underline{x})$-free resolution of $M/\underline{x}M$ ([10], Proposition 1.1.5). Thus, we may assume from the outset that R is an artinian complete intersection of codimension c and M has finite length. By passing to a high syzygy again, and perhaps after relabeling the generators of the ideal $(f_1, ..., f_c)$, we may use Theorem 3.2.12 to assume M is a stable syzygy with respect to $f_1, ..., f_c$ and that $M \cong \mathrm{Syz}_j^R(N)$ with $j > c$ and N a finite length R-module without free summand.

We have the equality $1 = \min\{p \mid B_0(p), B_1(p) \neq 0\}$. Moreover, by Theorem 3.2.15 there is an isomorphism $M(1) \cong \mathrm{Syz}_j^{R(1)}(N)$, with $R(1) = Q/(f_1)$. But now we have that $M(1)$ is a stable syzygy of complexity one that is a high syzygy of a finite length module over a dimension $d-1$ hypersurface. Thus, the Matrix Factorization Conjecture implies that $\beta_0^{R(1)}(M(1)) = \mathrm{rank}_R(B_0(1)) \geqslant 2^{d-1}$. But now using the note after Theorem 3.2.20, we get $\beta\text{-}\deg_R(M) \geqslant 2^{d-1}$, which is the desired result. $\square$

We close by noting that Avramov-Buchweitz showed that the Betti Degree Conjecture implies the Total Rank Conjecture ([5], Example 7.4).

3.3.2 Examples of Small Dimension and Loewy Length

In this section we collect some examples in which the Matrix Factorization Conjecture holds. We begin with hypersurfaces of small dimension:

Proposition 3.3.3. *Let $(R, \mathfrak{m}, k)$ be a local hypersurface of dimension $d = 0$ or $d = 1$. Then the Matrix Factorization Conjecture holds for R.*

Proof. If $d = 0$ the result is trivial: the i^{th} Betti number of any module of infinite projective dimension is at least $2^d = 1$ for all $i \geqslant 0$.

Now suppose $d = 1$, and let M be a finite length R-module of infinite projective dimension. Then $\operatorname{rank}_R(M) = 0$. Thus, $\gamma_i^R(M) = \operatorname{rank}_R(\operatorname{Syz}_i^R(M))$ is well defined for all $i \geqslant 0$ ([10], Proposition 1.4.5). Suppose $\beta_i^R(M) = 1$ for some $i > 2$. This yields an exact sequence

$$0 \to \operatorname{Syz}_{i+1}^R(M) \to R \to \operatorname{Syz}_i^R(M) \to 0.$$

It follows that one of $\operatorname{Syz}_i^R(M)$ or $\operatorname{Syz}_{i+1}^R(M)$ has rank 0, and so one of these modules is annihilated by an R-regular element, which is a contradiction. $\square$

Our final proposition in this section shows that for a fixed hypersurface $R = Q/(f)$ with a residue field k of characteristic different from 2, there is a class of finite length R-modules for which the Matrix Factorization Conjecture holds. We recall that for a finitely generated module M over a local ring $(R, \mathfrak{m}, k)$, the *Loewy length* of M is given by

$$\ell\ell_R(M) = \inf\{n \in \mathbb{N} \mid \mathfrak{m}^n M = 0\}.$$

Proposition 3.3.4. *Let $(Q, \mathfrak{n}, k)$ be a regular local ring of dimension $d+1$, with char $k \neq 2$, let $f \in \mathfrak{n}^j$, and set $R = Q/(f)$. If $M \neq 0$ is an R-module such that $\ell\ell_R(M) = m < j$, then the Matrix Factorization Conjecture holds for M.*

Proof. Since $\ell\ell_Q(M) = \ell\ell_R(M) = m$, we see that $f \in \mathfrak{n}^j = \mathfrak{n}\mathfrak{n}^{j-1} \subseteq \mathfrak{n}(0 :_Q M)$. Letting $U_\bullet$ be a minimal Q-free resolution of M and choosing a system of higher homotopies $\boldsymbol{\sigma}$ of f, we know that $\mathrm{Sh}(U_\bullet, \boldsymbol{\sigma})$ is a minimal R-free resolution of M by Proposition 3.1.2. Because char $k \neq 2$ and Q is regular, we know from [31] that the Total Rank Conjecture holds for Q. Hence, we have an inequality

$$\sum_{n=0}^{d+1} \beta_n^Q(M) \geqslant 2^{d+1}.$$

By the construction of the Shamash resolution, we obtain the inequality

$$\beta_n^R(M) = \frac{1}{2}\left(\sum_{n=0}^{d+1} \beta_n^Q(M)\right) \geqslant \frac{1}{2} \cdot 2^{d+1} = 2^d$$

for all $n > d$, which is the desired result. □

3.3.3 Knörrer Periodicity

In this section we use the Buchweitz Equivalence ([12], Theorem 4.4.1) to show that the validity of the Matrix Factorization Conjecture is preserved under change of rings in certain situations. To do this, it will be necessary to recall the theory of Knörrer Periodicity from [25].

For the rest of the section we will use the following notation. Let $(Q, \mathfrak{n}, k)$ be a regular local ring of dimension d with char $k \neq 2$, let $f \in \mathfrak{n}^2$ be nonzero, and set $R = Q/(f)$. Following [25], we define $R^{\#} = Q[[z]]/(f + z^2)$ to be the *double branched cover* of R, which is a hypersurface ring of dimension d. We can of course repeat this

procedure, which yields $R^{\#\#} = Q[[u,v]]/(f+uv)$, a hypersurface ring of dimension $d+1$ (here we use that char $k \neq 2$ to replace $f+z^2+w^2$ with $f+uv$). Note that any R-module admits an $R^{\#}$-module structure via the natural projection $R^{\#} \twoheadrightarrow R$ sending z to 0, and similarly for $R^{\#\#}$.

Next we define a functor $F : \mathrm{mf}(Q,f) \to \mathrm{mf}(Q[[u,v]], f+uv)$ from ([25], Section 3). Given $(\varphi : F \to G, \psi : G \to F) \in \mathrm{mf}(Q,f)$, set $F(\varphi,\psi)$ to be

$$\left(\begin{pmatrix} u & \psi \\ \varphi & -v \end{pmatrix} : H \to H, \begin{pmatrix} v & \psi \\ \varphi & -u \end{pmatrix} : H \to H\right).$$

where $H = (F \oplus G) \otimes_Q Q[[u,v]]$. From now on we omit the free modules in the description of the matrix factorizations. For a morphism (α,β) in $\mathrm{mf}(Q,f)$, set $F(\alpha,\beta)$ to be

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right).$$

It can be shown that F induces a functor $\overline{F} : [\mathrm{mf}(Q,f)] \to [\mathrm{mf}(Q[[u,v]], f+uv)]$. Moreover, we have the following result:

Theorem 3.3.5. *([25], Theorem 3.1) The functor $\overline{F}$ is an equivalence of categories between $[mf(Q,f)]$ and $[mf(Q[[u,v]], f+uv)]$.*

In this situation we say that the hypersurface rings R and $R^{\#\#}$ are *Knörrer equivalent.*

Although we will have a result stated in terms of Knörrer equivalence, we will actually be more interested in a functor $\# : [\mathrm{mf}(Q,f)] \to [\mathrm{mf}(Q[[z]], f+z^2)]$, which we now recall.

Proposition 3.3.6. *([32], Chapter 12) Let $(Q,\mathfrak{n},k)$ be a regular local ring, let $f \in \mathfrak{n}^2$*

be a Q-regular element, and set $R = Q/(f)$. Then there is a functor $\# : [mf(Q, f)] \to [mf(Q[[z]], f + z^2)]$ given on objects by

$$(\varphi, \psi) \mapsto \left(\begin{pmatrix} \psi & -z \\ z & \varphi \end{pmatrix}, \begin{pmatrix} \varphi & z \\ -z & \psi \end{pmatrix} \right).$$

Remark 3.3.7. *As it will come in handy later, we will note that*

$$\left(\begin{pmatrix} \psi & -z \\ z & \varphi \end{pmatrix}, \begin{pmatrix} \varphi & z \\ -z & \psi \end{pmatrix} \right) \cong \left(\begin{pmatrix} \varphi & z \\ -z & \psi \end{pmatrix}, \begin{pmatrix} \psi & -z \\ z & \varphi \end{pmatrix} \right).$$

Note that $\#$ also yields a functor $\epsilon = \operatorname{cok}_{R^\#} \circ \# \circ \operatorname{cok}_R^{-1} : \underline{\mathrm{MCM}}(R) \to \underline{\mathrm{MCM}}(R^\#)$. A nice feature of the $\#$ functor is that there is a sense in which it preserves syzygies. We make this notion precise with the following two results, the second of which appears in the argument of a more general result in [19].

Proposition 3.3.8. *([32], Lemma 12.3) If M is a MCM R-module with no nontrivial free summands, then*

$$\epsilon(M) = Syz_1^{R^\#}(M),$$

where M is regarded as an $R^\#$-module via restriction of scalars along $R^\# \twoheadrightarrow R$.

Proposition 3.3.9. *([19]) If M is a finitely generated R-module and $M' = Syz_1^R(M)$, then $Syz_i^{R^\#}(M') \cong Syz_{i+1}^{R^\#}(M)$ for all $i \geqslant 1$.*

The second Proposition yields the following:

Corollary 3.3.10. *If M is a finitely generated R-module and $M' = Syz_j^R(M)$ for some $j > 0$, then $Syz_i^{R^\#}(M') = Syz_{i+j}^{R^\#}(M)$ for all $i \geqslant 1$.*

Proof. We proceed by induction on j. The case $j = 1$ is the previous proposition. If $j > 1$, then for all $i \geqslant 1$ we have

$$\mathrm{Syz}_i^{R^\#}(M') = \mathrm{Syz}_i^{R^\#}(\mathrm{Syz}_{j-1}^R(\mathrm{Syz}_1^R(M))) = \mathrm{Syz}_{i+j-1}^{R^\#}(\mathrm{Syz}_1^R(M)) = \mathrm{Syz}_{i+j}^{R^\#}(M).$$

□

Now suppose M is a finitely generated R-module of infinite projective dimension. Then we can view M as a (nonzero) object in $D^b_{\mathrm{sing}}(R)$. In [12], Buchweitz uses the theory of MCM approximations to show that in $D^b_{\mathrm{sing}}(R)$, the module M is equivalent to $\mathrm{Syz}_{2j}^R(M)$ for some (and hence for all) $j > \frac{d}{2}$. Now consider the following diagram:

$$\begin{array}{ccccc}
D^b_{\mathrm{sing}}(R) & \xrightarrow{i_R^{-1}} & \underline{MCM}(R) & \xrightarrow{\mathrm{cok}_R^{-1}} & [\mathrm{mf}(Q,f)] \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\#} \\
D^b_{\mathrm{sing}}(R^\#) & \xrightarrow{i_{R^\#}^{-1}} & \underline{MCM}(R^\#) & \xrightarrow{\mathrm{cok}_{R^\#}^{-1}} & [\mathrm{mf}(Q[[z]], f+z^2)]
\end{array}$$

where α is the restriction of scalars functor induced by the ring map $R^\# \twoheadrightarrow R$.

First, recall that for all $i \gg 0$, $\mathrm{Syz}_{i+2j}^{R^\#}(M) \cong \mathrm{Syz}_i^{R^\#}(\mathrm{Syz}_{2j}^R(M))$. Moreover, by Remark 3.3.7 and Proposition 3.3.8 we see that

$$\begin{aligned}
\epsilon(i_R^{-1}(M)) &= \epsilon(\mathrm{Syz}_{2j}^R(M)) \\
&= \mathrm{Syz}_1^{R^\#}(\mathrm{Syz}_{2j}^R(M)) \\
&= \mathrm{Syz}_{2j+1}^{R^\#}(M) \\
&\cong \mathrm{Syz}_{2j+2}^{R^\#}(M) \\
&= i_{R^\#}^{-1}(\alpha(M)).
\end{aligned}$$

By definition, the second square commutes, and so we see that the diagram commutes.

Sending M through the top row, we see that it corresponds to a unique reduced matrix factorization; namely $(\varphi, \psi) = \mathrm{cok}_R^{-1}(i_R^{-1}(M))$. Thus, we can read off the eventual Betti numbers of M from its image in $[\mathrm{mf}(Q, f)]$. Repeating the same process on the bottom row, we see that

$$\left(\begin{pmatrix} \psi & -z \\ z & \varphi \end{pmatrix}, \begin{pmatrix} \varphi & z \\ -z & \psi \end{pmatrix} \right) = \mathrm{cok}_{R^\#}^{-1}(i_{R^\#}^{-1}(\alpha(M)))$$

is the reduced matrix factorization which describes the asymptotic structure of the minimal $R^\#$-free resolution of M. This leads to the following result:

Proposition 3.3.11. *Let $(Q, \mathfrak{n}, k)$ be a regular local ring of dimension $d+1$, and let $f \in \mathfrak{n}^2$ be a Q-regular element. If the Matrix Factorization Conjecture holds for the ring $R^\# = Q[[z]]/(f + z^2)$, then the Matrix Factorization Conjecture holds for the ring $R = Q/(f)$.*

Note: Iterating this result and taking the contrapositive yields Proposition 1.0.13 from the Introduction.

Proof. Let M be a finite length R-module of infinite projective dimension, and suppose its minimal R-free resolution is eventually desribed by the reduced matrix factorization (φ, ψ). Viewing M as an $R^\#$-module, we see from the preceding discussion that its minimal $R^\#$-free resolution is eventually described by the reduced matrix factorization

$$\left(\begin{pmatrix} \psi & -z \\ z & \varphi \end{pmatrix}, \begin{pmatrix} \varphi & z \\ -z & \psi \end{pmatrix} \right).$$

Since we are assuming that the Matrix Factorization Conjecture holds over $R^{\#}$, it follows that the size of these matrices is at least $2^{d+1} \times 2^{d+1}$. This means that the size of the matrices φ and ψ is at least $2^d \times 2^d$, which gives the desired result. $\square$

We close by remarking that there is an alternate proof of the proposition that goes as follows: since dim $R^{\#} = d+1$, $R = R^{\#}/zR^{\#}$, and dim $R = d$, it follows that z is $R^{\#}$-regular ([10], Proposition A.4). Thus, we have an equality

$$\mathcal{P}_M^R(t) = \frac{\mathcal{P}_M^{R^{\#}}(t)}{(1+t)}$$

([4], Proposition 3.3.5). In particular, $\beta_n^{R^{\#}}(M) = \beta_n^R(M) + \beta_{n-1}^R(M)$ for all $n > 0$. Applying the hypothesis yields an inequality

$$2^{d+1} \leqslant \beta_n^{R^{\#}}(M) = \beta_n^R(M) + \beta_{n-1}^R(M)$$

for $n > d+1$. But for $n > d+1$ we also have $\beta_n^R(M) = \beta_{n-1}^R(M)$, and so $2^{d+1} \leqslant 2\beta_n^R(M)$ for $n > d+1$. Dividing both sides by two gives the desired result.

Although the second proof requires less machinery, the first proof is useful in the sense that if we are given explicit matrix factorizations describing the eventual structure of an R-resolution of M, we can produce explicit matrix factorizations describing the eventual structure of an $R^{\#}$-resolution of M.

3.3.4 Counterexamples

In this section we recall a construction of Iyengar-Walker from [24] which produces counterexamples to a variety of conjectures pertaining to complexes of free modules. For our purposes, we will be the most interested in their construction providing a

counterexample to the Betti Degree Conjecture, which will in turn provide a counterexample to the Matrix Factorization Conjecture. We then build off of their construction to produce additional counterexamples, both to the Matrix Factorization Conjecture and the Betti Degree Conjecture.

3.3.4.1 The Iyengar-Walker Construction

In what follows, $(Q, \mathfrak{n}, k)$ is a regular local ring of dimension $d \geqslant 8$ with char $k \neq 2$ and $R = Q/(f_1, ..., f_c)$ is a complete intersection of codimension c satisfying the condition that $f_i \in \mathfrak{n}^3$ for $1 \leqslant i \leqslant c$. Let Λ be an exterior algebra on a k-vector space with basis $e_1, ..., e_d$. In [24], Iyengar-Walker produce an endomorphism on Λ having small rank.

Proposition 3.3.12. *([24], Corollary 2.3) There is an element $\omega \in \Lambda^2$ such that*

$$dim_k Ker(\phi) + dim_k Coker(\phi) < 2^d,$$

where $\phi : \Lambda \to \Lambda$ denotes multiplication by ω.

Let M be a complex of R-modules with finitely generated total homology. As in [24], we will define the Betti numbers of M by taking a semiprojective resolution, say F, of M over R, and setting

$$\beta_i^R(M) = \dim_k \operatorname{Ext}^i_R(M, k) = \dim_k \mathrm{H}_{-i}(\operatorname{Hom}_R(F, k)).$$

Because M has finitely generated total homology, there is Laurent polynomial $p_M(t)$

such that

$$\mathcal{P}_M^R(t) := \sum_{i\in\mathbb{Z}} \beta_i^R(M)t^i = \frac{p_M(t)}{(1-t^2)^c},$$

(see [4], Remark 9.2.6). Also, $p_M(1) \neq 0$ if and only if $\mathrm{cx}_R(M) = c$ (see [4], Theorem 9.2.1). Finally, if $c \geqslant 1$ and M has complexity c, then $p_M(1) = 2\beta\text{-}\deg_R(M)$ (see [24], Section 4).

We now recall the structure of the modules $\mathrm{Ext}_Q^*(k,k)$ and $\mathrm{Ext}_R^*(k,k)$ (see [4], Example 10.2.3). Since Q is a regular local ring, there is an isomorphism of k-algebras

$$\Lambda \cong \mathrm{Ext}_Q^*(k,k).$$

Because of our assumption that the regular sequence $f_1, ..., f_c$ is contained in $\mathfrak{n}^3$, we obtain an isomorphism of k-algebras

$$\mathrm{Ext}_R^*(k,k) \cong \Lambda \otimes_k S,$$

where $S = k[\chi_1, ..., \chi_c]$ is a polynomial ring such that each χ_i has cohomological degree 2. Using this structure, together with the previous Proposition, one obtains the following result:

Theorem 3.3.13. *([24], Theorem 4.1) Let $(R, \mathfrak{m}, k)$ be a complete intersection of codimension c with char $k \neq 2$ such that $R = Q/(f_1, ..., f_c)$ with $(Q, \mathfrak{n}, k)$ a regular local ring and $f_i \in \mathfrak{n}^3$ for all $1 \leqslant i \leqslant c$. If $e =$ embdim $R \geqslant 8$, then there exists a fintely generated R-module N such that*

$$0 < p_N(1) < 2^e.$$

In particular, if R is artinian then we have $c = e$. Thus, the module N from the Theorem is a counterexample to the Betti Degree Conjecture. By taking the contrapositive of Proposition 3.3.2, we immediately obtain the following corollary:

Corollary 3.3.14. *For each $d \geqslant 7$, there is a local hypersurface ring $(R, \mathfrak{m}, k)$ with char $k \neq 2$ and dim $R = d$ for which the Matrix Factorization Conjecture fails.*

3.3.4.2 Extending to the Quadratic Case

The methods of the previous section produce counterexamples to the Betti Degree Conjecture for certain complete intersections $R = Q/(f_1, ..., f_c)$ satisfying $f_1, ..., f_c \in \mathfrak{n}_Q^3$. By extension, the corresponding counterexamples to the Matrix Factorization Conjecture all come from hypersurfaces of the form $R = Q/(f)$ with $f \in \mathfrak{n}_Q^3$. In this section we will produce counterexamples to the Matrix Factorization Conjecture with $f \in \mathfrak{n}_Q^2 \setminus \mathfrak{n}_Q^3$, and we will produce counterexamples to the Betti Degree Conjecture without assuming that each f_i is in $\mathfrak{n}_Q^3$.

In the case of the Matrix Factorization Conjecture, we have already done the necessary work. By Proposition 3.3.11, if $R = Q/(f)$ is a hypersurface for which the Matrix Factorization Conjecture fails, then $R^{\#} = Q[[z]]/(f + z^2)$ is again a hypersurface for which the Matrix Factorization Conjecture fails. Making the necessary adjustments for the Betti Degree Conjecture is a slightly more involved endeavor, which we will now begin.

Let $(Q, \mathfrak{n}, k)$ be a regular local ring of dimension d. Let $R = Q/(f_1, f_2, ..., f_c)$ be a complete intersection of codimension c such that $f_1, ..., f_c \in \mathfrak{n}_Q^3$, and set $R_1^{\#} = Q[[z]]/(f_1 + z^2, f_2, ..., f_c)$. The next two results serve to illustrate the relationship between homological invariants over R and homological invariants over $R_1^{\#}$.

Lemma 3.3.15. *$R_1^{\#}$ is a complete intersection of codimension c. Moreover, the image*

of z in $R_1^{\#}$ is a regular element.

Proof. By ([10], Proposition A.4), the first claim would follow if we could demonstrate an equality dim $R_1^{\#} := j = d - c + 1$. We first note that dim $R = d - c$ and that $R \cong R_1^{\#}/zR_1^{\#}$. We now obtain

$$\begin{aligned} \dim R &\geqslant \dim R_1^{\#} - 1 \quad \text{([10], Proposition A.4)} \\ &= j - 1 \\ &\geqslant (d - c + 1) - 1 \quad \text{([10], Proposition A.4)} \\ &= d - c. \end{aligned}$$

This now forces $j - 1 = d - c$, and so we have dim $R_1^{\#} = d - c + 1$. From this we also see that dim $R =$ dim $R_1^{\#} - 1$, from which we conclude that the image of z in $R_1^{\#}$ is a regular element. □

Lemma 3.3.16. *Let M be a finitely generated R-module. Then there are equalities*

$$cx_R(M) = cx_{R_1^{\#}}(M) \quad \text{and} \quad 2\beta\text{-}deg_R(M) = \beta\text{-}deg_{R_1^{\#}}(M).$$

Proof. From ([4], Proposition 3.3.5) we have an equality

$$\mathcal{P}_M^R(t) = \frac{\mathcal{P}_M^{R_1^{\#}}(t)}{1+t},$$

which yields

$$\beta_n^{R_1^{\#}}(M) = \beta_n^R(M) + \beta_{n-1}^R(M) \quad \text{for all } n \geqslant 1.$$

Both desired equalities now follow immediately. □

Having established these preliminaries, we can state our main result for the section.

Theorem 3.3.17. *Let $(Q, \mathfrak{n}, k)$ be a regular local ring of dimension $2n$ with $n \geqslant 20$, let $R = Q/(f_1, f_2, ..., f_{2n})$ be a complete intersection of codimension $2n$ with $f_i \in \mathfrak{n}^3$ for $1 \leqslant i \leqslant 2n$, and set $R_1^{\#} = Q[[z]]/(f_1 + z^2, f_2, ..., f_{2n})$. If char $k = 0$, then there exists a finitely generated $R_1^{\#}$-module M of complexity $2n$ such that*

$$\beta\text{-}deg_{R_1^{\#}}(M) < 2^{2n-1}.$$

Proof. By ([24], Theorem 4.1), there is finitely generated R-module M of complexity $2n$ such that

$$\beta\text{-deg}_R(M) = \frac{1}{2} \cdot \binom{2(n+1)}{n+1}.$$

By ([24], Remark 2.2), there is an inequality

$$\frac{1}{2}\binom{2(n+1)}{n+1} < 2^{2n-1} \cdot \frac{4}{\sqrt{\pi(n+1)}} < 2^{2n-2},$$

where the last inequality comes from the assumption that $n \geqslant 20$. Lemma 3.3.16 now implies that

$$\beta\text{-deg}_{R_1^{\#}}(M) = 2\beta\text{-deg}_R(M) < 2 \cdot 2^{2n-2} = 2^{2n-1},$$

which is the desired result. □

Earlier we saw explicitly how to take reduced matrix factorizations over a hypersurface $R = Q/(f)$ that were counterexamples to the Matrix Factorizations and turn them into counterexamples over $R^{\#} = Q[[z]]/(f + z^2)$. It would be interesting to

see if there was a similar correspondence between higher matrix factorizations that yielded counterexamples to the Betti Degree Conjecture over R and $R_1^{\#}$.

3.3.5 Establishing a Linear Lower Bound

We have now seen that the Matrix Factorization Conjecture need not hold in general. This leaves us with the following question: is there any general lower bound, depending only on the dimension of the underlying hypersurface, for the size of matrix factorizations appearing in the minimal free resolutions of finite length modules of infinite projective dimension? And if so, what is the optimal lower bound? In this section we establish a lower bound that is linear with respect to dimension. This bound follows from a general principle regarding free complexes having finite length total homology:

Proposition 3.3.18. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension $d > 0$ and let $F_\bullet$ be a complex of finitely generated free R-modules such that $0 < \ell(H(F_\bullet)) < \infty$. Also suppose there is a $j > 0$ such that for every $x \in \mathfrak{m}^j$, multiplication on $F_\bullet$ by x is nullhomotopic. If $F_i \neq 0$, then there is an inequality*

$$\mathit{rank}_R(F_{i-1}) + \mathit{rank}_R(F_{i+1}) \geqslant d.$$

Proof. Let $F_\bullet$, with differential ∂, be a complex as described in the statement, and suppose $F_j = R^{n_j}$ for each j. We may assume, without loss of generality, that $F_\bullet$ is a minimal complex. Write $\partial_i = (a_{ij})$ and $\partial_{i+1} = (b_{ij})$ as matrices with $a_{ij}, b_{ij} \in R$. Now let $x \in \mathfrak{m}^j$ be nonzero. By assumption, there is a nullhomotopy between multiplication

by x and 0. This gives a diagram

$$\begin{tikzcd}
R^{n_{i+1}} \arrow[r, "\partial_{i+1}"] \arrow[d, "x"] & R^{n_i} \arrow[r, "\partial_i"] \arrow[d, "x"] \arrow[dl, "\beta"'] & R^{n_{i-1}} \arrow[d, "x"] \arrow[dl, "\alpha"'] \\
R^{n_{i+1}} \arrow[r, "\partial_{i+1}"] & R^{n_i} \arrow[r, "\partial_i"] & R^{n_{i-1}}
\end{tikzcd}$$

in which

$$\partial_{i+1}\beta + \alpha\partial_i = x \cdot \mathrm{Id}_{R^{n_i}}$$

for some matrices $\alpha = \left(c_{ij}^x\right)$ and $\beta = \left(d_{ij}^x\right)$ with entries in R (the superscripts merely indicate that these ring elements may depend on the choice of x). Equating the top left entries in the matrices coming from this equality yields

$$\sum_{\ell=1}^{n_{i+1}} b_{1\ell} d_{\ell 1}^x + \sum_{\ell=1}^{n_{i-1}} c_{1\ell}^x a_{\ell 1} = x.$$

Letting $I := \left(a_{11}, ..., a_{n_{i-1}1}, b_{11}, ..., b_{1n_{i+1}}\right) \subseteq R$ we see that $x \in I$. As $x \in \mathfrak{m}^j$ was an arbitrary nonzero element, we see that $\mathfrak{m}^j \subseteq I \subseteq \mathfrak{m}$ (where the last containment follows from the minimality of $F_\bullet$). Hence, $\mathrm{ht}(I) = \mathrm{ht}(\mathfrak{m}) = \dim R = d$. On the other hand, we know from Krull's Principal Ideal Theorem that

$$\mathrm{ht}(I) \leqslant \mu(I) \leqslant n_{i-1} + n_{i+1},$$

which yields the desired result. □

Corollary 3.3.19. *Let $(R, \mathfrak{m}, k)$ be a local hypersurface ring of dimension d, and let M be a finite length R-module of infinite projective dimension. Then for all $n > d+1$ there is an inequality $\beta_n^R(M) \geqslant \frac{d}{2}$.*

Proof. We may assume $d > 0$. Let $F_\bullet$ be a minimal R-free resolution of M. Then $0 < \ell(H(F_\bullet)) = \ell(M) < \infty$, and for $j = \ell\ell_R(M) > 0$, multiplication on $F_\bullet$ by each $x \in \mathfrak{m}^j$ is nullhomotopic. Thus, Proposition 3.3.18 applies to $F_\bullet$. Choosing $n > d+1$, we see that $\beta^R_{n-1}(M) = \beta^R_n(M) = \beta^R_{n+1}(M)$ ([18], Theorem 6.1). The previous Proposition yields an inequality $\beta^R_{n-1}(M) + \beta^R_{n+1}(M) \geqslant d$. Thus,

$$\beta^R_n(M) = \frac{1}{2}(\beta^R_{n-1}(M) + \beta^R_{n+1}(M)) \geqslant \frac{d}{2},$$

which is the desired result. □

Proposition 3.3.18 also has applications to finite free resolutions:

Corollary 3.3.20. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d, and let $M \neq 0$ be a finite length R-module of finite projective dimension. Then there is an inequality*

$$\sum_{i=0}^{d} \beta^R_i(M) \geqslant \frac{1}{2}(d^2 + d).$$

Proof. We may assume $d > 0$. Let $F_\bullet$ be a minimal R-free resolution of M. Since M is nonzero and has finite length, we know that $\mathrm{pd}_R(M) = d$. Proposition 3.3.18 now yields

$$\begin{aligned}
2\sum_{i=0}^{d} \beta^R_i(M) &= 2\sum_{i=0}^{d} \mathrm{rank}_R(F_i) \\
&\geqslant \mathrm{rank}_R(F_0) + 2\sum_{i=1}^{d-1} \mathrm{rank}_R(F_i) + \mathrm{rank}_R(F_d) \\
&= \sum_{i=0}^{d} (\mathrm{rank}_R(F_{i-1}) + \mathrm{rank}_R(F_{i+1})) \\
&\geqslant (d+1)d.
\end{aligned}$$

□

We remark that this result also follows from Theorem 3.3.22 below. The bound from the previous result is in the same vein as one found in [6], where they demonstrate a lower bound

$$\sum_{i=0}^{d} \beta_i^R(M) \geqslant \frac{3}{2}(d-1)^2 + 8$$

for $d \geqslant 5$ (which is strictly greater than the lower bound from Corollary 3.3.20).

We conclude with a more general application of Proposition 3.3.18. First we require a definition. For a ring R, we say that a *perfect complex* is a bounded complex of finitely generated projective R-modules (we take this as the "non-derived" definition of perfect).

Lemma 3.3.21. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let $F_\bullet$ be a perfect complex such that $0 < \ell(H(F)) < \infty$. Then $Hom_R(F_\bullet, F_\bullet)$ is a perfect complex with $\ell(H(F)) < \infty$.*

Proof. Clearly $\operatorname{Hom}_R(F_\bullet, F_\bullet)$ is a perfect complex. As for the statement about its homology, let $\mathfrak{p}$ be a prime ideal of R different from $\mathfrak{m}$. First, we have the usual localization isomorphism

$$\operatorname{Hom}_R(F_\bullet, F_\bullet)_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}((F_\bullet)_\mathfrak{p}, (F_\bullet)_\mathfrak{p}).$$

Note that $(F_\bullet)_\mathfrak{p}$ is a perfect complex over $R_\mathfrak{p}$, and in particular it is semiprojective ([17], Proposition 5.2.9). $(F_\bullet)_\mathfrak{p}$ is also acyclic by assumption, and so it follows that $\operatorname{Hom}_{R_\mathfrak{p}}((F_\bullet)_\mathfrak{p}, (F_\bullet)_\mathfrak{p})$ is acyclic as well. Hence the homology of $\operatorname{Hom}_R(F_\bullet, F_\bullet)$ is supported at the maximal ideal, and so it has finite length. $\square$

Theorem 3.3.22. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d, and let $F_\bullet$ be a perfect complex such that $0 < \ell(H(F)) < \infty$. Then $\sum_{i \in \mathbb{N}} rank_R(F_i) \geqslant \frac{1}{2}(d^2 + d)$.*

Proof. As before, we may assume without loss of generality that $d > 0$ and that $F_\bullet$ is a minimal complex. Also, perhaps after a suspension, we may assume that $F_\bullet$ is concentrated in homological degrees 0 through n for some $0 < n < \infty$. The New Intersection Theorem (see [27]) implies that $n \geqslant d$. By Lemma 3.3.21, $\mathrm{Hom}_R(F_\bullet, F_\bullet)$ has finite length homology. Hence, there is a power $j > 0$ of $\mathfrak{m}$ such that $\mathfrak{m}^j \cdot [\mathrm{id}_{F_\bullet}] = 0$ in $H_0(\mathrm{Hom}_R(F_\bullet, F_\bullet))$. Thus, there is a positive integer j such that multiplication by x on $F_\bullet$ is nullhomotopic for every $x \in \mathfrak{m}^j$.

Using Proposition 3.3.18, we see that for each $0 \leqslant i \leqslant n$ there is an inequality

$$\mathrm{rank}_R(F_{i-1}) + \mathrm{rank}_R(F_{i+1}) \geqslant d.$$

But then we have

$$\begin{aligned} 2\sum_{i=0}^{n} \mathrm{rank}_R(F_i) &\geqslant \mathrm{rank}_R(F_0) + 2\sum_{i=1}^{n-1} \mathrm{rank}_R(F_i) + \mathrm{rank}_R(F_n) \\ &= \sum_{i=0}^{n} (\mathrm{rank}_R(F_{i-1}) + \mathrm{rank}_R(F_{i+1})) \\ &\geqslant (n+1)d \\ &\geqslant (d+1)d. \end{aligned}$$

Dividing by two yields the desired result. □

References

[1] Y. André, La conjecture du facteur direct, arXiv preprint 1609.00345 (2016).

[2] L. Avramov, Modules of finite virtual projective dimension, *Invent. Math.* **96** (1989), 71-101.

[3] L. Avramov, Local rings over which all modules have rational Poincaré series, *J. Pure Appl. Algebra* **91**, (1994), 29-48.

[4] L. Avramov, Infinite free resolutions, in *Six lectures on commutative algebra*, Mod Birkhäuser Class., 1-118, Birkhäuser Verlag, Basel, 2010.

[5] L. Avramov and R.-O. Buchweitz, Homological algebra modulo a regular sequence with special attention to codimension two, *J. Algebra* **230**, (2000), 24-67.

[6] L. Avramov and R.-O. Buchweitz, Lower bounds for betti numbers, *Compositio Math* **86** (1993), no. 2, 147-158.

[7] L. Avramov, V. Gasharov, and I. Peeva, Complete intersection dimension, *Publ. Math. IHES* **86** (1997), 67-114.

[8] P. Bergh, D. Jorgensen, and F. Moore, Totally acyclic approximations, arXiv preprint 1606.07976v1 (2016).

[9] B. Bhatt, On the direct summand conjecture and its derived invariant, arXiv preprint 1608.08882 (2016).

[10] W. Bruns and J. Herzog, Cohen-Macaulay rings, *Camridge studies in advanced mathematics* **39**, Cambridge University Press, 1993.

[11] D. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, *Amer. J. Math.* **99** (1977), 447-485.

[12] R. O. Buchweitz, Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, Preprint, University of Hannover, 1986.

[13] J. Burman, Chang's theorem on betti numbers of exponent-two modules over regular local rings, *Comm. Algbera* **39** (2011), no. 2, 718-729.

[14] S.-T. Chang, Betti numbers of modules of exponent two over regular local rings, *J. Algebra* **193** (1997), no. 2, 640-659.

[15] H. Charalambous, Betti numbers of multigraded modules, *J. Algebra* **137** (1991), no. 2, 491-500.

[16] H. Charalambous, E.G. Evans, and M. Miller, Betti numbers for modules of finite length, *Proc. Amer. Math. Soc.* **109** (1990), no. 1, 63-70.

[17] L.W. Christensen, H.-B. Foxby, H. Holm, Derived category methods in commutative algebra, Preprint, 2012.

[18] D. Eisenbud, Homological algebra on a complete intersection, with an application to group representations, *Trans. Amer. Math. Soc.* **260** (1980), 35-64.

[19] D. Eisenbud and I. Peeva, Minimal free resolutions over complete intersections, *Lecture Notes in Mathematics* **2152** (2016).

[20] E.G. Evans and P. Griffiths, Syzygies, *London Math. Soc. Lecture Note Series* **106**, Cambridge University Press, 1985.

[21] S.I. Gelfand and Yu.I. Manin, Methods of homological algebra, *Springer Monographs in Mathematics*, 2003.

[22] T. Gulliksen, A change of ring theorem with applications to Poincaré series and intersection multiplicity, *Math. Scand.* **34** (1974), 167-183.

[23] T. Gulliksen, On the deviations of a local ring, *Math. Scand.* **47** (1980), 5-20.

[24] S. Iyengar and M. Walker, Examples of finite free complexes of small rank and homology, arXiv preprint 1706.02156 (2017).

[25] H. Knörrer, Cohen-Macaulay modules on hypersurface singularities I, *Invent. Math* **99** (1987), no. 1, 153-164.

[26] M. Nagata, Local rings, *Wiley*, New York, 1962.

[27] P. Roberts, Multiplicities and Chern classes in local algebra, *Cambridge Tracts Math. 133*, Cambridge University Press, 1998.

[28] L. Santoni, Horrocks' question for monomially graded modules, *Pac. J. Math* **141** (1990), no.1, 105-124.

[29] J. Shamash, The Poincaré series of a local ring, *J. Algebra* **12** (1969), 453-470.

[30] J. Tate, Homology of Noetherian rings and local rings, *Illinois J. Math* **1** (1957), 14-25.

[31] M. Walker, Total betti numbers of modules of finite projective dimension, arXiv preprint 1702.02560 (2017).

[32] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, *Cambridge University Press* vol. 146, (1990).