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A Frailty Model for Projection of Human Mortality Improvements

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Abstract‡

Based on the everyday observations that individual human beings vary significantly in their capacity to combat death, we adopt a so-called frailty model of human mortality. This frailty model assumes that each individual in a given population is endowed with his or her own frailty index, r, which remains constant for life. In addition, we assume that the individual's force of mortality (hazard rate function) at age x, $\mu_x(r)$, satisfies $\mu_x(r) = r\mu_x$ where μ_x is the population's base force of mortality at age x. Given the probability distribution of the frailty index among the newborns in the population, an expression is given for the distribution of the frailty index among the survivors reaching age

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x in the population. Finally, assuming that (i) the rate of mortality improvement for any age is proportional to the average frailty level of the individuals at that age, (ii) a gamma distribution for the frailty index, and (iii) a Gompertz form for the population's base force of mortality, we graduate (smooth) the observed mortality improvement factors in the published Society of Actuaries' GAR-94 Table.

Key words and phrases: *force of mortality, hazard rate, gamma distribution, Gompertz law*

1 A Review of Actuarial Mortality Projection

Throughout most of the twentieth century (except during periods of famine, war, and other civil strife), there has been a long and consistent trend of mortality improvement. Lancaster (1990, Chapter 3.6, Table 3.6.1) shows the persistent decline in the overall mortality in several western countries. The reason for this decline is largely because of improvements in public health, improvements in the production and distribution of food, and advances in medicine and technology.

Interestingly, Vaupel and Yashin (1987, pp. 123) note that progress in reducing mortality can be conceived in two ways. Demographers generally view mortality change as change in the force of mortality and associated life table statistics for a population. Most relative laypersons, on the other hand, especially physicians and other health and safety personnel, perceive a reduction in mortality as being achieved by saving the lives of individuals faced with death. A demographer might report that the force of mortality at age fifty among U.S. males was cut in half from 1900 to 1980, from 1.6 percent to 0.8 percent. A public health specialist might focus attention on the lives that were saved in 1980 compared with 1900 because of new surgical and medical procedures, the introduction of penicillin, polio vaccines, and other pharmaceuticals, better nutrition and sanitation, improved automotive safety, a decrease in cigarette smoking, faster and more effective ambulance service, and so on.

Actuaries, like demographers, generally view mortality change as change in the force of mortality and associated life table statistics for a population. In fact, the projection of mortality improvement has been an important subject to actuaries. For example, in the first issue of the *Transactions of the Society of Actuaries* Jenkins and Lew (1949) give a lengthy discussion on this subject. Over the past few decades, various methods have been suggested by actuaries and demographers to project age-specific mortality rates. Pollard (1987) gives an excellent review of these methods. We only summarize methods adopted by actuaries in North America and the United Kingdom in the projection of future mortality rates.

1.1 The American Approach

The Society of Actuaries 1994 Group Annuity Reserving Table (GAR-94) [1] has adopted a generation life table approach to project mortality improvement.

Let q_x^z be the mortality rate observed at age x in calendar year z. Mortality improvement implies that the mortality rates for age x in future years form a non-increasing sequence in z. In the GAR-94 Table this implies that:

$$q_x^{1994} \geq q_x^{1995} \geq q_x^{1996} \geq q_x^{1997} \geq \cdots.$$

Let AA_x^z denote the annual improvement factor in the mortality rate for age x from calendar year z to $z+1$, i.e.,

$$AA_x^z = 1 - \frac{q_x^{z+1}}{q_x^z}.$$

The GAR-94 Table assumes that at each age the $AA_x^z = AA_x$, a constant, as z increases:

$$\frac{q_x^{1995}}{q_x^{1994}} = \frac{q_x^{1996}}{q_x^{1995}} = \frac{q_x^{1997}}{q_x^{1996}} = \cdots = 1 - AA_x. \tag{1}$$

To produce the mortality rate for a person age x in year $(1994+n)$, the following formula is used:

$$q_x^{1994+n} = q_x^{1994}(1 - AA_x)^n. \tag{2}$$

To assist in mortality projections using equation (2), the Society of Actuaries published the 1994 mortality rates as the base table, coupled with the improvement factors AA_x. Some values of q_x and AA_x for

[1]The 1994 Group Annuity Mortality Table and the 1994 Group Annuity Reserving Table are published by the Society of Actuaries Group Annuity Valuation Table Task Force in *Transactions of the Society of Actuaries*, 47 (1995): 865-913.

males at ages $x = 50, \cdots, 99$ are listed in Table 1 (where the values of q_x contain no margin). Specifically, Table 1 is an extract of the Society of Actuaries 1994 Group Annuity Reserving Table (the SOA GAR-94 Table) with: (i) base mortality rates q_x, (ii) improvement factors AA_x, and (iii) implied rate of improvement $\hat{E}_x$.

1.2 The British Approach

Based on the mortality experience in the United Kingdom, British actuaries have developed a more sophisticated method of projecting mortality.[2] Using the 1980 mortality rates as the base table, continuing improvement in mortality beyond 1980 is modeled as:

$$q_x^z = q_x^{1980}\{a(x) + [1 - a(x)](0.4)^{\frac{z-1980}{20}}\}, \tag{3}$$

where z is the calendar year, and

$$a(x) = \begin{cases} 0.5, & x < 60 \\ \frac{x-10}{100}, & 60 \le x \le 110 \\ 1 & x > 110. \end{cases} \tag{4}$$

Note that

$$\lim_{z \to \infty} q_x^z = a(x) q_x^{1980},$$

and

$$AA_x^z = 1 - \frac{a(x) + [1 - a(x)](0.4)^{\frac{z+1-1980}{20}}}{a(x) + [1 - a(x)](0.4)^{\frac{z-1980}{20}}} \tag{5}$$

is a decreasing function of z.

Equation (3) has three characteristics: (i) mortality improvement declines with advancing age; (ii) the mortality rate declines exponentially with the passage of time to a long-term limiting value; and (iii) the mortality improvement exhibits a decelerating trend.

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[2] See Continuous Mortality Investigation Bureau (CMIB). "Standard Tables of Mortality Based on the 1979-82 Experiences." *Continuous Mortality Investigation Reports*, 10 (1990): 1-138.

Table 1
Excerpt of the Society of Actuaries' Male 1994 Group Annuity Reserving Table (SOA GAR-94)

x	q_x	AA_x	$\hat{E}_x$	x	q_x	AA_x	$\hat{E}_x$
50	0.002773	0.018	0.01802	75	0.040012	0.014	0.01429
51	0.003088	0.019	0.01903	76	0.043933	0.014	0.01431
52	0.003455	0.020	0.02003	77	0.048570	0.013	0.01332
53	0.003854	0.020	0.02004	78	0.053991	0.012	0.01234
54	0.004278	0.020	0.02004	79	0.060066	0.011	0.01134
55	0.004758	0.019	0.01904	80	0.066696	0.010	0.01035
56	0.005322	0.018	0.01805	81	0.073780	0.009	0.009351
57	0.006001	0.017	0.01705	82	0.081217	0.008	0.008346
58	0.006774	0.016	0.01605	83	0.088721	0.008	0.008380
59	0.007623	0.016	0.01606	84	0.096358	0.007	0.007364
60	0.008576	0.016	0.01607	85	0.104559	0.007	0.007398
61	0.009663	0.015	0.01507	86	0.113755	0.007	0.007437
62	0.010911	0.015	0.01508	87	0.124377	0.006	0.006414
63	0.012335	0.014	0.01409	88	0.136537	0.005	0.005384
64	0.013914	0.014	0.01410	89	0.149949	0.005	0.005427
65	0.015629	0.014	0.01411	90	0.164442	0.004	0.004380
66	0.017462	0.013	0.01311	91	0.179849	0.004	0.004422
67	0.019391	0.013	0.01313	92	0.196001	0.003	0.003351
68	0.021354	0.014	0.01415	93	0.213325	0.003	0.003389
69	0.023364	0.014	0.01416	94	0.231936	0.003	0.003432
70	0.025516	0.015	0.01519	95	0.251189	0.002	0.002319
71	0.027905	0.015	0.01521	96	0.270441	0.002	0.002350
72	0.030625	0.015	0.01523	97	0.289048	0.002	0.002383
73	0.033549	0.015	0.01525	98	0.306750	0.001	0.001207
74	0.036614	0.015	0.01528	99	0.323976	0.001	0.001224

1.3 The Frailty Approach

Most actuarial and demographic techniques for projecting mortality rates are based on the extrapolation of past mortality rates. Few mathematical formulations are based on the underlying biological mechanism of mortality improvement.

Traditional life table methods, after accounting for factors such as race, gender, and smoking status, implicitly assume that the population is homogeneous, an assumption that is usually unrealistic. Empirical evidence shows that the following factors significantly affect mortality rates: genetics, economic status, education, marital status, and lifestyle. If mortality is not classified according to these additional risk factors, then the group's mortality characteristic will be heterogeneous.

For practical reasons not all of the above risk factors are usually included in mortality estimates. Thus, it is important to examine the consequences of heterogeneity when interpreting observed mortality rates and mortality improvements (Vaupel et al., 1979; Hougaard, 1991). A formal mathematical account of the treatment of heterogeneity can be found in Hougaard (1984, 1995) and the text of Namboodiri and Suchindran (1987).

Vaupel et al. (1979) propose a frailty model to study the effect of heterogeneity on cohort mortality rates.[3] In their model, each individual in a given population is endowed with his or her intrinsic *frailty index*, r, which is assumed to remain constant for life. An individual age x with frailty index r has force of mortality (hazard rate function), $\mu_x(r)$, which is assumed to satisfy

$$\mu_x(r) = r\mu(x) \tag{6}$$

where μ_x is the population's base force of mortality at age x. Weak (strong) individuals are associated with high (low) values of r.

2 Measurement of Mortality Improvement

To facilitate an easier discussion of mortality improvements, the following notational style is used:

[3] This frailty model can be viewed as a special version of the Cox (1972) proportional hazard model in the context of an unobserved covariate. Norberg (1989) uses a proportional hazard model for the heterogeneity in group life insurance. Two early actuarial applications of the frailty model that merit mentioning are Redington (1969) where there is a range of sample calculations, and Beard (1971) where the Gamma-Gompertz model is analysed.

- x denotes the current age and is placed at the right subscript;
- z denotes the current calendar year and is placed at the right superscript;
- A *bar* ($\bar{}$) placed on top of a quantity indicates that it is for a group of individuals; and
- A *hat* ($\hat{}$) placed on top of a quantity indicates that it is the estimated or observed value.

For example, $\hat{\mu}_x^z$ represents the estimated or observed hazard rate for an individual at age x, at calendar time z.

Customary measures of progress in mortality consider only changes in mortality rates $\overline{q}_x^z$ over different calendar years. Vaupel et al. (1979) argue that this may not be the most informative measure for mortality improvement. Instead of measuring progress in terms of mortality rates, Vaupel et al. state that it may be more appropriate to measure such progress in terms of the hazard rate (force of mortality) for standard individuals. Vaupel et al. (1979) give two main reasons:

1. For the frailty model of equation (6), the ratio of the μ's measures mortality progression at any level of frailty because the ratio is independent of r:

$$\frac{\mu_x^{z+n}(r)}{\mu_x^z(r)} = \frac{\mu_x^{z+n}(r')}{\mu_x^z(r')}.$$

   However, this is not true for the ratio of the q's, i.e., the ratio depends on r:

$$\frac{q_x^{z+n}(r)}{q_x^z(r)} \neq \frac{q_x^{z+n}(r')}{q_x^z(r')}.$$

2. In youth and middle age, when μ_x and q_x are close to zero, μ_x is approximately equal to q_x. At the elderly ages, however, μ_x, which is not bounded by 1, can greatly exceed q_x. As a result, progress that substantially reduces μ_x may have much less effect on q_x. For example, consider a reduction in μ_x from 2 to 1: if these values of μ_x stayed constant over the course of a year, q_x would only be reduced from 0.86 to 0.63.

For an integer age x, we define annual improvement factors E_x^z in terms of hazard rates

$$E_x^z = 1 - \frac{\overline{\mu}_{x+0.5}^{z+1}}{\overline{\mu}_{x+0.5}^{z}} = 1 - \frac{\log(1 - \overline{q}_x^{z+1})}{\log(1 - \overline{q}_x^{z})} \tag{7}$$

where a constant hazard rate function is assumed for the age interval $(x, x+1)$.

In the GAR-94 Table, the improvement factor AA_x is measured by the ratio of the observed mortality rates:

$$\hat{q}_x^{1994+n} = \hat{q}_x^{1994}(1 - AA_x)^n. \tag{8}$$

The implied improvement factor $\hat{E}_x^{1994}$ is

$$\hat{E}_x^{1994} = 1 - \frac{\log[1 - \hat{q}_x^{1994}(1 - AA_x)]}{\log[1 - \hat{q}_x^{1994}]}. \tag{9}$$

Table 1 shows the values of $\hat{E}_x^{1994}$ for comparison with the values of AA_x. From Table 1, one can see that the values of $\hat{E}_x^{1994}$ do not deviate much from AA_x for ages below 85. However, the relative difference becomes significant beyond age 85 and may affect our estimate of annuity costs (as they are based on mortality projections many years into the future).

3 A Mathematical Model for Frailty

3.1 The Basic Model

Consider a cohort of newborns (age exactly 0) where their survival capacity varies across individuals. A *standard* newborn is one whose future lifetime, X, has a force of mortality μ_x and cumulative force of mortality

$$H_x = \int_0^x \mu_t dt. \tag{10}$$

Each individual has his/her *unknown* constant frailty index r with force of mortality given in equation (6). Thus a standard newborn has $r = 1$.

To model the heterogeneity of frailty for the cohort of newborns age exactly 0, let R_0 be the unknown frailty index of an individual chosen at random from the cohort of newborns. Assume that R_0 has a probability density function (pdf) $g_0(r)$ for $r > 0$.

For a newborn with frailty r, the (conditional) survivor function and (conditional) pdf are:

$$\begin{aligned} \Pr[X > x | R_0 = r] &= S(x|r) = e^{-rH_x}, \\ f(x|r) &= -\frac{\partial}{\partial r} S(x|r) = r\,\mu_x\, e^{-rH_x}. \end{aligned}$$

The joint density of X and R_0 is

$$f(x,r) = f(x|r)\, g_0(r) = r\,\mu_x\, e^{-rH_x}\, g_0(r),$$

and the unconditional probability of a newborn chosen at random surviving to age x is

$$\begin{aligned} \Pr[X > x] &= \overline{S}(x) \\ &= \int_0^\infty S(x|r)\, g_0(r) dr \\ &= \int_0^\infty e^{-rH_x}\, g_0(r) dr \\ &= M_{g_0}(-H_x), \end{aligned} \tag{11}$$

where $M_{g_0}(\theta)$ is the moment generating function R_0, i.e.,

$$M_{g_0}(\theta) = E[e^{\theta R_0}] = \int_0^\infty e^{-r\theta}\, g_0(r) dr.$$

From $\overline{S}(x)$ we can get $\overline{f}(x)$, the pdf of X,

$$\overline{f}(x) = \mu_x \int_0^\infty r e^{-rH_x} g_0(r) dr, \tag{12}$$

and $\overline{\mu}_x$, the force of mortality associated with $\overline{S}(x)$,

$$\overline{\mu}_x = \frac{\overline{f}(x)}{\overline{S}(x)} = \frac{\mu_x \int_0^\infty r e^{-rH_x} g_0(r) dr}{M_{g_0}(-H_x)}. \tag{13}$$

Next we turn our attention to the survivors age exactly x from the cohort of newborns. Clearly the distribution of the frailty index among these survivors will not necessarily be the same as at age 0 because one would expect more of the weaker ones to have died earlier. So the population at age x should have a larger percentage of stronger individuals.

Let R_x be the frailty variable for the survivor cohort at age x chosen at random. R_x has pdf $g_x(r)$ given by

$$g_x(r) = \frac{S(x|r)g_0(r)}{\overline{S}(x)} = \frac{e^{-rH_x}\, g_0(r)}{M_{g_0}(-H_x)}. \tag{14}$$

Thus the average frailty for the survivor cohort at age x is

$$\overline{R}_x = E[R_x] = \frac{\int_0^\infty r e^{-rH_x} g_0(r)dr}{M_{g_0}(-H_x)} = \frac{\overline{\mu}_x}{\mu_x}. \tag{15}$$

Note that $\overline{\mu}_x = \mu_x \overline{R}_x$, i.e., the force of mortality for $[X > x]$ is always equal to the force of mortality of the standard individual multiplied by the average frailty among the survivors.

Among those who die at age x (i.e., in $(x, x+dx)$), the frailty index has a conditional density:

$$\frac{f(x,r)}{\overline{f}(x)} = \frac{r\, e^{-rH_x}\, g_0(r)}{\int_0^\infty r e^{-rH_x} g_0(r)dr}. \tag{16}$$

3.2 Gamma Frailty Density

Because of its mathematical tractability and its flexible shape, the gamma distribution has been used by many authors (including Vaupel et al., 1979) to model the frailty variable. Specifically we assume that R_0 has a gamma density:

$$g_0(r) = \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}, \quad r > 0 \tag{17}$$

where $\alpha > -1$ is a shape parameter and $\beta > 0$ is a scale parameter. The moment generating function is

$$M_{g_0}(\theta) = \left(\frac{\beta}{\beta - \theta}\right)^\alpha.$$

The first two moments of R_0 are:

$$\overline{R}_0 = E[R_0] = \frac{\alpha}{\beta}, \quad \text{and} \quad \sigma^2(R_0) = \frac{\alpha}{\beta^2}.$$

From equation (14) R_x is also gamma distributed with shape parameter α and a different scale parameter $\beta + H_x$.

In this case, the mean frailty of the survivors at age x is

$$\overline{R}_x = \frac{\alpha}{\beta + H_x}.$$

From equation (16), the frailty index for those who die in $(x, x+dx)$ has a conditional density that is also gamma distributed, with a shape parameter $\alpha + 1$ and a scale parameter $\beta + H_x$. In this case, the mean frailty of those who die is:

$$\overline{R}_x \times \frac{\alpha + 1}{\alpha},$$

which is greater than the mean frailty of the survivors.

3.3 Gompertz's Law

Assume that the standard individual's lifetime follows Gompertz's law:

$$\mu_x = bc^x \log(c), \qquad H_x = b(c^x - 1). \tag{18}$$

Gompertz's law has been used by actuaries since 1825. Several biological theories of aging have been developed that imply a Gompertz form of hazard rates (see Strehler, 1977, Chapter 5). Brillinger (1961) argues that if the human body is considered as a series system of independent components, then the hazard rate function may follow the Gompertz law (also see Carriere, 1992).

If the frailty variable R_0 is assumed to have a gamma density in equation (17), then the birth cohort has an unconditional survivor function

$$\overline{S}(x) = \left(\frac{\beta}{\beta + b(c^x - 1)}\right)^{\alpha},$$

and hazard rate function

$$\overline{\mu}_x = \frac{\alpha\, b\, c^x \log(c)}{\beta + b(c^x - 1)}. \tag{19}$$

Equation (19) is derived in the manner used by Beard (1971), and is one of the "laws" of mortality originally proposed by Perks (1932).

Pollard (1980, 1993) studies the case where each individual in a population has a hazard rate function of the Gompertz type. Note that the cohort hazard rate function in equation (19) increases exponentially initially, but the growth rate decreases with advancing age. Pollard points out that this is a phenomenon observed in many populations.

Among the survivors age x, R_x has a gamma distribution with a shape parameter α and a scale parameter $\beta + b(c^x - 1)$. The mean frailty for the survivors age x is

$$\overline{R}_x = \frac{\alpha}{\beta + b(c^x - 1)}.$$

4 A Model for Mortality Improvement

In a given calendar year, the overall level of mortality improvement depends on the marginal changes of many external factors such as medical technology and its availability to the general public. In general, projection of these external factors for future years is a difficult task and requires more detailed (perhaps non-actuarial) investigation. In this paper we are concerned mainly with the rates of mortality improvement among different cohorts in a given calendar year, where the same underlying external factors apply to all ages.

We hypothesize that mortality improvements due to the marginal advancement of life-saving techniques progress as follows:

Hypoyhesis 1. *For each age x the rate of improvement in terms of the force of mortality (hazard rate) is proportional to the average frailty $\overline{R}_x$.*

This hypothesis is based on the argument that marginal improvements in life-saving techniques have relatively larger effect on frailer individuals with higher than average values of r. Most deaths of strong individuals with lower than average values of r are due to natural aging; thus, improvements in life-saving techniques or better health-practices would have relatively smaller effects on healthier individuals, i.e., those with lower than average values of r.

Assume that, for each birth cohort at time y, frailty has a gamma density with:

$$g_0(r) = \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}, \qquad \text{where } \alpha = \beta.$$

If we assume Gompertz mortality for a standard individual, the mean frailty of the survivors at age x, at calendar time $z = y + x$, is

$$\overline{R}_x^z = \frac{\alpha}{\alpha + b(c^x - 1)}.$$

Based on the above hypothesis, the improvement factor E_x^z is proportional to the average frailty of the survivor cohort at age x:

$$E_x^z = \kappa \overline{R}_{x+0.5}^z = \kappa \frac{\alpha}{\alpha + b(c^{x+0.5} - 1)}, \tag{20}$$

where κ is a constant, and an adjustment of a half year is used because E_x^z is measured by the ratios of mid-year hazard rates.

Equation (20) of E_x^z implies that, at any fixed calendar time, the mortality improvement decreases rapidly at advanced ages, due to the exponential growth in $H_x = b(c^x - 1)$ with age.

5 Fitting the Gamma-Gompertz Model

Now we will fit the Gamma-Gompertz model to the GAR-94 Base Table, which gives the cohort age-specific mortality rates. Neither frailty nor heterogeneity was discussed in the GAR-94 Table. Specifically, we assume Gompertz's law for each individual's force of mortality.

Suppose that the cohort is homogeneous and each individual's lifetime follows the Gompertz law with $\mu_x = B c^x$. We would expect that

$$c[x] = [\frac{\mu_{x+20.5}}{\mu_{x+0.5}}]^{\frac{1}{20}}$$

be approximately constant.

From the GAR-94 mortality rates, we have calculated the values of $c[x]$ at different ages (see Table 2). As shown in Table 2, the values of $c[x]$ exhibit a gradual decreasing trend as age increases. Although there are many possible explanations to this observed pattern, we will try to fit the mortality rates to a frailty (heterogeneous) model.

Table 2
Calculated Values of $c[x]$

x	$c[x]$	x	$c[x]$
50	1.1180	60	1.1097
51	1.1171	61	1.1088
52	1.1160	62	1.1076
53	1.1151	63	1.1059
54	1.1142	64	1.1040
55	1.1133	65	1.1023
56	1.1124	66	1.1010
57	1.1114	67	1.1005
58	1.1107	68	1.1006
59	1.1102	69	1.1012

5.1 Fitting the 1994 Base Mortality Rates

Based on considerations that mortality rates at advanced ages may not be as accurate due to smaller sample sizes, we suggest using some representative age range, say, from 50 to 75. For many populations, from the mortality rates at ages 50 and 70, one can get a good approximation of the shape of the mortality curve at all ages (Benjamin, 1982; Pollard, 1991 and 1993).

We assume that the standard force of mortality follows the Gompertz law with

$$\mu_x = bc^x \log(c), \qquad H_x = b(c^x - 1).$$

Furthermore, we will choose μ_x such that $\bar{R}_0 = 1$.

We define a measure for goodness of fit by using the sum of squared errors for ages from 50 to 75:

$$\text{DIST}_{50:75} = \sum_{x=50}^{75} (\overline{\mu}_{x+0.5} - \hat{\mu}_{x+0.5})^2$$

where

$$\hat{\mu}_{x+0.5} = -\log(1 - q_x)$$

can be obtained from the mortality rates in Table 1.

Assume that R_0 has a gamma density, then $\bar{R}_0 = 1$ implies $\alpha = \beta$ so that

$$\overline{\mu}_x = \frac{\alpha b c^x \log(c)}{\alpha + b(c^x - 1)}. \tag{21}$$

By minimizing $\mathrm{DIST}_{50:75}$, we get the following estimate of the unknown parameters:

$$c = 1.1248, \quad b = 0.66 \times 10^{-4}, \quad \alpha = 1.306, \tag{22}$$

with the minimum distance being

$$\min\{\mathrm{DIST}_{50:75}\} = 0.2238 \times 10^{-5}.$$

We have noticed that the estimation of parameters for the frailty distribution (mixing density) is not very robust, depending on the age range used in the estimation of parameters. This is a common phenomenon in many mixture models (Chan, 1995; Manton et al., 1986; Everitt and Hand, 1981).

5.2 Fitting the GAR-94 Mortality Improvement Factors

We shall use a Gamma-Gompertz model for the 1994 base mortality rates and adopt the particular set of estimated parameters in equation (22) in the Gamma-Gompertz model:

$$c = 1.1248, \quad b = 0.66 \times 10^{-4}, \quad \alpha = 1.306.$$

The frailty model of mortality improvement in equation (20) suggests the following pattern for the improvement factors:

$$E_x^{1994} = \kappa \times \frac{1.306}{1.306 + 0.66 \times 10^{-4} \times (1.1248^{x+0.5} - 1)}.$$

Now we use this frailty model of mortality improvement to fit the empirical improvement factors $\hat{E}_x^{1994}$ in Table 1. We are mainly interested in the mortality improvement at senior ages, say, 50 and above. We first choose an age range from 50 through 95 and define a loss measure:

$$M_{50:95} = \sum_{x=50}^{95} (E_x^{1994} - \hat{E}_x^{1994})^2.$$

The ages below 50 are excluded because of the sudden dip in the observed improvement factors (see Figure 1) which may be a result of other exogenous factors (e.g., accident, AIDS). The ages beyond 95 are not included because of the scarcity of available data for extreme ages 95 and above.

By minimizing the loss measure $M_{50:95}$, we get a least square estimate for κ:

$$\kappa = \frac{\sum_{50}^{95}(E_x^{1994})(\hat{E}_x^{1994})}{\sum_{50}^{95}(E_x^{1994})^2} = 0.01769.$$

Table 3 compares the Gamma-Gompertz frailty model improvement factors E_x and the empirical improvement factors $\hat{E}_x$ in the GAR-94 table. Figure 1 also displays these improvement factors. Note that in Figure 1 the $\hat{E}_x$s in the GAR-94 Table do not follow a smooth pattern. Also, there is insignificant mortality improvement in the 25–45 age group.[4] Beyond age 50 the frailty model seems to be an acceptable fit and may provide a theoretical basis for the observed improvement factors. The frailty model of mortality improvement has the definite advantage that the projected mortality rates are smooth.

The choices of the age range, from 50 to 95, and the loss measure (i.e., the squared error) are arbitrary and are for illustration purposes only. One may use other age ranges or weighted squared error, as appropriate.

6 Other Evidence

According to United Nations 1991,[5] in developed countries, one half of female and one-third of male deaths now occur after age 80. The mortality reductions within this age range are crucial in determining changes in life expectancy and actuarial annuity values.

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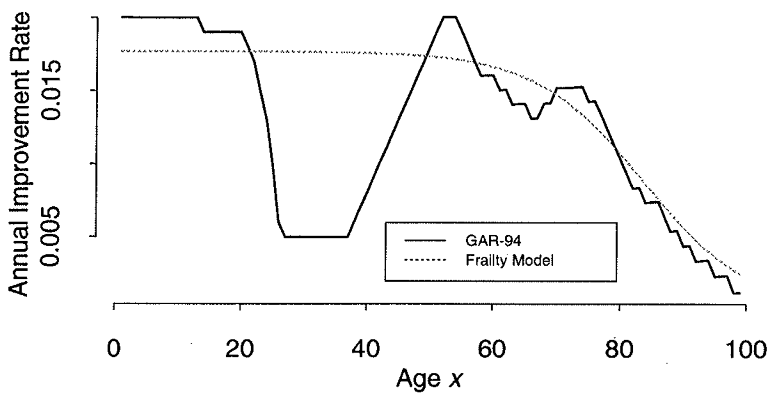
[4]This could be attributed to extra AIDS deaths in the 25–45 age group.

[5]United Nations Demographic Yearbook 1991. New York: United Nations

Table 3
Mortality Improvement Rates

x	E_x	$\hat{E}_x$	x	E_x	$\hat{E}_x$
50	0.017410	0.018030	75	0.013520	0.014290
51	0.017370	0.019030	76	0.013130	0.014320
52	0.017330	0.020030	77	0.012720	0.013330
53	0.017290	0.020040	78	0.012290	0.012340
54	0.017240	0.020040	79	0.011840	0.011340
55	0.017180	0.019050	80	0.011370	0.010350
56	0.017120	0.018050	81	0.010890	0.0093500
57	0.017050	0.017050	82	0.010390	0.0083500
58	0.016980	0.016050	83	0.0098800	0.0083800
59	0.016890	0.016060	84	0.0093700	0.0073600
60	0.016800	0.016070	85	0.0088500	0.0074000
61	0.016690	0.015070	86	0.0083300	0.0074400
62	0.016580	0.015080	87	0.0078100	0.0064100
63	0.016450	0.014090	88	0.0073100	0.0053800
64	0.016310	0.014100	89	0.0068100	0.0054300
65	0.016150	0.014110	90	0.0063200	0.0043800
66	0.015980	0.013110	91	0.0058500	0.0044200
67	0.015790	0.013130	92	0.0054000	0.0033500
68	0.015580	0.014150	93	0.0049700	0.0033900
69	0.015350	0.014160	94	0.0045600	0.0034300
70	0.015100	0.015190	95	0.0041800	0.0023200
71	0.014830	0.015210	96	0.0038200	0.0023500
72	0.014530	0.015230	97	0.0034800	0.0023800
73	0.014220	0.015260	98	0.0031600	0.0012100
74	0.013880	0.015280	99	0.0028700	0.0012200

Figure 1
Mortality Improvement Factors $\hat{E}_x$ for GAR-94
And E_x for the Frailty Model



Kannisto et al., (1994) study the reduction in mortality at advanced ages based on a large and reliable database for 27 countries, 1960s through 1980s. The following is cited from Kannisto et al., (1994, pp. 801):

> For nine countries - Austria, Belgium, England and Wales, West Germany, France, Japan, Scotland, Sweden, and Switzerland - data are available through 1991. A glimpse at the most recent trends is provided by calculating the annual average rate of mortality improvement between 1982-86 and 1987-91 for this aggregate of nine countries. For males the rate of improvement was 1.7 percent for octogenarians and 1.2 percent for nonagenarians; for females the corresponding rates were 2.5 percent and 1.6 percent.

Even though the magnitude of the mortality improvement at advanced ages is higher than those in the GAR-94 Table, the general pattern of deceleration of mortality improvement at advanced ages is consistent with our frailty model of mortality improvement.

In a panel discussion of mortality trends, Moriyama (1967) also provides evidence that the rate of improvement in mortality rates decreases at advanced ages.

7 Closing Comments

The main contribution of this paper is the utilization of a frailty model to derive mathematical formulae for mortality improvement factors. As marginal advancement in life-saving techniques determines the pace of mortality improvement, we assume that weaker individuals are more likely to benefit from these advances than are stronger individuals. This assumption is supported in the demography literature (Vaupel and Yashin, 1985). To project the future trend of mortality improvement, one needs to assess carefully the future advancement in medical technology. A major breakthrough in medical technology or an unexpected new epidemic may have a sudden impact on the mortality improvement.

Several authors, including Bowers et al., (1986) and London (1985), have discussed the importance of smoothness in mortality rates. Their arguments for smoothness can be extended to mortality improvement factors. Our frailty model provides useful mathematical formulae for graduation of empirical improvement factors.

One potential shortcoming of our model is that the frailty index is assumed to be determined at birth and remains constant for life. Intuition suggests, however, that this assumption may be overly simplistic. In future studies, the concept of frailty may be modeled as a variable dependent upon exogenous observable factors such as lifestyle, environment, economic status, or marital status.

We hope this paper stimulates further research on this important subject.

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