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DECOMPOSITIONS OF MODULES AND MATRICES

BY THOMAS S. SHORES AND ROGER WIEGAND¹

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ABSTRACT A canonical form for a module M over a commutative ring R is a decomposition $M \cong R/I_1 \oplus \cdots \oplus R/I_n$, where the I_j are ideals of R and $I_1 \subseteq \cdots \subseteq I_n$. A complete structure theory is developed for those rings for which every finitely generated module has a canonical form. The (possibly larger) class of rings, for which every finitely generated module is a direct sum of cyclics, is also considered, and partial results are obtained for rings with fewer than 2^c prime ideals. For example, if R is countable and every finitely generated R -module is a direct sum of cyclics, then R is a principal ideal ring. Finally, some topological criteria are given for Hermite rings and elementary divisor rings.

All rings in this announcement are commutative with 1, and all modules are unital. A *canonical form* for an R -module M is a decomposition $M \cong R/I_1 \oplus \cdots \oplus R/I_n$, where $I_1 \subseteq \cdots \subseteq I_n \neq R$. If M has a canonical form, the ideals I_j are uniquely determined [K]. A CF-ring is a ring for which every finitely generated direct sum of cyclics has a canonical form. It can be shown that R is CF if and only if

$$R/I \oplus R/J \cong R/(I \cap J) \oplus R/(I + J)$$

for every pair of ideals I, J .

By a valuation ring we shall mean a ring, possibly with zero-divisors, whose lattice of ideals is totally ordered. A ring R is arithmetical, provided the local ring $R_{\mathfrak{m}}$ is a valuation ring for each maximal ideal \mathfrak{m} . Finally, an h -local domain [M1] is an integral domain such that (1) every nonzero ideal is contained in only finitely many maximal ideals, and (2) every nonzero prime ideal is contained in a unique maximal ideal.

THEOREM 1. *Every CF-ring is a finite direct product of indecomposable CF-rings. The indecomposable CF-rings are precisely the rings R such that (i) R is arithmetical, (ii) R has a unique minimal prime P , (iii) R/P is an h -local domain, and (iv) every ideal contained in P is comparable with every ideal of R .*

Thus valuation rings and arithmetical h -local domains are CF-rings.

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The next theorem gives a fairly complete picture of the remaining indecomposable CF-rings.

THEOREM 2. *Let R be an indecomposable CF-ring with minimal prime P . Suppose R is neither a valuation ring nor an h -local domain. Then $P^2 = 0$, P is an indecomposable, torsion, divisible R/P -module, and $P = P_{\mathfrak{m}}$ for a unique maximal ideal \mathfrak{m} .*

EXAMPLE. Let A be an arithmetical h -local domain (for example a Dedekind domain) with more than one maximal ideal, and let K be the quotient field of A . Let \mathfrak{m} be any maximal ideal of A , let $P = K/A_{\mathfrak{m}}$, and make $R = A \oplus P$ into a ring by setting $(a, p)(a', p') = (aa', ap' + a'p)$. Then R is a CF-ring which is neither a domain nor a valuation ring.

Let us define an FGC-ring to be a ring R such that every finitely generated R -module is a direct sum of cyclic modules. If, in addition, R is CF, we say R is an FGCF-ring. The local FGC-rings have been characterized as the almost maximal valuation rings [G]. Thus, if R is FGC, it follows that $R_{\mathfrak{m}}$ is almost maximal for every \mathfrak{m} , and it is not hard to see that R is Bezout, that is, every finitely generated ideal of R is principal. In general, these two conditions are far from sufficient. Surprisingly, CF-rings have the requisite “finiteness” conditions:

THEOREM 3. *A CF-ring R is FGCF if and only if R is Bezout and $R_{\mathfrak{m}}$ is almost maximal for each maximal ideal \mathfrak{m} .*

If, in the example above, $A_{\mathfrak{m}}$ is maximal and all the other localizations of A are almost maximal, the ring $R = A \oplus P$ can be shown to have almost maximal localizations. If, in addition, A is Bezout (for example, a ring of type I [M2]) then R is an FGCF-ring.

A direct proof of Theorem 3 would appear to be difficult. Theorems 1 and 2, however, reduce the task to consideration of the three types of indecomposable CF-rings. Only the third type presents any difficulty, and it is handled by techniques similar to those in [G].

The problem of classifying FGC-rings seems to be much harder. It would be a great help to know that an FGC-ring has only finitely many minimal primes. (Indeed, this was a major step in the characterization of CF-rings.) We have some partial results in this direction:

THEOREM 4. *Let R be an FGC-ring. Then every compact set of minimal primes is finite. If R has fewer than 2^c prime ideals, then R has only finitely many minimal primes.*

COROLLARY. *If R is an FGC-ring with fewer than 2^c prime ideals then R has noetherian maximal ideal space.*

COROLLARY. *Every countable FGC-ring is a principal ideal ring.*

To prove Theorem 4, one uses methods similar to those in [P] to show that a fairly benign condition on $\text{spec}(R)$, the prime spectrum of R , prevents FGC. The statement of this condition involves two topologies on $\text{spec}(R)$ —the usual (Zariski) topology, and the patch topology, which is generated by declaring that the quasicompact open sets in the usual topology shall be both open and closed.

THEOREM 5. *Suppose $\text{spec}(R)$ has a point in the patch closure of each of three pairwise disjoint Zariski open sets. Then R is not an FGC-ring.*

We know of no examples of FGC-rings which are not already FGCF. Nonetheless it seems plausible that a semilocal, locally almost maximal domain might have FGC without being h -local.

When we restrict our attention to decompositions of finitely presented modules, we are led inevitably to questions about diagonalization of matrices. A module M with m generators and n relations is isomorphic to R^m/K , where K is generated by the columns of an $m \times n$ matrix A . In this case we say M is *presented* by A . It is well known that if A and B are equivalent then they present isomorphic modules, but that the converse fails, even if A and B have the same size. It has recently been shown, however, that if every finitely presented R -module is a direct sum of cyclics then every matrix over R is equivalent to a diagonal matrix [LLS].

The situation is somewhat different for canonical forms. We say a matrix A has a canonical form provided A is equivalent to a diagonal matrix $[d_{ij}]$ in which $d_{i+1,i+1}$ divides $d_{i,i}$ for all i . In [LLS] it was proved that R is Bezout if and only if every diagonal matrix over R has a canonical form. On the other hand, R is arithmetical if and only if every finitely presented direct sum of cyclics has a canonical form.

An elementary divisor ring is a ring for which every matrix has a canonical form. (By [LLS] it is enough to check that every finitely presented module is a direct sum of cyclics.) In the standard examples of elementary divisor rings (for example, the adequate rings [LLS]), the dimension of the maximal ideal space is at most one. The following theorem, and a construction due to Heinzer [H], show that higher-dimensional examples exist.

THEOREM 6. *Every Bezout ring with noetherian maximal ideal space is an elementary divisor ring.*

Finally, we turn our attention to an intermediate class of rings, between Bezout rings and elementary divisor rings. A ring R is *Hermite* provided every matrix is equivalent to a triangular matrix. (Equivalently, every 1 by 2 matrix is equivalent to a diagonal matrix [K].) M. Henriksen has raised the following question: If R is a Bezout ring with compact minimal prime spectrum, is R Hermite? While we suspect that the answer is “no”,

the following results indicate that a counterexample would have to have a rather perverse nilradical.

THEOREM 7. *Let R be a Bezout ring with compact minimal prime spectrum. Then R is Hermite if and only if every 1 by 2 matrix with nilpotent entries is equivalent to a diagonal matrix.*

COROLLARY. *Let R be a Bezout ring with compact minimal prime spectrum. If the nilradical of R is T -nilpotent, then R is Hermite.*

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