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Green's Functions and Lyapunov Inequalities for Nabla Caputo Boundary Value Problems

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GREEN'S FUNCTIONS AND LYAPUNOV INEQUALITIES FOR NABLA CAPUTO BOUNDARY VALUE PROBLEMS

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Lyapunov inequalities have many applications for studying solutions to boundary value problems. In particular, they can be used to give existence-uniqueness results for certain nonhomogeneous boundary value problems, study the zeros of solutions, and obtain bounds on eigenvalues in certain eigenvalue problems. In this work, we will establish uniqueness of solutions to various boundary value problems involving the nabla Caputo fractional difference under a general form of two-point boundary conditions and give an explicit expression for the Green's functions for these problems. We will then investigate properties of the Green's functions for specific cases of these boundary value problems. Using these properties, we will develop Lyapunov inequalities for certain nabla Caputo BVPs. Further applications and extensions will be explored, including applications of the Contraction Mapping Theorem to nonlinear versions of the BVPs and a development of Green's functions for a more general linear nabla Caputo fractional operator.

To my mother, Majda,

for nurturing the joy of mathematics,

thinking, and learning.

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Table of Contents

Chapter 1

Introduction

In his 1695 letter to Leibniz, L'Hôpital posed the question, *what does it mean to take a half-derivative of a function?* This question is said to have launched fractional calculus, which has since developed many ways of extending derivatives to be of any complex order. Fractional calculus has been applied to fields ranging from signal processing to hydraulics of dams, temperature field problems in oil strata, diffusion problems, and waves in liquids and gases [11]. Fractional differential equations have also been used in modeling porous materials, which is extensively studied in [9]. Also, fractional differential systems have been used in 3D printing and oil drilling [55]. In [44, 54], the authors study a thermostat model by means of a fractional boundary value problem.

Additionally, fractional calculus has been applied in biophysics and blood flow phenomena. For example, the authors in [59, 60] model $CD4^{+}T$ cells' infections with a fractional system given by

$$\begin{cases} D^{\alpha_1}(T) = s - KVT - dT + bI \\ D^{\alpha_2}(I) = KVT - (b+\delta)I \\ D^{\alpha_3}(V) = N\delta I - cV, \end{cases}$$

where D^{α_i}, $i = 1, 2, 3$ are fractional derivatives with $\alpha_i > 0$; T, I and V represent the concentration of uninfected cells, infected cells, and free HIV particles in the blood respectively, and the parameters represent physical phenomena.

Discrete fractional calculus consists of studying fractional derivatives of functions defined on a discrete domain. Applications of discrete fractional calculus include modeling tumor growth, which is explored by Atici and Sengul in [8].

1.1 Background for Nabla Fractional Calculus

1.1.1 Nabla Whole Order Differences and Integrals

In the setting of discrete fractional calculus, functions are defined on a discrete domain, namely either

$$\mathbb{N}_a := \{a, a+1, a+2, \ldots\} \text{ or } \mathbb{N}_a^b := \{a, a+1, \ldots, b\},$$

where $a, b \in \mathbb{R}$ such that $b - a$ is a positive integer. There are two main ways of defining a difference of a function, $f : \mathbb{N}_a \to \mathbb{R}$: the delta, or forward, difference, and the nabla, or backward, difference. Both of these approaches are presented by Goodrich and Peterson in [29].

One undesirable peculiarity of the delta fractional difference operator is that the resulting function's domain has empty intersection with the domain of the original function. The issue of shifting of domains is not as problematic in the case of the nabla fractional difference [29, p. 149].

In this work, we will focus on the nabla difference. The *nabla difference* of a function $f : \mathbb{N}_a \to \mathbb{R}$ is defined to be

$$\nabla f(t) := f(t) - f(t-1),$$

for $t \in \mathbb{N}_{a+1}$. We also define the *backwards jump operator*, $\rho : \mathbb{N}_a \to \mathbb{N}_a$, to be $\rho(t) := \max\{a, t-1\}$. We define nabla differences of any higher order $N \in \mathbb{N}$ recursively; i.e.,

$$\nabla^N f(t) := \nabla(\nabla^{N-1} f)(t), \ \ t \in \mathbb{N}_{a+N}.$$

Additionally, we take by convention $\nabla^0 f(t) := f(t)$.

The discrete nabla counterpart to the definite integral in single variable calculus is the *nabla definite integral* of a function $f : \mathbb{N}_{a+1}^b \to \mathbb{R}$, which, for $c, d \in \mathbb{N}_a^b$, is defined to be

$$\int_c^d f(t)\nabla t := \begin{cases} \sum_{t=c+1}^{d} f(t), & d > c \\ 0, & d \leq c. \end{cases}$$

Many results of single variable calculus have analogous counterparts in the whole order discrete case. In particular, the Fundamental Theorem of Nabla Calculus, which is given below, provides a relationship between the nabla difference and the nabla definite integral.

Theorem 1.1 *[29, Theorem 3.37, Goodrich and Peterson]* ***(Fundamental Theorem of Nabla Calculus)*** *If $f : \mathbb{N}_{a+1}^b \to \mathbb{R}$ and F is any nabla antidifference of f on $\mathbb{N}_a^b$ (i.e., $\nabla F(t) = f(t)$, for $t \in \mathbb{N}_{a+1}^b$), then*

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

Next, we will develop a nabla fractional sum which directly leads to a nabla fractional difference.

1.1.2 Nabla Fractional Sums and Differences

The *whole order sum* of a function f defined on $\mathbb{N}_{a+1}$, denoted by $\nabla_a^{-n} f$ where $n \in \mathbb{N}_1$, is defined by repeated integration; namely,

$$\nabla_a^{-n} f(t) := \int_a^t \int_a^{\tau_n} \int_a^{\tau_{n-1}} \cdots \int_a^{\tau_2} f(\tau_1) \nabla\tau_1 \nabla\tau_2 \cdots \nabla\tau_{n-1} \nabla\tau_n.$$

Moreover, we define $\nabla_a^{-0} f(t) := f(t)$. The $-n$ in this operator signifies that we are performing an operation reverse of taking a difference n times. Later, we will state a result that condenses this repeated integration into an expression involving a single integral, which allows us to generalize to a sum of any positive order. We will then define a fractional difference in terms of this fractional sum.

The expression for a fractional sum involves a nabla Taylor monomial, so we will first introduce nabla Taylor monomials, which are analogous to the Taylor monomials in continuous whole order calculus. The Taylor monomials in the nabla fractional context are defined in terms of the *rising function*, which, in turn, is defined in terms of the *Gamma function*.

Definition 1.2 [29, Definition 1.6] The **Gamma function** is defined by

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt,$$

for $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$. The Gamma function can then be extended to be an analytic function on $\mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$.

We have the following useful property of the Gamma function.

Proposition 1.3 [29, p. 3] For $z \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$, we have

$$\Gamma(z+1) = z\Gamma(z).$$

The rising function for $x, r \in \mathbb{N}_1$ is defined to be

$$x^{\overline{r}} := x(x+1)\cdots(x+r-1) = \frac{(x+r-1)!}{(x-1)!}. \tag{1.1}$$

This definition of the rising function is generalized for $x, r \in \mathbb{R}$ using the Gamma function, which extends the factorial function using Proposition 1.3.

Some conventions involving the Gamma function are given in the remark below.

Remark 1.4 [29, p. 4, 152] Let n and N be nonnegative integers. Then,

$$\frac{\Gamma(-n)}{\Gamma(-N)} = (-1)^{N-n}\frac{N!}{n!}.$$

Also, if t is a nonpositive integer and $t+r$ is not a nonpositive integer, then, by convention,

$$\frac{\Gamma(t+r)}{\Gamma(t)} := 0.$$

Motivated by (1.1), we define the generalized rising function as follows.

Definition 1.5 [29, Definition 3.4] The **generalized rising function** is defined by

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)},$$

for values of t and r so that the given expression is defined, following the conventions given in Remark 1.4.

Definition 1.6 [29, Definition 3.56] For $\nu \in \mathbb{R}$, the ν-th order **nabla Taylor monomial**, based at $s \in \mathbb{N}_a$, is defined to be

$$H_\nu(t,s) := \frac{(t-s)^{\overline{\nu}}}{\Gamma(\nu+1)}, \tag{1.2}$$

for $t \in \mathbb{N}_a$, using the conventions in Remark 1.4 when appropriate. In particular, if $t \leq s$ and $\nu \neq 0$, then $H_\nu(t,s) = 0$.

The following generalized power rules show that rising functions can be thought of as the discrete counterpart to power functions from single variable calculus.

Theorem 1.7 *[29, Theorem 3.5] The following equalities hold for values of $t, r,$ and α such that the expressions make sense:*

$$\nabla(t+\alpha)^{\overline{r}} = r(t+\alpha)^{\overline{r-1}},$$

$$\nabla(\alpha - t)^{\overline{r}} = -r(\alpha - \rho(t))^{\overline{r-1}}.$$

Next, we state several properties of the nabla Taylor monomials.

Theorem 1.8 *[29, Theorem 3.57] For $t \in \mathbb{N}_a$ and $\mu \in \mathbb{R}$,*

(i) for $\mu \neq 0$, $H_\mu(a,a) = 0$ and $H_0(t,a) \equiv 1$;

(ii) $\nabla H_\mu(t,a) = H_{\mu-1}(t,a)$;

(iii) for $\mu \neq -1$, $\int_a^t H_\mu(s,a)\nabla s = H_{\mu+1}(t,a)$;

(iv) for $\mu \neq -1$, $\int_a^t H_\mu(t,\rho(s))\nabla s = H_{\mu+1}(t,a)$;

(v) for $k \in \mathbb{N}_1$, $s \in \{a+n \mid n \in \mathbb{Z}\}$, and $t \in \mathbb{N}_{s+k+1}$, $H_{-k}(t,s) = 0$;

provided the expressions above are defined.

The following proposition gives a formula for the n-th order sum, which motivates the subsequent definition of a nabla sum of a function of any arbitrary positive order.

Proposition 1.9 [29, p. 185] Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $n \in \mathbb{N}_1$. Then,

$$\nabla_a^{-n} f(t) = \int_a^t H_{n-1}(t,\rho(s)) f(s) \nabla s,$$

for $t \in \mathbb{N}_{a+1}$.

Definition 1.10 [29, Definition 3.54, 3.58] Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. Then, the **nabla fractional sum** of f of order ν, based at a, is defined by

$$\nabla_a^{-\nu} f(t) := \int_a^t H_{\nu-1}(t, \rho(s)) f(s) \nabla s, \tag{1.3}$$

for $t \in \mathbb{N}_{a+1}$. Also recall, we define $\nabla_a^{-0} f(t) := f(t)$.

The Caputo fractional difference is defined in terms of the fractional sum.

Definition 1.11 [29, Definition 3.117] Let $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$ and $\nu > 0$. Let $N := \lceil \nu \rceil$; i.e., N is the *ceiling* of ν, or the smallest integer greater than or equal to ν. Then, the ν-th order **nabla Caputo fractional difference** of f is defined to be

$$\nabla_{a*}^{\nu} f(t) := \nabla_a^{-(N-\nu)} \nabla^N f(t), \tag{1.4}$$

for $t \in \mathbb{N}_{a+1}$. By convention, $\nabla_{a*}^{\nu} f(t) = 0$ for $t \in \{a - k \mid k \in \mathbb{N}_0\}$.

Note that, for $N \in \mathbb{N}_1$ and $t \in \mathbb{N}_{a+1}$,

$$\nabla_{a*}^{N} f(t) = \nabla_a^{-(N-N)} \nabla^N f(t) = \nabla_a^{-0} \nabla^N f(t) = \nabla^N f(t),$$

by Definition 1.10.

Remark 1.12 Note that the operator ∇_{a*}^{ν} takes functions defined on $\mathbb{N}_{a-N+1}$ as input and outputs functions defined on $\mathbb{N}_{a+1}$. We will use the notation $\nabla_{a*}^{\nu} x(t)$ throughout to mean $(\nabla_{a*}^{\nu} x)(t)$; i.e., t is the argument of the nabla Caputo fractional difference of the function x. To avoid ambiguity, we will at times use the notation $\nabla_{a*}^{\nu}[x(\cdot)](t)$ to mean the nabla Caputo fractional difference of x evaluated at t. Likewise, in the case of the whole order nabla difference, $\nabla x(t)$ means $(\nabla x)(t) = \nabla[x(\cdot)](t)$.

We note here an alternative definition of a fractional difference as given in [29, Definition 3.61]: the *Riemann-Louiville* fractional difference is defined with the operators in the reverse order from the Caputo difference, meaning the $(N-\nu)$-th order sum is followed by the N-th order nabla difference. Namely, if $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$, we have the well studied *nabla Riemann-Louiville fractional difference* of f defined to be

$$\nabla_a^{\nu} f(t) := \nabla^N \nabla_a^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+N+1},$$

where $N := \lceil \nu \rceil$. However, defining a fractional difference in this way results in nonzero fractional derivatives of constant functions, which is resolved with the Caputo definition.

The next proposition gives a binomial formula for the N-th whole order difference and will be used frequently when expanding the nabla Caputo operator.

Proposition 1.13 [29, p. 190] Let $f : \mathbb{N}_a \to \mathbb{R}$. Then,

$$\nabla^N f(t) = \sum_{i=0}^{N} (-1)^i \binom{N}{i} x(t-i),$$

for $t \in \mathbb{N}_{a+N}$.

In this work, we will be considering fractional difference equations and boundary value problems which involve the nabla Caputo difference operator.

Remark 1.14 The nabla Caputo difference operator is a linear operator.

Next, we will state some composition rules involving fractional operators.

Theorem 1.15 *[29, Lemma 3.108, Corollary 3.122, Theorem 3.109] Let* $k \in \mathbb{N}_0$, $\mu, \nu > 0$, $N := \lceil \mu \rceil$, *and* $f : \mathbb{N}_{a+1} \to \mathbb{R}$. *Then,*

$$\nabla_a^{\nu} \nabla_a^{-\mu} f(t) = \nabla_a^{\nu-\mu} f(t), \text{ and, in particular, } \nabla^k \nabla_a^{-\mu} f(t) = \nabla_a^{k-\mu} f(t); \tag{1.5}$$

$$\nabla^k \nabla_a^\mu f(t) = \nabla_a^{k+\mu} f(t), \tag{1.6}$$

for $t \in \mathbb{N}_{a+k}$. *Moreover,*

$$\nabla_a^{-(N-\mu)} \nabla_a^{N-\mu} f(t) = f(t), \tag{1.7}$$

for $t \in \mathbb{N}_{a+1}$.

A variation of constants formula for a nabla Caputo initial value problem is given in the next theorem.

Theorem 1.16 *[29, Theorem 3.120] Consider the IVP*

$$\begin{cases} \nabla_{a*}^\nu x(t) = h(t), & t \in \mathbb{N}_{a+1} \\ \nabla^k x(a) = c_k, & k \in \mathbb{N}_0^{N-1}, \end{cases} \tag{1.8}$$

where $\nu > 0$, $N := \lceil \nu \rceil$, $h : \mathbb{N}_{a+1} \to \mathbb{R}$, *and* $c_k \in \mathbb{R}$ *for* $k \in \mathbb{N}_0^{N-1}$. *Then, the unique solution to the IVP* (1.8) *is given by*

$$x(t) = \sum_{k=0}^{N-1} H_k(t,a) c_k + \nabla_a^{-\nu} h(t), \ \ t \in \mathbb{N}_{a-N+1}.$$

By comparison, consider the corresponding variation of constants formula for the integer order case.

Theorem 1.17 *[29, Theorem 3.51] Assume* $h : \mathbb{N}_{a+1} \to \mathbb{R}$ *and* $n \in \mathbb{N}_1$. *Consider the IVP*

$$\begin{cases} \nabla^n x(t) = h(t), & t \in \mathbb{N}_{a+1} \\ \nabla^k x(a) = c_k, & k \in \mathbb{N}_0^{n-1}, \end{cases} \tag{1.9}$$

where $c_k \in \mathbb{R}$ *for* $k \in \mathbb{N}_0^{n-1}$. *Then, the unique solution to the IVP* (1.9) *is given by*

$$x(t) = \sum_{k=0}^{n-1} H_k(t,a) c_k + \nabla_a^{-n} h(t), \ \ t \in \mathbb{N}_{a-n+1}.$$

Notice that the whole order IVP (1.9) and the Caputo fractional IVP (1.8) have practically the same formula for the respective solutions. This is a nice property of the Caputo fractional difference, which results from taking the whole order difference first, and then the fractional sum. Because of the similarity between the solutions to the IVPs for the nabla whole order case and the nabla Caputo fractional case, we can develop analogues for BVPs in the nabla Caputo fractional case that closely resemble the whole order continuous case, as will be seen in the development of Green's functions for two-point boundary value problems in Chapter 2.

The next theorem gives generalized power rules for the nabla Taylor monomials.

Theorem 1.18 *[29, Theorem 3.93] Let $\nu, \mu \in \mathbb{R}$, and $\nu > 0$ such that $\mu, \nu + \mu$, and $\mu - \nu$ are not negative integers. Then, for $t \in \mathbb{N}_a$,*

(i) $\nabla_a^{-\nu} H_\mu(t,a) = H_{\mu+\nu}(t,a)$;

(ii) $\nabla_a^{\nu} H_\mu(t,a) = H_{\mu-\nu}(t,a)$,

where ∇_a^ν is the nabla Riemann-Louiville fractional difference.

We will also state the following Leibniz formula, which is useful when showing that integral expressions satisfy nabla difference equations.

Theorem 1.19 *[29, Theorem 3.41]* ***(Nabla Leibniz Formula).*** *Assume $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$. Then, for $t \in \mathbb{N}_{a+1}$,*

$$\nabla \left(\int_a^t f(t,\tau)\nabla\tau \right) = \int_a^t \nabla_t f(t,\tau)\nabla\tau + f(\rho(t),t). \tag{1.10}$$

1.2 Lyapunov Inequalities

Lyapunov-type inequalities are useful tools for studying solutions to boundary value problems. In particular, they can be applied to give nonexistence results for certain homogeneous boundary value problems and to give existence-uniqueness results for corresponding nonhomogeneous boundary value problems. They can also be used to obtain bounds on eigenvalues in certain eigenvalue problems and to consider oscillation and stability criteria of solutions.

In addition, the "zeros" of solutions and in particular, the distance between consecutive "zeros" of solutions may be analyzed using Lyapunov inequalities. In the discrete case, a solution changing sign or being zero is analogous to solutions in the continuous case having zeros. Adopting the terminology in [45], we can say that $y : \mathbb{N}_a^b \to \mathbb{R}$ has a *generalized zero* at $c \in \mathbb{N}_{a+1}^b$ provided $y(c)y(\rho(c)) \leq 0$ and $y(\rho(c)) \neq 0$. The concept of generalized zeros can be used to study disconjugacy for the discrete case and is considered for systems involving the whole order delta difference in [47].

Lyapunov inqualities have been extended and generalized in a variety of directions due to their many applications. For example, see [4], in which Lyapunov inequalities for differential equations of higher order are considered. See also [16] and [17] for generalizations to fractional BVPs involving the Riemann-Louiville derivative and extensions including fractional BVPs with solutions defined on multivariate domains. In [22] and [31], Lyapunov inequalities for fractional differential equations involving the continuous Caputo fractional derivative are investigated.

1.2.1 Ordinary Differential Equations Case

The original Lyapunov inequality result [39] can be stated as follows.

Theorem 1.20 *Let $q : [a, b] \to \mathbb{R}$ continuous. If the boundary value problem*

$$\begin{cases} x'' + q(t)x = 0, \quad t \in [a, b] \\ x(a) = x(b) = 0 \end{cases} \tag{1.11}$$

has a nontrivial solution; i.e., $x(t) \not\equiv 0$, *then*

$$\int_a^b |q(t)|dt > \frac{4}{b-a}.$$

If the BVP (1.11) *has a nontrivial solution such that* $x(t) \neq 0$ *for all* $t \in (a,b)$, *then*

$$\int_a^b q_+(t)dt > \frac{4}{b-a},$$

where $q_+(t) := \max\{q(t), 0\}$, *for* $t \in [a,b]$.

Further developments in Lyapunov inequalities for higher integer order differential equations have been made in [4]. These involve third order linear differential equations with three-point boundary conditions.

1.2.2 Recent Developments in Lyapunov Inequalities for the Fractional Case

In fractional order differential equations, a number of recent developments have been made involving Lyapunov inequalities, which are similar to the original Lyapunov inequality in Theorem 1.20. We will cite and name a few of these developments here. In [16], boundary value problems involving the equation $\left(D^\alpha_{a^+}x\right)(t) + q(t)x = 0$, for $2 < \alpha \leq 3$, are studied and several Lyapunov-type inequalities are derived. Here, $D^\alpha_{a^+}x$ denotes the continuous Riemann-Louiville fractional derivative of x. The same equation is also studied in [17] under different boundary conditions. Additional work on Lyapunov inequalities has been done for fractional differential equations in [22, 31, 32, 41], which all involve the Caputo fractional derivative. In [31] and [32], fractional equations of order between one and two are considered under Robin boundary conditions and Sturm-Louiville boundary conditions, respectively. In [22], conjugate boundary conditions for fractional equations of order between one and two and applications of Lyapunov inequalities to zeros of a Mittag-Leffler function are considered. Also, [41] involves fractional equations of order between two and three and applications including a Mittag-Leffler function and an eigenvalue problem are discussed.

In particular, we will make note the following result from [17]. The methods used to obtain this result with be adapted to the nabla Caputo fractional case in Chapter 4.

Theorem 1.21 *[17, Theorem 2.1] Assume*

$$\left(D^{\alpha}_{a+}x\right)(t) + q(t)x = 0, \;\; 2 < \alpha \leq 3,$$

where $q \in C([a,b],\mathbb{R})$*, has a nontrivial solution* $x(t)$ *satisfying either of the boundary conditions*

$$x(a) = 0 \text{ and } x'(a) = x'(b) = 0$$

or

$$x(a) = x(b) = 0 \text{ and } x'(a) = 0,$$

and $x(t)$ *does not change sign on* $[a,b]$*. Then,*

$$\int_a^b q_+(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$$

In discrete fractional calculus, Lyapunov-type inequalities for two-point conjugate and right-focal boundary value problems involving a delta fractional difference equation of order between one and two are considered in [23]. In [26], Lyapunov inequalities for delta fractional equations are used to study disconjugacy and oscillation of solutions. In the nabla Riemann-Louiville case, a Lyapunov inequality for a boundary value problem of order α, where $2 < \alpha \leq 3$, is given in [1]. There still remains much to be explored in Lyapunov inequalities for fractional difference equations, and we will develop some results in the nabla Caputo fractional case in Chapter 4.

1.3 Green's Functions for Nabla Fractional Boundary Value Problems

Green's functions were first named by Riemann after the mathematician George Green, who introduced the idea in his 1828 mathematical physics paper [19]. In the context of ordinary differential equations, fractional calculus, or discrete fractional calculus, a Green's function is used to give a formula of a solution to an equation of the form $(Lu)(t) = h(t)$ subject to given boundary conditions, where L is a general linear difference or differential operator, assuming that the solution to a boundary value problem is unique. Since the Green's function does not depend on the non-homogeneous term $h(t)$, in some sense, it can be thought of as an "inverse" to the operator L, and the solution, u, is given by an integral equation where the Green's function is the integral kernel [14]. In Chapter 2, we will consider the case where L is the nabla Caputo operator.

1.3.1 Role of Green's Functions in Lyapunov Inequalities

Green's functions play an essential role in deriving Lyapunov inequalities for boundary value problems. A general method of obtaining Lyapunov inequalities involves converting a given boundary value problem to an equivalent integral equation involving a Green's function and then using bounds on the Green's function [24].

1.4 Further Reading

For other related results, see [2, 5–7, 10, 12, 14, 18, 20, 21, 25, 28, 30, 33, 35–37, 40, 46, 48, 50–53, 56, 58, 61–63]. In particular, [37, 51, 52] are general texts in continuous fractional calculus, and [53] provides an introduction to fractional order operators with engineering applications. Also, [48] provides a survey and history of Lyapunov inequalities in the whole order continuous setting, and a survey on recent developments in Lyapunov inequalities is given in [56]. For a view of Lyapunov inequalities in the context of time scales, see [12]. In [6], Lyapunov inequalities for nonlinear ordinary difference equations are studied. For results involving Lyapunov inequalities in continuous fractional calculus, see [5, 18, 21, 25, 30, 40, 46, 50]. Other results involving boundary value problems in continuous fractional calculus include [10, 36, 63]. Discrete fractional boundary value problems are studied in [20, 28]. Lyapunov inequalities in the delta whole order case are considered in [58, 61, 62]. For results involving the nabla Riemann-Louiville fractional difference, see [2, 7]. Linear fractional nabla difference equations are considered in [33].

Chapter 2

Green's Functions for Two-Point Boundary Value Problems Involving a Nabla Caputo Operator

In this chapter, we will consider the Green's functions for several cases of $(k, N-k)$ boundary value problems, including those involving conjugate and right-focal boundary conditions. Analogous Green's functions in the ordinary differential equations context are given in [35, Chapter 6], and the formulas for the Green's functions in terms of determinants are strikingly similar in the nabla Caputo fractional case. The boundary value problems studied in this chapter have more general boundary conditions at the right endpoint than those given in [35, Chapter 6] for the differential equations case. Considering the boundary conditions in this way allows us to obtain a result in Theorem 2.12 which gives a single explicit expression for Green's functions encompassing the cases of conjugate, right-focal, and even more general boundary conditions. In Subsection 2.3.2, we obtain the Green's functions considered in [27, Theorems 3.9, 4.6] as special cases of Theorem 2.12.

The results for the more general version of the boundary conditions at the right endpoint, with suitable adjustments, may also be employed to give counterparts in the ordinary differential equations setting or the continuous fractional setting. The authors in [14] use methods similar to those in [35, Chapter 6] for the ordinary differential equations case with general two-point boundary conditions of the form

$$\sum_{j=0}^{n-1} \alpha_j^i y^{(j)}(a) + \beta_j^i y^{(j)}(b) = \gamma_i, \;\; i \in \mathbb{N}_1^n,$$

where α_j^i, β_j^i and γ_i are real constants, and they develop an algorithm to compute Green's functions for n-th order linear differential equations with constant coefficients by solving an $n \times n$ linear system using the existence of n linearly independent solutions to the homogeneous differential equation. In this chapter, we will use a related approach for the nabla Caputo fractional case, using closed form expressions of N linearly independent solutions in terms of nabla Taylor monomials to the nabla Caputo difference equation $\nabla_{a*}^{\nu} x(t) = 0$, $t \in \mathbb{N}_{a+1}$.

2.1 Preliminaries

In the following theorem, we will establish a form for a general solution to $\nabla_{a*}^{\nu} x(t) = h(t)$ in terms of Taylor monomials based at modified points. This form will be useful when considering $(k, N-k)$ boundary value problems.

Theorem 2.1 *Let $\nu > 0$ and $N := \lceil \nu \rceil$. A general solution to*

$$\nabla_{a*}^{\nu} x(t) = h(t), \;\; t \in \mathbb{N}_{a+1} \tag{2.1}$$

is given by

$$x(t) = \sum_{p=0}^{N-1} c_p H_p(t, a-N+p) + \nabla_a^{-\nu} h(t), \tag{2.2}$$

for $t \in \mathbb{N}_{a-N+1}$, where c_p for $p \in \mathbb{N}_0^{N-1}$ are arbitrary constants.

Proof. First, we will show that $x(t)$, given by (2.2), satisfies the equation (2.1) on $\mathbb{N}_{a+1}$. We will use the notation from Remark 1.12 to avoid ambiguity about domains.

For $t \in \mathbb{N}_{a+1}$, consider

$$\begin{aligned}\nabla_{a*}^{\nu} x(t) &= \nabla_{a*}^{\nu}\left[\sum_{p=0}^{N-1} c_p H_p(\cdot, a-N+p) + \nabla_a^{-\nu} h(\cdot)\right](t)\\ &= \sum_{p=0}^{N-1} c_p \nabla_{a*}^{\nu}[H_p(\cdot, a-N+p)](t) + \nabla_{a*}^{\nu}\nabla_a^{-\nu}[h(\cdot)](t),\end{aligned}$$

where we have made use of the linearity of the operator ∇_{a*}^{ν}. Note that, for $t \in \mathbb{N}_{a+1}$ and $p \in \mathbb{N}_0^{N-1}$,

$$\begin{aligned}\nabla_{a*}^{\nu}[H_p(\cdot, a-N+p)](t) &\overset{(1.4)}{=} \nabla_a^{-(N-\nu)}\nabla^N[H_p(\cdot, a-N+p)](t)\\ &= \nabla_a^{-(N-\nu)}\nabla^{N-p-1}\nabla^{p+1}[H_p(\cdot, a-N+p)](t)\\ &\;\;\vdots\\ &= \nabla_a^{-(N-\nu)}\nabla^{N-p-1}\nabla^{2}[H_1(\cdot, a-N+p)](t)\\ &= \nabla_a^{-(N-\nu)}\nabla^{N-p-1}\nabla(1)\\ &= \nabla_a^{-(N-\nu)}\nabla^{N-p-1}0 = 0, \qquad (2.3)\end{aligned}$$

where in the second equality, the operator $\nabla^{N-p-1}\nabla^{p+1}$ makes sense because $p \leq N-1$, and the third equality follows by repeated applications of Theorem 1.8, part (ii). Also,

$$\begin{aligned}\nabla_{a*}^{\nu}\nabla_a^{-\nu}h(t) &\overset{(1.4)}{=} \nabla_a^{-(N-\nu)}\nabla^N\nabla_a^{-\nu}h(t)\\ &\overset{(1.5)}{=} \nabla_a^{-(N-\nu)}\nabla_a^{N-\nu}h(t)\\ &\overset{(1.7)}{=} h(t), \qquad (2.4)\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$. Hence, by (2.3) and (2.4) we have that

$$\sum_{p=0}^{N-1} c_p \nabla_{a*}^{\nu} \left[H_p(t, a-N+p)\right] + \nabla_{a*}^{\nu} \nabla_a^{-\nu} h(t) = h(t),$$

for $t \in \mathbb{N}_{a+1}$. Thus, $x(t)$ is a solution to (2.1) and is defined on $\mathbb{N}_{a-N+1}$.

Now we will show that $x(t)$, given by (2.2), is a general solution to (2.1). First, suppose $y(t)$ is any solution to $\nabla_{a*}^{\nu} y(t) = 0$. Then, $y(t)$ is determined by its initial values, $\nabla^k y(a)$ for $k \in \mathbb{N}_0^{N-1}$, by Theorem 1.16. Let the initial values of $H_p(t, a-N+k)$ for $p \in \mathbb{N}_0^{N-1}$ be given by the vector

$$\mathbf{v}_p := \langle H_p(t, a-N+k)|_{t=a}, \nabla H_p(t, a-N+k)|_{t=a}, \ldots, \nabla^{N-1} H_p(t, a-N+k)|_{t=a}\rangle. \quad (2.5)$$

Then, for $k \in \mathbb{N}_0^{N-1}$,

$$\begin{aligned} \nabla^k H_p(t, a-N+p)\Big|_{t=a} &= H_{p-k}(a, a-N+p), \text{ by Theorem 1.8, part (ii)} \\ &= \begin{cases} \frac{(N-p)^{\overline{p-k}}}{(p-k)!} = \frac{(N-k-1)!}{(N-p-1)!(p-k)!}, & \text{if } k \leq p \\ 0, & \text{if } k > p, \end{cases} \end{aligned}$$

by Theorem 1.8, part (v). Hence, the first p coordinates of the vector (2.5) are nonzero, and the remaining $N-p$ are zero. Therefore, the vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{N-1} \in \mathbb{R}^N$ are linearly independent, giving a basis for $\mathbb{R}^N$. Hence,

$$\langle y(a), \nabla y(a), \ldots, \nabla^{N-1} y(a)\rangle = \sum_{p=0}^{N-1} c_p \mathbf{v}_p,$$

for some $c_p \in \mathbb{R}$, for $p \in \mathbb{N}_0^{N-1}$. It follows that $y(t) = \sum_{p=0}^{N-1} c_p H_p(t, a-N+p)$.

Next, suppose $w(t)$ is a solution to (2.1). Since $\nabla_a^{-\nu} h(t)$ is a particular solution to (2.1), as shown in (2.4), and by the linearity of the Caputo difference, we have $w(t)-\nabla_a^{-\nu}h(t)$ is a solution to $\nabla_{a*}^{\nu}y(t)=0$. Then, by the above argument, $w(t)-\nabla_a^{-\nu}h(t)=\sum_{p=0}^{N-1}c_pH_p(t,a-N+p)$. Thus, $w(t)=\sum_{p=0}^{N-1}c_pH_p(t,a-N+p)+\nabla_a^{-\nu}h(t)$, so (2.2) gives a general solution to (2.1). ♢

Remark 2.2 In the continuous case, for each $p\in\mathbb{N}_1^{n-1}$, $x_p(t):=\frac{(t-a)^p}{p!}$ is a solution to the equation $x^{(n)}=0$ satisfying the initial conditions $x^{(i)}(a)=0$ for $i\in\mathbb{N}_0^{p-1}$. In particular, we say $x_p(t)$ has a zero of multiplicity p at $t=a$. In an analogous way, for each $p\in\mathbb{N}_1^{N-1}$, $H_p(t,a-N+p)$ is a solution to $\nabla_{a*}^{\nu}x(t)=0$ satisfying $\nabla^i x(a-N+p)=0$ for $i\in\mathbb{N}_0^{p-1}$. Moreover, $H_p(t,a-N+p)$ has p consecutive zeros on the domain $\mathbb{N}_{a-N+1}$ at $t=a-N+1,\dots,a-N+p$.

Next, we get a form of any solution $x(t)$ to the homogeneous equation $\nabla_{a*}^{\nu}x(t)=0$ which satisfies k homogeneous initial conditions, for any fixed $k\in\mathbb{N}_1^{N-1}$ with $N:=\lceil\nu\rceil$.

Lemma 2.3 *Let $\nu>1$, $N:=\lceil\nu\rceil$, $k\in\mathbb{N}_1^{N-1}$, and suppose $x:\mathbb{N}_{a-N+1}\to\mathbb{R}$ is a solution to the equation*

$$\nabla_{a*}^{\nu}x(t)=0,\quad t\in\mathbb{N}_{a+1}. \tag{2.6}$$

Moreover, assume that x satisfies the conditions

$$\nabla^i x(a-N+k)=0,\quad i\in\mathbb{N}_0^{k-1}.$$

Then,

$$x(t)=\sum_{p=k}^{N-1}c_pH_p(t,a-N+p),$$

where $c_k,c_{k+1},\dots,c_{N-1}\in\mathbb{R}$.

Proof. Let $x(t)$ be a solution to (2.6). Then, by Theorem 2.1, we have

$$x(t) = \sum_{p=0}^{N-1} c_p H_p(t, a-N+p), \quad t \in \mathbb{N}_{a-N+1},$$

where c_p for $p \in \mathbb{N}_0^{N-1}$ are constants. Note that $\nabla^i x$, for $i \in \mathbb{N}_0^{k-1}$, is defined on $\mathbb{N}_{a-N+1+i}$, and we have $a-N+k \in \mathbb{N}_{a-N+1+i}$. Let $i \in \mathbb{N}_0^{k-1}$ and consider

$$\begin{aligned} \nabla^i x(a-N+k) &= \sum_{p=0}^{N-1} c_p \nabla^i H_p(t, a-N+p) \mid_{t=a-N+k} \\ &= \sum_{p=0}^{N-1} c_p H_{p-i}(a-N+k, a-N+p). \end{aligned} \tag{2.7}$$

Note that

$$H_{p-i}(a-N+k, a-N+p) = \frac{(k-p)^{\overline{p-i}}}{\Gamma(p-i+1)} = \frac{\Gamma(k-i)}{\Gamma(k-p)\Gamma(p-i+1)}.$$

Also,

$$\frac{\Gamma(k-i)}{\Gamma(p-i+1)} = \begin{cases} 0, & \text{if } p-i+1 \leq 0 \\ \frac{(k-i-1)!}{(p-i)!}, & \text{if } p-i+1 > 0, \end{cases}$$

and

$$\frac{1}{\Gamma(k-p)} = \begin{cases} 0, & \text{if } k-p \leq 0 \\ \frac{1}{(k-p-1)!}, & \text{if } k-p > 0. \end{cases}$$

Hence, $H_{p-i}(a-N+k, a-N+p) \neq 0$ if and only if $p < k$ and $p > i-1$. From the conditions $\nabla^i x(a-N+k)=0$ for $i\in\mathbb{N}_0^{k-1}$ and (2.7), we then have

$$\sum_{p=0}^{N-1} c_p H_{p-i}(a-N+k, a-N+p) = \sum_{p=i}^{k-1} c_p \frac{(k-i-1)!}{(p-i)!(k-p-1)!} = 0, \tag{2.8}$$

for each $i\in\mathbb{N}_0^{k-1}$. Letting $i=k-1,\ k-2,\ \ldots,\ 0$ in (2.8) with the given order implies $c_{k-1}=c_{k-2}=\cdots=c_0=0$. Hence, $x(t)=\sum\limits_{p=k}^{N-1} c_p H_p(t,a-N+p)$. $\Diamond$

2.2 Green's Functions

Next, we give an existence-uniqueness result for two-point boundary value problems involving the operator ∇_{a*}^{ν}. This type of existence-uniqueness result is often referred to as Fredholm's Alternative Theorem [14]. The standard argument of the proof can be found in [35] for the ordinary differential equations case, in [3] for a nabla fractional self-adjoint equation, and in [27] for $(N-1,1)$ right-focal BVPs involving the equation $\nabla_{a*}^{\nu}x(t)=0$. The argument could also be formulated more generally for linear fractional differential or difference operators under general multipoint boundary conditions.

Theorem 2.4 *(Existence-Uniqueness Theorem) Let $\nu>1$, $N:=\lceil\nu\rceil$, $k\in\mathbb{N}_1^{N-1}$, and $h:\mathbb{N}_{a+1}^{b}\to\mathbb{R}$. Furthermore, let $j_m\in\mathbb{N}_0^{N-1}$ for $m\in\mathbb{N}_1^{N-k}$, with $j_1<j_2<j_3<\cdots<j_{N-k}$, and assume $b-a\in\mathbb{N}_{\max\{1,j_{N-k}-N+k+1\}}$. Then, the homogeneous $(k,N-k)$ BVP*

$$\begin{cases} \nabla_{a*}^{\nu}y(t)=0, & t\in\mathbb{N}_{a+1}^{b} \\ \nabla^{i}y(a-N+k)=0, & i\in\mathbb{N}_0^{k-1} \\ \nabla^{j_m}y(b)=0, & m\in\mathbb{N}_1^{N-k} \end{cases} \tag{2.9}$$

has only the trivial solution if and only if the nonhomogeneous $(k, N-k)$ BVP

$$\begin{cases} \nabla_{a*}^{\nu} w(t) = h(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{i} w(a-N+k) = A_i, & i \in \mathbb{N}_0^{k-1} \\ \nabla^{j_m} w(b) = B_{j_m}, & m \in \mathbb{N}_1^{N-k}, \end{cases} \tag{2.10}$$

for all choices of A_i and B_{j_m}, where $A_i, B_{j_m} \in \mathbb{R}$, for $i \in \mathbb{N}_0^{k-1}$ and $m \in \mathbb{N}_1^{N-k}$, has a unique solution.

Proof. By Theorem 2.1, a general solution to $\nabla_{a*}^{\nu} y(t) = 0$ is given by

$$y(t) = c_0 H_0(t, a-N) + c_1 H_1(t, a-N+1) + \cdots + c_{N-1} H_{N-1}(t, a-1).$$

Fix $k \in \mathbb{N}_1^{N-1}$ and let $\alpha := a - N + k$. Then, y satisfies the boundary conditions in (2.9) if and only if

$$\begin{aligned} c_0 H_0(\alpha, a-N) + c_1 H_1(\alpha, a-N+1) + \cdots + c_{N-1} H_{N-1}(\alpha, a-1) &= 0 \\ c_0 \nabla H_0(\alpha, a-N) + c_1 \nabla H_1(\alpha, a-N+1) + \cdots + c_{N-1} \nabla H_{N-1}(\alpha, a-1) &= 0 \\ &\vdots \\ c_0 \nabla^{k-1} H_0(\alpha, a-N) + c_1 \nabla^{k-1} H_1(\alpha, a-N+1) + \cdots + c_{N-1} \nabla^{k-1} H_{N-1}(\alpha, a-1) &= 0 \end{aligned}$$

and

$$\begin{aligned} c_0 \nabla^{j_1} H_0(b, a-N) + c_1 \nabla^{j_1} H_1(b, a-N+1) + \cdots + c_{N-1} \nabla^{j_1} H_{N-1}(b, a-1) &= 0 \\ c_0 \nabla^{j_2} H_0(b, a-N) + c_1 \nabla^{j_2} H_1(b, a-N+1) + \cdots + c_{N-1} \nabla^{j_2} H_{N-1}(b, a-1) &= 0 \\ &\vdots \\ c_0 \nabla^{j_{N-k}} H_0(b, a-N) + c_1 \nabla^{j_{N-k}} H_1(b, a-N+1) + \cdots + c_{N-1} \nabla^{j_{N-k}} H_{N-1}(b, a-1) &= 0. \end{aligned}$$

Let $x_p(t) := H_p(t, a-N+p)$. This linear system is equivalent to the vector equation

$$\underbrace{\begin{pmatrix} x_0(\alpha) & x_1(\alpha) & \cdots & x_{N-1}(\alpha) \\ \nabla x_0(\alpha) & \nabla x_1(\alpha) & \cdots & \nabla x_{N-1}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{k-1} x_0(\alpha) & \nabla^{k-1} x_1(\alpha) & \cdots & \nabla^{k-1} x_{N-1}(\alpha) \\ \nabla^{j_1} x_0(b) & \nabla^{j_1} x_1(b) & \cdots & \nabla^{j_1} x_{N-1}(b) \\ \nabla^{j_2} x_0(b) & \nabla^{j_2} x_1(b) & \cdots & \nabla^{j_2} x_{N-1}(b) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}} x_0(b) & \nabla^{j_{N-k}} x_1(b) & \cdots & \nabla^{j_{N-k}} x_{N-1}(b) \end{pmatrix}}_{=:M} \underbrace{\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \\ c_k \\ c_{k+1} \\ \vdots \\ c_{N-1} \end{pmatrix}}_{=:\mathbf{c}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since, by hypothesis, the homogeneous BVP (2.9) has only the trivial solution, the above vector equation has only the trivial solution $\mathbf{c} = \mathbf{0}$. Since $\det M \neq 0$ if and only if the vector equation has only the trivial solution, this implies that $\det M \neq 0$.

Now suppose w is a solution to the nonhomogeneous equation $\nabla_{a*}^{\nu} w(t) = h(t)$. Then, by Theorem 2.1, we have

$$w(t) = d_0 H_0(t, a-N) + d_1 H_1(t, a-N+1) + \cdots + d_{N-1} H_{N-1}(t, a-1) + \nabla_a^{-\nu} h(t),$$

for some constants $d_0, d_1, \ldots, d_{N-1}$. Then, the boundary value problem (2.10) has a solution if and only if the vector equation

$$M\mathbf{d} = \begin{pmatrix} A_0 - \nabla_a^{-\nu} h(a-N+k) \\ \vdots \\ A_{k-1} - \nabla^{k-1}\nabla_a^{-\nu} h(a-N+k) \\ B_{j_1} - \nabla^{j_1}\nabla_a^{-\nu} h(b) \\ \vdots \\ B_{j_{N-k}} - \nabla^{j_{N-k}}\nabla_a^{-\nu} h(b) \end{pmatrix},$$

where $\mathbf{d} := \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{N-1} \end{pmatrix}$, has a solution. Since $\det M \neq 0$, this vector equation has a unique solution $\mathbf{d}$, so the BVP (2.10) has a unique solution.

Conversely, suppose the boundary value problem (2.10) has a unique solution. Since (2.10) has a unique solution for the particular homogeneous case $h(t) \equiv 0$, $A_i = 0$, and $B_{j_m} = 0$ in (2.10), we have that (2.9) has a unique solution. Since the trivial solution satisfies the boundary conditions and the difference equation in (2.9), we have that the homogeneous BVP (2.9) has only the trivial solution. $\Diamond$

Remark 2.5 Note that, from the proof of Theorem 2.4, a necessary and sufficient condition for the nonhomogeneous BVP (2.10) to have a unique solution is $\det M \neq 0$, where M is given as above.

Let $\alpha := a - N + k$. In the remainder of this section, we let $D :=$

$$\begin{pmatrix} \nabla^{j_1} H_k(b,\alpha) & \nabla^{j_1} H_{k+1}(b,\alpha+1) & \cdots & \nabla^{j_1} H_{N-1}(b,a-1) \\ \nabla^{j_2} H_k(b,\alpha) & \nabla^{j_2} H_{k+1}(b,\alpha+1) & \cdots & \nabla^{j_2} H_{N-1}(b,a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}} H_k(b,\alpha) & \nabla^{j_{N-k}} H_{k+1}(b,\alpha+1) & \cdots & \nabla^{j_{N-k}} H_{N-1}(b,a-1) \end{pmatrix}. \tag{2.11}$$

Theorem 2.6 *A necessary and sufficient condition for uniqueness of solutions to the nonhomogeneous BVP* (2.10) *is* $\det D \neq 0$*, where* D *is given by* (2.11)*.*

Proof. By Lemma 2.3, a solution to $\nabla^\nu_{a*}x(t) = 0$, for $t \in \mathbb{N}^b_{a+1}$, which satisfies the conditions $\nabla^i x(a-N+k) = 0$, for $i \in \mathbb{N}^{k-1}_0$ where $k \in \mathbb{N}^{N-1}_1$ is fixed, is given by

$$x(t) = c_k H_k(t, a-N+k) + c_{k+1}H_{k+1}(t, a-N+k+1) + \cdots + c_{N-1}H_{N-1}(t, a-1).$$

Using the boundary conditions at $t = b$ in (2.9) in the last equation, we get the vector equation

$$D \begin{pmatrix} c_k \\ c_{k+1} \\ \vdots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where D is given by (2.11). This vector equation has only the trivial solution if and only if $\det D \neq 0$. It follows by Theorem 2.4 that the nonhomogeneous BVP (2.10) has a unique solution if and only if $\det D \neq 0$. $\Diamond$

The next theorem follows directly from Lemma A.1 and Lemma A.2, given in Appendix A.

Theorem 2.7 *The matrix* D*, given by* (2.11)*, has a nonzero determinant.*

By Theorem 2.6 and Theorem 2.7, we get that the nonhomogeneous BVP (2.10) has a unique solution, which is stated in the next theorem.

Theorem 2.8 *The nonhomogeneous BVP* (2.10) *has a unique solution.*

As a special case of Theorem 2.8, we get that the nonhomogeneous $(N-1,1)$ BVP

$$\begin{cases} \nabla_{a*}^{\nu} x(t) = h(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{i} x(a-1) = A_i, & i \in \mathbb{N}_{0}^{N-2} \\ \nabla^{j} x(b) = B, \end{cases}$$

where $\nu > 1$, $N := \lceil \nu \rceil$, $b - a \in \mathbb{N}_{N-1}$, $j \in \mathbb{N}_0^{N-1}$ is fixed, $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, and A_i, $B \in \mathbb{R}$ for $i \in \mathbb{N}_0^{N-2}$, has a unique solution, which confirms [27, Theorem 4.3].

The next example shows that two-point boundary value problems involving the operator ∇_{a*}^{ν} need not always have unique solutions.

Example 2.9 Let $\nu > 3$, $N := \lceil \nu \rceil$, and fix $k \in \mathbb{N}_3^{N-1}$. For any $c_0, c_1 \in \mathbb{R}$,

$$x(t) = c_0 + c_1 H_1(t, a-N+1), \;\; t \in \mathbb{N}_{a-N+1}^{b}$$

is a solution to the equation $\nabla_{a*}^{\nu} x(t) = 0$. One may verify that $x(t)$ satisfies the N boundary conditions

$$\begin{cases} \nabla^{i} x(a-N+k) = 0, & i \in \mathbb{N}_2^{k-1} \\ \nabla^{j} x(b) = 0, & j \in \mathbb{N}_2^{N-k+3}. \end{cases}$$

The function $G : \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ given in the next theorem is called the *Green's function* for the homogeneous BVP (2.9). Note that the Green's function is used to find the unique solution to the nonhomogeneous BVP (2.10).

Theorem 2.10 *Let* $\nu > 1$ *and* $N := \lceil \nu \rceil$. *Assume* $k \in \mathbb{N}_1^{N-1}$, $j_m \in \mathbb{N}_0^{N-1}$ *for* $m \in \mathbb{N}_1^{N-k}$, *with* $j_1 < j_2 < \cdots < j_{N-k}$, *and* $b-a \in \mathbb{N}_{\max\{1, j_{N-k}-N+k+1\}}$. *For each fixed* $s \in \mathbb{N}_{a+1}^b$, *let* $u(t,s)$ *be defined as the solution to the BVP*

$$\begin{cases} \nabla_{a*}^{\nu} u(t,s) = 0, & t \in \mathbb{N}_{a+1}^b \\ \nabla^i u(a-N+k, s) = 0, & i \in \mathbb{N}_0^{k-1} \\ \nabla^{j_m} u(b,s) = -\nabla^{j_m} H_{\nu-1}(b, \rho(s)), & m \in \mathbb{N}_1^{N-k}. \end{cases} \tag{2.12}$$

Define

$$G(t,s) := \begin{cases} u(t,s), & \text{if } t \leq \rho(s) \\ v(t,s), & \text{if } t \geq \rho(s), \end{cases} \tag{2.13}$$

where $v(t,s) := u(t,s) + H_{\nu-1}(t, \rho(s))$ *and* $(t,s) \in \mathbb{N}_{a-N+1}^b \times \mathbb{N}_{a+1}^b$. *Then,*

$$w(t) := \int_a^b G(t,s)h(s)\nabla s$$

is the unique solution to the nonhomogeneous $(k, N-k)$ *BVP* (2.10) *with* $A_i, B_{j_m} = 0$. *Note that in the case* $t = \rho(s)$, *we have* $u(t,s) = v(t,s)$.

Proof. By Theorem 2.8, the BVP (2.12), for each fixed $s \in \mathbb{N}_{a+1}^b$, has a unique solution, so $u(t,s)$ is well defined. Let $G(t,s)$ be defined as in (2.13) and $w(t) := \int_a^b G(t,s)h(s)\nabla s$. First, for $t \in \mathbb{N}_a^b$,

$$\begin{aligned} w(t) &= \int_a^t G(t,s)h(s)\nabla s + \int_t^b G(t,s)h(s)\nabla s \\ &= \int_a^t v(t,s)h(s)\nabla s + \int_t^b u(t,s)h(s)\nabla s \\ &= \int_a^t [u(t,s) + H_{\nu-1}(t,\rho(s))]h(s)\nabla s + \int_t^b u(t,s)h(s)\nabla s. \end{aligned}$$

Hence,

$$\begin{aligned} w(t) &= \int_a^b u(t,s)h(s)\nabla s + \int_a^t H_{\nu-1}(t,\rho(s))h(s)\nabla s \\ &\overset{(1.3)}{=} \int_a^b u(t,s)h(s)\nabla s + \nabla_a^{-\nu}h(t). \end{aligned}$$

Now for the case when $t \in \mathbb{N}_{a-N+1}^{a-1}$, note that $G(t,s) = u(t,s)$ by (2.13), since $s \in \mathbb{N}_{a+1}^b$. Hence, in this case, we have $w(t) = \int_a^b u(t,s)h(s)\nabla s$. Noting that $\nabla_a^{-\nu}h(t) = 0$ for $t \in \mathbb{N}_{a-N+1}^{a-1}$ by convention, we have that $w(t) = \int_a^b u(t,s)h(s)\nabla s + \nabla_a^{-\nu}h(t)$ holds for all $t \in \mathbb{N}_{a-N+1}^b$.

We have

$$\begin{aligned} \nabla_{a*}^{\nu}w(t) &= \nabla_{a*}^{\nu}\left[\int_a^b u(t,s)h(s)\nabla s + \nabla_a^{-\nu}h(t)\right] \\ &= \sum_{s=a+1}^{b} \nabla_{a*}^{\nu}u(t,s)h(s) + \nabla_{a*}^{\nu}\nabla_a^{-\nu}h(t) \\ &\overset{(2.12),(2.4)}{=} h(t). \end{aligned}$$

Since $\nabla_a^{-\nu}h(a-N+1) = \cdots = \nabla_a^{-\nu}h(a) = 0$ by convention, in particular, we get $\nabla^i(\nabla_a^{-\nu}h)(a-N+k) = 0$ for $i \in \mathbb{N}_0^{k-1}$. Thus,

$$\begin{aligned} \nabla^i w(t)|_{t=a-N+k} &= \int_a^b \nabla^i u(a-N+k,s)h(s)\nabla s + \nabla^i(\nabla_a^{-\nu}h)(a-N+k) \\ &= 0 \end{aligned}$$

for $i \in \mathbb{N}_0^{k-1}$, since, for each fixed $s \in \mathbb{N}_{a+1}^b$, $u(t,s)$ satisfies the boundary conditions at $t = a - N + k$ given in (2.12). Moreover, for $j_m \in \mathbb{N}_0^{N-1}$, $m \in \mathbb{N}_1^{N-k}$,

$$
\begin{aligned}
\nabla^{j_m} w(t)|_{t=b} &= \int_a^b \nabla_t^{j_m} u(t,s)h(s)\nabla s\Big|_{t=b} + \nabla^{j_m}\left[\int_a^t H_{\nu-1}(t,\rho(s))h(s)\nabla s\right]\Big|_{t=b} \\
&\overset{(1.10)}{=} \int_a^b \nabla_t^{j_m} u(t,s)h(s)\nabla s\Big|_{t=b} \\
&\quad + \left[\nabla^{j_m-1}\int_a^t \nabla_t H_{\nu-1}(t,\rho(s))h(s)\nabla s + H_{\nu-1}(\rho(t),\rho(t))h(t)\right]\Big|_{t=b} \\
&= \int_a^b \nabla_t^{j_m} u(t,s)h(s)\nabla s\Big|_{t=b} + \left[\nabla^{j_m-1}\int_a^t \nabla_t H_{\nu-1}(t,\rho(s))h(s)\nabla s\right]\Big|_{t=b} \\
&\overset{(1.10)}{=} \int_a^b \nabla_t^{j_m} u(t,s)h(s)\nabla s\Big|_{t=b} \\
&\quad + \left[\nabla^{j_m-2}\int_a^t \nabla_t^2 H_{\nu-1}(t,\rho(s))h(s)\nabla s + \nabla_t H_{\nu-1}(\rho(t),\rho(t))h(t)\right]\Big|_{t=b} \\
&= \int_a^b \nabla_t^{j_m} u(t,s)h(s)\nabla s\Big|_{t=b} + \left[\nabla^{j_m-2}\int_a^t \nabla_t^2 H_{\nu-1}(t,\rho(s))h(s)\nabla s\right]\Big|_{t=b} \\
&\ \vdots \\
&\overset{(1.10)}{=} \int_a^b \nabla_t^{j_m} u(t,s)h(s)\nabla s\Big|_{t=b} \\
&\quad + \left[\int_a^t \nabla_t^{j_m} H_{\nu-1}(t,\rho(s))h(s)\nabla s + \nabla_t^{j_m-1} H_{\nu-1}(\rho(t),\rho(t))h(t)\right]\Big|_{t=b} \\
&\overset{(2.12)}{=} \int_a^b -\nabla_t^{j_m}\left[H_{\nu-1}(t,\rho(s))\right]\Big|_{t=b} h(s)\nabla s \\
&\quad + \int_a^b \nabla_t^{j_m}\left[H_{\nu-1}(t,\rho(s))\right]\Big|_{t=b} h(s)\nabla s \\
&= 0.
\end{aligned}
$$

$\Diamond$

The proof of the following corollary is standard and follows in a straightforward manner from Theorem 2.10.

Corollary 2.11 *Assume that the hypotheses of Theorem 2.10 hold. Also, let* $h : \mathbb{N}_{a+1}^b \to \mathbb{R}$, $G(t,s)$ *be as defined in Theorem 2.10, and* w *be the unique solution to the*

BVP

$$\begin{cases} \nabla^{\nu}_{a*} w(t) = 0, & t \in \mathbb{N}^{b}_{a+1} \\ \nabla^{i} w(a-N+k) = A_i, & i \in \mathbb{N}^{k-1}_{0} \\ \nabla^{j_m} w(b) = B_{j_m}, & m \in \mathbb{N}^{N-k}_{1}. \end{cases} \tag{2.14}$$

Then, the unique solution to the nonhomogeneous BVP

$$\begin{cases} \nabla^{\nu}_{a*} y(t) = h(t), & t \in \mathbb{N}^{b}_{a+1} \\ \nabla^{i} y(a-N+k) = A_i, & i \in \mathbb{N}^{k-1}_{0} \\ \nabla^{j_m} y(b) = B_{j_m}, & m \in \mathbb{N}^{N-k}_{1}, \end{cases}$$

is given by

$$y(t) := w(t) + \int_a^b G(t,s)h(s)\nabla s,$$

where $w(t)$ satisfies the homogeneous equation and the nonhomogeneous boundary conditions in (2.14)*.*

Theorem 2.12 *Assume that the hypotheses of Theorem 2.10 hold. Then, the Green's function for the $(k, N-k)$ BVP* (2.9) *is given by* (2.13)*, where $u(t,s) =$*

$$\frac{1}{\beta}\begin{vmatrix} 0 & H_k(t,\alpha) & \cdots & H_{N-1}(t,a-1) \\ \nabla^{j_1} H_{\nu-1}(b,\rho(s)) & \nabla^{j_1} H_k(b,\alpha) & \cdots & \nabla^{j_1} H_{N-1}(b,a-1) \\ \nabla^{j_2} H_{\nu-1}(b,\rho(s)) & \nabla^{j_2} H_k(b,\alpha) & \cdots & \nabla^{j_2} H_{N-1}(b,a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}} H_{\nu-1}(b,\rho(s)) & \nabla^{j_{N-k}} H_k(b,\alpha) & \cdots & \nabla^{j_{N-k}} H_{N-1}(b,a-1) \end{vmatrix}, \tag{2.15}$$

for $(t,s) \in \mathbb{N}^{b}_{a-N+1} \times \mathbb{N}^{b}_{a+1}$, with $\beta := \det D$, where D is given by (2.11)*, $v(t,s) := u(t,s) + H_{\nu-1}(t,\rho(s))$, and $\alpha := a - N + k$.*

Proof. Let $u(t,s)$ be given by (2.15). By Theorem 2.7, $\beta \neq 0$, so u is well defined. Then, expanding $u(t,s)$ along the first row, for each fixed s, $u(t,s)$ is a linear combination of $H_k(t,a-N+k), H_{k+1}(t,a-N+k+1), \ldots, H_{N-1}(t,a-1)$. Hence, for each fixed s, $u(t,s)$ is a solution to $\nabla^\nu_{a*}x(t)=0$. To show $\nabla^i u(a-N+k,s)=0$ for each $i \in \mathbb{N}_0^{k-1}$, it suffices to show $\nabla^i H_k(a-N+k,a-N+k) = \nabla^i H_{k+1}(a-N+k,a-N+k+1) = \cdots = \nabla^i H_{N-1}(a-N+k,a-1) = 0$ for each $i \in \mathbb{N}_0^{k-1}$. Consider, for $r \in \mathbb{N}_k^{N-1}$,

$$\begin{aligned}
\nabla^i H_r(t,a-N+r)|_{t=a-N+k} &= H_{r-i}(a-N+k,a-N+r)\\
&= \frac{(k-r)^{\overline{r-i}}}{\Gamma(r-i+1)}\\
&= \frac{\Gamma(k-i)}{\Gamma(k-r)\Gamma(r-i+1)}\\
&= 0,
\end{aligned}$$

since $k-i>0$ and $k-r$ is a nonpositive integer. Hence, we have that $u(t,s)$ satisfies the boundary conditions at $t=a-N+k$ given in (2.12).

Next, define $z(t,s) :=$

$$\frac{1}{\beta}\begin{vmatrix}
H_{\nu-1}(t,\rho(s)) & H_k(t,\alpha) & \cdots & H_{N-1}(t,a-1)\\
\nabla^{j_1}H_{\nu-1}(b,\rho(s)) & \nabla^{j_1}H_k(b,\alpha) & \cdots & \nabla^{j_1}H_{N-1}(b,a-1)\\
\nabla^{j_2}H_{\nu-1}(b,\rho(s)) & \nabla^{j_2}H_k(b,\alpha) & \cdots & \nabla^{j_2}H_{N-1}(b,a-1)\\
\vdots & \vdots & \ddots & \vdots\\
\nabla^{j_{N-k}}H_{\nu-1}(b,\rho(s)) & \nabla^{j_{N-k}}H_k(b,\alpha) & \cdots & \nabla^{j_{N-k}}H_{N-1}(b,a-1)
\end{vmatrix},$$

where $\alpha = a - N + k$. Expanding $z(t,s)$ along the first row, we have $z(t,s) =$

$$\frac{1}{\beta}H_{\nu-1}(t,\rho(s))\begin{vmatrix} \nabla^{j_1}H_k(b,\alpha) & \nabla^{j_1}H_{k+1}(b,\alpha+1) & \cdots & \nabla^{j_1}H_{N-1}(b,a-1) \\ \nabla^{j_2}H_k(b,\alpha) & \nabla^{j_2}H_{k+1}(b,\alpha+1) & \cdots & \nabla^{j_2}H_{N-1}(b,a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}}H_k(b,\alpha) & \nabla^{j_{N-k}}H_{k+1}(b,\alpha+1) & \cdots & \nabla^{j_{N-k}}H_{N-1}(b,a-1) \end{vmatrix}$$

$$+\frac{1}{\beta}\begin{vmatrix} 0 & H_k(t,\alpha) & \cdots & H_{N-1}(t,a-1) \\ \nabla^{j_1}H_{\nu-1}(b,\rho(s)) & \nabla^{j_1}H_k(b,\alpha) & \cdots & \nabla^{j_1}H_{N-1}(b,a-1) \\ \nabla^{j_2}H_{\nu-1}(b,\rho(s)) & \nabla^{j_2}H_k(b,\alpha) & \cdots & \nabla^{j_2}H_{N-1}(b,a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}}H_{\nu-1}(b,\rho(s)) & \nabla^{j_{N-k}}H_k(b,\alpha) & \cdots & \nabla^{j_{N-k}}H_{N-1}(b,a-1) \end{vmatrix}.$$

Hence, we have

$$z(t,s) = H_{\nu-1}(t,\rho(s)) + u(t,s).$$

Next, for $m \in \mathbb{N}_1^{N-k}$, $\nabla^{j_m}z(b,s) =$

$$\frac{1}{\beta}\begin{vmatrix} \nabla^{j_m}H_{\nu-1}(b,\rho(s)) & \nabla^{j_m}H_k(b,\alpha) & \cdots & \nabla^{j_m}H_{N-1}(b,a-1) \\ \nabla^{j_1}H_{\nu-1}(b,\rho(s)) & \nabla^{j_1}H_k(b,\alpha) & \cdots & \nabla^{j_1}H_{N-1}(b,a-1) \\ \nabla^{j_2}H_{\nu-1}(b,\rho(s)) & \nabla^{j_2}H_k(b,\alpha) & \cdots & \nabla^{j_2}H_{N-1}(b,a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}}H_{\nu-1}(b,\rho(s)) & \nabla^{j_{N-k}}H_k(b,\alpha) & \cdots & \nabla^{j_{N-k}}H_{N-1}(b,a-1) \end{vmatrix}.$$

Hence, the first row in the determinant will be the same as the $(m+1)$-st row, giving $\nabla^{j_m}z(b,s) = 0$ for each $m \in \mathbb{N}_1^{N-k}$. But since $z(t,s) = H_{\nu-1}(t,\rho(s)) + u(t,s)$, this means for each $m \in \mathbb{N}_1^{N-k}$,

$$\nabla^{j_m} H_{\nu-1}(b, \rho(s)) + \nabla^{j_m} u(b, s) = 0;$$

i.e.,

$$\nabla^{j_m} u(b, s) = -\nabla^{j_m} H_{\nu-1}(b, \rho(s)).$$

Hence, we have that $u(t, s)$ satisfies the boundary conditions at $t = b$ in (2.12). Thus, the result follows by Theorem 2.10. $\Diamond$

2.3 $(k, N-k)$ Conjugate and Right-Focal Cases

In this section, we will focus on Green's functions for the special cases of conjugate and right-focal BVPs, which have signficant analogues in the ordinary differential equations context. We begin by introducing the following definition of a Wronskian in order to make the analogues with the ordinary differential equations case explicit. An analogous determinant for the delta case is referred to as the *Casoratian* and is given in [34, Chapter 3].

Definition 2.13 For functions $x_i : \mathbb{N}_{a-N+1} \to \mathbb{R}$, $i \in \mathbb{N}_1^k$, we define an analogue version of the Wronskian of $x_1, x_2, \ldots, x_k$ for the nabla case to be

$$W(t) = W[x_1, x_2, \ldots, x_k](t) := \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_k(t) \\ \nabla x_1(t) & \nabla x_2(t) & \cdots & \nabla x_k(t) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{k-1} x_1(t) & \nabla^{k-1} x_2(t) & \cdots & \nabla^{k-1} x_k(t) \end{vmatrix},$$

for $t \in \mathbb{N}_{a-N+k}$.

The next two theorems give the cases of $(k, N-k)$ conjugate and $(k, N-k)$

right-focal boundary conditions, respectively, and follow from Theorem 2.12.

Theorem 2.14 *Let $\nu > 1$, $N := \lceil \nu \rceil$, $b - a \in \mathbb{N}_1$, $k \in \mathbb{N}_1^{N-1}$, and consider the homogeneous $(k, N-k)$ conjugate BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{i} x(a-N+k) = 0, & i \in \mathbb{N}_0^{k-1} \\ \nabla^{j} x(b) = 0, & j \in \mathbb{N}_0^{N-k-1}. \end{cases} \tag{2.16}$$

Then, the Green's function for the BVP (2.16) *is given by* (2.13)*, where $u(t,s) =$*

$$\frac{1}{\beta}\begin{vmatrix} 0 & H_k(t, a-N+k) & \cdots & H_{N-1}(t, a-1) \\ H_{\nu-1}(b, \rho(s)) & H_k(b, a-N+k) & \cdots & H_{N-1}(b, a-1) \\ \nabla H_{\nu-1}(b, \rho(s)) & \nabla H_k(b, a-N+k) & \cdots & \nabla H_{N-1}(b, a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{N-k-1} H_{\nu-1}(b, \rho(s)) & \nabla^{N-k-1} H_k(b, a-N+k) & \cdots & \nabla^{N-k-1} H_{N-1}(b, a-1) \end{vmatrix},$$

and $v(t,s) := u(t,s) + H_{\nu-1}(t, \rho(s))$, for $(t,s) \in \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b}$. In this case, $\beta = W[H_k(\cdot, a-N+k), H_{k+1}(\cdot, a-N+k+1), \ldots, H_{N-1}(\cdot, a-1)](b)$, noting $\beta \neq 0$.

Theorem 2.15 *Let $\nu > 1$, $N := \lceil \nu \rceil$, $b - a \in \mathbb{N}_k$, $k \in \mathbb{N}_1^{N-1}$, and consider the homogeneous $(k, N-k)$ right-focal BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{i} x(a-N+k) = 0, & i \in \mathbb{N}_0^{k-1} \\ \nabla^{j} x(b) = 0, & j \in \mathbb{N}_k^{N-1}. \end{cases} \tag{2.17}$$

The Green's function for the BVP (2.17) *is given by* (2.13)*, where*

$$u(t,s)=\frac{1}{\beta}\begin{vmatrix} 0 & H_k(t,a-N+k) & \cdots & H_{N-1}(t,a-1) \\ \nabla^k H_{\nu-1}(b,\rho(s)) & \nabla^k H_k(b,a-N+k) & \cdots & \nabla^k H_{N-1}(b,a-1) \\ \nabla^{k+1} H_{\nu-1}(b,\rho(s)) & \nabla^{k+1} H_k(b,a-N+k) & \cdots & \nabla^{k+1} H_{N-1}(b,a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{N-1} H_{\nu-1}(b,\rho(s)) & \nabla^{N-1} H_k(b,a-N+k) & \cdots & \nabla^{N-1} H_{N-1}(b,a-1) \end{vmatrix},$$

and $v(t,s):=u(t,s)-H_{\nu-1}(t,\rho(s))$*, for* $(t,s)\in\mathbb{N}_{a-N+1}^{b}\times\mathbb{N}_{a+1}^{b}$*. In this case,* $\beta:=W[\nabla^k H_k(\cdot,a-N+k),\nabla^k H_{k+1}(\cdot,a-N+k+1),\ldots,\nabla^k H_{N-1}(\cdot,a-1)](b)$*, noting* $\beta\neq 0$*.*

2.3.1 $(N-1,1)$ Right-Focal Case

In the next corollary to Theorem 2.15, we see that the Green's function matches the one given in [27, Theorem 3.9] for the homogeneous $(N-1,1)$ right-focal BVP

$$\begin{cases} \nabla_{a*}^{\nu}x(t)=0, & t\in\mathbb{N}_{a+1}^{b} \\ \nabla^i x(a-1)=0, & i\in\mathbb{N}_0^{N-2} \\ \nabla^{N-1}x(b)=0, \end{cases} \tag{2.18}$$

where $\nu>1$, $N:=\nu$, and $b-a\in\mathbb{N}_{N-1}$.

Corollary 2.16 *Assume* $\nu>1$*,* $N:=\lceil\nu\rceil$*, and* $b-a\in\mathbb{N}_{N-1}$*. Then, the Green's function for the* $(N-1,1)$ *right-focal BVP* (2.18) *is given by*

$$G(t,s)=\begin{cases} -H_{N-1}(t,a-1)H_{\nu-N}(b,\rho(s)), & \text{if } t\leq\rho(s) \\ -H_{N-1}(t,a-1)H_{\nu-N}(t,\rho(s))+H_{\nu-1}(b,\rho(s)), & \text{if } t\geq\rho(s), \end{cases}$$

for $(t,s) \in \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b}$.

Proof. We will apply Theorem 2.15 in the case $k = N-1$. In this case, we have

$$\begin{aligned} \beta &= \nabla^{N-1} H_{N-1}(t, a-1)|_{t=b} \\ &= H_0(b, a-1) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} u(t,s) &= \frac{1}{\beta} \begin{vmatrix} 0 & H_{N-1}(t,a-1) \\ \nabla^{N-1} H_{\nu-1}(b,\rho(s)) & \nabla^{N-1} H_{N-1}(b,a-1) \end{vmatrix} \\ &= \begin{vmatrix} 0 & H_{N-1}(t,a-1) \\ H_{\nu-1-(N-1)}(b,\rho(s)) & 1 \end{vmatrix} \\ &= -H_{N-1}(t,a-1) H_{\nu-N}(b,\rho(s)). \end{aligned}$$

◊

2.3.2 A More General $(N-1,1)$ Problem and $(N-1,1)$ Conjugate Case

Let $\nu > 1$, $N := \lceil \nu \rceil$, $j \in \mathbb{N}_0^{N-1}$ be fixed, and consider the $(N-1,1)$ BVP

$$\begin{cases} \nabla_{a*}^{\nu} x(t) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{i} x(a-N+k) = 0, & i \in \mathbb{N}_0^{N-2} \\ \nabla^{j} x(b) = 0. \end{cases} \tag{2.19}$$

Note that the boundary conditions in (2.19) are not of the form for $(k, N-k)$ right-focal or $(k, N-k)$ conjugate boundary conditions when $j \in \mathbb{N}_1^{N-2}$.

In the next theorem, we apply Theorem 2.12 to the the special case of the BVP (2.19), which confirms [27, Theorem 4.6].

Theorem 2.17 *Let $\nu > 1$, $N := \lceil \nu \rceil$, $j \in \mathbb{N}_0^{N-1}$ be fixed, and $b-a \in \mathbb{N}_{\max\{1,j\}}$. Then, the Green's function for the BVP* (2.19) *is given by*

$$G(t,s) = \begin{cases} -\frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)}, & \text{if } t \leq \rho(s) \\ -\frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)} + H_{\nu-1}(t,\rho(s)), & \text{if } t \geq \rho(s), \end{cases}$$

for $(t,s) \in \mathbb{N}_{a-N+1}^b \times \mathbb{N}_{a+1}^b$.

Proof. By Theorem 2.12, we have

$$\begin{aligned} u(t,s) &:= \frac{1}{\beta}\begin{vmatrix} 0 & H_{N-1}(t,a-1) \\ \nabla^j H_{\nu-1}(b,\rho(s)) & \nabla^j H_{N-1}(b,a-1) \end{vmatrix} \\ &= \frac{1}{\beta}\begin{vmatrix} 0 & H_{N-1}(t,a-1) \\ H_{\nu-j-1}(b,\rho(s)) & H_{N-j-1}(b,a-1) \end{vmatrix} \\ &= \frac{1}{\beta}\left[-H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))\right] \end{aligned}$$

and $v(t,s) = u(t,s) - H_{\nu-1}(t,\rho(s))$. In this case, we have

$$\beta = \nabla^j H_{N-1}(b,a-1) = H_{N-j-1}(b,a-1).$$

$\Diamond$

Corollary 2.18 *Let $\nu > 1$, $N := \lceil \nu \rceil$, $b-a \in \mathbb{N}_1$, and consider the conjugate*

$(N-1,1)$ BVP

$$\begin{cases} \nabla_{a*}^{\nu} x(t) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{i} x(a-1) = 0, & i \in \mathbb{N}_{0}^{N-2} \\ x(b) = 0. \end{cases} \tag{2.20}$$

The Green's function for the BVP (2.20) *is given by*

$$G(t,s) = \begin{cases} -\frac{H_{N-1}(t,a-1)H_{\nu-1}(b,\rho(s))}{H_{N-1}(b,a-1)}, & \text{if } t \le \rho(s) \\ -\frac{H_{N-1}(t,a-1)H_{\nu-1}(b,\rho(s))}{H_{N-1}(b,a-1)} + H_{\nu-1}(t,\rho(s)), & \text{if } t \ge \rho(s), \end{cases}$$

for $(t,s) \in \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b}$.

Note also that $j = N-1$ in (2.19) gives the $(N-1,1)$ right-focal boundary conditions, and the formula for the Green's function given in Theorem 2.17 with $j = N-1$ matches the one in Corollary 2.16.

2.4 Further Work

Example 2.9 provided two-point boundary conditions under which BVPs involving the operator ∇_{a*}^{ν} do not have unique solutions. Naturally, a next step would be to consider additional two-point boundary conditions which can guarantee uniqueness of solutions to BVPs involving the operator ∇_{a*}^{ν}. Moreover, one may develop an analogue in the nabla Caputo context of the general two-point boundary conditions considered in [14] for the differential equations case which were stated at the beginning of this chapter. It can then be investigated whether a complete classification on these analogous general two-point boundary conditions giving unique and nonunique solutions to BVPs involving the operator ∇_{a*}^{ν} can be attained. Another possible direction for future work could be to explore multipoint boundary conditions. For example, Green's functions for k-point focal BVPs in the ordinary differential equations case are studied in [57].

Additionally, more general linear operators involving the nabla Caputo fractional difference may be considered, and one could search for Green's functions for two-point boundary value problems involving these more general operators. A start in this direction will be explored in Chapter 5, where we will consider the operator $L_a x(t) := \nabla_{a*} x(t) + cx(t) = 0$ with $|c| < 1$. Yet another possibility is to consider Green's functions for the nabla Riemann-Louiville fractional difference operator under two-point boundary conditions.

Chapter 3

Properties of Specific Green's Functions

In this chapter, we will focus specifically on Green's functions for $(N-1,1)$ boundary value problems and examine properties and bounds of these Green's functions. In particular, since we have the eventual goal of using these bounds to establish Lyapunov inequalities for nabla Caputo boundary value problems involving the equation $-\nabla^{\nu}_{a*}x(t) = q(t)x(t-1)$ in Chapter 4, the Green's functions in this chapter will give solutions to boundary value problems involving the equation

$$-\nabla^{\nu}_{a*}x(t) = h(t), \quad t \in \mathbb{N}_{a+1},$$

as opposed to the equation $\nabla^{\nu}_{a*}x(t) = h(t)$ from Chapter 2. Note that if $G : \mathbb{N}^{b}_{a-N+1} \times \mathbb{N}^{b}_{a+1} \to \mathbb{R}$ is the Green's function for the boundary value problem (2.9), then $-G(t,s)$ is the Green's function for the boundary value problem

$$\begin{cases} -\nabla^{\nu}_{a*}x(t) = 0, & t \in \mathbb{N}^{b}_{a+1} \\ \nabla^{i}x(a-N+k) = 0, & i \in \mathbb{N}^{k-1}_{0} \\ \nabla^{j_m}x(b) = 0, & m \in \mathbb{N}^{N-k}_{1}, \end{cases}$$

where $\nu > 1$, $N := \lceil \nu \rceil$, $k \in \mathbb{N}_1^{N-1}$, $j_m \in \mathbb{N}_0^{N-1}$ for $m \in \mathbb{N}_1^{N-k}$, such that $j_1 < j_2 < j_3 < \cdots < j_{N-k}$, and $b - a \in \mathbb{N}_{\max\{1, j_{N-k}-N+k+1\}}$.

3.1 Green's Function for an $(N-1, 1)$ Conjugate BVP

In this section, we will focus on the Green's function for the $(N-1, 1)$ conjugate BVP,

$$\begin{cases} -\nabla_{a*}^{\nu} x(t) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^i x(a-1) = 0, & i \in \mathbb{N}_0^{N-2} \\ x(b) = 0, \end{cases} \tag{3.1}$$

where $\nu > 1$, $N := \lceil \nu \rceil$, and $b - a \in \mathbb{N}_1$.

The next theorem follows by Theorem 2.10 and Corollary 2.18.

Theorem 3.1 *Consider the nabla Caputo* $(N-1, 1)$ *conjugate BVP*

$$\begin{cases} -\nabla_{a*}^{\nu} x(t) = h(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^i x(a-1) = 0, & i \in \mathbb{N}_0^{N-2} \\ x(b) = 0, \end{cases} \tag{3.2}$$

with $\nu > 1$, $N := \lceil \nu \rceil$, $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, *and* $b - a \in \mathbb{N}_1$. *Then,* $x : \mathbb{N}_{a-N+1}^{b} \to \mathbb{R}$ *is a solution to the* $(N-1, 1)$ *conjugate BVP* (3.2) *if and only if* $x(t)$ *satisfies the integral equation*

$$x(t) = \int_a^b G_\nu(t, s) h(s) \nabla s, \tag{3.3}$$

for $t \in \mathbb{N}_{a-N+1}^{b}$, *where* $G_\nu : \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ *is the Green's function for the* *homogeneous BVP* (3.1) *and is given by*

$$G_\nu(t,s) = \begin{cases} \frac{H_{N-1}(t,a-1)H_{\nu-1}(b,\rho(s))}{H_{N-1}(b,a-1)}, & t \leq \rho(s) \\ \frac{H_{N-1}(t,a-1)H_{\nu-1}(b,\rho(s))}{H_{N-1}(b,a-1)} - H_{\nu-1}(t,\rho(s)), & t \geq \rho(s). \end{cases} \tag{3.4}$$

For comparison, we make note of the Green's function for the analogous BVP in the whole order continuous case. It follows by [29, Example 6.21 on p. 293] that the Green's function for the BVP

$$\begin{cases} -x^{(n)} = 0 \\ x^{(i)}(a) = 0, \quad i \in \mathbb{N}_0^{n-2} \\ x(b) = 0 \end{cases}$$

is given by

$$G(t,s) = \begin{cases} \dfrac{(t-a)^{n-1}}{(b-a)^{n-1}} \dfrac{(b-s)^{n-1}}{(n-1)!}, & a \leq t \leq s \leq b \\ \dfrac{(t-a)^{n-1}}{(b-a)^{n-1}} \dfrac{(b-s)^{n-1}}{(n-1)!} - \dfrac{(t-s)^{n-1}}{(n-1)!}, & a \leq s \leq t \leq b. \end{cases}$$

To see the similarity more explicitly, note that the Green's function for the nabla Caputo fractional case may be rewritten as

$$G_\nu(t,s) = \begin{cases} \frac{(t-\rho(a))^{\overline{N-1}}}{(b-\rho(a))^{\overline{N-1}}} \frac{(b-\rho(s))^{\overline{N-1}}}{(N-1)!}, & t \leq \rho(s) \\ \frac{(t-\rho(a))^{\overline{N-1}}}{(b-\rho(a))^{\overline{N-1}}} \frac{(b-\rho(s))^{\overline{N-1}}}{(N-1)!} - \frac{(t-\rho(s))^{\overline{N-1}}}{(N-1)!}, & t \geq \rho(s), \end{cases}$$

where $(t,s) \in \mathbb{N}_{a-N+1}^b \times \mathbb{N}_{a+1}^b$.

3.1.1 The Case $1 < \nu \leq 2$

The following corollary is a special case of Theorem 3.1 for $1 < \nu \leq 2$, where $N := \lceil \nu \rceil = 2$.

Corollary 3.2 *Consider the nabla Caputo BVP*

$$\begin{cases} -\nabla_{a*}^{\nu} x(t) = h(t), & t \in \mathbb{N}_{a+1}^{b} \\ x(a-1) = x(b) = 0, \end{cases} \tag{3.5}$$

with $1 < \nu \leq 2$*,* $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$*, and* $b - a \in \mathbb{N}_1$*. Then,* $u : \mathbb{N}_{a-1}^{b} \to \mathbb{R}$ *is a solution to the BVP* (3.5) *if and only if* $x(t)$ *satisfies the integral equation*

$$x(t) = \int_{a}^{b} G_{\nu}(t,s)h(s)\nabla s, \tag{3.6}$$

for $t \in \mathbb{N}_{a-1}^{b}$*, where* $G_{\nu} : \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ *is given by*

$$G_{\nu}(t,s) = \begin{cases} \frac{H_1(t,\rho(a))}{H_1(b,\rho(a))} H_{\nu-1}(b,\rho(s)), & t \leq \rho(s) \\ \frac{H_1(t,\rho(a))}{H_1(b,\rho(a))} H_{\nu-1}(b,\rho(s)) - H_{\nu-1}(t,\rho(s)), & t \geq \rho(s). \end{cases} \tag{3.7}$$

Example 3.3 will use Corollary 3.2 to find the solution to a specific case of the BVP (3.5).

Example 3.3 Consider the BVP (3.5), where $a = 1$, $b = 6$, $\nu = 1.9$, and $h(t) \equiv 1$; i.e.,

$$\begin{cases} -\nabla_{1*}^{1.9} x(t) = 1, & t \in \mathbb{N}_{2}^{6} \\ x(0) = x(6) = 0. \end{cases} \tag{3.8}$$

By Corollary 3.2,

$$G_{1.9}(t,s) = \begin{cases} \frac{t}{6}H_{0.9}(6,\rho(s)), & t \le \rho(s) \\ \frac{t}{6}H_{0.9}(6,\rho(s)) - H_{0.9}(t,\rho(s)), & t \ge \rho(s), \end{cases}$$

for $(t,s) \in \mathbb{N}_0^6 \times \mathbb{N}_2^6$, and the solution to the BVP (3.8) is given by

$$\begin{aligned} x(t) &= \int_1^6 G_{1.9}(t,s)h(s)\nabla s \\ &= \int_1^t G_{1.9}(t,s)\nabla s + \int_t^6 G_{1.9}(t,s)\nabla s, \text{ for } t \in \mathbb{N}_0^6 \text{ fixed} \\ &= \int_1^t \left(\frac{t}{6}H_{0.9}(6,\rho(s)) - H_{0.9}(t,\rho(s))\right)\nabla s + \int_t^6 \frac{t}{6}H_{0.9}(6,\rho(s))\nabla s \\ &= \frac{t}{6}\int_1^6 H_{0.9}(6,\rho(s))\nabla s - \int_1^t H_{0.9}(t,\rho(s))\nabla s \\ &= \frac{t}{6}H_{1.9}(6,1) - H_{1.9}(t,1), \text{ by Theorem 1.8, part (iv)} \\ &= \frac{t}{6}\frac{(6-1)^{\overline{1.9}}}{\Gamma(2.9)} - \frac{(t-1)^{\overline{1.9}}}{\Gamma(2.9)} \\ &= \frac{t}{6}\frac{\Gamma(6.9)}{\Gamma(5)\Gamma(2.9)} - \frac{\Gamma(t+0.9)}{\Gamma(t-1)\Gamma(2.9)}, \end{aligned}$$

for $t \in \mathbb{N}_0^6$.

Using the last expression for $x(t)$ above, we obtain a graphical solution to the BVP (3.8), as shown in Figure 3.1.

The next theorem shows that when $\nu = 2$, the Green's function is of constant sign. This is not always the case for all ν such that $1 < \nu < 2$.

Theorem 3.4 *For $\nu = 2$ in (3.7), we have $G_\nu(t,s) \ge 0$ for all $(t,s) \in \mathbb{N}_{a-1}^b \times \mathbb{N}_{a+1}^b$. In this case,*

$$G_2(t,s) = \begin{cases} \frac{(t-a+1)(b-\rho(s))}{b-a+1}, & t \le \rho(s) \\ \frac{(t-a+1)(b-\rho(s))}{b-a+1} - (t-\rho(s)), & t \ge \rho(s). \end{cases} \quad (3.9)$$

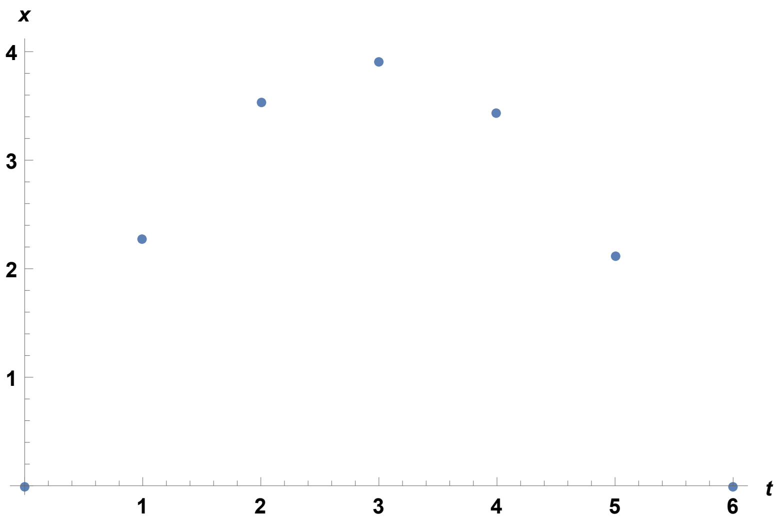

Figure 3.1: A solution to the BVP (3.8).

Proof. For $t \leq \rho(s)$, $G_2(t,s) \geq 0$ since $t \geq a-1$ and $\rho(s) \leq b$. Next, for $t \geq \rho(s)$ and $s \in \mathbb{N}_{a+1}^{b}$ fixed,

$$\begin{aligned}\nabla_t G_2(t,s) &= \nabla_t \left[\frac{(t-a+1)(b-\rho(s))}{b-a+1} - (t-\rho(s))\right] \\ &= \frac{b-s+1}{b-a+1} - 1 \\ &< 0,\end{aligned}$$

where the last inequality follows since $0 < b-s+1 < b-a+1$. Hence, f is a decreasing function of t for $t \geq \rho(s)$, so

$$\begin{aligned}G_2(t,s) &\geq G_2(b,s) \\ &= \frac{(b-a+1)(b-\rho(s))}{b-a+1} - (b-\rho(s)) = 0.\end{aligned}$$

$\Diamond$

The following example shows that, in general, the Green's function for the BVP

(3.5) need not be of constant sign.

Example 3.5 Consider the BVP (3.5), where $a = 1$, $b = 9$, and $\nu = 1.5$. In this case,

$$G_{1.5}(t,s) = \begin{cases} \dfrac{H_1(t,0)}{H_1(9,0)} H_{0.5}(9,\rho(s)), & t \le \rho(s) \\ \dfrac{H_1(t,0)}{H_1(9,0)} H_{0.5}(9,\rho(s)) - H_{0.5}(t,\rho(s)), & \rho(s) \le t, \end{cases}$$

for $(t,s) \in \mathbb{N}_0^9 \times \mathbb{N}_2^9$. Then,

$$\begin{aligned} G_{1.5}(3,2) &= \frac{H_1(3,0)}{H_1(9,0)} H_{0.5}(9,1) - H_{0.5}(3,1) \\ &= \frac{1}{\Gamma(1.5)} \left(\frac{(3)(8)^{\overline{0.5}}}{9} - (2)^{\overline{0.5}} \right) \\ &= \frac{1}{\Gamma(1.5)} \left(\frac{\Gamma(8.5)}{3 \cdot 7!} - \Gamma(2.5) \right) \\ &= \frac{7.5 \cdot 6.5 \cdot 5.5 \cdot 4.5 \cdot 3.5 \cdot 2.5 \cdot 1.5}{3 \cdot 7!} - 1.5 \\ &= -\frac{927}{2048}. \end{aligned}$$

However, it is always the case that when $t \le \rho(s)$, we have $G_\nu(t,s) \ge 0$, since $\frac{H_1(t,\rho(a))}{H_1(b,\rho(a))} H_{\nu-1}(b,\rho(s)) \ge 0$ follows from the next proposition.

The following proposition will be used in the subsequent lemmas. We will make frequent use of the fact that $\Gamma(t) > 0$ for $t > 0$, and $\Gamma(t) < 0$ for $-1 < t < 0$.

Proposition 3.6 Let $\alpha > -1$ and $s \in \mathbb{N}_a$. Then, the following hold:

(i) If $t \in \mathbb{N}_{\rho(s)}$, then $H_\alpha(t,\rho(s)) \ge 0$, and if $t \in \mathbb{N}_s$, then $H_\alpha(t,\rho(s)) > 0$.

(ii) If $t \in \mathbb{N}_{\rho(s)}$ and $\alpha > 0$, then $H_\alpha(t,\rho(s))$ is a decreasing function of s, and if $t \in \mathbb{N}_s$ and $-1 < \alpha < 0$, then $H_\alpha(t,\rho(s))$ is an increasing function of s.

(iii) If $t \in \mathbb{N}_{\rho(s)}$ and $\alpha \ge 0$, then $H_\alpha(t,\rho(s))$ is a nondecreasing function of t, and if $\alpha > 0$ and $t \in \mathbb{N}_s$, then $H_\alpha(t,\rho(s))$ is an increasing function of t. Also, if

$t \in \mathbb{N}_{s+1}$ and $-1 < \alpha < 0$, then $H_\alpha(t, \rho(s))$ is a decreasing function of t.

Proof. (i) First, consider

$$\begin{aligned} H_\alpha(t, \rho(s)) &= \frac{(t - \rho(s))^{\overline{\alpha}}}{\Gamma(\alpha + 1)} \\ &= \frac{\Gamma(t - \rho(s) + \alpha)}{\Gamma(t - \rho(s))\Gamma(\alpha + 1)}. \end{aligned} \tag{3.10}$$

If $t = \rho(s)$, then $H_\alpha(t, \rho(s)) = 0$. Otherwise, if $t \in \mathbb{N}_s$, then $t - \rho(s) + \alpha > 0$, $t - \rho(s) > 0$, and $\alpha + 1 > 0$. By (3.10), $H_\alpha(t, \rho(s)) > 0$.

(ii) Next, consider

$$\begin{aligned} \nabla_s H_\alpha(t, \rho(s)) &= \nabla_s \frac{(t - \rho(s))^{\overline{\alpha}}}{\Gamma(\alpha + 1)} \\ &= \nabla_s \frac{(t + 1 - s)^{\overline{\alpha}}}{\Gamma(\alpha + 1)} \\ &= -\alpha \frac{(t + 1 - \rho(s))^{\overline{\alpha - 1}}}{\Gamma(\alpha + 1)}, \text{ by Theorem 1.7} \\ &= \frac{-(t + 1 - \rho(s))^{\overline{\alpha - 1}}}{\Gamma(\alpha)} \\ &= -\frac{\Gamma(t + 1 - s + 1 + \alpha - 1)}{\Gamma(t + 1 - s + 1)\Gamma(\alpha)} \\ &= -\frac{\Gamma(t - s + 1 + \alpha)}{\Gamma(t - s + 2)\Gamma(\alpha)}. \end{aligned} \tag{3.11}$$

First, suppose $t \in \mathbb{N}_{\rho(s)}$ and $\alpha > 0$. Then, $t - s + 1 \geq 0$, so $t - s + 2 > 0$, and $t - s + 1 + \alpha > 0$. Thus, from (3.11), we have $\nabla_s H_\alpha(t, \rho(s)) < 0$, and it follows that $H_\alpha(t, \rho(s))$ is a decreasing function of s when $\alpha > 0$ and $t \in \mathbb{N}_{\rho(s)}$.

Now let $-1 < \alpha < 0$ and $t \in \mathbb{N}_s$. Then, $t - s + 2 > t - s + 1 + \alpha > 0$, so in this case $\nabla_s H_\alpha(t, \rho(s)) > 0$.

(iii) Lastly, consider

$$\begin{aligned}\nabla_t H_\alpha(t,\rho(s)) &= H_{\alpha-1}(t,\rho(s)), \text{ by Theorem 1.8, part (ii)}\\ &= \frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}\\ &= \frac{\Gamma(t-\rho(s)+\alpha-1)}{\Gamma(t-\rho(s))\Gamma(\alpha)}. \end{aligned} \tag{3.12}$$

When $\alpha = 0$, $H_\alpha(t,\rho(s)) \equiv 1$ is nondecreasing in t. Now let $\alpha > 0$, and first consider the case $t \in \mathbb{N}_s$. Then, $t-\rho(s)+\alpha-1 = t-s+\alpha > 0$ and $t-\rho(s) > 0$. Hence, by (3.12), $\nabla H_\alpha(t,\rho(s)) > 0$, so $H_\alpha(t,\rho(s))$ is an increasing function of t. Next, letting $t = s-1$ in (3.12),

$$\begin{aligned}\nabla_t H_\alpha(\rho(s),\rho(s)) &= \frac{\Gamma(\rho(s)-\rho(s)+\alpha-1)}{\Gamma(\rho(s)-\rho(s))\Gamma(\alpha)}\\ &= \frac{\Gamma(\alpha-1)}{\Gamma(0)\Gamma(\alpha)}\\ &= \begin{cases} 0, & \text{if } \alpha \neq 1\\ 1, & \text{if } \alpha = 1, \end{cases}\end{aligned}$$

using the conventions involving the Gamma function stated in Remark 1.4. Hence, we have $\nabla_t H_\alpha(t,\rho(s)) \geq 0$, so $H_\alpha(t,\rho(s))$ is a nondecreasing function of t when $t \in \mathbb{N}_{\rho(s)}$ and $\alpha \geq 0$.

Now suppose $-1 < \alpha < 0$ and $t \in \mathbb{N}_{s+1}$. Then, $t-\rho(s)+\alpha-1 = t-s+\alpha > 0$, since $t \geq s+1$. Then by (3.12), $\nabla_t H_\alpha(t,\rho(s)) < 0$, so $H_\alpha(t,\rho(s))$ is a decreasing function of t. $\Diamond$

The next theorem will give sufficient conditions on a, b, and ν, where $1 < \nu < 2$, so that G_ν is nonnegative for all $(t,s) \in \mathbb{N}_{a-1}^b \times \mathbb{N}_{a+1}^b$.

Theorem 3.7 *Suppose* $1 < \nu < 2$ *and fix* $s \in \mathbb{N}_{a+1}^b$.

(i) If $\nu \geq \frac{b-a}{b-a+1} + 1$ *and* $t \in \mathbb{N}_{s+1}^{b}$*, then* $G_\nu(t,s) \geq 0$*.*

(ii) If $t \in \mathbb{N}_{a-1}^{s}$*, then* $G(t,s) \geq 0$*.*

Proof. (i) Fix $s \in \mathbb{N}_{a+1}^{b}$ and let $t \in \mathbb{N}_{s+1}^{b}$. We will show that G_ν is decreasing with respect to t for $t \in \mathbb{N}_{s+1}^{b}$. Then, since $G_\nu(b,s) = 0$, it follows that $G_\nu(t,s) \geq 0$ for $t \in \mathbb{N}_{s+1}^{b}$. Consider

$$\nabla_t G_\nu(t,s) = \frac{1}{b-a+1} H_{\nu-1}(b,\rho(s)) - H_{\nu-2}(t,\rho(s)). \tag{3.13}$$

Note that

$$\begin{aligned}
H_{\nu-1}(b,\rho(s)) &= \frac{(b-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} \\
&= \frac{\Gamma(b-\rho(s)+\nu-1)}{\Gamma(b-\rho(s))\Gamma(\nu)} \\
&= \frac{(b-\rho(s)+\nu-2)\Gamma(b-\rho(s)+\nu-2)}{\Gamma(b-\rho(s))(\nu-1)\Gamma(\nu-1)} \\
&= \frac{(b-\rho(s)+\nu-2)}{\nu-1} \frac{(b-\rho(s))^{\overline{\nu-2}}}{\Gamma(\nu-1)} \\
&= \frac{(b-\rho(s)+\nu-2)}{\nu-1} H_{\nu-2}(b,\rho(s)).
\end{aligned}$$

Then, from (3.13), we have

$$\nabla_t G_\nu(t,s) = \frac{(b-\rho(s)+\nu-2)}{(b-a+1)(\nu-1)} H_{\nu-2}(b,\rho(s)) - H_{\nu-2}(t,\rho(s)). \tag{3.14}$$

Since $\nu \geq \frac{b-a}{b-a+1} + 1$, we have

$$\begin{aligned}
\frac{b-a}{b-a+1} &\leq \nu - 1 \\
\frac{b-a}{(b-a+1)(\nu-1)} &\leq 1.
\end{aligned}$$

Since $0 < b - s + \nu - 1 < b - (a+1) + 1 = b - a$ for $s \in \mathbb{N}_{a+1}^{b}$ and $0 < \nu - 1 < 1$, it follows that

$$\frac{b - s + \nu - 1}{(b - a + 1)(\nu - 1)} < \frac{b - a}{(b - a + 1)(\nu - 1)}.$$

Hence,

$$0 < \frac{b - \rho(s) + \nu - 2}{(b - a + 1)(\nu - 1)} < 1. \tag{3.15}$$

By Proposition 3.6, part (iii), $H_{\nu-2}(t, \rho(s))$ is a decreasing function of t for $t \in \mathbb{N}_{s+1}^{b}$. Also, $H_{\nu-2}(b, \rho(s)) > 0$ by Proposition 3.6, part (i). Hence, for $t \in \mathbb{N}_{s+1}^{b}$,

$$0 < H_{\nu-2}(b, \rho(s)) \leq H_{\nu-2}(t, \rho(s)). \tag{3.16}$$

Then, from (3.14),

$$\begin{aligned}
\nabla_t G_\nu(t, s) &= \frac{(b - \rho(s) + \nu - 2)}{(b - a + 1)(\nu - 1)} H_{\nu-2}(b, \rho(s)) - H_{\nu-2}(t, \rho(s)) \\
&\overset{(3.16)}{\leq} \frac{(b - \rho(s) + \nu - 2)}{(b - a + 1)(\nu - 1)} H_{\nu-2}(b, \rho(s)) - H_{\nu-2}(b, \rho(s)) \\
&\overset{(3.15)}{<} 0.
\end{aligned}$$

This shows that G_ν is decreasing with respect to t for $t \in \mathbb{N}_{s+1}^{b}$, which implies $G_\nu(t, s) \geq 0$ for $t \in \mathbb{N}_{s+1}^{b}$.

(ii) Note that, for $t \leq \rho(s)$, we have $G_\nu(t, s) \geq 0$. Next, suppose $s = t$. Then, for $t \in \mathbb{N}_{a+1}^{b}$,

$$
\begin{aligned}
\nabla_t G_\nu(t,s)|_{s=t} &\overset{(3.13)}{=} \left[\frac{1}{b-a+1} H_{\nu-1}(b,\rho(s)) - H_{\nu-2}(t,\rho(s))\right]\Big|_{s=t} \\
&= \frac{1}{b-a+1} H_{\nu-1}(b,\rho(t)) - H_{\nu-2}(t,\rho(t)) \\
&= \frac{1}{b-a+1} \frac{(b-\rho(t))^{\overline{\nu-1}}}{\Gamma(\nu)} - 1 \\
&= \frac{1}{b-a+1} \frac{\Gamma(b-\rho(t)+\nu-1)}{\Gamma(b-\rho(t))\Gamma(\nu)} - 1 \\
&= \frac{1}{b-a+1} \frac{\Gamma(b-t+\nu)}{\Gamma(b-t+1)\Gamma(\nu)} - 1. \qquad (3.17)
\end{aligned}
$$

Now fix $t \in \mathbb{N}_{a+1}^{b}$. By the property of the Gamma function given in Proposition 1.3,

$$
\begin{aligned}
\frac{1}{b-a+1} \frac{\Gamma(b-t+\nu)}{\Gamma(b-t+1)\Gamma(\nu)} &= \frac{(b-t-1+\nu)\cdots\nu}{(b-a+1)(b-t)!} \\
&= \left(\frac{b-t-1+\nu}{b-a+1}\right)\left(\frac{b-t-2+\nu}{b-t}\right)\left(\frac{b-t-3+\nu}{b-t-1}\right)\cdots\frac{\nu}{2}.
\end{aligned}
\qquad (3.18)
$$

Note that $0 < \frac{b-t-1+\nu}{b-a+1} < 1$, $0 < \frac{b-t-2+\nu}{b-t} < 1$, $0 < \frac{b-t-3+\nu}{b-t-1} < 1$, $\ldots$, and $0 < \frac{\nu}{2} < 1$; i.e., each factor in the product (3.18) is strictly between zero and one. Hence,

$$
\frac{1}{b-a+1} \frac{\Gamma(b-t+\nu)}{\Gamma(b-t+1)\Gamma(\nu)} - 1 < 0,
$$

so we have $\nabla_t G_\nu(t,s)|_{s=t} < 0$. Thus, $G_\nu(t,t)$ is a decreasing function of t. This means that $G_\nu(t,t) \geq G_\nu(b,b)$, so

$$\begin{aligned}
G_\nu(t,t) &= \frac{t-a+1}{b-a+1} H_{\nu-1}(b,\rho(t)) - H_{\nu-1}(t,\rho(t)) \\
&= \frac{t-a+1}{b-a+1} \frac{(b-\rho(t))^{\overline{\nu-1}}}{\Gamma(\nu)} - 1 \\
&\geq \frac{b-a+1}{b-a+1} \frac{(b-\rho(b))^{\overline{\nu-1}}}{\Gamma(\nu)} - 1 \\
&= 0.
\end{aligned}$$

Therefore, we have $G_\nu(t,s) \geq 0$ for all $(t,s) \in \mathbb{N}_{a-1}^b \times \mathbb{N}_{a+1}^b$. ◊

Example 3.8 Recall that in Example 3.3 we had $a = 1, b = 6$, and $\nu = 1.9$. Then,

$$\begin{aligned}
\nu = 1.9 &\geq \frac{b-a}{b-a+1} + 1 = \frac{6-1}{6-1+1} + 1 = \frac{5}{6} + 1 \\
&\approx 1.8333,
\end{aligned}$$

so the condition in Theorem 3.7, part (i) is satisfied. Thus, in this case, we have $G_{1.9}(t,s) \geq 0$ for all $(t,s) \in \mathbb{N}_0^6 \times \mathbb{N}_2^6$. Fixing $s = 5$, a graph of the Green's function is given in Figure 3.2.

3.2 Green's Function Bounds for More $(N-1,1)$ BVPs

The next theorem follows by Theorem 2.10 and Theorem 2.17. When $j = 0$ in the theorem below, the result corresponds to Theorem 3.1, and when $j = N-1$, (3.19) is the $(N-1,1)$ right-focal boundary value problem.

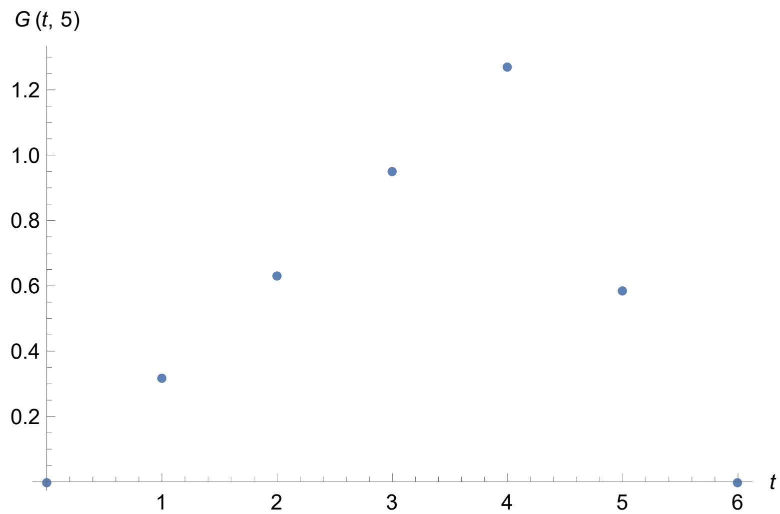

Figure 3.2: Green's function as a function of t, with $a = 1, b = 6$, and $\nu = 1.9$, and $s = 5$ fixed.

Theorem 3.9 *Consider the nabla Caputo* $(N-1, 1)$ *BVP*

$$\begin{cases} -\nabla^{\nu}_{a*}x(t) = h(t), & t \in \mathbb{N}^{b}_{a+1} \\ \nabla^{i}x(a-1) = 0, & i \in \mathbb{N}^{N-2}_{0} \\ \nabla^{j}x(b) = 0, \end{cases} \tag{3.19}$$

with $\nu > 1$, $N := \lceil \nu \rceil$, $j \in \mathbb{N}^{N-1}_{0}$ *fixed,* $b - a \in \mathbb{N}_{\max\{1,j\}}$, *and* $h : \mathbb{N}^{b}_{a+1} \to \mathbb{R}$. *Then,* $x : \mathbb{N}^{b}_{a-N+1} \to \mathbb{R}$ *is a solution to the* $(N-1, 1)$ *BVP* (3.19) *if and only if* $x(t)$ *satisfies the integral equation*

$$x(t) = \int_{a}^{b} G_{\nu}(t, s)h(s)\nabla s, \tag{3.20}$$

for $t \in \mathbb{N}^{b}_{a-N+1}$, *where* $G_{\nu} : \mathbb{N}^{b}_{a-N+1} \times \mathbb{N}^{b}_{a+1} \to \mathbb{R}$ *is given by*

$$G_{\nu}(t, s) = \begin{cases} \frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)}, & t \leq \rho(s) \\ \frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)} - H_{\nu-1}(t, \rho(s)), & t \geq \rho(s). \end{cases} \tag{3.21}$$

The following two propositions will be used in the next theorem, which gives a bound on the Green's function given by (3.21). The statements of these propositions were communicated by Scott Gensler.

Proposition 3.10 Let f, g be real-valued functions on a set S, such that $f(t), g(t) \geq 0$ for all $t \in S$. Moreover, assume there exists $s_0, s_1 \in S$ where $\max_{s \in S} f(s) = f(s_0)$ and $\max_{s \in S} g(s) = g(s_1)$; i.e., f and g attain their maximum in S. Then, for each fixed $t \in S$,

$$\begin{aligned}|f(t) - g(t)| &\leq \max\{f(t), g(t)\} \\ &\leq \max\{\max_{t \in S} f(t), \max_{t \in S} g(t)\}.\end{aligned}$$

Proof. Let $t \in S$ be fixed, and consider the case $f(t) \geq g(t)$, so $\max\{f(t), g(t)\} = f(t)$. Then,

$$|f(t) - g(t)| = f(t) - g(t) \leq f(t),$$

since $f(t), g(t) \geq 0$. Hence, $|f(t) - g(t)| \leq \max\{f(t), g(t)\}$. Switching the roles of f and g in the above argument gives the proof for the case $g(t) \geq f(t)$.

Finally, since for each fixed t, $f(t) \leq \max_{t \in S} f(t)$ and $g(t) \leq \max_{t \in S} g(t)$, we have

$$\max\{f(t), g(t)\} \leq \max\{\max_{t \in S} f(t), \max_{t \in S} g(t)\}.$$

$\Diamond$

Proposition 3.11 If $0 < \nu \leq \mu$, then

$$H_\nu(t, a) \leq H_\mu(t, a),$$

for each fixed $t \in \mathbb{N}_a$.

Proof. Consider $0 < \nu \leq \mu$ and $t \in \mathbb{N}_a$ fixed. Note that $H_\nu(a,a) = H_\mu(a,a) = 0$, so $H_\nu(t,a) \leq H_\mu(t,a)$ holds in the case $t = a$. Now suppose $t \in \mathbb{N}_{a+1}$. Consider

$$\begin{aligned} H_\nu(t,a) &= \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)} \\ &= \frac{\Gamma(t-a+\nu)}{\Gamma(t-a)\Gamma(\nu+1)} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} H_\mu(t,a) &= \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} \\ &= \frac{\Gamma(t-a+\mu)}{\Gamma(t-a)\Gamma(\mu+1)}. \end{aligned} \tag{3.23}$$

Hence, to show $H_\nu(t,a) \leq H_\mu(t,a)$, from (3.22) and (3.23), it suffices to show that

$$\frac{\Gamma(t-a+\nu)}{\Gamma(\nu+1)} \leq \frac{\Gamma(t-a+\mu)}{\Gamma(\mu+1)}.$$

If $t = a+1$, then $\frac{\Gamma(t-a+\nu)}{\Gamma(\nu+1)} = \frac{\Gamma(1+\nu)}{\Gamma(\nu+1)} = 1$ and $\frac{\Gamma(t-a+\mu)}{\Gamma(\mu+1)} = \frac{\Gamma(1+\mu)}{\Gamma(\mu+1)} = 1$, so the inequality holds. For each fixed $t \in \mathbb{N}_{a+2}$, consider

$$\begin{aligned} \frac{\Gamma(t-a+\nu)}{\Gamma(\nu+1)} &= \frac{(t-a+\nu-1)\cdots(\nu+1)\Gamma(\nu+1)}{\Gamma(\nu+1)}, \text{ by Proposition 1.3} \\ &= \underbrace{(t-a+\nu-1)\cdots(\nu+1)}_{t-a-1 \text{ factors}} \end{aligned} \tag{3.24}$$

and similarly,

$$\frac{\Gamma(t-a+\mu)}{\Gamma(\mu+1)} = \underbrace{(t-a+\mu-1)\cdots(\mu+1)}_{t-a-1 \text{ factors}}. \tag{3.25}$$

Now since $\nu \leq \mu$, each of the $t-a+1$ factors in the product (3.24) is less than or equal to each of the corresponding factors in (3.25). It follows that

$$(t-a+\nu-1)\cdots(\nu+1) \le (t-a+\mu-1)\cdots(\mu+1),$$

so $H_\nu(t,a) \le H_\mu(t,a)$. ◊

The following theorem will give bounds on the absolute value of the Green's function given by (3.21). These bounds will be used in the next chapter to derive Lyapunov inequalities.

Theorem 3.12 *Let G_ν be as in* (3.21)*. Then,*

$$|G_\nu(t,s)| \le H_{N-1}(b, a-1),$$

for $(t,s) \in \mathbb{N}^b_{a-N+1} \times \mathbb{N}^b_{a+1}$. In particular, when $j=0$ in (3.21)*, we have*

$$|G_\nu(t,s)| \le H_{\nu-1}(b,a).$$

Proof. Let $s \in \mathbb{N}^b_{a+1}$ be fixed. From (3.21), for $t \in \mathbb{N}^b_s$,

$$|G_\nu(t,s)| = \left| \frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)} - H_{\nu-1}(t,\rho(s)) \right|. \tag{3.26}$$

First, we will show $\frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)} \ge 0$ and $H_{\nu-1}(t,\rho(s)) \ge 0$. By Proposition 3.6, part (i), the following hold:

1. $H_{\nu-1}(t,\rho(s)) > 0$, for $t \in \mathbb{N}^b_s$;

2. $H_{\nu-j-1}(b,\rho(s)) > 0$, since $\nu - j - 1 \ge \nu - N > -1$;

3. $H_{N-1}(t, a-1) > 0$, for $t \in \mathbb{N}^b_s$;

4. $H_{N-j-1}(b, a-1) > 0$, since $N - j - 1 \geq 0$.

Hence, $\frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)} \geq 0$ and $H_{\nu-1}(t, \rho(s)) \geq 0$ for $t \in \mathbb{N}_s^b$. From (3.26), using Proposition 3.10, we have

$$|G_\nu(t,s)| \leq \max\left\{ \max_{\substack{t\in\mathbb{N}_s^b \\ s\in N_{a+1}^b}} \frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)}, \max_{\substack{t\in\mathbb{N}_s^b \\ s\in N_{a+1}^b}} H_{\nu-1}(t,\rho(s)) \right\}. \tag{3.27}$$

Now we will find an upper bound on $\max_{\substack{t\in\mathbb{N}_s^b \\ s\in N_{a+1}^b}} \frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)}$. For the case $\nu - j - 1 > 0$,

$$\begin{aligned} H_{\nu-j-1}(b,\rho(s)) &\leq H_{\nu-j-1}(b,a), \text{ by Proposition 3.6, part (ii)} \\ &\leq H_{N-j-1}(b,a), \text{ by Proposition 3.11} \\ &\leq H_{N-j-1}(b+1,a), \text{ by Proposition 3.6, part (iii)} \\ &= H_{N-j-1}(b,a-1). \end{aligned} \tag{3.28}$$

Note also that if $\nu - j - 1 = 0$, then (3.28) still holds. Hence,

$$\begin{aligned} \frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)} &\leq H_{N-1}(b,a-1)\frac{H_{N-j-1}(b,a)}{H_{N-j-1}(b,a-1)} \\ &\leq H_{N-1}(b,a-1). \end{aligned} \tag{3.29}$$

If $\nu - j - 1 < 0$, then $\nu < j + 1$. Therefore, we must have $j = N - 1$ because $j \in \mathbb{N}_0^{N-1}$. Since $-1 < \nu - N < 0$, by Proposition 3.6, part (ii), $\max_{s\in\mathbb{N}_{a+1}^b} H_{\nu-N}(b, \rho(s)) =$

$H_{\nu-N}(b,\rho(b)) = 1$. Therefore,

$$H_{N-1}(t,a-1)H_{\nu-N}(b,\rho(s)) \leq H_{N-1}(b,a-1)H_{\nu-N}(b,\rho(b))$$
$$\leq H_{N-1}(b,a-1).$$

Thus, in the case $\nu - j - 1 < 0$, (3.29) also holds.

Also, since $H_{\nu-1}(t,\rho(s))$ is an increasing function of t and decreasing function of s by Proposition 3.6, part (ii) and part (iii),

$$H_{\nu-1}(t,\rho(s)) \leq H_{\nu-1}(b,a). \tag{3.30}$$

Thus, from (3.27),

$$|G_\nu(t,s)| \leq \max\left\{H_{N-1}(b,a-1), H_{\nu-1}(b,a)\right\}.$$

In the case $t \leq \rho(s)$ and $s \in \mathbb{N}_{a+1}^b$, note that from (3.21)

$$|G_\nu(t,s)| = \left|\frac{H_{N-1}(t,a-1)H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)}\right|$$
$$\leq H_{N-1}(b-1,a-1),$$

since $H_{N-1}(t,a-1) = 0$ for $t \in \mathbb{N}_{a-N+1}^a$, and $H_{N-1}(t,a-1) \leq H_{N-1}(b-1,a-1)$ for $t \in \mathbb{N}_{a+1}^{\rho(s)}$ and $s \in \mathbb{N}_{a+1}^b$. Also, we have that $\frac{H_{\nu-j-1}(b,\rho(s))}{H_{N-j-1}(b,a-1)} \leq 1$ follows from (3.28). Hence, for $(t,s) \in \mathbb{N}_{a-N+1}^b \times \mathbb{N}_{a+1}^b$, $G_\nu(t,s) \leq \max\{H_{\nu-1}(b,a), H_{N-1}(b,a-1)\}$.

We will now show that $H_{\nu-1}(b,a) \leq H_{N-1}(b+1,a)$, from which the first result will follow. Since $0 < \nu - 1 \leq N - 1$, from Proposition 3.11,

$$H_{\nu-1}(b,a) \leq H_{N-1}(b,a), \tag{3.31}$$

and from Proposition 3.6, part (iii), we have

$$H_{N-1}(b,a) \leq H_{N-1}(b+1,a). \tag{3.32}$$

Thus from (3.31) and (3.32), it follows that $H_{\nu-1}(b,a) \leq H_{N-1}(b+1,a)$.

For the case $j = 0$, we can improve the bound on $|G_\nu(t,s)|$, so we will consider this case separately. In the case $j = 0$, $H_{N-j-1}(t,a-1) = H_{N-1}(t,a-1)$ is a nondecreasing function of t, for $t \in \mathbb{N}_{a-1}$, by Proposition 3.6, part (iii), so $\frac{H_{N-1}(t,a-1)}{H_{N-1}(b,a-1)} \leq 1$. Also, using conventions, $H_{N-1}(t,a-1) = 0$, for $t \in \mathbb{N}_{a-N+1}^{a}$. It follows that $\max_{t\in\mathbb{N}_{a-N+1}^{b}} \left[\frac{H_{N-1}(t,a-1)H_{\nu-1}(b,\rho(s))}{H_{N-1}(b,a-1)}\right] \leq H_{\nu-1}(b,\rho(s))$. Thus, from (3.26) with $j = 0$ and Proposition 3.10,

$$\begin{aligned}|G_\nu(t,s)| &\leq \max\left\{\max_{t\in\mathbb{N}_s^b}\left[\frac{H_{N-1}(t,a-1)H_{\nu-1}(b,\rho(s))}{H_{N-1}(b,a-1)}\right], \max_{t\in\mathbb{N}_s^b}\left[H_{\nu-1}(t,\rho(s))\right]\right\}\\ &\leq H_{\nu-1}(b,\rho(s))\\ &\overset{(3.30)}{\leq} H_{\nu-1}(b,a).\end{aligned}$$

Also, when $t \leq \rho(s)$ for the case $j = 0$,

$$\begin{aligned}|G_\nu(t,s)| &= \left|\frac{H_{N-1}(t,a-1)H_{\nu-1}(b,\rho(s))}{H_{N-1}(b,a-1)}\right|\\ &\leq |H_{\nu-1}(b,\rho(s))|\\ &\leq |H_{\nu-1}(b,a)|.\end{aligned}$$

Hence, we have that $|G_\nu(t,s)| \leq H_{\nu-1}(b,a)$ for all $(t,s) \in \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b}$ when $j = 0$. $\Diamond$

3.2.1 The Cases $1 < \nu \leq 2$ and $2 \leq \nu < 3$ for an $(N-1, 1)$ Right-Focal BVP

Much work for the specific cases $1 < \nu \leq 2$ and $2 \leq \nu < 3$ of the BVP (3.19) where $j = N - 1$ has been done in [27]. We will expand on that work involving bounds of the Green's functions for the specific cases in this section.

We will now consider the Green's function for the BVP

$$\begin{cases} -\nabla^{\nu}_{a*}x(t) = 0, & t \in \mathbb{N}^{b}_{a+1} \\ x(a-1) = \nabla x(b) = 0, \end{cases} \tag{3.33}$$

where $1 < \nu \leq 2$.

From Theorem 3.9, the Green's function for the BVP (3.33), is given by

$$G_{\nu}(t,s) = \begin{cases} H_1(t, a-1)H_{\nu-2}(b, \rho(s)), & t \leq \rho(s) \\ H_1(t, a-1)H_{\nu-2}(b, \rho(s)) - H_{\nu-1}(t, \rho(s)), & t \geq \rho(s), \end{cases} \tag{3.34}$$

and $(t, s) \in \mathbb{N}^{b}_{a-1} \times \mathbb{N}^{b}_{a+1}$. In particular, we consider the case $1 \leq b - a \leq \frac{1}{2-\nu}$.

Theorem 3.13 *Let* $1 < \nu < 2$ *and* $1 \leq b - a \leq \frac{1}{2-\nu}$. *Then,*

$$0 \leq G_{\nu}(t,s) \leq b - a, \tag{3.35}$$

where G_{ν} *is given by* (3.34).

Proof. From [27, Theorem 3.11], it follows that

$$\max_{t \in \mathbb{N}^{b}_{a-1}} G_{\nu}(t,s) = G_{\nu}(\rho(s), s),$$

for each fixed $s \in \mathbb{N}_{a+1}^{b}$. From (3.34), we see that

$$G_\nu(\rho(s), s) = H_1(\rho(s), a-1)H_{\nu-2}(b, \rho(s)), \tag{3.36}$$

for $s \in \mathbb{N}_{a+1}^{b}$. From Proposition 3.6, since $-1 < \nu - 2 < 0$, we have that $H_{\nu-2}(b, \rho(s))$ is an increasing function of s for $s \in \mathbb{N}_{a+1}^{b}$, so $H_{\nu-2}(b, \rho(s)) \leq 1$. Then,

$$\begin{aligned} G_\nu(t, s) &\leq H_1(\rho(s), a-1) \\ &\leq H_1(b-1, a-1) = b - a. \end{aligned}$$

It follows directly from [27, Theorem 3.11(i)] that $G_\nu(t, s) \geq 0$. $\Diamond$

We will now consider the Green's function for the BVP

$$\begin{cases} -\nabla_{a*}^{\nu} x(t) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ x(a-1) = \nabla x(a-1) = 0 \\ \nabla^2 x(b) = 0, \end{cases} \tag{3.37}$$

where $2 < \nu \leq 3$. In particular, we will consider bounds on the Green's function for the BVP (3.37) for the case $\frac{5}{2} \leq \nu \leq 3$ and $2 \leq b - a \leq \frac{1}{3-\nu}$. Note that we obtain a better bound than the one given in Theorem 3.12 for the case $N = 3$.

Theorem 3.14 *Let $\frac{5}{2} \leq \nu \leq 3$ and $2 \leq b - a \leq \frac{1}{3-\nu}$. Then, the Green's function, $G_\nu : \mathbb{N}_{a-2}^{b} \times \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, for the BVP (3.37) satsifies the inequality*

$$0 \leq G_\nu(t, s) \leq \frac{(b-a+1)^{\overline{2}}}{2} - 1 \tag{3.38}$$

for all $(t, s) \in \mathbb{N}_{a-2} \times \mathbb{N}_{a+1}^{b}$.

Proof. From [27, Theorem 4.2], it follows that the Green's function $G_\nu(t,s)$ for the BVP (3.37) satsifies $G_\nu(t,s) \geq 0$ and $\max_{t\in\mathbb{N}_{a-2}^b} G_\nu(t,s) = G_\nu(b,s)$. We will now calculate $\max_{\substack{t\in\mathbb{N}_{a-2}^b \\ s\in N_{a+1}^b}} G_\nu(t,s)$. Note that it follows from Theorem 3.9 that

$$G_\nu(t,s) = \begin{cases} H_2(t,a-1)H_{\nu-3}(b,\rho(s)), & t \leq \rho(s) \\ H_2(t,a-1)H_{\nu-3}(b,\rho(s)) - H_{\nu-1}(t,\rho(s)), & t \geq \rho(s). \end{cases} \tag{3.39}$$

Hence,

$$G_\nu(b,s) = H_2(b,a-1)H_{\nu-3}(b,\rho(s)) - H_{\nu-1}(b,\rho(s)).$$

Now for $s \in \mathbb{N}_{a+1}^b$, $H_{\nu-3}(b,\rho(s)) \geq 0$ and $H_{\nu-1}(b,\rho(s)) \geq 0$, by Proposition 3.6, part (i). Then, for $s \in \mathbb{N}_{a+1}^b$,

$$G_\nu(b,s) \leq H_2(b,a-1)\left(\max_{s\in\mathbb{N}_{a+1}^b} H_{\nu-3}(b,\rho(s))\right) - \min_{s\in\mathbb{N}_{a+1}^b} H_{\nu-1}(b,\rho(s)). \tag{3.40}$$

Since $-1 < \nu - 3 \leq 0$ by Proposition 3.6, part (ii), $H_{\nu-3}(b,\rho(s))$ is an increasing function of s for $s \in \mathbb{N}_{a+1}^b$. Hence, $\max_{s\in\mathbb{N}_{a+1}^b} H_{\nu-3}(b,\rho(s)) = 1$. Also, $H_{\nu-2}(b,\rho(s))$ is a decreasing function of s, so $\min_{s\in\mathbb{N}_{a+1}^b} H_{\nu-2}(b,\rho(s)) = 1$. Then, from (3.40), we obtain $G_\nu(t,s) \leq H_2(b,a-1) - 1 = \frac{(b-a+1)^{\overline{2}}}{2} - 1$. ♢

3.3 Further Work

The formula given in Chapter 2, Theorem 2.12 for computing a Green's function can be used to examine more cases of Green's functions and establish their properties. For example, we can consider $(N-2,2)$ boundary value problems and the corresponding Green's functions or investigate more general results for $(k, N-k)$ BVPs. There remains to be found a sufficient condition for which any particular $(N-1,1)$ Green's function from this chapter is of constant sign for general ν. In Theorem 3.7, we have a sufficient condition for which the Green's function for the conjugate case when $1 < \nu \leq 2$ is of constant sign. However, we continue to seek improvements on this condition. Having the Green's function be of constant sign is useful in obtaining Lyapunov inequalities that give information regarding sign change of solutions. Since, in the discrete case, a solution changing sign can be viewed as the analogue to solutions in the continuous case having zeros, the concept can be used to study disconjugacy for the discrete setting. Also, applying fixed point theorems often heavily relies on the Green's function being of constant sign, as in the case of Contraction Mapping Theorem used in [3] and Krasnoselskii's Theorem in [27]. In [15], an alternative approach using spectral theory that does not require computation of the Green's function is used for finding conditions under which Green's functions are of constant sign that perhaps could be considered in the nabla Caputo context.

Additionally, improvements can be made on the bounds on Green's functions, with the hope of eventually obtaining sharp bounds.

Chapter 4

Lyapunov Inequalities for Nabla Caputo Boundary Value Problems

In this chapter, we will give Lyapunov inequalities for $(N-1,1)$ boundary value problems using the results of the previous chapter. In particular, using the Green's functions for the equation $-\nabla_{a*}^{\nu}x(t)=0$, we will obtain integral equations for solutions to BVPs involving $\nabla_{a*}^{\nu}x(t)+q(t)x(t-1)=0$. We will also consider boundary conditions distinct from the $(k,N-k)$ boundary conditions from Chapter 2 and obtain Lyapunov inequalities for $2<\nu\leq 3$ using a method similar to the one used in [16]. We will then show how to generalize this method for higher order BVPs. In the last section, we will state and prove Lyapunov inequalities for a nabla Caputo self-adjoint equation, as studied in [3] and [27]. Throughout this chapter, we assume $\nu>1$.

4.1 Initial Value Problems and Boundary Value Problems Involving $\nabla_{a*}^{\nu}x(t)+q(t)x(t-1)=f(t)$

In this section, we will show some standard results that apply in the context of the equation $\nabla_{a*}^{\nu}x(t)+q(t)x(t-1)=f(t)$. These results include uniqueness of solutions to IVPs, existence of N linearly independent solutions, where $N:=\lceil\nu\rceil$, and representing general solutions as a linear combination of the linearly independent solutions. These results lead to existence-uniqueness results for BVPs. Note that the second term in the last equation involves $x(t-1)$ instead of $x(t)$ to ensure uniqueness of solutions to initial value problems without imposing restrictions on $q(t)$.

Theorem 4.1 *Let $\nu > 1$, $N := \lceil \nu \rceil$, and $f : \mathbb{N}_{a+1} \to \mathbb{R}$. Then, the initial value problem*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = f(t), \quad t \in \mathbb{N}_{a+1} \\ x(a-N+1) = A_{N-1}, \quad x(a-N+2) = A_{N-2}, \cdots, x(a-1) = A_1, \quad x(a) = A_0 \end{cases} \tag{4.1}$$

has a unique solution defined on $\mathbb{N}_{a-N+1}$.

Proof. By the initial conditions in (4.1), $x(t)$ is uniquely defined for $t \in \mathbb{N}_{a-N+1}^{a}$. We will show by induction on k, where $t = a + k$, that x is uniquely defined for $t \in \mathbb{N}_{a+1}$. Expanding the operator ∇_{a*}^{ν} gives

$$\begin{aligned} \nabla_{a*}^{\nu} x(t) &= \nabla_a^{-(N-\nu)} \nabla^N x(t) \\ &= \int_a^t H_{N-\nu-1}(t, \rho(s)) \nabla^N x(s) \nabla s \\ &= \int_a^t H_{N-\nu-1}(t, \rho(s)) \sum_{i=0}^{N} (-1)^i \binom{N}{i} x(s-i) \nabla s \\ &= \sum_{s=a+1}^{t} H_{N-\nu-1}(t, \rho(s)) \sum_{i=0}^{N} (-1)^i \binom{N}{i} x(s-i). \end{aligned}$$

Hence, the equation in (4.1) is equivalent to

$$\sum_{s=a+1}^{t} H_{N-\nu-1}(t, \rho(s)) \sum_{i=0}^{N} (-1)^i \binom{N}{i} x(s-i) + q(t)x(t-1) = f(t), \quad t \in \mathbb{N}_{a+1}. \tag{4.2}$$

For the base case, letting $t = a + 1$ in (4.2), we get

$$\sum_{i=0}^{N}(-1)^i\binom{N}{i}x(a+1-i) + q(a+1)x(a) = f(a+1).$$

Plugging in the initial values given in (4.1) and solving for $x(a+1)$ in the above equation gives

$$x(a+1) = -\sum_{i=1}^{N}(-1)^i\binom{N}{i}A_{i-1} - q(a+1)A_0 + f(a+1).$$

Now for the strong inductive hypothesis assume that x is defined for $t = a - N + 1, \dots, a, \dots, a+k$. We will show that from (4.2), we can compute $x(a+k+1)$ uniquely. Letting $t = a + k + 1$ in (4.2), we get

$$\left(\sum_{s=a+1}^{a+k+1} H_{N-\nu-1}(a+k+1, \rho(s)) \sum_{i=0}^{N}(-1)^i\binom{N}{i}x(s-i)\right) + q(a+k+1)x(a+k)$$
$$= f(a+k+1). \qquad (4.3)$$

Solving for $x(a+k+1)$ in (4.3) gives

$$\begin{aligned} x(a+k+1) = &- \sum_{s=a+1}^{a+k} H_{N-\nu-1}(a+k+1, \rho(s)) \sum_{i=0}^{N}(-1)^i\binom{N}{i}x(s-i) \\ &- q(a+k+1)x(a+k) - \sum_{i=1}^{N}(-1)^i\binom{N}{i}x(a+k+1-i) + f(a+k+1). \end{aligned}$$

Hence, x is uniquely defined on $\mathbb{N}_{a-N+1}$. ◊

Theorem 4.2 *Let $N := \lceil \nu \rceil$. Then, there exist N linearly independent solutions to*

$$\nabla_{a*}^{\nu}x(t) + q(t)x(t-1) = 0, \quad t \in \mathbb{N}_{a+1} \qquad (4.4)$$

defined on $\mathbb{N}_{a-N+1}$.

Proof. For $k \in \mathbb{N}_0^{N-1}$, let x_k be defined to be the unique solution satisfying $\nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = 0$, for $t \in \mathbb{N}_{a+1}$ and the initial conditions

$$x(a-N+1+i) = \begin{cases} 0, & \text{if } k \neq i \\ 1, & \text{if } k = i, \end{cases}$$

for $i \in \mathbb{N}_0^{N-1}$, which is guaranteed to exist by Theorem 4.1. Then, suppose there exist $c_0, \cdots, c_{N-1} \in \mathbb{R}$ such that

$$c_0 x_0(t) + \cdots + c_{N-1} x_{N-1}(t) = 0 \tag{4.5}$$

for all $t \in \mathbb{N}_{a-N+1}$. Letting $t = a - N + 1 + i$ in (4.5) for each fixed $i \in \mathbb{N}_0^{N-1}$ gives

$$c_0 x_0(a-N+1+i) + \cdots + c_i x_i(a-N+1+i) + \cdots + c_{N-1} x_{N-1}(a-N+1+i) = 0,$$

which implies $c_i = 0$ for each fixed $i \in \mathbb{N}_0^{N-1}$. Hence, we get that the solutions x_k for $k \in \mathbb{N}_0^{N-1}$ are N linearly independent solutions to (4.4) on $\mathbb{N}_{a-N+1}$. $\Diamond$

The argument in the proof of the following theorem is standard and is also used in [27, Chapter 3].

Theorem 4.3 *Suppose* $x_0, \cdots, x_{N-1}$ *are linearly independent solutions to* (4.4) *on* $\mathbb{N}_{a-N+1}$. *Then,* $x(t) := c_0 x_0(t) + \cdots + c_{N-1} x_{N-1}(t)$ *is a general solution to* (4.4).

Proof. By the linearity of the Caputo fractional difference, we get that $x(t) := c_0 x_0(t) + \cdots + c_{N-1} x_{N-1}(t)$ is a solution. Now suppose we have that $y(t)$ is a solution to (4.4) and define $A_0 = y(a), \cdots, A_{N-1} = y(a-N+1)$. Note that plugging $t = a, \cdots, a-N+1$ into $x(t)$ and setting it equal to $y(t)$ evaluated at

$t = a, \cdots, a - N + 1$, respectively, we get the system of equations

$$\begin{aligned} c_0 x_0(a) + \cdots + c_{N-1} x_{N-1}(a) &= A_0 \\ c_0 x_0(a+1) + \cdots + c_{N-1} x_{N-1}(a+1) &= A_1 \\ &\vdots \\ c_0 x_0(a-N+1) + \cdots + c_{N-1} x_{N-1}(a-N+1) &= A_{N-1}. \end{aligned}$$

This system is equivalent to the vector equation

$$\begin{pmatrix} x_0(a) & \cdots & x_{N-1}(a) \\ x_0(a+1) & \cdots & x_{N-1}(a+1) \\ \vdots & \ddots & \vdots \\ x_0(a-N+1) & \cdots & x_{N-1}(a-N+1) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{N-1} \end{pmatrix}. \tag{4.6}$$

We claim that since x_k, for $k \in \mathbb{N}_0^{N-1}$, are linearly independent, the determinant of the above matrix is nonzero. Suppose for contradiction that the determinant is zero. Then, we can write one column as a linear combination of the other columns; i.e., without loss of generality,

$$\alpha_0 \begin{pmatrix} x_0(a) \\ x_0(a+1) \\ \vdots \\ x_0(a-N+1) \end{pmatrix} + \cdots + \alpha_{N-2} \begin{pmatrix} x_{N-2}(a) \\ x_{N-2}(a+1) \\ \vdots \\ x_{N-2}(a-N+1) \end{pmatrix} = \begin{pmatrix} x_{N-1}(a) \\ x_{N-1}(a+1) \\ \vdots \\ x_{N-1}(a-N+1) \end{pmatrix}$$

for some $\alpha_0, \alpha_1, \cdots, \alpha_{N-2} \in \mathbb{R}$. In particular, $\alpha_1, \ldots, \alpha_{N-2}$ are not all zero, since otherwise $x_{N-1}(t) \equiv 0$, contradicting that $x_0, \ldots, x_{N_1}$ are linearly independent.

Define $Y(t) := \alpha_0 x_0(t) + \alpha_1 x_1(t) + \cdots + \alpha_{N-2} x_{N-2}(t)$ for $t \in \mathbb{N}_{a-N+1}$. Then, note that $Y(t)$ and $x_{N-1}(t)$ both solve the IVP

$$\begin{cases} \nabla_{a*}^{\nu} w(t) = 0, & t \in \mathbb{N}_{a+1} \\ w(a-i) = x_{N-1}(a-i), & i \in \mathbb{N}_0^{N-1}. \end{cases}$$

Therefore, by uniqueness of solutions to IVPs, $x_{N-1}(t) = \alpha_0 x_0(t) + \alpha_1 x_1(t) + \cdots + \alpha_{N-2} x_{N-2}(t)$, for $t \in \mathbb{N}_{a-N+1}$. Hence, $\alpha_0 x_0(t) + \alpha_1 x_1(t) + \cdots + \alpha_{N-2} x_{N-2}(t) - x_{N-1}(t) = 0$ for all $t \in \mathbb{N}_{a-N+1}$. This contradicts that $x_0, \ldots, x_{N-1}$ are linearly independent, so we must have that

$$\begin{vmatrix} x_0(a) & \cdots & x_{N-1}(a) \\ x_0(a+1) & \cdots & x_{N-1}(a+1) \\ \vdots & \ddots & \vdots \\ x_0(a-N+1) & \cdots & x_{N-1}(a-N+1) \end{vmatrix} \neq 0.$$

Hence, we can uniquely solve for $c_0, \cdots, c_{N-1}$ in (4.6), which shows that $y(t) = c_0 x_0(t) + \cdots + c_{N-1} x_{N-1}(t)$ for some $c_i \in \mathbb{R}$, so $x(t) = c_0 x_0(t) + \cdots + c_{N-1} x_{N-1}(t)$ is a general solution to (4.4). $\Diamond$

Corollary 4.4 *A general solution to the nonhomogeneous equation* (4.1)*, defined on* $\mathbb{N}_{a-N+1}$*, is given by* $x(t) = c_0 x_0(t) + \cdots + c_{N-1} x_{N-1}(t) + x_p(t)$*, where* $x_0, \cdots, x_{N-1}$ *are linearly independent solutions to the corresponding homogeneous equation,* (4.4)*, and* x_p *is a particular solution to* (4.1)*.*

Proof. By linearity, $x(t)$ is a solution. Suppose $y(t)$ is any solution to the nonhomogeneous equation, (4.1). Then, using linearity, $y(t) - x_p(t)$ is a solution to the homogeneous equation (4.4). Hence, from Theorem 4.3, $y(t) - x_p(t) = c_0 x_0(t) + \cdots + c_{N-1} x_{N-1}(t)$ for some $c_i \in \mathbb{R}$, and the result follows. $\Diamond$

4.1.1 Existence-Uniqueness of Solutions to BVPs

The theorems in this subsection give versions of Fredholm's alternative for boundary value problems involving the equation $\nabla_{a*}^{\nu}x(t) + q(t)x(t-1) = f(t)$. The proofs use Theorem 4.3 and Corollary 4.4 and are nearly identical to the proof of Theorem 2.4, so they are omitted. The Lyapunov inequalities developed in this chapter will give sufficient conditions for the homogeneous BVPs to have only the trivial solution. Then, using the theorems below, the same conditions give sufficient conditions for the nonhomogenous BVPs to have unique solutions.

Theorem 4.5 *Let $\nu > 1$, $N := \lceil \nu \rceil$, $j \in \mathbb{N}_0^{N-1}$ be fixed, $b - a \in \mathbb{N}_{\max\{1,j\}}$ and $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$. If the homogeneous BVP*

$$\begin{cases} \nabla_{a*}^{\nu}x(t) + q(t)x(t-1) = 0, & t \in \mathbb{N}_{a+1}^b \\ \nabla^i x(a-1) = 0, & i \in \mathbb{N}_0^{N-2} \\ \nabla^j x(b) = 0, & \end{cases}$$

has only the trivial solution, then the nonhomogeneous BVP

$$\begin{cases} \nabla_{a*}^{\nu}x(t) + q(t)x(t-1) = f(t), & t \in \mathbb{N}_{a+1}^b \\ \nabla^i x(a-1) = A_i, & i \in \mathbb{N}_0^{N-2} \\ \nabla^j x(b) = B, & \end{cases}$$

where $f : \mathbb{N}_{a+1}^b \to \mathbb{R}$ and $A_i, B \in \mathbb{R}$ for $i \in \mathbb{N}_0^{N-2}$, has a unique solution defined on $\mathbb{N}_{a-N+1}^b$.

Theorem 4.6 *Let $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$, $\nu > 2$, $N := \lceil \nu \rceil$, $b - a \in \mathbb{N}_{N-1}$, and $r \in \{1, 2\}$ be* *fixed. If the homogeneous BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = 0, \;\; t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{N-2} x(a-1) = 0 \\ \nabla^{N-r} x(b) = 0 \\ \nabla^{i} x(c_i) = 0, \;\; i \in \mathbb{N}_0^{N-3}, \end{cases}$$

where $c_i \in \{a-1, b\}$*, has only the trivial solution, then the nonhomogenous BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = f(t), \;\; t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{N-2} x(a-1) = A_0 \\ \nabla^{N-r} x(b) = B_0 \\ \nabla^{i} x(c_i) = C_i, \;\; i \in \mathbb{N}_0^{N-3}, \end{cases}$$

where $f : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ *and* A_0*,* B_0*,* $C_i \in \mathbb{R}$ *for* $i \in \mathbb{N}_0^{N-3}$*, has a unique solution defined on* $\mathbb{N}_{a-N+1}^{b}$*.*

Corollary 4.7 *Let* $q : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$*,* $2 < \nu \leq 3$*,* $b - a \in \mathbb{N}_2$*, and* $r \in \{1, 2\}$ *be fixed. If the homogeneous BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = 0, \;\; t \in \mathbb{N}_{a+1}^{b} \\ \nabla x(a-1) = 0, \;\; \nabla^{r} x(b) = 0, \;\; x(c) = 0, \end{cases}$$

where $c \in \{a-1, b\}$ *has only the trivial solution, then the nonhomogenous BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = f(t), \;\; t \in \mathbb{N}_{a+1}^{b} \\ \nabla x(a-1) = A_0, \;\; \nabla^{r} x(b) = B_0, \;\; x(c) = C_0, \end{cases}$$

where $f : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ *and* $A_0,\ B_0,\ C_0 \in \mathbb{R}$, *has a unique solution defined on* $\mathbb{N}_{a-2}^{b}$.

Theorem 4.8 *Let* $q : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, $\nu > 3$, $N := \lceil \nu \rceil$, *and* $b - a \in \mathbb{N}_{N-1}$. *If the homogeneous BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{N-3} x(a-1) = 0, \ \ \nabla^{N-2} x(a-1) = 0 \\ \nabla^{N-1} x(b) = 0 \\ \nabla^{i} x(c_i) = 0, & i \in \mathbb{N}_0^{N-4}, \end{cases}$$

where $c_i \in \{a-1, b\}$ *for* $i \in \mathbb{N}_0^{N-4}$, *has only the trivial solution, then the nonhomogeneous BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = f(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{N-3} x(a-1) = A_0, \ \ \nabla^{N-2} x(a-1) = A_1 \\ \nabla^{N-1} x(b) = B_0 \\ \nabla^{i} x(c_i) = C_i, & i \in \mathbb{N}_0^{N-4}, \end{cases}$$

where $f : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ *and* $A_0,\ A_1,\ B_0,\ C_i \in \mathbb{R}$ *for* $i \in \mathbb{N}_0^{N-4}$, *has a unique solution defined on* $\mathbb{N}_{a-N+1}^{b}$.

4.2 Lyapunov Inequalities for $(N-1, 1)$ BVPs

4.2.1 Conjugate BVP

The following theorem gives a necessary condition for a boundary value problem with $(N-1, 1)$ conjugate boundary conditions to have a nontrivial solution. This means that from the contrapositive statement of the theorem, we obtain a sufficient condition for the BVP to have only the trivial solution. With Theorem 4.5, this gives a sufficient condition for a corresponding nonhomogeneous BVP to have a unique solution.

Theorem 4.9 *Let* $\nu > 1$, $N := \lceil \nu \rceil$, $q : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, *and* $b - a \in \mathbb{N}_1$. *Consider the conjugate boundary value problem*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = 0, & t \in \mathbb{N}_{a+1}^{b}, \\ \nabla^{i} x(a-1) = 0, & i \in \mathbb{N}_{0}^{N-2} \\ x(b) = 0. \end{cases} \tag{4.7}$$

If the BVP (4.7) *has a nontrivial solution* $x : \mathbb{N}_{a-N+1}^{b} \to \mathbb{R}$, *then*

$$\int_a^b |q(s)| \nabla s \geq \frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)}. \tag{4.8}$$

Proof. From Theorem 3.1, we have that a solution $x : \mathbb{N}_{a-N+1}^{b} \to \mathbb{R}$ satisfies the integral equation

$$x(t) = \int_a^b G_\nu(t,s) q(s) x(s-1) \nabla s, \ \ t \in \mathbb{N}_{a-N+1}^{b}, \tag{4.9}$$

where $G_\nu(t,s)$ is given by (3.4). Assume that x is a nontrivial solution to the BVP (4.7). Then,

$$|x(t)| \leq \int_a^b |G_\nu(t,s)||q(s)||x(s-1)| \nabla s, \ \ t \in \mathbb{N}_{a-N+1}^{b}.$$

Let $M := \max_{t \in \mathbb{N}_{a-N+1}^b} |x(t)|$. Then, in particular, if $t = t_0$ such that $|x(t_0)| = M$, we have

$$\begin{aligned}|x(t_0)| &\leq \int_a^b |G_\nu(t_0,s)||q(s)||x(s-1)|\nabla s\\ &= \sum_{s=a+1}^{b} |G_\nu(t_0,s)||q(s)||x(s-1)|.\end{aligned}$$

Note that $x(a) \neq 0$ since otherwise, by the uniqueness of solutions to IVPs shown in Theorem 4.1, we would have that x is the trivial solution. Thus, x is not identically zero on $\mathbb{N}_a^{b-1}$, and $b - a \in \mathbb{N}_1$ implies $\mathbb{N}_a^{b-1} \neq \emptyset$, so the inequality

$$|x(t_0)| \leq \sum_{s=a+1}^{b} |G_\nu(t_0,s)||q(s)|M$$

holds, noting that $|x(s-1)| \leq M$ for $s - 1 \in \mathbb{N}_{a+1}^b$. Hence, we have

$$M \leq \int_a^b |G_\nu(t_0,s)||q(s)|M\nabla s;$$

i.e.,

$$\begin{aligned}1 &\leq \int_a^b |G_\nu(t_0,s)||q(s)|\nabla s\\ &\leq \int_a^b \frac{\Gamma(b-a+\nu-1)}{\Gamma(b-a)\Gamma(\nu)}|q(s)|\nabla s,\end{aligned}$$

by the bound on $|G_\nu(t,s)|$ given in Theorem 3.12. Thus, we have

$$\int_a^b |q(s)|\nabla s \geq \frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)}.$$

$\Diamond$

Corollary 4.10 *Let $1 < \nu \leq 2$ and $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$. Assume $b - a \in \mathbb{N}_1$. Consider the* *second order conjugate boundary value problem*

$$\begin{cases} \nabla^{\nu}_{a*}x(t) + q(t)x(t-1) = 0, \quad t \in \mathbb{N}^{b}_{a+1} \\ x(a-1) = x(b) = 0. \end{cases} \tag{4.10}$$

If the BVP (4.10) *has a nontrivial solution* $x : \mathbb{N}^{b}_{a-1} \to \mathbb{R}$*, then*

$$\int_{a}^{b} |q(s)| \nabla s \geq \frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)}. \tag{4.11}$$

Using the Lyapunov inequality in Theorem 4.9 along with Theorem 4.5, we get the following corollary.

Corollary 4.11 *Let* $\nu > 1$*,* $N := \lceil \nu \rceil$*, and* $q : \mathbb{N}^{b}_{a+1} \to \mathbb{R}$*. Assume* $b - a \in \mathbb{N}_1$*. Consider the nonhomogeneous boundary value problem*

$$\begin{cases} \nabla^{\nu}_{a*}x(t) + q(t)x(t-1) = f(t), & t \in \mathbb{N}^{b}_{a+1} \\ \nabla^{i}x(a-1) = A_i, & i \in \mathbb{N}^{N-2}_{0} \\ \nabla x(b) = B, \end{cases} \tag{4.12}$$

where $f : \mathbb{N}^{b}_{a+1} \to \mathbb{R}$*. If* $q(t)$ *satisfies*

$$\int_{a}^{b} |q(t)| \nabla t < \frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)},$$

then the BVP (4.12) *has a unique solution defined on* $\mathbb{N}^{b}_{a-N+1}$*.*

For the case $1 < \nu \leq 2$ in (4.7), the results of Theorem 3.4 and Theorem 3.7 give sufficient conditions such that the Green's function does not change sign. We use this property to obtain another Lyapunov inequality. These results involve a sufficient condition for a nontrivial solution to the BVP to change sign on its domain or, in other words, have a generalized zero.

Theorem 4.12 *Assume* $b-a \in \mathbb{N}_1$. *Suppose either* $1 < \nu < 2$ *and* $\nu \geq \frac{b-a}{b-a+1} + 1$ *holds; or* $\nu = 2$. *If the BVP* (4.10) *has a nontrivial solution which does not change sign on* $\mathbb{N}_a^{b-1}$, *then we have*

$$\int_a^b q_+(s)\nabla s \geq \frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)}.$$

Proof. A solution, $x : \mathbb{N}_{a-1}^b \to \mathbb{R}$, to the BVP (4.10) satisfies the integral equation

$$x(t) = \int_a^b G_\nu(t,s)q(s)x(s-1)\nabla s,$$

where $G_\nu(t,s)$ is given by (3.4). Assume, without loss of generality, that $x(t) \geq 0$ for $t \in \mathbb{N}_a^{b-1}$, and let $M = \max_{t \in \mathbb{N}_a^{b-1}} x(t)$. Since $G_\nu \geq 0$ by Theorem 3.7 and Theorem 3.4,

$$x(t) \leq \int_a^b G_\nu(t,s)q_+(s)x(s-1)\nabla s,$$

for $t \in \mathbb{N}_a^{b-1}$. Let $t = t_0$ such that $x(t_0) = M$. Then,

$$x(t_0) \leq \int_a^b G_\nu(t_0,s)q_+(s)x(s-1)\nabla s.$$

By the bound on $|G_\nu(t,s)|$ given in Theorem 3.12, we have

$$\begin{aligned} x(t_0) &\leq \int_a^b H_{\nu-1}(b,a)q_+(s)x(s-1)\nabla s \\ &\leq \int_a^b H_{\nu-1}(b,a)q_+(s)M\nabla s, \end{aligned}$$

so

$$M \leq \int_a^b H_{\nu-1}(b,a) q_+(s) M \nabla s.$$

Since x is nontrivial, $x(a) \neq 0$, so $M > 0$. Dividing both sides of the last inequality by M gives

$$1 \leq \int_a^b H_{\nu-1}(b,a) q_+(s) \nabla s,$$

from which the result follows. ◊

The following example illustrates an application of Theorem 4.9 to an eigenvalue problem and is similar to one given in [23] for the delta fractional second order case.

Example 4.13 Assume $b-a \in \mathbb{N}_1$. Let $q(t) \equiv -\lambda \in \mathbb{R}$, $\nu > 1$, and $N := \lceil \nu \rceil$. Then, if

$$\begin{cases} \nabla_{a*}^{\nu} x(t) = \lambda x(t-1), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^i x(a-1) = 0, & i \in \mathbb{N}_0^{N-2} \\ x(b) = 0 \end{cases} \tag{4.13}$$

has a nontrivial solution, we have

$$|\lambda| \geq \frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)(b-a)}.$$

In other words, any eigenvalue of the BVP (4.13) must satisfy this last inequality.

Proof. By Theorem 4.9, a necessary condition for the BVP (4.13) to have a nontrivial solution is

$$\int_a^b |\lambda| \nabla s \geq \frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)}.$$

Since

$$\int_a^b |\lambda|\nabla s = \sum_{s=a+1}^{b} |\lambda| = (b-a)|\lambda|,$$

we must have $|\lambda| \geq \frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)(b-a)}$. ◊

If $a = 1$, $b = 6$, $\nu = 1.2$ in (4.13), we must have that any eigenvalue λ satisfies $|\lambda| \geq .135281$.

We can also consider a more general, possibly nonlinear, equation for which we obtain the following result using the same method of proof as Theorem 4.9. A similar result is given in [41, Theorem 3.1].

Theorem 4.14 *Let $\nu > 1$, $N := \lceil \nu \rceil$, and $q : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$. Assume $b - a \in \mathbb{N}_1$ and suppose $f : \mathbb{R} \to \mathbb{R}$ and $|f(x)| \leq B|x|$. Consider the conjugate boundary value problem*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t) f(x(t-1)) = 0, & t \in \mathbb{N}_{a+1}^{b}, \\ \nabla^i x(a-1) = 0, & i \in \mathbb{N}_0^{N-2} \\ x(b) = 0. \end{cases} \tag{4.14}$$

If the BVP (4.14) has a nontrivial solution $x : \mathbb{N}_{a-N+1}^{b} \to \mathbb{R}$, then

$$\int_a^b |q(s)|\nabla s \geq \frac{\Gamma(b-a)\Gamma(\nu)}{B\Gamma(b-a+\nu-1)}. \tag{4.15}$$

In the next example, we consider a nonlinear BVP and the resulting Lyapunov inequality.

Example 4.15 Consider the following continuous piecewise-defined function:

$$f(x) = \begin{cases} 4x, & x < -1 \\ 4x^3, & -1 \le x \le 1 \\ 4x, & x > 1. \end{cases} \tag{4.16}$$

Then, $|f(x)| \le 4|x|$, so f satisfies the condition given in Theorem 4.14 with $B = 4$. Hence, if the BVP (4.14) with f given by (4.16) has a nontrivial solution, then

$$\int_a^b |q(s)|\nabla s \ge \frac{\Gamma(b-a)\Gamma(\nu)}{4\Gamma(b-a+\nu-1)}.$$

4.2.2 Additional $(N-1, 1)$ BVPs

We obtain the following Lyapunov inequality from the Green's function bound given in Theorem 3.12. Note that $j = 0$ in (4.17) below gives the conjugate BVP, but for that specific case, we have a stronger Lyapunov inequality in Theorem 4.9.

Theorem 4.16 *Let $\nu > 1, N := \lceil \nu \rceil$, $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$, and consider the BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = 0, & t \in \mathbb{N}_{a+1}^b \\ \nabla^i x(a-1) = 0, & i \in \mathbb{N}_0^{N-1} \\ \nabla^j x(b) = 0, \end{cases} \tag{4.17}$$

where $j \in \mathbb{N}_0^{N-1}$ is fixed. Assume $b-a \in \mathbb{N}_{\max\{1,j\}}$. If the BVP (4.17) has a nontrivial solution $x : \mathbb{N}_{a-N+1}^b \to \mathbb{R}$, then

$$\int_a^b |q(s)|\nabla s \ge \frac{(N-1)!}{(b-a+N-1)(b-a+N-2)\cdots(b-a+1)}. \tag{4.18}$$

Proof. A solution $x(t)$ of the BVP (4.17) satisfies the integral equation

$$x(t) = \int_a^b G_\nu(t,s)q(s)x(s-1)\nabla s, \tag{4.19}$$

where G_ν is given by (3.21). Then, by (4.19) and following the same steps as in the proof of Theorem 4.9, we arrive at

$$1 \leq \int_a^b |G_\nu(t_0,s)||q(s)|\nabla s,$$

where t_0 is such that $\max_{t\in\mathbb{N}_{a-N+1}^b} |x(t)| = |x(t_0)|$. Then, from the bound on $|G_\nu(t,s)|$ given in Theorem 3.12,

$$1 \leq \int_a^b H_{N-1}(b,a-1)|q(s)|\nabla s$$

from which it follows that

$$\int_a^b |q(s)|\nabla s \geq \frac{1}{H_{N-1}(b,a-1)}.$$

Since

$$\begin{aligned}
\frac{1}{H_{N-1}(b,a-1)} &= \frac{1}{\frac{(b-a+1)^{\overline{N-1}}}{\Gamma(N)}} \\
&= \frac{1}{\frac{\Gamma(b-a+1+N-1)}{\Gamma(b-a+1)\Gamma(N)}} \\
&= \frac{\Gamma(N)\Gamma(b-a+1)}{\Gamma(b-a+1+N-1)} \\
&= \frac{(N-1)!}{(b-a+N-1)(b-a+N-2)\cdots(b-a+1)},
\end{aligned}$$

(4.18) follows. ◊

Using the Lyapunov inequality in Theorem 4.16 along with Theorem 4.5, we get the following corollary.

Corollary 4.17 *Let* $\nu > 1$, $N := \lceil \nu \rceil$, *and* $q : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$. *Assume* $b - a \in \mathbb{N}_{\max\{1,j\}}$. *Consider the nonhomogeneous boundary value problem*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = f(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{i} x(a-1) = A_i, & i \in \mathbb{N}_0^{N-2} \\ \nabla^{j} x(b) = B, & \end{cases} \tag{4.20}$$

where $f : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ *and* $j \in \mathbb{N}_0^{N-1}$ *is fixed. If* $q(t)$ *satisfies*

$$\int_a^b |q(t)|\nabla t < \frac{(N-1)!}{(b-a+N-1)(b-a+N-2)\cdots(b-a+1)},$$

then the BVP (4.20) *has a unique solution defined on* $\mathbb{N}_{a-N+1}^{b}$.

Remark 4.18 It follows by the contrapositive of the statement of Theorem 4.16 that, if

$$\int_a^b |q(s)|\nabla s < \frac{(N-1)!}{(b-a+N-1)(b-a+N-2)\cdots(b-a+1)},$$

then the BVP (4.17) has no nontrivial solution.

Remark 4.18 is used in the next example.

Example 4.19 Suppose $a = 2,\ \ b = 20$. First note that

(i) If $1 < \nu \le 2$, then $\frac{(N-1)!}{(b-a+N-1)(b-a+N-2)\cdots(b-a+1)} = \frac{1}{19} \approx 0.0526$.

(ii) If $2 < \nu \le 3$, then $\frac{(N-1)!}{(b-a+N-1)(b-a+N-2)\cdots(b-a+1)} = \frac{2}{20\cdot 19} \approx 0.00526$.

(iii) If $3 < \nu \le 4$, then $\frac{(N-1)!}{(b-a+N-1)(b-a+N-2)\cdots(b-a+1)} = \frac{6}{21\cdot 20\cdot 19} \approx 0.00075$.

Let $q(t) := \frac{1}{10t^3} - \frac{1}{10(t-1)^3} < 0$ for $t \in \mathbb{N}_3$. Then,

$$\int_2^{20} |q(s)|\nabla s = -\frac{1}{10}\int_2^{20}\left(\frac{1}{s^3} - \frac{1}{(s-1)^3}\right)\nabla s$$
$$= -\frac{1}{10s^3}\bigg|_{s=2}^{20} \approx 0.0124875.$$

Thus, for $j \in \mathbb{N}_0^1$ fixed, the boundary value problem

$$\begin{cases} \nabla_{2*}^{\nu} x(t) + \left(\frac{1}{10t^3} - \frac{1}{10(t-1)^3}\right) x(t-1) = 0, \quad t \in \mathbb{N}_3^{20} \\ x(1) = \nabla^j x(20) = 0, \end{cases}$$

for $1 < \nu \leq 2$, has no nontrivial solution by (i). Similarly, for $2 < \nu \leq 3$ and $j \in \mathbb{N}_0^2$ fixed, the BVP

$$\begin{cases} \nabla_{2*}^{\nu} x(t) + \frac{1}{100}\left(\frac{1}{10t^3} - \frac{1}{10(t-1)^3}\right) x(t-1) = 0, \quad t \in \mathbb{N}_3^{20} \\ x(1) = \nabla x(1) = \nabla^j x(20) = 0 \end{cases}$$

has no nontrivial solution by (ii), and, for $3 < \nu \leq 4$ and $j \in \mathbb{N}_0^3$ fixed, the BVP

$$\begin{cases} \nabla_{2*}^{\nu} x(t) + \frac{1}{1000}\left(\frac{1}{10t^3} - \frac{1}{10(t-1)^3}\right) x(t-1) = 0, \quad t \in \mathbb{N}_3^{20} \\ x(1) = \nabla x(1) = \nabla^2 x(1) = \nabla^j x(20) = 0 \end{cases}$$

has no nontrivial solution by (iii).

When $j = N-1$, for certain cases of the BVP (4.17), the Green's function does not change sign, and we obtain the following further Lyapunov inequality results. These results give a sufficient condition for a nontrivial solution to have generalized zeros.

Theorem 4.20 *Let $1 < \nu < 2$ and $1 \leq b-a \leq \frac{1}{2-\nu}$. Assume $b-a \in \mathbb{N}_1$ and consider* *the BVP*

$$\begin{cases} \nabla^{\nu}_{a*}x(t) + q(t)x(t-1) = 0, & t \in \mathbb{N}^{b}_{a+1} \\ x(a-1) = 0, \ \ \nabla x(b) = 0. \end{cases} \tag{4.21}$$

Assume the BVP (4.21) *has a nontrivial solution* $x : \mathbb{N}^{b}_{a-1} \to \mathbb{R}$ *which does not change sign on* $\mathbb{N}^{b-1}_{a}$*. Then,*

$$\int_{a}^{b} q_{+}(s)\nabla s \geq \frac{1}{b-a}. \tag{4.22}$$

Proof. Using the bound on the Green's function given Theorem 3.14, in a manner similar to the proof of Theorem 4.12, we arrive at

$$1 \leq \int_{a}^{b} (b-a)q_{+}(s)\nabla s,$$

from which (4.22) follows. $\Diamond$

Similar to Theorem 4.20, we also obtain the following theorem for the case $2 < \nu \leq 3$.

Theorem 4.21 *Let* $\frac{5}{2} \leq \nu \leq 3$ *and* $2 \leq b-a \leq \frac{1}{3-\nu}$*. Assume* $b-a \in \mathbb{N}_2$ *and consider the BVP*

$$\begin{cases} \nabla^{\nu}_{a*}x(t) + q(t)x(t-1) = 0, & \in \mathbb{N}^{b}_{a+1} \\ x(a-1) = \nabla x(a-1) = 0 \\ \nabla^2 x(b) = 0. \end{cases} \tag{4.23}$$

Assume the BVP (4.23) *has a nontrivial solution* $x : \mathbb{N}^{b}_{a-2} \to \mathbb{R}$*, which does not change sign on* $\mathbb{N}^{b-1}_{a}$*. Then,*

$$\int_{a}^{b} q_{+}(s)\nabla s \geq \frac{2}{(b-a+1)(b-a+2)-2}.$$

4.3 A Reduction of Order Technique for Obtaining Lyapunov Inequalities

4.3.1 $2 < \nu \leq 3$ Case

The following lemma gives bounds on an integrals involving a Green's function and will be used to obtain Lyapunov inequalities for boundary value problems involving $\nabla^{\nu}_{a*}x(t) + q(t)x(t-1) = 0$, where $2 < \nu \leq 3$, with four distinct boundary conditions.

Throughout this section, we define

$$\begin{aligned} A &:= \max\left\{\frac{H_{\gamma-1}(b,a)}{H_1(b,a-1)}H_2(b,a-1), H_\gamma(b,a)\right\} \\ &= \max\left\{\frac{(b-a)^{\overline{\gamma-1}}(b-a+2)}{2\Gamma(\gamma)}, \frac{(b-a)^{\overline{\gamma}}}{\Gamma(\gamma+1)}\right\} \\ &= \max\left\{\frac{\Gamma(b-a+\gamma-1)(b-a+2)}{2\Gamma(b-a)\Gamma(\gamma)}, \frac{\Gamma(b-a+\gamma)}{\Gamma(b-a)\Gamma(\gamma+1)}\right\}. \end{aligned} \tag{4.24}$$

Lemma 4.22 *Let $s \in \mathbb{N}^b_{a+1}$ and $1 < \gamma \leq 2$. Then, for $j = 0$ in* (3.21),

$$\left|\int_{a-1}^{t} G_\gamma(\tau, s)\nabla\tau\right| \leq A \tag{4.25}$$

and

$$\left|\int_{t}^{b} G_\gamma(\tau, s)\nabla\tau\right| \leq A, \tag{4.26}$$

and, for $j = 1$ in (3.21),

$$\left|\int_{a-1}^{t} G_\gamma(\tau, s)\nabla\tau\right| \leq H_2(b, a-1) \tag{4.27}$$

and

$$\left|\int_{t}^{b} G_\gamma(\tau, s)\nabla\tau\right| \leq H_2(b, a-1), \tag{4.28}$$

where G_γ is defined by (3.21) *with $N = 2$.*

Proof. For $(t,s) \in \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b}$, by (3.21),

$$\begin{aligned}\int_{a-1}^{t} G_\gamma(\tau,s)\nabla\tau &= \int_{a-1}^{\rho(s)} \frac{H_1(\tau,a-1)H_{\gamma-j-1}(b,\rho(s))}{H_{1-j}(b,a-1)}\nabla\tau \\ &\quad + \int_{\rho(s)}^{t} \left[\frac{H_1(\tau,a-1)H_{\gamma-j-1}(b,\rho(s))}{H_{1-j}(b,a-1)} - H_{\gamma-1}(\tau,\rho(s))\right]\nabla\tau \\ &= \frac{H_{\gamma-j-1}(b,\rho(s))}{H_{1-j}(b,a-1)}\int_{a-1}^{t} H_1(\tau,a-1)\nabla\tau - \int_{\rho(s)}^{t} H_{\gamma-1}(\tau,\rho(s))\nabla\tau.\end{aligned}$$

Evaluating the integral from the first term, by Theorem 1.8, part (iii),

$$\int_{a-1}^{t} H_1(\tau,a-1)\nabla\tau = H_2(t,a-1).$$

Next, for $t \geq \rho(s)$,

$$\int_{\rho(s)}^{t} H_{\gamma-1}(\tau,\rho(s))\nabla\tau = H_\gamma(t,\rho(s)).$$

Note that if $\rho(s) \geq t$, $\int_{\rho(s)}^{t} H_{\gamma-1}(\tau,\rho(s))\nabla\tau = 0$. Thus,

$$\int_{a-1}^{t} G_\gamma(\tau,s)\nabla\tau = \frac{H_{\gamma-j-1}(b,\rho(s))}{H_{1-j}(b,a-1)}H_2(t,a-1) - H_\gamma(t,\rho(s)),$$

so

$$\left|\int_{a-1}^{t} G_\gamma(\tau,s)\nabla\tau\right| = \left|\frac{H_{\gamma-j-1}(b,\rho(s))}{H_{1-j}(b,a-1)}H_2(t,a-1) - H_\gamma(t,\rho(s))\right|, \tag{4.29}$$

for $t \in \mathbb{N}_s^b$, and

$$\left|\int_{a-1}^{t} G_\gamma(\tau,s)\nabla\tau\right| = \left|\frac{H_{\gamma-j-1}(b,\rho(s))}{H_{1-j}(b,a-1)}H_2(t,a-1)\right|, \tag{4.30}$$

for $t \in \mathbb{N}_{a-1}^{s-1}$.

We will now examine the first term from the right hand side of (4.29) for the case

$j = 0$. By Proposition 3.6, parts (i)-(ii), for $s \in \mathbb{N}_{a+1}^{b}$

$$0 \leq H_{\gamma-1}(b, \rho(s)) \leq H_{\gamma-1}(b, a), \tag{4.31}$$

since $H_{\gamma-1}(b, \rho(s))$ is a decreasing function of s. Also, by Proposition 3.6, parts (i)-(ii), since $H_2(t, a-1)$ is nondecreasing function of t for $t \in \mathbb{N}_{a-1}^{b}$,

$$0 \leq H_2(t, a-1) \leq H_2(b, a-1). \tag{4.32}$$

Hence, by (4.31) and (4.32),

$$0 \leq \frac{H_{\gamma-1}(b, \rho(s))}{H_1(b, a-1)} H_2(t, a-1) \leq \frac{H_{\gamma-1}(b, a)}{H_1(b, a-1)} H_2(b, a-1). \tag{4.33}$$

Now we consider the second term in (4.29). By Proposition 3.6, it follows that

$$0 \leq H_{\gamma}(t, \rho(s)) \leq H_{\gamma}(b, a). \tag{4.34}$$

From (4.29), (4.30), (4.33), (4.34), and Proposition 3.10, for the case $j = 0$, we obtain

$$\left| \int_{a-1}^{t} G_{\gamma}(\tau, s) \nabla \tau \right| \leq \max \left\{ \frac{H_{\gamma-1}(b, a)}{H_1(b, a-1)} H_2(b, a-1), H_{\gamma}(b, a) \right\},$$

so (4.25) holds.

Consider the case $j = 1$. Then, $H_{\gamma-2}(b, \rho(s))$ in (4.29) is an increasing function of s by Proposition 3.6, parts (i)-(ii), since $-1 < \gamma - 2 \leq 0$. Hence, for $s \in \mathbb{N}_{a+1}^{b}$,

$$0 \leq H_{\gamma-2}(b, \rho(s)) \leq H_{\gamma-2}(b, \rho(b)) = 1.$$

Therefore,

$$0 \leq H_{\gamma-2}(b, \rho(s)) H_2(t, a-1) \leq H_2(b, a-1). \tag{4.35}$$

Then, by (4.34), (4.35), and Proposition 3.10, we obtain

$$\left| \int_{a-1}^{t} G_\gamma(\tau, s) \nabla \tau \right| \leq \max \left\{ H_2(b, a-1), H_\gamma(b, a) \right\}.$$

Since $1 < \gamma \leq 2$, we have $H_\gamma(b, a) \leq H_2(b, a) \leq H_2(b+1, a) = H_2(b, a-1)$ by Proposition 3.6, part (iii) and Proposition 3.11, so, in the case $j = 1$, we have

$$\left| \int_{a-1}^{t} G_\gamma(\tau, s) \nabla \tau \right| \leq H_2(b, a-1).$$

Thus, (4.27) holds.

Similarly, by (3.21), we consider for $(t, s) \in \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b}$,

$$\begin{aligned} \int_{t}^{b} G_\gamma(\tau, s) \nabla \tau = \int_{t}^{\rho(s)} & \frac{H_1(\tau, a-1) H_{\gamma-j-1}(b, \rho(s))}{H_{1-j}(b, a-1)} \nabla \tau \\ & + \int_{\rho(s)}^{b} \left[\frac{H_1(\tau, a-1) H_{\gamma-j-1}(b, \rho(s))}{H_{1-j}(b, a-1)} - H_{\gamma-1}(\tau, \rho(s)) \right] \nabla \tau \\ = & \frac{H_{\gamma-j-1}(b, \rho(s))}{H_{1-j}(b, a-1)} \int_{t}^{b} H_1(\tau, a-1) \nabla \tau - \int_{\rho(s)}^{b} H_{\gamma-1}(\tau, \rho(s)) \nabla \tau. \end{aligned}$$

Evaluating the integral in the first term,

$$\begin{aligned} \int_{t}^{b} H_1(\tau, a-1) \nabla \tau &= H_2(\tau, a-1) \Big|_{\tau=t}^{b} \\ &= H_2(b, a-1) - H_2(t, a-1). \end{aligned}$$

Next, for $t > \rho(s)$,

$$\int_{\rho(s)}^{t} H_{\gamma-1}(\tau,\rho(s))\nabla\tau = H_{\gamma}(\tau,\rho(s))\Big|_{\tau=\rho(s)}^{b}$$
$$= H_{\gamma}(b,\rho(s)).$$

Note that, if $\rho(s) \geq t$, then $\int_t^{\rho(s)} H_{\gamma-1}(\tau,\rho(s))\nabla\tau = 0$.

Thus, we have

$$\int_t^b G_{\gamma}(\tau,s)\nabla\tau = \frac{H_{\gamma-j-1}(b,\rho(s))}{H_{1-j}(b,a-1)}\left[H_2\left(b,a-1\right) - H_2\left(t,a-1\right)\right] - H_{\gamma}(b,\rho(s)),$$

so

$$\left|\int_t^b G_{\gamma}(\tau,s)\nabla\tau\right| = \left|\frac{H_{\gamma-j-1}(b,\rho(s))}{H_{1-j}(b,a-1)}\left[H_2\left(b,a-1\right) - H_2\left(t,a-1\right)\right] - H_{\gamma}(b,\rho(s))\right|, \tag{4.36}$$

for $t \in \mathbb{N}_s^b$, and

$$\left|\int_t^b G_{\gamma}(\tau,s)\nabla\tau\right| = \left|\frac{H_{\gamma-j-1}(b,\rho(s))}{H_{1-j}(b,a-1)}\left[H_2(b,a-1) - H_2(t,a-1)\right]\right|, \tag{4.37}$$

for $t \in \mathbb{N}_{a-1}^{s-1}$. Consider the case $j = 0$. Noting that $0 \leq H_2(b,a-1) - H_2(t,a-1) \leq H_2(b,a-1)$, we obtain

$$\left|\int_t^b G_{\gamma}(\tau,s)\nabla\tau\right| \leq A$$

from Proposition 3.10, (4.34), (4.36), and (4.37). Hence, (4.26) holds. For the case $j = 1$, using Proposition 3.11 in a similar manner as above, we obtain

$$\left|\int_t^b G_{\gamma}(\tau,s)\nabla\tau\right| \leq H_2(b,a-1).$$

Thus, (4.28) holds. $\Diamond$

We will now consider, for $2 < \nu \leq 3$ and $q : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$,

$$\nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = 0, \quad t \in \mathbb{N}_{a+1}^{b} \tag{4.38}$$

subject to one of the following sets of boundary conditions:

$$x(a-1) = \nabla x(a-1) = 0, \;\; \nabla x(b) = 0, \tag{4.39}$$

$$\nabla x(a-1) = 0, \quad x(b) = \nabla x(b) = 0, \tag{4.40}$$

$$x(a-1) = \nabla x(a-1) = 0, \quad \nabla^2 x(b) = 0, \tag{4.41}$$

or

$$\nabla x(a-1) = 0, \quad x(b) = \nabla^2 x(b) = 0. \tag{4.42}$$

Note that the boundary conditions given by (4.40) and (4.42) are not of the same form as the boundary conditions considered in Chapter 2. The next theorem uses a change of variable to reduce the third order fractional boundary value problems (4.38)-(4.39), (4.38)-(4.40), (4.38)-(4.41), and (4.38)-(4.42) to a second order fractional boundary value problem, so that the bounds on the integral of the Green's function from the previous theorem for the second order boundary value problem can be applied to obtain a Lyapunov inequality. This method is used in [16] for continuous fractional BVPs. Since a slight change in boundary conditions can result in a big change in the Green's function, this technique avoids the need to determine the existence of and to compute a new Green's function for each higher order BVPs, and the same bound can be used to obtain Lyapunov inequalities for more than one BVP. Establishing conditions under which Green's functions are of constant sign is not straightforward, so yet another advantage of this technique is that properties regarding sign change of a lower order Green's function can be used to derive Lyapunov inequalities which involve conclusions concerning solutions changing sign in higher order problems.

We will first show in the next lemma that, if there is a nontrivial solution, it is not identically zero on the domain $\mathbb{N}_a^{b-1}$.

Lemma 4.23 *Let $\nu > 1$ and suppose $x : \mathbb{N}_{a-N+1}^b \to \mathbb{R}$ is a solution to the equation*

$$\nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = 0, \quad t \in \mathbb{N}_{a+1}^b, \tag{4.43}$$

where $\nu > 1$ and $N := \lceil \nu \rceil$. Assume $b - a \in \mathbb{N}_{N-1}$. If $x(t) = 0$ for all $t \in \mathbb{N}_a^{b-1}$, then $x(t) \equiv 0$ on $\mathbb{N}_{a-N+1}^b$.

Proof. Expanding the nabla Caputo operator, we obtain

$$\begin{aligned}
\nabla_{a*}^{\nu} x(t) &= \nabla_a^{-(N-\nu)} \nabla^N x(t) \\
&= \int_a^t H_{N-\nu-1}(t, \rho(s)) \nabla^N x(s) \nabla s \\
&= \int_a^t H_{N-\nu-1}(t, \rho(s)) \sum_{i=0}^{N} (-1)^i \binom{N}{i} x(s-i) \nabla s \\
&= \sum_{s=a+1}^{t} H_{N-\nu-1}(t, \rho(s)) \sum_{i=0}^{N} (-1)^i \binom{N}{i} x(s-i).
\end{aligned}$$

Hence, equation (4.43) becomes

$$\sum_{s=a+1}^{t} H_{N-\nu-1}(t, \rho(s)) \sum_{i=0}^{N} (-1)^i \binom{N}{i} x(s-i) + q(t)x(t-1) = 0, \quad t \in \mathbb{N}_{a+1}^b. \tag{4.44}$$

Assuming $x(t) = 0$ for $t \in \mathbb{N}_a^{b-1}$, we will show by induction on $k \in \mathbb{N}_1^{N-1}$ that $\sum_{i=k+1}^{N} (-1)^i \binom{N}{i} x(a+k-i) = 0$.

Since $b-a \in \mathbb{N}_{N-1}$, we have $b \geq a+1$, so for the base case, we consider $t = a+1$ in (4.44), giving

$$\sum_{i=0}^{N}(-1)^i\binom{N}{i}x(a+1-i) + q(a+1)x(a) = \sum_{i=2}^{N}(-1)^i\binom{N}{i}x(a+1-i) = 0.$$

Next, assume for the inductive hypothesis that $\sum\limits_{i=0}^{N}(-1)^i\binom{N}{i}x(a+m-i) = 0 = \sum\limits_{i=m+1}^{N}(-1)^i\binom{N}{i}x(a+m-i) = 0$ for $m = 1, 2, \ldots, k-1$, where $k \in \mathbb{N}_2^{N-1}$. Since $b-a \in \mathbb{N}_{N-1}$, we have $b \geq a+k$, so evaluating (4.44) at $t = a+k$ and simplifying the left hand side, we obtain

$$\begin{aligned}
&\sum_{s=a+1}^{a+k} H_{N-\nu-1}(a+k,\rho(s))\sum_{i=0}^{N}(-1)^i\binom{N}{i}x(s-i) + q(a+k)x(a+k-1) \\
&= \sum_{s=a+1}^{a+k-1} H_{N-\nu-1}(a+k,\rho(s))\sum_{i=0}^{N}(-1)^i\binom{N}{i}x(s-i) \\
&\qquad + H_{N-\nu-1}(a+k,\rho(a+k))\sum_{i=0}^{N}(-1)^i\binom{N}{i}x(a+k-i) \\
&= H_{N-\nu-1}(a+k,\rho(a+k))\sum_{i=0}^{N}(-1)^i\binom{N}{i}x(a+k-i), \qquad (4.45)
\end{aligned}$$

where $\sum\limits_{s=a+1}^{a+k-1} H_{N-\nu-1}(a+k,\rho(s))\sum\limits_{i=0}^{N}(-1)^i\binom{N}{i}x(s-i) = 0$ by the inductive hypothesis. Then, by (4.44) and (4.45), it follows that $\sum\limits_{i=0}^{N}(-1)^i\binom{N}{i}x(a+k-i) = 0$. Since $x(t) = 0$ for all $t \in \mathbb{N}_a^{b-1}$, we have $\sum\limits_{i=0}^{N}(-1)^i\binom{N}{i}x(a+k-i) = \sum\limits_{i=k+1}^{N}(-1)^i\binom{N}{i}x(a+k-i) = 0$. Therefore, by induction, we have

$$\sum_{i=k+1}^{N}(-1)^i\binom{N}{i}x(a+k-i) = 0 \qquad (4.46)$$

holds for all $k \in \mathbb{N}_1^{N-1}$. Now we will show that it follows from (4.46) that $x(a-N+1) =$

$\cdots = x(a-2) = x(a-1) = 0$. First, notice that for $k = N-1$, (4.46) gives

$$\sum_{i=N}^{N}(-1)^i\binom{N}{i}x(a+(N-1)-i) = (-1)^N x(a-1) = 0.$$

Then, for $k = N-2$ in (4.46), we have that the left hand side of (4.46) is

$$\begin{aligned}\sum_{i=N-1}^{N}(-1)^i\binom{N}{i}x(a+(N-2)-i) &= (-1)^{N-1}\binom{N}{N-1}x(a-1) + (-1)^N x(a-2)\\ &= (-1)^N x(a-2),\end{aligned}$$

which implies $x(a-2) = 0$. Proceeding in this manner, we get $x(a-N+1) = \cdots = x(a-2) = x(a-1) = 0$. Additionally, we have $x(a) = 0$, which means by the uniqueness of solutions to IVPs given in Theorem 4.1, $x(t) \equiv 0$ on its entire domain $\mathbb{N}_{a-N+1}^{b}$. ♢

In the remainder of this section, we define

$$\begin{aligned}C &:= \min\left\{\frac{H_1(b,a-1)}{H_{\nu-2}(b,a)H_2(b,a-1)}, \frac{1}{H_{\nu-1}(b,a)}\right\}\\ &= \min\left\{\frac{2\Gamma(\nu-1)\Gamma(b-a)}{\Gamma(b-a+\nu-2)(b-a+2)}, \frac{\Gamma(\nu)\Gamma(b-a)}{\Gamma(b-a+\nu-1)}\right\}. \qquad (4.47)\end{aligned}$$

We will use in the proof of the next theorem that, with $\gamma = \nu - 1$ in A given by (4.24),

$$\begin{aligned}\frac{1}{A} &= \frac{1}{\max\left\{\frac{H_{\nu-2}(b,a)H_2(b,a-1)}{H_1(b,a-1)}, H_{\nu-1}(b,a)\right\}}\\ &= \min\left\{\frac{H_1(b,a-1)}{H_{\nu-2}(b,a)H_2(b,a-1)}, \frac{1}{H_{\nu-1}(b,a)}\right\}\\ &= C.\end{aligned}$$

Theorem 4.24 *Let $2 < \nu \leq 3$ and C be as defined by (4.47). Assume $b - a \in \mathbb{N}_2$.*

(i) If the BVP (4.38), (4.39) or (4.38), (4.40) has a nontrivial solution $x : \mathbb{N}_{a-2}^b \to \mathbb{R}$, then

$$\int_a^b |q(s)| \, \nabla s \geq C.$$

(ii) If the BVP (4.38), (4.41) or (4.38), (4.42) has a nontrivial solution $x : \mathbb{N}_{a-2}^b \to \mathbb{R}$, then

$$\int_a^b |q(s)| \, \nabla s \geq \frac{1}{H_2(b, a-1)}.$$

Proof. First, note that for $t \in \mathbb{N}_{a+1}$,

$$\begin{aligned} \nabla_{a*}^{\nu} x(t) &= \nabla_a^{-(3-\nu)} \nabla^3 x(t) \\ &= \nabla_a^{-(3-\nu)} \nabla^2 \nabla x(t) \\ &= \nabla_a^{-(2-(\nu-1))} \nabla^2 \nabla x(t) \\ &\overset{(1.4)}{=} \nabla_{a*}^{\nu-1} (\nabla x(t)). \end{aligned}$$

Hence, we can write (4.38) as

$$\nabla_{a*}^{\nu-1} \nabla x(t) + q(t) x(t-1) = 0, \quad t \in \mathbb{N}_{a+1}^b.$$

Let $y(t) := \nabla x(t)$ for $t \in \mathbb{N}_{a-1}^b$. Then, the BVP (4.38), (4.39) becomes

$$\begin{cases} -\nabla_{a*}^{\nu-1} y(t) = q(t) x(t-1), & t \in \mathbb{N}_{a+1}^b \\ y(a-1) = y(b) = 0, \end{cases}$$

and the BVP (4.38), (4.41) becomes

$$\begin{cases} -\nabla_{a*}^{\nu-1} y(t) = q(t)x(t-1), \quad t \in \mathbb{N}_{a+1}^{b} \\ y(a-1) = \nabla y(b) = 0. \end{cases}$$

Consider the case of the boundary conditions (4.39) or (4.41). By Theorem 3.9 with $N = 2$, we have by (3.21)

$$y(t) = \int_a^b G_{\nu-1}(t,s)q(s)x(s-1)\nabla s,$$

for $t \in \mathbb{N}_{a-1}^{b}$, where $G_{\nu-1}$ is given by (3.21) with $j = 0$ for the boundary conditions (4.39) and $j = 1$ for the boundary conditions (4.41). Since $\nabla x(t) = y(t)$, for $t \in \mathbb{N}_{a-1}^{b}$, we have

$$\nabla x(t) = \int_a^b G_{\nu-1}(t,s)q(s)x(s-1)\nabla s. \tag{4.48}$$

Thus, for the BVP (4.38), (4.39) or the BVP (4.38), (4.41), using the boundary condition from (4.39) or (4.41) for x at $t = a-1$ and applying the Fundamental Theorem of Nabla Calculus given in Theorem 1.1,

$$\begin{aligned} x(t) &= x(t) - x(a-1) \\ &= \int_{a-1}^{t} \nabla x(\tau)\nabla\tau \\ &\overset{(4.48)}{=} \int_{a-1}^{t}\int_a^b G_{\nu-1}(\tau,s)q(s)x(s-1)\nabla s\nabla\tau, \end{aligned}$$

for $t \in \mathbb{N}_{a-1}^{b}$. Then,

$$\begin{aligned} x(t) &= \int_{a-1}^{t} \sum_{s=a+1}^{b} G_{\nu-1}(\tau,s)q(s)x(s-1)\nabla\tau \\ &= \sum_{s=a+1}^{b} \int_{a-1}^{t} G_{\nu-1}(\tau,s)q(s)x(s-1)\nabla\tau \\ &= \sum_{s=a+1}^{b} q(s)x(s-1) \int_{a-1}^{t} G_{\nu-1}(\tau,s)\nabla\tau, \end{aligned}$$

where the second equality follows from linearity of the nabla integral. It follows that

$$x(t) = \int_{a}^{b} q(s)x(s-1) \left(\int_{a-1}^{t} G_{\nu-1}(\tau,s)\nabla\tau \right) \nabla s. \tag{4.49}$$

Similarly, for the BVP (4.38), (4.40), or the BVP (4.38), (4.42), we get from (4.48) that

$$\begin{aligned} -x(t) &= x(b) - x(t) \\ &= \int_{t}^{b} \nabla x(\tau)\nabla\tau \\ &\overset{(4.48)}{=} \int_{t}^{b} \int_{a}^{b} G_{\nu-1}(\tau,s)q(s)x(s-1)\nabla s\nabla\tau. \end{aligned}$$

Also,

$$\begin{aligned} -x(t) &= \int_{t}^{b} \sum_{s=a+1}^{b} G_{\nu-1}(\tau,s)q(s)x(s-1)\nabla\tau \\ &= \sum_{s=a+1}^{b} \int_{t}^{b} G_{\nu-1}(\tau,s)q(s)x(s-1)\nabla\tau \\ &= \sum_{s=a+1}^{b} q(s)x(s-1) \int_{t}^{b} G_{\nu-1}(\tau,s)\nabla\tau. \end{aligned}$$

Therefore, in this case,

$$x(t) = \int_a^b (-q(s))x(s-1)\left(\int_t^b G_{\nu-1}(\tau, s)\nabla\tau\right)\nabla s. \tag{4.50}$$

Next, it follows by taking the absolute value of both sides of the equations (4.49) and (4.50) and using Lemma 4.22 with $\gamma = \nu - 1$ in the case $j = 0$, that

$$|x(t)| \leq \int_a^b |q(s)||x(s-1)|A\nabla s, \tag{4.51}$$

for all $t \in \mathbb{N}_{a-2}^b$, where A is given by (4.24) with $\gamma = \nu - 1$.

Let $M := \max_{t \in N_{a-2}^b} |x(t)|$. Then, in particular, we get from (4.51) that

$$\begin{aligned} M &\leq \int_a^b |q(s)||x(s-1)|A\nabla s \\ &\leq \int_a^b |q(s)|MA\nabla s. \end{aligned}$$

Since $x(t)$ is a nontrivial solution, we know $M > 0$. Moreover, x is not identically zero on $\mathbb{N}_a^{b-1}$ by Theorem 4.23, so the right hand side of (4.51) is nonzero. Hence, we get

$$1 \leq \int_a^b |q(s)|A\nabla s.$$

Finally,

$$\frac{1}{A} \leq \int_a^b |q(s)|\nabla s.$$

Thus, we get

$$\int_a^b |q(s)|\nabla s \geq C, \tag{4.52}$$

so (i) holds.

Similarly, in the case $j = 1$, we get from (4.49), (4.50), and Lemma 4.22, that

$$\begin{aligned} M &\leq \int_a^b |q(s)||x(s-1)|H_2(b, a-1)\nabla s \\ &\leq \int_a^b |q(s)|MH_2(b, a-1)\nabla s. \end{aligned}$$

Since $x(t)$ is a nontrivial solution, we know $M > 0$ and x is not identically zero on $\mathbb{N}_a^{b-1}$, so we get

$$1 \leq \int_a^b |q(s)|H_2(b, a-1)\nabla s.$$

Finally,

$$\frac{1}{H_2(b, a-1)} \leq \int_a^b |q(s)|\nabla s,$$

so (ii) holds. ◊

For the case $\nu = 3$, we get the following corollary to Theorem 4.24.

Corollary 4.25 *Let $b - a \in \mathbb{N}_2$. For $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$, consider*

$$\nabla^3 x(t) + q(t)x(t-1) = 0, \quad t \in \mathbb{N}_{a+1}^b, \tag{4.53}$$

and the boundary conditions

$$x(a-1) = \nabla x(a-1) = 0, \;\; \nabla x(b) = 0, \tag{4.54}$$

$$\nabla x(a-1) = 0, \quad x(b) = \nabla x(b) = 0, \tag{4.55}$$

$$x(a-1) = \nabla x(a-1) = 0, \;\; \nabla^2 x(b) = 0, \tag{4.56}$$

or

$$\nabla x(a-1)=0, \quad x(b)=\nabla^2 x(b)=0. \tag{4.57}$$

If the BVP (4.53), (4.54) *or* (4.53), (4.55) *has a nontrivial solution* $x:\mathbb{N}_{a-2}^{b}\to\mathbb{R}$, *then*

$$\int_a^b |q(s)|\,\nabla s \geq \frac{2}{(b-a)(b-a+2)}.$$

Also, if the BVP (4.53), (4.56) *or* (4.53), (4.57) *has a nontrivial solution* $x:\mathbb{N}_{a-2}^{b}\to\mathbb{R}$, *then*

$$\int_a^b |q(s)|\,\nabla s \geq \frac{2}{(b-a+1)(b-a+2)}.$$

Proof. Using $\nu=3$ in Theorem 4.24 for the boundary conditions (4.54) or (4.55), we have

$$\begin{aligned}
&\min\left\{\frac{2\Gamma(\nu-1)\Gamma(b-a)}{\Gamma(b-a+\nu-2)(b-a+2)},\frac{\Gamma(\nu)\Gamma(b-a)}{\Gamma(b-a+\nu-1)}\right\}\\
&=\min\left\{\frac{2\Gamma(3-1)\Gamma(b-a)}{\Gamma(b-a+3-2)(b-a+2)},\frac{\Gamma(3)\Gamma(b-a)}{\Gamma(b-a+3-1)}\right\}\\
&=\min\left\{\frac{2\Gamma(2)\Gamma(b-a)}{\Gamma(b-a+1)(b-a+2)},\frac{\Gamma(3)\Gamma(b-a)}{\Gamma(b-a+2)}\right\}\\
&=\min\left\{\frac{2\Gamma(b-a)}{\Gamma(b-a+1)(b-a+2)},\frac{2\Gamma(b-a)}{\Gamma(b-a+2)}\right\}.
\end{aligned}$$

So,

$$\begin{aligned}
&\min\left\{\frac{2\Gamma(\nu-1)\Gamma(b-a)}{\Gamma(b-a+\nu-2)(b-a+2)},\frac{\Gamma(\nu)\Gamma(b-a)}{\Gamma(b-a+\nu-1)}\right\}\\
&=\min\left\{\frac{2\Gamma(b-a)}{\Gamma(b-a)(b-a)(b-a+2)},\frac{2\Gamma(b-a)}{\Gamma(b-a)(b-a)(b-a+1)}\right\}\\
&=\min\left\{\frac{2}{(b-a)(b-a+2)},\frac{2}{(b-a)(b-a+1)}\right\}\\
&=\frac{2}{(b-a)(b-a+2)}.
\end{aligned}$$

The statement for the boundary conditions (4.56) or (4.57) follows directly from Theorem 4.24. ◊

Using the Lyapunov inequalities in Theorem 4.24 along with Corollary 4.7, we get the following corollary.

Corollary 4.26 *Let* $q : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, $2 < \nu \leq 3$, *and* $r \in \{1, 2\}$ *be fixed. Assume* $b - a \in \mathbb{N}_2$. *Consider the nonhomogenous BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = f(t), \;\; t \in \mathbb{N}_{a+1}^{b} \\ \nabla x(a-1) = A_0, \;\; \nabla^r x(b) = B_0, \;\; x(c) = C_0, \end{cases} \tag{4.58}$$

where $c \in \{a-1, b\}$, $f : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, *and* A_0, B_0, $C_0 \in \mathbb{R}$. *If* $q(t)$ *satisfies*

$$\int_a^b |q(t)|\nabla t < C,$$

then the BVP (4.58) *with* $r = 1$ *has a unique solution defined on* $\mathbb{N}_{a-2}^{b}$. *If* $q(t)$ *satisfies*

$$\int_a^b |q(t)|\nabla t < \frac{1}{H_2(b, a-1)},$$

then the BVP (4.58) *with* $r = 2$ *has a unique solution defined on* $\mathbb{N}_{a-2}^{b}$.

Note that by the contrapositive of Theorem 4.24, we have a sufficient condition for the BVP (4.38), (4.39) to have only the trivial solution. The following example uses the contrapositive of the statement of Theorem 4.24, part (i) to show that a certain BVP of the form (4.38), (4.39) has no nontrivial solution. In fact, we may replace the boundary conditions of the form (4.39) in the next example by the form (4.40) to obtain the same results.

The next example involves the *nabla exponential function*. Motivated by the exponential function in the continuous setting, for $p_0 \in \mathbb{R}$ such that $p_0 \neq 1$, we define $E_{p_0}(t,a)$ to be the unique solution to the initial value problem

$$\begin{cases} y(t) = p_0 \nabla y(t), \quad t \in \mathbb{N}_{a+1} \\ y(a) = 1. \end{cases}$$

Then, it can be shown that $E_{p_0}(t,a) = (1-p_0)^{a-t}$ for $t \in \mathbb{N}_a$, and $\int_a^t p_0 E_{p_0}(s,a)\nabla s = E_{p_0}(s,a)\Big|_{s=a}^{t} = E_{p_0}(t,a) - 1$ [29, p. 153; Example 3.7; Theorem 3.36, part (iv)], which will be used in the next example. The nabla exponential function will be defined more generally in Chapter 5.

Example 4.27 Consider the BVP (4.38), (4.39) with $a = 2$, $b = 10$, $q(t) = (0.005)E_{0.01}(t,2) = 0.005(0.99)^{2-t}$, and $2 < \nu \leq 3$. We will determine for which values of ν the contrapositive of Theorem 4.24 can be applied. In this case, we have the BVP

$$\begin{cases} \nabla_{2*}^{\nu} x(t) + (0.005)E_{0.01}(t,2)x(t-1) = 0, \quad t \in \mathbb{N}_3^{10} \\ x(1) = \nabla x(1) = 0, \;\; \nabla x(10) = 0. \end{cases} \tag{4.59}$$

Note that $q(t) \geq 0$. Then, we have

$$\begin{aligned} \int_2^{10} |q(s)|\nabla s &= \int_2^{10} \frac{0.01}{2} E_{0.01}(s,2)\nabla s \\ &= \frac{1}{2}E_{0.01}(s,2)\Big|_{s=2}^{10} \\ &= \frac{1}{2}E_{0.01}(10,2) - \frac{1}{2}E_{0.01}(2,2) \\ &= \frac{1}{2}(1-0.01)^{2-10} - \frac{1}{2}; \end{aligned}$$

hence,

$$\int_2^{10} |q(s)|\nabla s = \frac{1}{2}(0.99)^{-8} - \frac{1}{2} \approx 0.0418617. \tag{4.60}$$

Also, in this case we have

$$\begin{aligned}
\frac{2\Gamma(b-a)\Gamma(\nu-1)}{\Gamma(b-a+\nu-2)(b-a+2)} &= \frac{2\Gamma(10-2)\Gamma(\nu-1)}{\Gamma(10-2+\nu-2)(10-2+2)} \\
&= \frac{2\Gamma(8)\Gamma(\nu-1)}{\Gamma(6+\nu)(10)} \\
&= \frac{7!\Gamma(\nu-1)}{5\Gamma(6+\nu)} \\
&= \frac{1}{5(\nu-1)} \frac{7!}{(5+\nu)(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)} &= \frac{\Gamma(10-2)\Gamma(\nu)}{\Gamma(10-2+\nu-1)} \\
&= \frac{\Gamma(8)\Gamma(\nu)}{\Gamma(7+\nu)} \\
&= \frac{7!\Gamma(\nu)}{\Gamma(7+\nu)} \\
&= \frac{1}{6+\nu} \frac{7!}{(5+\nu)(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu}.
\end{aligned}$$

For $\nu \in (2,3]$, we have that $5(\nu-1) \geq 6+\nu$ for $2.75 < \nu \leq 3$, and $5(\nu-1) < 6+\nu$ for $2 < \nu \leq 2.75$. Hence, we have

$$\min\left\{\frac{2\Gamma(b-a)\Gamma(\nu-1)}{\Gamma(b-a+\nu-2)(b-a+2)}, \frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)}\right\}$$

$$= \begin{cases} \frac{1}{6+\nu} \frac{7!}{(5+\nu)(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu} & 2 < \nu \leq 2.75 \\ \frac{1}{5(\nu-1)} \frac{7!}{(5+\nu)(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu} & 2.75 < \nu \leq 3 \end{cases}$$

Define $F:(2,2.75]\to\mathbb{R}$ by $F(\nu):=\frac{1}{6+\nu}\frac{7!}{(5+\nu)(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu}$. Then, F is a decreasing function of ν. Hence, to find the largest possible ν for which the Lyapunov inequality in Theorem 4.24, part (i) can be applied, we will solve for ν in the equation

$$F(\nu)=\int_2^{10}|q(s)|\nabla s;$$

i.e., we will find ν such that

$$\frac{1}{6+\nu}\frac{7!}{(5+\nu)(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu}=\frac{1}{2}(0.99)^{-8}-\frac{1}{2}.$$

Solving for ν in the previous equation via WolframAlpha gives $\nu\approx 2.44888$. Define $\nu'=2.44888$. Now for $2<\nu<\nu'$, since F is a decreasing function of ν,

$$\int_a^b|q(t)|\nabla t<F(\nu).$$

Hence, for $2<\nu<\nu'$, by Theorem 4.24, the BVP (4.59) has only the trivial solution on $\mathbb{N}_0^{10}$. It follows that for $2<\nu<\nu'$, by Corollary 4.26, for $f:\mathbb{N}_{a+1}^b\to\mathbb{R}$ the nonhomogeneous BVP

$$\begin{cases}\nabla_{2*}^{\nu}x(t)+(0.005)E_{0.01}(t,2)x(t-1)=f(t), & t\in\mathbb{N}_3^{10}\\ \nabla x(1)=A_0,\ \ \nabla x(1)=B_0,\ \ \nabla x(10)=C_0,\end{cases}$$

where A_0, B_0, $C_0\in\mathbb{R}$, has a unique solution defined on $\mathbb{N}_0^{10}$.

Next, define $H:(2.75,3]\to\mathbb{R}$ by $H(\nu):=\frac{1}{5(\nu-1)}\frac{7!}{(5+\nu)(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu}$. Then, notice that H is a decreasing function of ν. Hence, to find the largest possible ν for which the Lyapunov inequality in Theorem 4.24 can be applied, we will solve for ν in the equation

$$H(\nu) = \int_2^{10} |q(s)|\nabla s;$$

i.e., we will find ν such that

$$\frac{1}{5(\nu-1)}\frac{7!}{(5+\nu)(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu} = \frac{1}{2}(0.99)^{-8} - \frac{1}{2}.$$

Solving for ν in the previous equation via WolframAlpha gives $\nu \approx 2.71$. Since H is a decreasing function of ν,

$$\int_a^b |q(s)|\nabla s \geq H(\nu)$$

for $2.75 < \nu \leq 3$. Hence, for $\nu' \leq \nu \leq 3$, Theorem 4.24 gives us no information.

4.3.2 Higher Order BVPs

In this section, we will generalize the method used in Theorem 4.24 to boundary value problems of higher order.

Theorem 4.28 *Let $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$, $\nu > 2$, and $N := \lceil \nu \rceil$. Assume $b - a \in \mathbb{N}_{N-1}$, and consider the BVP* (4.43),

$$\nabla^{N-2}x(a-1) = 0, \;\; \nabla^{N-2}x(b) = 0, \;\; \nabla^i x(c_i) = 0, \;\; i \in \mathbb{N}_0^{N-3}, \tag{4.61}$$

where $c_i \in \{a-1, b\}$. Let A be as defined in (4.24) *with $\gamma = \nu - N + 2$. If the boundary value problem* (4.43), (4.61) *has a nontrivial solution, $x : \mathbb{N}_{a-N+1}^b \to \mathbb{R}$, then*

$$\int_a^b |q(s)|\nabla s \geq \frac{1}{A} \cdot \frac{1}{(b-a+1)^{N-2}}.$$

Furthermore, consider the BVP (4.43),

$$\nabla^{N-2}x(a-1)=0,\ \ \nabla^{N-1}x(b)=0,\ \ \nabla^{i}x(c_i)=0,\ \ i\in\mathbb{N}_0^{N-3}, \tag{4.62}$$

where $c_i \in \{a-1, b\}$. *If the boundary value problem* (4.43), (4.62) *has a nontrivial solution,* $x : \mathbb{N}_{a-N+1}^{b} \to \mathbb{R}$, *then*

$$\int_a^b |q(s)|\nabla s \geq \frac{1}{H_2(b,a-1)} \cdot \frac{1}{(b-a+1)^{N-2}}.$$

Proof. First, note that for $t \in \mathbb{N}_{a+1}$,

$$\begin{aligned}
\nabla_{a*}^{\nu}x(t) &\overset{(1.4)}{=} \nabla_a^{-(N-\nu)}\nabla^N x(t)\\
&= \nabla_a^{-(N-\nu)}\nabla^2\nabla^{N-2}x(t)\\
&= \nabla_a^{-(2-(\nu-N+2))}\nabla^2\nabla^{N-2}x(t)\\
&= \nabla_{a*}^{\nu-N+2}\nabla^{N-2}x(t),
\end{aligned}$$

where the last equality follows since $-1<\nu-N\leq 0$ implies $1<\nu-N+2\leq 2$, so $\lceil \nu-N+2\rceil = 2$. Hence, we can write (4.43) as

$$\nabla_{a*}^{\nu-N+2}\nabla^{N-2}x(t)+q(t)x(t-1)=0,\ \ t\in\mathbb{N}_{a+1}^{b}. \tag{4.63}$$

Let $y(t) := \nabla^{N-2}x(t)$ for $t \in \mathbb{N}_{a-N+1+2}^{b}$. Then, the BVP (4.43), (4.61) becomes

$$\begin{cases} -\nabla_{a*}^{\nu-N+2}y(t) = q(t)x(t-1), \quad t\in\mathbb{N}_{a+1}^{b}\\ y(a-1)=y(b)=0, \end{cases} \tag{4.64}$$

and the BVP (4.43), (4.62) becomes

$$\begin{cases} -\nabla_{a*}^{\nu-N+2}y(t) = q(t)x(t-1), \quad t \in \mathbb{N}_{a+1}^{b} \\ y(a-1) = \nabla y(b) = 0. \end{cases} \tag{4.65}$$

Then, by Theorem 3.9, we have

$$y(t) = \int_a^b G_{\nu-N+2}(t,s)q(s)x(s-1)\nabla s,$$

for $t \in \mathbb{N}_{a-1}^{b}$, where $G_{\nu-N+2}$ is given by (3.21) with $j = 0$ in the case of the BVP (4.64), and $j = 1$ in the case of the BVP (4.65). Since $\nabla^{N-2}x(t) = y(t)$, we have

$$\nabla^{N-2}x(t) = \int_a^b G_{\nu-N+2}(t,s)q(s)x(s-1)\nabla s. \tag{4.66}$$

Applying the Fundamental Theorem of Nabla Calculus, Theorem 1.1, with the appropriate boundary conditions given by (4.61) or (4.62), for $t \in \mathbb{N}_{a-1}^{b}$, we get either

$$\begin{aligned} \nabla^{N-3}x(t) &= \nabla^{N-3}x(t) - \nabla^{N-3}x(a-1) \\ &\overset{(4.66)}{=} \int_{a-1}^{t}\int_a^b G_{\nu-N+2}(\tau,s)q(s)x(s-1)\nabla s\nabla\tau \\ &= \int_{a-1}^{t}\left(\sum_{s=a+1}^{b} G_{\nu-N+2}(\tau,s)q(s)x(s-1)\right)\nabla\tau \\ &= \sum_{s=a+1}^{b}\int_{a-1}^{t} G_{\nu-N+2}(\tau,s)q(s)x(s-1)\nabla\tau \\ &= \sum_{s=a+1}^{b} q(s)x(s-1)\int_{a-1}^{t} G_{\nu-N+2}(\tau,s)\nabla\tau \\ &= \int_a^b q(s)x(s-1)\left(\int_{a-1}^{t} G_{\nu-N+2}(\tau,s)\nabla\tau\right)\nabla s, \end{aligned} \tag{4.67}$$

or similarly

$$\begin{aligned}-\nabla^{N-3}x(t) &= \nabla^{N-3}x(b) - \nabla^{N-3}x(t)\\ &= \int_a^b q(s)x(s-1)\left(\int_t^b G_{\nu-N+2}(\tau,s)\nabla\tau\right)\nabla s, \end{aligned} \tag{4.68}$$

where we have interchanged the order of integration by using linearity of the nabla integral. Define $F(t_0) := \int_{a-1}^{t_0} G_{\nu-N+2}(\tau,s)\nabla\tau$ for the case (4.67) or $F(t_0) := \int_{t_0}^{b} G_{\nu-N+2}(\tau,s)\nabla\tau$ for the case (4.68).

Similarly, assuming $N \geq 4$, from the boundary conditions, proceeding in this manner and continuing to integrate, and then taking the absolute value of both sides, we get

$$|x(t)| = \left|\int_{A_{N-2}}\int_{A_{N-3}}\cdots\int_{A_2}\int_{A_1}\int_a^b q(s)x(s-1)F(t_0)\nabla s\nabla t_0\nabla t_1\nabla t_2\cdots\nabla t_{N-3}\right| \tag{4.69}$$

for $t \in \mathbb{N}_{a-1}^b$, where $\int_{A_i}$ denotes $\int_{a-1}^{t_i}$ if the boundary condition in (4.61) or (4.62) is $\nabla^i x(a-1) = 0$ or $\int_{t_i}^b$ if the boundary condition is $\nabla^i x(b) = 0$, for $i \in \mathbb{N}_0^{N-4}$. Then, by (4.69), for $t \in \mathbb{N}_{a-1}^b$ we obtain

$$\begin{aligned}|x(t)| &\leq \int_{A_{N-2}}\int_{A_{N-3}}\cdots\int_{A_2}\int_{A_1}\int_a^b |q(s)||x(s-1)|\,|F(t_0)|\,\nabla s\nabla t_0\nabla t_1\nabla t_2\cdots\nabla t_{N-3}\\ &\leq \int_{a-1}^{b}\int_{a-1}^{b}\cdots\int_{a-1}^{b}\int_{a-1}^{b}\int_a^b |q(s)||x(s-1)|\,|F(t_0)|\,\nabla s\nabla t_0\nabla t_1\nabla t_2\cdots\nabla t_{N-3},\end{aligned}$$

where the last inequality holds since $\int_a^b |q(s)||x(s-1)|\,|F(t_0)|\,\nabla s \geq 0$. Let $t = t'$ such that $x(t') = \max_{t\in\mathbb{N}_{a-1}^b} |x(t)|$, and define $B := x(t')$. By Lemma 4.23, since by assumption x is a nontrivial solution, $x(t) \not\equiv 0$ on $\mathbb{N}_a^{b-1}$. In particular, $B \neq 0$. Letting $t = t'$ in the last inequality, we get

$$|x(t')| \leq \int_{a-1}^{b}\int_{a-1}^{b}\cdots\int_{a-1}^{b}\int_{a-1}^{b}\int_{a}^{b}|q(s)||x(s-1)|\,|F(t_0)|\,\nabla s\nabla t_0\nabla t_1\nabla t_2\cdots\nabla t_{N-3}$$
$$\leq \int_{a-1}^{b}\int_{a-1}^{b}\cdots\int_{a-1}^{b}\int_{a-1}^{b}\int_{a}^{b}|q(s)|B\,|F(t_0)|\,\nabla s\nabla t_0\nabla t_1\nabla t_2\cdots\nabla t_{N-3},$$

so

$$B \leq \int_{a-1}^{b}\int_{a-1}^{b}\cdots\int_{a-1}^{b}\int_{a-1}^{b}\int_{a}^{b}|q(s)|B\,|F(t_0)|\,\nabla s\nabla t_0\nabla t_1\nabla t_2\cdots\nabla t_{N-3},$$

which implies

$$1 \leq \int_{a-1}^{b}\int_{a-1}^{b}\cdots\int_{a-1}^{b}\int_{a-1}^{b}\int_{a}^{b}|q(s)|\,|F(t_0)|\,\nabla s\nabla t_0\nabla t_1\nabla t_2\cdots\nabla t_{N-3}.$$

For the boundary conditions given by (4.61), $|F(t_0)| \leq A$ by Lemma 4.22, where A is defined by (4.24), so

$$1 \leq \int_{a-1}^{b}\int_{a-1}^{b}\cdots\int_{a-1}^{b}\int_{a-1}^{b}\int_{a}^{b}|q(s)|A\nabla s\nabla t_0\nabla t_1\nabla t_2\cdots\nabla t_{N-3}.$$

After integrating the right hand side of the last inequality, we have

$$\frac{1}{A(b-a+1)^{N-2}} \leq \int_{a}^{b}|q(s)|\nabla s.$$

Similarly, for the boundary conditions (4.62), $|F(t_0)| \leq H_2(t, a-1)$ by Lemma

4.22, so

$$\frac{1}{H_2(b,a-1)(b-a+1)^{N-2}} \leq \int_a^b |q(s)|\nabla s.$$

$\Diamond$

Using the Lyapunov inequalities in Theorem 4.28 along with Theorem 4.6, we get the following corollary.

Corollary 4.29 *Let $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$, $\nu > 2$, $N := \lceil \nu \rceil$, and $r \in \{1,2\}$ be fixed. Assume $b - a \in \mathbb{N}_{N-1}$. Consider the nonhomogeneous boundary value problem*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = f(t), \;\; t \in \mathbb{N}_{a+1}^b \\ \nabla^{N-2} x(a-1) = A_0 \\ \nabla^{N-r} x(b) = B_0 \\ \nabla^i x(c_i) = C_i, \;\; i \in \mathbb{N}_0^{N-3}, \end{cases} \tag{4.70}$$

where $c_i \in \{a-1, b\}$, $f : \mathbb{N}_{a+1}^b \to \mathbb{R}$ and $A_0, B_0, C_i \in \mathbb{R}$ for $i \in \mathbb{N}_0^{N-3}$. If $q(t)$ satisfies

$$\int_a^b |q(t)|\nabla t < \frac{1}{A} \cdot \frac{1}{(b-a+1)^{N-2}},$$

then the BVP (4.70) with $r = 2$ has a unique solution defined on $\mathbb{N}_{a-N+1}^b$. If $q(t)$ satisfies

$$\int_a^b |q(t)|\nabla t < \frac{1}{H_2(b,a-1)} \cdot \frac{1}{(b-a+1)^{N-2}},$$

then the BVP (4.70) with $r = 1$ has a unique solution defined on $\mathbb{N}_{a-N+1}^b$.

Employing the same methodology as in the proof of Theorem 4.28, we obtain the following result using the previously established conditions in Theorem 3.4, Theorem 3.7, and Theorem 3.13 under which the given Green's functions are nonnegative.

Theorem 4.30 *Let $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$, $\nu > 2$, $N := \lceil \nu \rceil$, and $b - a \in \mathbb{N}_{N-1}$.*

(i) *Assume* $\nu - N + 2 \geq \frac{b-a}{b-a+1} + 1$. *If the boundary value problem* (4.43), (4.61) *has a nontrivial solution,* $x : \mathbb{N}_{a-N+1}^{b} \to \mathbb{R}$, *which does not change sign on* $\mathbb{N}_a^{b-1}$ *then*

$$\int_a^b q_+(s)\nabla s \geq \frac{1}{A} \cdot \frac{1}{(b-a+1)^{N-2}},$$

where A *is defined by* (4.24).

(ii) *Assume* $1 \leq b - a \leq \frac{1}{N-\nu}$. *If the boundary value problem* (4.43), (4.62) *has a nontrivial solution,* $x : \mathbb{N}_{a-N+1}^{b} \to \mathbb{R}$, *which does not change sign on* $\mathbb{N}_a^{b-1}$ *then*

$$\int_a^b q_+(s)\nabla s \geq \frac{1}{H_2(b, a-1)} \cdot \frac{1}{(b-a+1)^{N-2}}.$$

Example 4.31 Let $q : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ be such that $q(t) < 0$ for all $t \in \mathbb{N}_{a+1}^{b}$ and assume $\nu - N + 2 \geq \frac{b-a}{b-a+1} + 1$. By Theorem 4.30, part (i), it follows that any nontrivial solution to the BVP (4.43), (4.61) must change sign on $\mathbb{N}_a^{b-1}$. In other words, under the given assumptions, any nontrivial solution to (4.43), (4.61) has a generalized zero on the domain $\mathbb{N}_a^{b-1}$.

We can also exploit the property of the $(2, 1)$ right-focal Green's function being nonnegative when $\frac{5}{2} \leq \nu \leq 3$ and $2 \leq b - a \leq \frac{1}{3-\nu}$ by Theorem 3.14 to get results similar to Theorem 4.28 and Theorem 4.30. First, we will give bounds on integrals involving the Green's function for the case $2 < \nu \leq 3$, which can be obtained in a manner similar to Lemma 4.22.

Lemma 4.32 *Let* $s \in \mathbb{N}_{a+1}^{b}$ *and* $2 < \nu \leq 3$. *Then,*

$$\left| \int_{a-1}^{t} G_\nu(\tau, s)\nabla\tau \right| \leq H_3(b, a-1)$$

and

$$\left|\int_t^b G_\nu(\tau, s)\nabla\tau\right| \leq H_3(b, a-1),$$

where G_ν *is as given in* (3.21) *with* $N = 3$ *and* $j = 2$.

Theorem 4.33 *Let* $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$, $t \in \mathbb{N}_{a+1}^b$, $\nu > 3$, $N := \lceil \nu \rceil$, *and* $b - a \in \mathbb{N}_{N-1}$. *Consider the BVP* (4.43),

$$\nabla^{N-3}x(a-1) = 0, \;\; \nabla^{N-2}x(a-1) = 0, \;\; \nabla^{N-1}x(b) = 0, \;\; \nabla^i x(c_i) = 0, \;\; i \in \mathbb{N}_0^{N-4}, \tag{4.71}$$

where $c_i \in \{a-1, b\}$. *If the boundary value problem* (4.43), (4.71) *has a nontrivial solution,* $x : \mathbb{N}_{a-N+1}^b \to \mathbb{R}$, *which does not change sign on* $\mathbb{N}_a^{b-1}$, *then*

$$\int_a^b q_+(s)\nabla s \geq \frac{1}{H_3(b, a-1)} \cdot \frac{1}{(b-a+1)^{N-2}}.$$

Moreover, if the boundary value problem (4.43), (4.71) *has a nontrivial solution,* $x : \mathbb{N}_{a-N+1}^b \to \mathbb{R}$, *then*

$$\int_a^b |q(s)|\nabla s \geq \frac{1}{H_3(b, a-1)} \cdot \frac{1}{(b-a+1)^{N-2}}.$$

Using the second Lyapunov inequality from Theorem 4.33 along with Theorem 4.8, we get the following corollary.

Corollary 4.34 *Let* $q : \mathbb{N}_{a+1}^b \to \mathbb{R}$, $\nu > 3$, $N := \lceil \nu \rceil$, *and* $b - a \in \mathbb{N}_{N-1}$. *Consider* *the nonhomogeneous boundary value problem*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = f(t), \;\; t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{N-3} x(a-1) = A_0, \;\; \nabla^{N-2} x(a-1) = A_1 \\ \nabla^{N-1} x(b) = B_0 \\ \nabla^{i} x(c_i) = C_i, \;\; i \in \mathbb{N}_0^{N-4}, \end{cases} \tag{4.72}$$

where $c_i \in \{a-1, b\}$, $f : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ *and* A_0, A_1, B_0, $C_i \in \mathbb{R}$ *for* $i \in \mathbb{N}_0^{N-3}$. *If* $q(t)$ *satisfies*

$$\int_a^b |q(t)| \nabla t < \frac{1}{H_3(b, a-1)} \cdot \frac{1}{(b-a+1)^{N-2}},$$

then the BVP (4.72) *has a unique solution defined on* $\mathbb{N}_{a-N+1}^{b}$.

4.4 Lyapunov Inequalities Involving a Self-Adjoint Equation

In this section, first we will consider for $0 < \nu < 1$, the self-adjoint equation

$$\nabla \nabla_{a^*}^{\nu} x(t) + q(t)x(t-1) = 0, \;\; t \in \mathbb{N}_{a+2}^{b}, \tag{4.73}$$

subject to the conjugate boundary conditions

$$x(a) = x(b) = 0, \tag{4.74}$$

or the right-focal boundary conditions

$$x(a) = \nabla x(b) = 0. \tag{4.75}$$

The following theorems will give Lyapunov inequalities for the boundary value problems (4.73), (4.74) and (4.73), (4.75). The technique used in the proofs is the same as used earlier in the chapter, so proofs are omitted.

Theorem 4.35 *Assume the BVP* (4.73), (4.74) *has a nontrivial solution* $x : \mathbb{N}_a^b \to \mathbb{R}$, *and let* $b - a \in \mathbb{N}_2$. *Then,*

$$\int_{a+1}^{b} |q(s)|\nabla s \geq \frac{4(b-a)^{\overline{\nu}}\Gamma(\nu+1)}{(b-a)^2}. \tag{4.76}$$

Moreover, if x *does not change sign on* $\mathbb{N}_{a+1}^{b-1}$, *then*

$$\int_{a+1}^{b} q_+(s)\nabla s \geq \frac{4(b-a)^{\overline{\nu}}\Gamma(\nu+1)}{(b-a)^2}. \tag{4.77}$$

Note that taking $\nu = 1$ in the above inequalities gives the right hand side

$$\frac{4(b-a)^{\overline{1}}\Gamma(1+1)}{(b-a)^2} = \frac{4}{b-a},$$

which is the same as in the inequality for the second order continuous case of the self-adjoint equation in Theorem 1.20.

Theorem 4.36 *Assume the BVP* (4.73), (4.75) *has a nontrivial solution* $x : \mathbb{N}_a^b \to \mathbb{R}$, *and let* $b - a \in \mathbb{N}_2$. *Then,*

$$\int_{a+1}^{b} |q(s)|\nabla x \geq \frac{\nu}{b-a+\nu-1}.$$

Moreover, if x *does not change sign on* $\mathbb{N}_{a+1}^{b-1}$, *then*

$$\int_{a+1}^{b} q_+(s)\nabla s \geq \frac{\nu}{b-a+\nu-1}.$$

Recall that an essential part of the proof for Lyapunov inequalities makes use of bounds on Green's functions. We give results on Green's function bounds for the self-adjoint equation case below. The proof of Theorem 4.35 makes use of the following bound on the Green's function, which follows directly from [3, Theorem 61].

Theorem 4.37 *The Green's function $G(t,s)$ for the boundary value problem*

$$\begin{cases} -\nabla\nabla_{a^*}^{\nu}x(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b} \\ x(a) = 0 \\ x(b) = 0 \end{cases}$$

satisfies $0 \leq G(t,s) \leq \frac{(b-a)^2}{4(b-a)^{\nu}\Gamma(\nu+1)}$.

The proof of Theorem 4.36 makes use of the next bound on the Green's function, which follows directly from [27, Theorem 5.11].

Theorem 4.38 *We have that the Green's function for the BVP*

$$\begin{cases} -\nabla\nabla_{a^*}^{\nu}x(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b} \\ x(a) = 0 \\ \nabla x(b) = 0 \end{cases}$$

satisfies $0 \leq G(t,s) \leq \frac{b-a+v-1}{\nu}$ *for* $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$.

The next example uses the Lyapunov inequality for the conjugate BVP involving the self-adjoint equation given in Theorem 4.37.

Example 4.39 Consider the BVP (4.73), (4.74) with $a = 1$, $b = 20$, $\nu = \frac{1}{2}$; i.e.,

$$\begin{cases} \nabla[\nabla^{\frac{1}{2}}_{1*}x(t)] + \left[\frac{1}{10t^3} - \frac{1}{10(t-1)^3}\right] x(t-1) = 0, \quad \in \mathbb{N}_3^{20} \\ x(1) = 0 \\ x(20) = 0. \end{cases}$$

Note that $q(t) = \frac{1}{10t^3} - \frac{1}{10(t-1)^3}$, and $q(t) < 0$ for $t \in \mathbb{N}_3$. Hence, $\int_2^{20} |q(s)|\nabla s = -\frac{1}{10}\int_2^{20}(\frac{1}{s^3} - \frac{1}{(s-1)^3})\nabla s = -\frac{1}{10s^3}\ |_{s=2}^{s=20} = .0124875$. Also, $\frac{4(b-a)^{\overline{\nu}}\Gamma(\nu+1)}{(b-a)^2} = \frac{4\cdot 18^{\overline{\frac{1}{2}}}\Gamma(\frac{3}{2})}{18^2\Gamma(18)} \approx .0461$, so $\int_2^{20} |q(s)|\nabla s < \frac{4(20-2)^{\overline{\frac{1}{2}}}\Gamma(\frac{1}{2}+1)}{(20-2)^2}$. Thus, the given BVP has only the trivial solution.

The next theorem, which follows from [3, Theorem 50], demonstrates a useful consequence of the conclusion in Example 4.39.

Theorem 4.40 *Assume* (4.73), (4.74) *has only the trivial solution. Then, the BVP*

$$\begin{cases} \nabla\nabla^{\nu}_{a*}x(t) + q(t)x(t-1) = f(t), \quad t \in \mathbb{N}_{a+2}^{b} \\ x(a) = A \\ x(b) = B \end{cases}$$

has a unique solution.

Example 4.41 By Example 4.39 and Theorem 4.40, the boundary value problem

$$\begin{cases} \nabla[\nabla^{\frac{1}{2}}_{1*}x(t)] + \left[\frac{1}{10t^3} - \frac{1}{10(t-1)^3}\right] x(t-1) = f(t), \quad t \in \mathbb{N}_3^{20} \\ x(1) = A \\ x(20) = B. \end{cases}$$

has a unique solution for any given fixed $A, B \in \mathbb{R}$ and $f : \mathbb{N}_3^{20} \to \mathbb{R}$.

Now we will consider a three-point boundary problem involving the self-adjoint equation. The following theorem follows by [3, Theorem 74] and the proof of [3, Theorem 78].

Theorem 4.42 *Let* $0 < \nu < 1$*,* $b - a \in \mathbb{N}_2$*,* $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$*,* $0 \leq \alpha \leq 1$*, and* $k \in \mathbb{N}_1^{(b-a)-1}$*. The Green's function for the homogeneous BVP*

$$\begin{cases} -\nabla\nabla_{a*}^{\nu} x(t) = 0, & t \in \mathbb{N}_{a+2}^{b} \\ x(a) = 0, & \\ x(b) - \alpha x(a+k) = 0, & \end{cases}$$

satisfies

$$0 \leq G(t,s) \leq H_\nu(b,a)\left[\frac{b-a+\nu-1}{\nu}\right]$$

for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$.

Proof. From the proof of Theorem 78 of [3], we have

$$G(t,s) \leq \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)} - H_\nu(t,\rho(s)),$$

for $t \in \mathbb{N}_{a+1}^b$ and $s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}$. Then, by Proposition 3.6, we have $\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)} \geq 0$ and $H_\nu(t,\rho(s)) \geq 0$. Hence, it follows that $G(t,s) \leq \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)}$ for $t \in \mathbb{N}_{a+1}^b$ and $s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}$. Moreover, by the proof of Theorem 78 of [3], we have $G(t,s) \leq \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)}$ for $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{\max\{t+1,a+2\}}^b$. Hence, we have

$$G(t,s) \leq \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)},$$

for all $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$. Now by Proposition 3.6, it follows that

$$G(t,s) \leq \frac{H_\nu(b,a)H_\nu(b,a)}{H_{\nu-1}(b,a)} = H_\nu(b,a)\left[\frac{b-a+\nu-1}{\nu}\right].$$

$\Diamond$

The next theorem gives a Lyapunov inequality for a three-point BVP using the bounds given Theorem 4.42.

Theorem 4.43 *Let* $0 < \nu < 1$, $b - a \in \mathbb{N}_2$, $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$, $0 \leq \alpha \leq 1$, *and* $k \in \mathbb{N}_1^{(b-a)-1}$. *Assume the BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + q(t)x(t-1) = 0, \quad t \in \mathbb{N}_{a+2}^{b} \\ x(a) = 0, \\ x(b) - \alpha x(a+k) = 0, \end{cases}$$

has a nontrivial solution $x : \mathbb{N}_a^b \to \mathbb{R}$. *Then,*

$$\int_{a+1}^{b} |q(s)| \nabla s \geq \frac{\nu}{(b-a+\nu-1)H_{\nu}(b,a)}.$$

Moreover, if x *does not change sign on* $\mathbb{N}_{a+1}^{b-1}$, *then*

$$\int_{a+1}^{b} q_{+}(s) \nabla s \geq \frac{\nu}{(b-a+\nu-1)H_{\nu}(b,a)}.$$

Chapter 5

Some Applications of Contraction Mapping Theorem and Green's Functions Involving the Nabla Mittag-Leffler Function

In this chapter, we present some further applications and extensions of the results in the previous chapters, which will be elaborated further in future work.

5.1 Applications of Contraction Mapping Theorem to Nonlinear BVPs

In this section, we will use the bounds on Green's functions established in Chapter 3 to study existence and uniqueness of solutions to nonlinear BVPs. The Contraction Mapping Theorem has been applied to boundary value problems involving nabla fractional self-adjoint equations in [3] and [13].

Definition 5.1 [35, Definition 7.2] A **contraction mapping**, T, on a complete metric space, (X, d), is a function, $T : X \to X$, which satisfies $d(Tx, Ty) \leq \alpha d(x, y)$ where $\alpha \in (0, 1)$ is a constant, which is referred to as the *contraction constant*.

Theorem 5.2 ***(Contraction Mapping Theorem)*** *If T is a contraction mapping on a complete metric space (X, d) with contraction constant $\alpha \in (0, 1)$, then T has a unique fixed point $\overline{x} \in X$; i.e $T\overline{x} = \overline{x}$.*

Remark 5.3 Define $X := \{x \mid x : \mathbb{N}_{a-N+1}^{b} \to \mathbb{R}\}$ and $d(x,y) := ||x-y||$ where $||x|| := \max_{t\in\mathbb{N}_{a-N+1}^{b}} |x(t)|$. Then, note that (X,d) is a complete metric space.

Theorem 5.4 *Assume $b-a \in \mathbb{N}_1$ and consider the BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) = F(t, x(t-1)), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{i} x(a-1) = A_i, & i \in \mathbb{N}_0^{N-2} \\ x(b) = B, \end{cases} \tag{5.1}$$

where $\nu > 1$, $N := \lceil \nu \rceil$, and $F : \mathbb{N}_{a+1}^{b} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to its second variable and satisfies a uniform Lipschitz condition with constant K; i.e.,

$$|F(t,x) - F(t,y)| \le K|x-y|$$

for all $(t,x), (t,y) \in \mathbb{N}_{a+1}^{b} \times \mathbb{R}$. If

$$K(b-a)\frac{\Gamma(b-a+\nu-1)}{\Gamma(b-a)\Gamma(\nu)} < 1,$$

then the BVP (5.1) has a unique solution.

Proof. If x is a solution to (5.1), then, by Corollary 2.11, x satisfies the integral equation

$$x(t) = w(t) + \int_a^b G(t,s) F(s, x(s-1)) \nabla s,$$

where $G : \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ is as given in Corollary 2.18 and w is the solution to the homogeneous equation satisfying nonhomogeneous boundary conditions. Let (X,d) be as defined in Remark 5.3. Define the operator T on X by

$$Tx(t) := w(t) + \int_a^b G(t,s) q(s) F(s, x(s-1)) \nabla s.$$

Note that $T : X \to X$. We will show that T is a contraction mapping. Consider, for $t \in \mathbb{N}_{a-N+1}^{b}$ fixed and $x, y \in X$,

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \left| \int_a^b G(t,s)F(s,x(s-1))\nabla s - \int_a^b G(t,s)F(s,y(s-1))\nabla s \right| \\
&= \left| \int_a^b G(t,s)\left[F(s,x(s-1)) - F(s,y(s-1))\right]\nabla s \right| \\
&\le \int_a^b |G(t,s)|\,|F(s,x(s-1)) - F(s,y(s-1))|\,\nabla s \\
&\le K \int_a^b |G(t,s)||x(s-1) - y(s-1)|\nabla s,
\end{aligned}$$

where the last inequality follows by the Lipschitz condition on F. Then,

$$\begin{aligned}
|Tx(t) - Ty(t)| &\le K||x-y|| \int_a^b |G(t,s)|\nabla s \\
&\le K||x-y|| \int_a^b H_{\nu-1}(b,a)\nabla s,
\end{aligned}$$

by the bound on $|G(t,s)|$ given in Theorem 3.12. Since t is fixed and arbitrary, we have

$$|Tx(t) - Ty(t)| \le K||x-y||H_{\nu-1}(b,a)(b-a)$$

is true for all $t \in \mathbb{N}_{a-N+1}^{b}$, so

$$||Tx - Ty|| \le \alpha||x-y||,$$

where $\alpha := K(b-a)\frac{\Gamma(b-a+\nu-1)}{\Gamma(b-a)\Gamma(\nu)} < 1$. Hence, we have that T is a contraction mapping on X. Hence, by the Contraction Mapping Theorem, T has a unique fixed point $x_0 \in X$, such that

$$x_0(t) = Tx_0(t) = \int_a^b G(t,s)F(s,x_0(s-1))\nabla s,$$

for $t \in \mathbb{N}_{a-N+1}^{b}$. Hence, x_0 is the unique solution to the nonlinear BVP (5.1). $\Diamond$

The proof of the next theorem follows by Theorem 2.17 and Corollary 2.11, and it is similar to the proof of Theorem 5.4 with the bound $H_{\nu-1}(b,a)$ replaced with $H_{N-1}(b,a-1)$ from the Green's function bound given in Theorem 3.12. Hence, we omit the proof.

Theorem 5.5 *Assume $b-a \in \mathbb{N}_{\max\{1,j\}}$ and consider the BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) = F(t, x(t-1)), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{i} x(a-1) = A_i, & i \in \mathbb{N}_0^{N-2} \\ \nabla^{j} x(b) = B, \end{cases} \tag{5.2}$$

where $\nu > 1$, $N := \lceil \nu \rceil$, $j \in \mathbb{N}_0^{N-1}$ is fixed, and $F : \mathbb{N}_{a+1}^{b} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to its second variable and satisfies a uniform Lipschitz condition with constant K. If

$$K(b-a)\binom{b-a+N-1}{N-1} < 1,$$

then the BVP (5.2) *has a unique solution.*

The following theorems give conditions for the existence of positive solutions to some nonlinear BVPs with homogeneous boundary conditions, using the fact that the Green's functions are nonnegative under the given conditions on ν. This allows the operator T to be defined on $X = \{x \mid x : \mathbb{N}_{a-N+1}^{b} \to [0,\infty)\}$. Note that, in this case, (X,d) is a complete metric space, where $d(x,y)$ is defined previously.

Theorem 5.6 *Let $1 < \nu \leq 2$ and $b-a \in \mathbb{N}_1$. Suppose either $\nu \geq \frac{b-a}{b-a+1} + 1$ and*

$1 < \nu < 2$; or $\nu = 2$ holds. Consider the BVP

$$\begin{cases} -\nabla_{a*}^{\nu} x(t) = F(t, x(t-1)), \quad t \in \mathbb{N}_{a+1}^{b} \\ x(a-1) = 0, \ \ x(b) = 0, \end{cases} \tag{5.3}$$

where $F : \mathbb{N}_{a+1}^{b} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to its second variable and satisfies a uniform Lipschitz condition with constant K, If

$$K(b-a)\frac{\Gamma(b-a+\nu-1)}{\Gamma(b-a)\Gamma(\nu)} < 1,$$

then the BVP (5.3) *has a unique positive solution.*

Proof. If x is a solution to (5.1), then, by Theorem 3.1, x satisfies the integral equation

$$x(t) = \int_a^b G(t,s)F(s, x(s-1))\nabla s,$$

where $G : \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ is as given in Corollary 2.18. Let
$X := \{x \mid x : \mathbb{N}_{a-N+1}^{b} \to [0,\infty)\}$ and $d(x,y)$ be as defined in Remark 5.3. Define the operator T on X by

$$Tx(t) := \int_a^b G(t,s)q(s)F(s, x(s-1))\nabla s.$$

Note that $T : X \to X$ since $G(t,s) \geq 0$ by Theorem 3.4 and Theorem 3.7. The remainder of the proof follows in the same manner as the proof of Theorem 5.4. $\Diamond$

Existence of any number of positive solutions to the boundary value problems considered in the next two theorems is studied in detail in [27] by means of the Guo-Krasnoselskii fixed point theorem. Although the conditions in [27, Theorem

3.16] on the the nonlinear term may be less restrictive than being Lipschitz, and the conclusions in [27, Theorem 3.16] give more information regarding the solution, it may be more difficult to find examples of a nonlinear term, as can be seen by [27, Example 3.19]. Hence, when only a unique solution is desired, the next theorems may be easier to apply.

The proof of the next theorem uses the Green's function bounds in Theorem 3.13 and follows in manner similar to the proof above.

Theorem 5.7 *Let $1 < \nu < 2$, $1 \leq b - a \leq \frac{1}{2-\nu}$ and consider the BVP*

$$\begin{cases} -\nabla_{a*}^{\nu} x(t) = F(t, x(t-1)), \quad t \in \mathbb{N}_{a+1}^{b} \\ x(a-1) = 0, \;\; \nabla x(b) = 0, \end{cases} \tag{5.4}$$

where $F : \mathbb{N}_{a+1}^{b} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to its second variable and satisfies a uniform Lipschitz condition with constant K. If

$$(b-a) < \frac{1}{\sqrt{K}},$$

then the BVP (5.4) *has a unique positive solution.*

The bound in Theorem 3.14 is used to prove the next theorem.

Theorem 5.8 *Let $\frac{5}{2} \leq \nu \leq 3$ and $2 \leq b - a < \frac{1}{3-\nu}$. Consider the BVP*

$$\begin{cases} -\nabla_{a*}^{\nu} x(t) = F(t, x(t-1)), \quad t \in \mathbb{N}_{a+1}^{b} \\ x(a-1) = \nabla x(a-1) = 0 \\ \nabla^2 x(b) = 0, \end{cases} \tag{5.5}$$

where $F : \mathbb{N}_{a+1}^{b} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to its second variable and satisfies *a uniform Lipschitz condition with constant K. If*

$$K(b-a)\left(\frac{(b-a+1)^{\overline{2}}}{2}-1\right)<1,$$

then the BVP (5.5) *has a unique positive solution.*

5.2 Green's Functions for BVPs Involving $\nabla^{\nu}_{a*}x(t)+cx(t)=0$

In this section, we will expand upon the results of Chapter 2 for BVPs involving the equation $\nabla^{\nu}_{a*}x(t)+cx(t)=0$, where $|c|<1$. The results of Chapter 2 can be viewed as a special case of $c\equiv 0$ in this more general context.

5.2.1 IVPs and General Solution Involving $\nabla^{\nu}_{a*}x(t)+q(t)x(t)=0$

In this subsection, we present some standard results for the equation $\nabla^{\nu}_{a*}x(t)+q(t)x(t)=0$ when $1+q(t)\neq 0$ for all $t\in\mathbb{N}_{a+1}$. Note that, in particular, the results of Theorem 5.9, Theorem 5.10, and Theorem 5.11 will apply for the special case $q(t)\equiv c$, where $|c|<1$, which will be considered in the next subsection.

The proof of the following theorem uses the same method as the proof of Theorem 4.1. In this case, we must use the fact that $1+q(t)\neq 0$ to solve for x uniquely on the domain $\mathbb{N}_{a+1}$. Note that initial conditions given by $\nabla^i x(a)=A_i,\ \ i\in\mathbb{N}_0^{N-1}$ are equivalent to having initial conditions of the form $x(a-i)=A_i',\ \ i\in\mathbb{N}_0^{N-1}$, which can be shown using the binomial expansion of $\nabla^i x(a)$ as given in Proposition 1.13. This fact will be used in the proof of the next theorem.

Theorem 5.9 *Let $q:\mathbb{N}_{a+1}\to\mathbb{R}$, $\nu>0$, and $N:=\lceil\nu\rceil$. The initial value problem*

$$\begin{cases}\nabla^{\nu}_{a*}x(t)+q(t)x(t)=f(t), & t\in\mathbb{N}_{a+1}\\ \nabla^i x(a)=A_i, & i\in\mathbb{N}_0^{N-1}\end{cases}\tag{5.6}$$

has a unique solution defined on $\mathbb{N}_{a-N+1}$ *if and only if* $1+q(t) \neq 0$ *for all* $t \in \mathbb{N}_{a+1}$.

Proof. Expanding the fractional equation in (5.6) gives

$$\sum_{s=a+1}^{t} H_{N-\nu-1}(t,\rho(s)) \left(\sum_{j=0}^{N} (-1)^j \binom{N}{j} x(s-j) \right) + q(t)x(t) = f(t), \quad t \in \mathbb{N}_{a+1}, \quad (5.7)$$

as given in the proof of Theorem 4.1, where we now have $q(t)x(t)$ in place of $q(t)x(t-1)$. From the initial conditions in (5.6), we have equivalent initial conditions given by $x(a-i) = A_i'$ for $i \in \mathbb{N}_0^{N-1}$, so x is uniquely defined on $\mathbb{N}_{a-N+1}^{a}$. Letting $t = a+1$ in (5.7), we obtain

$$x(a+1) + \sum_{j=1}^{N} (-1)^j \binom{N}{i} x(a+1-j) + q(a+1)x(a+1) = f(a+1).$$

Then, we have

$$x(a+1)(1+q(a+1)) = -\sum_{j=1}^{N} (-1)^j \binom{N}{i} A_{j-1}' + f(a+1). \qquad (5.8)$$

We can uniquely solve for $x(a+1)$ in (5.8) if and only if $(1+q(a+1)) \neq 0$. We will proceed by induction, assuming x is uniquely determined on $\mathbb{N}_{a-N+1}^{a+k}$, in a manner similar to the proof of Theorem 4.1. We can solve for $x(a+k+1)$ if and only if $q(a+k+1)+1 \neq 0$. Hence, we get that x is uniquely determined on $\mathbb{N}_{a-N+1}$ if and only if $1+q(t) \neq 0$, for all $t \in \mathbb{N}_{a+1}$. ◊

Note that the Lyapunov inequality results of Chapter 4 may be applied to the equation $\nabla_{a*}^{\nu} x(t) + q(t)x(t) = 0$, provided $1+q(t) \neq 0$ for all $t \in \mathbb{N}_{a+1}$.

The proof of the next theorem follows in the same manner as the proof of Theorem 4.2, so it is omitted.

Theorem 5.10 *Assume* $1 + q(t) \neq 0$ *for all* $t \in \mathbb{N}_{a+1}$. *Let* $N := \lceil \nu \rceil$. *Then, there exist* N *linearly independent solutions to*

$$\nabla_{a*}^{\nu} x(t) + q(t)x(t) = 0, \ \ t \in \mathbb{N}_{a+1}$$

defined on $\mathbb{N}_{a-N+1}$.

The same argument for the proof of Theorem 4.3 applies in the case of the next theorem.

Theorem 5.11 *Assume* $1 + q(t) \neq 0$ *for all* $t \in \mathbb{N}_{a+1}$. *If* $y_0, y_1, \dots, y_{N-1}$ *are* N *linearly independent solutions to the equation*

$$\nabla_{a*}^{\nu} x(t) + q(t)x(t) = 0, \ \ t \in \mathbb{N}_{a+1} \tag{5.9}$$

defined on $\mathbb{N}_{a-N+1}$ *where* $N := \lceil \nu \rceil$, *then a general solution to the equation* (5.9) *is given by*

$$y(t) = c_0 y_0(t) + c_1 y_1(t) + \cdots + c_{N-1} y_{N-1}(t),$$

for $t \in \mathbb{N}_{a-N+1}$ *where* $c_0, c_1, \dots, c_{N-1}$ *are arbitrary constants.*

5.2.2 General Solution to $\nabla_{a*}^{\nu} x(t) + cx(t) = 0$ in Terms of Nabla Mittag-Leffler Functions

In the remainder of this section, we will focus on the case $q(t) \equiv c$, where $|c| < 1$ and define the operator $L_a x(t) := \nabla_{a*}^{\nu} x(t) + cx(t)$ for $t \in \mathbb{N}_{a+1}$.

In this subsection, we are interested in a general solution to the equation

$$L_a x(t) = 0, \ \ t \in \mathbb{N}_{a+1},$$

where $|c| < 1$, $\nu > 0$ and $N := \lceil \nu \rceil$. The general solution will be given in terms of the nabla Mittag-Leffler function, which is a generalization of the nabla exponential function. To motivate the definition of the nabla Mittag-Leffler function, we will first define the nabla exponential function and give its nabla Taylor series.

Definition 5.12 [29, p. 153] For $p : \mathbb{N}_{a+1} \to \mathbb{R}$ such that $1 - p(t) \neq 0$ for $t \in \mathbb{N}_{a+1}$, the **nabla exponential function**, based at a, denoted by $E_p(t, a)$ and defined on $\mathbb{N}_a$, is defined to be the unique solution to the initial value problem

$$\begin{cases} \nabla y(t) = p(t)y(t), \quad t \in \mathbb{N}_{a+1} \\ y(a) = 1. \end{cases}$$

Theorem 5.13 *[29, Theorem 3.50] Assume $|p| < 1$ is a constant. Then,*

$$E_p(t, a) = \sum_{k=0}^{\infty} p^k H_k(t, a),$$

for $t \in \mathbb{N}_a$.

Definition 5.14 [29, Definition 3.98] (Nabla Mittag-Leffler Function). For $|p| < 1, \ \alpha > 0, \ \beta \in \mathbb{R}$,

$$E_{p,\alpha,\beta}(t, a) := \sum_{k=0}^{\infty} p^k H_{\alpha k+\beta}(t, a), \ \ t \in \mathbb{N}_a.$$

Remark 5.15 Let $\nu > 0$ and $N := \lceil \nu \rceil$. Note that for each $i \in \mathbb{N}_0^{N-1}$, the domain of the Mittag-Leffler function $E_{-c,\nu,i}(t, a - N + i)$, where $|c| < 1$, can be extended to $\mathbb{N}_{a-N+1}$ using the fact that for any $\nu > 0$, $k \in \mathbb{N}_0$, and $i \in \mathbb{N}_0^{N-1}$, $H_{\nu k+i}(t, a - N + i)$ is defined on $\mathbb{N}_{a-N+1}$.

The authors in [43] study an initial value problem related to Theorem 5.19 and arrive at a solution using transform methods. In [49], this problem is studied in the context of the (q,h)-discrete time scale and also uses transform methods. In contrast, we will give a direct proof and avoid the use of transform methods, which must assume a priori that the transform exists. Also, we will make explicit the domain of the solution. Moreover, the Mittag-Leffler functions here will be based at modified points, keeping in line with our work in Chapter 2.

For comparison, we will next state the parallel result in the case of the nabla Riemann-Louiville fractional difference.

Theorem 5.16 *[29, Theorem 3.101] Assume* $N := \lceil \nu \rceil$ *and* $|c| < 1$*. Then,*

$$E_{-c,\nu,\nu-i}(t, \rho(a)), \quad i \in \mathbb{N}_1^N$$

are N *linearly independent solutions defined on* $\mathbb{N}_a$ *of*

$$\nabla^{\nu}_{\rho(a)} x(t) + cx(t) = 0, \quad t \in \mathbb{N}_{a+N}.$$

In particular, a general solution to the fractional equation $\nabla^{\nu}_{\rho(a)} x(t) + cx(t) = 0$ *is given by*

$$x(t) = c_1 E_{-c,\nu,\nu-1}(t, \rho(a)) + c_2 E_{-c,\nu,\nu-2}(t, \rho(a)) + \cdots + c_N E_{-c,\nu,\nu-N}(t, \rho(a)),$$

for $t \in \mathbb{N}_a$*.*

Remark 5.17 Let $t \in \mathbb{N}_{a-N+1}$ be fixed and $c \in \mathbb{R}$ such that $|c| < 1$. For each $i \in \mathbb{N}_0^{N-1}$, it can be shown using the ratio test that the series $\sum_{k=0}^{\infty} (-c)^k H_{\nu k+i}(t, a-N+i)$ converges absolutely.

We will use the following remark in the proof of the next theorem.

Remark 5.18 Let $f_k : \mathbb{N}_{a-N+1} \to \mathbb{R}$ for each $k \in \mathbb{N}_0$ and suppose $\sum_{k=0}^{\infty} f_k(t)$ is absolutely convergent for each $t \in \mathbb{N}_{a-N+1}$. Then, it can be shown using properties of absolutely convergent series that,

$$\nabla_{a*}^{\nu} \sum_{k=0}^{\infty} f_k(t) = \sum_{k=0}^{\infty} \nabla_{a*}^{\nu} f_k(t),$$

for $t \in \mathbb{N}_{a+1}$.

Theorem 5.19 *Let $\nu > 0$, $N := \lceil \nu \rceil$, and $|c| < 1$. Then,*

$$E_{-c,\nu,i}(t, a-N+i), \ \ i \in \mathbb{N}_0^{N-1}$$

are N linearly independent solutions defined on $\mathbb{N}_{a-N+1}$ to

$$\nabla_{a*}^{\nu} x(t) + cx(t) = 0, \ \ t \in \mathbb{N}_{a+1}. \tag{5.10}$$

Also, a general solution is given by

$$x(t) = c_0 E_{-c,\nu,0}(t, a-N) + c_1 E_{-c,\nu,1}(t, a-N+1) + \cdots + c_{N-1} E_{-c,\nu,N-1}(t, a-1). \tag{5.11}$$

Proof. Note that for the case $c = 0$, we have $E_{0,\nu,i}(t, a-N+i) = H_i(t, a-N+i)$ for $i \in \mathbb{N}_0^{N-1}$, and it follows from Theorem 2.1 that $H_i(t, a-N+i)$ for $i \in \mathbb{N}_0^{N-1}$ give N linearly independent solutions to $\nabla_{a*}^{\nu} x(t) = 0$.

Now consider $|c| < 1$ and $c \neq 0$. Let $i \in \mathbb{N}_0^{N-1}$ be fixed. Then,

$$\begin{aligned} \nabla_{a*}^{\nu} E_{-c,\nu,i}(t, a-N+i) &= \nabla_{a*}^{\nu} \sum_{k=0}^{\infty} (-c)^k H_{\nu k+i}(t, a-N+i) \\ &= \sum_{k=0}^{\infty} (-c)^k \nabla_{a*}^{\nu} H_{\nu k+i}(t, a-N+i), \end{aligned}$$

by Remark 5.17 and Remark 5.18. Then,

$$\begin{aligned}\nabla_{a*}^{\nu} E_{-c,\nu,i}(t, a-N+i) &= \sum_{k=0}^{\infty} (-c)^k \nabla_a^{-(N-\nu)} \nabla^N H_{\nu k+i}(t, a-N+i) \\ &= \nabla_a^{-(N-\nu)} \nabla^N H_i(t, a-N+i) \\ &\quad + \sum_{k=1}^{\infty} (-c)^k \nabla_a^{-(N-\nu)} \nabla^N H_{\nu k+i}(t, a-N+i) \\ &= \sum_{k=1}^{\infty} (-c)^k \nabla_a^{-(N-\nu)} \nabla^N H_{\nu k+i}(t, a-N+i), \qquad (5.12)\end{aligned}$$

since $\nabla^N H_i(t, a-N+i) = 0$ for each $i \in \mathbb{N}_0^{N-1}$. Now, consider $k \in \mathbb{N}_1$ and $i \in \mathbb{N}_0^{N-1}$. Next, we will show that $\nu k + i - N$ is not a negative integer. Note that $\nu \neq \frac{1}{k}, \frac{2}{k}, \ldots, \frac{N-1}{k}$ for any $k \in \mathbb{N}_1$ since $\frac{1}{k} < \frac{2}{k} < \cdots < \frac{N-1}{k} \leq N-1 < \nu$ for any $k \in \mathbb{N}_1$. We have

$$\nu k + i - N \in \{n \mid n \in \mathbb{Z} \text{ and } n < 0\}$$

if and only if

$$\nu k \in \{n + (N-i) \mid n \in \mathbb{Z} \text{ and } n < 0\}.$$

Note that $\nu k \notin \{n \mid n \in \mathbb{Z} \text{ and } n \leq N-1\}$ because $\nu k > 0$ and $\nu \neq \frac{N-1}{k}, \ldots, \frac{1}{k}$ for all $k \in \mathbb{N}_1$. Then, since $\{n + (N-i) \mid n \in \mathbb{Z} \text{ and } n < 0\} \subseteq \{n \mid n \in \mathbb{Z} \text{ and } n \leq N-1\}$, we have $\nu k \notin \{n + (N-i) \mid n \in \mathbb{Z} \text{ and } n < 0\}$, so $\nu k + i - N \notin \{n \mid n \in \mathbb{Z} \text{ and } n < 0\}$. Thus, by Theorem 1.18, part (i) and since $\nu k + i - N$ and $\nu k + i - N + (N-\nu)$ are not negative integers,

$$\begin{aligned}\nabla_a^{-(N-\nu)} \nabla^N H_{\nu k+i}(t, a-N+i) &= \nabla_a^{-(N-\nu)} H_{\nu k+i-N}(t, a-N+i) \\ &= H_{\nu k+i-N+(N-\nu)}(t, a-N+i) \\ &= H_{\nu k+i-\nu}(t, a-N+i). \qquad (5.13)\end{aligned}$$

Then, we have by (5.12) and (5.13),

$$\begin{aligned}\nabla_{a*}^{\nu}E_{-c,\nu,i}(t,a-N+i) &= \sum_{k=1}^{\infty}(-c)^k H_{\nu k+i-\nu}(t,a-N+i)\\ &= \sum_{k=1}^{\infty}(-c)^k H_{\nu(k-1)+i}(t,a-N+i)\\ &= \sum_{k=0}^{\infty}(-c)^{k+1} H_{\nu k+i}(t,a-N+i)\\ &= -c\sum_{k=0}^{\infty}(-c)^k H_{\nu k+i}(t,a-N+i)\\ &= -cE_{-c,\nu,i}(t,a-N+i),\end{aligned}$$

which shows that $E_{-c,\nu,i}(t,a-N+i)$ is a solution to $L_a x(t)=0$ for each $i\in\mathbb{N}_0^{N-1}$.

Next, we will show that $E_{-c,\nu,i}(t,a-N+i)$, for $i\in\mathbb{N}_0^{N-1}$, are linearly independent. Suppose $c_0,c_1,\ldots,c_{N-1}$ are constants such that, for all $t\in\mathbb{N}_{a-N+1}$,

$$c_0E_{-c,\nu,0}(t,a-N)+c_1E_{-c,\nu,1}(t,a-N+1)+\cdots+c_{N-1}E_{-c,\nu,N-1}(t,a-1)=0. \quad (5.14)$$

Note that letting $t=a-N+1$ in (5.14) in $E_{-c,\nu,i}(t,a-N+i)$ for $i\in\mathbb{N}_0^{N-1}$ gives

$$\begin{aligned}E_{-c,\nu,i}(a-N+1,a-N+i) &= \sum_{k=0}^{\infty}(-c)^k H_{\nu k+i}(a-N+1,a-N+i)\\ &= \sum_{k=0}^{\infty}(-c)^k \frac{\Gamma(\nu k+1)}{\Gamma(1-i)\Gamma(\nu k+i+1)}.\end{aligned}$$

We have $\frac{\Gamma(\nu k+1)}{\Gamma(1-i)\Gamma(\nu k+i+1)}=0$ for $i\in\mathbb{N}_1^{N-1}$, so letting $t=a-N+1$ in (5.14) implies

$c_0 = 0$. Next, if $t = a - N + 2$,

$$E_{-c,\nu,i}(a-N+2, a-N+i) = \sum_{k=0}^{\infty}(-c)^k H_{\nu k+i}(a-N+2, a-N+i)$$
$$= \sum_{k=0}^{\infty}(-c)^k \frac{\Gamma(\nu k+2)}{\Gamma(2-i)\Gamma(\nu k+i+1)}.$$

Since $\frac{\Gamma(\nu k+2)}{\Gamma(2-i)\Gamma(\nu k+i+1)} = 0$ for $i \in \mathbb{N}_2^{N-1}$, letting $t = a - N + 2$ in (5.14) implies $c_1 = 0$. Proceeding in this manner by letting $t = a - N + 3, \cdots, a-1$ in (5.14) implies $c_2 = \cdots = c_{N-1} = 0$, respectively. Hence, $E_{-c,\nu,0}(t, a-N), E_{-c,\nu,1}(t, a-N+1), \ldots, E_{-c,\nu,N-1}(t, a-1)$ give N linearly independent solutions to (5.10), and by Theorem 5.11, it follows that (5.11) gives a general solution to (5.10). ♢

5.2.3 Development of Green's Functions

5.2.3.1 Variation of Constants Formula for an IVP

First, we have the following variation of constants formula, giving a particular solution to the nonhomogeneous equation $L_a y(t) = h(t)$.

Theorem 5.20 *(Variation of Constants). Assume $\nu > 0$ and $N := \lceil \nu \rceil$. Then, the solution to the initial value problem*

$$\begin{cases} L_a y(t) = h(t), & t \in \mathbb{N}_{a+1} \\ \nabla^i y(a) = 0, & i \in \mathbb{N}_0^{N-1} \end{cases} \tag{5.15}$$

is given by

$$y(t) = \int_a^t E_{-c,\nu,\nu-1}(t, \rho(s)) h(s) \nabla s, \tag{5.16}$$

for $t \in \mathbb{N}_{a-N+1}$.

Proof. The initial conditions $\nabla^i y(a) = 0, \ i \in \mathbb{N}_0^{N-1}$ are equivalent to $y(a-i) =$

$0,\ \ i \in \mathbb{N}_0^{N-1}$. By the convention on the nabla definite integral, we have that $y(t)$, as given by (5.16), satisfies the initial conditions $y(a-i) = 0$ for $i \in \mathbb{N}_0^{N-1}$.

Using that $\sum\limits_{k=0}^{\infty}(-c)^k H_{\nu k+\nu-1}(t,\rho(s))$ is absolutely convergent for each fixed $t \in \mathbb{N}_{a-N+1}$ and $s \in \mathbb{N}_{a+1}$, next consider

$$\begin{aligned}
&\nabla_{a*}^{\nu} \int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s \\
&\qquad = \nabla_{a*}^{\nu} \int_a^t \left(\sum_{k=0}^{\infty}(-c)^k H_{\nu k+\nu-1}(t,\rho(s))\right) h(s)\nabla s \\
&\qquad = \nabla_{a*}^{\nu} \sum_{s=a+1}^{t} \left(\sum_{k=0}^{\infty}(-c)^k H_{\nu k+\nu-1}(t,\rho(s))\right) h(s) \\
&\qquad = \nabla_{a*}^{\nu} \sum_{k=0}^{\infty}(-c)^k \sum_{s=a+1}^{t} H_{\nu k+\nu-1}(t,\rho(s))h(s) \\
&\qquad = \sum_{k=0}^{\infty}(-c)^k \nabla_{a*}^{\nu} \sum_{s=a+1}^{t} H_{\nu k+\nu-1}(t,\rho(s))h(s) \\
&\qquad = \nabla_{a*}^{\nu} \int_a^t H_{\nu-1}(t,\rho(s))h(s)\nabla s \\
&\qquad\qquad\quad + \sum_{k=1}^{\infty}(-c)^k \nabla_{a*}^{\nu} \sum_{s=a+1}^{t} H_{\nu k+\nu-1}(t,\rho(s))h(s) \\
&\qquad \overset{(1.3)}{=} \nabla_{a*}^{\nu}\nabla_a^{-\nu}h(t) + \sum_{k=1}^{\infty}(-c)^k \nabla_{a*}^{\nu} \sum_{s=a+1}^{t} H_{\nu k+\nu-1}(t,\rho(s))h(s) \\
&\qquad \overset{(2.4)}{=} h(t) + \sum_{k=1}^{\infty}(-c)^k \nabla_a^{-(N-\nu)}\nabla^N \int_a^t H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s. \qquad (5.17)
\end{aligned}$$

Note that, by the Leibniz formula given in Theorem 1.19, we have

$$\begin{aligned}
&\nabla^N \int_a^t H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s \\
&\qquad = \nabla^{N-1} \int_a^t \nabla_t H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s + H_{\nu k+\nu-1}(\rho(t),\rho(t))h(t) \\
&\qquad = \nabla^{N-1} \int_a^t \nabla_t H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s.
\end{aligned}$$

So,

$$
\begin{aligned}
&\nabla^N \int_a^t H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s \\
&\qquad = \nabla^{N-2} \int_a^t \nabla_t^2 H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s + \nabla_t H_{\nu k+\nu-1}(\rho(t),\rho(t))h(t) \\
&\qquad = \nabla^{N-2} \int_a^t \nabla_t^2 H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s \\
&\qquad\ \ \vdots \\
&\qquad = \int_a^t \nabla_t^N H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s + \nabla_t^{N-1} H_{\nu k+\nu-1}(\rho(t),\rho(t))h(t) \\
&\qquad = \int_a^t \nabla_t^N H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s. \qquad (5.18)
\end{aligned}
$$

Then, from (5.17), we have

$$
\begin{aligned}
&\nabla_{a*}^{\nu} \int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s \\
&\qquad = h(t) + \sum_{k=1}^{\infty} (-c)^k \nabla_a^{-(N-\nu)} \nabla^N \int_a^t H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s \\
&\qquad \overset{(5.18)}{=} h(t) + \sum_{k=1}^{\infty} (-c)^k \nabla_a^{-(N-\nu)} \int_a^t \nabla_t^N H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s. \qquad (5.19)
\end{aligned}
$$

Next, using the definition of the nabla fractional sum, consider for $k \in \mathbb{N}_1$,

$$
\begin{aligned}
&\nabla_a^{-(N-\nu)} \int_a^t \nabla_t^N H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s \\
&\qquad \overset{(1.3)}{=} \int_a^t H_{N-\nu-1}(t,\rho(s)) \left[\int_a^s \nabla_s^N H_{\nu k+\nu-1}(s,\rho(\xi))h(\xi)\nabla \xi \right] \nabla s \\
&\qquad = \sum_{s=a+1}^{t} \sum_{\xi=a+1}^{s} H_{N-\nu-1}(t,\rho(s)) \nabla_s^N H_{\nu k+\nu-1}(s,\rho(\xi))h(\xi) \\
&\qquad = \sum_{\xi=a+1}^{t} \sum_{s=\xi}^{t} H_{N-\nu-1}(t,\rho(s)) \nabla_s^N H_{\nu k+\nu-1}(s,\rho(\xi))h(\xi),
\end{aligned}
$$

where in the last equality, we have interchanged the order of summations. Hence,

$$\nabla_a^{-(N-\nu)} \int_a^t \nabla_t^N H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s$$
$$= \int_a^t \int_{\xi-1}^t H_{N-\nu-1}(t,\rho(s))\nabla_s^N H_{\nu k+\nu-1}(s,\rho(\xi))h(\xi)\nabla s\nabla\xi. \tag{5.20}$$

Now we have

$$\int_a^t \int_{\xi-1}^t H_{N-\nu-1}(t,\rho(s))\nabla_s^N H_{\nu k+\nu-1}(s,\rho(\xi))h(\xi)\nabla s\nabla\xi$$
$$= \int_a^t h(\xi) \int_{\xi-1}^t H_{N-\nu-1}(t,\rho(s))\nabla_s^N H_{\nu k+\nu-1}(s,\rho(\xi))\nabla s\nabla\xi$$
$$\overset{(1.3)}{=} \int_a^t h(\xi)\nabla_{(\xi-1)}^{-(N-\nu)}\nabla_t^N H_{\nu k+\nu-1}(t,\rho(\xi))\nabla\xi$$
$$= \int_a^t h(\xi)\nabla_{(\xi-1)}^{-(N-\nu)} H_{\nu k+\nu-1-N}(t,\rho(\xi))\nabla\xi$$
$$= \int_a^t h(\xi)H_{\nu k+\nu-1-N+N-\nu}(t,\rho(\xi))\nabla\xi, \tag{5.21}$$

where the second to last equality follows from Theorem 1.8, part (ii), and the last equality follows from Theorem 1.18, part (i), noting that $\nu k+\nu-1-N$ is not a negative integer for $k \in \mathbb{N}_1$. Hence, from (5.20) and (5.21), we have

$$\nabla_a^{-(N-\nu)} \int_a^t \nabla_t^N H_{\nu k+\nu-1}(t,\rho(s))h(s)\nabla s = \int_a^t h(\xi)H_{\nu k-1}(t,\rho(\xi))\nabla\xi. \tag{5.22}$$

Therefore,

$$\nabla_{a*}^{\nu} \int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s$$
$$\overset{(5.19),(5.22)}{=} h(t) + \sum_{k=1}^{\infty}(-c)^k \int_a^t h(\xi)H_{\nu k-1}(t,\rho(\xi))\nabla\xi.$$

Then, using absolute convergence of the series $\sum_{k=1}^{\infty}(-c)^k\int_a^t h(\xi)H_{\nu k-1}(t,\rho(\xi))\nabla\xi$ for each fixed $t\in\mathbb{N}_{a-N+1}$ in the next step, we have

$$\begin{aligned}
&\nabla_{a*}^{\nu}\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s\\
&\qquad = h(t)+\int_a^t\left[\sum_{k=1}^{\infty}(-c)^k H_{\nu k-1}(t,\rho(\xi))\right]h(\xi)\nabla\xi\\
&\qquad = h(t)+\int_a^t\left[\sum_{k=0}^{\infty}(-c)^{k+1} H_{\nu(k+1)-1}(t,\rho(\xi))\right]h(\xi)\nabla\xi\\
&\qquad = h(t)+(-c)\int_a^t\left[\sum_{k=0}^{\infty}(-c)^{k} H_{\nu k+\nu-1}(t,\rho(\xi))\right]h(\xi)\nabla\xi\\
&\qquad = h(t)+(-c)\int_a^t E_{-c,\nu,\nu-1}(t,\rho(\xi))h(\xi)\nabla\xi.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\nabla_{a*}^{\nu}\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s+c\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s\\
&\qquad = h(t)+(-c)\int_a^t E_{-c,\nu,\nu-1}(t,\rho(\xi))h(\xi)\nabla\xi+c\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s\\
&\qquad = h(t).
\end{aligned}$$

Thus, $\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s$ is the solution to the initial value problem (5.15). $\Diamond$

Corollary 5.21 *The solution to the initial value problem*

$$\begin{cases} L_a y(t)=h(t), & t\in\mathbb{N}_{a+1}\\ \nabla^i y(a)=A_i, & i\in\mathbb{N}_0^{N-1}\end{cases}$$

is given by

$$y(t)=w(t)+\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s,$$

where $w(t)$ is the unique solution to the initial value problem

$$\begin{cases} L_a y(t) = 0, & t \in \mathbb{N}_{a+1} \\ \nabla^i y(a) = A_i, & i \in \mathbb{N}_0^{N-1}. \end{cases}$$

5.2.3.2 Green's Functions

Throughout the remainder of this section, we will assume $\nu > 1$, $N := \lceil \nu \rceil$, $k \in \mathbb{N}_1^{N-1}$ is fixed, $j_m \in \mathbb{N}_0^{N-1}$ for $m \in \mathbb{N}_1^{N-k}$, with $j_1 < j_2 < j_3 < \cdots < j_{N-k}$, and $b - a \in \mathbb{N}_{\max\{1, j_{N-k}-N+k+1\}}$.

Next, we give an existence-uniqueness result for two-point boundary value problems involving the operator L_a. The proof is practically the same as the proof of Theorem 2.4, so it is omitted.

Theorem 5.22 *(Existence-Uniqueness Theorem) Let $\nu > 1$, $N := \lceil \nu \rceil$, $k \in \mathbb{N}_1^{N-1}$, and $h : \mathbb{N}_{a+1}^b \to R$. Furthermore, let $j_m \in \mathbb{N}_0^{N-1}$ for $m \in \mathbb{N}_1^{N-k}$, with $j_1 < j_2 < j_3 < \cdots < j_{N-k}$, and assume $b - a \in \mathbb{N}_{\max\{1, j_{N-k}-N+k+1\}}$. The homogeneous $(k, N-k)$ BVP*

$$\begin{cases} L_a y(t) = 0, & t \in \mathbb{N}_{a+1}^b \\ \nabla^i y(a-N+k) = 0, & i \in \mathbb{N}_0^{k-1} \\ \nabla^{j_m} y(b) = 0, & m \in \mathbb{N}_1^{N-k}, \end{cases}$$

has only the trivial solution if and only if the nonhomogeneous $(k, N-k)$ BVP

$$\begin{cases} L_a w(t) = h(t), & t \in \mathbb{N}_{a+1}^b \\ \nabla^i w(a-N+k) = A_i, & i \in \mathbb{N}_0^{k-1} \\ \nabla^{j_m} w(b) = B_{j_m}, & m \in \mathbb{N}_1^{N-k}, \end{cases}$$

where $A_i, B_{j_m} \in \mathbb{R}$, has a unique solution.

Remark 5.23 In the remainder of this section, we let $x_p(t) := E_{-c,\nu,p}(t, a-N+p)$, for each $p \in \mathbb{N}_0^{N-1}$.

Similar to Lemma 2.3, we get the following lemma.

Lemma 5.24 *Let $\nu > 1$, $N := \lceil \nu \rceil$, and suppose $x : \mathbb{N}_{a-N+1} \to \mathbb{R}$ is a solution to the equation*

$$L_a x(t) = 0, \quad t \in \mathbb{N}_{a+1}. \tag{5.23}$$

Moreover, assume that x satisfies the conditions

$$\nabla^i x(a-N+k) = 0, \quad i \in \mathbb{N}_0^{k-1}.$$

Then,

$$x(t) = \sum_{p=k}^{N-1} c_p x_p(t),$$

where $c_k, c_{k+1}, \ldots, c_{N-1} \in \mathbb{R}$.

Proof. First, note that from the binomial expansion of ∇^i given in Proposition 1.13, it follows from $\nabla^i x(a-N+k) = 0$, $i \in \mathbb{N}_0^{k-1}$, that $x(a-N+1) = x(a-N+2) = \cdots = x(a-N+k) = 0$. By Theorem 5.19, $x(t) = c_0 E_{-c,\nu,0}(t, a-N) + c_1 E_{-c,\nu,1}(t, a-N+1) + \cdots + c_{N-1} E_{-c,\nu,N-1}(t, a-1)$. Then, from the condition $x(a-N+1) = 0$, we get

$$c_0 E_{-c,\nu,0}(a-N+1, a-N) + c_1 E_{-c,\nu,1}(a-N+1, a-N+1) \\ + \cdots + c_{N-1} E_{-c,\nu,N-1}(a-N+1, a-1) = 0.$$

Then, as shown in the proof of Theorem 5.19, we get $c_0 = 0$. Similarly, as shown in the proof of Theorem 5.19, next we have that $x(a-N+2)=0$ implies

$$c_1E_{-c,\nu,1}(a-N+2,a-N+1)+\cdots+c_{N-1}E_{-c,\nu,N-1}(a-N+2,a-1)=0,$$

from which we get $c_1=0$. Continuing in this manner, from $x(a-N+3)=\cdots=x(a-N+k)=0$, we obtain $c_2=\cdots=c_{k-1}=0$, respectively. Hence, $x(t)=\sum_{p=k}^{N-1}c_pE_{-c,\nu,p}(t,a-N+p)$. ♢

In the remainder of this section, we let $D:=$

$$\begin{pmatrix} \nabla^{j_1}x_k(b) & \nabla^{j_1}x_{k+1}(b) & \cdots & \nabla^{j_1}x_{N-1}(b) \\ \nabla^{j_2}x_k(b) & \nabla^{j_2}x_{k+1}(b) & \cdots & \nabla^{j_2}x_{N-1}(b) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}}x_k(b) & \nabla^{j_{N-k}}x_{k+1}(b) & \cdots & \nabla^{j_{N-k}}x_{N-1}(b) \end{pmatrix}. \tag{5.24}$$

Using the previous lemma, we get the following theorem. The proof follows in a manner similar to the proof of Theorem 2.6, so we omit it.

Theorem 5.25 *A necessary and sufficient condition for uniqueness of solutions to the nonhomogeneous BVP in Theorem 5.22 is* $\det D\neq 0$*, where* D *is given by* (5.24).

The next lemma will be used in the following theorem.

Lemma 5.26 *Let* $\nu>1$ *and* $N:=\lceil\nu\rceil$. *Suppose* $j\in\mathbb{N}_0^{N-1}$, $|c|<1$, *and* $t\in\mathbb{N}_{a-N+1}$. *Then,*

$$\nabla^jE_{-c,\nu,\nu-1}(\rho(t),\rho(t))=0. \tag{5.25}$$

Proof. Consider

$$\nabla^j E_{-c,\nu,\nu-1}(t,\rho(s))\Big|_{\substack{t=\rho(t)\\ s=t}} = \left[\nabla^j \sum_{r=0}^{\infty}(-c)^r H_{\nu r+\nu-1}(t,\rho(s))\right]\Big|_{\substack{t=\rho(t)\\ s=t}}$$
$$= \sum_{r=0}^{\infty}(-c)^r \left[\nabla^j H_{\nu r+\nu-1}(t,\rho(s))\right]\Big|_{\substack{t=\rho(t)\\ s=t}},$$

where the last equality holds since $E_{-c,\nu,\nu-1}(t,\rho(s))$ is absolutely convergent for each fixed $t \in \mathbb{N}_{a-N+1}$ and $s \in \mathbb{N}_{a+1}$. Then,

$$\nabla^j E_{-c,\nu,\nu-1}(t,\rho(s))\Big|_{\substack{t=\rho(t)\\ s=t}} = \sum_{r=0}^{\infty}(-c)^r H_{\nu r+\nu-1-j}(\rho(t),\rho(t)),$$

which follows from Theorem 1.8, part (ii). Note that, for $r \in \mathbb{N}_0$,

$$\begin{aligned} H_{\nu r+\nu-1-j}(\rho(t),\rho(t)) &= \frac{0^{\overline{\nu r+\nu-1-j}}}{\Gamma(\nu r+\nu-j)} \\ &= \frac{\Gamma(\nu r+\nu-1-j)}{\Gamma(0)\Gamma(\nu r+\nu-j)} \\ &= 0, \end{aligned}$$

since $\nu r+\nu-1-j$ is not a nonpositive integer for all $r \in \mathbb{N}_0$ and $j \in \mathbb{N}_0^{N-1}$. Hence, (5.25) holds. ♢

Theorem 5.27 *(Green's Function Theorem). Assume* $k \in \mathbb{N}_1^{N-1}$, $j_m \in \mathbb{N}_0^{N-1}$ *for* $m \in \mathbb{N}_1^{N-k}$, $b-a \in \mathbb{N}_{\max\{1,j_{N-k}-N+k+1\}}$, *and that the homogeneous BVP in Theorem 5.22 has only the trivial solution. For each fixed* $s \in \mathbb{N}_{a+1}^b$, *let* $u(t,s)$ *be defined to be the* *solution to the BVP*

$$\begin{cases} L_a u(t,s) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^i u(a-N+k,s) = 0, & i \in \mathbb{N}_0^{k-1} \\ \nabla^{j_m} u(b,s) = -\nabla^{j_m} E_{-c,\nu,\nu-1}(b,\rho(s)), & m \in \mathbb{N}_1^{N-k}. \end{cases} \tag{5.26}$$

Define

$$G(t,s) := \begin{cases} u(t,s), & \text{if } t \le \rho(s) \\ v(t,s), & \text{if } t \ge \rho(s), \end{cases} \tag{5.27}$$

where $v(t,s) := u(t,s) + E_{-c,\nu,\nu-1}(t,\rho(s))$ *and* $(t,s) \in \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b}$. *Then,*

$$y(t) := \int_a^b G(t,s)h(s)\nabla s$$

is the unique solution to the nonhomogeneous $(k, N-k)$ *BVP in Theorem 5.22 with* $A_i, B_{j_m} = 0$. *Note that in the case* $t = \rho(s)$, *we have* $u(t,s) = v(t,s)$.

Proof. By Theorem 5.22, the BVP (5.26) for each fixed $s \in \mathbb{N}_{a+1}^{b}$ has a unique solution, so $u(t,s)$ is well defined. Let $G(t,s)$ be defined as in (5.27) and $y(t) := \int_a^b G(t,s)h(s)\nabla s$. First, for $t \in \mathbb{N}_{a-N+1}^{b}$,

$$\begin{aligned} y(t) &= \int_a^t G(t,s)h(s)\nabla s + \int_t^b G(t,s)h(s)\nabla s \\ &= \int_a^t v(t,s)h(s)\nabla s + \int_t^b u(t,s)h(s)\nabla s \\ &= \int_a^t [u(t,s) + E_{-c,\nu,\nu-1}(t,\rho(s))]h(s)\nabla s + \int_t^b u(t,s)h(s)\nabla s \\ &= \int_a^b u(t,s)h(s)\nabla s + \int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s \end{aligned}$$

By Theorem 5.20, $\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s$ is a solution to $L_a y(t) = h(t)$, so

$$\begin{aligned} L_a y(t) &= L_a \left[\int_a^b u(t,s)h(s)\nabla s + \int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s \right] \\ &= \sum_{s=a+1}^{b} L_a u(t,s)h(s) + L_a \left(\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s \right) \\ &= h(t), \end{aligned}$$

using the linearity of the operator L_a and where $L_a u(t,s) = 0$ by (5.26). We have

$$\begin{aligned} \nabla^i y(t)|_{t=a-N+k} &= \int_a^b \nabla^i u(a-N+k,s)h(s)\nabla s \\ &\quad + \nabla^i \left(\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s) \right) \nabla s \Big|_{t=a-N+k} \\ &= 0, \end{aligned}$$

for $i \in \mathbb{N}_0^{k-1}$, since for each fixed $s \in \mathbb{N}_{a+1}^b$, $u(t,s)$ satisfies the boundary conditions at $t = a-N+k$ in (5.26). Also, $\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s\Big|_{t=a-i} = 0$, for $i \in \mathbb{N}_0$, by the convention on nabla integrals. Moreover, for $j_m \in \mathbb{N}_0^{N-1}$, $m \in \mathbb{N}_1^{N-k}$, in a similar manner to the proof of Theorem 2.10,

$$\begin{aligned} \nabla^{j_m} y(t)|_{t=b} &= \int_a^b \nabla^{j_m} u(b,s)h(s)\nabla s + \nabla^{j_m} \left[\int_a^t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s \right] \Big|_{t=b} \\ &\overset{(1.10)}{=} \int_a^b \nabla^{j_m} u(b,s)h(s)\nabla s \\ &\quad + \left[\nabla^{j_m-1} \int_a^t \nabla_t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s + E_{-c,\nu,\nu-1}(\rho(t),t)h(t) \right] \Big|_{t=b} \\ &\overset{(5.25)}{=} \int_a^b \nabla^{j_m} u(b,s)h(s)\nabla s + \left[\nabla^{j_m-1} \int_a^t \nabla_t E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s \right] \Big|_{t=b} \\ &\vdots \end{aligned}$$

So,

$$\begin{aligned}
\nabla^{j_m} y(t)\big|_{t=b} &\overset{(1.10)}{=} \int_a^b \nabla^{j_m} u(b,s)h(s)\nabla s + \int_a^t \nabla_t^{j_m} E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s\Big|_{t=b} \\
&\qquad + \nabla^{j_m-1} E_{-c,\nu,\nu-1}(\rho(t),\rho(t))h(t)\Big|_{t=b} \\
&\overset{(5.25)}{=} \int_a^b \nabla^{j_m} u(b,s)h(s)\nabla s + \left[\int_a^t \nabla_t^{j_m} E_{-c,\nu,\nu-1}(t,\rho(s))h(s)\nabla s\right]\Big|_{t=b} \\
&\overset{(5.26)}{=} \int_a^b \left[-\nabla^{j_m} E_{-c,\nu,\nu-1}(b,\rho(s))\right] h(s)\nabla s \\
&\qquad + \int_a^b \nabla^{j_m} E_{-c,\nu,\nu-1}(b,\rho(s))h(s)\nabla s \\
&= 0.
\end{aligned}$$

◊

The proof of the following corollary is standard and follows in a straightforward manner from the previous theorem.

Corollary 5.28 *Assume that the hypotheses of Theorem 5.27 hold. Also, let $G(t,s)$ be as defined by (5.27), and w be the unique solution to the BVP*

$$\begin{cases} L_a w(t) = 0, & t \in \mathbb{N}_{a+1}^b \\ \nabla^i w(a-N+k) = A_i, & i \in \mathbb{N}_0^{k-1} \\ \nabla^{j_m} w(b) = B_{j_m}, & m \in \mathbb{N}_1^{N-k}. \end{cases}$$

Then, the unique solution to the nonhomogeneous BVP

$$\begin{cases} L_a y(t) = h(t), & t \in \mathbb{N}_{a+1}^b \\ \nabla^i y(a-N+k) = A_i, & i \in \mathbb{N}_0^{k-1} \\ \nabla^{j_m} y(b) = B_{j_m}, & m \in \mathbb{N}_1^{N-k}, \end{cases}$$

is given by

$$y(t) := w(t) + \int_a^b G(t,s)h(s)\nabla s.$$

Theorem 5.29 *Assume that the hypotheses of Theorem 5.27 hold. Then, the Green's function for the* $(k, N-k)$ *homogeneous BVP is given by* (5.27)*, where* $u(t,s) =$

$$\frac{1}{\beta}\begin{vmatrix} 0 & x_k(t) & x_{k+1}(t) & \cdots & x_{N-1}(t) \\ \nabla^{j_1}E_{-c,\nu,\nu-1}(b,\rho(s)) & \nabla^{j_1}x_k(b) & \nabla^{j_1}x_{k+1}(b) & \cdots & \nabla^{j_1}x_{N-1}(b) \\ \nabla^{j_2}E_{-c,\nu,\nu-1}(b,\rho(s)) & \nabla^{j_2}x_k(b) & \nabla^{j_2}x_{k+1}(b) & \cdots & \nabla^{j_2}x_{N-1}(b) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}}E_{-c,\nu,\nu-1}(b,\rho(s)) & \nabla^{j_{N-k}}x_k(b) & \nabla^{j_{N-k}}x_{k+1}(b) & \cdots & \nabla^{j_{N-k}}x_{N-1}(b) \end{vmatrix}, \quad (5.28)$$

for $(t,s) \in \mathbb{N}_{a-N+1}^b \times \mathbb{N}_{a+1}^b$*, with* $\beta := \det D$*, where* D *given by* (5.24)*,* $v(t,s) := u(t,s) + E_{-c,\nu,\nu-1}(t,\rho(s))$*.*

Proof. Let $u(t,s)$ be given by (5.28). By Theorem 5.25, $\beta \neq 0$. Also, note that each entry in D and in the determinant given in (5.28) is an absolutely convergent series. Since the Cauchy product of absolutely convergent series is absolutely convergent, we have that u is well defined. Then, expanding $u(t,s)$ along the first row, for each fixed s, $u(t,s)$ is a linear combination of $x_k(t), x_{k+1}(t), \ldots, x_{N-1}(t)$. Hence, for each fixed s, $u(t,s)$ is a solution to $L_a x(t) = 0$. To show $\nabla^i u(a-N+k,s) = 0$ for each $i \in \mathbb{N}_0^{k-1}$, it suffices to have $\nabla^i x_k(a-N+k) = \nabla^i x_{k+1}(a-N+k) = \cdots = \nabla^i x_{N-1}(a-N+k) = 0$, for each $i \in \mathbb{N}_0^{k-1}$. Note that $\nabla^i x_p(a-N+k) = 0$ for $i \in \mathbb{N}_0^{k-1}$ is equivalent to $x(a-N+1) = \cdots = x(a-N+k) = 0$. As shown in the proof of Theorem 5.19, we have $E_{-c,\nu,p}(a-N+1, a-N+p) = \cdots = E_{-c,\nu,p}(a-N+k, a-N+p) = 0$, for each $p \in \mathbb{N}_k^{N-1}$. Hence, we have that $u(t,s)$ satisfies the boundary conditions at $t = a-N+k$ given in (5.26).

Next, define $z(t,s)$ to be

$$\frac{1}{\beta}\begin{vmatrix} E_{-c,\nu,\nu-1}(t,\rho(s)) & x_k(t) & x_{k+1}(t) & \cdots & x_{N-1}(t) \\ \nabla^{j_1}E_{-c,\nu,\nu-1}(b,\rho(s)) & \nabla^{j_1}x_k(b) & \nabla^{j_1}x_{k+1}(b) & \cdots & \nabla^{j_1}x_{N-1}(b) \\ \nabla^{j_2}E_{-c,\nu,\nu-1}(b,\rho(s)) & \nabla^{j_2}x_k(b) & \nabla^{j_2}x_{k+1}(b) & \cdots & \nabla^{j_2}x_{N-1}(b) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}}E_{-c,\nu,\nu-1}(b,\rho(s)) & \nabla^{j_{N-k}}x_k(b) & \nabla^{j_{N-k}}x_{k+1}(b) & \cdots & \nabla^{j_{N-k}}x_{N-1}(b) \end{vmatrix}.$$

Expanding $z(t,s)$ along the first row, we get

$$z(t,s) = E_{-c,\nu,\nu-1}(t,\rho(s)) + u(t,s),$$

in a similar manner to the proof of Theorem 2.12.

Next, for each $m \in \mathbb{N}_1^{N-k}$, we have $\nabla^{j_m}z(b,s) = 0$, implying

$$\nabla^{j_m}E_{-c,\nu,\nu-1}(b,\rho(s)) + \nabla^{j_m}u(b,s) = 0;$$

i.e.,

$$\nabla^{j_m}u(b,s) = -\nabla^{j_m}E_{-c,\nu,\nu-1}(b,\rho(s)).$$

Hence, we have that $u(t,s)$ satisfies the boundary conditions at $t = b$ in (5.26). Thus, the result follows by Theorem 5.27. $\Diamond$

5.2.3.3 Examples of $(N-1,1)$ Green's Functions

Lemma 5.30 *Let $\nu > 1$, $N := \lceil \nu \rceil$, $j \in \mathbb{N}_0^{N-1}$ fixed, $b-a \in \mathbb{N}_{\max\{1,j\}}$, $-1 < c \le 0$, and $h : \mathbb{N}_{a+1}^b \to \mathbb{R}$. Then, the BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + cx(t) = 0, & t \in \mathbb{N}_{a+1}^b \\ \nabla^i x(a-1) = 0, & i \in \mathbb{N}_0^{N-2} \\ \nabla^j x(b) = 0, & \end{cases}$$

has only the trivial solution.

Proof. First, the conditions $\nabla^i x(a-1) = 0$ for $i \in \mathbb{N}_0^{N-2}$ hold if and only if $x(a-N+1) = \cdots = x(a-1) = 0$. Plugging in $t = a-N+1, \ldots, a-1$ into the general solution $x(t) = c_0 E_{-c,\nu,0}(t, a-N) + c_1 E_{-c,\nu,1}(t, a-N+1) + \cdots + c_{N-1} E_{-c,\nu,N-1}(t, a-1)$ implies $c_0 = \cdots = c_{N-2} = 0$, as shown at the end of the proof of Theorem 5.19. Hence, $x(t) = c_{N-1} E_{-c,\nu,N-1}(t, a-1)$. Next, we have

$$\begin{aligned} \nabla^j x(b) &= c_{N-1} \nabla^j E_{-c,\nu,N-1}(t, a-1)\Big|_{t=b} \\ &= c_{N-1} \sum_{p=0}^{\infty} (-c)^p \nabla^j H_{\nu p+N-1}(t, a-1)\Big|_{t=b}, \text{ using absolute convergence} \\ &= c_{N-1} \sum_{p=0}^{\infty} (-c)^p H_{\nu p+N-1-j}(b, a-1) \\ &= 0, \end{aligned}$$

which implies $c_{N-1} = 0$ since $(-c)^p H_{\nu p+N-1-j}(b, a-1) \ge 0$ for $p \in \mathbb{N}_0$ follows by our assumption that $-1 < c \le 0$. $\Diamond$

Using the previous lemma, we obtain the following theorem, which applies Theorem 5.29 in the special cases of $(N-1,1)$ BVPs when $-1 < c \le 0$.

Theorem 5.31 *Let $\nu > 1$, $N := \lceil \nu \rceil$, $j \in \mathbb{N}_0^{N-1}$ fixed, $b - a \in \mathbb{N}_{\max\{1,j\}}$, $-1 < c \leq 0$, and $h : \mathbb{N}_{a+1}^b \to \mathbb{R}$. Consider the BVP*

$$\begin{cases} \nabla_{a*}^{\nu} x(t) + cx(t) = h(t), & t \in \mathbb{N}_{a+1}^b \\ \nabla^i x(a-1) = 0, & i \in \mathbb{N}_0^{N-2} \\ \nabla^j x(b) = 0. \end{cases} \tag{5.29}$$

Then, the unique solution to the BVP (5.29) is given by $\int_a^b G(t,s)h(s)\nabla s$, for $t \in \mathbb{N}_{a-N+1}^b$, where G is defined by

$$G(t,s) := \begin{cases} \frac{-E_{-c,\nu,N-1}(t,a-1)E_{-c,\nu,\nu-j-1}(b,\rho(s))}{E_{-c,\nu,N-j-1}(b,a-1)}, & \text{if } t \leq \rho(s) \\ \frac{-E_{-c,\nu,N-1}(t,a-1)E_{-c,\nu,\nu-j-1}(b,\rho(s))}{E_{-c,\nu,N-j-1}(b,a-1)} + E_{-c,\nu,\nu-1}(t,\rho(s)), & \text{if } t \geq \rho(s). \end{cases} \tag{5.30}$$

The next example shows that if $0 < c < 1$, then the BVP (5.29) need not have a unique solution.

Example 5.32 Consider the BVP

$$\begin{cases} \nabla_{0*}^{2.8} x(t) + cx(t) = 0, & t \in \mathbb{N}_1^{10} \\ x(-1) = \nabla x(-1) = 0 \\ x(10) = 0, \end{cases} \tag{5.31}$$

where $0 < c < 1$. Then, $x(t) = E_{-c,2.8,2}(t,-1)$, $t \in \mathbb{N}_{-2}^{10}$ is a solution to the equation in (5.31) satisfying $x(-1) = \nabla x(-1) = 0$. Next, setting $E_{-c,2.8,2}(t,-1)|_{t=10} = \sum_{p=0}^{\infty} (-c)^p H_{2.8p+2}(t,-1)|_{t=10} = 0$ and solving for c via WolframAlpha gives $c \approx 0.0509$, 0.2378, 0.7376. Hence, $E_{-0.0509,2.8,2}(t,-1)$, $E_{-0.2378,2.8,2}(t,-1)$, and $E_{-.7376,2.8,2}(t,-1)$ are nontrivial solutions to the BVP (5.31).

Adapting the Lyapunov inequality results from Corollary 4.11 and Corollary 4.17 for the equation $\nabla_{a*}^{\nu}x(t)+cx(t)=0$, we obtain the following theorem, which applies Theorem 5.29 in the special cases of $(N-1,1)$ BVPs where we do not have to assume $-1<c\leq 0$.

Theorem 5.33 *Let $\nu>1$, $N:=\lceil\nu\rceil$, $j\in\mathbb{N}_0^{N-1}$ fixed, $b-a\in\mathbb{N}_{\max\{1,j\}}$, $|c|<1$, and $h:\mathbb{N}_{a+1}^{b}\to\mathbb{R}$. Consider the BVP* (5.29)*.*

(i) If $|c|<\frac{\Gamma(b-a-1)\Gamma(\nu)}{\Gamma(b-a+\nu-1)}$, then the unique solution to the BVP (5.29) *with $j=0$ is given by $\int_a^b G(t,s)h(s)\nabla s$, for $t\in\mathbb{N}_{a-N+1}^{b}$, where G is defined by* (5.30) *with $j=0$.*

(ii) If $|c|<\frac{(N-1)!}{(b-a)(b-a+N-1)(b-a+N-2)\cdots(b-a+1)}$, then the unique solution to the BVP (5.29)*, with $j\in\mathbb{N}_1^{N-1}$ fixed, is given by $\int_a^b G(t,s)h(s)\nabla s$, for $t\in\mathbb{N}_{a-N+1}^{b}$, where G is defined by* (5.30) *with the corresponding fixed j.*

5.3 Further Work

For future directions, one may investigate if the results of Section 5.2 can be generalized to consider an analogue of the n-th order linear ordinary differential equation [35, p. 281], $x^{(n)}(t)+p_{n-1}(t)x^{(n-1)}(t)+\cdots+p_1(t)x'(t)+p_0(t)x(t)=h(t)$. One such analogue in the nabla Caputo case may be

$$\nabla_{a*}^{\nu}x(t)+p_{N-1}(t)\nabla_{a*}^{\nu-1}x(t)+\cdots+p_1(t)\nabla_{a*}^{\nu-N+1}x(t)+p_0(t)x(t)=h(t),$$

where $N:=\lceil\nu\rceil$. Alternatively, one may consider an "n-th order" linear sequential equation, where the nabla Caputo operator of order ν, for $0<\nu\leq 1$ is applied n times. For the nabla Riemann-Louiville case, sequential difference equations are studied in [2], and Lyapunov inequalities involving sequential equations in the continuous fractional case are studied in [25]. In [38], an explicit representation of Green's functions for a linear fractional Riemann-Louiville operator with variable coefficients in the continuous case is given and involves a multivariate Mittag-Leffler function.

To extend the results of Section 5.2 in these cases, linearity of the equation and uniqueness of solutions to IVPs is essential, so that we may use a suitable set of linearly independent solutions for the construction of Green's functions. Moreover, one must obtain an appropriate variation of constants formula defined in terms of a suitable Cauchy function, as is done for ordinary differential equations case in [35, Chapter 6]. In this more general context, the Cauchy function is defined as a solution to an appropriate initial value problem. As a starting point, the equation $\nabla^{\nu}_{a*}x(t) + q(t)x(t) = h(t)$, where $1 + q(t) \neq 0$ may be considered, utilizing the results of Subsection 5.2.1.

Another future direction may include studying generalized zeros of nabla Mittag-Leffler functions using Lyapunov inequalities. In [22], zeros of a Mittag-Leffler function are considered using Lyapunov inequalities for a conjugate fractional BVP.

Lastly, note that in Theorem 5.29, we have assumed that the homogeneous BVP has only the trivial solution. In certain cases, in particular when $-1 < c \leq 0$, it may be possible to remove this assumption by showing that $\det D \neq 0$, where D is as defined in Section 5.2, using a methodology similar to the proofs given in Appendix A.

Appendix A

Nonzero Determinant Calculation

Lemma A.1 *Let* $D :=$

$$\begin{pmatrix} \nabla^{j_1} H_k(b, a-N+k) & \nabla^{j_1} H_{k+1}(b, a-N+k+1) & \cdots & \nabla^{j_1} H_{N-1}(b, a-1) \\ \nabla^{j_2} H_k(b, a-N+k) & \nabla^{j_2} H_{k+1}(b, a-N+k+1) & \cdots & \nabla^{j_2} H_{N-1}(b, a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{j_{N-k}} H_k(b, a-N+k) & \nabla^{j_{N-k}} H_{k+1}(b, a-N+k+1) & \cdots & \nabla^{j_{N-k}} H_{N-1}(b, a-1) \end{pmatrix}, \tag{A.1}$$

where $k \in \mathbb{N}_1^{N-1}$ *is fixed,* $j_1 < j_2 < \cdots < j_{N-k}$*, and* $j_m \in \mathbb{N}_0^{N-1}$*, for* $m \in \mathbb{N}_1^{N-k}$*. Then,* $\det D \neq 0$ *if and only if* $\det \hat{D} \neq 0$*, where* $\hat{D} :=$

$$\begin{pmatrix} \prod_{i=k+1}^{N-1}(i-j_1) & \prod_{i=k+2}^{N-1}(i-j_1) & \prod_{i=k+3}^{N-1}(i-j_1) & \cdots & \prod_{i=N-1}^{N-1}(i-j_1) & 1 \\ \prod_{i=k+1}^{N-1}(i-j_2) & \prod_{i=k+2}^{N-1}(i-j_2) & \prod_{i=k+3}^{N-1}(i-j_2) & \cdots & \prod_{i=N-1}^{N-1}(i-j_2) & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \prod_{i=k+1}^{N-1}(i-j_{N-k}) & \prod_{i=k+2}^{N-1}(i-j_{N-k}) & \prod_{i=k+3}^{N-1}(i-j_{N-k}) & \cdots & \prod_{i=N-1}^{N-1}(i-j_{N-k}) & 1 \end{pmatrix}. \tag{A.2}$$

Proof. Let $p \in \mathbb{N}_k^{N-1}$ and $m \in \mathbb{N}_1^{N-k}$. Then, the entry in row m and column $p-k+1$

of the matrix D is

$$H_{p-j_m}(b, a-N+p) = \frac{(b-a+N-p)^{\overline{p-j_m}}}{\Gamma(p-j_m+1)} = \frac{\Gamma(b-a+N-j_m)}{\Gamma(b-a+N-p)\Gamma(p-j_m+1)}.$$

Since $j_m \in \mathbb{N}_0^{N-1}$, we have $N - j_m \geq 1$. Hence, each entry of D is well defined, and $D =$

$$\begin{pmatrix} \frac{\Gamma(b-a+N-j_1)}{\Gamma(b-a+N-k)\Gamma(k-j_1+1)} & \frac{\Gamma(b-a+N-j_1)}{\Gamma(b-a+N-k-1)\Gamma(k-j_1+2)} & \cdots & \frac{\Gamma(b-a+N-j_1)}{\Gamma(b-a+1)\Gamma(N-j_1)} \\ \frac{\Gamma(b-a+N-j_2)}{\Gamma(b-a+N-k)\Gamma(k-j_2+1)} & \frac{\Gamma(b-a+N-j_2)}{\Gamma(b-a+N-k-1)\Gamma(k-j_2+2)} & \cdots & \frac{\Gamma(b-a+N-j_2)}{\Gamma(b-a+1)\Gamma(N-j_2)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(b-a+N-j_{N-k})}{\Gamma(b-a+N-k)\Gamma(k-j_{N-k}+1)} & \frac{\Gamma(b-a+N-j_1)}{\Gamma(b-a+N-k-1)\Gamma(k-j_{N-k}+2)} & \cdots & \frac{\Gamma(b-a+N-j_{N-k})}{\Gamma(b-a+1)\Gamma(N-j_{N-k})} \end{pmatrix}.$$

Then,

$$\det D = \Gamma(b-a+N-j_1)\Gamma(b-a+N-j_2)\cdots\Gamma(b-a+N-j_{N-k}) \det E_1,$$

where $E_1 :=$

$$\begin{pmatrix} \frac{1}{\Gamma(b-a+N-k)\Gamma(k-j_1+1)} & \frac{1}{\Gamma(b-a+N-k-1)\Gamma(k-j_1+2)} & \cdots & \frac{1}{\Gamma(b-a+1)\Gamma(N-j_1)} \\ \frac{1}{\Gamma(b-a+N-k)\Gamma(k-j_2+1)} & \frac{1}{\Gamma(b-a+N-k-1)\Gamma(k-j_2+2)} & \cdots & \frac{1}{\Gamma(b-a+1)\Gamma(N-j_2)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\Gamma(b-a+N-k)\Gamma(k-j_{N-k}+1)} & \frac{1}{\Gamma(b-a+N-k-1)\Gamma(k-j_{N-k}+2)} & \cdots & \frac{1}{\Gamma(b-a+1)\Gamma(N-j_{N-k})} \end{pmatrix}.$$

Now, $\det E_1 = \frac{1}{\Gamma(b-a+N-k)\Gamma(b-a+N-k-1)\cdots\Gamma(b-a+1)} \det E_2$, where $E_2 :=$

$$\begin{pmatrix} \frac{1}{\Gamma(k-j_1+1)} & \frac{1}{\Gamma(k-j_1+2)} & \cdots & \frac{1}{\Gamma(N-j_1)} \\ \frac{1}{\Gamma(k-j_2+1)} & \frac{1}{\Gamma(k-j_2+2)} & \cdots & \frac{1}{\Gamma(N-j_2)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\Gamma(k-j_{N-k}+1)} & \frac{1}{\Gamma(k-j_{N-k}+2)} & \cdots & \frac{1}{\Gamma(N-j_{N-k})} \end{pmatrix}.$$

Note $\frac{1}{\Gamma(b-a+N-k)\Gamma(b-a+N-k-1)\cdots\Gamma(b-a+1)} \neq 0$. Hence, $\det E_2 \neq 0$ if and only if $\det D \neq 0$. Next, consider the matrix obtained by multiplying row m of the matrix E_2 by $\Gamma(N - j_m)$ for each $m \in \mathbb{N}_1^{N-k}$, which we define to be $E_3 :=$

$$\begin{pmatrix} \frac{\Gamma(N-j_1)}{\Gamma(k-j_1+1)} & \frac{\Gamma(N-j_1)}{\Gamma(k-j_1+2)} & \cdots & \frac{\Gamma(N-j_1)}{\Gamma(N-j_1)} \\ \frac{\Gamma(N-j_2)}{\Gamma(k-j_2+1)} & \frac{\Gamma(N-j_2)}{\Gamma(k-j_2+2)} & \cdots & \frac{\Gamma(N-j_2)}{\Gamma(N-j_2)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(N-j_{N-k})}{\Gamma(k-j_{N-k}+1)} & \frac{\Gamma(N-j_{N-k})}{\Gamma(k-j_{N-k}+2)} & \cdots & \frac{\Gamma(N-j_{N-k})}{\Gamma(N-j_{N-k})} \end{pmatrix}.$$

Then, using the property of the Gamma function given in Proposition 1.3, $E_3 =$

$$\begin{pmatrix} (k+1-j_1)\cdots(N-1-j_1) & (k+2-j_1)\cdots(N-1-j_1) & \cdots & 1 \\ (k+1-j_2)\cdots(N-1-j_2) & (k+2-j_2)\cdots(N-1-j_2) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (k+1-j_{N-k})\cdots(N-1-j_{N-k}) & (k+2-j_{N-k})\cdots(N-1-j_{N-k}) & \cdots & 1 \end{pmatrix},$$

and $\hat{D} = E_3$. Moreover, $\det E_3 \neq 0$ if and only if $\det D \neq 0$. $\Diamond$

The next lemma will use the formula for the Vandermonde determinant given in [42].

Lemma A.2 *Let $k \in \mathbb{N}_1^{N-1}$ be fixed, $j_1 < j_2 < \cdots < j_{N-k}$, and $j_m \in \mathbb{N}_0^{N-1}$, for*

$m \in \mathbb{N}_1^{N-k}$. *Then,* $\det \hat{D} \neq 0$*, where* $\hat{D}$ *is given by* (A.2).

Proof. By the Vandermonde determinant formula [42, p. 17],

$$\begin{vmatrix} (j_1)^{N-k-1} & (j_1)^{N-k-2} & \cdots & j_1 & 1 \\ (j_2)^{N-k-1} & (j_2)^{N-k-2} & \cdots & j_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \\ (j_{N-k})^{N-k-1} & (j_{N-k})^{N-k-2} & \cdots & j_{N-k} & 1 \end{vmatrix} = \prod_{1 \le p < r \le N-k} (j_p - j_r).$$

Let $E :=$

$$\begin{pmatrix} (-1)^{N-k-1}(j_1)^{N-k-1} & (-1)^{N-k-2}(j_1)^{N-k-2} & \cdots & (-1)j_1 & 1 \\ (-1)^{N-k-1}(j_2)^{N-k-1} & (-1)^{N-k-2}(j_2)^{N-k-2} & \cdots & (-1)j_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \\ (-1)^{N-k-1}(j_{N-k})^{N-k-1} & (-1)^{N-k-2}(j_{N-k})^{N-k-2} & \cdots & (-1)j_{N-k} & 1 \end{pmatrix}.$$

Next, we will show that the matrix $\hat{D}$ can be obtained by elementary column operations on the matrix E.

Denote an entry in the first column of $\hat{D}$ by $f(j_m) := \prod_{i=k+1}^{N-1} (i - j_m)$, for each $m \in \mathbb{N}_1^{N-k}$. Note that f is a polynomial of degree $N-k-1$ in j_m, and the coefficient of j_m is $(-1)^{N-k-1}$. Then, $f(j_m) = (-1)^{N-k-1}(j_m)^{N-k-1} + \hat{f}(j_m)$, where $\hat{f}$ is a polynomial in j_m of degree less than or equal to $N-k-2$. Hence, we can write $\hat{f}(j_m)$ as a linear combination of $(-1)^{N-k-2}(j_m)^{N-k-2}, \ldots, (-1)j_m$, and 1; i.e., the entries in the remaining $N-k-1$ columns. Therefore, we can perform elementary column operations on E to obtain the first column of $\hat{D}$. Similarly, denoting an entry in the second column of $\hat{D}$ by $g(j_m)$, for each $m \in \mathbb{N}_1^{N-k}$, we have $g(j_m) = (-1)^{N-k-2}(j_m)^{N-k-2} + \hat{g}(j_m)$, where $\hat{g}$ is a polynomial in j_m of degree less than or equal to $N-k-3$ and hence can be written as a linear combination of

$(-1)^{N-k-3}(j_m)^{N-k-3}, \ldots,$ and 1. Proceeding in this manner, we get that $\hat{D}$ can be obtained by elementary column operations on E. Note that, using properties of determinants, $\det E =$

$$= (-1)^{\frac{(N-k)(N-k-1)}{2}} \begin{vmatrix} (j_1)^{N-k-1} & (j_1)^{N-k-2} & \cdots & j_1 & 1 \\ (j_2)^{N-k-1} & (j_2)^{N-k-2} & \cdots & j_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \\ (j_{N-k})^{N-k-1} & (j_{N-k})^{N-k-2} & \cdots & j_{N-k} & 1 \end{vmatrix}.$$

Hence, $\det \hat{D} = \det E = (-1)^{\frac{(N-k)(N-k-1)}{2}} \prod_{1 \leq p < r \leq N-k} (j_p - j_r) \neq 0$ since $j_1 < j_2 < \cdots < j_{N-k}$. $\diamondsuit$

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