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Properties and Convergence of State-based Laplacians

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PROPERTIES AND CONVERGENCE OF STATE-BASED LAPLACIANS

by

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The classical Laplace operator is a vital tool in modeling many physical behaviors, such as elasticity, diffusion and fluid flow. Incorporated in the Laplace operator is the requirement of twice differentiability, which implies continuity that many physical processes lack. In this thesis we introduce a new nonlocal Laplace-type operator, that is capable of dealing with strong discontinuities. Motivated by the state-based peridynamic framework, this new nonlocal Laplacian exhibits double nonlocality through the use of iterated integral operators. The operator introduces additional degrees of flexibility that can allow better representation of physical phenomena at different scales and in materials with different properties. We obtain explicit rates of convergence for this doubly nonlocal operator to the classical Laplacian as the radii for the horizons of interaction kernels shrink to zero. We study mathematical properties of this state-based Laplacian, including connections with other nonlocal and local counterparts. Finally, we study the solutions of the state-based Laplacian, and use the structure of the solutions to further exhibit the connections between other nonlocal and local Laplacians.

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Table of Contents

Notation Summary

1. Point and set notation

- Vectors in $\mathbb{R}^n$ are denoted by bold letters, i.e. $\mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{p}, \mathbf{q}$.
- Vector valued functions are denoted by bold letters, i.e. $\mathbf{u}, \boldsymbol{\nu}, \boldsymbol{\alpha}$.
- Ω is a domain of $\mathbb{R}^n$, possibly unbounded, unless otherwise specified.
- $\Omega' \subseteq \Omega$.
- $\partial\Omega$ is the boundary of the region Ω.
- $\mathcal{H}_x$ is the horizon of $\mathbf{x}$, taken to be the ball some length δ centered at $\mathbf{x}$. See Figure 1.1
- $\mathcal{B}$ the body in the reference configuration. See section (2.1).
- $\boldsymbol{\xi}, \boldsymbol{\zeta}$ are bonds, often written as $\boldsymbol{\xi} = \mathbf{p} - \mathbf{x}$. See Section 1.1.
- $\mathcal{B}_\delta(x)$ is the ball of radius δ centered at 0.
- δ_0 is the Dirac mass measure centered at the origin.
- $\delta_\mu, \delta_\gamma, \delta_\eta$ are the length scales associated with the support of each kernel, μ, γ, η respectively.
- $\mathcal{B}_\gamma$ is the support of γ, i.e. the ball of radius δ_γ centered at 0. Similarly for $\mathcal{B}_\mu$ and $\mathcal{B}_\eta$.
- w_n is the volume of the unit ball in n dimensions.
- $L^p(\Omega) = \{\mathbf{f} : \Omega \to \mathbb{R}^n | \mathbf{f} \text{ is Lebesgue measurable}, ||\mathbf{f}||_{L^p(\Omega)} < \infty\}$, where

$$||\mathbf{f}||_{L^p(\Omega)} = \left(\int_\Omega |\mathbf{f}|^p d\mathbf{x} \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty.$$

- $L^\infty(\Omega) = \{\mathbf{f} : \Omega \to \mathbb{R}^n | \mathbf{f}$ is Lebesgue measurable, $||\mathbf{f}||_{L^\infty(\Omega)} < \infty\}$, where

$$||\mathbf{f}||_{L^\infty(\Omega)} < \infty = \operatorname*{ess\,sup}_{\Omega} |\mathbf{f}|.$$

- For $1 \leq p \leq \infty, L^p = L^p(\mathbb{R}^n)$.

- For $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is defined to be

$$W^{k,p}(\Omega) = \{\mathbf{f} : \Omega \to \mathbb{R}^n | \mathbf{f} \in L^1_{loc}(\Omega), D^\alpha \mathbf{f} \in L^p(\Omega) \text{ for all } |\alpha| \leq k\},$$

  where L^1_{loc} is all locally summable functions, α is a multiindex, and $D^\alpha \mathbf{f} = \partial^{\alpha_1}_{x_1} \mathbf{f} + \cdots + \partial^{\alpha_n}_{x_n} \mathbf{f}$. Then

$$||\mathbf{f}||_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha \leq k} \int_\Omega |D^\alpha \mathbf{f}|^p dx \right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha \leq k} \operatorname*{ess\,sup}_{\Omega} |D^\alpha \mathbf{f}| & p = \infty. \end{cases}$$

2. Functions

- $\mathbf{F}$ is the force in Example 1 and the flux in Example 2.

- $\boldsymbol{\nu}$ is the unit outward normal.

- $\rho(\mathbf{x})$ is the mass density at $\mathbf{x}$.

- $\mathbf{u} * \gamma = (u_1 * \gamma, u_2 * \gamma, \ldots)$, is the convolution of a vector and a scalar function. See equation (4.2).

- $\langle \mathbf{u}, \varphi \rangle = (\langle u_1, \varphi \rangle, \ldots, \langle u_n, \varphi \rangle)$ is the duality product of a vector and a scalar function.

- $\mathbf{f}$ is a pairwise force vector. See equation (1.15).

- $\mathbf{b}$ is a prescribed body force density. See equation (1.15).
- dV indicates a volume integral, whereas $d\sigma$ indicates a surface integral.
- We denote $\hat{f}$ to be the Fourier transform of f, and $\check{f}$ of $\mathcal{F}^{-1}(f)$ to be the inverse Fourier transform of f.
- μ, γ, η are symmetric radial functions, called kernels. Prototypical forms of these kernels are given in (2.8).
- Scaling factors, $\pi_\mu, \pi_\gamma, \pi_\eta$ given in (3.3), (2.10), and (2.11) respectively, with prototypical forms given in (3.6), (2.13), and (2.14).
- $\sigma(\delta_\mu)$ is the scaling of the bond-based Laplacian, $\mathcal{L}^b_\mu[\mathbf{u}]$, given in (3.2).
- $\sigma(\gamma, \eta)$ or $\sigma(\delta_\gamma, \delta_\eta)$ is the scaling of the state-based Laplacian, $\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]$, given in (3.9).

3. Local operators

- $\nabla \mathbf{u}$ is the classical gradient.
- $\Delta \mathbf{u}$ is the classical Laplacian.
- $\mathrm{div}(\mathbf{v})$ is the classical divergence operator.

4. Nonlocal operators and nonlocal calculus

- $\mathcal{L}^b_\mu[\mathbf{u}]$ is the bond-based Laplacian with kernel μ. See equation (1.19).
- $\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{v}]$ is the nonlocal divergence operator. See equation (1.20).
- $\mathcal{G}_{\boldsymbol{\alpha}}[\mathbf{u}]$ is the nonlocal gradient operator. See equation (1.21).
- $\underline{\mathbf{A}}\langle \boldsymbol{\xi} \rangle$ for $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{x}}$, the image of a peridynamic state acting on the bond $\boldsymbol{\xi}$. See section (2.1).
- $\underline{\mathbb{D}}\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle$ is a double state, which acts on two bonds. See section (2.1).

- $\underline{\mathbf{A}} \bullet \underline{\mathbf{B}}$ is the dot product for states. See section (2.1).
- $(\underline{\mathbf{A}} \bullet \underline{\mathbb{D}})_j \langle \boldsymbol{\xi} \rangle$ is the left product of vector state $\underline{\mathbf{A}}$ and double state $\underline{\mathbb{D}}$. See section (2.1).
- $(\underline{\mathbb{D}} \bullet \underline{\mathbf{B}})_i \langle \boldsymbol{\xi} \rangle$ is the right product of $\underline{\mathbf{B}}$ and $\underline{\mathbb{D}}$. See section (2.1).
- $\underline{\mathbb{D}}^{\dagger}$ is the adjoint of $\underline{\mathbb{D}}$
- $\nabla\psi(\underline{\mathbf{A}})$ is the Frechet derivative of ψ, and $\nabla\nabla\psi = \nabla(\nabla\psi)$ is the second Fréchet derivative of ψ. See section (2.1).
- $\underline{\mathbf{T}}$ is a vector state that describes the force. See equation (2.1)
- $\underline{\mathbb{K}}$ is the double state-kernel of the state-based Laplacian $\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]$. See Section 2.2.

Chapter 1

Introduction

The focus of this thesis is an introduction and analysis of a new nonlocal Laplace-type operator. The classical Laplacian is used in a variety of models for physical phenomena such as elasticity and diffusion, including heat and fluid flow. Mathematically, the classical Laplacian requires input functions that are twice differentiable. This requirement inhibits the ability to accurately represent discontinuities, which often appear in models for cracks, anomalous diffusion, or transport. To address some of these issues we introduce a new Laplacian, an operator that behaves like the Laplacian on smooth functions, but requires much less regularity than the classical Laplacian. We begin with a short overview of the derivation of classical elasticity and diffusion equations to illustrate how the Laplacian incorporates physical properties in these key models. The derivations can be found in traditional texts such as [18].

Example 1 (Elasticity)**.**

In a deformation process, let $u(x,t)$ be the displacement at (x,t) and $\mathbf{F}$ be the force acting on Ω', a subregion of a larger domain Ω. The net force through $\partial\Omega'$ is given by

$$-\int_{\partial\Omega'} \mathbf{F}\cdot\boldsymbol{\nu}\,d\sigma. \tag{1.1}$$

Using the Green-Gaus theorem we have

$$-\int_{\partial\Omega'} \mathbf{F}\cdot\vec{\nu}d\sigma = -\int_{\Omega'} \mathrm{div}\mathbf{F}dx. \tag{1.2}$$

For elastic bodies we can take $\mathbf{F} = \mathbf{F}(\nabla u)$, where ∇u is the elastic force, so that the higher magnitude of the gradient, the larger the force. Assuming linear elasticity, we use Hooke's Law to obtain

$$\mathbf{F} = -c\nabla u. \tag{1.3}$$

Hence we get

$$-\int_{\Omega'} \mathrm{div}\mathbf{F}dx = c\int_{\Omega'} \mathrm{div}(\nabla u)dx = c\int_{\Omega'} \Delta u dx, \tag{1.4}$$

where Δ is the classical Laplace operator. The acceleration in a subregion Ω' of Ω is given by

$$\int_{\Omega'} u_{tt}(x,t)dx. \tag{1.5}$$

Using Newton's law we find

$$\int_{\Omega'} \big(\rho(x)u_{tt}(x,t) - c\Delta u\big)dx = 0, \qquad \text{for all } \Omega' \subseteq \Omega, \tag{1.6}$$

where $\rho(x)$ is the density at x. Thus

$$\rho(x)u_{tt}(x,t) - c\Delta u = 0. \tag{1.7}$$

Equation (1.7) is the classical wave equation.

Example 2 (Diffusion)**.**

If we instead let $u(x,t)$ be the density of some physical quantity at (x,t), $\mathbf{F}(x,t)$

be the flux, and assume Fick's law for diffusion, we have

$$\mathbf{F} = -c\nabla u, \tag{1.8}$$

which again produces (1.4). If we assume that no creation or destruction of mass happens inside Ω, then the change in total mass, with respect to time, is given by the flux through $\partial\Omega'$. Hence we have,

$$\frac{d}{dt}\int_{\Omega'} u(x,t)dx = -\int_{\partial\Omega'} \mathbf{F}\cdot\boldsymbol{\nu}d\sigma = c\int_{\Omega'} \Delta u dx, \tag{1.9}$$

thus producing

$$\int_{\Omega'} \big(u_t(x,t) - c\Delta u\big)dx = 0, \qquad \text{for all } \Omega' \subseteq \Omega. \tag{1.10}$$

Finally, we obtain

$$u_t - c\Delta u = 0, \tag{1.11}$$

the classical diffusion equation. If we assume that the process has finished, i.e. taking $t \to \infty$, or that changes in time are very small, we have that $u_t = 0$. Thus with $c = 1$, we get Laplace's equation

$$-\Delta u = 0. \tag{1.12}$$

Remark 1.1. Note that the diffusion equation also governs other physical processes, where Fick's law for diffusion is replaced by Fourier's law for thermodynamics, or Ohm's law in electrostatics.

These models, or variations of them, are used in classical continuum mechanics where material is assumed to be continuously distributed instead of consisting of discrete particles. As mentioned previously, models of elasticity and diffusion, as written, necessitate our function u to be twice differentiable in space. This becomes a glaring shortcoming as we often encounter materials and situations which are not C^2. We can loosen regularity requirements by considering the weak versions of theses models

$$\langle \rho u_{tt}, \varphi \rangle - c \langle \Delta u, \varphi \rangle = 0, \tag{1.13}$$

and

$$\langle u_t, \varphi \rangle - c \langle \Delta u, \varphi \rangle = 0, \tag{1.14}$$

where $\varphi \in C_c^\infty(\Omega)$. In these weak forms, we are not viewing the models at a point (x, t), but rather through a test function φ. We are in essence gathering information about what is happening around the point (x, t). Although these weak forms reduce the amount of regularity we require on u, we still need $u \in W^{k,p}(\Omega)$ for some p and k.

While weak forms often prove useful and satisfactory, there are still situations when the capabilities of these models fall short of encompassing all of the physical relevance we may want or need. Modeling a material that develops a crack, or modeling at the discrete or atomistic level is impossible in a differential framework. Other methods have been developed to compensate for these problems. For example, we could use fractional mechanics theories which typically include a separate set of equations to predict where a crack may develop and how fast it may grow (see [6] as an example). While these other methods work well in some circumstances, it is not clear that they are sufficient in every situation and with every length scale.

A relatively recent method of dealing with these types of discontinuities has been the incorporation of nonlocal operators into physical models. By nonlocal, we mean that the operator takes into account neighbors of every point. Nonlocal operators often arise as integral or fractional-derivative operators, allowing a reduction of the smoothness required on the functions of which the operators are to be applied. Models which use these nonlocal operators have been termed nonlocal models and they allow the study of solutions and domains beset by discontinuities.

Over the past decades nonlocal theories have been successfully employed in modeling various types of phenomena, including nonlocal diffusion [1], fractional kinetics and anomalous transport, [54], speech signal modeling [2], viscoelasticity [47], image processing [24, 35], fluid mechanics [32], and phase separation [23, 21]. In addition, the theory of peridynamics, introduced by Stewart Silling [43, 46], incorporates nonlocal models in dynamic fracture; see also [13].

Mathematically, fractional Laplacians have been studied by many authors, see [8], [10], and [53] as examples, and other fractional operators have have been studied in papers such as [48]. An introductory text for fractional calculus is available in [37].

Integro-differential Laplacians have been studied mathematically in [9],[36], [27], [19], and [49]. In [25] and [15] the authors develop a nonlocal (integral) vector calculus that mimics classical calculus.

For this thesis the motivation behind considering a nonlocal framework comes from the theory of peridynamics, introduced by Silling in [43], which is a reformulation of classical continuum mechanics. The first applications were to dynamic fracture, see [3] and thermal diffusion see [4, 38]. In addition, the peridynamic theory has been used to model composite laminates [29], concrete [30], porous media flow [31], and tumor growth [33]. The main focus here is on a new diffusion-type operator, which we will label the state-based Laplacian. The operator is introduced in Section 2.3 and it is inspired by the most general formualtion of peridynamics, the state-based theory. The purpose of this chapter is to give the reader sufficient motivation for the nonlocal framework of peridynamics, and to introduce that framework. We will further highlight the relevance of peridynamics as a type of nonlocal model, and as the framework in which the state-based Laplacian arises naturally.

1.1 Bond-based peridynamics

The power and convenience of having a cohesive set of equations, together with the ability to track dynamic fracture growth are some of the most significant benefits of Silling's reformulation of classical continuum mechanics in [43]. In this paper, Silling names his theory Peridynamics for the greek words *peri* meaning near, and *dynamics* meaning force. He presents a nonlocal, unified approach to modeling discrete, discontinuous and continuous material. In this theory, the spatial derivatives in classical theory have essentially been replaced by integrals, thus allowing for lower regularity of solutions. In addition, the authors of [42] illustrate the relationship between peridynamics and classical molecular dynamics, particularly how peridynamics can be seen as an upscaling of molecular dynamics. In the nearly two-decades since the release of this new formulation, peridynamics has been shown to be extremely effective at tracking dynamic fracture in different materials (homogeneous or heterogeneous); see fiber-reinforced composites [28], composite laminates [29], orthotropic material [22], layered glass [5], concrete [30].

In the original formulation of peridynamics each point $\mathbf{x}$ interacts with all its neighbors within a domain $\mathcal{H}_{\mathbf{x}}$, called the horizon of $\mathbf{x}$, taken to be a ball of radius δ centered at $\mathbf{x}$. If $\mathbf{p} \in \mathcal{H}_{\mathbf{x}}$ then $\boldsymbol{\zeta} = \mathbf{p} - \mathbf{x}$ is called a bond for the point $\mathbf{x}$ (see Figure 1.1). In this context consider the cumulative force that is acting on $\mathbf{x}$ through its neighbors inside the horizon, a force that is expressed through integral operators. By replacing differential operators with integral operators we allow low-regularity solutions to satisfy elasticity models. The bond-based peridynamics equation of elasticity, as introduced by Silling [43], is given by

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{H}_{\mathbf{x}}} \mathbf{f}(\mathbf{u}(\mathbf{q},t) - \mathbf{u}(\mathbf{x},t), \mathbf{q} - \mathbf{x}) dV_{\mathbf{x}} + \mathbf{b}(\mathbf{x},t), \tag{1.15}$$

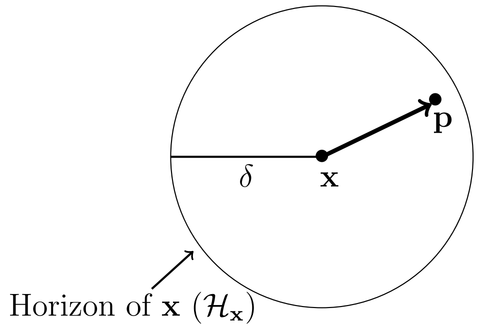

Figure 1.1: The interaction of the points $\mathbf{p}$ has on $\mathbf{x}$, in the bond-based theory of peridynamics.

where ρ is material density, and $\mathbf{b}$ is a prescribed body force density field. The displacement vector field is given by $\mathbf{u}$, and $\mathbf{f}$ gives the force vector that the particle $\mathbf{q}$ exerts on the particle $\mathbf{x}$. The form of $\mathbf{f}$ embodies the constitutive information of the material. From Newton's third law, we require that

$$\mathbf{f}(-(\mathbf{u}(\mathbf{q},t)-\mathbf{u}(\mathbf{x},t)),-(\mathbf{q}-\mathbf{x}))=-\mathbf{f}(\mathbf{u}(\mathbf{q},t)-\mathbf{u}(\mathbf{x},t),\mathbf{q}-\mathbf{x}). \tag{1.16}$$

For linear peridynamic material, the right had side of (1.15) becomes

$$\int_{\mathcal{H}_{\mathbf{x}}}(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\,\mu(\mathbf{y}-\mathbf{x})d\mathbf{y}+\mathbf{b}(\mathbf{x},t), \tag{1.17}$$

where from (1.16) we must have $\mu(\boldsymbol{\xi})=\mu(-\boldsymbol{\xi}),\forall\boldsymbol{\xi}$. If μ is a function such that $\mu\equiv 0$ when $|\boldsymbol{\xi}|>\delta$, we obtain the protagonist of the bond-based formulation, the nonlocal Laplacian,

$$\mathcal{L}_{\mu}[\mathbf{u}](\mathbf{x})=\int_{\mathcal{B}_{\delta}(x)}(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\,\mu(\mathbf{y}-\mathbf{x})d\mathbf{y}. \tag{1.18}$$

In the above formula, $\mathcal{B}_{\delta}(0)$ is the ball of radius δ centered at $\mathbf{x}$, while the kernel μ measures the strength of the bond $\mathbf{y}-\mathbf{x}$. Observe that this operator is well-defined even for very rough functions $\mathbf{u}:\mathbb{R}^n\to\mathbb{R}^k, n,k\geq 1$, as long as the integration for each component of $\mathbf{u}$ is valid, ($\mathcal{L}_\mu^b \in \mathbb{R}^k$). The constant $\delta > 0$ is the radius of the horizon, and describes the range of nonlocality. It can vary from very small values (peridynamics) to very large ones ($\delta = \infty$ in nonlocal diffusion [1]). Of interest to us is the case of a finite horizon as well as the transition to infinitesimal values; in other words, we study the limiting behavior of nonlocal operators as δ goes to zero. In order to obtain convergence, we will consider the scaled nonlocal Laplacian with scaling $\sigma(\delta)$, which for clarity we label as the bond-based Laplacian with kernel μ:

$$\mathcal{L}_\mu^b[\mathbf{u}](\mathbf{x}) = \sigma(\delta) \int_{\mathcal{B}_\delta(0)} (\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\, \mu(\mathbf{y} - \mathbf{x}) d\mathbf{y}. \tag{1.19}$$

The convergence of the bond-based Laplacian to the classical Laplacian, along with the scaling $\sigma(\delta)$, will be discussed in Chapter 3.

To highlight an important structural connection between the classical Laplacian and the bond-based Laplacian we present definitions of nonlocal gradient and divergence operators. These operators were formulated in [15] to provide the framework for a nonlocal vector calculus.

Definition 1.2 (Nonlocal operators)**.** Let $\mathbf{v} : \Omega \times \Omega \to \mathbb{R}^k$ be a vector two-point function, $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{\alpha}(\mathbf{x}, \mathbf{y}) : \Omega \times \Omega \to \mathbb{R}^k$, be an antisymmetric vector two-point function. The nonlocal divergence operator $\mathcal{D}_{\boldsymbol{\alpha}}$ on $\mathbf{v}$ is defined as

$$\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{v}](\mathbf{x}) := \int_\Omega (\mathbf{v}(\mathbf{x}, \mathbf{y}) + \mathbf{v}(\mathbf{y}, \mathbf{x})) \cdot \boldsymbol{\alpha}(\mathbf{x}, \mathbf{y}) d\mathbf{y} \quad \text{for } \mathbf{x} \in \Omega, \tag{1.20}$$

where $\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{v}] : \Omega \to \mathbb{R}$. The nonlocal two-point gradient operator is defined as

$$\mathcal{G}_{\boldsymbol{\alpha}}[\mathbf{u}](\mathbf{x}) := (\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\, \boldsymbol{\alpha}(\mathbf{x}, \mathbf{y}) \quad \text{for } (\mathbf{x}, \mathbf{y}) \in \Omega \times \Omega, \tag{1.21}$$

where $\mathcal{G}_{\boldsymbol{\alpha}} : \Omega \to \mathbb{R}^k$.

Taking $\mu = \boldsymbol{\alpha}^2/2$ we find

$$\mathcal{L}_\mu[\mathbf{u}](\mathbf{x}) = \mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{G}_{\boldsymbol{\alpha}}[\mathbf{u}]](\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega. \tag{1.22}$$

Thus, the nonlocal bond-based Laplacian can be written in the familiar "divergence of the gradient" from that appears in classical models, i.e. equation (1.4).

Bond-based peridynamics and the bond-based Laplacian have been well studied, and are continuing to be studied. However, in bond-based models particles interact through a central potential, thus "seeing" only neighbors in their horizon [46, Item 1 in list on pg. 153]. A consequence of this formulation gives a restriction on the Poisson ratio of 1/4 in 3D or 2D plane strain, and 1/3 in 2D plane stress, see [50]. Moreover, the bond-based systems lack the generality of stress tensors that are usually considered in continuum mechanics as they impose only a pairwise force interaction on particles [46, Item 2 in list on pg. 153]. For a more detailed discussion of these aspects and the motivation for a more general theory, see [46]. The state-based theory of peridynamics, introduced in the next section, overcomes these issues and generalizes the bond-based theory. The connection between the Laplace type operators that appear in each of these formulations is one of the goals of this work and is studied further in Chapter 4.

1.2 State-based peridynamics

To overcome the deficiencies of the bond-based model, Silling et al. in [46] introduced the theory of state-based peridynamics, in which the force between points are expressed through general operators called states. A discussion of these states, as

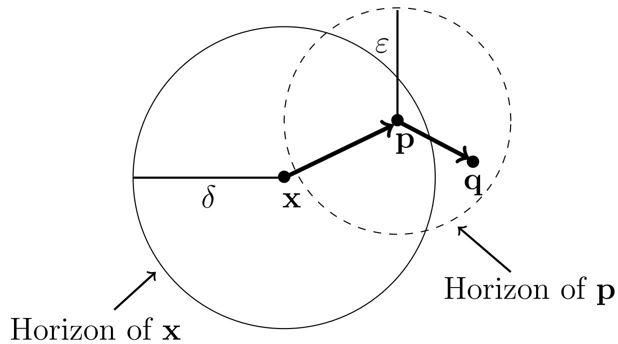

Figure 1.2: The indirect interaction that the point **q** has on **x** through their common neighbor **p**.

relevant to this thesis, is given in Section (2.1). These state operators allow indirect force interactions of a neighbor with its neighbor's neighbors. A given point **x** will be affected directly by its neighbors **p**, as well as indirectly, by the neighbors **q** of **p** through the point **p** (see Figure 1.2). Mathematically, the interactions of the point **x** will be expressed through double integrals over the product space $\mathcal{B}_\delta(\mathbf{x}) \times \mathcal{B}_\varepsilon(\mathbf{p})$, for every point **p** in the horizon of **x**. Thus, the points affecting the behavior at **x** can be $\varepsilon + \delta$ distance away from **x**.

This setting allows a very general approach to modeling that can incorporate a wide variety of physical behavior. The bond-based theory then, becomes a particular case of the state-based setting where points interact not only with their immediate neighbors (direct interactions), but also with neighbors of the neighbors (indirect interactions). This composition of interactions could also be extended to model a broader range of phenomena, such as seen in nonlinear elasticity, viscoelasticity, and viscoplasticity (for bond-based formulations of these models see [17], [52], and respectively [20]).

In particular the state-based equation of motion is given by

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int\limits_{\mathcal{H}_{\mathbf{x}}} \left\{ \underline{\mathbf{T}}[\mathbf{x},t]\left\langle \mathbf{q}-\mathbf{x}\right\rangle - \underline{\mathbf{T}}[\mathbf{q},t]\left\langle \mathbf{x}-\mathbf{q}\right\rangle \right\} dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x},t), \tag{1.23}$$

where ρ is material density, and $\mathbf{b}$ is a prescribed body force density field. Above the operator $\underline{\mathbf{T}}$ is called a vector state which when computed at the point $\mathbf{x}$ is applied to a bond $\mathbf{q}-\mathbf{x}$ whose resulting action is the force which $\mathbf{q}$ exerts on $\mathbf{x}$. Thus the right hand side of (1.23) describes the cumulative effect of all action-reaction forces between $\mathbf{x}$ and its neighbors, and provides a very general framework for incorporating the material constitutive restrictions.

The focus of this work is on the study of a newly introduced state-based Laplacian operator:

$$\begin{aligned}\mathcal{L}^{s}_{\gamma\eta}[\mathbf{u}](\mathbf{x}) = &\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \left(\gamma(\mathbf{p}-\mathbf{x})+\gamma(\mathbf{q}-\mathbf{x})\right)\eta(\mathbf{q}-\mathbf{p})[\mathbf{u}(\mathbf{q})-\mathbf{u}(\mathbf{x})]d\mathbf{q}d\mathbf{p}\\ &-\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \left(\gamma(\mathbf{x}-\mathbf{p})+\gamma(\mathbf{q}-\mathbf{p})\right)\eta(\mathbf{q}-\mathbf{x})[\mathbf{u}(\mathbf{q})-\mathbf{u}(\mathbf{p})]d\mathbf{q}d\mathbf{p},\end{aligned}$$

that arises naturally in the state-based formulation of peridynamics (again, the integration is performed on each component of $\mathbf{u}$). As motivated by the physical considerations above, this operator captures effects from a wider and more diverse range of interactions by looking at cumulative effects modeled through two integral operators with two (possibly different) kernels, γ and η. By incorporating two kernels the operator gains an additional degree of flexibility that is important in applications, thus increasing the physical relevance of the model. The engineering and computational communities have provided us with many studies for state-based models ([51], [41], [34], and [26]; also, see the overview paper [44]), but the theoretical investigations of these doubly nonlocal operators are still in their early stages.

Chapter 2

The state-based Laplacian

As mentioned perviously, the state-based Laplacian arises naturally from the state-based formulation of peridynamics. The introduction of this new operator is the focus of this chapter. We will start with a discussion of the importance of a new nonlocal Laplacian. Then, we will build up the tools needed to present the state-based peridynamics theory. From there, we show how the state-based Laplacian originates from the linearized state-based peridynamic equation of motion. We conclude this chapter with a conversation about the kernels, γ and η, of the state-based Laplacian, and end the with a rewriting of the Laplacian which is more convenient to work with.

This new nonlocal Laplacian was inspired by three particular choices for kernels given by Silling in [45]; the examples concern elastic materials (in bond-based framework), linear fluids, and linear isotropic solids. At a mathematical level the state-based Laplacian is a double convolution-type operator, which generalizes the operator (1.19), while also providing a "decomposition" of the operator with respect to the kernels γ and η. The role of each kernel will be discussed from a physical, as well as a mathematical point of view. Additionally, by writing the state-based Laplacian in convolution form we obtain an operator that is well-defined on spaces of very irregular functions, even on the space of distributions; see Chapter 4.

To summarize, the main contributions of this thesis are:

- At a **physical level** we introduce a mathematical operator that captures nonlocal effects in materials that are more general than the ones modeled with the single integral, bond-based operator. While this operator appears naturally in the (very) general state-based formulation, it allows us through its specific form involving two kernels to incorporate a variety of examples. Thus, we introduce a framework in which the nonlocal Laplacian can model very different materials, or even different behavior. In this more general context we have the ability to study transitional behavior from one class of phenomena to another, as well as the transition from one type of material to another.

- At a **mathematical level** the double convolution operator gives us a novel way to model physical behavior in the space of discontinuous functions or distributions. The transition to "smooth" behavior can be studied through convergence results of the nonlocal operator to the classical Laplacian as the horizons of interaction shrink to zero. We obtain explicit rates of convergence and we discuss the importance of regularity for functions on which the nonlocal operator is applied.

Finally, we make note of a couple of distinctions between this operator and other operators. First, the structure of the state-based Laplacian resembles the nonlocal biharmonic introduced in [39], due to the presence of the double nonlocality. However, we show that the doubly nonlocal Laplacian approaches a second, and not a fourth-order differential operator. This aspect will be discussed in more detail in Section 4.1. Also, we take a scalar-valued kernel rather than a tensor-valued kernel, (see (2.5)) hence, in the one dimensional case $\mathcal{L}^s_{\gamma\eta}$ is a nonlocal version of the Navier operator from elasticity, but in higher dimensions $\mathcal{L}^s_{\gamma\eta}$ is a nonlocal version of a Laplace type operator, and not the Navier operator. Indeed, $\mathcal{L}^s_{\gamma\eta}$ is missing the nonlocal counterpart of the $\nabla \mathrm{div}\mathbf{u}$ term (see [36]).

2.1 Notation and peridynamic states

In this section we introduce notation that is used in the state-based peridynamic formulation. We follow the notation given in [45].

Let $\mathcal{B}$ be the body in the reference configuration, and let $\mathbf{x} \in \mathcal{B}$. We let $\delta > 0$ be the horizon of $\mathbf{x}$, and for $\mathbf{q} \in \mathcal{B}$ such that $|\mathbf{q} - \mathbf{x}| \leq \delta$, as before, call $\boldsymbol{\xi} = \mathbf{q} - \mathbf{x}$ a bond. Let $\mathcal{H}_{\mathbf{x}}$ be the set of all such bonds of $\mathbf{x}$. A peridynamic state $\underline{\mathbf{A}}$ is a mapping from $\mathcal{H}_{\mathbf{x}}$. Denote by $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$ for $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{x}}$, the image of state acting on the bond $\boldsymbol{\xi}$. If $\underline{\mathbf{A}}$ is a scalar state then the value of $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$ is a scalar. Similarly, $\underline{\mathbf{A}}$ is a vector state when $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$ is a vector. A double state, $\underline{\mathbb{D}}$, takes two bonds $\boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathcal{H}_{\mathbf{x}}$ and $\underline{\mathbb{D}}\langle\boldsymbol{\xi}, \boldsymbol{\zeta}\rangle$ is a second order tensor.

The dot product of two vector states $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ is

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}_{\mathbf{x}}} \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}}.$$

The left product of vector state $\underline{\mathbf{A}}$ and double state $\underline{\mathbb{D}}$ is the vector state defined by

$$(\underline{\mathbf{A}} \bullet \underline{\mathbb{D}})_j \langle\boldsymbol{\xi}\rangle = \int_{\mathcal{H}_{\mathbf{x}}} \underline{A}_j \langle\boldsymbol{\xi}\rangle \, \underline{D}_{ij} \langle\boldsymbol{\zeta}, \boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\zeta}}, \quad \forall \boldsymbol{\xi} \in \mathcal{H}_{\mathbf{x}}.$$

Then the right product of $\underline{\mathbf{B}}$ and $\underline{\mathbb{D}}$ is the vector state defined by

$$(\underline{\mathbb{D}} \bullet \underline{\mathbf{B}})_i \langle\boldsymbol{\xi}\rangle = \int_{\mathcal{H}_{\mathbf{x}}} \underline{D}_{ij} \langle\boldsymbol{\xi}, \boldsymbol{\zeta}\rangle \, \underline{B}_j \langle\boldsymbol{\zeta}\rangle \, dV_{\boldsymbol{\zeta}}, \quad \forall \boldsymbol{\xi} \in \mathcal{H}_{\mathbf{x}}.$$

The adjoint of $\underline{\mathbb{D}}$ is denoted by $\underline{\mathbb{D}}^{\dagger}$ and defined by

$$\underline{D}^{\dagger}_{ij} \langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle = \underline{D}_{ji} \langle \boldsymbol{\zeta}, \boldsymbol{\xi} \rangle ,$$

with $\underline{\mathbb{D}}$ being self-adjoint if $\underline{\mathbb{D}} = \underline{\mathbb{D}}^{\dagger}$.

Let $\mathcal{V}$ be the set of all vector states, and $\psi : \mathcal{V} \to \mathbb{R}$. If ψ is Fréchet Differentiable at $\underline{\mathbf{A}} \in \mathcal{V}$, then for any $\underline{\mathbf{a}} \in \mathcal{V}$

$$\psi(\underline{\mathbf{A}} + \underline{\mathbf{a}}) = \psi(\underline{\mathbf{A}}) + \nabla\psi(\underline{\mathbf{A}}) \bullet \underline{\mathbf{a}} + o(||\underline{\mathbf{a}}||)$$

where the Fréchet derivative $\nabla\psi(\underline{\mathbf{A}})$ is a vector state. If $\underline{\mathbf{S}} : \mathcal{V} \to \mathcal{V}$ then

$$\underline{\mathbf{S}}(\underline{\mathbf{A}} + \underline{\mathbf{a}}) = \underline{\mathbf{S}}(\underline{\mathbf{A}}) + \nabla\underline{\mathbf{S}}(\underline{\mathbf{A}}) \bullet \underline{\mathbf{a}} + o(||\underline{\mathbf{a}}||)$$

where $\nabla\underline{\mathbf{S}}(\underline{\mathbf{A}})$ is a double state. If $\nabla\psi$ is Fréchet differentiable, then the second Fréchet derivative of ψ is a double state defined by $\nabla\nabla\psi = \nabla(\nabla\psi)$ on $\mathcal{V}$.

2.2 State-based equation of motion and its linearization

We restate the state-based equation of motion from (1.23). The displacement from the equilibrium position of a point $\mathbf{x}$ in the body $\mathcal{B}$ at time $t \geq 0$, denoted by $\mathbf{u}(\mathbf{x}, t)$, is described by the equation

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t) = \int_{\mathcal{H}_{\mathbf{x}}} \{\underline{\mathbf{T}}[\mathbf{x}, t] \langle \mathbf{q} - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{q}, t] \langle \mathbf{x} - \mathbf{q} \rangle\} \, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x}, t), \qquad (2.1)$$

where ρ is material density, and $\mathbf{b}$ is a prescribed body force density field. Again, the resulting action of $\underline{\mathbf{T}}$ is the force which $\mathbf{q}$ exerts on $\mathbf{x}$. Thus the right hand side of

(1.23) describes the cumulative effect of all action-reaction forces between $\mathbf{x}$ and its neighbors. We now describe the linearization of (2.1), as can be found in [45]. For further discussion on linearized state-based peridynamics, see [40].

Let $\mathbf{y}$ be the motion of the peridynamic body $\mathcal{B}$. The position of a point $\mathbf{x} \in \mathcal{B}$ at time $t \geq 0$ is $\mathbf{y}(\mathbf{x}, t)$. Let $\underline{\mathbf{Y}}[\mathbf{x}, t]$ be the deformation state, the vector state defined by

$$\underline{\mathbf{Y}}[\mathbf{x}, t] \langle \mathbf{q} - \mathbf{x} \rangle = \mathbf{y}(\mathbf{q}, t) - \mathbf{y}(\mathbf{x}, t), \quad (\mathbf{q} - \mathbf{x}) \in \mathcal{H}_{\mathbf{x}}.$$

A material is simple if $\underline{\mathbf{T}}$ only depends on $\underline{\mathbf{Y}}$. Hence, if we assume the material is simple, then

$$\underline{\mathbf{T}}[\mathbf{x}, t] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}[\mathbf{x}, t], \mathbf{x}).$$

If the material is also elastic, then there exists a function $\hat{W} : \mathcal{V} \times \mathbb{R}^3 \to \mathbb{R}^3$ such that for any $\underline{\mathbf{Y}}$

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x}) = \nabla \hat{W}(\underline{\mathbf{Y}}, \mathbf{x}).$$

Now, deriving the linearized state-based model will give a linear integro-differential equation expressed in terms of the displacement. We define $\underline{\mathbf{T}}(\underline{\mathbf{Y}}^0) = \underline{\mathbf{T}}^0$, and

$$\nabla \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}^0) = \underline{\mathbb{K}}, \tag{2.2}$$

which is a double state, where $\underline{\mathbf{T}}^0$ is the initial force and $\underline{\mathbf{Y}}^0[\mathbf{x}]\langle \mathbf{q} - \mathbf{x} \rangle = \mathbf{y}^0(\mathbf{q}) - \mathbf{y}^0(\mathbf{x})$. Thus, linearizing $\underline{\mathbf{T}}$ gives

$$\underline{\mathbf{T}}(\underline{\mathbf{U}}) = \underline{\mathbf{T}}^0 + \underline{\mathbb{K}} \bullet \underline{\mathbf{U}},$$

where $\underline{\mathbf{U}}$ is the displacement. If the material is elastic, then

$$\underline{\mathbb{K}} = \nabla \underline{\mathbf{T}}(\hat{\underline{\mathbf{Y}}}^0) = \nabla \nabla \hat{W}(\underline{\mathbf{Y}}^0),$$

which implies that $\underline{\mathbb{K}}^{\dagger} = \underline{\mathbb{K}}$, hence $\underline{\mathbb{K}}$ is symmetric.

Assume $\mathbf{b}^0$ is a time independent body force density field on $\mathcal{B}$, which gives an equilibrated deformation $\mathbf{y}^0$. Then subject the body to another body force density field $\mathbf{b}$, so that

$$\hat{\mathbf{b}} = \mathbf{b}^0 + \mathbf{b}.$$

The change in the displacement field is denoted $\mathbf{u}$ hence

$$\mathbf{y} = \mathbf{y}^0 + \mathbf{u}.$$

Finally, assume $\underline{\mathbf{T}}[\mathbf{x}]\langle\mathbf{p}-\mathbf{x}\rangle = 0$ whenever $|\mathbf{p}-\mathbf{x}| > \delta$. Linearizing the possibly nonlinear equation of motion (1.23) produces

$$\begin{aligned}\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = &\int_{\mathcal{B}} \left(\underline{\mathbf{T}}^0[\mathbf{x}] + \underline{\mathbb{K}}[\mathbf{x}] \bullet \underline{\mathbf{U}}[\mathbf{x}]\right)\langle\mathbf{p}-\mathbf{x}\rangle\, dV_{\mathbf{p}} \\ &- \int_{\mathcal{B}} \left(\underline{\mathbf{T}}^0[\mathbf{p}] + \underline{\mathbb{K}}[\mathbf{p}] \bullet \underline{\mathbf{U}}[\mathbf{p}]\right)\langle\mathbf{x}-\mathbf{p}\rangle\, dV_{\mathbf{p}} + \hat{b}(\mathbf{x},t).\end{aligned} \tag{2.3}$$

Since $\mathbf{y}^0$ is equilibrated

$$\int_{\mathcal{B}} \left\{\underline{\mathbf{T}}^0[\mathbf{x}]\langle\mathbf{p}-\mathbf{x}\rangle - \underline{\mathbf{T}}^0[\mathbf{p}]\langle\mathbf{x}-\mathbf{p}\rangle\right\} + \mathbf{b}^0(\mathbf{x}) = 0.$$

Thus

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}} \left\{(\underline{\mathbb{K}}[\mathbf{x}] \bullet \underline{\mathbf{U}}[\mathbf{x}])\langle\mathbf{p}-\mathbf{x}\rangle - (\underline{\mathbb{K}}[\mathbf{p}] \bullet \underline{\mathbf{U}}[\mathbf{p}])\langle\mathbf{x}-\mathbf{p}\rangle\right\} dV_{\mathbf{q}} dV_{\mathbf{p}} + \mathbf{b}(\mathbf{x},t).$$

Using the definition of dot product of two vector states we have

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\mathbb{K}}[\mathbf{x}]\,\langle \mathbf{p}-\mathbf{x},\mathbf{q}-\mathbf{x}\rangle \cdot \underline{\mathbf{U}}[\mathbf{x}]\,\langle \mathbf{q}-\mathbf{x}\rangle\, dV_{\mathbf{q}} dV_{\mathbf{p}} - \int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\mathbb{K}}[\mathbf{p}]\,\langle \mathbf{x}-\mathbf{p},\mathbf{q}-\mathbf{p}\rangle \cdot \underline{\mathbf{U}}[\mathbf{p}]\,\langle \mathbf{q}-\mathbf{p}\rangle\, dV_{\mathbf{q}} dV_{\mathbf{p}} + \mathbf{b}(\mathbf{x},t).$$

Finally, writing out the definition of a dot product we obtain

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\mathbb{K}}[\mathbf{x}]\,\langle \mathbf{p}-\mathbf{x},\mathbf{q}-\mathbf{x}\rangle\,(\mathbf{u}(\mathbf{q},t)-\mathbf{u}(\mathbf{x},t)) dV_{\mathbf{q}} dV_{\mathbf{p}} - \int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\mathbb{K}}[\mathbf{p}]\,\langle \mathbf{x}-\mathbf{p},\mathbf{q}-\mathbf{p}\rangle\,(\mathbf{u}(\mathbf{q},t)-\mathbf{u}(\mathbf{p},t)) dV_{\mathbf{q}} dV_{\mathbf{p}} + \mathbf{b}(\mathbf{x},t). \tag{2.4}$$

The double state-kernel $\mathbb{K}$, at a point $\mathbf{x}$, scalar valued, essentially weighs the interactions between two bonds, $\mathbf{p}-\mathbf{x}$ and $\mathbf{q}-\mathbf{x}$, whose output is denoted by $\mathbb{K}[\mathbf{x}]\,\langle \mathbf{p}-\mathbf{x},\mathbf{q}-\mathbf{x}\rangle$. For a simple material, equation (2.4) above represents a linearized state-based model for an elastic material if and only if $\mathbb{K}$ is symmetric, i.e. for any two bonds $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ which share the same application point, $\mathbb{K}[\mathbf{x}]\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle = \mathbb{K}[\mathbf{x}]\langle\boldsymbol{\zeta},\boldsymbol{\xi}\rangle$. See the discussion in [45, Section 4.2, Proposition 4.1].

In [45] several choices of the state-kernel $\mathbb{K}$ are considered, each of them leading to a different physical model. For $\mathbb{K}[\mathbf{x}]\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle$ given in terms of a Dirac mass supported at $\boldsymbol{\zeta}=\boldsymbol{\xi}$, one recovers the peridynamic bond-based formulation, which we will discuss in further detail in Chapter 4. Below, we consider a particular scalar-valued, choice of the state kernel $\mathbb{K}$ which will give rise to a new Laplacian-type operator. The definition and properties of this new operator, together with the connections to nonlocal and local Laplacians are discussed.

2.3 A doubly nonlocal Laplacian operator

Motivated by the discussion in the previous section, we consider the state-kernel $\mathbb{K}$ given by

$$\mathbb{K}[\mathbf{x}]\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle := [\gamma(\boldsymbol{\xi})+\gamma(\boldsymbol{\zeta})]\eta(\boldsymbol{\zeta}-\boldsymbol{\xi}), \tag{2.5}$$

where γ and η are symmetric functions, i.e. $\gamma(-\boldsymbol{\zeta})=\gamma(\boldsymbol{\zeta})$, and $\eta(-\boldsymbol{\xi})=\eta(\boldsymbol{\xi})$. Taking $\boldsymbol{\xi}=\mathbf{p}-\mathbf{x}$ and $\boldsymbol{\zeta}=\mathbf{q}-\mathbf{x}$, (2.5) becomes

$$\mathbb{K}[\mathbf{x}]\langle\mathbf{p}-\mathbf{x},\mathbf{q}-\mathbf{x}\rangle = [\gamma(\mathbf{p}-\mathbf{x})+\gamma(\mathbf{q}-\mathbf{x})]\eta(\mathbf{q}-\mathbf{p}). \tag{2.6}$$

We are now in position to formally introduce the new Laplace-type operator.

Definition 2.1. We define the nonlocal state-based Laplace operator $\mathcal{L}^s_{\gamma\eta}$ with kernels γ and η, to be the operator given by

$$\begin{aligned}\mathcal{L}^s_{\gamma\eta}[\mathbf{u}](\mathbf{x}) =&\sigma(\gamma,\eta)\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\left(\gamma(\mathbf{p}-\mathbf{x})+\gamma(\mathbf{q}-\mathbf{x})\right)\eta(\mathbf{q}-\mathbf{p})[\mathbf{u}(\mathbf{q})-\mathbf{u}(\mathbf{x})]d\mathbf{q}d\mathbf{p} \\ &-\sigma(\gamma,\eta)\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\left(\gamma(\mathbf{x}-\mathbf{p})+\gamma(\mathbf{q}-\mathbf{p})\right)\eta(\mathbf{q}-\mathbf{x})[\mathbf{u}(\mathbf{q})-\mathbf{u}(\mathbf{p})]d\mathbf{q}d\mathbf{p},\end{aligned} \tag{2.7}$$

where $\sigma(\gamma,\eta)$ is a normalizing factor which is given by (3.9).

In section 3.2 the scaling $\sigma(\gamma,\eta)$ will be determined for kernels γ,η with finite radii of interaction, $\delta_\gamma,\delta_\eta$ such that

$$|\mathcal{L}^s_{\gamma\eta}[\mathbf{u}](\mathbf{x})-\Delta\mathbf{u}(\mathbf{x})| \to 0 \text{ as } \delta_\gamma,\delta_\eta\to 0,$$

for $\mathbf{u}$ sufficiently smooth, and for every point $\mathbf{x}$ in the domain.

2.4 Kernels of the state-based Laplacian

Note from (2.5) that while $\mathbb{K}$ is symmetric with respect to the bonds $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, i.e. $\mathbb{K}[\mathbf{x}]\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle = \mathbb{K}[\mathbf{x}]\langle\boldsymbol{\zeta},\boldsymbol{\xi}\rangle$, the kernels γ, η play different roles in describing the dynamics. Indeed, the kernel elongations of the bonds $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are measured by the kernel γ, while η accounts for the interdependence between $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. Thus the choice $\eta(\boldsymbol{\zeta}-\boldsymbol{\xi}) = \delta_0(\boldsymbol{\zeta}-\boldsymbol{\xi})$, where δ_0 is the Dirac mass centered at the origin, will yield the bond-based model, [45]. With the same choice for η, and γ given by two derivatives of the Dirac mass, we obtain the classical Laplacian, [14]. These connections are strengthened as we show convergence of the operator to the classical Laplacian in Chapter 3, and are made explicit in Chapter 4 when we introduce the convolution form of the operator (2.7).

As previously done for bond-based peridynamics models, we will consider bounded regions of interactions for both stretching and bond interdependence effects, as given by γ, respectively η. Our specific assumptions for the kernels are given below.

Assumption 1. Assume that γ and η are nonnegative radial and integrable ($\gamma, \eta \in L^1$) functions, so with an abuse of notation we write $\gamma(\boldsymbol{\xi}) = \gamma(|\boldsymbol{\xi}|)$ and $\eta(\boldsymbol{\zeta}) = \eta(|\boldsymbol{\zeta}|)$. Assume that γ is supported inside the ball of radius δ_γ, and η is supported inside the ball of radius δ_η so that we have

$$\gamma(|\boldsymbol{\xi}|) = 0 \text{ for } |\boldsymbol{\xi}| > \delta_\gamma, \quad \text{and} \quad \eta(|\boldsymbol{\zeta}|) = 0 \text{ for } |\boldsymbol{\zeta}| > \delta_\eta.$$

Assumption 2. We consider specific rational forms for γ and η that allow us to explicitly compute the scaling for the operator $\mathcal{L}^s_{\gamma\eta}$ which gives the convergence to the classical Laplacian. For $\delta_\eta, \delta_\gamma > 0$ and $\alpha, \beta < n$, the choices

$$\gamma(\boldsymbol{\xi}) = \begin{cases} \dfrac{1}{|\boldsymbol{\xi}|^\alpha}, & |\boldsymbol{\xi}| \leq \delta_\gamma \\ 0, & |\boldsymbol{\xi}| > \delta_\gamma \end{cases}, \quad \eta(\boldsymbol{\zeta}) = \begin{cases} \dfrac{1}{|\boldsymbol{\zeta}|^\beta}, & |\boldsymbol{\zeta}| \leq \delta_\eta \\ 0, & |\boldsymbol{\zeta}| > \delta_\eta \end{cases}, \tag{2.8}$$

produce the state-kernel $\mathbb{K}$

$$\mathbb{K}[\mathbf{x}]\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle = \begin{cases} \left(\dfrac{1}{|\boldsymbol{\xi}|^\alpha} + \dfrac{1}{|\boldsymbol{\zeta}|^\alpha} \right) \dfrac{1}{|\boldsymbol{\zeta} - \boldsymbol{\xi}|^\beta}, & |\boldsymbol{\xi}| < \delta_\gamma, \text{ and } |\boldsymbol{\zeta} - \boldsymbol{\xi}| < \delta_\eta \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Remark 2.2. The cases where $\alpha, \beta = n$ is not considered in this thesis because of the lack of integrability of the kernels this produces, indeed, we would need to use the principal value of the integrals. Moreover, $\alpha, \beta = n$ require smoother functions $\mathbf{u}$ to be taken in the operator. For a discussion on bond-based kernels see [12, 11].

From this point on, to avoid confusion, we will refer to the horizon length of the bond-based Laplacian with kernel μ as δ_μ. Thus, every horizon length scale is immediately identifiable with the kernel it is associated with.

Next we introduce two functions $\pi_\gamma, \pi_\eta : (0, \infty) \to [0, \infty)$ related to our kernels γ and respectively η, which will be needed for the proof of our convergence result to allow us to move the derivatives on the function $\mathbf{u}$ through integration by parts. They are selected such that they satisfy

$$\nabla_{\mathbf{y}} \pi_\gamma(|\mathbf{y}|) = \mathbf{y}\gamma(\mathbf{y}), \quad \pi_\gamma(\delta_\gamma) = 0, \tag{2.10}$$

and

$$\nabla_{\mathbf{r}} \pi_\eta(|\mathbf{r}|) = \mathbf{r}\eta(\mathbf{r}), \quad \pi_\eta(\delta_\eta) = 0. \tag{2.11}$$

With the same abuse of notation for radial functions, we have that π_γ, and π_η are given explicitly by

$$\pi_\gamma(\mathbf{y}) = \pi_\gamma(|\mathbf{y}|) := \int_{\delta_\gamma}^{|\mathbf{y}|} \lambda\gamma(\lambda)d\lambda, \quad \text{and} \quad \pi_\eta(\mathbf{r}) = \pi_\eta(|\mathbf{r}|) := \int_{\delta_\eta}^{|\mathbf{r}|} \rho\eta(\rho)d\rho. \tag{2.12}$$

Under Assumption 2 we obtain

$$\pi_\gamma(\xi) = \begin{cases} \dfrac{|\xi|^{2-\alpha} - \delta_\gamma^{2-\alpha}}{2-\alpha}, & \text{if } \alpha \neq 2 \\ \ln(|\xi|/\delta_\gamma), & \text{if } \alpha = 2, \end{cases} \tag{2.13}$$

and

$$\pi_\eta(\zeta) = \begin{cases} \dfrac{|\zeta|^{2-\beta} - \delta_\eta^{2-\beta}}{2-\beta}, & \text{if } \beta \neq 2 \\ \ln(|\zeta|/\delta_\eta), & \text{if } \beta = 2. \end{cases} \tag{2.14}$$

For notational simplicity, when we write $\mathcal{B}_\gamma$ we mean the ball of radius δ_γ centered at zero. Similarly, $\mathcal{B}_\eta$ is the ball of radius δ_η centered at zero.

Lemma 2.3. *Under Assumption 1 with π_γ and π_η satisfying* (2.10) *and* (2.11)*, we have*

$$\frac{1}{n}\int_{\mathcal{B}_\gamma} |\mathbf{y}|^2\gamma(\mathbf{y})d\mathbf{y} = -\int_{\mathcal{B}_\gamma} \pi_\gamma(\mathbf{y})d\mathbf{y}, \tag{2.15}$$

and

$$\frac{1}{n}\int_{\mathcal{B}_\eta} |\mathbf{r}|^2\eta(\mathbf{r})d\mathbf{r} = -\int_{\mathcal{B}_\eta} \pi_\eta(\mathbf{r})d\mathbf{r}. \tag{2.16}$$

Proof. We prove the first equality, the second follows in a similar fashion. By taking the inner product of (2.10) with $\mathbf{y}$ we obtain

$$\mathbf{y}\cdot\nabla_{\mathbf{y}}\pi_\gamma(\mathbf{y})d\mathbf{y} = |\mathbf{y}|^2\gamma(\mathbf{y}).$$

Integration with respect to $\mathbf{y}$ on $\mathcal{B}_\gamma$ yields

$$\int_{\mathcal{B}_\gamma} \mathbf{y} \cdot \nabla_{\mathbf{y}} \pi_\gamma(\mathbf{y}) d\mathbf{y} = \int_{\mathcal{B}_\gamma} |\mathbf{y}|^2 \gamma(\mathbf{y}) d\mathbf{y}.$$

By performing an integration by parts on the left side, where ν is the normal derivative in the $\mathbf{y}$ direction, we have

$$\begin{aligned} \int_{\mathcal{B}_\gamma} \mathbf{y} \cdot \nabla_{\mathbf{y}} \pi_\gamma(\mathbf{y}) d\mathbf{y} &= \int_{\partial\mathcal{B}_\gamma} \pi_\gamma(\mathbf{y}) \mathbf{y} \cdot \nu \, d\mathbf{y} - \int_{\mathcal{B}_\gamma} \operatorname{div}(\mathbf{y}) \cdot \pi_\gamma(\mathbf{y}) d\mathbf{y} \\ &= \int_{\partial\mathcal{B}_\gamma} \pi_\gamma(\mathbf{y}) \mathbf{y} \cdot \nu \, d\mathbf{y} - n \int_{\mathcal{B}_\gamma} \pi_\gamma(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

For $\mathbf{y} \in \partial\mathcal{B}_\gamma$, $\pi_\gamma(\mathbf{y}) = \pi_\gamma(\delta_\gamma) = 0$ thus (2.15) holds. □

In order to employ the functions π_γ and π_η in the proof of our convergence result, we will need a different formulation for the state-based Laplacian, which is obtained in the next section.

2.5 Second formulation for the state-based Laplacian

The new expression for the state-based Laplacian will more easily allow us to identify the domains of integration for the variables, and simplify the integrand. This more convenient form is given by Proposition 2.4.

Proposition 2.4. *Under Assumption 1, the state-based Laplacian can be written in* *the following form:*

$$\mathcal{L}^s_{\gamma\eta}[\mathbf{u}](\mathbf{x}) = 2\sigma(\delta_\gamma, \delta_\eta) \int\limits_{\mathcal{B}_\gamma} \int\limits_{\mathcal{B}_\eta} \gamma(\mathbf{y})\eta(\mathbf{r})[\mathbf{u}(\mathbf{x}+\mathbf{y}+\mathbf{r}) - \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}+\mathbf{r}) + \mathbf{u}(\mathbf{x}+\mathbf{y})] d\mathbf{r} d\mathbf{y}. \quad (2.17)$$

Proof. By rearranging (2.7) we obtain

$$\begin{aligned} \frac{\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]}{2\sigma(\delta_\gamma, \delta_\eta)}(\mathbf{x}) = & \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \gamma(\mathbf{p}-\mathbf{x})\eta(\mathbf{q}-\mathbf{p})[\mathbf{u}(\mathbf{q}) - \mathbf{u}(\mathbf{x})] d\mathbf{q} d\mathbf{p} \\ & - \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \gamma(\mathbf{x}-\mathbf{p})\eta(\mathbf{q}-\mathbf{x})[\mathbf{u}(\mathbf{q}) - \mathbf{u}(\mathbf{p})] d\mathbf{q} d\mathbf{p} \\ & + \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \gamma(\mathbf{q}-\mathbf{x})\eta(\mathbf{q}-\mathbf{p})[\mathbf{u}(\mathbf{q}) - \mathbf{u}(\mathbf{x})] d\mathbf{q} d\mathbf{p} \\ & - \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \gamma(\mathbf{q}-\mathbf{p})\eta(\mathbf{q}-\mathbf{x})[\mathbf{u}(\mathbf{q}) - \mathbf{u}(\mathbf{p})] d\mathbf{q} d\mathbf{p}. \quad (2.18) \end{aligned}$$

We perform a change of variables in each of the above integrals

- $\mathbf{y} := \mathbf{p} - \mathbf{x}$ and $\mathbf{r} := \mathbf{q} - \mathbf{p}$, in the first integral,
- $\mathbf{y} := \mathbf{p} - \mathbf{x}$ and $\mathbf{r} := \mathbf{q} - \mathbf{x}$, in the second integral,
- $\mathbf{y} := \mathbf{q} - \mathbf{x}$ and $\mathbf{r} := \mathbf{p} - \mathbf{q}$, in the third integral,
- and $\mathbf{y} := \mathbf{q} - \mathbf{p}$ and $\mathbf{r} := \mathbf{q} - \mathbf{x}$ in the fourth integral.

The resulting form is

$$
\begin{aligned}
&\frac{\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]}{2\sigma(\delta_\gamma,\delta_\eta)}(\mathbf{x}) \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \gamma(\mathbf{y})\eta(\mathbf{r})[\mathbf{u}(\mathbf{y}+\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}+\mathbf{r}) + \mathbf{u}(\mathbf{x}+\mathbf{y})]d\mathbf{r}d\mathbf{y} \\
&+ \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \gamma(\mathbf{y})\eta(\mathbf{r})[\mathbf{u}(\mathbf{y}+\mathbf{x}) - \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{r}+\mathbf{x}) + \mathbf{u}(\mathbf{r}+\mathbf{x}-\mathbf{y})]d\mathbf{r}d\mathbf{y}.
\end{aligned}
$$

A final change of variables in

$$
\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \gamma(\mathbf{y})\eta(\mathbf{r})\mathbf{u}(\mathbf{r}+\mathbf{x}-\mathbf{y})d\mathbf{r}d\mathbf{y},
$$

and the fact that $\gamma(\mathbf{y}) = \gamma(-\mathbf{y})$, together with Assumption 1 give (2.17). □

We will use this form of the state-based Laplacian to prove the results of Chapter 3.

Chapter 3

Convergence

In the previous chapter we introduced a new nonlocal state-based Laplacian. We would like to spend some time analyzing the properties of this new operator. In particular, we would like to understand the operator as the nonlocality vanishes, i.e. we are interested in the results of the state-based Laplacian when we shrink the horizons of the state-based theory to zero. We start by reproving a result from [19] in Section 3.1. This result illustrates the convergence and rate of the bond-based Laplacian to the classical Laplacian, each applied to C^4 functions, as the length of the horizon δ_μ goes to zero, where δ_μ is the horizon of the bond-based Laplacian associated with the kernel μ. The rate of convergence is shown to be δ_μ^2. In Section 3.2 we prove two convergence results for the state-based Laplacian to the classical Laplacian as the state-based horizon lengths, δ_γ and δ_μ, go to zero. The first result gives convergence for analytical functions on a bounded interval obtaining a convergence rate of $\delta_\gamma^2 + \delta_\eta^2$. The final and main result of the chapter is convergence for C^4 functions on possibly unbounded domains in $\mathbb{R}^n$, with a convergence rate of δ_γ^2.

3.1 Convergence of the bond-based Laplacian

The convergence of the bond-based Laplacian with kernel μ to the classical Laplacian as the horizon δ_μ shrinks to zero has been studied in several papers. It has been shown that the rate of convergence for the nonlocal Laplacian to the classical Laplacian, whenever applied to a sufficiently smooth function u, is proportional to δ_μ^2 (the proportionality constant depends on bounds for the fourth derivative of u); see [14],[19],[36] where the arguments are based on the work in [7]; see also [49] where the analysis for numerical error is performed. Furthermore, in [36] the authors have shown strong convergence in L^2 of nonlocal L^2 solutions to classical solutions with H_0^1 Sobolev regularity.

In [19] the authors produce a new technique to prove that the bond-based Laplacian converges to the classical Laplacian. They show the bond-based Laplacian applied to sufficiently smooth functions provides an approximation for the classical Laplacian for δ_μ near zero. In particular they show that the rate of convergence is δ_μ^2, where δ_μ is the length scale for the horizon of the bond-based model. In this section we reprove this result using the method in [19] in order to keep the presentation self contained, and as preface to the next section (3.2), where we will utilize this technique to prove our main result.

Theorem 3.1 (Foss, Radu). *[19] Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, possibly unbounded, and let $\mathbf{u} \in C^4(\Omega)$ with*

$$M_4 := \sup_{\mathbf{x}\in\Omega} |\mathbf{u}^{(4)}(\mathbf{x})| < \infty. \tag{3.1}$$

Assume that μ is a nonnegative radial function and is supported inside the ball of radius δ_μ. Then, with scaling factor $\sigma(\delta_\mu)$ given by (3.2), $\mathcal{L}_\mu^b$ converges to $\Delta\mathbf{u}$ at a rate of δ_μ^2.

The scaling $\sigma(\delta_\mu)$ of the bond-based Laplacian is shown in the proof of the theorem to be

$$\sigma(\delta_\mu) = -\frac{2}{\int\limits_{\mathcal{B}_\mu} \pi_\mu(\mathbf{y})d\mathbf{y}}, \tag{3.2}$$

where π_μ is the function associated with μ taking a similar form as π_γ and π_η in (2.10) and (2.11). That is

$$\nabla_{\mathbf{y}}\pi_\mu(|\mathbf{y}|) = \mathbf{y}\mu(\mathbf{y}), \quad \pi_\mu(\delta_\mu) = 0, \tag{3.3}$$

with the same abuse of notation for radial functions, π_μ is given explicitly by

$$\pi_\mu(\mathbf{y}) = \pi_\mu(|\mathbf{y}|) := \int\limits_{\delta_\mu}^{|\mathbf{y}|} \lambda\mu(\lambda)d\lambda. \tag{3.4}$$

In particular, for the specific kernels

$$\mu(\boldsymbol{\xi}) = \begin{cases} \dfrac{1}{|\boldsymbol{\xi}|^\alpha}, & |\boldsymbol{\xi}| \leq \delta_\mu \\ 0, & |\boldsymbol{\xi}| > \delta_\mu \end{cases}, \tag{3.5}$$

we have

$$\pi_\mu(\xi) = \begin{cases} \dfrac{|\xi|^{2-\alpha} - \delta_\mu^{2-\alpha}}{2-\alpha}, & \text{if } \alpha \neq 2 \\ \ln(|\xi|/\delta_\mu), & \text{if } \alpha = 2, \end{cases} \tag{3.6}$$

and

$$\sigma(\delta_\mu) = \frac{2(2-\kappa+n)\delta_\mu^{\kappa-n-2}}{\omega_{n-1}}, \tag{3.7}$$

where w_{n-1} is the volume of the unit ball in $n-1$ dimensions.

We now prove Theorem 3.1.

Proof. From a change of variables of (1.19) we have

$$\frac{\mathcal{L}^b_\mu[\mathbf{u}]}{\sigma(\delta_\mu)}(\mathbf{x}) = \int\limits_{\mathcal{B}_\mu} \mu(\mathbf{y})\big(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x})\big)d\mathbf{y}.$$

Applying the fundamental theorem of calculus we obtain

$$\frac{\mathcal{L}^b_\mu[\mathbf{u}]}{\sigma(\delta_\mu)}(\mathbf{x}) = \int\limits_{\mathcal{B}_\mu} \mu(\mathbf{y}) \int\limits_0^1 \nabla\mathbf{u}(\mathbf{x}+s\mathbf{y})\mathbf{y}dsd\mathbf{y},$$

where $\nabla\mathbf{u}$ is the Jacobian matrix for $\mathbf{u}$. Using π_μ as defined in (3.3) we have

$$\frac{\mathcal{L}^b_\mu[\mathbf{u}]}{\sigma(\delta_\mu)}(\mathbf{x}) = \int\limits_{\mathcal{B}_\mu} \int\limits_0^1 \nabla_y \pi_\mu(\mathbf{y})\nabla\mathbf{u}(\mathbf{x}+s\mathbf{y})dsd\mathbf{y}.$$

Integrating by parts gives

$$\frac{\mathcal{L}^b_\mu[\mathbf{u}]}{\sigma(\delta_\mu)}(\mathbf{x}) = \int\limits_0^1 \int\limits_{\partial\mathcal{B}_\mu} \pi_\mu(\mathbf{y})\nabla\mathbf{u}(\mathbf{x}+s\mathbf{y})\frac{\mathbf{y}}{\delta_\mu}dsd\mathbf{y} - \int\limits_0^1 \int\limits_{\mathcal{B}_\mu} \pi_\mu(\mathbf{y})\mathrm{div}_{\mathbf{y}}\left(\nabla\mathbf{u}(\mathbf{x}+s\mathbf{y})\right)dsd\mathbf{y}.$$

For $\mathbf{y} \in \partial\mathcal{B}_\mu$, we have that $\pi_\mu(\mathbf{y}) = \pi_\mu(\delta_\mu) = 0$, thus the first term vanishes and we obtain

$$\frac{\mathcal{L}^b_\mu[\mathbf{u}]}{\sigma(\delta_\mu)}(\mathbf{x}) = -\int\limits_0^1 \int\limits_{\mathcal{B}_\mu} \pi_\mu(\mathbf{y})s\Delta\mathbf{u}(\mathbf{x}+s\mathbf{y})dsd\mathbf{y}.$$

Adding and subtracting $\Delta\mathbf{u}(\mathbf{x})$ we have

$$\frac{\mathcal{L}^b_\mu[\mathbf{u}]}{\sigma(\delta_\mu)}(\mathbf{x}) = -\int_0^1\int_{\mathcal{B}_\mu} \pi_\mu(\mathbf{y})s\big(\Delta\mathbf{u}(\mathbf{x}+s\mathbf{y}) - \Delta\mathbf{u}(\mathbf{x})\big)dsd\mathbf{y} - \int_0^1\int_{\mathcal{B}_\mu} \pi_\mu(\mathbf{y})s\Delta\mathbf{u}(\mathbf{x})dsd\mathbf{y}.$$

In order to make the coefficient of the classical Laplacian one in the second term, we take $\sigma(\delta_\mu)$ as given in (3.2). With this choice of scaling we can write

$$\mathcal{L}^b_\mu[\mathbf{u}](\mathbf{x}) - \Delta\mathbf{u}(\mathbf{x}) = -\sigma(\delta_\mu)\int_0^1\int_{\mathcal{B}_\mu} \pi_\mu(\mathbf{y})s\big(\Delta\mathbf{u}(\mathbf{x}+s\mathbf{y}) - \Delta\mathbf{u}(\mathbf{x})\big)dsd\mathbf{y}.$$

Integrating by parts with respect to s on the right hand side produces

$$\mathcal{L}^b_\mu[\mathbf{u}](\mathbf{x}) - \Delta\mathbf{u}(\mathbf{x}) = \sigma(\delta_\mu)\int_{\mathcal{B}_\mu} \pi_\mu(\mathbf{y})\left(\frac{1-s^2}{2}\right)\big(\Delta\mathbf{u}(\mathbf{x}+s\mathbf{y}) - \Delta\mathbf{u}(\mathbf{x})\big)\Big|_{s=0}^{s=1}d\mathbf{y} - \sigma(\delta_\mu)\int_{\mathcal{B}_\mu}\int_0^1 \pi_\mu(\mathbf{y})\left(\frac{1-s^2}{2}\right)\Delta\nabla\mathbf{u}(\mathbf{x}+s\mathbf{y})\mathbf{y}dsd\mathbf{y}.$$

Evaluating the first integral at $s=0$ and $s=1$ we obtain

$$\mathcal{L}^b_\mu[\mathbf{u}](\mathbf{x}) - \Delta\mathbf{u}(\mathbf{x}) = -\sigma(\delta_\mu)\int_{\mathcal{B}_\mu}\int_0^1 \pi_\mu(\mathbf{y})\left(\frac{1-s^2}{2}\right)\Delta\nabla\mathbf{u}(\mathbf{x}+s\mathbf{y})\mathbf{y}dsd\mathbf{y}.$$

Once again, we integrate by parts with respect to s to get

$$\mathcal{L}^b_\mu[\mathbf{u}](\mathbf{x}) - \Delta\mathbf{u}(\mathbf{x}) = \sigma(\delta_\mu)\int\limits_{\mathcal{B}_\mu} \pi_\mu(\mathbf{y})\left(\frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right)\Delta\nabla\mathbf{u}(\mathbf{x}+s\mathbf{y})\mathbf{y}\Big|_{s=0}^{s=1} d\mathbf{y}$$
$$- \sigma(\delta_\mu)\int\limits_{\mathcal{B}_\mu}\int\limits_0^1 \pi_\mu(\mathbf{y})\left(\frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right)\Delta\nabla^2\mathbf{u}(\mathbf{x}+s\mathbf{y})\mathbf{y}\cdot\mathbf{y}ds d\mathbf{y}.$$

Evaluating at $s = 0$ and $s = 1$ we find that the first integral vanishes, leaving

$$\mathcal{L}^b_\mu[\mathbf{u}](\mathbf{x}) - \Delta\mathbf{u}(\mathbf{x}) = -\sigma(\delta_\mu)\int\limits_{\mathcal{B}_\mu}\int\limits_0^1 \pi_\mu(\mathbf{y})\left(\frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right)\Delta\nabla^2\mathbf{u}(\mathbf{x}+s\mathbf{y})\mathbf{y}\cdot\mathbf{y}ds d\mathbf{y}.$$

With M_4 as defined in (3.1), we estimate

$$\begin{aligned}|\mathcal{L}^b_\mu[\mathbf{u}](\mathbf{x}) - \Delta\mathbf{u}(\mathbf{x})| &\leq M_4\sigma(\delta_\mu)\int\limits_{\mathcal{B}_\mu}\int\limits_0^1 \pi_\mu(\mathbf{y})\left(\frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right)|\mathbf{y}|^2 ds d\mathbf{y} \\ &= \frac{M_4\sigma(\delta_\mu)}{8}\int\limits_{\mathcal{B}_\mu}\pi_\mu(\mathbf{y})|\mathbf{y}|^2 d\mathbf{y} \\ &= \frac{M_4\sigma(\delta_\mu)n\omega_{n-1}}{8}\int\limits_0^{\delta_\mu}\pi_\mu(\lambda)\lambda^{n+1}d\lambda \\ &\leq \frac{M_4\sigma(\delta_\mu)\delta_\mu^2}{8}\int\limits_{\mathcal{B}_\mu}\pi_\mu(\mathbf{y})d\mathbf{y},\end{aligned}$$

where in the last two steps we used the coarea formula and the fact that

$$\lambda^n \leq \lambda^{n-1}\delta_\mu.$$

Finally, taking $\sigma(\delta_\mu)$ to be as defined in (3.2) we obtain

$$|\mathcal{L}^b_\mu[\mathbf{u}](\mathbf{x}) - \Delta\mathbf{u}(\mathbf{x})| \le \frac{M_4}{4}\delta_\mu^2. \tag{3.8}$$

□

We see that the bond-based Laplacian applied to $\mathbf{u}$ approaches the classical Laplacian applied to $\mathbf{u}$ at a rate that is quadratically dependent on the length scale of the horizon, δ_μ. One might hope that a similar rate, one that depends explicitly on the length scales of *both* horizons, would hold in the convergence of the state-based Laplacian. We will show that the state-based Laplacian has a rate of convergence $\delta_\gamma^2 + \delta_\eta^2$ only under strict conditions. Then we will loosen restrictions to show that we can still obtain convergence, but at a rate that is explicitly dependent only on the length scale associated with γ, i.e. δ_γ^2, but with the condition that $\delta_\eta \le \delta_\gamma$.

3.2 Convergence of the state-based Laplacian

The main result of this section shows that the state-based Laplacian applied to sufficiently smooth functions provides an approximation for the classical Laplacian applied to the same function, for δ_γ and δ_η close to zero. In fact, we exhibit a rate of convergence for the error of this approximation that is quadratic with respect to the kernel horizons. The scaling of the state-based Laplacian needed for this approximation will be shown to satisfy

$$\sigma(\delta_\gamma, \delta_\eta) = -\frac{1}{2\displaystyle\int_{\mathcal{B}_\eta} \eta(\mathbf{r})d\mathbf{r} \int_{\mathcal{B}_\gamma} \pi_\gamma(\mathbf{y})d\mathbf{y}}, \tag{3.9}$$

where η is the kernel in $\mathcal{L}^s_{\gamma\eta}$, and π_γ is the function associated with γ given by (2.10). From Lemma (2.3) we find that this scaling is equivalent to

$$\sigma(\delta_\gamma, \delta_\eta) = \frac{n}{2\int\limits_{\mathcal{B}_\eta} \eta(\mathbf{r})d\mathbf{r} \int\limits_{\mathcal{B}_\gamma} \mathbf{y}^2\gamma(\mathbf{y})d\mathbf{y}}. \tag{3.10}$$

In particular, for the specific kernels of (2.8), if $\alpha \neq 2$, we have

$$\sigma(\delta_\gamma, \delta_\eta) = \frac{(n-\beta)(n-\alpha+2)n\delta_\eta^{\beta-n}\delta_\gamma^{\alpha-n-2}}{2w_{n-1}^2}, \tag{3.11}$$

where w_{n-1} is the volume of the ball in $n-1$ dimensions. We begin by showing that in one dimension the rate of the difference between the nonlocal Laplacian and the classical Laplacian, when applied to analytic functions is of order $\delta_\eta^2 + \delta_\gamma^2$.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}$ be a bounded interval, and let u be analytic in Ω, with*

$$M := \sup_{x\in\Omega} |u^{(k)}(x)| < \infty,\ k \geq 4. \tag{3.12}$$

For γ and η satisfying Assumption 1 above, and the scaling $\sigma(\delta_\gamma, \delta_\eta)$ given by (3.10) *with $n = 1$, we have*

$$\|\mathcal{L}^s_{\gamma\eta}[u] - \Delta u\|_{L^\infty(\Omega)} < C(\delta_\eta^2 + \delta_\gamma^2), \tag{3.13}$$

as $\delta_\gamma, \delta_\eta \to 0$, where the constant C depends on M given by (3.12).

Proof. From (2.17) we have

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\delta_\gamma, \delta_\eta)} = \int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta} \gamma(y)\eta(r)[u(x+y+r) - u(x) - [u(x+r) - u(x+y)]]\,drdy.$$

Using the analytic expansion for u around x in the first term, and around $x+y$ in the third term we obtain

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\delta_\gamma,\delta_\eta)} = \int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[u'(x)(y+r) + u''(x)\frac{(y+r)^2}{2}\right. \tag{3.14}$$

$$\left. + u'''(x)\frac{(y+r)^3}{3!} + u^{(4)}(x)\frac{(y+r)^4}{4!} + \sum_{n=5}^{\infty} u^{(n)}(x)\frac{(y+r)^n}{n!}\right] drdy$$

$$-\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[u'(x+y)(r-y) + u''(x+y)\frac{(r-y)^2}{2} + u'''(x+y)\frac{(r-y)^3}{3!}\right.$$

$$\left. + u^{(4)}(x+y)\frac{(r-y)^4}{4!} + \sum_{n=5}^{\infty} u^{(n)}(x+y)\frac{(r-y)^n}{n!}\right] drdy.$$

Since $\gamma(y)$ and $\eta(r)$ are symmetric, each of the terms that is an odd power in y or r in the first integral on the right hand side of (3.14) is antisymmetric, with respect to y, or respectively r; hence, they vanish after integration. Similarly, in the second integral the terms containing odd powers of r are antisymmetric and therefore they also disappear (note that the same does not hold for y due to the presence of y in $u(x+y)$). We obtain:

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\delta_\gamma,\delta_\eta)} = \int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[u''(x)\frac{y^2+r^2}{2} + u^{(4)}(x)\frac{y^4+6y^2r^2+r^4}{4!}\right.$$

$$\left. + \sum_{n=3}^{\infty} u^{(2n)}(x)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right] drdy$$

$$-\int_{\mathcal{B}_\gamma}\int_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[-u'(x+y)y + u''(x+y)\frac{r^2+y^2}{2}\right.$$

$$-u'''(x+y)\frac{3r^2y+y^3}{3!} + u^{(4)}(x+y)\frac{r^4+6r^2y^2+y^4}{4!}$$

$$-\sum_{n=3}^{\infty} u^{(2n-1)}(x+y)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}$$

$$\left.+\sum_{n=3}^{\infty} u^{(2n)}(x+y)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right] drdy.$$

Gathering the even derivative terms we have

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\delta_\gamma,\delta_\eta)} = \int_{\mathcal{B}_\gamma}\int_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[\left(u''(x)-u''(x+y)\right)\frac{y^2+r^2}{2}\right.$$

$$+\left(u^{(4)}(x)-u^{(4)}(x+y)\right)\frac{y^4+6y^2r^2+r^4}{4!}$$

$$\left.+\sum_{n=3}^{\infty}\left(u^{(2n)}(x)-u^{(2n)}(x+y)\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right] drdy$$

$$+\int_{\mathcal{B}_\gamma}\int_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[u'(x+y)y + u'''(x+y)\frac{3r^2y+y^3}{3!}\right.$$

$$\left.+\sum_{n=3}^{\infty} u^{(2n-1)}(x+y)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\right] drdy.$$

Employing analytic expansions in each of the even derivative terms near x gives

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\delta_\gamma,\delta_\eta)} = -\int_{\mathcal{B}_\gamma}\int_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[\left(u'''(x)y + \sum_{j=2}^{\infty} u^{(2+j)}(x)\frac{y^j}{j!}\right)\frac{y^2+r^2}{2}\right.$$

$$+\left(u^{(5)}(x)y + \sum_{j=2}^{\infty} u^{(4+j)}(x)\frac{y^j}{j!}\right)\frac{y^4+6y^2r^2+r^4}{4!}$$

$$\left.+\sum_{n=3}^{\infty}\left(\sum_{j=1}^{\infty} u^{(2n+j)}(x)\frac{y^j}{j!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right] drdy$$

$$+\int_{\mathcal{B}_\gamma}\int_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[u'(x+y)y + u'''(x+y)\frac{3r^2y+y^3}{3!}\right.$$

$$\left.+\sum_{n=3}^{\infty} u^{(2n-1)}(x+y)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\right] drdy.$$

As before, each of the odd power terms (in y) in the first integral are antisymmetric and vanish. Simplifying produces

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\delta_\gamma,\delta_\eta)}$$

$$= -\int_{\mathcal{B}_\gamma}\int_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} u^{(2n+2j)}(x)\frac{y^{2j}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right] drdy$$

$$+\int_{\mathcal{B}_\gamma}\int_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[u'(x+y)y + u'''(x+y)\frac{3r^2y+y^3}{3!}\right.$$

$$\left.+\sum_{n=3}^{\infty} u^{(2n-1)}(x+y)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\right] drdy.$$

Next, we perform an analytic expansion around x for each of the odd derivatives in the second integral to obtain

$$\frac{\mathcal{L}^s_{\gamma\eta}[u]}{2\sigma(\delta_\gamma,\delta_\eta)}$$

$$= -\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} u^{(2n+2j)}(x)\frac{y^{2j}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right]drdy$$

$$+\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[\left(u'(x)+u''(x)y+\sum_{j=2}^{\infty}u^{(1+j)}(x)\frac{y^j}{j!}\right)y\right.$$

$$+\left(u'''(x)+u^{(4)}(x)y+\sum_{j=2}^{\infty}u^{(3+j)}(x)\frac{y^j}{j!}\right)\frac{(3r^2y+y^3)}{3!}$$

$$\left.+\sum_{n=3}^{\infty}\left(\sum_{j=0}^{\infty}u^{(2n-1+j)}(x)\frac{y^j}{j!}\right)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\right]drdy.$$

Once again, the odd power terms (in y) in the second integral are antisymmetric and vanish. Simplifying and moving $2\sigma(\delta_\gamma,\delta_\eta)$ to the right side of the equation we obtain

$$\mathcal{L}^s_{\gamma\eta}[u](x)$$

$$= -2\sigma(\delta_\gamma,\delta_\eta)\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta} \gamma(y)\eta(r)\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} u^{(2n+2j)}(x)\frac{y^{2j}}{(2j)!}\right)\right.$$

$$\left.\cdot\sum_{i=0}^{n}\binom{2n}{2i}\frac{y^{2n-2i}r^{2i}}{(2n)!}\right]drdy$$

$$+\left(2\sigma(\delta_\gamma,\delta_\eta)\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(y)\eta(r)y^2drdy\right)u''(x)$$

$$+2\sigma(\delta_\gamma,\delta_\eta)\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(y)\eta(r)\sum_{j=1}^{\infty}u^{(2+2j)}(x)\frac{y^{2j+1}}{(2j+1)!}ydrdy$$

$$+\,2\sigma(\delta_\gamma,\delta_\eta)\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(y)\eta(r)\left[\sum_{n=2}^{\infty}\left(\sum_{j=0}^{\infty}u^{(2n+2j)}(x)\frac{y^{2j+1}}{(2j+1)!}\right)\right.$$
$$\left.\cdot\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{y^{2n-1-2i}r^{2i}}{(2n-1)!}\right]drdy.$$

The scaling given by (3.10) normalizes the coefficient of u'', so the error of the approximation $|\mathcal{L}^s_{\gamma\eta}[u](x)-u''(x)|$ is given by the remaining terms:

$$|\mathcal{L}^s_{\gamma\eta}[u](x)-u''(x)|$$
$$\leq 2M\sigma(\delta_\gamma,\delta_\eta)\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(y)\eta(r)\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{|y|^{2j}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{|y|^{2n-2i}|r|^{2i}}{(2n)!}\right]drdy$$

$$+2M\sigma(\delta_\gamma,\delta_\eta)\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(y)\eta(r)\sum_{j=1}^{\infty}\frac{|y|^{2j+1}}{(2j+1)!}\,|y|\,drdy$$

$$+\,2M\sigma(\delta_\gamma,\delta_\eta)\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(y)\eta(r)\left[\sum_{n=2}^{\infty}\left(\sum_{j=0}^{\infty}\frac{|y|^{2j+1}}{(2j+1)!}\right)\right.$$
$$\left.\cdot\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{|y|^{2n-1-2i}|r|^{2i}}{(2n-1)!}\right]drdy,$$

where M is defined in (3.12). Since $|y|<\delta_\gamma$, and $|r|<\delta_\eta$, we get

$$|\mathcal{L}^s_{\gamma\eta}[u](x)-u''(x)|$$
$$\leq 2M\sigma(\delta_\gamma,\delta_\eta)\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{\delta_\gamma^{2j-2}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{\delta_\gamma^{2n-2i}\delta_\eta^{2i}}{(2n)!}\right]\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(y)\eta(r)y^2drdy$$

$$+2M\sigma(\delta_\gamma,\delta_\eta)\sum_{j=1}^{\infty}\frac{\delta_\gamma^{2j}}{(2j+1)!}\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(y)\eta(r)y^2drdy$$

$$+\,2M\sigma(\delta_\gamma,\delta_\eta)\left[\sum_{n=2}^{\infty}\left(\sum_{j=0}^{\infty}\frac{\delta_\gamma^{2j}}{(2j+1)!}\right)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{\delta_\gamma^{2n-2-2i}\delta_\eta^{2i}}{(2n-1)!}\right]$$
$$\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(y)\eta(r)y^2drdy.$$

Using $\sigma(\delta_\gamma,\delta_\eta)$ as given in (3.10) we have

$$|\mathcal{L}^s_{\gamma\eta}[u](x)-u''(x)|\le M\left[\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{\delta_\gamma^{2j-2}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{\delta_\gamma^{2n-2i}\delta_\eta^{2i}}{(2n)!}\right]$$

$$+M\sum_{j=1}^{\infty}\frac{\delta_\gamma^{2j}}{(2j+1)!}+M\left[\sum_{n=2}^{\infty}\left(\sum_{j=0}^{\infty}\frac{\delta_\gamma^{2j}}{(2j+1)!}\right)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{\delta_\gamma^{2n-2-2i}\delta_\eta^{2i}}{(2n-1)!}\right].$$

Separating the $n=1$ terms in the first set of summations, the $j=1$ in the second summation, and the $n=2$ terms in the third set of summations, we obtain

$$|\mathcal{L}^s_{\gamma\eta}[u](x)-u''(x)|$$
$$\le M\left(\frac{\delta_\gamma^2+\delta_\eta^2}{2}\right)\sum_{j=1}^{\infty}\frac{\delta_\gamma^{2j-2}}{(2j)!}+M\left[\sum_{n=2}^{\infty}\left(\sum_{j=1}^{\infty}\frac{\delta_\gamma^{2j-2}}{(2j)!}\right)\sum_{i=0}^{n}\binom{2n}{2i}\frac{\delta_\gamma^{2n-2i}\delta_\eta^{2i}}{(2n)!}\right]$$

$$+M\frac{\delta_\gamma^2}{6}+M\sum_{j=2}^{\infty}\frac{\delta_\gamma^{2j-2}}{(2j)!}$$

$$+M\left(\frac{\delta_\gamma^2+3\delta_\eta^2}{6}\right)\sum_{j=0}^{\infty}\frac{\delta_\gamma^{2j}}{(2j+1)!}+M\left[\sum_{n=3}^{\infty}\left(\sum_{j=0}^{\infty}\frac{\delta_\gamma^{2j}}{(2j+1)!}\right)\sum_{i=0}^{n-1}\binom{2n-1}{2i}\frac{\delta_\gamma^{2n-2-2i}\delta_\eta^{2i}}{(2n-1)!}\right].$$

Since all of the above series are convergent for $\delta_\gamma,\delta_\eta<1$ we note that each term is of order δ_γ^2 or δ_η^2. Thus as δ_γ and δ_η shrink to zero, $\mathcal{L}^s_{\gamma\eta}[u]$ converges to u'' at a rate of $\delta_\gamma^2 + \delta_\eta^2$. □

Next we will present a much more general convergence result that holds in any dimension, under less regularity for $\mathbf{u}$, however we add an additional restriction on the support of the kernels. The ideas follow the method developed in [19] and illustrated in the proof of Theorem 3.1 to show convergence of the bond-based Laplacian to the classical Laplacian.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, possibly unbounded, and let $\mathbf{u} \in C^4(\Omega)$ with*

$$M_4 := \sup_{\mathbf{x}\in\Omega} |\mathbf{u}^{(4)}(\mathbf{x})| < \infty. \tag{3.15}$$

Let γ and η satisfy Assumption 1, with the restriction that $\delta_\eta \leq \delta_\gamma$. If

$$c_1|\mathbf{y}|^{-\alpha} \leq \gamma(\mathbf{y}) \leq c_2|\mathbf{y}|^{-\alpha} \quad \textit{for} \quad 0 \leq \alpha < n, \ \textit{and} \ 0 < c_1 \leq c_2,$$

then $\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]$ with scaling factor $\sigma(\delta_\gamma, \delta_\eta)$ given by (3.9) converges to $\Delta\mathbf{u}$ at a rate of δ_γ^2.

Proof. From (2.17) we have

$$\frac{\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]}{2\sigma(\delta_\gamma,\delta_\eta)}(\mathbf{x}) = \int_{\mathcal{B}_\gamma}\int_{\mathcal{B}_\eta} \gamma(\mathbf{y})\eta(\mathbf{r})\Big(\mathbf{u}(\mathbf{x}+\mathbf{y}+\mathbf{r}) - \mathbf{u}(\mathbf{x}) - [\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}+\mathbf{y})]\Big)d\mathbf{r}d\mathbf{y}.$$

Applying the fundamental theorem of calculus we obtain

$$\frac{\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]}{2\sigma(\delta_\gamma,\delta_\eta)}(\mathbf{x}) = \int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(\mathbf{y})\eta(\mathbf{r})\int\limits_0^1[\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))](\mathbf{y}+\mathbf{r})ds d\mathbf{r} d\mathbf{y}$$

$$-\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(\mathbf{y})\eta(\mathbf{r})\int\limits_0^1[\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))](\mathbf{r}-\mathbf{y})ds d\mathbf{r} d\mathbf{y},$$

where $\nabla\mathbf{u}$ is the Jacobian matrix for $\mathbf{u}$. Expanding and collecting similar terms produces

$$\frac{\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]}{2\sigma(\delta_\gamma,\delta_\eta)}(\mathbf{x})$$
$$=\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(\mathbf{y})\eta(\mathbf{r})\int\limits_0^1\left(\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))+\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))\right)\mathbf{y} ds d\mathbf{r} d\mathbf{y}$$

$$+\int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\gamma(\mathbf{y})\eta(\mathbf{r})\int\limits_0^1(\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))-\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y})))\mathbf{r} ds d\mathbf{r} d\mathbf{y}$$

Using π_γ and π_η as defined in (2.10) and (2.11) we obtain

$$\frac{\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]}{2\sigma(\delta_\gamma,\delta_\eta)} =: I_1 + I_2 \tag{3.16}$$

where

$$I_1 := \int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\int\limits_0^1\eta(\mathbf{r})\Big(\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))$$
$$+\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))\Big)\nabla_{\mathbf{y}}\pi_\gamma(\mathbf{y}) ds d\mathbf{r} d\mathbf{y},$$

$$I_2 := \int\limits_{\mathcal{B}_\gamma}\int\limits_{\mathcal{B}_\eta}\int\limits_0^1 \gamma(\mathbf{y})\Big(\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))$$

$$-\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))\Big)\nabla_{\mathbf{r}}\pi_\eta(\mathbf{r})dsd\mathbf{r}d\mathbf{y}.$$

Note that I_1 and I_2 are vector valued quantities. Integration by parts in I_1 yields

$$I_1 = -\int\limits_{\mathcal{B}_\eta}\int\limits_0^1\int\limits_{\mathcal{B}_\gamma} \eta(\mathbf{r})\pi_\gamma(\mathbf{y})\mathrm{div}_{\mathbf{y}}[\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))]d\mathbf{y}dsd\mathbf{r}$$

$$-\int\limits_{\mathcal{B}_\eta}\int\limits_0^1\int\limits_{\mathcal{B}_\gamma} \eta(\mathbf{r})\pi_\gamma(\mathbf{y})\mathrm{div}_{\mathbf{y}}[\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))]d\mathbf{y}dsd\mathbf{r}$$

$$+\int\limits_{\mathcal{B}_\eta}\int\limits_0^1\int\limits_{\mathcal{B}_{\partial\delta_\gamma(0)}} \eta(\mathbf{r})\pi_\gamma(\mathbf{y})\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))\frac{\mathbf{y}}{\delta_\gamma}d\mathbf{y}dsd\mathbf{r}$$

$$+\int\limits_{\mathcal{B}_\eta}\int\limits_0^1\int\limits_{\mathcal{B}_{\partial\delta_\gamma(0)}} \eta(\mathbf{r})\pi_\gamma(\mathbf{y})\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))\frac{\mathbf{y}}{\delta_\gamma}d\mathbf{y}dsd\mathbf{r}.$$

For $\mathbf{y}\in\partial\mathcal{B}_\gamma$, we have $\pi_\gamma(\mathbf{y}) = \pi_\gamma(\delta_\gamma) = 0$ thus the last two terms vanish, so I_1 becomes

$$I_1 = -\int\limits_{\mathcal{B}_\eta}\int\limits_0^1\int\limits_{\mathcal{B}_\gamma} \eta(\mathbf{r})s\Delta\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))\pi_\gamma(\mathbf{y})d\mathbf{y}dsd\mathbf{r}$$

$$-\int_{\mathcal{B}_\eta}\int_0^1\int_{\mathcal{B}_\gamma}\eta(\mathbf{r})(1-s)\Delta\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))\pi_\gamma(\mathbf{y})d\mathbf{y}dsd\mathbf{r},$$

which, after adding and subtracting $\Delta\mathbf{u}(\mathbf{x})$ we can write as

$$I_1 = -\int_{\mathcal{B}_\eta}\int_0^1\int_{\mathcal{B}_\gamma}\eta(\mathbf{r})s[\Delta\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))-\Delta\mathbf{u}(\mathbf{x})]\pi_\gamma(\mathbf{y})d\mathbf{y}dsd\mathbf{r}$$

$$-\int_{\mathcal{B}_\eta}\int_0^1\int_{\mathcal{B}_\gamma}\eta(\mathbf{r})(1-s)[\Delta\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))-\Delta\mathbf{u}(\mathbf{x})]\pi_\gamma(\mathbf{y})d\mathbf{y}dsd\mathbf{r}$$

$$-\Delta\mathbf{u}(\mathbf{x})\int_{\mathcal{B}_\eta}\int_0^1\int_{\mathcal{B}_\gamma}\eta(\mathbf{r})\pi_\gamma(\mathbf{y})d\mathbf{y}dsd\mathbf{r}.$$

We use the same approach for I_2; we first integrate by parts, using the fact that $\pi_\eta(\mathbf{r}) = \pi_\eta(\delta_\eta) = 0$ for $r \in \partial\mathcal{B}_\eta$, and then add and subtract $\Delta\mathbf{u}(\mathbf{x})$ to obtain

$$I_2 = -\int_{\mathcal{B}_\gamma}\int_0^1\int_{\mathcal{B}_\eta}\gamma(\mathbf{y})s[\Delta\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))-\Delta\mathbf{u}(\mathbf{x})]\pi_\eta(\mathbf{r})d\mathbf{r}dsd\mathbf{y}$$

$$+\int_{\mathcal{B}_\gamma}\int_0^1\int_{\mathcal{B}_\eta}\gamma(\mathbf{y})s[\Delta\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))-\Delta\mathbf{u}(\mathbf{x})]\pi_\eta(\mathbf{r})d\mathbf{r}dsd\mathbf{y}.$$

In order to make the coefficient of the Laplacian in the third integral of I_1 equal to 1, we take $\sigma(\delta_\gamma,\delta_\eta)$ as given by (3.9). With this choice of scaling we write

$$\mathcal{L}^s_{\gamma\eta}[\mathbf{u}](\mathbf{x}) - \Delta\mathbf{u}(\mathbf{x}) =: 2\sigma(\delta_\gamma,\delta_\eta)\left(J_1+J_2+J_3+J_4\right),$$

where

$$J_1 = -\int_{\mathcal{B}_\eta}\int_0^1\int_{\mathcal{B}_\gamma} \eta(\mathbf{r})\pi_\gamma(\mathbf{y})s[\Delta\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r})) - \Delta\mathbf{u}(\mathbf{x})]d\mathbf{y}dsd\mathbf{r},$$

$$J_2 = -\int_{\mathcal{B}_\eta}\int_0^1\int_{\mathcal{B}_\gamma} \eta(\mathbf{r})\pi_\gamma(\mathbf{y})(1-s)[\Delta\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y})) - \Delta\mathbf{u}(\mathbf{x})]d\mathbf{y}dsd\mathbf{r},$$

$$J_3 = -\int_{\mathcal{B}_\gamma}\int_0^1\int_{\mathcal{B}_\eta} \gamma(\mathbf{y})\pi_\eta(\mathbf{r})s[\Delta\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r})) - \Delta\mathbf{u}(\mathbf{x})]d\mathbf{r}dsd\mathbf{y},$$

and

$$J_4 = \int_{\mathcal{B}_\gamma}\int_0^1\int_{\mathcal{B}_\eta} \gamma(\mathbf{y})\pi_\eta(\mathbf{r})s[\Delta\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y})) - \Delta\mathbf{u}(\mathbf{x})]d\mathbf{r}dsd\mathbf{y}.$$

Again, J_1, J_2, J_3, and J_4 are vector valued. We now look to bound each integral; we begin with J_1. Integrating by parts with respect to s, and using antisymmetry of the integrands we obtain

$$J_1 = \int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma} \frac{(1-s^2)}{2}(\Delta\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r})) - \Delta\mathbf{u}(\mathbf{x}))\eta(\mathbf{r})\pi_\gamma(\mathbf{y})\Big|_{s=0}^{s=1} d\mathbf{y}d\mathbf{r}$$

$$-\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\int_0^1 \frac{(1-s^2)}{2}\Delta\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))(\mathbf{y}+\mathbf{r})\eta(\mathbf{r})\pi_\gamma(\mathbf{y})dsd\mathbf{y}d\mathbf{r}.$$

After evaluating at $s=0$ and $s=1$ in the first term gives,

$$J_1 = -\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\int_0^1 \frac{(1-s^2)}{2}\Delta\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))(\mathbf{y}+\mathbf{r})\eta(\mathbf{r})\pi_\gamma(\mathbf{y})dsd\mathbf{y}d\mathbf{r}.$$

Integrating by parts with respect to s again we obtain

$$J_1 = \int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma} \left(\frac{1}{3} - \frac{s}{2} + \frac{s^3}{6}\right) \Delta\nabla\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))(\mathbf{y}+\mathbf{r})\eta(\mathbf{r})\pi_\gamma(\mathbf{y})\Big|_{s=0}^{s=1} d\mathbf{y}d\mathbf{r}$$

$$-\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\int\limits_0^1 \left(\frac{1}{3} - \frac{s}{2} + \frac{s^3}{6}\right) \Delta\left[\nabla^2\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))(\mathbf{y}+\mathbf{r})\right](\mathbf{y}+\mathbf{r})\eta(\mathbf{r})\pi_\gamma(\mathbf{y})dsd\mathbf{y}d\mathbf{r},$$

where ∇^2 is the Hessian tensor. Evaluating the first integral at $s = 1$ yields a factor of zero, while evaluating at $s = 0$ produces an antisymmetric function which vanishes after integration. Hence, we have

$$J_1 = -\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\int\limits_0^1 \left(\frac{1}{3} - \frac{s}{2} + \frac{s^3}{6}\right) \Delta[\nabla^2\mathbf{u}(\mathbf{x}+s(\mathbf{y}+\mathbf{r}))(\mathbf{y}+\mathbf{r})](\mathbf{y}+\mathbf{r})\eta(\mathbf{r})\pi_\gamma(\mathbf{y})dsd\mathbf{y}d\mathbf{r}.$$

Taking M_4 as defined in (3.15) we estimate the magnitude of J_1 as follows

$$|J_1| \le M_4 \int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\int\limits_0^1 \left(\frac{1}{3} - \frac{s}{2} + \frac{s^3}{6}\right) \left(|\mathbf{y}|^2 + |\mathbf{r}|^2 + 2|\mathbf{y}\mathbf{r}|\right)\eta(\mathbf{r})|\pi_\gamma(\mathbf{y})|dsd\mathbf{y}d\mathbf{r}$$

$$\le \frac{M_4}{4}\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma} \left(|\mathbf{y}|^2 + |\mathbf{r}|^2\right)\eta(\mathbf{r})|\pi_\gamma(\mathbf{y})|d\mathbf{y}d\mathbf{r}.$$

Using the coarea formula we obtain

$$|J_1| \leq \frac{M_4}{4}\left(n\omega_{n-1}\int\limits_{\mathcal{B}_\eta}\eta(\mathbf{r})d\mathbf{r}\int_0^{\delta_\gamma}\lambda^{n+1}|\pi_\gamma(\lambda)|d\lambda\right.$$

$$\left.+n\omega_{n-1}\int_0^{\delta_\eta}\rho^{n+1}\eta(\rho)d\rho\int\limits_{\mathcal{B}_\gamma}|\pi_\gamma(\mathbf{y})|d\mathbf{y}\right)$$

$$\leq \frac{M_4}{4}\left(\delta_\gamma^2+\delta_\eta^2\right)\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\eta(\mathbf{r})|\pi_\gamma(\mathbf{y})|d\mathbf{y}d\mathbf{r},$$

where in the last equality we use the coarea formula and the fact that

$$\lambda^n \leq \lambda^{n-1}\delta_\gamma \text{ and } \rho^n \leq \rho^{n-1}\delta_\eta. \tag{3.17}$$

Multiplying by $2\sigma(\delta_\gamma, \delta_\eta)$ given by (3.9) gives the bound

$$|2\sigma(\delta_\gamma,\delta_\eta)J_1| \leq \frac{M_4}{4}\left(\delta_\gamma^2+\delta_\eta^2\right).$$

To find a bound on J_2, we first integrate by parts with respect to s to obtain

$$J_2 = -\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\frac{2s-s^2-1}{2}\left(\Delta\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))-\Delta\mathbf{u}(\mathbf{x})\right)\eta(\mathbf{r})\pi_\gamma(\mathbf{y})\Big|_{s=0}^{s=1}d\mathbf{y}d\mathbf{r}$$

$$+\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\int\limits_0^1\frac{2s-s^2-1}{2}\left(\Delta\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))(\mathbf{r}-\mathbf{y})\right)\eta(\mathbf{r})\pi_\gamma(\mathbf{y})d\mathbf{y}d\mathbf{r},$$

after which evaluation at $s=0$ and $s=1$ gives

$$J_2 = -\frac{1}{2}\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma} \left(\Delta\mathbf{u}(\mathbf{x}+\mathbf{y}) - \Delta\mathbf{u}(\mathbf{x})\right)\eta(\mathbf{r})\pi_\gamma(\mathbf{y})d\mathbf{y}d\mathbf{r}$$

$$+\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\int_0^1 \frac{2s-s^2-1}{2}\Big(\Delta\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))$$

$$-\Delta\nabla\mathbf{u}(\mathbf{x})\Big)(\mathbf{r}-\mathbf{y})\eta(\mathbf{r})\pi_\gamma(\mathbf{y})dsd\mathbf{y}d\mathbf{r}.$$

In the last line we have added the last term which is zero by the antisymmetry of the integrand. Using the fundamental theorem of calculus for the first integral and and integrating by parts with respect to s in the second integral we have

$$J_2 = -\frac{1}{2}\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\int_0^1 \left(\Delta\nabla\mathbf{u}(\mathbf{x}+s\mathbf{y})\mathbf{y}\right)\eta(\mathbf{r})\pi_\gamma(\mathbf{y})dsd\mathbf{y}d\mathbf{r}$$

$$+\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\left(\frac{s^2-s}{2}+\frac{1-s^3}{6}\right)\Big(\Delta\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))$$

$$-\Delta\nabla\mathbf{u}(\mathbf{x})\Big)(\mathbf{r}-\mathbf{y})\eta(\mathbf{r})\pi_\gamma(\mathbf{y})\Big|_{s=0}^{s=1}d\mathbf{y}d\mathbf{r}$$

$$-\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\int_0^1\left(\frac{s^2-s}{2}+\frac{1-s^3}{6}\right)\Delta[\nabla^2\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))(\mathbf{r}-\mathbf{y})](\mathbf{r}-\mathbf{y})$$

$$\cdot\,\eta(\mathbf{r})\pi_\gamma(\mathbf{y})dsd\mathbf{y}d\mathbf{r}.$$

Without changing the value of the integral we can insert again an antisymmetric integrand in the first integral. We also evaluate the second integral at $s=0$ and $s=1$, to produce

$$J_2 = -\frac{1}{2}\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\int\limits_0^1 \left(\Delta\nabla\mathbf{u}(\mathbf{x}+s\mathbf{y}) - \Delta\nabla\mathbf{u}(\mathbf{x})\right)\mathbf{y}\eta(\mathbf{r})\pi_\gamma(\mathbf{y})ds d\mathbf{y} d\mathbf{r}$$

$$-\frac{1}{6}\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma} \left(\Delta\nabla\mathbf{u}(\mathbf{x}+\mathbf{y}) - \Delta\nabla\mathbf{u}(\mathbf{x})\right)(\mathbf{r}-\mathbf{y})\eta(\mathbf{r})\pi_\gamma(\mathbf{y}) d\mathbf{y} d\mathbf{r}$$

$$-\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\int\limits_0^1 \left(\frac{s^2-s}{2}+\frac{1-s^3}{6}\right)\Delta[\nabla^2\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))(\mathbf{r}-\mathbf{y})](\mathbf{r}-\mathbf{y})$$

$$\cdot\,\eta(\mathbf{r})\pi_\gamma(\mathbf{y})ds d\mathbf{y} d\mathbf{r}.$$

Now, integrating by parts with respect to s in the first integral and applying the fundamental theorem of calculus in the second integral produces

$$J_2 = \frac{1}{2}\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}(1-s)\left(\Delta\nabla\mathbf{u}(\mathbf{x}+s\mathbf{y}) - \Delta\nabla\mathbf{u}(\mathbf{x})\right)\mathbf{y}\eta(\mathbf{r})\pi_\gamma(\mathbf{y})\Big|_{s=0}^{s=1} d\mathbf{y} d\mathbf{r}$$

$$-\frac{1}{2}\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\int\limits_0^1 (1-s)\Delta[\nabla^2\mathbf{u}(x+s\mathbf{y})\mathbf{y}]\mathbf{y}\eta(\mathbf{r})\pi_\gamma(\mathbf{y}) d\mathbf{y} d\mathbf{r}$$

$$-\frac{1}{6}\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\int\limits_0^1 \Delta[\nabla^2\mathbf{u}(\mathbf{x}+s\mathbf{y})\mathbf{y}](\mathbf{r}-\mathbf{y})\eta(\mathbf{r})\pi_\gamma(\mathbf{y}) d\mathbf{y} d\mathbf{r}$$

$$-\int\limits_{\mathcal{B}_\eta}\int\limits_{\mathcal{B}_\gamma}\int\limits_0^1 \left(\frac{s^2-s}{2}+\frac{1-s^3}{6}\right)\Delta[\nabla^2\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))(\mathbf{r}-\mathbf{y})](\mathbf{r}-\mathbf{y})$$

$$\cdot\,\eta(\mathbf{r})\pi_\gamma(\mathbf{y})ds d\mathbf{y} d\mathbf{r}.$$

Evaluating the first integral at $s = 0$ and $s = 1$ gives

$$J_2 = -\frac{1}{2}\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\int_0^1 (1-s)\Delta[\nabla^2\mathbf{u}(x+s\mathbf{y})\mathbf{y}]\mathbf{y}\eta(\mathbf{r})\pi_\gamma(\mathbf{y})d\mathbf{y}d\mathbf{r}$$

$$-\frac{1}{6}\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\int_0^1 \Delta[\nabla^2\mathbf{u}(\mathbf{x}+s\mathbf{y})\mathbf{y}](\mathbf{r}-\mathbf{y})\eta(\mathbf{r})\pi_\gamma(\mathbf{y})d\mathbf{y}d\mathbf{r}$$

$$-\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\int_0^1 \left(\frac{s^2-s}{2}+\frac{1-s^3}{6}\right)\Delta[\nabla^2\mathbf{u}(\mathbf{x}+\mathbf{y}+s(\mathbf{r}-\mathbf{y}))(\mathbf{r}-\mathbf{y})](\mathbf{r}-\mathbf{y})$$

$$\cdot\,\eta(\mathbf{r})\pi_\gamma(\mathbf{y})dsd\mathbf{y}d\mathbf{r}.$$

By bounding the fourth order derivatives as we did with J_1, and using $|\mathbf{y}| < \delta_\gamma$ and $|\mathbf{r}| < \delta_\eta$ we obtain

$$|J_2| \leq \frac{7M_4}{12}\left(\delta_\gamma^2+\delta_\eta^2\right)\int_{\mathcal{B}_\eta}\int_{\mathcal{B}_\gamma}\eta(\mathbf{r})|\pi_\gamma(\mathbf{y})|d\mathbf{y}d\mathbf{r},$$

hence,

$$|2\sigma(\delta_\gamma,\delta_\eta)J_2| \leq \frac{7M_4}{12}\left(\delta_\gamma^2+\delta_\eta^2\right).$$

Using approaches similar to the ones employed to bound J_1 and J_2, we find the following bounds for J_3, respectively J_4:

$$|J_3| \leq \frac{M_4}{4}\int_{\mathcal{B}_\gamma}\int_{\mathcal{B}_\eta}\left(|\mathbf{y}|^2+|\mathbf{r}|^2\right)|\pi_\eta(\mathbf{r})|\gamma(\mathbf{y})d\mathbf{r}d\mathbf{y},$$

and,

$$|J_4| \leq \frac{5M_4}{6} \int\limits_{\mathcal{B}_\eta} \int\limits_{\mathcal{B}_\gamma} \left(|\mathbf{y}|^2 + |\mathbf{r}|^2\right) |\pi_\eta(\mathbf{r})| \gamma(\mathbf{y}) d\mathbf{y} d\mathbf{r}.$$

Using Lemma 2.3 we have that

$$|J_3| \leq \frac{M_4}{4} \int\limits_{\mathcal{B}_\eta} \int\limits_{\mathcal{B}_\gamma} |\mathbf{r}|^2 \eta(\mathbf{r}) |\pi_\gamma(\mathbf{y})| d\mathbf{y} d\mathbf{r} + \frac{M_4}{4n} \int\limits_{\mathcal{B}_\eta} \int\limits_{\mathcal{B}_\gamma} |\mathbf{r}|^4 \eta(\mathbf{r}) \gamma(\mathbf{y}) d\mathbf{y} d\mathbf{r}.$$

Then using (3.17) we find

$$|J_3| \leq \frac{M_4}{4} \delta_\eta^2 \int\limits_{\mathcal{B}_\eta} \int\limits_{\mathcal{B}_\gamma} \eta(\mathbf{r}) |\pi_\gamma(\mathbf{y})| d\mathbf{y} d\mathbf{r} + \frac{M_4}{4n} \delta_\eta^4 \int\limits_{\mathcal{B}_\eta} \int\limits_{\mathcal{B}_\gamma} \eta(\mathbf{r}) \gamma(\mathbf{y}) d\mathbf{y} d\mathbf{r},$$

thus,

$$|2\sigma(\delta_\gamma, \delta_\eta) J_3| \leq \frac{M_4}{4} \delta_\eta^2 + \frac{M_4}{4n} \delta_\eta^4 \frac{\int\limits_{\mathcal{B}_\gamma} \gamma(\mathbf{y}) d\mathbf{y}}{\left| \int\limits_{\mathcal{B}_\gamma} \pi_\gamma(\mathbf{y}) d\mathbf{y} \right|}. \tag{3.18}$$

Using the assumption that $c_1|y|^{-\alpha} \leq \gamma(|y|) \leq c_2|y|^{-\alpha}$ where $0 \leq \alpha < n$, and $0 < c_1 \leq c_2$, we get

$$|2\sigma(\delta_\gamma, \delta_\eta) J_3| \leq \frac{M_4}{4} \delta_\eta^2 + \frac{M_4 c_2 n(n+2-\alpha)}{4c_1 n(n-\alpha)} \frac{\delta_\eta^4}{\delta_\gamma^2}.$$

Hence, under the assumption $\delta_\eta \leq \delta_\gamma$, we have

$$|2\sigma(\delta_\gamma, \delta_\eta) J_3| \leq \frac{M_4 c_1 n(n-\alpha) + M_4 c_2 n(n+2-\alpha)}{4c_1 n(n-\alpha)} \delta_\eta^2.$$

Similarly, we find the bound on J_4 to be

$$|2\sigma(\delta_\gamma, \delta_\eta)J_4| \leq \frac{5M_4c_1n(n-\alpha) + 5M_4c_2n(n+2-\alpha)}{6c_1n(n-\alpha)}\delta_\eta^2.$$

Putting all of these together we find

$$\begin{aligned}|\mathcal{L}^s_{\gamma\eta}(\mathbf{u}) - \Delta\mathbf{u}(x)| &\leq |2\sigma(\delta_\gamma, \delta_\eta)J_1| + |2\sigma(\delta_\gamma, \delta_\eta)J_2| + |2\sigma(\delta_\gamma, \delta_\eta)J_3| + |2\sigma(\delta_\gamma, \delta_\eta)J_4| \\ &\leq C\left(\delta_\gamma^2 + \delta_\eta^2\right) \leq C\delta_\gamma^2,\end{aligned}$$

where the value of the constant C changes from line to line, and it depends on M_4, n, α, c_1 and c_2. This estimate shows that our nonlocal state-based Laplacian with the scaling of (3.11) converges to the classical Laplacian at a rate of δ_γ^2. Note that the quadratic rate is independent of the dimension, but the constant C does depend on the dimension. □

Remark 3.4. In Theorem 3.3 we can relax the growth restrictions on γ by assuming instead that there exists a $C_1 > 0$ such that

$$\int\limits_{\mathcal{B}_\gamma} \gamma(\mathbf{y})d\mathbf{y} \leq \frac{C_1}{\delta_\gamma^2}\left|\int\limits_{\mathcal{B}_\gamma} \pi_\gamma(\mathbf{y})d\mathbf{y}\right|. \tag{3.19}$$

Since $\delta_\eta \leq \delta_\gamma$, (3.18) combined with (3.19) implies

$$|\sigma(\delta_\gamma, \delta_\eta)J_3| \leq \frac{M_4(n+C_1)}{8n}\delta_\eta^2.$$

Similarly,

$$|\sigma(\delta_\gamma, \delta_\eta)J_4| \leq \frac{5M_4(n+C_1)}{6n}\delta_\eta^2,$$

and the rate of convergence in the theorem holds.

Furthermore, we can replace the condition $\delta_\eta \leq \delta_\gamma$ by the assumption that there exists a $C_2 > 0$ such that

$$\int\limits_{\mathcal{B}_\gamma} \gamma(\mathbf{y}) d\mathbf{y} \leq \frac{C_2}{\delta_\eta^2} \left| \int\limits_{\mathcal{B}_\gamma} \pi_\gamma(\mathbf{y}) d\mathbf{y} \right|,$$

which becomes a condition that links the growth of γ with the growth of η. We then obtain from (3.18) that

$$|\sigma(\delta_\gamma, \delta_\eta) J_3| \leq \frac{M_4(n + C_2)}{8n} \delta_\eta^2,$$

and

$$|\sigma(\delta_\gamma, \delta_\eta) J_4| \leq \frac{5M_4(n + C_2)}{6n} \delta_\eta^2.$$

The resulting rate of convergence will be $\delta_\gamma^2 + \delta_\eta^2$.

Chapter 4

Convolutions and the Fourier Transform

In the previous chapter we focused on showing that the state-based Laplacian applied to $\mathbf{u}$ converges to the classical Laplacian applied $\mathbf{u}$, whenever $\mathbf{u}$ is sufficiently smooth. We will continue the course of making connections between the state-based Laplacian and the classical Laplacian. In addition, similarities between the bond-based and the state-based Laplacians are studied. From the geometrical description of the state-based peridynamics formulation given in Figure 4.1, one can imagine that if we shrunk δ_η to zero, we should return the bond-based formulation of peridynamics. In [45], Silling highlights this idea by presenting an example, in the state-based formulation, using the Dirac mass centered at zero to show that this produces the bond-based formulation. The focus of this chapter will be on emphasizing these connections.

We begin by showing that the state-based Laplacian (2.7) can be written as convolutions of the kernels of $\mathcal{L}^s_{\gamma\eta}$ with $\mathbf{u}$, and discuss the properties emphasized by the convolution structure. As the bond-based Laplacian (1.19) can also be written as a convolution of the kernel of $\mathcal{L}^b_\mu$ and $\mathbf{u}$ we again see how the state-based, bond-based and classical Laplacians are connected. We illustrate these connections by taking the kernels to be Dirac mass measures, or combinations of derivatives of the Dirac mass measure. We conclude by studying the solutions to the Cauchy problem using the Fourier transform, and show improved regularity of the solutions.

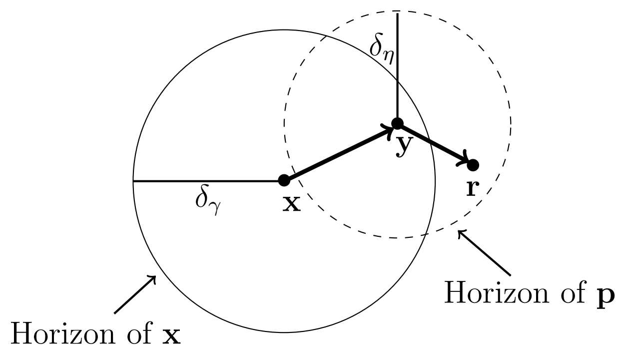

Figure 4.1: The indirect interaction that the point $\mathbf{x}$ has on $\mathbf{y}$ through their common neighbor $\mathbf{r}$. Relabeled here, from Figure 1.2, for convenience.

4.1 Convolution form of the state-based Laplacian

Assuming that γ and η are L^1 integrable, the state-based Laplacian defined in (2.7) can be expressed in terms of double and single convolutions as follows:

$$\frac{\mathcal{L}^s_{\gamma\eta}[\mathbf{u}]}{2\sigma(\gamma,\eta)} = (\gamma * \eta * \mathbf{u}) - (\eta * \mathbf{u}) \left\|\gamma\right\|_{L^1} + (\gamma * \mathbf{u}) \left\|\eta\right\|_{L^1} - \mathbf{u} \left\|\gamma\right\|_{L^1} \left\|\eta\right\|_{L^1}, \tag{4.1}$$

where we mean $L^1 = L^1(\mathbb{R}^n)$. The convolution above is performed component wise, so each component of a vector is convolved with the scalar kernels:

$$\mathbf{u} * \gamma = (u_1 * \gamma, u_2 * \gamma, \ldots). \tag{4.2}$$

The expression (4.1) is similar to the convolution form of the bond-based Laplacian from (1.19) as expressed by

$$\frac{\mathcal{L}_\mu[\mathbf{u}]}{\sigma(\mu)} = \mu * \mathbf{u} - \mathbf{u} \left\|\mu\right\|_{L^1}. \tag{4.3}$$

For a physical interpretation of the operator $\mathcal{L}_\mu$ when μ is a probability measure, in the context of nonlocal diffusion, see [1].

Also, note that although the double integral form of the operator in its (2.7) or (4.1) form implies similarity to the biharmonic operator

$$\mathcal{B}_\mu[\mathbf{u}] = \mathcal{L}_\mu^2[\mathbf{u}] = \mathbf{u} * \mu * \mu - 2\mathbf{u} * \mu \left\|\mathbf{u}\right\|_{L^1} + \mathbf{u} \left\|\mu\right\|_{L^1}^2 ,$$

introduced in [39], the convolution formulation of $\mathcal{L}^s_{\gamma\eta}$ clearly shows that no choice of kernels γ and η will yield the nonlocal biharmonic. Indeed, in order to eliminate the single convolution term one would have to choose a kernel that would also eliminate the double convolution term (single convolution is associated with bond-based Laplacian, while double convolution is associated with the state-based Laplacian). Finally, the doubly nonlocal state-based Laplacian was shown in Chapter 3 to converge to a second-order differential operator, while the nonlocal biharmonic provides and approximation to the classical biharmonic operator Δ^2.

4.1.1 The state-based Laplacian on the space of distributions

The convolution formulation (4.1) shows that the operator $\mathcal{L}^s_{\gamma\eta}$ can be conveniently defined for functions $\mathbf{u}$ of different smoothness levels depending on choices of γ and η. In particular, γ and η in C^∞ will allow choosing $\mathbf{u}$ less smooth and vice-versa. The support for each of the kernels γ and η could be taken to be unbounded, but for applications linked to peridynamics, the finite horizon is the relevant choice. In fact, both the state-based and bond-based Laplacians can be defined in the weak sense on the space of distributions. Recall the definition of a convolution between an L^1 function and a distribution: Let $f \in \mathcal{D}'(\mathbb{R}^n)$, and $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\tilde{\varphi}(x) = (-x)$, then for $g \in L^1(\mathbb{R}^n)$,

$$\langle f * g, \varphi \rangle = \langle f, g * \tilde{\varphi} \rangle . \tag{4.4}$$

Hence, if $\gamma, \eta \in L^1(\mathbb{R}^n)$, then $\mathbf{u}$ can be taken in $\mathcal{D}'(\mathbb{R}^n)$, the space of distributions. In addition, γ and η can be chosen to be Dirac masses, or derivatives of Dirac masses, as shown in the next section.

4.2 Connections between the state-based, bond-based and classical Laplacians

From the geometrical view of the state-based peridynamic formulation in Figure 4.1, we note that if we "shrink" δ_η to zero, we should get back the bond-based peridynamic formulation. This provides the visual and physical motivation for what we prove in this section. We will show that if η is taken to be the Dirac mass, in the sense of distributions, we will indeed get back the bond-based Laplacian with kernel γ, $\mathcal{L}^b_\gamma$. Similarly, one can imagine shrinking the horizon of the bond-based Laplacian to a point and recovering the classical Laplacian. In fact, it has been shown in [25] that taking μ to be a combination of derivatives of the Dirac mass measure in the bond-based Laplacian will give back the classical Laplacian. Taking these two together, by letting $\mu = \gamma$ in the bond-based Laplacian, we recover the classical Laplacian from the state-based Laplacian. We start by proving the following Lemma, which demonstrates how derivatives of the Dirac mass apply to functions in L^1.

Lemma 4.1. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$, and δ_0 the Dirac mass measure centered at the origin. For a function $\mathbf{v} \in L^1(\mathbb{R}^n)$, in the sense of distributions*

$$\left\langle \frac{\partial^{(k)}}{\partial x_j^{(k)}} \delta_0 * \mathbf{v}, \varphi \right\rangle = (-1)^k \left\langle \frac{\partial^{(k)}}{\partial x_j^{(k)}} \mathbf{v}, \varphi \right\rangle. \tag{4.5}$$

Proof. Let $v \in L^1(\mathbb{R})$, and $\varphi \in C_c^\infty(\mathbb{R})$. Taking $\tilde{\varphi}(x) = \varphi(-x)$, we have

$$\left\langle \frac{\partial^{(k)}}{\partial x_j^{(k)}} \delta_0 * \mathbf{v}, \varphi \right\rangle = \left\langle \frac{\partial^{(k)}}{\partial x_j^{(k)}} \delta_0, \underline{\mathrm{v}} * \tilde{\varphi} \right\rangle, \tag{4.6}$$

where $v * \tilde{\varphi} \in C_c^\infty(\mathbb{R})$. When δ_0 is the Dirac mass measure centered at zero we obtain

$$\begin{aligned} \left\langle \frac{\partial^{(k)}}{\partial x_j^{(k)}} \delta_0, \mathbf{v} * \tilde{\varphi} \right\rangle &= (-1)^k \left\langle \delta_0, \frac{\partial^{(k)}}{\partial x_j^{(k)}} \mathbf{v} * \tilde{\varphi} \right\rangle \\ &= (-1)^k \left(\frac{\partial^{(k)}}{\partial x_j^{(k)}} \mathbf{v} * \tilde{\varphi} \right) \Bigg|_{\mathbf{x}=0}. \end{aligned} \tag{4.7}$$

Then using the definition of the convolution and $\tilde{\varphi}$ we get

$$\begin{aligned} \left(\frac{\partial^{(k)}}{\partial x_j^{(k)}} \mathbf{v} * \tilde{\varphi} \right) \Bigg|_{\mathbf{x}=0} &= \int_{\mathbb{R}} \left(\frac{\partial^{(k)}}{\partial x_j^{(k)}} \mathbf{v} \right) (y) \tilde{\varphi}(x-y) dy \Bigg|_{\mathbf{x}=0} \\ &= \int_{\mathbb{R}} \left(\frac{\partial^{(k)}}{\partial x_j^{(k)}} \mathbf{v} \right) (y) \varphi(y-x) dy \Bigg|_{\mathbf{x}=0} \\ &= \int_{\mathbb{R}} \left(\frac{\partial^{(k)}}{\partial x_j^{(k)}} \mathbf{v} \right) (y) \varphi(y) dy. \end{aligned} \tag{4.8}$$

Finally, using (4.6), (4.7) and (4.8) we have

$$\left\langle \frac{\partial^{(k)}}{\partial x_j^{(k)}} \delta_0 * \mathbf{v}, \varphi \right\rangle = (-1)^k \left\langle \frac{\partial^{(k)}}{\partial x_j^{(k)}} \mathbf{v}, \varphi \right\rangle.$$

□

We now show that if we let η converge to the Dirac mass measure in state-based Laplacian formula, we recover the bond-based Laplacian with kernel γ.

Proposition 4.2 (State-based to bond-based)**.** *Let* $\mathbf{u} \in L^1(\mathbb{R}^n)$*. If* $||\eta_i||_{L^1(\mathbb{R}^n)} = 1$ *with* $\eta_i \to \delta_0$ *as* $i \to \infty$*, in the sense of distributions, where* δ_0 *is the Dirac mass* *measure centered at the origin, then* $\mathcal{L}^s_{\gamma\eta_i}[\mathbf{u}] \to \mathcal{L}_\gamma[\mathbf{u}]$, *as* $i \to \infty$, *in the sense of distributions.*

Proof. Let $\mathbf{u} \in L^1(\mathbb{R}^n)$, and $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $||\eta_i||_{L^1} = 1$, and $\langle \eta_i, \varphi\rangle \to \langle \delta_0, \varphi\rangle$ in the sense of distributions. We have

$$\langle \eta_i * \mathbf{u}, \varphi\rangle = \langle \eta_i, \mathbf{u} * \tilde{\varphi}\rangle ,$$

where $\tilde{\varphi}(\mathbf{x}) = \varphi(-\mathbf{x})$. Since, $\mathbf{u} * \tilde{\varphi} \in C_c^\infty(\mathbb{R}^n)$,

$$\langle \eta_i, \mathbf{u} * \tilde{\varphi}\rangle \to \langle \delta_0, \mathbf{u} * \tilde{\varphi}\rangle . \tag{4.9}$$

Then by Lemma 4.1 and (4.9)

$$\langle \eta_i * \mathbf{u}, \varphi\rangle \to \langle \mathbf{u}, \varphi\rangle , \qquad \text{as } i \to \infty.$$

Using the formulation in (4.1), we have

$$\left\langle \frac{\mathcal{L}^s_{\gamma\eta_i}[\mathbf{u}]}{2\sigma(\gamma,\eta_i)}, \varphi \right\rangle = \langle \mathbf{u} * \eta_i * \gamma, \varphi\rangle - \langle \mathbf{u} * \eta_i, \varphi\rangle + \langle ||\eta_i||_{L^1}\mathbf{u} * \gamma, \varphi\rangle - \langle \mathbf{u}||\gamma||_{L^1}, \varphi\rangle .$$

By the definition of η_i we get

$$2\sigma(\gamma,\eta_i) = -\frac{2}{2\int\limits_{\mathbb{R}^n} \eta_i(\mathbf{r})d\mathbf{r} \int\limits_{\mathbb{R}^n} \pi_\gamma(\mathbf{y})d\mathbf{y}} = -\frac{1}{\int\limits_{\mathbb{R}^n} \pi_\gamma(\mathbf{y})d\mathbf{y}}. \tag{4.10}$$

As $i \to \infty$ we obtain

$$-\int\limits_{\mathbb{R}^n} \pi_\gamma(\mathbf{y})d\mathbf{y} \left\langle \mathcal{L}^s_{\gamma\eta_i}[\mathbf{u}], \varphi \right\rangle \to \langle \mathbf{u} * \gamma, \varphi \rangle - \langle \mathbf{u}, \varphi \rangle + \langle \mathbf{u} * \gamma, \varphi \rangle - \langle \mathbf{u}, \varphi \rangle$$

$$= 2\big(\langle \mathbf{u} * \gamma, \varphi \rangle - \langle \mathbf{u}, \varphi \rangle \big)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Thus

$$\left\langle \mathcal{L}^s_{\gamma\eta_i}[\mathbf{u}], \varphi \right\rangle \to -\frac{2}{\int\limits_{\mathbb{R}^n} \pi_\gamma(\mathbf{y})d\mathbf{y}} \big(\langle \mathbf{u} * \gamma, \varphi \rangle - \langle \mathbf{u}, \varphi \rangle \big) \tag{4.11}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then we can rewrite the right hand side of (4.11) to give

$$-\frac{2}{\int\limits_{\mathbb{R}^n} \pi_\gamma(\mathbf{y})d\mathbf{y}} \big(\langle \mathbf{u} * \gamma, \varphi \rangle - \langle \mathbf{u}, \varphi \rangle \big) = -\frac{2}{\sigma(\gamma) \int\limits_{\mathbb{R}^n} \pi_\gamma(\mathbf{y})d\mathbf{y}} \langle \mathcal{L}_\gamma[\mathbf{u}], \varphi \rangle . \tag{4.12}$$

Using $\sigma(\gamma)$ as defined in (3.2), we obtain

$$\left\langle \mathcal{L}^s_{\gamma\eta_i}[\mathbf{u}], \varphi \right\rangle \to \langle \mathcal{L}_\gamma[\mathbf{u}], \varphi \rangle , \tag{4.13}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. □

It has been shown in [25] that if μ is taken to be a combination of derivatives of the Dirac mass, then then the bond-based Laplacian becomes the classical Laplacian. We restate and prove this result here for completeness and to study in full detail the connections between the state-based, bond-based, and classical Laplacians.

Proposition 4.3 (Bond-based to classical)**.** *[25] Let* $\mathbf{u} \in L^1(\mathbb{R}^n)$*, and* $|\mathbf{y}|^2\mu_i \in L^1(\mathbb{R}^n)$*, supp*$(\mu_i) \subseteq \mathcal{B}_{\delta(0)}$*. If* $||\mu_i||_{L^1(\mathbb{R}^n)} = 1$ *with* $\mu_i \to \Delta\delta_0 + \delta_0$ *as* $i \to \infty$ *in the* *sense of distributions, where* δ_0 *is the Dirac mass measure centered at the origin, then* $\mathcal{L}_{\mu_i}[\mathbf{u}] \to \Delta \mathbf{u}$*, as* $i \to \infty$*, in the sense of distributions.*

Proof. Let $\mathbf{u} \in L^1(\mathbb{R}^n)$, $||\mu_i||_{L^1} = 1$, $\varphi \in C_c^\infty(\mathbb{R}^n)$, and $\langle \mu_i, \varphi \rangle \to \langle \Delta\delta_0, \varphi \rangle + \langle \delta_0, \varphi \rangle$ in the sense of distributions. We have

$$\langle \mu_i * \mathbf{u}, \varphi \rangle = \langle \mu_i, \mathbf{u} * \tilde{\varphi} \rangle \,, \tag{4.14}$$

where $\tilde{\varphi}(\mathbf{x}) = \varphi(-\mathbf{x})$. Since, $\mathbf{u} * \tilde{\varphi} \in C_c^\infty(\mathbb{R}^n)$, we get

$$\langle \mu_i, \mathbf{u} * \tilde{\varphi} \rangle \to \langle \Delta\delta_0, \mathbf{u} * \tilde{\varphi} \rangle + \langle \delta_0, \mathbf{u} * \tilde{\varphi} \rangle \,. \tag{4.15}$$

Then by Lemma 4.1,

$$\langle \mu_i * \mathbf{u}, \varphi \rangle \to \langle \Delta \mathbf{u}, \varphi \rangle + \langle \mathbf{u}, \varphi \rangle \,, \qquad \text{as } i \to \infty. \tag{4.16}$$

Using the formulation in (4.3), we have

$$\left\langle \frac{\mathcal{L}_{\mu_i}[\mathbf{u}]}{\sigma(\mu_i)}, \varphi \right\rangle = \langle \mathbf{u} * \mu_i, \varphi \rangle - \langle \mathbf{u} ||\mu_i||_{L^1}, \varphi \rangle \,. \tag{4.17}$$

As $i \to \infty$ we obtain

$$\begin{aligned} \left\langle \frac{\mathcal{L}_{\mu_i}[\mathbf{u}]}{\sigma(\mu_i)}, \varphi \right\rangle &\to \langle \Delta \mathbf{u}, \varphi \rangle + \langle \mathbf{u}, \varphi \rangle - \langle \mathbf{u}, \varphi \rangle \\ &= \langle \Delta \mathbf{u}, \varphi \rangle \,, \end{aligned} \tag{4.18}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Lemma 2.3 with γ replaced by μ_i gives

$$\sigma(\mu_i) = \frac{-2}{\int\limits_{\mathbb{R}^n} \pi_{\mu_i}(\mathbf{y})d\mathbf{y}} = \frac{2n}{\int\limits_{\mathbb{R}^n} |\mathbf{y}|^2\mu_i(\mathbf{y})d\mathbf{y}}. \tag{4.19}$$

Now, we take $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi(\mathbf{y}) = |\mathbf{y}|^2$ for $\mathbf{y}$ in $\mathcal{B}_{\delta(0)}$. Then

$$\int\limits_{\mathbb{R}^n} |\mathbf{y}|^2\mu_i(\mathbf{y})d\mathbf{y} = \int\limits_{\mathbb{R}^n} \varphi(\mathbf{y})\mu_i(\mathbf{y})d\mathbf{y}.$$

Then, in the sense of distributions

$$\left\langle \mu_i, |\mathbf{y}|^2 \right\rangle_{\mathbb{R}^n} = \left\langle \mu_i, |\mathbf{y}|^2 \right\rangle_{\mathcal{B}_{\delta(0)}} \to \left\langle \Delta\delta_0, |\mathbf{y}|^2 \right\rangle + \left\langle \delta_0, |\mathbf{y}|^2 \right\rangle \tag{4.20}$$

as $i \to \infty$. Thus, we obtain

$$\begin{aligned} \left\langle \Delta\delta_0, |\mathbf{y}|^2 \right\rangle + \left\langle \delta_0, \mathbf{y}^2 \right\rangle &= \left\langle \delta_0, \Delta|\mathbf{y}|^2 \right\rangle \\ &= \left\langle \delta_0, \operatorname{div}(\nabla|\mathbf{y}|^2) \right\rangle \\ &= \langle \delta_0, \operatorname{div}(2\mathbf{y}) \rangle \\ &= \langle \delta_0, 2n \rangle \\ &= 2n. \end{aligned} \tag{4.21}$$

Hence, in the sense of distributions, from (4.19) and (4.21) we have that

$$\sigma(\mu_i) \to 1. \tag{4.22}$$

□

From Propositions 4.2 and 4.3 we immediately get the following corollary. In the state-based Laplacian, if η is taken to be the Dirac mass measure, and γ is taken to be the combination of two derivatives of the Dirac mass and the Dirac mass, we return the classical Laplacian.

Corollary 4.4 (State-based to classical)**.** *Let* $\mathbf{u} \in L^1(\mathbb{R}^n)$ *and supp*$(\mu_i) \subseteq \mathcal{B}_{\delta(0)}$. *If* $||\eta_i||_{L^1(\mathbb{R}^n)} = 1$, $||\gamma_i||_{L^1(\mathbb{R}^n)} = 1$ *with* $\eta_i \to \delta_0$, $\gamma_i \to \Delta\delta_0 + \delta_0$ *as* $i \to \infty$, *in the sense of distributions, and* $|\mathbf{y}|^2\mu_i \in L^1(\mathbb{R}^n)$, *where* δ_0 *is the Dirac mass measure centered at the origin, then* $\mathcal{L}^s_{\gamma_i\eta_i}[\mathbf{u}] \to \Delta\mathbf{u}$, *as* $i \to \infty$, *in the sense of distributions.*

We end this section with the observation that taking γ to be only the Dirac mass, results in the state-based Laplacian being zero. Thus, in some sense, how “fast” we shrink γ is important.

Proposition 4.5. *Let* $\mathbf{u} \in L^1(\mathbb{R}^n)$. *If* $||\gamma_i||_{L^1} = 1$ *with* $\gamma_i \to \delta_0$ *as* $i \to \infty$, *in the sense of distributions, where* δ_0 *is the Dirac mass measure centered at the origin, then* $\mathcal{L}^s_{\gamma_i\eta}[\mathbf{u}] \to 0$, *as* $i \to \infty$, *in the sense of distributions.*

Proof. Let $\mathbf{u} \in L^1(\mathbb{R}^n)$, and $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $||\gamma_i||_{L^1} = 1$, and $\langle \gamma_i, \varphi \rangle \to \langle \delta_0, \varphi \rangle$ in the sense of distributions. We have

$$\langle \gamma_i * \mathbf{u}, \varphi \rangle = \langle \gamma_i, \mathbf{u} * \tilde{\varphi} \rangle \,, \tag{4.23}$$

where $\tilde{\varphi}(\mathbf{x}) = \varphi(-\mathbf{x})$. Since, $\mathbf{u} * \tilde{\varphi} \in C_c^\infty(\mathbb{R}^n)$ we get

$$\langle \gamma_i, \mathbf{u} * \tilde{\varphi} \rangle \to \langle \delta_0, \mathbf{u} * \tilde{\varphi} \rangle \,. \tag{4.24}$$

Then by Lemma 4.1

$$\langle \gamma_i * \mathbf{u}, \varphi \rangle \to \langle \mathbf{u}, \varphi \rangle \qquad \text{as } i \to \infty. \tag{4.25}$$

The formulation in (4.1) gives

$$\left\langle \frac{\mathcal{L}^s_{\gamma_i \eta}[\mathbf{u}]}{2\sigma(\gamma_i,\eta)}, \varphi \right\rangle = \langle \mathbf{u} * \eta * \gamma_i, \varphi\rangle - \langle \mathbf{u} * \eta, \varphi\rangle + \langle ||\eta||_{L^1}\mathbf{u} * \gamma_i, \varphi\rangle - \langle \mathbf{u}||\eta||_{L^1}, \varphi\rangle .$$

As $n \to \infty$, we obtain

$$\begin{aligned} \left\langle \frac{\mathcal{L}^s_{\gamma_i \eta}[\mathbf{u}]}{2\sigma(\gamma_i,\eta)}, \varphi \right\rangle &\to \langle \mathbf{u} * \eta, \varphi\rangle - \langle \mathbf{u} * \eta, \varphi\rangle + \langle ||\eta||_{L^1}\mathbf{u}, \varphi\rangle - \langle \mathbf{u}||\eta||_{L^1}, \varphi\rangle \\ &= 0, \end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

□

4.2.1 Solutions of the state-based Laplacian

In the previous section we studied in detail the convolution form (4.1) of the state-based Laplacian (2.7). The ability to write the state-based Laplacian in a combination of convolutions hints at using the Fourier transform to study the solutions of the operator. We would be remiss not to take this opportunity, as properties of convolutions can provide a rich analysis in the study of regularity and integrability properties of functions. In the previous section we showed how closely the state-based and bond-based Laplacians are related, and one might wonder if and how their solutions are related. We indeed find that the state-based solutions can be written as a convolution of the bond-based solution and another function. We begin with the solutions to the bond-based Laplacian (1.19). We will use $\hat{v}$ to denote the Fourier transform of a function, v.

Proposition 4.6 (Solution of the Cauchy problem for the bond-based Laplacian)**.**

The solution $\mathbf{u}_\mu^b$ *of the problem*

$$\mathcal{L}_\mu^b[\mathbf{u}](x) = f(x), \quad \mathbf{x} \in \mathbb{R}^n, \tag{4.26}$$

is given by

$$\hat{\mathbf{u}}_\mu^b = \frac{\hat{f}}{\sigma(\mu)\big(\hat{\mu} - \|\mu\|_{L^1}\big)}. \tag{4.27}$$

Proof. If $\mathcal{L}_\mu^b[\mathbf{u}] = f$, then using the convolution structure given in (4.3), we have

$$\sigma(\mu)\big(\mu * \mathbf{u} - \mathbf{u}\,\|\mu\|_{L^1}\big) = f. \tag{4.28}$$

Applying the Fourier transform we obtain

$$\sigma(\mu)\big(\hat{\mu}\hat{\mathbf{u}} - \hat{\mathbf{u}}\,\|\mu\|_{L^1}\big) = \hat{f}. \tag{4.29}$$

Thus we find that

$$\hat{\mathbf{u}}_\mu^b = \frac{\hat{f}}{\sigma(\mu)\big(\hat{\mu} - \|\mu\|_{L^1}\big)}. \tag{4.30}$$

□

Proposition 4.7 (Solution of the Cauchy problem for the state-based Laplacian)**.** *The solution* $\mathbf{u}_{\gamma\eta}^s$ *of the problem*

$$\mathcal{L}_{\gamma\eta}^s[\mathbf{u}](x) = f(x), \quad \mathbf{x} \in \mathbb{R}^n, \tag{4.31}$$

is given by

$$\hat{\mathbf{u}}_{\gamma\eta}^s = \frac{\hat{f}}{2\sigma(\gamma,\eta)(\hat{\eta} + ||\eta||_{L^1})(\hat{\gamma} - ||\gamma||_{L^1})}. \tag{4.32}$$

Proof. If $\mathcal{L}^s_{\gamma\eta}[\mathbf{u}] = f$ then using the convolution structure given in (4.1), we have

$$2\sigma(\gamma,\eta)\big(\,(\gamma * \eta * \mathbf{u}) - (\eta * \mathbf{u})\,\|\gamma\|_{L^1} + (\gamma * \mathbf{u})\,\|\eta\|_{L^1} - \mathbf{u}\,\|\gamma\|_{L^1}\,\|\eta\|_{L^1}\,\big) = f. \quad (4.33)$$

Applying the Fourier transform we obtain

$$2\sigma(\gamma,\eta)\big(\hat{\mathbf{u}}\hat{\eta}\hat{\gamma} - \hat{\mathbf{u}}\hat{\eta}||\gamma||_{L^1} + \hat{\mathbf{u}}\hat{\gamma}||\eta||_{L^1} - \hat{\mathbf{u}}||\gamma||_{L^1}||\eta||_{L^1}\big) = \hat{f}. \quad (4.34)$$

Then we find that

$$\begin{aligned}\hat{\mathbf{u}} &= \frac{\hat{f}}{2\sigma(\gamma,\eta)\big(\hat{\eta}\hat{\gamma} - \hat{\eta}||\gamma||_{L^1} + \hat{\gamma}||\eta||_{L^1} - ||\gamma||_{L^1}||\eta||_{L^1}\big)} \\ &= \frac{\hat{f}}{2\sigma(\gamma,\eta)(\hat{\eta} + ||\eta||_{L^1})(\hat{\gamma} - ||\gamma||_{L^1})}. \end{aligned} \quad (4.35)$$

□

Proposition 4.8 (Structure of the solutions of the state-based Laplacian)**.** *The solution of the state-based Laplacian Cauchy problem stated in* (4.31)*, can be written as*

$$\hat{\mathbf{u}}^s_{\gamma\eta} = 2||\eta||_{L^1}\hat{\mathbf{u}^b_\gamma} \cdot \frac{1}{(\hat{\eta} + ||\eta||_{L^1})} \quad (4.36)$$

where $\mathbf{u}^b_\gamma$ *is the solution to the bond-based Cauchy problem* $\mathcal{L}^b_\gamma u = f$*. Thus, the state solution* $\mathbf{u}^s_{\gamma\eta}$ *can be written as*

$$\mathbf{u}^s_{\gamma\eta} = 2||\eta||_{L^1}\mathbf{u}^\gamma_b * N \quad (4.37)$$

where $\hat{N} = \frac{1}{(\hat{\eta}+||\eta||_{L^1})}$.

In Proposition 4.2, we showed that taking η to be δ_0 in the state-based Laplacian produced the bond-based Laplacian, and thus we would hope that doing so would produce solutions to the bond-based Laplacian. Indeed, if $\eta = \delta_0$ in the sense of distributions, then $\hat{N} = \frac{1}{2}$. Thus we have that

$$2\mathbf{u}_b^\gamma * N = \mathbf{u}_b^\gamma * \delta_0 = \mathbf{u}_b^\gamma,$$

in the sense of distributions.

Similarly, in Proposition 4.3, we showed that taking μ to be $\Delta\delta_0 + \delta_0$ in the bond-based Laplacian produced the classical Laplacian. Recall that the solution to the classical Cauchy problem $\Delta\mathbf{u} = f$, can be written as

$$\mathbf{u}^c = \mathcal{F}^{-1}\left(\frac{-\hat{f}}{|\xi|^2}\right).$$

If $\mu = \Delta\delta_0 + \delta_0$, in the sense of distributions, then

$$\hat{\mathbf{u}}^b = \frac{\hat{f}}{\sigma(\mu)\big(\hat{\mu} - \|\mu\|_{L^1}\big)} = \frac{\hat{f}}{-|\xi|^2 + 1 - 1} = \frac{-\hat{f}}{|\xi|^2}.$$

Hence $\mathbf{u}_\mu^b$ becomes $\mathbf{u}^c$.

Putting these last two statements together, in the state-based Laplacian, if we take $\eta = \delta_0$ and $\gamma = \Delta\delta_0 + \delta_0$ in the sense of distributions, we have that

$$\mathbf{u}_{\gamma\eta}^s = \mathbf{u}^c.$$

If we have convergence to the Dirac masses instead of equality, then we get comparable convergence results.

Our final two propositions involve using the convolution form of the solutions to the state-based Cauchy problem to improve the integrability and regularity of the solutions.

Proposition 4.9 (Improved integrability of solutions to the state-based Cauchy problem)**.** *Taking N to be such that $\hat{N} = \frac{1}{(\hat{\eta}+||\eta||_{L^1})}$, the integrability of the solution to the state-based Laplacian is given by*

$$||\mathbf{u}^s_{\gamma\eta}||_{L^r} \leq c||\mathbf{u}^b_\gamma||_{L^p}||N||_{L^q} \quad \text{for } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,$$

where $c = 2||\eta||_{L^1}$.

In addition, if $\mathbf{u}^b_\gamma \in C^k(\mathbb{R}^n)$ and $N \in C^m(\mathbb{R}^n)$ then $\mathbf{u}^s_{\gamma\eta} \in C^{k+m}(\mathbb{R}^n)$, for $k, m \leq \infty$.

Proof. Apply Young's inequality for convolutions:

$$||f * g||_{L^r} \leq ||f||_{L^p}||g||_{L^q} \quad \text{for } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,$$

to (4.37). □

We give the following corollary as examples of this proposition.

Corollary 4.10. *The following are all examples of Proposition 4.9*

1. *If $\mathbf{u}^b_\gamma, N \in L^1(\mathbb{R}^n)$, then $\mathbf{u}^s_{\gamma\eta} \in L^1(\mathbb{R}^n)$.*

2. *If $\mathbf{u}^b_\gamma, N \in L^2(\mathbb{R}^n)$, then $\mathbf{u}^s_{\gamma\eta} \in L^\infty(\mathbb{R}^n)$.*

3. *If $\mathbf{u}^b_\gamma \in L^1(\mathbb{R}^n)$, $N \in L^\infty(\mathbb{R}^n)$, or $\mathbf{u}^b_\gamma \in L^\infty(\mathbb{R}^n)$, $N \in L^1(\mathbb{R}^n)$, then $\mathbf{u}^s_{\gamma\eta} \in L^\infty(\mathbb{R}^n)$.*

4. *If $\mathbf{u}^b_\gamma \in L^1(\mathbb{R}^n)$, $N \in L^2(\mathbb{R}^n)$, or $\mathbf{u}^b_\gamma \in L^2(\mathbb{R}^n)$, $N \in L^1(\mathbb{R}^n)$, then $\mathbf{u}^s_{\gamma\eta} \in L^2(\mathbb{R}^n)$.*

5. *If $\mathbf{u}^b_\gamma \in L^1(\mathbb{R}^n)$, $N \in L^q(\mathbb{R}^n)$ or $\mathbf{u}^b_\gamma \in L^q(\mathbb{R}^n)$, $N \in L^1(\mathbb{R}^n)$, for $q \geq 1$, then $\mathbf{u}^s_{\gamma\eta} \in L^q(\mathbb{R}^n)$.*

6. *If* $\mathbf{u}_\gamma^b \in L^2(\mathbb{R}^n)$*, and* $N \in L^q(\mathbb{R}^n)$ *for* $2/3 \leq q \leq 2$*, then* $\mathbf{u}_{\gamma\eta}^s \in L^p(\mathbb{R}^n)$*, where* $p = \frac{2q}{2-q}$.

7. *If* $N \in L^2(\mathbb{R}^n)$*, and* $\mathbf{u}_\gamma^b \in L^q(\mathbb{R}^n)$ *for* $2/3 \leq q \leq 2$*, then* $\mathbf{u}_{\gamma\eta}^s \in L^p(\mathbb{R}^n)$*, where* $p = \frac{2q}{2-q}$.

The following proposition gives an increase in regularity of solutions to the state-based problem, by increasing the differentiability of N.

Proposition 4.11 (Improved regularity of solutions to the state-based Cauchy problem)**.** *Take* N *to be such that* $\hat{N} = \frac{1}{(\hat{\eta}+||\eta||_{L^1})}$*. If* $N \in W^{k,q}(\mathbb{R}^n)$ *and* $\mathbf{u}_\gamma^b \in L^p(\mathbb{R}^n)$*, then*

$$\mathbf{u}_{\gamma\eta}^s \in W^{k,r}(\mathbb{R}^n) \quad \textit{where } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Proof. If $N \in W^{k,q}(\mathbb{R}^n)$ and $\mathbf{u}_\gamma^b \in L^p(\mathbb{R}^n)$, then $\partial^j N \in L^q(\mathbb{R}^n)$ for all $j \leq k$. Then

$$\partial^j \mathbf{u}_{\gamma\eta}^s = 2\mathbf{u}_\gamma^b * \partial^j N,$$

for all $j \leq k$. Hence, for $j \leq k$, by Young's inequality we obtain

$$||\partial^j \mathbf{u}_{\gamma\eta}^s||_{L^r} \leq 2||\eta||_{L^1}||\mathbf{u}_\gamma^b||_{L^p}||\partial^j N||_{L^q} \leq \infty, \quad \text{for } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Thus, $\mathbf{u}_{\gamma\eta}^s \in W^{k,r}(\mathbb{R}^n)$. □

Examples similar to Corollary 4.10 can be extrapolated for Proposition 4.11.

Chapter 5

Final Remarks and Future Directions

To summarize, the main ideas of this thesis revolve around the introduction of a new nonlocal Laplace-type operator, which is intimately connected to the state-based theory of peridynamics. The newly introduced state-based Laplacian offers an approximation of the classical Laplace operator for functions that are sufficiently smooth, however, it can be applied even to discontinuous functions or distributions.

5.1 Physical Aspects

As a first observation, the operator provides a lot more flexibility in modeling diffusion-type phenomena. Indeed, this flexibility is achieved through two (possibly different) kernels, each with its own rate of growth, and its own horizon. Since $\Delta = \mathrm{div}(\nabla)$, we do not have much control on either one of the differential operators, div or ∇, possibly just incorporate some variable coefficients. The behavior, however, is prescribed in a restrictive way, not allowing adjustment for the rates of the change. In contrast γ and η have the freedom to be taken from a large variety of spaces, including distributions, or even non-integrable kernels (which would give rise to fractional derivatives). In addition, our operator can be applied to vector-valued functions, as it acts on each component.

The state-based Laplacian is intimately connected to the theory of elasticity as given by state-based peridynamics. However, note that the operator was not designed to approximate the Navier operator (except for in the one dimensional case) from the system of classical elasticity [36], as we do not recover the term $\nabla \mathrm{div}\, \mathbf{u}$ in the limit as the horizon goes to zero; we only get $\Delta \mathbf{u}$ in the limit. This is a result of taking scalar valued kernels rather than tensor valued. Thus, one direction of future research is to develop a double-kernel Navier-type operator.

Of interest to the dynamic fracture community would be an operator that incorporates fracture in a time-dependent state-based model. In peridynamics, fracture appears as a result of the modeling and it is not introduced through ad hoc systems of equations. In fact, every bond gets broken when the energy sustained surpasses a critical level. This bond-breaking condition could be given in implicit or explicit ways. The mathematical analysis prefers the second option, thus it would be desirable to incorporate fracture in the state-based model in an explicit way as done in [16] for a bond-based model.

5.2 Mathematical Aspects

The error bounds between the nonlocal and the local Laplacians

$$(\mathcal{L}^s_{\gamma\eta} - \Delta)(\mathbf{u})$$

are the main focus of this work. The quadratic dependence on the horizon, for the error, agrees with results obtained from the bond-based framework. Moreover, our proof also shows that the quadratic rate of this convergence with respect to the horizons is optimal for functions $\mathbf{u} \in C^4$. The C^4 level of regularity required for the functions that are considered for these error bound is the same as in the bond-based formulation, which also needs four derivatives on the input function. Also, our bounds are obtained in L^∞ for functions $\mathbf{u}$ that live in a much smaller space (C^4).

An interesting aspect worthy of future exploration is that in the proof of Theorem 3.3 it is not clear if $\delta_\eta \leq \delta_\gamma$ is a necessary condition for convergence. Our conjecture is that for $\delta_\eta > \delta_\gamma$ the state Laplacian will not converge to the classical Laplacian, so we are working on producing a counterexample to this convergence result. A numerical convergence study for different choices of the two kernels, would be helpful in highlighting this relationship. This work will provide insight to the applied community regarding the optimal use of Laplacians (local vs. nonlocal) depending on the physical model.

Recall that in Section 1.1 we showed that the bond-based Laplacian (1.19) can be written as the nonlocal divergence of the nonlocal gradient (see (1.22)). This structure is nice as it aligns with the classical definition of the Laplacian (see (1.4)). We have yet to find state-based versions of the divergence and the gradient that align mathematically to give us state-based Laplacian, or to be physically relevant. As this structure arises in the bond-based Laplacian, it would be ideal if we could extend that idea to the state-based Laplacian. A deeper study of the structure of this operator would be valuable in gaining a better understanding of the operator, and further illustrate its connections to the bond-based can classical Laplacians.

In Chapter 4 we studied the Cauchy problem $\mathcal{L}^s_{\gamma\eta}[\mathbf{u}] = f$ on unbounded domains, but we have yet to produce promising results on the bounded domain. This, of course, is of high interest as we would like to prove well-posedness of state-based models.

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