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A Tensor's Torsion

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A TENSOR'S TORSION

by

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While tensor products are quite prolific in commutative algebra, even some of their most basic properties remain relatively unknown. We explore one of these properties, namely a tensor's torsion. In particular, given any finitely generated modules, M and N over a ring R, the tensor product $M \otimes_R N$ almost always has nonzero torsion unless one of the modules M or N is free. Specifically, we look at which rings guarantee nonzero torsion in tensor products of non-free modules over the ring. We conclude that a specific subclass of one-dimensional Gorenstein rings will have this property.

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Table of Contents

Chapter 1

Introduction

Tensor products are one of the most prolific tools used in commutative algebra. They are, for instance, a necessity in computing Ext and Tor of modules. The definition is as follows.

Definition 1.1. *Let M and N be modules over a commutative ring R. Let F be the free abelian group with basis $\{m \otimes n | m \in M, n \in N\}$ and H be the subgroup of F generated by $\{(m_1+m_2) \otimes n - m_1 \otimes n - m_2 \otimes n, m \otimes (n_1+n_2) - m \otimes n_1 - m \otimes n_2, (mr) \otimes n - m \otimes (rn) | m_1, m_2 \in M, n_1, n_2 \in N, r \in R\}$. We define the* tensor product *of M and N over R to be $M \otimes_R N := F/H$.*

Note that $M \otimes_R N$ is an R-module, where $r \cdot (m \otimes n) = (rm) \otimes n$ for $r \in R, m \in M, n \in N$.

While tensor products are widely used, even the most basic properties can be difficult to study. Of particular interest is torsion.

Definition 1.2. *Given a commutative ring R and an R-module M, we say M has* torsion *if there exists some non-zero m in M and non-zero-divisor r in R such that $rm = 0$. Define the torsion submodule of M to be*

$$\{m \in M | rm = 0 \textit{ for some non-zero-divisor } r \in R\}.$$

If a nonzero module M does not have torsion, or in other words if the torsion submodule is $\{0\}$, we say M is torsion-free.

Given a commutative ring R and R-modules M and N, it is likely the case that $M \otimes_R N$ has torsion, even if both M and N are torsion-free.

Example 1.1. *Let $R = \mathbb{C}[x,y]$ and $M = (x,y)$ and consider the tensor product $M \otimes_R M$. Then $x \otimes y - y \otimes x$ is a nonzero element of $M \otimes_R M$ and xy is a non-zero-divisor of R, but $xy(x \otimes y - y \otimes x) = 0$, so $M \otimes_R M$ has torsion, even though M is torsion-free.*

The most common exception to torsion in the tensor product occurs when either M or N is free. If for example M is a finitely generated free R-module, say $M \cong R^n$ for some n and N is a torsion-free R-module, then $M \otimes_R N \cong N^n$ which is torsion-free. For many rings, this is the only time a tensor product over the ring is torsion-free. We make the following definition.

Definition 1.3. *Let R be a commutative ring. We say R has the* torsion in tensor products property *if given any two finitely generated R-modules M and N, if $M \otimes_R N$ is torsion-free, then M or N is free.*

The main question this thesis seeks to answer is which rings have the torsion in tensor products property. It has been shown, for instance, that one-dimensional hypersurface domains satisfy this property. We wish to extend these results. While we can find examples of Gorenstein one-dimensional domains that do not satisfy the torsion in tensor products property, we do find a particular class of Gorenstein one-dimensional domains that satisfy it. We prove the following theorems.

Theorem 1.1. *Let R be a one-dimensional local Gorenstein domain. Suppose $M \otimes_R N$ is torsion-free for some nonzero finitely generated R-modules M and N. If $S = \mathfrak{m}^{-1}$ is not local, then either M or N is a free R-module.*

Theorem 1.2. *Let R be a one-dimensional local Gorenstein domain. Suppose $M \otimes_R N$ is torsion-free for some nonzero finitely generated R-modules M and N. If there is residue field growth from R to $S = \mathfrak{m}^{-1}$, then either M or N is a free R-module.*

To better understand which rings have the torsion in tensor products property, we first develop some background information.

1.1 Classifications of Rings

From here on, we assume all rings mentioned are commutative. We will also focus primarily on local rings.

Definition 1.4. *We say a ring is* local *if it is Noetherian and has only one maximal ideal. We write $(R, \mathfrak{m}, k)$ for a local ring R with maximal ideal $\mathfrak{m}$ and residue field k.*

Note that in the definition, we require a local ring to be Noetherian. While this isn't always standard, we will only be considering local rings that are Noetherian, and hence include it in our definition.

Definition 1.5. *For a ring R, the* total ring of quotients, *$Q(R)$, is defined to be $S^{-1}R$, where S is the set of non-zero-divisors of R. We define the integral closure of R in $Q(R)$ to be the subring of $Q(R)$ given by $\{x \in Q(R) \mid x^n + a_{n-1}x^{n-1} + \cdots a_0 = 0$ for some n and $a_0, a_1, ..., a_{n-1} \in R\}$. We write $\overline{R}$ for the integral closure of R in $Q(R)$.*

While commutative algebra aims to study commutative rings, it is often helpful to narrow the rings we study to ones with slightly nicer properties. The following classification of local rings is most often considered when studying commutative algebra [8].

Regular $\subseteq$ Hypersurface $\subseteq$ Complete Intersection $\subseteq$ Gorenstein $\subseteq$ Cohen Macaulay

To understand the definition of these rings, we first need to establish a few definitions.

Definition 1.6. *Let R be a ring and M an R-module. Given $r \in R$, we say r is M-*regular *if $r \cdot m \neq 0$ for all $m \in M$ with $m \neq 0$. We say the sequence $x_1, x_2, ..., x_n$ is an* M-regular sequence *(or M-sequence) if x_i is $M/(x_1, ..., x_{i-1})M$-regular for $1 \leq i \leq n$ and $M/(x_1, ..., x_n)M \neq 0$.*

Let R be a ring and I an ideal in R. We say $x_1, x_2, ..., x_n \in I$ is a maximal M-sequence in the ideal I if it is an M-sequence and $x_1, ..., x_{n+1}$ is not an M-sequence for any $x_{n+1} \in R$.

Theorem 1.3. *[8, Theorem 1.2.5] [17] Let R be a Noetherian, M a finitely generated R-module, and I an ideal. If $IM \neq M$, then every maximal M-sequence in I has the same length. Moreover, this length is equal to the least i such that $\operatorname{Ext}^i_R(R/I, M) \neq 0$.*

This theorem allows us to define the notion of grade.

Definition 1.7. *Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal of R such that $IM \neq M$. Then the* grade *of I on M, denoted* $\operatorname{grade}(I, M)$ *is the length of a maximal M-sequence in I. If $IM = M$ we define the grade of I on M be ∞.*

In this thesis we are particularly interested in a special case of the grade of a module over a local ring.

Definition 1.8. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring. The* depth *of an* R*-module,* M*, denoted* $\operatorname{depth}(M)$*, is equal to the grade of* $\mathfrak{m}$ *on* M*.*

One of the most important invariants of a ring is the dimension.

Definition 1.9. *The* Krull dimension *of a ring* R *is the supremum of the lengths* n *of strictly decreasing chains of prime ideals* $\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq ... \supsetneq \mathfrak{p}_n$*. We may further define the dimension of a module* M *to be the dimension of* $R/\operatorname{ann}(M)$ *where* $\operatorname{ann}(M) = \{r \in R \mid rM = 0\}$*.*

With the definitions of depth and dimension, we can now define Cohen-Macaulay rings and modules.

Definition 1.10. *Given a local ring* R *and a finitely generated* R*-module* M*,* $\operatorname{depth} M \leq \dim M \leq \dim R$*. If* $\operatorname{depth} M = \dim M$*, we say* M *is* Cohen-Macaulay*. If* $\operatorname{depth} R = \dim R$*, then we say* R *is a Cohen-Macaulay ring.*

Example 1.2. *Let* $R = \mathbb{C}[[x, y]]$*. It can be shown that the dimension of* R *is 2. Also, note that the maximal ideal of* R *is* (x, y) *and* x, y *is a maximal regular sequence in this maximal ideal, which means the depth of* R *is 2. Thus,* R *is Cohen-Macaulay. We may further notice that* $R/(x, y) \cong \mathbb{C}$*. Therefore,* $\dim R/(x, y) = 0$*. Also,* x, y *generate the maximal ideal. This tells us the ring* R *is not only Cohen-Macaulay, but also regular.*

Let $(R, \mathfrak{m})$ be a local ring with dimension n and $x_1, x_2, ..., x_n$ be elements of R. If $\dim R/(x_1, x_2, ..., x_n) = 0$ we say $x_1, x_2, ..., x_n$ is a system of parameters for R. If R has a system of parameters that generates $\mathfrak{m}$, we say R is regular.

In this thesis we will deal mostly with a specific type of Cohen-Macaulay ring known as a Gorenstein ring. In order to understand the definition of a Gorenstein ring, we first need to establish the definition of an injective resolution. Recall that an R-module I is injective if the functor $\operatorname{Hom}_R(\_, I)$ is exact.

Definition 1.11. *Let R be a ring. We say the exact sequence $0 \to M \to I^1 \to I^2 \to \cdots$ is an* injective resolution *of the R-module M if I^i is an injective R-module for all i.*

It is important to note that every module has an injective resolution. Given an R-module M, we define the injective dimension of M, written $\operatorname{injdim}_R M$, as the smallest integer n such that there is an injective resolution, I^* of M with $I^m = 0$ for all $m > n$. If no such n exists, we say the injective dimension is infinite.

Definition 1.12. *A Noetherian local ring is* Gorenstein *if $\operatorname{injdim}_R R < \infty$.*

Example 1.3. *Let $R = k[[x_1, x_2, ..., x_n]]$. Then $\dim R = n$ and $\dim(R/(x_1, x_2, ..., x_n)) = 0$, so $x_1, x_2, ..., x_n$ is a system of parameters for R. Also, the maximal ideal of R is generated by $x_1, x_2, ..., x_n$, so R is regular.*

We can generalize the notion of regular to hypersurfaces and complete intersections.

Definition 1.13. *We say a local ring R is a* hypersurface *if the completion of R is isomorphic to $S/(x)$ for some regular ring S and regular element x in S. We say the ring R is a complete intersection if the completion of R is isomorphic to $S/(x_1, x_2, ..., x_n)$ for some regular ring S and regular sequence $\{x_1, x_2, ..., x_n\}$ in S.*

Every complete intersection is Gorenstein, which gives us the classification of rings listed above:

$$\text{Regular} \subseteq \text{Hypersurface} \subseteq \text{Complete Intersection} \subseteq \text{Gorenstein} \subseteq \text{Cohen Macaulay}$$

1.2 Numerical Semigroups

In order to better understand torsion in tensor products, it is helpful to first look at an important class of rings known as numerical semigroup rings. We first define a numerical semigroup:

Definition 1.14. *A* numerical semigroup T *is a subset of the nonnegative integers,* $\mathbb{N}_0$*, that satisfies the following conditions:*

1. $0 \in T$
2. $\mathbb{N}_0 \setminus T$ *is finite*
3. T *is closed under addition*

We write $\langle \alpha_1, \alpha_2, ..., \alpha_n \rangle$ *for the numerical semigroup generated by the integers* $\alpha_1, \alpha_2, ..., \alpha_n$.

There are a few important definitions associated with these numerical semigroups.

Definition 1.15. *For a numerical semigroup T, we define the following:*

The gap set *of T is the set* $G(T) := \mathbb{N}_0 \setminus T$.

The genus *of T, denoted $g(T)$, is the number of elements in $G(T)$.*

The Frobenius number *of T, denoted $F(T)$, is the largest integer not contained in T.*

The conductor *of T is* $F(T) + 1$.

The multiplicity *of T, denoted $e(T)$, is the smallest nonzero integer in T.*

One particularly important definition we wish to study is the notion of a symmetric numerical semigroup.

Definition 1.16. *A numerical semigroup T is* symmetric *if the number of elements in T that are less than the conductor is equal to the genus of T. In other words, if c is the conductor of T, then T is symmetric if and only if $g(T) = \frac{c}{2}$.*

Example 1.4. *Consider the numerical semigroup T generated by $\langle 4, 5, 6 \rangle$. The gap set of T is $G(T) = \{1, 2, 3, 7\}$, which makes $F(T) = 7$. Since the conductor is 8 and $g(T) = 4$, we know T is symmetric. Also, it is clear that the multiplicity $e(T) = 4$.*

These numerical semigroups and subsequent definitions are of particular interest in commutative algebra due to their relation to numerical semigroup rings.

Definition 1.17. *Given a numerical semigroup $T = \langle \alpha_1, \alpha_2, ..., \alpha_n \rangle$, the ring $k[[T]] = k[[t^{\alpha_1}, t^{\alpha_2}, ..., t^{\alpha_n}]]$ is the* numerical semigroup ring *associated to T.*

Every numerical semigroup ring is Cohen-Macaulay of dimension 1. When needing to understand Cohen-Macaulay rings, especially those of dimension 1, it is often easiest to look towards examples from numerical semigroup rings. In particular, many important invariants of numerical semigroup rings relate to easily obtainable information in numerical semigroups. Consider, for example, the definition of multiplicity for local rings.

Notation 1.1. *For an R-module M, we write $\ell(M)$ to denote the length of M.*

Definition 1.18. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, and I an $\mathfrak{m}$-primary ideal of R. For a finitely generated R-module M, we define the* multiplicity *of I on M to be*

$$e_R(I, M) := \lim_{n \to \infty} \frac{d!}{n^d} \ell_R(M/I^n M).$$

We also define the multiplicity of a module to be $e_R(M) := e_R(\mathfrak{m}, M)$ *and the multiplicity of a ring to be* $e(R) := e_R(R)$.

For numerical semigroup rings, the multiplicity of the ring is easily determined.

Theorem 1.4. *[18] The multiplicity of a numerical semigroup ring is equal to the multiplicity of the associated numerical semigroup.*

Similarly, we can consider the following definition.

Definition 1.19. *Let* R *be a one-dimensional Noetherian ring. We define the* conductor ideal *of* R *to be*

$$C := \{x \in R : x\overline{R} \subseteq R\}.$$

To understand the conductor of a numerical semigroup ring, we first note the following important lemma.

Lemma 1.5. *For any numerical semigroup ring,* R*,* $\overline{R} = k[[t]]$.

Proof. Since R is a numerical semigroup ring, we can write R as $k[[t^{\alpha_1}, t^{\alpha_2}, ..., t^{\alpha_n}]]$. Let c be the conductor of the associated numerical semigroup. Then $t = \frac{t^{c+1}}{t^c} \in Q(R)$. Since t solves the equation $x^{\alpha_1} - t^{\alpha_1} = 0$ for x, we know t is integral over R. Therefore, $R \subseteq k[[t]] \subseteq \overline{R}$. Since $k[[t]]$ is integrally closed, $k[[t]] = \overline{R}$. $\square$

We can now see that conductors of numerical semigroup rings are also easy to compute.

Theorem 1.6. *The conductor ideal of a numerical semigroup ring* $R = k[[t^{\alpha_1}, t^{\alpha_2}, ..., t^{\alpha_n}]]$ *is the ideal* $(t^c, t^{c+1}, t^{c+2}, ...)$ *where* c *is the conductor of the associated numerical semigroup.*

Proof. Let C be the conductor of R. We first observe that $t^i \in R$ for all $i \geq c$ by definition of the conductor of a numerical semigroup. By Lemma 1.5, $\overline{R} = k[[t]]$ and since $t^i k[[t]] = (t^i, t^{i+1}, ...) \subseteq R$ for $i \geq c$, $(t^c, t^{c+1}, t^{c+2}, ...) \subseteq C$. Also, if $f(t) \in C$, then $f(t)k[[t]] \subseteq R$. Suppose $f(t) \notin (t^c, t^{c+1}, t^{c+2}, ...)$. Then we can write $f(t) = \sum_{i=0}^{r} a_i t^i$ for some integer r, and $a_i \in k$, with $a_j \neq 0$ for some $j < c$. Since $f(t)k[[t]] \subseteq R$, $f(t)t^{c-j-1} \in R$, but $f(t)t^{c-j-1} = \sum_{i=0}^{r} a_i t^i t^{c-j-1}$, which contains the term $a_j t^{c-1}$. Since $t^{c-1} \notin R$, $f(t)t^{c-j-1} \notin R$, which gives a contradiction. $\square$

Finally, we note that characterizing Gorenstein numerical semigroups is also quite straightforward.

Theorem 1.7. *[14] A numerical semigroup ring is Gorenstein if and only if its corresponding numerical semigroup is symmetric.*

Example 1.5. *Let $R = k[[t^4, t^5, t^6]]$. The corresponding numerical semigroup is $T = \langle 4, 5, 6 \rangle$, so from the previous example, we conclude that $\mathrm{e}(R) = 4$, the conductor ideal of R is (t^8), and R is in fact Gorenstein, since T is symmetric.*

Chapter 2

Main Results

2.1 Previous Results

In this thesis, we will mainly focus on one-dimensional Gorenstein rings and attempt to study the torsion in tensor products. There is, however, another property we can investigate in tensor products that acts as a sort of generalization of torsion-free modules in one-dimensional rings.

Definition 2.1. *Given a local ring R, we say an R-module M is* maximal Cohen-Macaulay*, or* MCM*, if* $\operatorname{depth}_R M = \dim_R R$.

Note that If $M = 0$, then $\operatorname{depth} M = \infty$, so maximal Cohen-Macaulay modules are always nonzero.

For Cohen-Macaulay one-dimensional rings, the notions of maximal Cohen-Macaulay and torsion-free modules are equivalent.

Theorem 2.1. *[12] Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay ring and M a finitely generated R-module. Then M is torsion-free if and only if M is maximal Cohen-Macaulay.*

Proof. First, suppose M is torsion-free. Since R is one-dimensional, M being maximal Cohen-Macaulay is equivalent to $\operatorname{depth}_R M > 0$. Since R is Cohen-Macaulay and

$M \neq 0$, there exists a non-zero-divisor $x \in \mathfrak{m}$. As M is torsion-free, x is M-regular, and thus $\operatorname{depth}_R M > 0$.

Now suppose M is maximal Cohen-Macaulay. The set of zero-divisors on M is equal to the union of associated primes of M. Note that $\mathfrak{m} \notin \operatorname{Ass}(M)$ since $\operatorname{depth}_R M > 0$, so as R is one-dimensional, all associated primes of M are minimal, and thus contained in the set of zero divisors of R. Hence M is torsion-free. □

While maximal Cohen-Macaulay is not equivalent to torsion-free in higher dimensional rings, there is a property known as Serre's condition that can relate a similar notion to torsion-free modules.

Definition 2.2. *Let R be a local ring and n a nonnegative integer. Then we say an R-module M satisfies* Serre's condition (S_n) *if for every prime p of R,*

$$\operatorname{depth}_R(M_p) \geq \min\{n, \dim(R_p)\}$$

It should be noted that there is some discrepancy between the definition of (S_n), with some sources replacing $\dim R$ with $\dim M$ in the definition (see for instance [8]). Throughout this paper, however, we will use the slightly stronger definition. Now, we can relate the notion of torsion-free to (S_1) for Gorenstein rings.

Theorem 2.2. *[8] Let R be a Cohen-Macaulay ring and M a finitely generated R-module. Then M is torsion-free if and only if M satisfies Serre's condition (S_1).*

There is also a nice relationship between reflexive modules and Serre's condition for Gorenstein rings.

Notation 2.1. *For an R-module M, we write M^* for* $\operatorname{Hom}_R(M, R)$.

Definition 2.3. *Let R be a ring and M an R-module. We say M is* reflexive *if the homomorphism $h : M \to M^{**}$ is an isomorphism, where h is induced by the map $M \times M^* \to R, (x, \phi) \mapsto \phi(x)$.*

Theorem 2.3. *[15, Corollary A.13] Let R be a Gorenstein domain. A finitely generated R-module M satisfies (S_2) if and only if M is reflexive.*

At the beginning of this thesis, we noted that with the exception of tensoring with free modules, tensor products are almost never torsion-free. This is especially true when dealing with one-dimensional rings, but with higher dimensional rings, there are some exceptions.

Example 2.1. *Let $R = k[[x, y, z, w]]/(xz - yw)$, where k is a field. Let $M = (x, y)$ and $N = (x, w)$. Then $M \otimes_R N$ is torsion-free, but not MCM. Since neither M nor N is free, the torsion in tensor products property fails to hold for R. Note, though, that in this case,* $\dim R = 3$.

In this example, we note that while the tensor product is torsion-free, it is not maximal Cohen-Macaulay. Just as we can ask when is a tensor product torsion-free, we may also wish to know when a tensor product is maximal Cohen-Macaulay. Unless we tensor with a free module, it is even more unlikely that our tensor product will in fact be maximal Cohen-Macaulay. This is, for instance, the case in hypersurfaces, provided the modules in the tensor product have rank.

Definition 2.4. *Let R be a Noetherian ring and M a finitely generated R-module. We say M has* rank *if there is some integer r such that $M_p = (R_p)^r$ for all $p \in \operatorname{Ass} R$.*

Theorem 2.4. *[12, Theorem 3.1] Let M and N be nonzero finitely generated modules over a hypersurface R, and assume that either M or N has rank. If $M \otimes_R N$ is MCM, then both M and N are MCM, and either M or N is free.*

This theorem does not hold, however, if we don't require the modules to have rank.

Example 2.2. *Let $R = k[[x, y]]/(xy)$, and $M = R/(x)$. Then M does not have rank. Also, $M \otimes M = M$, which is torsion-free over a one-dimensional ring. Hence it is maximal Cohen-Macaulay, but M is not free.*

One objective we have is to generalize this theorem to extend beyond hypersurfaces. Unfortunately, the theorem is also false if hypersurface is replaced with complete intersection.

Example 2.3. *Consider the one-dimensional complete intersection domain $R = k[[t^4, t^5, t^6]]$ and let $M = (t^4, t^5)$, $N = (t^4, t^6)$. Then $M \otimes_R N$ is torsion-free, but neither M nor N is free [12, Example 4.3].*

Thus, in generalizing the above theorem to other classes of rings like complete intersections, we must impose more conditions. One method is to place further conditions on the modules we use in the tensor product. The most noteworthy of these is proposed in the Huneke-Wiegand conjecture.

2.2 The Huneke-Wiegand Conjecture

In a 1994 paper, Craig Huneke and Roger Wiegand conjectured that for a special case of modules, the tensor product of said modules being torsion-free would imply that the modules in the tensor product are free. In its current form, the Huneke-Wiegand conjecture is stated as follows.

Conjecture 2.5. *Let R be a local Gorenstein domain and M a finitely generated R-module. If $M \otimes M^*$ is maximal Cohen-Macaulay, then M is free.*

To help prove or disprove this conjecture, we can make some simplifications. We define the following as the Huneke-Wiegand condition on the ring R:

HWC For a finitely generated module M, if $M \otimes_R M^*$ is MCM, then M is free.

In order to prove a ring has this condition, we need only look to torsion-free modules. We define a similar condition:

HWC1 For a finitely generated torsion-free module M, if $M \otimes_R M^*$ is MCM, then M is free.

To understand the relationship between HWC and HWC1, first consider the following lemma.

Lemma 2.6. *[2] Let R be a local ring, and M a finitely generated R-module. Let $t(M)$ denote the torsion submodule of M and $\perp M = M/t(M)$. Then $M \otimes_R M^* \cong \perp M \otimes_R (\perp M)^*$.*

Proof. Consider the exact sequence

$$0 \to t(M) \to M \to \perp M \to 0.$$

Tensoring with M^* gives the right exact sequence

$$t(M) \otimes_R M^* \xrightarrow{\alpha} M \otimes_R M^* \xrightarrow{\beta} \perp M \otimes_R M^* \to 0.$$

Note that $t(M) \otimes_R M^*$ is torsion and $M \otimes_R M^*$ is torsion-free. Thus α is the zero map and β is therefore an isomorphism. Note also that $(\perp M)^* = M^*$, since $t(M)^* = 0$,

giving us the desired isomorphism. $\square$

This allows us to prove the following.

Corollary 2.7. *[12, Lemma 1.1]*

For a local ring, HWC can be reduced to the case that M is torsion-free. That is, HWC1 $\implies$ HWC.

Proof. Suppose R satisfies HWC1. Let M be a finitely generated R-module and define $t(M)$ as the torsion submodule of M, $\perp M = M/t(M)$. By Lemma 2.6, if $M \otimes_R M$ is MCM, then $\perp M \otimes_R (\perp M)^*$ is also MCM, and thus by HWC1, $\perp M$ is free. If $M^* = 0$, then $M \otimes_R M^* = 0$, which is not MCM, so $M^* \neq 0$ and thus M must be free.

$\square$

We can also make further simplifications to the Huneke-Wiegand conjecture. In particular, we can often look to one-dimensional rings to help us understand rings of arbitrary dimension. In fact, if we can prove the Huneke-Wiegand conjecture for one-dimensional rings, then it will hold true for rings of any dimension. To see this we first consider the following lemmas.

Lemma 2.8. *[15, Lemma 5.11] Let R be Gorenstein, M and N be R-modules. A homomorphism $f : M \to N$ with M reflexive and N torsion-free is an isomorphism if and only if it is an isomorphism when localized at all prime ideals of height at most one.*

Proof. Let $K = Ker(f)$, and $C = coker(f)$. Then we have the exact sequence

$$0 \to K \to M \to N \to C \to 0.$$

Now, let $p \in Ass(R)$. As R is Gorenstein, p is minimal. Therefore f_p is an isomorphism and $K_p = 0$. Thus K must be torsion. But M is torsion-free, so $K = 0$. Now, suppose $C \neq 0$. Choose $p \in Ass(C)$. Then $\mathrm{ht}(p) \geq 2$, since $C_q = 0$ for all primes q of height less than 2. Therefore, as M is reflexive, M_p has depth at least 2, and, as N is torsion-free, N_p has depth at least 1. But C_p has depth 0, which contradicts the Depth Lemma [8, Proposition 1.2.9] when applied to the short exact sequence $0 \to M_p \to N_p \to C_p \to 0$. So $C = 0$, and f is an isomorphism. $\square$

Lemma 2.9. *[4, Proposition A.1] Let R be a local ring, M a finitely generated torsion-free R-module. The map $f : M \otimes_R M^* \to \mathrm{Hom}_R(M, M)$ taking $x \otimes g$ to the endomorphism $y \mapsto g(y)x$ is an isomorphism if and only if M is free.*

We can now use these lemmas to show why focusing on the one-dimensional case is sufficient for proving or disproving the Huneke-Wiegand conjecture.

Theorem 2.10. *The following conditions are equivalent:*

1. *All Gorenstein local domains R of dimension 1 satisfy the property that for an R-module M, if M and $M \otimes_R M^*$ are MCM then M is free.*

2. *All Gorenstein local domains R satisfy the property that for an R-module M, if $M \otimes_R M^*$ is non-zero reflexive, then M is free.*

Proof. (2) $\implies$ (1): This is clear, since in a Gorenstein ring, if a module is MCM, then it is reflexive [8].

(1) $\implies$ (2): Suppose condition (1) holds. Let R be a Gorenstein local domain and M an R-module. Suppose $M \otimes_R M^*$ is non-zero reflexive. Let $t(M)$ denote the torsion submodule of M and $\perp M = M/t(M)$. By Lemma 2.6, $M \otimes_R M^* \cong \perp M \otimes_R (\perp M)^*$. Note also that $(\perp M)^* = M^*$, since $t(M)^* = 0$. Thus, we have

$\perp M \otimes_R (\perp M)^*$ is also reflexive. Also, note that if $\perp M$ is free, then the short exact sequence

$$0 \to t(M) \to M \to \perp M \to 0$$

splits, showing that $M = \perp M \oplus t(M)$. Thus, $M \otimes_R M^*$ has a direct summand isomorphic to $t(M) \otimes_R M^*$, which is torsion. Thus, as $M \otimes_R M^*$ is nonzero and torsion-free, $t(M) = 0$, and $M = \perp M$. Hence, in general, we may assume M is torsion-free.

Now, consider the map $\phi : M \otimes_R M^* \to \mathrm{Hom}_R(M, M)$ taking $x \otimes f$ to the endomorphism $y \to f(y)x$. Note that as M is torsion-free, so is $\mathrm{Hom}_R(M, M)$. By Lemma 2.8, this map is an isomorphism if it is an isomorphism when localized at all height 1 primes. Let p be a height 1 prime. Note that R_p is a 1-dimensional Gorenstein domain. Also, as $M \otimes_R M^*$ is reflexive, it satisfies (S_2). Therefore $(M \otimes_R M^*)_p = M_p \otimes_{R_p} M_p^*$ is MCM, as R_p is 1-dimensional. Also, M_p is MCM, since M is torsion-free and thus satisfies (S_1). Therefore, by condition (1), M_p is free. By Lemma 2, this means that ϕ_p is an isomorphism. Then by Lemma 1, ϕ is an isomorphism and thus M is free.

$\square$

While the Huneke-Wiegand conjecture remains an open problem, much attention has been given to proving the conjecture. As a result, we do have some classes of rings for which we know the Huneke-Wiegand condition holds. We know from Theorem 2.4 that for a hypersurface R and module M that has rank, if $M \otimes_R M^*$ is maximal Cohen-Macaulay, then M is free. Thus, the Huneke-Wiegand condition holds for hypersurface domains. Furthermore, we have the following theorem for integrally closed domains.

Theorem 2.11. *[2, Proposition 3.3] Let R be a Noetherian integrally closed domain*

and let M be a torsion-free R-module. If $M \otimes M^*$ is reflexive, then M is projective.

Note that for local integrally closed Gorenstein domains, this theorem shows that such rings satisfy a stronger condition than the Huneke-Wiegand condition, as every maximal Cohen-Macaulay module is reflexive and projective modules are free.

One of the reasons the Huneke-Wiegand conjecture is particularly important is its connection to another conjecture by Auslander and Reiten [5].

Conjecture 2.12. *(The Auslander-Reiten Conjecture) Let R be an Artin algebra and M a finitely generated R-module such that $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ for $i > 0$. Then M is projective.*

In commutative algebra, this conjecture has been further generalized to all Noetherian rings. While this still remains an open problem, it has been proven for many classes of rings, including complete intersections [3, Proposition 1.9]. For clarity, we say a ring R satsifies the Auslander-Reiten condition if for any finitely generated R-module M with $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ for $i > 0$, M is projective. We have the following relationship between the Auslander-Reiten Conjecture and the Huneke-Wiegand Conjecture.

Theorem 2.13. *[9] If a local Gorenstein ring R satisfies the Huneke-Wiegand condition, then it satisfies the Auslander-Reiten condition.*

The converse to this theorem is false. In fact, while the Auslander-Reiten condition has been proven to hold for complete intersections, the Huneke-Wiegand condition does not even hold for arbitrary hypersurfaces.

Example 2.4. *Consider the hypersurface $R = k[[x, y]]/(xy)$, and R-module $M = R/(x)$. Since R is a hypersurface, the Auslander-Reiten condition holds. Note, how-*

ever, that $M^* = M$ and moreover, $M \otimes M^* = M$, which is torsion-free, but M is not free. So, the Huneke-Wiegand condition does not hold for R.

It is important to note that in this example, R is not a domain. It is still an open question whether or not the Auslander-Reiten condtion and Huneke-Wiegand condition are equivalent for Gorenstein domains.

2.3 Splitting, Residue Field Growth, and Ramification

A one-dimensional local ring that is integrally closed is a PID, and hence the torsion in tensor products property holds since all modules over such a ring are free. As we move away from integrally closed rings, this is no longer always true. We wish to be able to understand why the property fails in this case. When a local one-dimensional ring is not integrally closed, we can still relate the maximal ideal of the ring to the maximal ideal(s) of its integral closure.

Definition 2.5. *Let $(R, \mathfrak{m}, k)$ be a local ring and S be a ring that is integral over R. We say there is* splitting *from R to S if S is not local.*

We say there is residue field growth *from R to S if there is a maximal ideal $\mathfrak{n}$ of S such that $\dim_k S/\mathfrak{n} > 1$.*

Finally, we say there is ramification *from R to S if there is a maximal ideal $\mathfrak{n}$ of S such that $\mathfrak{m}S \subseteq \mathfrak{n}^2$.*

Theorem 2.14. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local domain with finite integral closure $\overline{R}$. If there is no splitting, residue field growth, or ramification from R to $\overline{R}$, then $R = \overline{R}$.*

Proof. As there is no splitting, $\overline{R}$ is local. Let $\mathfrak{n}$ be the maximal ideal of $\overline{R}$. As there is no residue field growth, $R/\mathfrak{m} = \overline{R}/\mathfrak{n}$, so $R + \mathfrak{n} = \overline{R}$. As there is no ramification,

$\mathfrak{m}\overline{R}$ is not contained in $\mathfrak{n}^2$. Since $\overline{R}$ is a DVR, every non-zero ideal of $\overline{R}$ is a power of $\mathfrak{n}$, and therefore $\mathfrak{m}\overline{R} = \mathfrak{n}$. Now we have $\overline{R} = R + \mathfrak{n} = R + \mathfrak{m}\overline{R}$. Since $\overline{R}$ is a finitely generated R-module, Nakayama's Lemma implies $R = \overline{R}$.

□

Note that in Example 2.3, we had a Gorenstein one-dimensional domain that fails to have the torsion in tensor products property. This ring, however, was a numerical semigroup ring. One important property of numerical semigroup rings is that there is only ramification from the ring to its integral closure. Thus, in looking for rings that satisfy the torsion in tensor products property, we may wish to exclude numerical semigroups where this fails, and further exclude rings that have ramification from the ring to its integral closure.

2.4 The Inverse of the Maximal Ideal

In order to connect the notions of splitting, residue field growth, and ramification from a ring to its integral closure with torsion in tensor products, it is useful to observe a particular intermediate ring, namely the inverse of the maximal ideal.

Definition 2.6. *Let R be a ring and I an ideal of R. We define the* inverse of I *as*

$$I^{-1} := \{x \in Q(R) : xI \subseteq R\}.$$

It is clear to see that the inverse of an ideal is an R-submodule of $Q(R)$. We, however, are particularly interested in the inverse of the maximal ideal, which has some special properties [7].

Lemma 2.15. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local domain that is not a PID. Then for any $x \in \mathfrak{m}^{-1}$, $x\mathfrak{m} \subset \mathfrak{m}$.*

Proof. Suppose $x\mathfrak{m} \nsubseteq \mathfrak{m}$. Since $x\mathfrak{m}$ is an ideal of R, $x\mathfrak{m} = R$. Therefore the homomorphism $f : \mathfrak{m} \to R$ given by multiplication by x is surjective. Also, since R is a domain, f must be injective. This means $\mathfrak{m}$ is isomorphic to R, which tells us $\mathfrak{m}$ is principal. This is a contradiction, since R is not a PID [13, Theorem 12.3]. □

Theorem 2.16. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local domain that is not a PID. Then $\mathfrak{m}^{-1}$ is a subring of $Q(R)$ that contains R.*

Proof. By lemma 2.15, we know $x\mathfrak{m} \subseteq \mathfrak{m}$ for all $x \in \mathfrak{m}^{-1}$. Then for any $y \in \mathfrak{m}^{-1}$, $yx\mathfrak{m} \subseteq R$. Thus, $yx \in \mathfrak{m}^{-1}$ for any $x, y \in \mathfrak{m}^{-1}$. It is clear that $\mathfrak{m}^{-1}$ is closed under subtraction and contains the multiplicative identity of $Q(R)$. Therefore $\mathfrak{m}^{-1}$ is a subring of $Q(R)$. □

We may also wish to characterize the inverse of the maximal ideal of a ring in a different way. Specifically, we can show in certain cases that the inverse of the maximal ideal is equal to the dual of the maximal ideal. In fact, we have the more general result.

Theorem 2.17. *Let R be a Noetherian ring, I an ideal of R such that there exists some $c \in I$ where c is a non-zero-divisor of R. Let J be an R-submodule of $Q(R)$. Then $\operatorname{Hom}_R(I, J) \cong \{q \in Q(R) \mid qI \subseteq J\}$.*

Proof. Suppose $q \in Q(R)$ with $qI \subseteq J$. Then multiplication by q is an element of $\operatorname{Hom}_R(I, J)$. Now, let $\phi \in \operatorname{Hom}_R(I, J)$ and let $d = \phi(c)$. Then for $x \in I$, $c\phi(x) = \phi(cx) = x\phi(c) = xd$. Since c is a non-zero-divisor of R, $\phi(x) = \frac{d}{c}x$ for all x in I, where $\frac{d}{c} \in Q(R)$. Thus, ϕ is multiplication by $\frac{d}{c}$, and clearly $\frac{d}{c}I \subseteq J$. □

Note that for a local ring $(R, \mathfrak{m}, k)$ with positive depth, (for example, Cohen-Macaulay rings), there is a non-zero-divisor of R in $\mathfrak{m}$, so Theorem 2.17 gives us that $\mathfrak{m}^* \cong \mathfrak{m}^{-1}$. Therefore, within this context, we may interchange $\mathfrak{m}^{-1}$ and $\mathfrak{m}^*$

freely. In fact, using the same theorem, we can also see that when $\mathfrak{m}^{-1}$ is a ring, then $\mathfrak{m}^{-1} \cong \mathrm{End}_R(\mathfrak{m})$.

Now, we know that for certain local rings $(R, \mathfrak{m}, k)$, that $\mathfrak{m}^{-1}$ is a ring, but it is useful to know how this ring relates to R and the integral closure of R. Consider the following propositions.

Proposition 2.18. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay ring that is not a PID and let $S = \mathfrak{m}^{-1}$. Then $R \subsetneq S \subseteq \overline{R}$.*

Proof. Since $\mathrm{End}_R(\mathfrak{m})$ is finitely generated as an R-module, and $\mathfrak{m}^{-1} \cong \mathrm{End}_R(\mathfrak{m})$, $\mathfrak{m}^{-1}$ is integral over R. Thus $\mathfrak{m}^{-1} \subseteq \overline{R}$. Also, $r\mathfrak{m} \subset \mathfrak{m}$ for all $r \in R$, so $R \subseteq S$. Now we have the following short exact sequence:

$$0 \to \mathfrak{m} \to R \to k \to 0.$$

Applying $\mathrm{Hom}_R(-, R)$, we get the exact sequence

$$0 \to k^* \to R^* \to \mathfrak{m}^* \to \mathrm{Ext}^1_R(k, R) \to 0.$$

Since $\mathrm{depth}(R) > 0$, $\mathrm{Ext}^1_R(k, R) \neq 0$ and $k^* = 0$. Therefore $R^* \subsetneq \mathfrak{m}^*$. In particular, note that R^* is exactly the set of maps that are multiplication by some element in R. The injection $i : R^* \hookrightarrow \mathfrak{m}^*$ from the short exact sequence takes f to $f|_{\mathfrak{m}}$, so i(f) is still multiplication by some element in R. It is clear by the isomorphism $\mathfrak{m}^{-1} \cong \mathfrak{m}^*$ that every element in $\mathfrak{m}^*$ is multiplication by some element in S. Since i is not surjective, there is some $x \in S \setminus R$ for which multiplication by x is in $\mathfrak{m}^*$. Therefore $R \neq S$.

$\square$

Proposition 2.19. *Let $(R, \mathfrak{m}, k)$ be a local ring and let S be a ring such that $R \subseteq S \subseteq \overline{R}$.*

1. *If there is no residue field growth from R to $\overline{R}$, then there is no residue field growth from R to S.*

2. *If there is no splitting from R to $\overline{R}$, then there is no splitting from R to S.*

3. *If there is no ramification from R to $\overline{R}$, then there is no ramification from R to S.*

Proof. 1. Suppose there is no residue field growth from R to $\overline{R}$. Then in particular, $\dim_k \overline{R}/\mathfrak{p} = 1$ for every maximal ideal $\mathfrak{p}$ in $\overline{R}$. Let $\mathfrak{n}$ be a maximal ideal in S. Suppose $\mathfrak{p} \subsetneq \mathfrak{q}$ for some prime $\mathfrak{q} \in \overline{R}$. Then $\mathfrak{q} \cap S \supseteq n$. By the incompatibility property [16, Theorem 9.3], this forces $\mathfrak{q} \cap S \supsetneq n$, but as n is maximal, this is not possible. So, $\mathfrak{p}$ must be maximal in $\overline{R}$. Since $(\dim_k S/\mathfrak{n}) \cdot (\dim_{S/\mathfrak{n}} \overline{R}/p) = \dim_k \overline{R}/p = 1$, $\dim_k S/\mathfrak{n} = 1$, meaning there is no residue field growth from R to S.

2. Suppose there is no splitting from R to $\overline{R}$. Then $\overline{R}$ is local. Let $\mathfrak{p}$ be the unique maximal ideal of $\overline{R}$. Suppose S is not local and let $\mathfrak{n}_1, \mathfrak{n}_2$ be maximal ideals of S. By the lying over property [16, Theorem 9.3], there exist primes $\mathfrak{p}_1, \mathfrak{p}_2 \in \overline{R}$ with $\mathfrak{p}_i \cap S = \mathfrak{n}_i$ for $i = 1, 2$. As we showed in 1, however, $\mathfrak{p}_1, \mathfrak{p}_2$ must be maximal, and thus $\mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p}$. Therefore, $n_1 = n_2$.

3. Suppose there is ramification from R to S. Then there is some maximal ideal $\mathfrak{n}$ of S with $mS \subseteq n^2$. Now, there exists some maximal ideal $\mathfrak{p} \in \overline{R}$ with $\mathfrak{n}\overline{R} \subseteq \mathfrak{p}$, so $m\overline{R} \subseteq n^2\overline{R} \subseteq p^2\overline{R}$. Therefore, there is ramification from R to $\overline{R}$.

□

Not only can we show that $R \subsetneq \mathfrak{m}^{-1} \subseteq \overline{R}$, but also that there is no intermediate module between R and $\mathfrak{m}^{-1}$. We first need the following lemma.

Lemma 2.20. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Gorenstein domain that is not a PID. Then $\mu_R(\mathfrak{m}^{-1}) = 2$. Specifically, $\mathfrak{m}^{-1} = (1, y)$ for some $y \in \mathfrak{m}^{-1} \setminus R$.*

Proof. From [7, Theorems 6.2 and 6.3], we know $\mu_R(\mathfrak{m}^{-1}) \leq 2$. Suppose $\mathfrak{m}^{-1}$ is principally generated, say by x. Since R is contained in $\mathfrak{m}^{-1}$, $1 = ax$ for some $a \in R$. From lemma 2.15, we know $x\mathfrak{m} \subset \mathfrak{m}$ for any $x \in \mathfrak{m}^{-1}$. Therefore a is not in $\mathfrak{m}$. This forces a to be a unit, meaning $x \in R$. But $R \subsetneq \mathfrak{m}^{-1}$, so x can't be in R. Thus $\mu_R(\mathfrak{m}^{-1}) = 2$.

Now, suppose R is generated by x and y. Then we can write $1 = ax + by$. Since 1 is not in $\mathfrak{m}$, either a or b must not be a unit. Without loss of generality, we may assume that a is a unit. This allows us to write $x = \frac{1}{a} - \frac{b}{a}y$, so $\mathfrak{m}^{-1} = (x, y) = (\frac{1}{a} - \frac{b}{a}y, y) = (\frac{1}{a}, y) = (1, y)$. Also, $y \notin R$, since $R \subsetneq \mathfrak{m}^{-1}$.

□

Proposition 2.21. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Gorenstein domain and suppose $R \subseteq M \subseteq \mathfrak{m}^{-1}$ for some R-module M. Then $M = R$ or $M = \mathfrak{m}^{-1}$.*

Proof. Suppose $M \neq R$. Let $x \in S \setminus R$ and let N be the R-module generated by 1 and x. Then $N \subseteq M$. Note that we can write $\mathfrak{m}^{-1} = (1, y)$ for some y. Thus $x = a + by$ for some $a, b \in R$. If $b \in \mathfrak{m}$, then $by \in R$, so $x \in R$. Since this can't happen, we know b must be a unit. This means we can write $y = \frac{1}{b}x - \frac{a}{b}$ where $\frac{1}{b}$ and $\frac{a}{b}$ are in R. This gives us that y is in N, so $\mathfrak{m}^{-1} = N$. Since $N \subseteq M \subseteq \mathfrak{m}^{-1}$, we must have $N = \mathfrak{m}^{-1}$. □

Given a local ring $(R, \mathfrak{m}, k)$, under certain circumstances we can consider R-modules as modules over the ring $\mathfrak{m}^{-1}$.

Theorem 2.22. *[7] Let $(R, \mathfrak{m}, k)$ be a local Gorenstein domain that is not a PID. Then any finitely generated torsion-free R-module M with no nonzero free direct sum-*

mand is also an S-module, where $S = m^{-1}$. Moreover, the R-module structure on M agrees with the S-module structure via the inclusion $R \subsetneq S$.

Proof. Recall that $S = \mathrm{End}_R(\mathfrak{m})$. There is no surjection $M \to R$, so $M^* = \mathrm{Hom}_R(M, \mathfrak{m})$, which is also a module over $\mathrm{End}_R(\mathfrak{m})$. Therefore, M^{**} is also a module over $\mathrm{End}_R(\mathfrak{m})$. Since R is Gorenstein, $M^{**} = M$, and hence M is an S-module. The "moreover" statement is clear from the naturality of the maps.

□

In determining whether or not a ring has the torsion in tensor products property, it is important to note that we do not need to look at all modules of the ring. In fact, we may exclude any module that has a nonzero free direct summand by the following lemma.

Lemma 2.23. *Let R be a local Noetherian ring. The following are equivalent:*

1. *For finitely generated R-modules M and N, if $M \otimes_R N$ is torsion-free, then either M or N is free.*

2. *For finitely generated torsion-free R-modules M and N that have no nonzero free direct summand, if $M \otimes_R N$ is torsion-free, then either M or N is free.*

Proof. Clearly (1) implies (2). To show the converse, let M and N be finitely generated R-modules with $M \otimes_R N$ torsion-free. Let $t(M)$ denote the torsion submodule of M and similarly $t(N)$ denote the torsion submodule of N. Also, let $\overline{M} = M/t(M)$ and $\overline{N} = N/t(N)$. Then by [12, Lemma 1.1], $\overline{M} \otimes_R \overline{N}$ is torsion-free. We may write $\overline{M} = M' \oplus F$ and $\overline{N} = N' \oplus G$, where M' and N' have no nonzero free direct summand, and F and G are free. Then $M \otimes_R N = (M' \oplus F) \otimes_R (N' \oplus G) = (M' \otimes_R N') \oplus (M' \otimes_R G) \oplus (F \otimes_R N') \oplus (F \otimes_R G)$. Since $M \otimes_R N$ is torsion-free, this implies $M' \otimes_R N'$ must be torsion-free. By (2), this means either M' or N' is free and

hence $\overline{M}$ or $\overline{N}$ are free. Without loss of generality, assume $\overline{M}$ is free. Again by [12, Lemma 1.1], either $N = 0$ or M is free.

□

Not only can we think of certain modules over a ring $(R, \mathfrak{m})$ as modules of the ring $\mathfrak{m}^{-1}$, but we also note that the tensor product remains unchanged whether we tensor over R or $\mathfrak{m}^{-1}$.

Lemma 2.24. *Let R be a local domain and $R \subset S \subset Q(R)$. Let M and N be nonzero R-modules that have no nonzero free direct summand. Suppose $M \otimes_R N$ is torsion-free. Then $M \otimes_R N$ is naturally isomorphic to $M \otimes_S N$.*

Proof. As R is a domain, $Q(R)$ is a field. Let $K = Q(R)$. Then $\dim_K K \otimes_R M$ is the rank of M as an R-module, which is the size of the maximal R-linearly independent set.

Claim: A set $x_1, ..., x_m \in M$ is linearly independent over R if and only if it is linearly independent over S.

If $x_1, ..., x_m \in M$ is linearly independent over S, then it is clearly linearly independent over R. To see the reverse direction, suppose $x_1, ..., x_m \in M$ is linearly independent in R and let $s_1x_1 + \cdots s_mx_m = 0$ for $s_i \in S$. We want to show $s_i = 0$ for all i. Choose $c \in R \setminus 0$ with $cs_i \in R$ for all i. Then $cs_1x_1 + \cdots cs_mx_m = 0$, so by linear independence of $x_1, ..., x_m \in R$, $cs_i = 0$ for all i. As R is a domain, $s_i = 0$ for all i.

This tells us that $\dim_K K \otimes_R M = \dim_K K \otimes_S M$, so the surjection $\beta : K \otimes_R M \twoheadrightarrow K \otimes_S M$ is an isomorphism. Similarly, we have the isomorphism $\gamma : K \otimes_R N \twoheadrightarrow K \otimes_S N$. Since $M \otimes_R N$ is torsion-free, there is an injection $\delta : M \otimes_R N \hookrightarrow K \otimes_R (M \otimes_R N) =$

$(K \otimes_R M) \otimes_K (K \otimes_R N) \cong (K \otimes_S M) \otimes_K (K \otimes_S N) = K \otimes_S (M \otimes_S N)$. We also have the surjection $\alpha : M \otimes_R N \twoheadrightarrow M \otimes_S N$, and injection $\epsilon : M \otimes_S N \hookrightarrow K \otimes_S (M \otimes_S N)$. Since $\epsilon \circ \alpha = \delta$, α must be an isomorphism.

□

We now have the tools to extend the class of rings that satisfy the torsion in tensor products property. First consider the following lemma.

Lemma 2.25. *Let R be a local PID and M and N torsion-free R-modules. If $M \otimes_R N$ is torsion-free, then M and N are free.*

Proof. By Lemma 2.23, we may assume M and N are torsion-free. Since torsion-free modules are free in a PID, both M and N must be free. □

Theorem 2.26. *Let R be a one-dimensional local Gorenstein domain. Suppose $M \otimes_R N$ is torsion-free for some nonzero finitely generated R-modules M and N. If $S = \mathfrak{m}^{-1}$ is not local, then either M or N is a free R-module.*

Proof. By Lemma 2.25, we need only consider the case when R is not a PID. Suppose neither M nor N is free. From Lemma 2.23, we may assume M and N are nonzero torsion-free modules that have no nonzero free direct summand. Then M and N are S-modules by Theorem 2.22. Let $A = S/\mathfrak{m}S = S/\mathfrak{m}$. By Lemma 2.20, we know $\mu_R(S) = 2$. This means that $\dim_k A = 2$. Since A is not local, there is a ring isomorphism $\psi : k \times k \to A$. Let $e = \psi((1,0))$. Then $e^2 = e$ and neither e nor $1 - e$ is a unit. Let $\overline{M} = M/\mathfrak{m}M$ and $\overline{N} = N/\mathfrak{m}N$.

Claim: $e\overline{M} \neq 0$.

Proof of Claim: Suppose $e\overline{M} = 0$. Consider the natural projection $\rho : S \to A$. There is some nonzero $\tilde{e} \in S$ such that $\rho(\tilde{e}) = e$. Then $\tilde{e}M \subseteq \mathfrak{m}M$. Clearly $\tilde{e}M + (1 - \tilde{e})M = M$. Now, as $\tilde{e}M \subset \mathfrak{m}M$, Nakayama's lemma implies that $(1 - \tilde{e})M = M$.

By the determinant trick, there is some $a \in (1-\tilde{e})S$ such that $(1+a)M = 0$. Since M is faithful as an R-module, M must be faithful as an S-module (S is contained in the quotient field of R), so $1 + a = 0$. Therefore $a^{-1} \in (1-\tilde{e})S$, so $1-\tilde{e}$ is a unit in S. But this means $\rho(1-\tilde{e}) = 1-e$ is a unit in A, which is a contradiction.

A similar argument shows that $(1-e)\overline{N} \neq 0$. Therefore, $e\overline{M} \otimes_k (1-e)\overline{N} \neq 0$. By Lemma 2.24 there is an isomorphism $\phi : M \otimes_R N \to M \otimes_S N$. This then induces an isomorphism $\overline{\phi} : \overline{M} \otimes_k \overline{N} \to \overline{M} \otimes_A \overline{N}$. But,$\overline{\phi}(e\overline{M} \otimes_k (1-e)\overline{N}) = e\overline{M} \otimes_A (1-e)\overline{N} = \overline{M} \otimes_A e(1-e)\overline{N} = 0$. As $e\overline{M} \otimes_k (1-e)\overline{N} \neq 0$, this is a contradiction. $\square$

Example 2.5. *Consider the Gorenstein ring $R = \mathbb{C}[x,y]_{(x,y)}/(y^2 - x^3 - x^2)$, and let $\mathfrak{m} = (x,y)$. Then $\mathfrak{m}^{-1} = R[\frac{y}{x}]$ which has 2 maximal ideals. Therefore, the torsion in tensor products property holds.*

Theorem 2.27. *Let R be a one-dimensional local Gorenstein domain. Suppose $M \otimes_R N$ is torsion-free for some nonzero finitely generated R-modules M and N. If there is residue field growth from R to $S = \mathfrak{m}^{-1}$, then either M or N is a free R-module.*

Proof. By Lemma 2.25, we need only consider the case when R is not a PID. Also, from Lemma 2.23, we may assume M and N are non-zero torsion-free modules that have no nonzero free direct summand. From Theorem 2.26, we may also assume S is local. Suppose neither M nor N is free. Since we know $\mu_R(S) = 2$ from Lemma 2.20, $\dim_k S/\mathfrak{m} = 2$. Let $\mathfrak{n}$ be the maximal ideal of S. We wish to show $\dim_k S/\mathfrak{n} = 1$. Suppose $\dim_k S/\mathfrak{n} > 1$. Then there is a surjection $S/\mathfrak{m} \twoheadrightarrow S/\mathfrak{n}$, so $2 = \dim_k S/\mathfrak{m} \geq \dim_k S/\mathfrak{n} \geq 2$. Therefore $\dim_k S/\mathfrak{n} = 2$, the surjection is an isomorphism, and $\mathfrak{n} = \mathfrak{m}$. Let $r_1 = \dim_k M/\mathfrak{m}M$ and $r_2 = \dim_k N/\mathfrak{m}N$. Then the dimensions of $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ over $S/\mathfrak{n}$ are $\frac{r_1}{2}$ and $\frac{r2}{2}$ respectively. So, $M/\mathfrak{m}M \otimes_{S/\mathfrak{m}S} N/\mathfrak{m}N$ has dimension $\frac{r_1 r_2}{4}$ over $S/\mathfrak{n}$. But, $M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} N/\mathfrak{m}N$ has dimension $r_1 r_2$ over k and therefore

dimension $\frac{r_1 r_2}{2}$ over $S/\mathfrak{n}$. Therefore $\frac{r_1 r_2}{2} = \frac{r_1 r_2}{4}$, and this forces either r_1 or r_2 to be 0, which (by Nakayama's Lemma) is a contradiction. □

Example 2.6. *Consider the Gorenstein ring $R = \mathbb{R}[[x, y]]/(x^2 + y^2)$. Let $\mathfrak{m} = (x, y)$. Then $S = \mathfrak{m}^{-1} = R[\frac{y}{x}]$. Note that in S, $(\frac{y}{x})^2 = -1$. Therefore, S is local with maximal ideal $\mathfrak{n} = (x, y)$. Then $S/\mathfrak{n} \cong \mathbb{C}$, so $\mathbb{R} \cong R/\mathfrak{m} \hookrightarrow S/\mathfrak{n} \cong \mathbb{C}$ is a field extension of degree 2. Therefore, R has the torsion in tensor products property.*

Now, we know that the existence of residue field growth or splitting from a ring to the inverse of its maximal ideal will guarantee the torsion in tensor products property holds under the conditions listed above. If there is ramification, however, this is not necessarily true. Our goal is to relate these theorems to the conjecture that if there is no ramification from the ring to its integral closure, then the torsion in tensor products property holds. To help us better understand why rings with ramification sometimes do not have the torsion in tensor products property, we return to studying numerical semigroups and numerical semigroup rings.

2.5 Numerical Semigroup Rings Revisited

We begin this section by noting that with the exception of the positive integers, numerical semigroups must be generated by at least 2 elements.

Lemma 2.28. *Let $T = \langle \alpha_1, \alpha_2, ..., \alpha_n \rangle$ be a numerical semigroup. If $n = 1$, then $T = \mathbb{N}_0$.*

Proof. Suppose $n = 1$ and $\alpha_1 \neq 1$. Then $n\alpha_1 + 1 \notin T$ for all $n \in \mathbb{N}$. This makes $G(T)$ infinite, which contradicts T being a numerical semigroup. Thus, if $n = 1$, $\alpha_1 = 1$, which makes $T = \mathbb{N}_0$. □

As the inverse of the maximal ideal is essential to many of our theorems, it is useful to determine the analogue of this concept in numerical semigroups.

Proposition 2.29. *Let $T \neq \mathbb{N}_0$ be a numerical semigroup. Then*

$T' = \{t \in \mathbb{Z} : t + (T \setminus \{0\}) \subseteq T\}$ is a numerical semigroup containing T.

Proof. We show first that $T' \subseteq \mathbb{N}_0$. Suppose $t \in T'$ and $t < 0$. Then $t + e(T) < e(T)$, so $t = -e(T)$. Since $T \neq \langle e(T) \rangle$, there is a least $\alpha \in T \setminus \{ne(T) \mid n \in \mathbb{N}\}$. Note that $t + \alpha \notin T$, since otherwise, as $t < 0$, $t + \alpha = n \cdot e(T)$ for some n, which means $\alpha = (n-1) \cdot e(T)$. This is a contradiction, since $\alpha \in T \setminus \{ne(T) \mid n \in \mathbb{N}\}$. Since $t + \alpha \notin T$, $t \notin T'$. Therefore $T \subseteq \mathbb{N}_0$. Also note that $T \subseteq T'$ by definition. Thus, $0 \in T'$ and $\mathbb{N}_0 \setminus T'$ is finite. Now if x and y are in T', then for $t \in T \setminus \{0\}$, $y + t \in T$. In particular, since $y \in T' \subseteq \mathbb{N}_0$, then $y + t \in T \setminus \{0\}$. So, $x + y + t \in T$. Thus, $x + y \in T'$. Therefore T' is a numerical semigroup. $\square$

Proposition 2.30. *Let $(R, \mathfrak{m}, k)$ be a numerical semigroup ring with associated numerical semigroup T. Then $S = \mathfrak{m}^{-1}$ is a numerical semigroup ring with associated numerical semigroup $T' = \{t \in \mathbb{Z} : t + (T \setminus \{0\}) \subseteq T\}$.*

Proof. We first show S is a numerical semigroup ring. As R is a numerical semigroup ring, $R \cong k[[t^{a_1}, t^{a_2}, ..., t^{a_n}]]$ for some $a_1, a_2, ..., a_n \in \mathbb{N}$. Since $\overline{R} \cong k[[t]]$ and $S \subseteq \overline{R}$, we may write an arbitrary element of S as $x = \sum_{i=0}^{\infty} c_i t^i$ where $c_i \in k$ for all i. Now, by definition of $\mathfrak{m}^{-1}$, we know $xt^{a_j} \in R$ for $1 \leq j \leq n$. So, $\sum_{i=0}^{\infty} c_i t^{i+a_j} \in R$. This means that $c_i t^{i+a_j} \in R$. Thus $c_i t^i \in S$, so each element of S must be a power series on some set of powers of t, each of which is contained in S. Therefore, S is a numerical semigroup ring.

Let Q be the numerical semigroup associated to S. Note that $\alpha \in Q$ if and only if $t^{\alpha+a_i} \in R$ for $1 \leq i \leq n$. This is true if and only if $\alpha + a_i \in T$ for all $1 \leq i \leq n$, which is true if and only if $\alpha \in T'$, since $T = \langle a_1, ..., a_n \rangle$. Therefore, $Q = T'$. □

We have seen that the multiplicity of a ring is related to the splitting, residue field growth, and ramification from the ring to its integral closure. As such, it may be useful to better understand both the multiplicity of a ring and the multiplicity of the inverse of its maximal ideal. We will further develop this concept in the following section, but we start by considering the multiplicity within numerical semigroups and numerical semigroup rings, which is simple to compute in this context.

Lemma 2.31. *Let T be a numerical semigroup. T is symmetric if and only if $F(T) - x \in T$ for all $x \in \mathbb{Z} \setminus T$.*

Proof. Suppose $F(T) - x \in T$ for all $x \in \mathbb{Z} \setminus T$. If $T = \mathbb{N}_0$, then T is symmetric. Assume $T \neq \mathbb{N}_0$. Let $y \in T$ with $y < F(T)$. Then $y = F(T) - x$ for some nonnegative integer x. Since $x + y = F(T) \notin T$ and $y \in T$, $x \notin T$. Therefore, each nonnegative integer in T that is less than the Frobenius number corresponds to a nonnegative integer not in T. Hence, T is symmetric.

Now suppose T is symmetric. Let $x \in \mathbb{Z} \setminus T$. We wish to show $F(T) - x \in T$. If $x < 0$, then $F(T) - x > F(T)$, so $x \in T$. Let $x > 0$. As above, if $y \in T$ with $y < F(T)$, then $y = F(T) - a$ for some $a \notin T$. As T is symmetric, there are as many nonnegative integers in T less than the conductor as there are elements not in T. Each element $y \in T$ that is less than the conductor corresponds to an element $a \notin T$, via the equality $y = F(T) - a$. Therefore, if $F(T) - x \notin T$, there would be more elements not in T than elements in T and less than the conductor. Hence, $F(T) - x \in T$. □

Theorem 2.32. *Let T be a symmetric numerical semigroup and*

$T' = \{t \in \mathbb{Z} : t + (T \setminus \{0\}) \subseteq T\}$. *Then $e(T') = e(T)$ if and only if $T \neq \langle 2, 3 \rangle$.*

Proof. If $T = \langle 2, 3 \rangle$, then $T' = \mathbb{N}_0$, so $e(T') = 1 \neq 2 = e(T)$.

Suppose $e(T') \neq e(T)$. Then $e(T') < e(T)$ since $T \subseteq T'$. Note that $F(T) \geq g(T)$. We show by way of contradiction that $F(T) = g(T)$. Suppose $F(T) > g(T)$. We know $F(T) \geq F(T')$ since $T \subseteq T'$.

Claim 1: If $F(T) > g(T)$, then $F(T) = F(T')$.

We know $F(T) \geq F(T')$ since $T \subseteq T'$. Suppose $F(T) > F(T')$. As T is symmetric, $F(T) - x \in T$ for all $x \in \mathbb{Z} \setminus T$ by Lemma 2.31. In particular, since $e(T') < e(T)$, then $e(T') \notin T$, and this gives $F(T) - e(T') \in T$. Note that $\{1, 2, ..., e(T) - 1\} \in G(T)$, so $g(T) \geq e(T) - 1$. Thus $F(T) > g(T) \geq e(T) - 1 \geq e(T')$. Therefore $F(T) - e(T') \in T \setminus \{0\}$ and $e(T') \in T'$, so by definition of T', $F(T) = e(T') + (F(T) - e(T')) \in T$, which is a contradiction.

Claim 2: If $F(T) > g(T)$, then T' is symmetric.

Since T is symmetric, we again note the fact that $F(T) - x \in T$ for all $x \in \mathbb{Z} \setminus T$. This then gives us $F(T') - x \in T'$ for all $x \in \mathbb{Z} \setminus T'$ since $T \subseteq T'$. Therefore T' is symmetric.

If $F(T) > g(T)$, T and T' are symmetric, so we must have $g(T') = \frac{F(T')+1}{2} = \frac{F(T)+1}{2} = g(T)$. But $e(T') \neq e(T)$ means T is properly contained in T', so $g(T) > g(T')$, a contradiction.

Therefore $F(T) = g(T)$. Since $g(T) = \frac{F(T)+1}{2}$, then $F(T) = \frac{F(T)+1}{2}$, so $F(T) = 1$. This tells us that $T = \{0, 2, 3, 4, ...\} =< 2, 3 >$.

□

Corollary 2.33. *If a Gorenstein numerical semigroup ring $(R, \mathfrak{m}, k)$ is not isomorphic to $k[[t^2, t^3]]$, then $e(R) = e(S)$, where $S = \mathfrak{m}^{-1}$.*

Proof. Let T be the numerical semigroup associated to R, and $T' = \{t \in \mathbb{Z} : t + (T \setminus \{0\}) \subseteq T\}$. Then by a previous lemma, T' is the numerical semigroup associated to S. If R is not isomorphic to $k[[t^2, t^3]]$, then $T \neq \langle 2, 3 \rangle$. So, by the theorem, $e(T) = e(T')$. Therefore, $e(R) = e(S)$. □

We know numerical semigroup rings have no splitting or ramification from the ring to its integral closure. We may, however, be interested in knowing whether or not there is ramification from a numerical semigroup ring to its integral closure. Consider the following example.

Example 2.7. *Let $R = k[[t^4, t^5, t^6]]$, where k is a field. Let $\mathfrak{m}$ be the maximal ideal of R, which is equal to (t^4, t^5, t^6). Then $S := \mathfrak{m}^{-1} = k[[t^4, t^5, t^6, t^7]]$. As R is a numerical semigroup ring, we know immediately that there is no residue field growth or splitting from R to S. Furthermore, we clearly see that there is no ramifcation from R to S, since the maximal ideal of S is (t^4, t^5, t^6, t^7). This is easily seen by the fact that $t^4 \in \mathfrak{m}S$, but $t^4 \notin (t^4, t^5, t^6, t^7)^2$.*

This example demonstrates that at least for numerical semigroups, there need not be splitting, residue field growth, or ramification from the ring to the inverse of its maximal ideal. As it turns out, this is actually more likely to happen for numerical semigroup rings than not. First note that the integral closure of a numerical semigroup ring is always the power series ring.

Theorem 2.34. *If a Gorenstein numerical semigroup ring* $(R, \mathfrak{m}, k)$ *is isomorphic to* $k[[t]]$ *or* $k[[t^2, t^3]]$*, then there is ramification from* R *to* $S = \mathfrak{m}^{-1}$*. Otherwise, there is no ramification, residue field growth or splitting from* R *to* S*.*

Proof. Since $R \subseteq S \subseteq \overline{R} = k[[t]]$, S must be local. Also, there can't be residue field growth from R to S since there is no residue field growth from R to $\overline{R}$. Now, if R is in fact isomorphic to $k[[t^2, t^3]]$, then $S = k[[t]]$, so S has ramification. Otherwise, we know from the previous corollary that $e(R) = e(S)$. We may write $R = k[[t^{\alpha_1}, t^{\alpha_2}, ..., t^{\alpha_n}]]$. Then $\mathfrak{m} = (t^{\alpha_1}, t^{\alpha_2}, ..., t^{\alpha_n})$. Let p be the maximal ideal of S. Then for ramification from R to S, we would need $p^2 \subseteq \mathfrak{m}S$. But this would imply that $t^a \in p$ for some $a < \min(\alpha_1, \alpha_2, ..., \alpha_n)$. This would then force $e(S) < e(R)$, which is a contradiction. $\square$

2.6 Multiplicity

Understanding the relationship between the multiplicity of a ring and the multiplicity of the inverse of its maximal ideal may be useful in determining the link between the splitting, residue field growth, and ramification from the ring to the inverse of its maximal ideal and that from the ring to its integral closure. Consider first the following definition.

Definition 2.7. *Let* R *be a one-dimensional semilocal Noetherian domain. Let* $m_1, ..., m_t$ *be the maximal ideals of* $\overline{R}$*. Then* $\overline{R}_{m_i}$ *is a DVR for each* m_i*. Let* v_i *be the valuation for* $\overline{R}_{m_i}$*. Then for* $r \in \overline{R}$*, we define* $v(r) := (v_1(r), ..., v_t(r))$ *and the value group* $v(R) = \{v(r) : r$ *is a non-zero-divisor in* $R\}$*.*

In the case that a ring has no splitting or residue field growth from the ring to its integral closure, we do in fact know the relationship between the multiplicity of the

ring and the multiplicity of the inverse of its maximal ideal.

Theorem 2.35. *Let R be a one-dimensional Gorenstein domain such that R is not integrally closed, but such that there is no splitting or residue field growth from R to $\overline{R}$. Let $S = m^{-1}$. Then $e(R) = e(S)$ if and only if $S \neq \overline{R}$. Moreover, if $S = \overline{R}$, then $e(R) = 2$.*

Proof. Note that $v(S) = \{x \in \mathbb{Z} : x + (v(R) \backslash \{0\}) \subseteq v(R)\}$ [6]. Then by Theorem 2.32, $v(R) = \langle 2, 3 \rangle$ or $e(v(R)) = e(v(S))$. If $v(R) = \langle 2, 3 \rangle$, then $v(S) = \mathbb{R}$, so $S = \overline{R}$. Also, in this case $e(v(R)) = 2$ so $e(R) = 2$ [6]. In the other case, since $e(v(R)) = e(v(S))$, then $e(R) = e(S)$. □

It is also interesting to note that for Gorenstein domains of low multiplicity, a version of the torsion in tensor products property also holds.

Proposition 2.36. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein one-dimensional domain such that $e(R) < 4$. If $I \otimes J$ is torsion-free for ideals I and J, then either I or J is principal.*

Proof. Suppose neither I nor J is principal. Then $\mu(I) \geq 2$ and $\mu(J) \geq 2$. If $I \otimes J$ is torsion-free, then $I \otimes J \cong IJ$. This tells us $\mu(IJ) = \mu(I)\mu(J) \geq 4$. But, $\mu(IJ) \leq e(R) < 4$ [1]. Therefore, either I or J must be principal. □

It is still an open question whether we can extend this proposition to all finitely generated modules of the ring.

2.7 Moving from $\mathfrak{m}^{-1}$ to $\overline{R}$

We have shown that for one-dimensional Gorenstein domains, if there is residue field growth or splitting from a ring to the inverse of its maximal ideal, then the torsion in tensor products property holds. We wish to be able to extend this idea to the

integral closure of the ring. In particular, for one-dimensional Gorenstein domains whose integral closure is finitely generated over the ring, we make the slightly stronger hypothesis that if there is no ramification from a ring to its integral closure, then the torsion in tensor products property holds. We have shown previously that if there is no ramification, there must be residue field growth or splitting from the ring to its integral closure. We also know that if there is splitting or residue field growth from the ring to the inverse of its maximal ideal, then there is residue field growth or splitting from the ring to its integral closure. The issue comes in the fact that while there must be splitting, residue field growth, or ramification from the ring to its integral closure (provided the ring is not already integrally closed), the same is not true when moving from the ring to the inverse of its maximal ideal. In other words, there are rings for which there is no splitting, residue field growth, or ramification from the ring to the inverse of its maximal ideal. Consider for instance Corollary 2.34, which states that with a single exception, there is no ramification, residue field growth, or splitting from a Gorenstein numerical semigroup ring to the inverse of its maximal ideal.

As numerical semigroup rings typically do not have splitting, residue field growth or ramification from the ring to the inverse of its maximal ideal, we might conjecture that under certain conditions, if there is no residue field growth, ramification, or splitting of the maximal ideal from a ring to the inverse of its maximal ideal, then there must be ramification in its integral closure. This would be a particularly nice result, since this would also prove the conjecture that if there is no ramification from the ring to its integral closure, then the torsion in tensor products property holds. Unfortunately, this turns out to be false. To see why, we need to first develop a way to generate examples with the necessary conditions. In particular, we utilize properties of the conductor ideal and the pullback. There is a particularly useful

characterization of Gorenstein rings that we can utilize to generate examples.

Theorem 2.37. *[7, Corollary 6.5] Let R be a one-dimensional reduced ring such that $\overline{R}$ is finitely generated as an R module, and let C be the conductor ideal of R. Then R is Gorenstein if and only if $\ell(\overline{R}/C) = 2\ell(R/C)$*

Now, note that for a Gorenstein one-dimensional domain R with conductor ideal C, R/C and $\overline{R}/C$ are Artinian rings. As it turns out, we can build a one-dimensional Gorenstein domain from a finite extension of Artinian rings under certain conditions using the pullback.

Theorem 2.38. *[19, Proposition 3.1]*

Let $i : A \hookrightarrow B$ be a module-finite extension of Artinian rings. Suppose the following conditions hold.

1. *B is a principal ideal ring*
2. *No nonzero ideal of B is contained in A*

Let D be a Dedekind domain admitting a surjective ring homomorphism $\pi : D \twoheadrightarrow B$. Let R be the pullback of i and π. Then R is a one-dimensional domain whose integral closure is D with D finitely generated as an R-module. Also, if C is the conductor ideal of R, then $A \cong R/C$ and $B \cong D/C$.

In finding a counterexample to the claim that there is always splitting or residue field growth from a local one-dimensional Gorenstein domain to the inverse of its maximal ideal, when there is no ramification from the ring to its integral closure, we wish to be able to find a reasonable example of an Artinian ring that we can pull back to a Gorenstein domain. The following example does in fact show that we can construct a Gorenstein one-dimensional domain that is not a PID with no residue

field growth, splitting, or ramification from the ring to the inverse of its maximal ideal, but such that there is also no ramification from the ring to its integral closure.

Example 2.8. *Let k be a field and $D = k[X]_{(X)\cup(X-1)}$, where X is an indeterminate. Then D is a PID with 2 maximal ideals. Let $A = k[T]/(T^2), B = k[X]/(X^2) \times k[X]/(X-1)^2$, and define $i : A \hookrightarrow B$ by $i(a+bt) = (a+bx, a+b(x-1))$ where $a, b \in k$ and decapitalization of the indeterminates indicates passage to cosets. Let $\pi : D \twoheadrightarrow B$ be the composition of the natural projection of $D \twoheadrightarrow D/(X^2(X-1)^2)$ and the isomorphism $D/(X^2(X-1)^2) \to B$ guaranteed by the Chinese Remainder Theorem. Define R to be the pullback of i and π. This gives us the following commutative diagram.*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d, two heads, "\pi"] \\ A \arrow[r, hook, "i"] & B \end{tikzcd}$$

Then R is a local one-dimensional domain whose integral closure is D with D finitely generated as an R-module. Furthermore, letting C be the conductor of R, we have $A \cong R/C$ and $B \cong D/C$. Since $\ell(A) = 2$ and $\ell(B) = 4$, R is Gorenstein.

Let $E = \{(a+bt, a+dt) | a, b, d \in k\}$. Then E is a ring strictly between A and B. Note that E is local with maximal ideal $\{(bt, dt) | b, d \in k\}$. If $\mathfrak{m}$ is the maximal ideal of R, then $\mathfrak{m}^{-1}/C$ is the smallest ring properly containing A. So, $\mathfrak{m}^{-1}/C \subseteq E$ which means $\mathfrak{m}^{-1}$ is local.

Theorem 2.39. *[11, Theorem 2.1] Let R be a one-dimensional, reduced, local ring whose integral closure is finitely generated as an R-module. Then $e(R) = v(R)$, where $v(R)$ is the embedding dimension of R (i.e. the number of generators of the maximal ideal).*

The following proposition is well known, but a proof is included for the convenience of the reader.

Proposition 2.40. *Let R be a ring whose completion is reduced and whose integral closure is finitely generated as an R-module. Assume* $\dim R = 1$ *and* $e(R) = 2$*. Then R is a hypersurface.*

Proof. We may assume R is complete. From Theorem 2.39, we know $v(R) = e(R) = 2$, so by the Cohen Structure Theorem [10], there is a surjection $\phi : S \twoheadrightarrow R$, where S is a two-dimensional regular local ring. Let $J = \operatorname{Ker}\phi$. We want to show J is principal. Since R is reduced, $J = P_1 \cap \cdots \cap P_n$ for P_i prime. Since the P_i are height one primes in a UFD, they are principal, say $P_i = (p_i)$ for all i. Then $J = (f)$ where $f = p_1 \cdots p_n$. □

Returning to Example 2.8, we note that R is a complete one-dimensional domain with multiplicity 2. Therefore, by Proposition 2.40, R is a hypersurface. This means that R still satisfies the torsion in tensor products property.

Chapter 3

Future Work

In this thesis, we have been able to extend the class of rings for which the torsion in tensor products property holds. Theorem 2.26 and Theorem 2.27 show that for Gorenstein one-dimensional domains, if there is residue field growth or splitting from the ring to the inverse of its maximal ideal, then the torsion in tensor products property does indeed hold. Our hope was to be able to extend this to say that if there is no ramification from the ring to its integral closure, then the torsion in tensor products property holds. One way to do this might have been to try to show that if there is no residue field growth or splitting from the ring to the inverse of its maximal ideal, then there is ramification from the ring to its integral closure. Unfortunately, we found a counterexample to this in Example 2.8. This ring, however, still satisfies the torsion in tensor products property. Thus, there is still hope that our initial conjecture holds. We formalize this conjecture as follows.

Conjecture 3.1. *Let (R, m, k) be a one-dimensional local Gorenstein reduced ring whose integral closure is finitely generated as an R-module. Suppose further that there is no ramification from R to $\overline{R}$. Given finitely generated R-modules M and N, if $M \otimes_R N$ is torsion-free, then either M or N is free.*

It still may be possible to utilize our results with the inverse of the maximal ideal

to prove this conjecture. We are still very confident that this conjecture is true. Of course, there are one-dimensional Gorenstein rings that have ramification from the ring to its integral closure, but for which the torsion in tensor products property still holds. It would be informative to be able to understand better why this property holds in these cases and classify which rings with ramification have the torsion in tensor products property.

Once we have established which one-dimensional Gorenstein domains have the torsion in tensor products property, our next goal would be to extend these results to higher dimensions. It is still of interest to know which higher dimensional rings satisfy the torsion in tensor products property, but recall that for higher dimensional rings, the analogue of torsion-free is maximal Cohen-Macaulay. We may define the following property.

Definition 3.1. *We say a local ring R satisfies the* MCM tensor products property *if given any finitely generated R-modules M and N, if $M \otimes_R N$ is maximal Cohen-Macaulay, then either M or N is free.*

Of course, our goal then would be to determine which rings satisfy the MCM tensor products property. Of particular interest would be looking at Gorenstein domains. When looking at one-dimensional rings, we consider ignoring those with ramification from the ring to its integral closure. We hypothesize that a similar set of higher dimensional rings will satisfy the MCM tensor products property.

Another case to consider would be extending the class of rings for which the torsion in tensor products property or MCM tensor products property holds beyond Gorenstein rings. For instance, which Cohen-Macaulay rings have this property? In order answer this, we may need to find a very different approach. Many of the theorems we use to prove our results rely on the properties of Gorenstein rings. There

are, however, many cases of non-Gorenstein rings that appear to satisfy the MCM tensor products property, so being able to classify these rings would be useful.

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