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## The algebraic structure of non-commutative analytic Toeplitz algebras

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In [6, 17, 18, 20], a good case is made that the appropriate analogue for the analytic Toeplitz algebra in  $n$  non-commuting variables is the WOT-closed algebra generated by the left regular representation of the free semigroup on  $n$  generators. The papers cited obtain a compelling analogue of Beurling's theorem and inner-outer factorization. In this paper, we add further evidence. The main result is a short exact sequence determined by a canonical homomorphism of the automorphism group onto this algebra onto the group of conformal automorphisms of the unit ball of  $\mathbb{C}^n$ . The kernel is the subgroup of *quasi-inner* automorphisms, which are trivial modulo the WOT-closed commutator ideal. Additional evidence of analytic properties comes from the structure of  $k$ -dimensional (completely contractive) representations, which have a structure very similar to the fibration of the maximal ideal space of  $H^\infty$  over the unit disk. An important tool in our analysis is a detailed structure theory for WOT-closed right ideals. Curiously, left ideals remain more obscure.

The non-commutative analytic Toeplitz algebra  $\mathfrak{L}_n$  is determined by the left regular representation of the free semigroup  $\mathcal{F}_n$  on  $n$  generators  $z_1, \dots, z_n$  which acts on  $\ell_2(\mathcal{F}_n)$  by  $\lambda(w)\xi_v = \xi_{wv}$  for  $v, w$  in  $\mathcal{F}_n$ . In particular, the algebra  $\mathfrak{L}_n$  is the unital, WOT-closed algebra generated by the isometries  $L_i = \lambda(z_i)$  for  $1 \leq i \leq n$ . This algebra and its norm-closed version (the non-commutative disk algebra) were introduced by Popescu [19] in an abstract sense in connection with a non-commutative von Neumann inequality and further studied in several papers [17, 19, 21, 20, 22]. For  $n = 1$ , we obtain the algebra generated by the unilateral shift, the analytic Toeplitz algebra. The corresponding algebra for the right regular representation is denoted  $\mathfrak{R}_n$ . This algebra is unitarily equivalent to

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$\mathfrak{L}_n$  and is also equal to the commutant of  $\mathfrak{L}_n$ . We will review these results and others from [6] which we will need in Sect. 1.

Section 2 contains the classification of the WOT-closed right and two-sided ideals of  $\mathfrak{L}_n$ . These ideals are determined by their range, which is always a subspace in  $\text{Lat } \mathfrak{R}_n$ ; and this pairing is a complete lattice isomorphism. The ideal is two-sided when the range is also in  $\text{Lat } \mathfrak{L}_n$ . This is the key tool needed to establish classify the weak- $*$  continuous multiplicative linear functionals on  $\mathfrak{L}_n$ . We obtain some factorization results for right ideals that allow us to show that a WOT-closed right ideal is finitely generated algebraically precisely when the wandering subspace of the range space is finite dimensional; and otherwise, they require a countably infinite set of generators even as a WOT-closed right ideal.

In Sect. 3, we examine the representation space of  $\mathfrak{L}_n$ . The multiplicative linear functionals have a structure that parallels the maximal ideal space of  $H^\infty$ . This provides a natural homomorphism of  $\mathfrak{L}_n$  into the space  $H^\infty(\mathbb{B}_n)$  of bounded analytic functions on the ball. Strikingly, the dilation theory for non-commuting  $n$ -tuples allows us to obtain an analogous structure for  $k$ -dimensional representations for every  $k < \infty$ . In particular, the open ball  $\mathbb{B}_{n,k}$  of strict contractions in  $\mathcal{M}_{k,nk}$  sits homeomorphically in a canonical way in this space.

The last section contains the main results of the paper. Automorphisms of  $\mathfrak{L}_n$  are shown to be automatically norm and WOT continuous. We show that there is a natural homomorphism from  $\text{Aut}(\mathfrak{L}_n)$  onto  $\text{Aut}(\mathbb{B}_n)$ , the group of conformal automorphisms of the ball of  $\mathbb{C}^n$ , determined by their action on the WOT-continuous linear functionals  $\varphi_\lambda$  for  $\lambda \in \mathbb{B}_n$ . The kernel of this map is the ideal of automorphisms which are trivial modulo the WOT-closed commutator ideal. In order to show that this homomorphism is surjective, we determine all automorphisms of  $\mathfrak{L}_n$  of the form  $\text{Ad}W$  for unitary  $W$ . Using certain automorphisms of the Cuntz–Toeplitz algebra found by Voiculescu [28], we are able to obtain an isomorphism of this subgroup  $\text{Aut}_u(\mathfrak{L}_n)$  with  $\text{Aut}(\mathbb{B}_n)$ . Thus the automorphism group of  $\mathfrak{L}_n$  is a semidirect product.

## 1. Background

For convenience of notation, we will write  $L = [L_1 \ \dots \ L_n]$  both for the  $n$ -tuple of isometries and the isometric operator from  $\mathcal{H}_n^{(n)}$  into  $\mathcal{H}_n$ . By  $L_v$  or  $v(L)$  we will denote the corresponding word  $\lambda(v)$  in the  $n$ -tuple. We allow  $n = \infty$ . In this case,  $\mathbb{C}^n$  is replaced by a separable Hilbert space  $\mathcal{H}$ , and the unit ball  $\mathbb{B}_n$  becomes the unit ball of  $\mathcal{H}$  endowed with the weak topology. This occasionally causes additional difficulties which will be pointed out as necessary.

The full Fock space of a Hilbert space  $\mathcal{H}$  is the Hilbert space

$$F(\mathcal{H}) = \sum_{k \geq 0} \oplus \mathcal{H}^{\otimes k}$$

where  $\mathcal{H}^{\otimes 0} = \mathbb{C}$  and  $\mathcal{H}^{\otimes k}$  is the tensor product of  $k$  copies of  $\mathcal{H}$ . When  $\mathcal{H} = \mathbb{C}^n$  with orthonormal basis  $\zeta_i$  for  $1 \leq i \leq n$ , the Fock space has an orthonormal

basis  $\zeta_w = \zeta_{i_1} \otimes \dots \otimes \zeta_{i_k}$  for all choices of  $w = (i_1, \dots, i_k)$  in  $\{1, \dots, n\}^k$  and  $k \geq 0$  (with the convention that  $\zeta_\emptyset$  spans  $\mathcal{H}^{\otimes 0}$ ). For each vector  $\zeta$  in  $\mathcal{H}$ , there is a left creation operator  $\ell(\zeta)\xi = \zeta \otimes \xi$ . Clearly, there is a natural isomorphism of Fock space onto  $\mathcal{H}_n$ , where  $n = \dim \mathcal{H}$ , given by sending  $\zeta_w$  to  $\xi_w$ . This unitary equivalence sends  $\ell(\zeta_i)$  to  $L_i$ .

The following heuristic is useful when working with operators in  $\mathfrak{L}_n$ . If  $A = \sum_w a_w L_w$  is a finite linear combination of the set  $\{L_w : w \in \mathcal{F}_n\}$ , then  $A\xi_1 = \sum_w a_w \xi_w$ ; conversely, given a finite linear combination of basis vectors  $\zeta = \sum_w a_w \xi_w$ , the operator  $A = \sum_w a_w L_w$  belongs to  $\mathfrak{L}_n$  and satisfies  $A\xi_1 = \zeta$ . Sometimes this operator will be denoted by  $L_\zeta$ . This correspondence of course cannot be extended to infinite combinations. However, notice that for an arbitrary element  $A$  of  $\mathfrak{L}_n$ ,  $A$  is completely determined by its action on  $\xi_1$ : indeed,  $A\xi_v = AR_v\xi_1 = R_v A\xi_1$ . So if  $A\xi_1 = \sum_w a_w \xi_w$ , we have

$$A\xi_v = \sum_w a_w \xi_{wv} = \sum_w a_w (L_w \xi_v).$$

It is useful to view the formal sum  $\sum_w a_w L_w$  as the Fourier expansion of  $A$ . In particular [6], the Cesaro sums

$$\Sigma_n(A) = \sum_{|w| < n} \left(1 - \frac{|w|}{n}\right) a_w L_w$$

converge in the strong-\* topology to  $A$ .

The algebra  $\mathfrak{L}_n$  contains no non-scalar normal elements. Moreover every non-scalar element of  $\mathfrak{L}_n$  is injective and has connected spectrum containing more than one point. So  $\mathfrak{L}_n$  contains no non-zero compact operators, quasinilpotent elements or non-scalar idempotents. In particular,  $\mathfrak{L}_n$  is semi-simple. (See Sect. 1 of [6].)

If  $\mathcal{M}$  is an invariant subspace for  $\mathfrak{L}_n$ , the wandering subspace is  $\mathcal{W} = \mathcal{M} \ominus \sum_{i=1}^n L_i \mathcal{M}$ . By the analogue of the Wold decomposition [16], it follows that  $\mathcal{M} = \sum_{w \in \mathcal{F}_n} \oplus L_w \mathcal{W}$ . The invariant subspaces of the analytic Toeplitz algebra are determined by Beurling's theorem [2] as the subspaces  $\omega H^2$  where  $\omega$  is an inner function in  $H^\infty$ . These subspaces are always cyclic with wandering subspace  $\omega H^2 \ominus z\omega H^2 = \mathbb{C}\omega$ . The subspace  $\omega H^2$  is the range of  $T_\omega$ , which is an isometry in  $H^\infty = \mathfrak{L}_1 = \mathfrak{R}_1$ . The analogue of Beurling's theorem is:

**Theorem 1.1 ([17, 6]).** *Every invariant subspace of  $\mathfrak{L}_n$  is generated by a wandering subspace. Thus it is the direct sum of cyclic subspaces. The cyclic invariant subspaces of  $\mathfrak{L}_n$  are precisely the ranges of isometries in  $\mathfrak{R}_n$ ; and the choice of isometry is unique up to a scalar.*

If  $\mathcal{M}$  is a cyclic invariant subspace for  $\mathfrak{L}_n$ , then its wandering subspace is 1-dimensional. If  $\zeta$  is a wandering vector for  $\mathcal{M}$ , then we denote the corresponding isometry in  $\mathfrak{R}_n$  by  $R_\zeta$ . Explicitly, we have the formula,  $R_\zeta \xi_w = L_w \zeta$ . Conversely, any isometry in  $\mathfrak{R}_n$  is an  $R_\zeta$  for some  $\mathfrak{L}_n$ -wandering vector  $\zeta$ . Similarly, we see that any isometry in  $\mathfrak{L}_n$  has the form  $L_\zeta$  for some  $\mathfrak{R}_n$ -wandering vector  $\zeta$ .

By analogy, the isometries of  $\mathfrak{L}_n$  are called *inner*; and the elements with dense range are called *outer*. An element  $A$  in  $\mathfrak{L}_n$  is inner if and only if  $\|A\| = \|A\xi_1\| = 1$ . As a corollary, one obtains the following analogue of inner–outer factorization:

**Corollary 1.2.** *Every  $A$  in  $\mathfrak{L}_n$  factors as  $A = L_\zeta B$  where  $L_\zeta$  is an isometry in  $\mathfrak{L}_n$  and  $B$  belongs to  $\mathfrak{L}_n$  and has dense range. This factorization is unique up to a scalar. The operator  $B$  is invertible if and only if  $A$  has closed range.*

We also need to understand the structure of the eigenvectors for the adjoint analogous to the point evaluations in the unit disk associated to eigenvalues of the backward shift.

**Theorem 1.3 (cf. [1, Example 8] and [6]).** *The eigenvectors for  $\mathfrak{L}_n^*$  are the vectors*

$$\nu_\lambda = (1 - \|\lambda\|^2)^{1/2} \sum_{w \in \mathcal{F}_n} \overline{w(\lambda)} \xi_w = (1 - \|\lambda\|^2)^{1/2} (I - \sum_{i=1}^n \overline{\lambda_i} L_i)^{-1} \xi_1$$

for  $\lambda$  in the unit ball  $\mathbb{B}_n$ . They satisfy

$$L_i^* \nu_\lambda = \overline{\lambda_i} \nu_\lambda$$

and  $(p(L)\nu_\lambda, \nu_\lambda) = p(\lambda)$  for every polynomial  $p = \sum_w a_w w$  in the semigroup algebra  $\mathbb{C}\mathcal{F}_n$ . This extends to the map  $\varphi_\lambda(A) = (A\nu_\lambda, \nu_\lambda)$ , which is a WOT-continuous multiplicative linear functional on  $\mathfrak{L}_n$ . The vector  $\nu_\lambda$  is cyclic for  $\mathfrak{L}_n$ . The subspace  $\mathcal{M}_\lambda = \{\nu_\lambda\}^\perp$  is  $\mathfrak{L}_n$  invariant, and its wandering subspace  $\mathcal{W}_\lambda$  is  $n$ -dimensional, spanned by

$$\zeta_{\lambda,i} = \lambda_i \xi_1 - (1 - \|\lambda\|^2)^{1/2} L_i \nu_\lambda \quad \text{for } 1 \leq i \leq n.$$

These results are used in [6] to show that  $\mathfrak{L}_n$  is hyper-reflexive. Moreover, for every weak-\* continuous linear functional  $f$  on  $\mathfrak{L}_n$  with  $\|f\| < 1$ , there are vectors  $\xi$  and  $\zeta$  such that  $f(A) = (A\xi, \zeta)$  for all  $A$  in  $\mathfrak{L}_n$  and  $\|\xi\| \|\zeta\| < 1$ . This yields the immediate consequence which will be important on several occasions.

**Corollary 1.4 ([6]).** *The weak-\* and WOT topologies on  $\mathfrak{L}_n$  coincide.*

## 2. WOT-closed ideals of $\mathfrak{L}_n$

In this section, we identify the WOT-closed right and two-sided ideals of  $\mathfrak{L}_n$ . Let  $\text{Id}_r(\mathfrak{L}_n)$ ,  $\text{Id}_\ell(\mathfrak{L}_n)$  and  $\text{Id}(\mathfrak{L}_n)$  denote the sets of all WOT-closed right, left and two-sided ideals respectively. The important observation is that these ideals can be identified by their ranges. If  $\mathfrak{J}$  belongs to  $\text{Id}_r(\mathfrak{L}_n)$ , then the subspace  $\overline{\mathfrak{J}\xi_1}$  belongs to  $\text{Lat } \mathfrak{R}_n$ . To see this, note that

$$\mathfrak{R}_n \overline{\mathfrak{J}\xi_1} = \overline{\mathfrak{J}\mathfrak{R}_n \xi_1} = \overline{\mathfrak{J}\mathcal{H}_n} = \overline{\mathfrak{J}\mathfrak{L}_n \xi_1} = \overline{\mathfrak{J}\xi_1}.$$

Thus  $\overline{\mathfrak{J}\xi_1} = \overline{\mathfrak{J}\mathcal{H}_n}$  is the range of  $\mathfrak{J}$  and is  $\mathfrak{R}_n$  invariant.

When  $\mathfrak{J}$  belongs to  $\text{Id}_\ell(\mathfrak{L}_n)$ , we have  $\mathfrak{L}_n \mathfrak{J} \xi_1 = \mathfrak{J} \xi_1$ ; so  $\overline{\mathfrak{J} \xi_1}$  is  $\mathfrak{L}_n$  invariant. Hence when  $\mathfrak{J}$  is a two-sided ideal,  $\overline{\mathfrak{J} \xi_1}$  belongs to  $\text{Lat}(\mathfrak{L}_n) \cap \text{Lat}(\mathfrak{R}_n)$ .

Conversely, if  $\mathcal{M}$  belongs to  $\text{Lat}(\mathfrak{R}_n)$ , we shall see during the proof of Theorem 2.1 that the set  $\{A \in \mathfrak{L}_n : A \xi_1 \in \mathcal{M}\}$  belongs to  $\text{Id}_r(\mathfrak{L}_n)$ . It will follow that when  $\mathfrak{J}$  is a right ideal, the subspace  $\overline{\mathfrak{J} \xi_1}$  determines  $\mathfrak{J}$  and moreover  $\mathfrak{J}$  is two-sided precisely when  $\overline{\mathfrak{J} \xi_1}$  is also  $\mathfrak{L}_n$  invariant.

We do not make the same claims for left ideals. One should note that when  $\mathfrak{J}$  is a left ideal,  $\overline{\mathfrak{J} \xi_1}$  is *not* equal to  $\overline{\mathfrak{J} \mathcal{H}_n}$ . The full range of the ideal is not a complete invariant. There are technical difficulties for left ideals that we were not able to resolve; but analogous results are plausible.

We remark that  $\text{Id}_r(\mathfrak{L}_n)$  and  $\text{Id}(\mathfrak{L}_n)$  form complete lattices with the operations of intersection and WOT-closed sum.

**Theorem 2.1.** *Let  $\mu : \text{Id}_r(\mathfrak{L}_n) \rightarrow \text{Lat}(\mathfrak{R}_n)$  be given by  $\mu(\mathfrak{J}) = \overline{\mathfrak{J} \xi_1}$ . Then  $\mu$  a complete lattice isomorphism. The restriction of  $\mu$  to the set  $\text{Id}(\mathfrak{L}_n)$  is a complete lattice isomorphism onto  $\text{Lat} \mathfrak{L}_n \cap \text{Lat} \mathfrak{R}_n$ . The inverse map  $\iota$  sends a subspace  $\mathcal{M}$  to*

$$\iota(\mathcal{M}) = \{J \in \mathfrak{L}_n : J \xi_1 \in \mathcal{M}\}.$$

*Proof.* We have seen above that  $\mathcal{M} = \mu(\mathfrak{J})$  is a subspace of the appropriate type for right and two-sided (and even left) ideals.

Conversely, we now check that  $\iota$  sends invariant subspaces to ideals of the appropriate type. So fix a subspace  $\mathcal{M}$  in  $\text{Lat}(\mathfrak{R}_n)$  and consider  $\iota(\mathcal{M})$ . It is clear that  $\iota(\mathcal{M})$  is a WOT-closed subspace of  $\mathfrak{L}_n$ . Suppose that  $J$  is in  $\iota(\mathcal{M})$  and  $A$  belongs to  $\mathfrak{L}_n$ . Then

$$JA \xi_1 \in \overline{J \mathcal{H}_n} = \overline{J \mathfrak{R}_n \xi_1} = \overline{\mathfrak{R}_n J \xi_1} \subset \mathcal{M}.$$

Whence  $\iota(\mathcal{M})$  is a right ideal.

And if  $\mathcal{M}$  is in  $\text{Lat} \mathfrak{L}_n$ , then for  $J$  in  $\iota(\mathcal{M})$  and  $A$  in  $\mathfrak{L}_n$ ,

$$AJ \xi_1 \in A \mathcal{M} \subset \mathcal{M}.$$

So  $\iota(\mathcal{M})$  is a left ideal. Thus  $\iota(\text{Lat} \mathfrak{L}_n \cap \text{Lat} \mathfrak{R}_n)$  is contained in  $\text{Id}(\mathfrak{L}_n)$ .

Next we show that  $\mu \iota$  is the identity map. Fix  $\mathcal{M}$  in  $\text{Lat}(\mathfrak{R}_n)$ . By the definitions of the maps involved, we have  $\mu \iota(\mathcal{M})$  is contained in  $\mathcal{M}$ . To see the opposite inclusion, let  $\{\zeta_j\}$  be an orthonormal basis for the  $\mathfrak{R}_n$  wandering subspace  $\mathcal{W} = \mathcal{M} \ominus \sum_{i=1}^n \oplus R_i \mathcal{M}$ . Then

$$\mathcal{M} = \sum_j \oplus \mathfrak{R}_n[\zeta_j] = \sum_j \oplus \text{Ran} L_{\zeta_j}.$$

Since  $L_{\zeta_j} \xi_1 = \zeta_j$  belongs to  $\mathcal{M}$ , it follows that  $L_{\zeta_j}$  lies in  $\iota(\mathcal{M})$ . So

$$\mathcal{M} = \sum_j \oplus \text{Ran} L_{\zeta_j} \subset \overline{\iota(\mathcal{M}) \mathcal{H}_n} = \overline{\iota(\mathcal{M}) \xi_1} = \mu(\iota(\mathcal{M})).$$

Therefore  $\mu \iota(\mathcal{M}) = \mathcal{M}$ .

Now fix  $\mathfrak{J}$  in  $\text{Id}_r(\mathfrak{L}_n)$ . As before, the definitions involved show that  $\mathfrak{J}$  is contained in  $\iota\mu(\mathfrak{J})$ . To see that this is an equality, we first show that for every  $\xi$  in  $\mathcal{H}_n$ ,

$$(1) \quad \overline{\mathfrak{J}\xi} = \overline{\iota\mu(\mathfrak{J})\xi}.$$

Since  $\mathfrak{L}_n[\xi]$  is a cyclic invariant subspace for  $\mathfrak{L}_n$ , it may be written as  $\mathfrak{L}_n[\xi] = \text{Ran}R_\eta$  where  $\eta$  is a wandering vector for  $\mathfrak{L}_n[\xi]$ . Thus

$$\overline{\mathfrak{J}\xi} = \overline{\mathfrak{J}\mathfrak{L}_n\xi} = \overline{\mathfrak{J}R_\eta\mathcal{H}_n} = \overline{R_\eta\mathfrak{J}\mathcal{H}_n} = R_\eta\mathcal{M}.$$

Evidently, the same computation for  $\iota\mu(\mathfrak{J})$  yields the same result; hence (1) holds.

Suppose that  $f$  is a WOT-continuous linear functional on  $\mathfrak{L}_n$  which annihilates the ideal  $\mathfrak{J}$ . By [6, Theorem 2.10], there are vectors  $\xi$  and  $\eta$  such that  $f(A) = (A\xi, \eta)$  for all  $A$  in  $\mathfrak{L}_n$ . Since  $f(\mathfrak{J}) = 0$ , it follows that  $\eta$  is orthogonal to  $\mathfrak{J}\xi$ . Then by the previous paragraph,  $\eta$  is also orthogonal to  $\iota\mu(\mathfrak{J})\xi$  and thus  $f$  also annihilates  $\iota\mu(\mathfrak{J})$ . By the Hahn-Banach Theorem, we therefore have  $\mathfrak{J} = \iota\mu(\mathfrak{J})$ .

Thus we have established that  $\mu$  is a bijective pairing between  $\text{Id}_r(\mathfrak{L}_n)$  and  $\text{Lat}\mathfrak{R}_n$  which carries  $\text{Id}(\mathfrak{L}_n)$  onto  $\text{Lat}\mathfrak{L}_n \cap \text{Lat}\mathfrak{R}_n$  and  $\iota = \mu^{-1}$ . If  $\mathfrak{J}_1$  and  $\mathfrak{J}_2$  are WOT-closed right ideals, then

$$\mu(\mathfrak{J}_1 + \mathfrak{J}_2) = \overline{(\mathfrak{J}_1 + \mathfrak{J}_2)\mathcal{H}_n} = \overline{\mathfrak{J}_1\mathcal{H}_n + \mathfrak{J}_2\mathcal{H}_n} = \mu(\mathfrak{J}_1) \vee \mu(\mathfrak{J}_2)$$

and hence sums are sent to spans. Similarly, if  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are subspaces in  $\text{Lat}\mathfrak{L}_n \cap \text{Lat}\mathfrak{R}_n$ , then

$$\begin{aligned} \iota(\mathcal{M}_1 \cap \mathcal{M}_2) &= \{J \in \mathfrak{L}_n : J\xi_1 \in \mathcal{M}_1 \cap \mathcal{M}_2\} \\ &= \{J \in \mathfrak{L}_n : J\xi_1 \in \mathcal{M}_1\} \cap \{J \in \mathfrak{L}_n : J\xi_1 \in \mathcal{M}_2\} \\ &= \iota(\mathcal{M}_1) \cap \iota(\mathcal{M}_2). \end{aligned}$$

It follows that  $\mu$  preserves intersections. Finally, to see that  $\mu$  is complete, note that if  $\mathfrak{J}_k$  is an increasing union (or decreasing intersection) of ideals, we have

$$\mu\left(\overline{\bigcup_k \mathfrak{J}_k}\right) = \overline{\bigcup_k \mathfrak{J}_k\mathcal{H}_n} = \bigvee_k \mu(\mathfrak{J}_k),$$

and similarly for intersections. Therefore  $\mu$  is a complete lattice isomorphism.  $\square$

**Corollary 2.2.** *If  $J$  belongs to  $\mathfrak{L}_n$ , then the WOT-closed (two-sided) ideal  $\langle J \rangle$  generated by  $J$  consists of all elements  $A$  in  $\mathfrak{L}_n$  such that  $A\xi_1$  lies in  $\overline{\mathfrak{L}_n J \mathcal{H}_n}$ .*

*Proof.* The ideal  $\langle J \rangle$  is determined by its range, and this must be the least element  $\mathcal{M}$  of  $\text{Lat}\mathfrak{L}_n \cap \text{Lat}\mathfrak{R}_n$  containing  $J\xi_1$ . Thus

$$\mathcal{M} = \overline{\mathfrak{L}_n \mathfrak{R}_n J \xi_1} = \overline{\mathfrak{L}_n J \mathfrak{R}_n \xi_1} = \overline{\mathfrak{L}_n J \mathcal{H}_n}.$$

By Theorem 2.1, it follows that  $\langle J \rangle = \iota(\mathcal{M})$ .  $\square$

Theorem 2.1 enables us to characterize the WOT-continuous multiplicative linear functionals on  $\mathfrak{L}_n$ .

**Theorem 2.3.** *Suppose  $\varphi$  is a (non-zero) WOT-continuous multiplicative linear functional on  $\mathfrak{L}_n$ . Then there exists  $\lambda$  in  $\mathbb{B}_n$  such that  $\varphi = \varphi_\lambda$ .*

*Proof.* Let  $\mathfrak{J} = \ker \varphi$ . Then  $\mathfrak{J}$  is a WOT-closed two sided maximal ideal of codimension one. Set  $\mathcal{M} = \mu(\mathfrak{J})$  and note that  $\xi_1 \notin \mathcal{M}$ . (If not, Theorem 2.1 implies  $I$  belongs to  $\iota(\mathcal{M}) = \mathfrak{J}$ , which is impossible.) In fact,  $\mathcal{M}$  has codimension one. To see this, let  $\mathcal{N} = \mathcal{M} + \mathbb{C}\xi_1$ . If  $\mathcal{N} \neq \mathcal{H}_n$ , let  $\zeta$  be a unit vector in  $\mathcal{N}^\perp$ . Choose  $A$  in  $\mathfrak{L}_n$  so that  $\|\zeta - A\xi_1\| < 1$  and set  $\alpha = \varphi(A)$ . Since  $A - \alpha I$  is in  $\mathfrak{J}$ ,  $A\xi_1 - \alpha\xi_1$  belongs to  $\mathcal{M}$ . Hence

$$\begin{aligned} 1 &= |(\zeta - \alpha\xi_1, \zeta)| = |(\zeta - A\xi_1, \zeta) + (A\xi_1 - \alpha\xi_1, \zeta)| \\ &= |(\zeta - A\xi_1, \zeta)| < 1, \end{aligned}$$

which is absurd. So  $\mathcal{N} = \mathcal{H}_n$  and hence  $\mathcal{M}$  belongs to  $\text{Lat } \mathfrak{L}_n \cap \text{Lat } \mathfrak{A}_n$  and has codimension one. Thus  $\mathcal{M}^\perp$  is a 1-dimensional invariant subspace for  $\mathfrak{L}_n^*$ . By Theorem 1.3, there is a point  $\lambda$  in  $\mathbb{B}_n$  such that  $\mathcal{M} = \{\nu_\lambda\}^\perp$ . By Theorem 2.1,  $\ker \varphi = \iota(\mathcal{M}) = \ker(\varphi_\lambda)$ . Therefore  $\varphi = \varphi_\lambda$ .  $\square$

We present, as an example, an ideal which will be important later. Let  $\overline{\mathfrak{C}}$  denote the WOT-closure of the commutator ideal of  $\mathfrak{L}_n$ . The space  $\mathcal{H}_n^s$  is the symmetric Fock space spanned by the vectors  $\frac{1}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(w)}$ , where  $w$  is in  $\mathcal{F}_n$ ,  $k = |w|$ ,  $S_k$  is the symmetric group on  $k$  elements, and  $\sigma(w)$  is the word with the terms in  $w$  permuted by  $\sigma$ . Also recall that for  $\lambda$  in  $\mathbb{B}_n$ ,  $\varphi_\lambda$  is the multiplicative linear functional on  $\mathfrak{L}_n$  given by  $\varphi_\lambda(A) = (A\nu_\lambda, \nu_\lambda)$  as in Theorem 1.3.

**Proposition 2.4.** *The WOT-closure of the commutator ideal is*

$$\overline{\mathfrak{C}} = \langle L_i L_j - L_j L_i : i \neq j \rangle = \bigcap_{\lambda \in \mathbb{B}_n} \ker \varphi_\lambda.$$

*The corresponding subspace in  $\text{Lat } \mathfrak{L}_n \cap \text{Lat } \mathfrak{A}_n$  is*

$$\begin{aligned} \mu(\overline{\mathfrak{C}}) &= \text{span}\{\xi_{uz_i z_j v} - \xi_{uz_j z_i v} : i \neq j, u, v \in \mathcal{F}_n\} \\ &= \mathcal{H}_n^{s\perp} = \text{span}\{\nu_\lambda : \lambda \in \mathbb{B}_n\}^\perp. \end{aligned}$$

*Proof.* Let  $\mathfrak{J}$  be the WOT-closed ideal generated by the set of commutators  $\{L_i L_j - L_j L_i : i \neq j\}$ . Clearly  $\overline{\mathfrak{C}} \supset \mathfrak{J}$ . On the other hand, consider the set of operators of the form  $A(BC - CB)D$  for  $A, B, C, D$  in  $\mathfrak{L}_n$ . These elements span a WOT-dense subset of  $\overline{\mathfrak{C}}$ . Moreover, since the polynomials in the  $L_i$  are WOT-dense in  $\mathfrak{L}_n$ , we may further suppose that each of  $A, B, C, D$  is such a polynomial. Thus by expanding, it suffices to show that operators of the form  $L_u(L_v L_w - L_w L_v)L_x$  belong to  $\mathfrak{J}$  for all words  $u, v, w, x$  in  $\mathcal{F}_n$ . Now, every permutation of  $k$  objects is the product of interchanges  $(i, i + 1)$  for some  $1 \leq i < k$ . Using this, it follows that  $L_w - L_{\sigma(w)}$  belongs to  $\mathfrak{J}$  for every  $w$  in  $\mathcal{F}_n$  and every  $\sigma$  in  $S_{|w|}$ . Therefore it follows that  $L_u(L_v L_w - L_w L_v)L_x$  belongs to  $\mathfrak{J}$ . Thus  $\mathfrak{J} = \overline{\mathfrak{C}}$ .



The subspace  $\mu(\mathfrak{C}) = \mu(\mathfrak{J})$  is the smallest  $\mathfrak{L}_n \mathfrak{R}_n$  invariant subspace containing  $\{\xi_{z_i z_j} - \xi_{z_j z_i} : i \neq j\}$ , which is the subspace spanned by the vectors of the form  $\xi_{uz_i z_j v} - \xi_{uz_j z_i v}$ .

It is now clear that  $\mathcal{H}_n^s$  is orthogonal to  $\mu(\mathfrak{C})$ . On the other hand, a vector  $\zeta = \sum_w a_w \xi_w$  is orthogonal to  $\mu(\mathfrak{C})$  if and only if it is orthogonal to every  $\xi_w - \xi_{\sigma(w)}$  for  $w \in \mathcal{F}_n$  and  $\sigma \in S_{|w|}$ . Hence  $a_{\sigma(w)} = a_w$ ; whence it follows that  $\zeta$  belongs to  $\mathcal{H}_n^s$ .

Next we show  $\text{span}\{\nu_\lambda : \lambda \in \mathbb{B}_n\} = \mathcal{H}_n^s$ . Evidently, each  $\nu_\lambda$  belongs to  $\mathcal{H}_n^s$ . Let  $Q_k$  denote the projection onto  $\text{span}\{\xi_w : |w| = k\}$ . For each  $\lambda$  in  $\mathbb{B}_n$  and  $z$  in  $\mathbb{T}$ ,

$$\nu_{z\lambda} = \sum_{m \geq 0} z^m Q_m \nu_\lambda.$$

Thus by considering the  $\mathcal{H}_n$ -valued integrals

$$\int_{\mathbb{T}} \bar{z}^k \nu_{z\lambda} dz$$

for  $k \geq 0$ , it follows that  $Q_k \nu_\lambda$  lies in  $\text{span}\{\nu_\lambda : \lambda \in \mathbb{B}_n\}$ . Now it is an easy exercise to show that the set of all  $Q_k \nu_\lambda$ 's contains each  $\frac{1}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(w)}$  for  $|w| = k$ . Hence  $\text{span}\{\nu_\lambda : \lambda \in \mathbb{B}_n\} = \mathcal{H}_n^s$ .

It is clear that the multiplicative linear functionals  $\varphi_\lambda$  vanish on the commutator, and hence on the WOT-closed ideal that it generates. Conversely, suppose that  $A$  in  $\mathfrak{L}_n$  is not in  $\bar{\mathfrak{C}}$ . Then  $A\xi_1$  is not contained in  $\mu(\mathfrak{C})$ . Therefore, since  $\mu(\mathfrak{C})$  is the orthogonal complement of the set  $\{\nu_\lambda : \lambda \in \mathbb{B}_n\}$ , there is a  $\lambda$  in  $\mathbb{B}_n$  such that

$$0 \neq (A\xi_1, \nu_\lambda) = (\xi_1, A^* \nu_\lambda) = \varphi_\lambda(A)(\xi_1, \nu_\lambda) = (1 - \|\lambda\|^2)^{1/2} \varphi_\lambda(A).$$

Thus  $\varphi_\lambda(A) \neq 0$ ; whence  $A$  is not in  $\ker \varphi_\lambda$ . □

Next we develop some useful lemmas about factorization in right ideals. In particular, they will allow us to determine when a right ideal is finitely generated. Recall from Theorem 1.1 that each isometry in  $\mathfrak{L}_n$  has the form  $L_\zeta$  for some  $\mathfrak{R}_n$ -wandering vector  $\zeta$ .

**Lemma 2.5.** *Let  $L_{\zeta_j}$ , for  $1 \leq j \leq k$ , be a finite set of isometries in  $\mathfrak{L}_n$  with pairwise orthogonal ranges  $\mathcal{M}_j$ . Let  $\mathcal{M} = \sum_{j=1}^k \mathcal{M}_j$  and  $\mathfrak{J} = \iota(\mathcal{M})$ . Then  $\mathfrak{J}$  equals  $\{A \in \mathfrak{L}_n : \text{Ran}(A) \subset \mathcal{M}\}$  and every element of  $\mathfrak{J}$  factors uniquely as*

$$A = \sum_{j=1}^k L_{\zeta_j} A_j \quad \text{for } A_j \in \mathfrak{L}_n.$$

*Thus the (algebraic) right ideal generated by  $\{L_{\zeta_j} : 1 \leq j \leq k\}$  equals  $\mathfrak{J}$ .*

*Proof.* Clearly each  $\mathcal{M}_j$  is  $\mathfrak{R}_n$  invariant, and thus so is  $\mathcal{M}$ . Hence if  $A$  in  $\mathfrak{L}_n$  satisfies  $A\xi_1 \in \mathcal{M}$ , then  $A\mathcal{H}_n$  is contained in  $\mathcal{M}$ . Thus

$$\mathfrak{J} = \{A \in \mathfrak{L}_n : \text{Ran}(A) \subset \mathcal{M}\}.$$

So  $\mathfrak{J}$  is a WOT-closed right ideal containing  $L_{\zeta_j}$  for  $1 \leq j \leq k$ .

Conversely, suppose that  $A$  belongs to  $\mathfrak{J}$ . Then since  $L_{\zeta_j}L_{\zeta_j}^*$  is the orthogonal projection onto  $\mathcal{M}_j$ , we obtain the factorization

$$A = \left( \sum_{j=1}^k L_{\zeta_j}L_{\zeta_j}^* \right) A = \sum_{j=1}^k L_{\zeta_j}A_j,$$

where  $A_j = L_{\zeta_j}^*A$ . This decomposition is unique because the  $L_{\zeta_j}$ 's are isometries with orthogonal ranges. We will show that each  $A_j$  belongs to  $\mathfrak{L}_n$ . As  $\zeta_j$  is a  $\mathfrak{R}_n$ -wandering vector for  $\mathcal{M}$ , it is orthogonal to  $\sum_{i=1}^n R_i\mathcal{M}$ . Now  $\mathcal{N} = \text{Ran}(A)$  is contained in  $\mathcal{M}$ ; whence  $\zeta_j$  is also orthogonal to  $\sum_{i=1}^n R_i\mathcal{N}$ . Therefore, for any word  $w$  in  $\mathcal{F}_n$ ,

$$(R_i^*A^*\zeta_j, \xi_w) = (\zeta_j, AR_i\xi_w) = (\zeta_j, R_iA\xi_w) = 0,$$

and so  $R_i^*A^*\zeta_j = 0$ . Now compute using [6, Lemma 1.11]

$$\begin{aligned} A_jR_i - R_iA_j &= L_{\zeta_j}^*AR_i - R_iL_{\zeta_j}^*A = (L_{\zeta_j}^*R_i - R_iL_{\zeta_j}^*)A \\ &= \xi_1(R_i^*L_{\zeta_j}\xi_1)^*A = \xi_1(A^*R_i^*\zeta_j)^* = \xi_1(R_i^*A^*\zeta_j)^* = 0. \end{aligned}$$

Therefore  $A_j$  belongs to  $\mathfrak{R}'_n = \mathfrak{L}_n$ . It is now evident that  $A$  belongs to the algebraic right ideal generated by  $\{L_{\zeta_j} : 1 \leq j \leq k\}$ .  $\square$

An important special case concerns the (two-sided) ideals  $\mathfrak{L}_n^{0,k}$  generated by  $\{L_w : |w| = k\}$ , which yields a useful decomposition of an arbitrary element of  $\mathfrak{L}_n$ . In particular, the ideal  $\mathfrak{L}_n^0 := \mathfrak{L}_n^{0,1}$  leads us to a unique decomposition of  $\mathfrak{L}_n$  as

$$\mathfrak{L}_n = \mathbb{C}I + \sum_{i=1}^n L_i\mathfrak{L}_n.$$

This provides a handle on the algebraic rigidity of  $\mathfrak{L}_n$  that will prove useful for analyzing the automorphism group.

**Corollary 2.6.** *For  $1 \leq n < \infty$  and  $k \geq 1$ , every  $A$  in  $\mathfrak{L}_n$  can be written uniquely as a sum*

$$A = \sum_{|w| < k} a_w L_w + \sum_{|w|=k} L_w A_w$$

where  $a_w \in \mathbb{C}$  and  $A_w \in \mathfrak{L}_n$ .

*Proof.* The isometries  $\{L_w : |w| = k\}$  have pairwise orthogonal ranges summing to  $\mathcal{M} = \text{span}\{\xi_v : |v| \geq k\}$ . This subspace is  $\mathfrak{L}_n\mathfrak{R}_n$  invariant, and thus by Theorem 2.1, the right ideal  $\iota(\mathcal{M})$  is in fact two-sided. Lemma 2.5 shows that  $\iota(\mathcal{M})$  coincides with  $\mathfrak{L}_n^{0,k}$ .

Given  $A$  in  $\mathfrak{L}_n$ , write  $A\xi_1 = \sum a_w \xi_w$ . The coefficients  $a_w$  for  $|w| < k$  are the unique constants such that  $(A - \sum_{|w| < k} a_w L_w)\xi_1$  lies in  $\mathcal{M}$ . Therefore by Lemma 2.5, this difference can be written uniquely as  $\sum_{|w|=k} L_w A_w$ .  $\square$

*Example 2.7.* Lemma 2.5 is not valid for countably many generators even with norm closure. Indeed, consider the isometries  $L_1^k L_2$  in  $\mathfrak{L}_2$  for  $k \geq 0$ . Their ranges are orthogonal, summing to the  $\mathfrak{L}_n \mathfrak{R}_n$ -invariant subspace generated by  $\xi_{z_2}$ . So the WOT-closed right ideal  $\mathfrak{J}$  that they generate is the two-sided WOT-closed ideal generated by  $L_2$ . Consider a sum of the form

$$A = \sum_{k \geq 0} L_1^k L_2 h_k(L_1)$$

where  $h_k$  will be functions in  $H^\infty$ . This will lie in  $\mathfrak{J}$  provided that  $A$  is a bounded operator. However, it is a norm limit of finite sums of this type only if the series converges in norm.

An easy computation shows that

$$A^* A = \sum_{k \geq 0} h_k(L_1)^* h_k(L_1).$$

As  $L_1$  is a unilateral shift of infinite multiplicity, this sums to an operator unitarily equivalent to the infinite ampliation of a Toeplitz operator with symbol  $\sum_{k \geq 0} |h_k|^2$ . Thus  $A$  is bounded precisely when this sum is bounded. The sum of operators is norm convergent exactly when this sum of functions is norm convergent. Constructing a sequence which is bounded but not norm convergent is easy.

The algebra  $H^\infty$  is logmodular [11], and so if  $f$  is a non-negative real function in  $L^\infty$  such that  $\log f$  is integrable, then there is a function  $h$  in  $H^\infty$  such that  $|h| = f$ . Choose a sequence of disjoint closed intervals  $J_k$  of the unit circle, each of positive length. Let  $f_k = 2^{-k} + \chi_{J_k}$  for  $k \geq 0$ , and let  $h_k$  be analytic functions with  $|h_k|^2 = f_k$ . Then

$$\sum_{k \geq 0} |h_k|^2 = 2 + \chi_J \quad \text{where} \quad J = \bigcup_{k \geq 0} J_k.$$

This sum is bounded. However,  $\|h_k\| > 1$  for all  $k$ , and thus this sum is not norm convergent.

Moreover, this ideal is not finitely generated as a right ideal because  $\mathcal{M}$  has an infinite dimensional  $\mathfrak{R}_n$  wandering space  $\mathcal{W}$ . Any element  $J$  in  $\mathfrak{J}$  has  $\text{Ran}(J)$  contained in  $\mathcal{M}$ , and its projection onto  $\mathcal{W}$  is a subspace of at most one dimension. The ranges of a set of generators must necessarily span  $\mathcal{M}$ ; and thus countably many are required.

We need a variant of Lemma 2.5 which is valid for countably generated ideals. Let  $\mathcal{C}_k(\mathfrak{L}_n)$  denote the *order  $k$  column space* of  $\mathfrak{L}_n$ , which is the set of all  $k$ -tuples of the form

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} \quad A_i \in \mathfrak{L}_n, \quad 1 \leq i \leq k$$

such that  $A$  is bounded with respect to the norm obtained by considering  $A$  as an element of  $\mathcal{B}(\mathcal{H}_n, \mathcal{H}_n^{(k)})$ . Similarly, let  $\mathcal{R}_k(\mathfrak{L}_n)$  denote the order  $k$  row space of  $\mathfrak{L}_n$  consisting of operators

$$A = [ A_1 \quad A_2 \quad \dots \quad A_k ] \quad A_i \in \mathfrak{L}_n, \quad 1 \leq i \leq k$$

such that  $A$  is bounded with respect to the norm obtained by considering  $A$  as an element of  $\mathcal{B}(\mathcal{H}_n^{(k)}, \mathcal{H}_n)$ . For  $k < \infty$ , this is all  $k$ -tuples, but the boundedness condition is non-trivial for  $k = \infty$ .

The following lemma shows, in particular, that the infinite row matrix,  $L = [L_1 \quad L_2 \quad L_3 \quad \dots]$ , maps  $\mathcal{C}_\infty(\mathfrak{L}_\infty)$  bijectively onto  $\mathfrak{L}_\infty^0$ . This result also applies to the two-sided WOT-closed ideal  $\mathfrak{J}$  generated by the set  $\{L_2, \dots, L_n\}$ . The range of this ideal is the sum of the pairwise orthogonal ranges of  $\{L_1^k L_j : k \geq 0, 2 \leq j \leq n\}$ .

**Lemma 2.8.** *Let  $L_{\zeta_j}$ , for  $j \geq 1$ , be a countably infinite set of isometries in  $\mathfrak{L}_n$  with pairwise orthogonal ranges  $\mathcal{M}_j$ . Let  $\mathcal{M} = \sum_{j=1}^\infty \mathcal{M}_j$ , and let  $\mathfrak{J}$  be the WOT-closed right ideal  $\iota(\mathcal{M})$ . Then every element of  $\mathfrak{J}$  factors uniquely as  $A = ZX$ , where  $Z$  is the fixed isometry in  $\mathcal{R}_\infty(\mathfrak{L}_n)$  given by  $Z = [L_{\zeta_1} \quad L_{\zeta_2} \quad \dots]$  and  $X$  is a bounded operator in  $\mathcal{C}_\infty(\mathfrak{L}_n)$ . Hence  $A$  can be written uniquely as the WOT limit*

$$A = \text{WOT-}\lim_{k \rightarrow \infty} \sum_{j=1}^k L_{\zeta_j} A_j.$$

*Proof.* The proof begins as in Lemma 2.5. There is a unique decomposition of  $A$  as a WOT-convergent sum,  $A = \text{WOT-}\sum_{j \geq 1} L_{\zeta_j} X_j$ , where  $X_j = L_{\zeta_j}^* A$ . The  $X_j$  are elements of  $\mathfrak{L}_n$  by the same computation. Thus defining  $X$  to be the column operator with entries  $X_j$ , we obtain a formal factorization  $A = ZX$ . To see that  $X$  is bounded, it suffices to compute that  $X^*X = A^*A$ . □

**Corollary 2.9.** *Every element  $A$  in  $\mathfrak{L}_\infty$  decomposes uniquely as*

$$A = \sum_{|w| < k} a_w L_w + W_k X_k$$

where  $(a_w)_{|w| < k}$  belongs to  $\ell^2$ ,  $W_k = [L_{w_{k,1}} \quad L_{w_{k,2}} \quad \dots]$  and  $\{w_{k,i}\}$  is a listing of all words of length  $k$ , and  $X_k$  belongs to  $\mathcal{C}_\infty(\mathfrak{L}_\infty)$ .

*Proof.* The identity  $A\xi_1 = \sum_{w \in \mathcal{F}_n} a_w \xi_w$  determines the coefficients  $a_w$  uniquely, and shows that they belong to  $\ell^2$ . For each  $j$ , the isometries  $L_{w_{j,i}}$  have pairwise orthogonal ranges, and hence the sum  $\sum_{|w|=j} a_w L_w$  is norm convergent. Summing this over  $j < k$  yields the unique operator of this form in the same coset of  $A + \mathfrak{L}_\infty^{0,k}$ . The remainder is factored by Lemma 2.8. □

These lemmas allow us to determine when a right ideal is finitely generated.

**Theorem 2.10.** *Let  $\mathfrak{J}$  be a WOT-closed right ideal. If  $\mathcal{M} = \mu(\mathfrak{J})$  in  $\text{Lat } \mathfrak{K}_n$  has a finite dimensional wandering space of dimension  $k$ , then  $\mathfrak{J}$  is generated by  $k$  isometries as an algebraic right ideal. When this wandering subspace is infinite dimensional,  $\mathfrak{J}$  is not finitely generated even as a WOT-closed right ideal. However, it is generated by countably many isometries as a WOT-closed right ideal.*

*Proof.* When the wandering space  $\mathcal{W}$  is finite dimensional, choose an orthonormal basis  $\{\zeta_j : 1 \leq j \leq k\}$ . Then  $\mathcal{M} = \sum_{j=1}^k \oplus \text{Ran} L_{\zeta_j}$ . Thus by Lemma 2.5, the isometries  $\{L_{\zeta_j} : 1 \leq j \leq k\}$  generate  $\mathfrak{J}$  as an algebraic right ideal. Similarly, when  $\mathcal{W}$  is infinite dimensional, Lemma 2.8 yields a countable set of isometries which generate  $\mathfrak{J}$  as a WOT-closed right ideal.

Finally, suppose that  $\mathfrak{J}$  is finitely generated as a WOT-closed right ideal, say by  $\{A_j : 1 \leq j \leq k\}$ . Then the operators of the form  $\sum_{j=1}^k A_j B_j$  for  $B_j$  in  $\mathfrak{L}_n$  are WOT-dense in  $\mathfrak{J}$ . Therefore

$$\mu(\mathfrak{J}) = \overline{\sum_{j=1}^k A_j \mathcal{H}_n} = \mathfrak{K}_n[\{A_j \xi_1 : 1 \leq j \leq k\}].$$

This subspace is finitely generated, and therefore has finite dimensional wandering space.  $\square$

### 3. Representations of $\mathfrak{L}_n$

In the category of unital operator algebras, we take the viewpoint that the natural representations are the completely contractive unital representations. Given an operator algebra  $\mathfrak{A}$ , for each  $1 \leq k \leq \aleph_0$  we let  $\text{Rep}_k(\mathfrak{A})$  denote the set of completely contractive representations of  $\mathfrak{A}$  into  $\mathcal{B}(\mathcal{H})$ , where  $\mathcal{H}$  is a fixed Hilbert space of dimension  $k$ . Put the topology of pointwise-weak-\* convergence on this space. When  $k < \infty$ , this is the topology of pointwise (norm) convergence. Since the unit ball of  $\mathcal{B}(\mathcal{H})$  is weak-\* compact (and norm compact when  $k < \infty$ ), Tychonoff's Theorem shows that the set of contractive maps from  $\mathfrak{A}$  into  $\mathcal{B}(\mathcal{H})$  is pointwise-weak-\* compact. When  $k < \infty$ , the collection of representations is closed in this topology, and thus is also compact. Unfortunately, the collection of representations is *not* closed when  $k = \infty$ . For an example, consider the direct sum  $\text{id}^{(n)}$  of  $n$  copies of the identity representation of  $\mathcal{B}(\mathcal{H})$  for  $n \geq 1$ . Since the direct sum of  $n$  copies of the unilateral shift  $S$  is unitarily equivalent to  $S^n$ , we may find representations  $\sigma_n$  of  $\mathcal{B}(\mathcal{H})$  on  $\mathcal{H}$  such that  $\sigma_n(S) = S^n$  for every  $n$ . Note that no pointwise-weak-\* limit point of this sequence of representations is multiplicative, and hence the space of representations is not closed when  $k = \infty$ .

The natural equivalence relation on representations is unitary equivalence. When  $k < \infty$ , the unitary group  $\mathcal{U}_k$  is compact and acts on  $\text{Rep}_k(\mathfrak{A})$ . Thus the quotient space is also compact and Hausdorff. This need not be the case for  $k = \aleph_0$  since unitary orbits of representations need not be closed in general.

*For these reasons, our standing assumption throughout this section is that all representations of  $\mathfrak{L}_n$  are on finite dimensional spaces.*

The familiar case of  $k = 1$  yields the set of multiplicative linear functionals. It is well known that multiplicative linear functionals are automatically completely contractive. In this case, unitary equivalence is the identity relation. Moreover, there is a bijective pairing between the multiplicative linear functional and its kernel, a maximal ideal of codimension 1. So  $\text{Rep}_1(\mathfrak{A})$  is the direct analogue of the maximal space of a commutative Banach algebra. Of course, in a non-abelian algebra, there may be many maximal ideals of other codimensions.

For  $k > 1$ , it is clear that two *similar* representations will have the same kernel. In the case of  $\mathfrak{L}_n$ , similar representations which are both completely contractive need not be unitarily equivalent. (Indeed, when  $n = 1$ , simply consider two similar, but non-unitarily equivalent, contractions.) When  $k < \infty$  and a representation  $\Phi$  in  $\text{Rep}_k(\mathfrak{A})$  is irreducible (no invariant subspaces), the range must be all of  $\mathcal{M}_k = \mathcal{B}(\mathcal{H})$ . This is because Burnside's Theorem [23] shows that every proper subalgebra of  $\mathcal{M}_k$  has a proper invariant subspace. Thus the kernel will be a maximal ideal of codimension  $k^2$ . Conversely, if  $\mathfrak{M}$  is a maximal ideal of  $\mathfrak{A}$  of finite codimension, then there is a finite dimensional representation of  $\mathfrak{A}$  on  $\mathfrak{A}/\mathfrak{M}$  with kernel  $\mathfrak{M}$ . This quotient is simple, and thus by Wedderburn's Theorem,  $\mathfrak{A}/\mathfrak{M}$  is isomorphic to  $\mathcal{M}_k$  for some positive integer  $k$ . In particular,  $\mathfrak{M}$  has codimension  $k^2$ . Restrict this representation to a minimal invariant subspace  $\mathcal{M}$  to obtain a representation  $\pi$  and note that  $\mathcal{M}$  must have dimension  $k$ . Now  $\pi$  does not act on a Hilbert space. However, it is clearly a completely contractive representation. Any Hilbert space norm on  $\mathfrak{M}$  is equivalent to the quotient norm, and thus will yield a completely bounded Hilbert space representation. Then by Paulsen's Theorem [13], this is similar to a completely contractive representation. This shows that the map from irreducible representations in  $\text{Rep}_k(\mathfrak{A})$  to the set of maximal ideals of codimension  $k^2$  is surjective.

The algebra  $\mathfrak{L}_n$  has many representations of every dimension. This will follow from Popescu's work on dilation theory for non-commuting  $n$ -tuples of operators. The case of  $k = 1$  is special and has some extra structure. So we will handle these special features separately.

Recall the situation for  $n = 1$  in which  $\mathfrak{L}_1$  is isomorphic to  $H^\infty$ . There is a natural continuous projection  $\pi_1$  of the maximal ideal space  $\mathfrak{M}_{H^\infty}$  of  $H^\infty$  onto the closed disk  $\overline{\mathbb{D}}$  given by evaluation at the coordinate function  $z$ . For each point  $\lambda$  in  $\mathbb{D}$ , there is a unique multiplicative linear functional  $\varphi_\lambda(h) = h(\lambda)$  extending evaluation of  $z$  at  $\lambda$ . But for  $|\lambda| = 1$ , there is a very large space  $\mathfrak{M}_\lambda$  of multiplicative linear functionals taking the value  $\lambda$  at  $z$ . (See Hoffman [11] or Garnett [10].) The famous corona theorem of Carleson [4] shows that the point evaluations in the open unit disk are dense in  $\mathfrak{M}_{H^\infty}$ .

Even though  $\mathfrak{L}_n$  is not commutative, the space  $\text{Rep}_1(\mathfrak{L}_n)$  of multiplicative linear functionals is very large. For representations of dimension greater than one, there are interesting parallels with the case of multiplicative linear functionals. The analysis is based on the extensive knowledge of dilation theory for non-commuting  $n$ -tuples. We reprise the results that we will need.

Recall that if  $\Phi$  is a linear map of an operator algebra  $\mathfrak{A}$  into  $\mathcal{B}(\mathcal{H})$ , then  $\Phi^{(k,\ell)}$  is the map from  $\mathcal{M}_{k,\ell}(\mathfrak{A})$  into  $\mathcal{M}_{k,\ell}(\mathcal{B}(\mathcal{H}))$ , each endowed with the usual

operator norms, given by  $\Phi^{(k,\ell)}([A_{ij}]) = [\Phi(A_{ij})]$ . When  $\ell = k$ , we write  $\Phi^{(k)}$  instead. The complete bound norm of  $\Phi$  is defined to be  $\|\Phi\|_{cb} = \sup_{k,\ell} \|\Phi^{(k,\ell)}\|$ . The map  $\Phi$  is completely contractive if  $\|\Phi\|_{cb} \leq 1$ . See Paulsen’s book [14] for details.

Let  $\overline{\mathbb{B}_{n,k}}$  denote the collection of all contractions in  $\mathcal{R}_n(\mathcal{B}(\mathcal{H}))$  where  $\dim \mathcal{H} = k$ ; namely all  $n$ -tuples  $T = [T_1 \ \dots \ T_n]$  in  $\mathcal{B}(\mathcal{H}^{(n)}, \mathcal{H})$ , such that  $\dim \mathcal{H} = k$  and  $\|T\| = \|\sum_{i=1}^n T_i T_i^*\|^{1/2} \leq 1$ . This is the higher dimensional analogue of the  $n$ -ball. It is endowed with the product norm topology when  $k < \infty$  and the product weak-\* topology when  $k = \aleph_0$ .

If  $\Phi$  is a (completely contractive) representation of  $\mathfrak{L}_n$  on a Hilbert space  $\mathcal{H}$ , then the  $n$ -tuple  $T = \Phi^{(1,n)}(L) = (\Phi(L_1), \dots, \Phi(L_n))$  is a contraction. Bunce [3], generalizing Frazho [8], showed that every contraction  $T$  has a dilation to an  $n$ -tuple of isometries  $S = (S_1, \dots, S_n)$  with orthogonal ranges. Popescu [16] extended this to  $n = \infty$  and showed that there is a unique minimal isometric dilation of  $T$ . This yields a representation of the *norm-closed* algebra generated by  $L$  because the map taking each  $L_i$  to  $S_i$  is a completely isometric isomorphism. Following this with the compression to the original space yields a homomorphism taking  $L_i$  to  $T_i$ . However, this map usually does not extend naturally to a continuous map from  $\mathfrak{L}_n$  into  $\text{Alg}(S)$ . Popescu [21] determines when this has a WOT-continuous extension to a representation of  $\mathfrak{L}_n$ . Nevertheless, when  $k = \dim \mathcal{H} < \infty$ , we shall see that norm-continuous extensions always exist.

The following is a technical lemma used in the proof of Theorem 3.2 below. Recall that  $\mathfrak{L}_n^{0,j}$  is the WOT-closed ideal of  $\mathfrak{L}_n$  generated by the set  $\{L_w : |w| = j\}$ .

**Lemma 3.1.** *Let  $\Phi$  belong to  $\text{Rep}_k(\mathfrak{L}_n)$ . If  $T := (\Phi(L_1), \dots, \Phi(L_n))$  satisfies  $\|T\| = r < 1$ , then  $\|\Phi(A)\| \leq r^j \|A\|$  for every  $A$  in  $\mathfrak{L}_n^{0,j}$ .*

*Proof.* Let  $W$  be the  $1 \times n^j$  row matrix with entries  $L_w$  for  $|w| = j$ . And let  $W(T)$  denote the row matrix with entries  $w(T)$  for  $|w| = j$ . By Corollary 2.6 for  $n < \infty$  and Corollary 2.9 for  $n = \infty$ , we may factor  $A = WX$  for some  $X$  in  $\mathcal{C}_{n^j}(\mathfrak{L}_n)$ . Notice that  $W$  is an isometry, and therefore  $\|A\| = \|X\|$ . By the Frazho–Bunce dilation result [8, 3] for  $n < \infty$  and Popescu [16] for  $n = \infty$ , the  $n$ -tuple  $r^{-1}T$  dilates to an  $n$ -tuple of isometries  $S$ , and therefore  $W_j(r^{-1}T)$  dilates to the isometry  $W_j(S)$ . Hence

$$\|W_j(T)\| = r^j \|W_j(r^{-1}T)\| \leq r^j \|W_j(S)\| = r^j.$$

Then since  $\Phi$  is completely contractive,

$$\begin{aligned} \|\Phi(A)\| &= \|\Phi^{(1,n^j)}(W) \Phi^{(n^j,1)}(X)\| \\ &\leq \|W_j(T)\| \|X\| \leq r^j \|A\|. \quad \square \end{aligned}$$

The first result of this section generalizes the fact that there is a natural map of  $\mathfrak{M}_{H^\infty}$  onto the closed unit disk. The uniqueness result appears to be new even for  $n = 1$  when  $k > 1$ . Recall that for  $n = \infty$ ,  $\mathbb{B}_\infty$  denotes the unit ball of Hilbert space with the weak topology.

**Theorem 3.2.** *For  $k < \infty$ , there is a natural continuous projection  $\pi_{n,k}$  of  $\text{Rep}_k(\mathfrak{L}_n)$  onto the closed unit ball  $\overline{\mathbb{B}_{n,k}}$  given by*

$$\pi_{n,k}(\Phi) = (\Phi(L_1), \dots, \Phi(L_n)).$$

*For each  $T$  in  $\mathbb{B}_{n,k}$ , the open unit ball, there is a unique representation in  $\pi_{n,k}^{-1}(T)$ . It is WOT-continuous and is given by Popescu's functional calculus. The restriction of  $\pi_{n,k}^{-1}$  to  $\mathbb{B}_{n,k}$  is a homeomorphism.*

*Proof.* Since  $\Phi$  in  $\text{Rep}_k(\mathfrak{L}_n)$  is completely contractive, it follows that

$$T = \Phi^{(1,n)}(L) = [ \Phi(L_1) \quad \dots \quad \Phi(L_n) ]$$

is a contraction. Hence  $\pi_{n,k}$  is a well defined map of  $\text{Rep}_k(\mathfrak{L}_n)$  into  $\overline{\mathbb{B}_{n,k}}$ . Since it is determined by evaluation at the points  $L_i$ , this is a continuous map from  $\text{Rep}_k(\mathfrak{L}_n)$  with the topology of pointwise convergence into the ball with the product topology. By Popescu's functional calculus, there is a representation  $\Phi_T$  for every  $T$  in the interior  $\mathbb{B}_{n,k}$  (and in fact, for every completely non-coisometric contraction). Since  $\text{Rep}_k(\mathfrak{L}_n)$  is compact, the image is compact and therefore maps onto  $\overline{\mathbb{B}_{n,k}}$ .

When  $\|T\| = r < 1$ , the WOT-continuous representation  $\Phi_T$  is defined as follows. Each  $A$  in  $\mathfrak{L}_n$  is determined by  $A\xi_1 = \sum_w a_w \xi_w$  as a formal sum  $A = \sum_w a_w L_w$ . The image  $\Phi_T(A)$  is determined as a norm convergent sum

$$\Phi_T(A) = \sum_w a_w w(T).$$

To see this, apply Lemma 3.1 for each  $j \geq 0$  to obtain

$$\left\| \sum_{|w|=j} a_w w(T) \right\| \leq r^j \left\| \sum_{|w|=j} a_w w(L) \right\| = r^j \left( \sum_{|w|=j} |a_w|^2 \right)^{1/2}.$$

Thus two partial sums of  $\sum_w a_w w(T)$  which both contain all words of length less than  $j$  will differ in norm by at most

$$(2) \quad \sum_{k \geq j} r^k \left( \sum_{|w|=k} |a_w|^2 \right)^{1/2} \leq \sum_{k \geq j} r^k \|A\| = r^j (1 - r)^{-1} \|A\|,$$

which tends to zero as  $j$  tends to infinity. Therefore this series is norm convergent. The fact that it is WOT-continuous was shown by Popescu in [21].

The proof of uniqueness follows similar lines. Let  $\Phi$  in  $\text{Rep}_k(\mathfrak{L}_n)$  be a completely contractive representation of  $\mathfrak{L}_n$  such that  $\pi_{n,k}(\Phi) = T$ , where  $\|T\| = r < 1$ . We shall show that  $\Phi = \Phi_T$ . So let  $A$  be an element of  $\mathfrak{L}_n$ . Then by Corollary 2.6 for  $n < \infty$  and Corollary 2.9 for  $n = \infty$ ,  $A$  can be written uniquely as

$$A = \sum_{|w| < j} a_w L_w + \sum_{|w|=j} L_w A_w \quad \text{with} \quad A_w \in \mathfrak{L}_n.$$

Therefore



$$\Phi(A) = \sum_{|w|<j} a_w w(T) + \sum_{|w|=j} w(T)\Phi(A_w).$$

Let  $\Sigma_k(A) = \sum_{|w|<k} (1 - \frac{|w|}{k}) a_w L_w$  denote the Cesaro sums. Recall that  $\|\Sigma_k(A)\| \leq \|A\|$ , and that they converge to  $A$  in the strong-\* operator topology. For each integer  $j$ , there is an integer  $k$  sufficiently large that

$$\left\| \sum_{|w|<j} \frac{|w|}{k} a_w L_w \right\| < r^j \|A\|.$$

Then  $A - \Sigma_k(A) = A_1 + \sum_{|w|<j} \frac{|w|}{k} a_w L_w$  where  $A_1$  belongs to  $\mathfrak{L}_n^{0,j}$ . Clearly,  $\|A_1\| < (2 + r^j)\|A\|$ . Hence, using the fact that  $\Phi$  is contractive and Lemma 3.1,

$$\begin{aligned} \|\Phi(A) - \Phi(\Sigma_k(A))\| &\leq \left\| \Phi\left(\sum_{|w|<j} \frac{|w|}{k} a_w L_w\right) \right\| + \|\Phi(A_1)\| \\ &\leq r^j \|A\| + r^j (2 + r^j)\|A\| < 4r^j \|A\|. \end{aligned}$$

Since  $\Phi$  and  $\Phi_T$  agree on polynomials in  $L$ , it follows that

$$\Phi(A) = \lim_{k \rightarrow \infty} \Phi(\Sigma_k(A)) = \lim_{k \rightarrow \infty} \Phi_T(\Sigma_k(A)) = \Phi_T(A).$$

Finally, we verify that the map sending  $T$  to  $\Phi_T$  maps  $\mathbb{B}_{n,k}$  homeomorphically onto the open set  $\pi_{n,k}^{-1}(\mathbb{B}_{n,k})$ . It is evident from the series representation of  $\Phi_T$  and estimate (2) above, that if  $\|T\| \leq r < 1$ ,  $\|T'\| \leq r$ , and  $A$  is in  $\mathfrak{L}_n$ ,

$$\|\Phi_T(A) - \Phi_{T'}(A)\| \leq \sum_{|w|\leq j} |a_w| \|w(T) - w(T')\| + 2r^j (1 - r)^{-1} \|A\|.$$

Thus as  $T'$  converges to  $T$ , it follows that  $\Phi_{T'}(A)$  converges to  $\Phi_T(A)$ . So this mapping of  $\mathbb{B}_{n,k}$  into  $\text{Rep}_k(\mathfrak{L}_n)$  is continuous.  $\square$

Now we specialize to 1-dimensional representations. In this case, each fibre over a point on the boundary of the ball is homeomorphic to every other because the gauge automorphisms act on the ball by the unitary group, and thus is transitive on the boundary. Moreover, this fibre is always very large based on the fact that it is known to be very large for  $n = 1$ .

**Theorem 3.3.** *There is a natural continuous projection  $\pi_{n,1}$  of the space  $\text{Rep}_1(\mathfrak{L}_n)$  onto the closed unit ball  $\overline{\mathbb{B}_n}$  in  $\mathbb{C}^n$  given by evaluation at the  $n$ -tuple  $(L_1, \dots, L_n)$ .*

*For each point  $\lambda$  in  $\mathbb{B}_n$ , there is a unique multiplicative linear functional in  $\pi_{n,1}^{-1}(\lambda)$ ; and it is given by  $\varphi_\lambda(A) = (A\nu_\lambda, \nu_\lambda)$ . The set  $\pi_{n,1}^{-1}(\mathbb{B}_n)$  is homeomorphic to  $\mathbb{B}_n$  and the restriction of the Gelfand transform to this ball is a contractive homeomorphism of  $\mathfrak{L}_n$  into  $H^\infty(\mathbb{B}_n)$ . The ball  $\mathbb{B}_n$  forms a Gleason part of  $\text{Rep}_1(\mathfrak{L}_n)$ . These are the only weak-\* continuous functionals on  $\mathfrak{L}_n$ .*

*For each point  $\lambda$  in  $\partial\mathbb{B}_n$ ,  $\pi_{n,1}^{-1}(\lambda)$  is homeomorphic to  $\pi_{n,1}^{-1}(1, 0, \dots, 0)$ . There is a canonical surjection of  $\pi_{n,1}^{-1}(\lambda)$  onto the fibre  $\mathfrak{M}_1$  of  $\mathfrak{M}_{H^\infty}$  given by restricting  $\varphi$  in  $\pi_{n,1}^{-1}$  to  $\text{Alg}(\sum_{i=1}^n \bar{\lambda}_i L_i) \simeq H^\infty$ . This map has a continuous section.*

*Proof.* By Theorem 3.2, the map  $\pi_{n,1}$  maps  $\text{Rep}_1(\mathfrak{L}_n)$  onto  $\overline{\mathbb{B}_n}$  by evaluation at  $L$ . For each point  $\lambda$  in the open ball, there is a unique preimage  $\pi_{n,1}^{-1}(\lambda)$  which is evidently  $\varphi_\lambda$ . Also, the preimage of  $\mathbb{B}_n$  is homeomorphic to the open ball. By Theorem 2.3, these are the only WOT-continuous multiplicative linear functionals on  $\mathfrak{L}_n$ . By Corollary 1.4, these coincide with the weak-\* continuous ones.

For each polynomial  $p(z) = \sum a_w w$  in  $\mathbb{C}\mathcal{F}_n$ , the Gelfand transform

$$\widehat{p(L)}(\lambda) = p(\lambda)$$

is evidently a contractive homomorphism of  $\mathbb{C}\mathcal{F}_n$  into  $\mathbb{C}[z]$  normed as a subset of  $H^\infty(\mathbb{B}_n)$ . Suppose that  $p_n(L)$  is a WOT-Cauchy sequence in  $\mathfrak{L}_n$ . Since the set  $\{\varphi_\lambda : \|\lambda\| \leq r\}$  is a compact set of WOT-continuous linear functionals for each  $0 < r < 1$ , the restriction of  $\widehat{p_n(L)}$  to this set converges uniformly. Thus the limit lies in  $H^\infty(\mathbb{B}_n)$ . This shows that the Gelfand map yields a contractive homomorphism into  $\mathcal{H}^\infty(\mathbb{B}_n)$  which carries WOT-convergent sequences to sequences converging uniformly on compact subsets of the ball.

Now recall that the Gleason part containing  $\varphi_0$  is the equivalence class

$$\{\varphi \in \text{Rep}_1(\mathfrak{L}_n) : \|\varphi - \varphi_0\| < 2\}.$$

Consider the positive linear functional  $\delta_\xi(T) = (T\xi, \xi)$  on  $\mathcal{B}(\mathcal{H})$  for a unit vector  $\xi$ . Let  $\zeta$  be another unit vector with  $|\langle \xi, \zeta \rangle| = \cos \theta$  for  $0 \leq \theta \leq \frac{\pi}{2}$ . It is a well known fact that

$$\|\delta_\xi - \delta_\zeta\| = \sup_{\|T\| \leq 1} |(T\xi, \xi) - (T\zeta, \zeta)| = 2(1 - \sin \theta).$$

Since  $(\nu_0, \nu_\lambda) = (1 - \|\lambda\|^2)^{1/2} \neq 0$ , it follows that  $\|\varphi_0 - \varphi_\lambda\| < 2$  for  $\lambda$  in  $\mathbb{B}_n$ . On the other hand, if  $\|\lambda\| = 1$ , then  $S = \sum_{i=1}^n \bar{\lambda}_i L_i$  is a proper isometry in  $\mathfrak{L}_n$  such that  $\varphi_0(S) = 0$  and  $\varphi_\lambda(S) = 1$ . So the Mobius map  $b_r(z) = \frac{z-r}{1-rz}$  for  $0 < r < 1$  can be used to obtain

$$\varphi_0(b_r(S)) = -r \quad \text{and} \quad \varphi_\lambda(b_r(S)) = 1.$$

Hence  $\|\varphi_0 - \varphi_\lambda\| = 2$ . So the Gleason part of  $\varphi_0$  is precisely  $\mathbb{B}_n$ .

Next consider the point  $\lambda = (1, 0, \dots, 0)$  in  $\partial\mathbb{B}_n$ . The algebra  $\text{Alg}(L_1)$  is isomorphic to  $H^\infty$ . For  $\varphi$  in  $\pi_{n,1}^{-1}(1, 0, \dots, 0)$ , let  $\rho(\varphi)$  be the restriction of  $\varphi$  to  $\text{Alg}(L_1)$ . Clearly,  $\rho(\varphi)$  belongs to  $\mathfrak{M}_1$ , the fibre of  $\mathfrak{M}_{H^\infty}$  over the point 1, and  $\rho$  is continuous. We now produce a right inverse for  $\rho$ .

Let  $P$  be the projection onto the subspace  $\text{span}\{\xi_{z_1^k} : k \geq 0\}$ ; and notice that  $P^\perp \mathcal{H}_n$  is an  $\mathfrak{L}_n$ -invariant subspace. So the map  $\Psi(A) = PA|_{P\mathcal{H}_n}$  is a homomorphism of  $\mathfrak{L}_n$ . In fact,  $P^\perp \mathcal{H}_n$  is also  $\mathfrak{K}_n$  invariant. Thus the kernel of this homomorphism is  $\mathfrak{J} = \{A \in \mathfrak{L}_n : PA\xi_1 = 0\}$ , which is the WOT-closed ideal generated by  $\{L_2, \dots, L_n\}$ .

The range of  $\Psi$  is contained in the WOT-closed algebra generated by the operators  $PL_iP$ , which are all 0 except for  $PL_1P$  which is a unilateral shift. The map taking  $L_1$  to  $PL_1P$  is isometric and WOT-continuous, and carries  $\text{Alg}(L_1)$

onto  $\mathcal{T}(H^\infty)$ . By composing  $\Psi$  with the isomorphism of  $\text{Alg}(L_1)$  onto  $H^\infty$ , we may regard  $\Psi$  as a surjection of  $\mathfrak{L}_n$  onto  $H^\infty$ .

Let  $\alpha := \Psi^*|_{\mathfrak{M}_1}$  be the restriction of the Banach space adjoint of  $\Psi$  to  $\mathfrak{M}_1$ . Clearly  $\alpha$  is a continuous map; and if  $\varphi = \alpha(\psi)$ , we have

$$\pi_{n,1}(\varphi) = (\pi_{1,1}(\psi), 0, \dots, 0).$$

So  $\alpha$  maps  $\mathfrak{M}_1$  into  $\pi_{n,1}^{-1}(1, 0, \dots, 0)$  and  $\rho \circ \alpha(\psi) = \rho(\psi\Psi) = \psi$  for  $\psi$  in  $\mathfrak{M}_1$ . Therefore this is a continuous section, and  $\rho$  is surjective. In particular, this yields a homeomorphism of  $\mathfrak{M}_1$  into  $\pi_{n,1}^{-1}(1, 0, \dots, 0)$ .

For any other  $\lambda$  with  $\|\lambda\| = 1$ , choose a unitary  $U = [u_{ij}]$  in  $\mathcal{M}_n$  such that  $u_{1j} = \lambda_j$ . We will show that the gauge automorphism  $\Theta_U$  maps  $\pi_{n,1}^{-1}(1, 0, \dots, 0)$  onto  $\pi_{n,1}^{-1}(\lambda)$ . Indeed, for any  $\varphi$  in  $\pi_{n,1}^{-1}(1, 0, \dots, 0)$ ,

$$\varphi\Theta_U(L_j) = \varphi\left(\sum_{i=1}^n u_{ij}L_i\right) = u_{1j} = \lambda_j.$$

It is evident that this map is continuous with inverse obtained by sending  $\varphi$  to  $\varphi\Theta_U^{-1}$ . So it induces a homeomorphism between  $\pi_{n,1}^{-1}(1, 0, \dots, 0)$  and  $\pi_{n,1}^{-1}(\lambda)$ . The role of  $L_1$  is played by  $\Theta_U^{-1}(L_1) = \sum_{i=1}^n \bar{\lambda}_i L_i$ . □

*Example 3.4.* This example is to illustrate some of the possibilities on the boundary when  $k > 1$ .

It is possible that the fibre over a boundary point is a singleton. Consider  $\text{Rep}_3(\mathfrak{L}_2)$ , the pair

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and a representation  $\Phi$  such that  $\Phi(L_i) = T_i$  for  $i = 1, 2$ . Then since  $T_1^2 = T_2^2 = T_1T_2 = T_2T_1 = 0$ , it follows from Lemma 2.5 that  $\ker \Phi$  contains the ideal  $\mathfrak{L}_2^{0,2}$ . Every element  $A$  in  $\mathfrak{L}_2$  can be represented uniquely as  $A = a_0I + a_1L_1 + a_2L_2 + A'$  where  $A'$  belongs to  $\mathfrak{L}_2^{0,2}$ . Therefore  $\Phi(A) = a_0I + a_1T_1 + a_2T_2$  is uniquely determined.

On the other hand, the fibre over  $T$  may be very large indeed. Let

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We consider a class of homomorphisms  $\Phi$  of  $\mathfrak{L}_2$  in  $\pi_{2,2}^{-1}(T)$ . Let  $\zeta_i$  denote the standard basis for  $\mathbb{C}^2$ . Both  $T_i$  are lower triangular, so we will consider those representations  $\Phi$  which map  $\mathfrak{L}_2$  into the algebra  $\mathcal{T}_2$  of  $2 \times 2$  lower triangular matrices.

The functionals  $\varphi_i(A) = (\Phi(A)\zeta_i, \zeta_i)$  are multiplicative since compression to the diagonal is multiplicative on  $\mathcal{T}_2$ . Moreover,  $\varphi_1(L_1) = 1$  and  $\varphi_1(L_2) = 0$ ,

and hence  $\varphi_1$  lies in  $\pi_{2,1}^{-1}(1, 0)$ . Likewise,  $\varphi_2(L_1) = \varphi_2(L_2) = 0$ . So  $\varphi_2 = \varphi_0$  is evaluation at 0 by Theorem 3.3. Recall from Corollary 2.6 that every  $A$  in  $\mathcal{L}_2$  can be uniquely written as  $A = a_0I + L_1A_1 + L_2A_2$ , where  $a_0 = \varphi_0(A)$  and  $A_i = L_i^*(A - a_0I)$ . Define  $\delta(A) = \varphi_1(A_2) = \varphi_1(L_2^*(A - \varphi_0(A)I))$ . Then

$$\begin{aligned} \Phi(A) &= a_0I + \Phi(L_1)\Phi(A_1) + \Phi(L_2)\Phi(A_2) \\ &= \begin{bmatrix} a_0 & 0 \\ 0 & a_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1(A_1) & 0 \\ * & * \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1(A_2) & 0 \\ * & * \end{bmatrix} \\ &= \begin{bmatrix} a_0 + \varphi_1(A_1) & 0 \\ \varphi_1(A_2) & a_0 \end{bmatrix} = \begin{bmatrix} \varphi_1(A) & 0 \\ \delta(A) & \varphi_0(A) \end{bmatrix}. \end{aligned}$$

In order to have a representation, it remains to verify complete contractivity.

An explicit family of such representations may be obtained as follows. Let  $\mathcal{M}_1 = \text{span}\{\xi_{z_1^k} : k \geq 0\}$  and  $\mathcal{M}_2 = \text{span}\{\xi_{z_2 z_1^k} : k \geq 0\}$ ; and set  $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ . Then  $\mathcal{M}^\perp$  is invariant for  $\mathcal{L}_2$  and  $\mathfrak{R}_2$ . Thus compression  $\Psi$  to  $\mathcal{M}$  is a WOT-continuous homomorphism. The compression to  $\mathcal{M}_1$  is a homomorphism onto  $H^\infty(L_1)$ , sending  $L_1$  to the unilateral shift as we have discussed before. The compressions of both  $L_i$  to  $\mathcal{M}$  vanish on  $\mathcal{M}_2$ , and  $L_2$  maps  $\mathcal{M}_1$  isometrically onto  $\mathcal{M}_2$ . Hence the compressions are

$$P_{\mathcal{M}}L_1|_{\mathcal{M}} \simeq \begin{bmatrix} T_z & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{M}}L_2|_{\mathcal{M}} \simeq \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}.$$

Thus  $\Psi$  maps  $\mathcal{L}_2$  onto the algebra of operators of the form

$$\begin{bmatrix} T_{h_0} & 0 \\ T_{h_1} & h_0(0) \end{bmatrix} \quad \text{for} \quad h_i \in H^\infty.$$

Indeed, this shows that every element of  $\mathcal{L}_2$  may be written uniquely as

$$A = h_0(L_1) + L_2h_1(L_1) + A' \quad \text{where} \quad h_i \in H^\infty \quad \text{and} \quad P_{\mathcal{M}}A' = 0.$$

Now let  $\psi$  be any multiplicative linear functional on  $H^\infty$  in the fibre  $\mathfrak{M}_1$ . Then  $\Phi = \psi^{(2)}\Psi$  is a completely contractive homomorphism of  $\mathcal{L}_2$  onto  $\mathcal{T}_2$  such that  $\Phi(L_i) = T_i$ . Indeed,

$$\Phi(A) = \psi^{(2)} \begin{bmatrix} T_{h_0} & 0 \\ T_{h_1} & h_0(0) \end{bmatrix} = \begin{bmatrix} \psi(h_0) & 0 \\ \psi(h_1) & h_0(0) \end{bmatrix}.$$

Hence we have shown that the fibre  $\pi_{2,2}^{-1}(T)$  is very large.

#### 4. Automorphisms of $\mathcal{L}_n$

In this section, we analyze the automorphism group of  $\mathcal{L}_n$ . The automorphisms of the algebra  $\mathcal{L}_1 = H^\infty$  are precisely the maps  $\Theta_\tau(h) = h(\tau)$  where  $\tau$  is a conformal automorphism of the unit disk. So  $\text{Aut}(\mathcal{L}_1)$  is isomorphic to  $\text{Aut}(\mathbb{B}_1)$ , the group of conformal automorphisms of the unit disk. In particular, they are

all norm and  $\text{WOT}$  continuous. See [11], where two proofs are given, both based on factorization of analytic functions. Our main result is Theorem 4.1, which is valid even for  $n = \infty$ . Our original proof of Theorem 4.1 failed when  $n = \infty$ . We are grateful to Palle Jorgensen for bringing the paper of Voiculescu [28] to our attention which enabled us to find an alternate proof which works for  $1 \leq n \leq \infty$ .

An automorphism of  $\mathfrak{L}_n$  will be called *quasi-inner* if it is trivial modulo the  $\text{WOT}$ -closed commutator ideal  $\overline{\mathfrak{C}}$  (see Proposition 2.4). Denote the set of all quasi-inner automorphisms by  $\text{q-Inn}(\mathfrak{L}_n)$ . In particular, this contains the subgroup  $\text{Inn}(\mathfrak{L}_n)$  of inner automorphisms.

**Theorem 4.1.** *There is a natural short exact sequence*

$$0 \longrightarrow \text{q-Inn}(\mathfrak{L}_n) \longrightarrow \text{Aut}(\mathfrak{L}_n) \xrightarrow{\tau} \text{Aut}(\mathbb{B}_n) \longrightarrow 0.$$

The map  $\tau$  takes  $\Theta$  to

$$\tau_\Theta(\lambda) = (\varphi_\lambda \Theta^{-1})^{(1,n)}(L) \quad \text{for } \lambda \in \mathbb{B}_n.$$

Moreover,  $\tau$  has a continuous section  $\kappa$  which carries  $\text{Aut}(\mathbb{B}_n)$  onto the subgroup  $\text{Aut}_u(\mathfrak{L}_n)$  of unitarily implemented automorphisms. Thus  $\text{Aut}(\mathfrak{L}_n)$  is a semidirect product.

The proof will be carried out in stages. First we establish an automatic continuity result for automorphisms.

**Lemma 4.2.** *Every automorphism  $\Theta$  of  $\mathfrak{L}_n$ , for  $n \geq 2$ , is continuous.*

*Proof.* The proof is a standard *gliding bump* argument. We define  $B_i = \Theta^{-1}(L_i)$  and set  $M = \max\{1, \|B_1\|, \|B_2\|\}$ . Suppose that  $\Theta$  is not continuous. Then there is a sequence  $A_k$  in  $\mathfrak{L}_n$  such that

$$\|A_k\| \leq (2M)^{-k} \quad \text{and} \quad \|\Theta(A_k)\| > k.$$

Let  $A$  be defined by the norm convergent sum

$$A = \sum_{k \geq 1} B_2^k B_1 A_k = \sum_{k=1}^m B_2^k B_1 A_k + B_2^{m+1} \sum_{k \geq 0} B_2^k B_1 A_{m+1+k}.$$

Set  $X_m = \sum_{k \geq 0} B_2^k B_1 A_{m+1+k}$ . Then for all  $k > 0$ ,

$$\begin{aligned} \|\Theta(A)\| &\geq \|L_1^* L_2^{*m} \Theta(A)\| \\ &= \left\| \sum_{k=1}^m L_1^* L_2^{*m} L_2^k L_1 \Theta(A_k) + L_1^* L_2^{*m} L_2^{m+1} \Theta(X_m) \right\| \\ &= \|\Theta(A_k)\| > k. \end{aligned}$$

This is absurd, and consequently  $\Theta$  must be continuous. □

Next we show that every automorphism determines a special point in the ball.

**Proposition 4.3.** *Let  $\Theta$  be an automorphism of  $\mathfrak{L}_n$ . Then there is a unique point  $\lambda$  in  $\mathbb{B}_n$  such that  $\Theta(\mathfrak{L}_n^0) = \ker \varphi_\lambda$ . Indeed,  $\varphi_\lambda = \varphi_0 \Theta^{-1}$ .*

*Proof.* Let

$$S = \Theta^{(1,n)}(L) := [ S_1 \quad \dots \quad S_n ] .$$

By Corollaries 2.6 and 2.9,  $\mathfrak{L}_n = \mathbb{C}I + \mathcal{L}\mathcal{C}_n(\mathfrak{L}_n)$ , and this decomposition is unique. Applying  $\Theta$  yields  $\mathfrak{L}_n = \mathbb{C}I + S\mathcal{C}_n(\mathfrak{L}_n)$ , and every  $A$  in  $\mathfrak{L}_n$  has a unique decomposition as  $A = \alpha I + SB$  for some  $B$  in  $\mathcal{C}_n(\mathfrak{L}_n)$ . Hence the continuous linear map  $T$  from  $\mathbb{C} \oplus \mathcal{C}_n(\mathfrak{L}_n)$  to  $\mathfrak{L}_n$  given by

$$T(\alpha, B) = \alpha I + SB$$

is a bijection. By Banach’s Isomorphism Theorem,  $T$  is invertible. So there is a constant  $c > 0$  so that

$$c^{-1} \|\alpha I + B^*B\|^{1/2} \leq \|\alpha I + SB\| \leq c \|\alpha I + B^*B\|^{1/2} .$$

Let  $\mathfrak{J} = S\mathcal{C}_n(\mathfrak{L}_n) = \Theta(\mathfrak{L}_n^0)$ . Since  $T$  maps a subspace of codimension one onto  $\mathfrak{J}$ , it also has codimension one. We claim that this ideal is WOT-closed. Suppose that  $J_\beta = SB_\beta$  is a bounded net in  $\mathfrak{J}$  which converges weak-\* to an operator  $X$  in  $\mathfrak{L}_n$ . Then the net  $B_\beta$  is bounded in  $\mathcal{C}_n(\mathfrak{L}_n)$  by the previous paragraph. Hence there is a cofinal subnet  $B_{\beta'}$  which converges weak-\* to an operator  $B$  in  $\mathcal{C}_n(\mathfrak{L}_n)$ . Consequently, it follows that  $X = SB$  belongs to  $\mathfrak{J}$ . This shows that the intersection of  $\mathfrak{J}$  with each closed ball is weak-\* closed. By the Krein–Smulian Theorem (c.f. [7, V.5.7]),  $\mathfrak{J}$  is weak-\* closed. By Corollary 1.4, the weak-\* and WOT topologies coincide on  $\mathfrak{L}_n$ . Hence  $\mathfrak{J}$  is WOT-closed.

Consider the multiplicative linear functional  $\varphi = \varphi_0 \Theta^{-1}$ , which yields the formula  $\varphi(\alpha I + SB) = \alpha$ . Since  $\mathfrak{J}$  is WOT-closed, this functional is WOT-continuous. Therefore by Theorem 2.3, there is a point  $\lambda$  in  $\mathbb{B}_n$  such that  $\varphi = \varphi_\lambda$ .  $\square$

Next we will show that automorphisms of  $\mathfrak{L}_n$  are automatically WOT-continuous. First we establish a criterion for WOT-convergence in  $\mathfrak{L}_n$ . Recall that  $\mathfrak{L}_n^{0,k}$  is the ideal generated by  $\{L_w : |w| = k\}$ .

**Lemma 4.4.** *For a bounded net  $A_\alpha$  in  $\mathfrak{L}_n$ ,  $n < \infty$ , the following are equivalent:*

- (i)  $\text{WOT-}\lim A_\alpha = 0$ .
- (ii)  $w\text{-}\lim_\alpha A_\alpha \xi_1 = 0$ .
- (iii)  $\lim_\alpha \text{dist}(A_\alpha, \mathfrak{L}_n^{0,k}) = 0$  for all  $k \geq 1$ .

*Proof.* It is evident that (i) implies (ii). If (ii) holds, then write

$$A_\alpha \xi_1 = \sum_w a_w^\alpha \xi_w .$$

Then  $A_{\alpha,k} := A_\alpha - \sum_{|w| < k} a_w^\alpha L_w$  belongs to  $\mathfrak{L}_n^{0,k}$  by Lemma 2.6. Condition (ii) clearly implies that  $\lim_\alpha a_w^\alpha = 0$  for every  $w$ . Hence

$$\limsup_{\alpha} \text{dist}(A_{\alpha}, \mathfrak{L}_n^{0,k}) \leq \limsup_{\alpha} \|A_{\alpha} - A_{\alpha,k}\| \leq \limsup_{\alpha} \sum_{|w|<k} |a_w^{\alpha}| = 0$$

for every  $k \geq 1$ . Now if (iii) holds, then

$$0 = \lim_{\alpha} \text{dist}(A_{\alpha}, \mathfrak{L}_n^{0,k}) \geq \limsup_{\alpha} \text{dist}(A_{\alpha}\xi_1, \mathfrak{L}_n^{0,k}\xi_1) = \left( \sum_{|w|<k} |a_w^{\alpha}|^2 \right)^{1/2}.$$

A fortiori,  $\lim_{\alpha} a_w^{\alpha} = 0$  for every  $w$  in  $\mathcal{F}_n$ . Therefore

$$\lim_{\alpha} (A_{\alpha}\xi_u, \xi_v) = \lim_{\alpha} (A_{\alpha}\xi_1, R_u^*\xi_v) = 0$$

for every pair of words  $u, v$  in  $\mathcal{F}_n$ . These vectors span a dense subset of  $\mathcal{H}_n$ . As the net  $A_{\alpha}$  is bounded, it converges WOT to 0.  $\square$

*Remark 4.5.* When  $n = \infty$ , it is easy to see that condition (iii) is stronger than WOT-convergence. However, conditions (i) and (ii) are still equivalent, and the proof is straight-forward.

**Theorem 4.6.** *Every automorphism  $\Theta$  of  $\mathfrak{L}_n$  is WOT-continuous.*

*Proof.* By Lemma 4.3, there is a point  $\lambda$  in  $\mathbb{B}_n$  such that  $\varphi_0\Theta^{-1} = \varphi_{\lambda}$ . Thus

$$\mathfrak{J} = \Theta(\mathfrak{L}_n^0) = \ker \varphi_{\lambda}.$$

Hence

$$\Theta(\mathfrak{L}_n^{0,k}) = \Theta(\mathfrak{L}_n^0)^k = \mathfrak{J}^k \quad \text{for all } k \geq 1.$$

Clearly  $\bigcap_{k \geq 1} \mathfrak{J}^k = \{0\}$  since

$$\Theta^{-1}(\bigcap_{k \geq 1} \mathfrak{J}^k) \subseteq \Theta^{-1}(\mathfrak{J}^k) = \mathfrak{L}_n^{0,k} \quad \text{for all } k \geq 1.$$

Thus by Theorem 2.1, we have

$$(3) \quad \bigcap_{k \geq 1} \overline{\mathfrak{J}^k \mathcal{H}_n} = \{0\}.$$

Set  $\zeta_w = \Theta(L_w)\xi_1$  for  $w \in \mathcal{F}_n$ . Fix an integer  $j$  and let  $\mathcal{M}_j$  and  $\mathcal{N}_j$  be the closed linear spans of  $\{\xi_w : |w| = j\}$  and  $\{\zeta_w : |w| = j\}$  respectively. If  $\beta = \sum_{|w|=j} b_w \xi_w$  is a finite linear combination of the  $\xi_w$ , put  $B = \sum_{|w|=j} b_w L_w$ . Then since  $\Theta$  is bounded,

$$\left\| \sum_{|w|=j} b_w \zeta_w \right\| = \|\Theta(B)\xi_1\| \leq \|\Theta\| \|B\| = \|\Theta\| \|\beta\|.$$

Thus the map  $\sum_{|w|=j} b_w \xi_w \mapsto \sum_{|w|=j} b_w \zeta_w$  extends to a bounded linear operator  $T_j : \mathcal{M}_j \rightarrow \mathcal{N}_j$ .

Now consider a bounded net  $B_{\alpha} = \sum_{|w|=j} b_w^{\alpha} L_w$  such that  $\lim_{\alpha} b_w^{\alpha} = 0$  for all  $w$ . Let  $\beta_{\alpha} = \sum_{|w|=j} b_w^{\alpha} \xi_w$ . It follows that

$$\text{w-}\lim_{\alpha} \Theta(B_{\alpha})\xi_1 = \text{w-}\lim_{\alpha} T_j \beta_{\alpha} = T_j \text{w-}\lim_{\alpha} \beta_{\alpha} = 0.$$

Hence by Remark 4.5,  $\Theta(B_\alpha)$  converges WOT to 0.

Again let  $A_\alpha$  be a bounded net converging WOT to 0 in  $\mathfrak{L}_n$  and let  $M = \sup_\alpha \|A_\alpha\|$ . By Remark 4.5, it suffices to show that

$$\lim_\alpha (\Theta(A_\alpha)\xi_1, \zeta) = 0$$

for  $\zeta$  in a dense subset of  $\mathcal{H}_n$ . A convenient choice is  $\cup_{k \geq 1} (\mathfrak{J}^k \mathcal{H}_n)^\perp$ , which is dense by the equality (3). Choose  $\zeta$  in  $(\mathfrak{J}^k \mathcal{H}_n)^\perp$ , and set  $p = k^2$ . Decompose  $A_\alpha = B_\alpha + C_\alpha$  where

$$B_\alpha = \Sigma_p(A_\alpha) + \sum_{|w| < k} \frac{|w|}{p} a_w^\alpha L_w \quad \text{and} \quad C_\alpha = A_\alpha - B_\alpha \in \mathfrak{L}_n^k.$$

Since the Cesaro mean  $\Sigma_p$  is contractive, it follows that  $\|\Sigma_p(A_\alpha)\| \leq M$ . Also the terms  $A_{\alpha,j} = \sum_{|w|=j} a_w^\alpha L_w$  are bounded by  $M$ , and thus

$$\|B_\alpha\| \leq M + p^{-1} \sum_{j=1}^{k-1} j \|A_{\alpha,j}\| \leq 2M.$$

Hence  $\|C_\alpha\| \leq 3M$ . Moreover each net  $B_\alpha$  and  $C_\alpha$  converge WOT to 0.

Now since  $B_\alpha$  is supported on words of length less than  $p$ , we have seen that  $\Theta(B_\alpha)$  converges WOT to 0. Finally by construction,

$$(\Theta(C_\alpha)\xi_1, \zeta) = 0$$

since  $\Theta(D_\alpha)\xi_1$  lies in  $\mathfrak{J}^k \mathcal{H}_n$ , which is orthogonal to  $\zeta$ .

The weak-\* topology on the unit ball of  $\mathcal{B}(\mathcal{H}_n)$  is metrizable, and the ball is compact. Thus it follows readily that a linear map  $\Theta$  is weak-\* continuous on a bounded convex set if and only if it takes every weak-\* null sequence to a weak-\* null sequence. Hence we see that  $\Theta$  is weak-\* continuous on every closed ball of  $\mathfrak{L}_n$ . Therefore by the Krein–Smulian Theorem (c.f [7, V.5.6]), it follows that  $\Theta$  is weak-\* continuous. By Corollary 1.4, the weak-\* and WOT topologies coincide on  $\mathfrak{L}_n$ . Thus  $\Theta$  is WOT-continuous.  $\square$

The tools are now available to define the map  $\tau$  given in Theorem 4.1. Using Theorem 3.3, we identify  $\mathbb{B}_n$  with  $\text{Rep}_1(\mathfrak{L}_n)$  by associating  $\lambda$  in  $\mathbb{B}_n$  with the multiplicative linear functional  $\varphi_\lambda$  in  $\text{Rep}_1(\mathfrak{L}_n)$ .

**Theorem 4.7.** *For each  $\Theta$  in  $\text{Aut}(\mathfrak{L}_n)$ , the dual map  $\tau_\Theta$  on  $\text{Rep}_1(\mathfrak{L}_n)$  given by  $\tau_\Theta(\varphi) := \varphi \circ \Theta^{-1}$  maps the open ball  $\mathbb{B}_n$  conformally onto itself. This determines a homomorphism of  $\text{Aut}(\mathfrak{L}_n)$  into the group  $\text{Aut}(\mathbb{B}_n)$  of conformal automorphisms. If  $\tau_\Theta(\varphi_0) = \varphi_0$ , then there is a unitary matrix  $U$  in  $\mathcal{U}_n$  such that  $\tau_\Theta(\varphi_\lambda) = \varphi_{U\lambda}$ .*

*Proof.* Since  $\Theta$  is WOT-continuous by Theorem 4.6, it follows that  $\tau_\Theta(\varphi_\lambda) = \varphi_\lambda \circ \Theta^{-1}$  is a WOT-continuous multiplicative linear functional. Hence by Theorem 2.3, this is a functional  $\varphi_\mu$ . Thus  $\tau_\Theta$  maps  $\mathbb{B}_n$  into itself. We obtain an explicit formula for this map using the fact that  $\varphi_\lambda^{(1,n)}(L) = \lambda$ , whence



$$\tau_{\Theta}(\lambda) = (\varphi_{\lambda} \Theta^{-1})^{(1,n)}(L) = \varphi_{\lambda}^{(1,n)}(T) = \widehat{T}(\lambda),$$

where

$$T = (\Theta^{-1})^{(1,n)}(L) = [ \Theta^{-1}(L_1) \quad \dots \quad \Theta^{-1}(L_n) ].$$

By Theorem 3.3,  $\widehat{T}$  is analytic and thus so is  $\tau_{\Theta}$ .

Next notice that the map  $\tau$  taking  $\Theta$  to  $\tau_{\Theta}$  is a homomorphism. It is evident that  $\tau_{\text{id}} = \text{id}$ ; that is, the identity automorphism induces the identity map on  $\mathbb{B}_n$ . Suppose that  $\Theta_j$  belong to  $\text{Aut}(\mathfrak{L}_n)$ , and  $\tau_j = \tau(\Theta_j)$  for  $j = 1, 2$ . Then

$$\begin{aligned} \tau(\Theta_1 \Theta_2)(\lambda) &= (\varphi_{\lambda} (\Theta_1 \Theta_2)^{-1})^{(1,n)}(L) = (\varphi_{\lambda} \Theta_2^{-1} \Theta_1^{-1})^{(1,n)}(L) \\ &= (\varphi_{\tau_2(\lambda)} \Theta_1^{-1})^{(1,n)}(L) = (\varphi_{\tau_1(\tau_2(\lambda))})^{(1,n)}(L) = \tau_1(\tau_2(\lambda)). \end{aligned}$$

Hence  $\tau(\Theta_1 \Theta_2) = \tau(\Theta_1) \circ \tau(\Theta_2)$ .

Consequently

$$\tau_{\Theta} \tau_{\Theta^{-1}} = \text{id} = \tau_{\Theta^{-1}} \tau_{\Theta};$$

from which we deduce that  $\tau_{\Theta}$  is a bijection. Therefore  $\tau_{\Theta}$  is a biholomorphic bijection (i.e. a conformal automorphism) of the ball.

If  $\tau$  is a conformal automorphism of  $\mathbb{B}_n$  such that  $\tau(0) = 0$ , then by Schwarz's Lemma, there is a unitary operator  $U$  in  $\mathcal{U}_n$  such that  $\tau(\lambda) = U\lambda$  [25, 2.2.5] for  $n < \infty$  and [12] for  $n = \infty$ .  $\square$

The following corollary characterizes the quasi-inner automorphisms.

**Corollary 4.8.** *For  $\Theta$  in  $\text{Aut}(\mathfrak{L}_n)$ , the following are equivalent:*

- (i)  $\Theta$  belongs to  $\ker \tau$ .
- (ii)  $\Theta(L_i) - L_i$  belongs to  $\overline{\mathfrak{C}}$  for  $1 \leq i \leq n$ .
- (iii)  $\Theta(A) - A$  belongs to  $\overline{\mathfrak{C}}$  for every  $A$  in  $\mathfrak{L}_n$ .

*Proof.* If  $\Theta$  belongs to  $\ker \tau$ , then so does  $\Theta^{-1}$ ; whence

$$\varphi_{\lambda}(\Theta(L_i) - L_i) = \tau_{\Theta^{-1}}(\lambda) - \lambda$$

is zero for every  $\lambda$  in  $\mathbb{B}_n$  if and only if  $\Theta(L_i) - L_i$  belongs to  $\bigcap_{\lambda \in \mathbb{B}_n} \ker \varphi_{\lambda}$  for  $1 \leq i \leq n$ . But this set equals  $\overline{\mathfrak{C}}$  by Proposition 2.4. So (i) and (ii) are equivalent.

Suppose that (ii) holds. As  $\overline{\mathfrak{C}}$  is an ideal, it readily follows that  $\Theta(p(L)) - p(L)$  belongs to  $\overline{\mathfrak{C}}$  for every polynomial in  $L$ . Then because  $\Theta$  is WOT-continuous and  $\overline{\mathfrak{C}}$  is WOT-closed, this extends to the WOT-closure of these polynomials, which is all of  $\mathfrak{L}_n$ . This establishes (iii). Clearly (iii) implies (ii).  $\square$

To complete the picture, we need to construct explicit automorphisms to demonstrate that the map  $\tau$  is surjective. In fact, much more will be established. An explicit section of  $\tau$  will be found that maps  $\text{Aut}(\mathbb{B}_n)$  onto the subgroup  $\text{Aut}_u(\mathfrak{L}_n)$  of unitarily implemented automorphisms. This will establish that  $\text{Aut}(\mathfrak{L}_n)$  actually has the structure of a semidirect product.

A certain class of unitarily implemented automorphisms of  $\mathfrak{L}_n$  are well known from quantum mechanics, and are called *gauge automorphisms*. Think of  $\mathcal{H}_n$  as

being identified with the Fock space  $F(\mathcal{H})$  with  $\dim \mathcal{H} = n$ . For any unitary  $U$  on  $\mathcal{H}$ , form the unitary operator

$$\tilde{U} = \sum_{k \geq 0} \oplus U^{\otimes k}$$

which acts on Fock space by acting as the  $k$ -fold tensor product of  $U$  on the  $k$ -fold tensor product of  $\mathcal{H}$ . It is evident that

$$\tilde{U} \ell(\zeta) \tilde{U}^* = \ell(U\zeta) \quad \text{for } \zeta \in \mathcal{H}.$$

Therefore  $\Theta_U = \text{Ad} \tilde{U}$  determines an automorphism of  $\mathcal{L}_n$ . If  $U = [u_{ij}]$  is an  $n \times n$  unitary matrix, this automorphism can also be seen to be given by

$$\Theta_U(L_j) = \sum_{i=1}^n u_{ij} L_i \quad \text{for } 1 \leq j \leq n.$$

An easy calculation shows that  $\Theta_U \Theta_V = \Theta_{UV}$ ; so this is a homomorphism of the unitary group  $\mathcal{U}_n$  into the automorphism group  $\text{Aut}(\mathcal{L}_n)$ . It follows from Lemma 4.10 below that  $\tau \Theta_U = \bar{U}$ , the coordinatewise conjugate of  $U$ . So  $\tau$  maps the group of gauge automorphisms onto the unitary group.

In [28], Voiculescu constructed a larger subgroup of automorphisms of the Cuntz–Toeplitz algebra  $\mathcal{E}_n$  which turn out to be the one we want. He starts with the group  $U(1, n)$  consisting of those  $(n + 1) \times (n + 1)$  matrices  $X$  such that  $X^* J X = J$ , where  $J = \begin{bmatrix} 1 & 0 \\ 0 & I_n \end{bmatrix}$ . One may compute that these matrices have the

form  $X = \begin{bmatrix} x_0 & \eta_1^* \\ \eta_2 & X_1 \end{bmatrix}$  where the coefficients satisfy the (redundant) relations:

- (i)  $\|\eta_1\|^2 = \|\eta_2\|^2 = |x_0|^2 - 1$
- (ii)  $X_1 \eta_1 = \bar{x}_0 \eta_2$  and  $X_1^* \eta_2 = x_0 \eta_1$
- (iii)  $X_1^* X_1 = I_n + \eta_1 \eta_1^*$  and  $X_1 X_1^* = I_n + \eta_2 \eta_2^*$ .

Let us write  $L_\zeta = \sum_{i=1}^n \zeta_i L_i$  for  $\zeta \in \mathbb{C}^n$ . Voiculescu shows that there is a (unique) automorphism  $\Theta_X$  of  $\mathcal{E}_n$  such that the restriction to the generators is given by

$$\Theta_X(L_\zeta) = (x_0 I - L_{\eta_2})^{-1} (L_{X_1 \zeta} - \langle \zeta, \eta_1 \rangle I).$$

It is easy to verify that the kernel of this map consists of the scalar matrices  $x_0 I_{n+1}$  for  $x_0$  in the circle  $\mathbb{T}$ . Moreover Voiculescu constructs a unitary operator  $U_X$  by

$$U_X(A \xi_1) = \Theta_X(A) (x_0 I - L_{\eta_2})^{-1} \xi_1 \quad \text{for all } A \in \mathcal{L}_n$$

so that  $\Theta_X(A) = U_X A U_X^*$  for all  $A$  in  $\mathcal{L}_n$ .

It is apparent that the norm-closed (nonself-adjoint) algebra  $\mathfrak{A}_n$  generated by  $\{L_i : 1 \leq i \leq n\}$  is mapped into itself by this map. Since this is a group homomorphism, it maps  $\mathfrak{A}_n$  onto itself. Then because  $\Theta_X$  is unitarily implemented, it is WOT-continuous and thus determines an automorphism of  $\mathcal{L}_n$ . We will provide

discussion below to indicate another method of obtaining these automorphisms that fits into our framework somewhat better.

There is also a natural map from  $U(1, n)$  onto  $\text{Aut}(\mathbb{B}_n)$  by fractional linear maps. This result must be well known. We do not have a reference, but the results of Phillips [15] on symplectic automorphisms of the ball of  $\mathcal{B}(\mathcal{H})$  may be modified to apply to the ball of  $\mathcal{B}(\mathcal{H}, \mathcal{K})$  for Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ . Taking  $\mathcal{H} = \mathbb{C}^n$  and  $\mathcal{K} = \mathbb{C}$  yields our map.

**Lemma 4.9.** *For  $X$  in  $U(1, n)$ , define a map  $\theta_X : \mathbb{B}_n \rightarrow \mathbb{C}^n$  by*

$$\theta_X(\lambda) = \frac{X_1 \lambda + \eta_2}{x_0 + \langle \lambda, \eta_1 \rangle} \quad \text{for } \lambda \in \mathbb{B}^n.$$

*Then  $\theta_X$  belongs to  $\text{Aut}(\mathbb{B}_n)$  and the associated map  $\theta : U(1, n) \rightarrow \text{Aut}(\mathbb{B}_n)$  is a surjective homomorphism with kernel equal to the scalars.*

*Proof.* First one computes using (i) and (ii) above:

$$\begin{aligned} & |x_0 + \langle \lambda, \eta_1 \rangle|^2 - \|X_1 \lambda + \eta_2\|^2 \\ &= |x_0|^2 + |\langle \lambda, \eta_1 \rangle|^2 - \|X_1 \lambda\|^2 - \|\eta_2\|^2 + 2\text{Re}(\langle \lambda, x_0 \eta_1 \rangle - \langle X_1 \lambda, \eta_2 \rangle) \\ &= (|x_0|^2 - \|\eta_2\|^2) - (\|X_1 \lambda\|^2 - |\langle \lambda, \eta_1 \rangle|^2) + 2\text{Re}\langle \lambda, x_0 \eta_1 - X_1^* \eta_2 \rangle \\ &= 1 - |\lambda|^2. \end{aligned}$$

Thus this map carries  $\mathbb{B}_n$  onto itself, and so belongs to  $\text{Aut}(\mathbb{B}_n)$ .

A straightforward calculation shows that this map is a group homomorphism. Again the kernel of this map is the circle of scalar matrices in  $U(1, n)$ . The unitary operator  $X = \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}$  is sent to  $U$ . Now  $\theta_X(0) = x_0^{-1} \eta_2$  is an arbitrary point in the ball. Hence the range of  $\theta$  is a transitive subgroup of  $\text{Aut}(\mathbb{B}_n)$  containing the unitary group. By Schwarz's lemma [25, 12], the range is the whole group of conformal automorphisms.  $\square$

To see the relationship between  $\theta$  and  $\tau$ , we make the following computation.

**Lemma 4.10.**  $\tau(\theta_X) = \theta(\bar{X})$  for all  $X$  in  $U(1, n)$ .

*Proof.* Compute for  $X = \begin{bmatrix} x_0 & \eta_1^* \\ \eta_2 & X_1 \end{bmatrix}$  that

$$X^{-1} = JX^*J = \begin{bmatrix} \bar{x}_0 & -\eta_2^* \\ -\eta_1 & X_1^* \end{bmatrix}.$$

Therefore if  $e_i$  form the standard basis for  $\mathbb{C}^n$ , then

$$\begin{aligned} \tau(\theta_X)(\lambda) &= [\varphi_\lambda \theta_{X^{-1}}(L_i)] \\ &= [\varphi_\lambda ((\bar{x}_0 I + L_{\eta_1})^{-1} (L_{X_1^* e_i} + \langle e_i, \eta_2 \rangle))] \\ &= (\bar{x}_0 + \langle \lambda, \bar{\eta}_1 \rangle)^{-1} \sum_{i=1}^n (\langle \lambda, \bar{X}_1^* e_i \rangle + \langle e_i, \eta_2 \rangle) e_i \end{aligned}$$

$$\begin{aligned}
 &= (\bar{x}_0 + \langle \lambda, \bar{\eta}_1 \rangle)^{-1} \sum_{i=1}^n (\langle \bar{X}_1 \lambda, e_i \rangle + \langle \bar{\eta}_2, e_i \rangle) e_i \\
 &= (\bar{x}_0 + \langle \lambda, \bar{\eta}_1 \rangle)^{-1} (\bar{X}_1 \lambda + \bar{\eta}_2) = \theta_{\bar{X}}(\lambda). \quad \blacksquare
 \end{aligned}$$

**Theorem 4.11.** *The restriction of  $\tau$  to the subgroup  $\text{Aut}_u(\mathfrak{L}_n)$  of unitarily implemented automorphisms is an isomorphism onto  $\text{Aut}(\mathbb{B}_n)$ .*

*Proof.* Define a map  $\kappa : \text{Aut}(\mathbb{B}_n) \rightarrow \text{Aut}_u(\mathfrak{L}_n)$  as follows: given  $\alpha$  in  $\text{Aut}(\mathbb{B}_n)$ , pick  $X \in U(1, n)$  belonging to  $\theta^{-1}(\alpha)$  and set  $\kappa(\alpha) = \Theta_{\bar{X}}$ . This is a well defined monomorphism because  $\Theta$  and  $\theta$  have the same kernel and complex conjugation is an automorphism of  $U(1, n)$ . By the previous lemma, it follows that  $\tau\kappa$  is the identity on  $\text{Aut}(\mathbb{B}_n)$ . In particular,  $\tau$  restricted to  $\text{Aut}_u(\mathfrak{L}_n)$  is a surjective homomorphism.

To prove that this map is injective, suppose that  $\Theta$  is a unitarily implemented automorphism such that  $\tau(\Theta) = \text{id}$ . A fortiori,  $\Theta$  is contractive. But  $\Theta(L_i) = L_i + C_i$  where  $C_i \in \bar{\mathfrak{C}}$ ; whence

$$1 \geq \|\Theta\|^2 \geq \|(L_i + C_i)\xi_1\|^2 = 1 + \|C_i\xi_1\|^2.$$

Consequently,  $C_i\xi_1 = 0$  which implies that  $C_i = 0$ . Therefore  $\Theta = \text{Id}$  and our map is an isomorphism.  $\square$

We record an immediate consequence of the proof.

**Corollary 4.12.** *Every contractive automorphism of  $\mathfrak{L}_n$  is unitarily implemented. In particular, it is completely isometric.*

It would be interesting to know if automorphisms of  $\mathfrak{L}_n$  are automatically completely bounded.

All the necessary parts for Theorem 4.1 have now been accumulated. The homomorphism  $\tau$  is now known to be surjective, with kernel  $\text{q-Inn}(\mathfrak{L}_n)$  and a continuous section  $\kappa$  onto  $\text{Aut}_u(\mathfrak{L}_n)$  as required.

*Remark 4.13.* We have another method for obtaining the unitarily implemented automorphisms which map onto the group of conformal automorphisms. We need to construct one such that  $\tau_{\Theta}(0) = \lambda$  for each  $\lambda$  in  $\mathbb{B}_n$ . This shows that the image group is transitive. Then since the gauge automorphisms map onto the unitary group, it will follow that the map is surjective.

Notice that if  $\Theta$  is unitarily implemented, then  $S_i = \Theta^{-1}(L_i)$  will be isometries with pairwise orthogonal ranges. They generate the ideal

$$\sum_{i=1}^n S_i \mathfrak{L}_n = \Theta^{-1}(\mathfrak{L}_n^0).$$

This is a WOT-closed two-sided ideal of codimension one, and thus by Theorem 2.1 its range is a  $\mathfrak{L}_n \mathfrak{R}_n$  invariant subspace of codimension one. The complement is a one-dimensional invariant subspace for  $\mathfrak{L}_n^*$ , and thus by Theorem 1.3 is spanned by  $\nu_\lambda$  for some  $\lambda$  in  $\mathbb{B}_n$ . It is easy to check that  $\tau_{\Theta}(0) = \lambda$ .

Conversely, given  $\lambda$ , we can construct such isometries. By Theorem 1.3, the subspace  $\{\nu_\lambda\}^\perp$  is  $\mathfrak{A}_n$  invariant and has an  $n$ -dimensional wandering space  $\mathcal{W}_\lambda$ . Let  $\zeta_i$  for  $1 \leq i \leq n$  be an orthonormal basis for  $\mathcal{W}_\lambda$ . Then by [6, Theorem 1.1], the operators  $S_i = L_{\zeta_i}$  are isometries in  $\mathfrak{L}_n$  with ranges summing to  $\{\nu_\lambda\}^\perp$ . We will sketch how to construct the automorphism  $\Theta$  which takes  $L_i$  to  $S_i$  for  $1 \leq i \leq n$ .

The first step is to show that  $\nu_\lambda$  is cyclic for the WOT-closed subalgebra  $\mathfrak{A}$  generated by  $\{S_1, \dots, S_n\}$ . This is established by showing that  $\zeta_w = w(S)\nu_\lambda$ ,  $w \in \mathcal{F}_n$ , is an orthonormal basis for  $\mathcal{H}_n$ . This immediately yields a unitary operator  $W$  such that  $WL_iW^* = S_i$  such that  $\text{Ad}W$  is an endomorphism of  $\mathfrak{L}_n$ .

The second step is to show that  $\mathfrak{A} = \mathfrak{L}_n$ . Since it is contained in  $\mathfrak{L}_n$ , we see that  $\nu_0 = \xi_1$  is an eigenvalue for  $\mathfrak{A}^*$ . Since  $\mathfrak{A}$  is unitarily equivalent to  $\mathfrak{L}_n$ , there is a non-zero  $\mu$  such that  $W\nu_\mu = \xi_1$ . Apply the argument again to obtain a second unitary  $W'$  so that  $\text{Ad}W'W(L_i) = S'_i = L_{\zeta'_i}$  where  $\zeta'_i$  form an orthonormal basis for the wandering space of  $\{\xi_1\}^\perp$ . But then (when  $n < \infty$ ) there is a unitary  $U$  in  $\mathcal{U}_n$  such that  $\zeta'_i = Ue_i = U\xi_{z_i}$ . Unfortunately, this argument fails for  $n = \infty$ . Consequently, it follows that  $\text{Ad}W'W = \Theta_U$ . Thus the two endomorphisms  $\text{Ad}W$  and  $\text{Ad}W'$  must have been automorphisms.

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