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The analytical solution for a monochromatic focused laser beam was recently published [Opt. Lett. **31**, 1447 (2006)]. The effect of introducing bandwidth by including a finite-length temporal pulse envelope is included exactly. This is done formally first in the frequency domain for an arbitrary pulse shape, and the specific case of a cosine-squared envelope is then solved analytically for all pulse lengths, thereby decreasing the computation time by 2 orders of magnitude. The inclusion of longer wavelengths reduces the fraction of laser energy in the focus from 86.5% to 83.5% for a 5 fs Ti:sapphire laser and 72.7% in a single-cycle pulse.

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Since the advent of chirped pulse amplification, ever shorter laser pulses—approaching the single-cycle limit—have been sought.[1] Research is ongoing on many fronts to generate ultrashort, high-energy beams, including through compression by ionization,[2] Raman amplification,[3] filamentation,[4] dielectric reflectors,[5] hollow-core fibers,[6] and even deformable mirrors.[7] Such ultrashort laser pulses find application in many areas of current interest, including the acceleration of high-energy, monoenergetic electrons, the generation of attosecond pulses, and photoionization.[8–11]

In this Letter the recently published solution for focused laser fields is used to develop an exact analytical solution to the vacuum Maxwell wave equation for tightly focused laser beams of arbitrary pulse length.[12] This saves a factor of 2 orders of magnitude in the time required for computing the field components. Another important feature of this solution is that the longitudinal fields are included, one of the most important features that has often been incorrectly neglected. The physical model of focused laser fields has been demonstrated many times to play a key role and must be considered to accurately model any experiment.[13–18] The longitudinal field strength is only a few percent of the transverse component but is necessary to accurately model vacuum acceleration (cf. Fig. 9 of Ref. 17).

The monochromatic solution for a laser polarized along $\hat{\mathbf{x}}$ and propagating along $\hat{\mathbf{z}}$ and having a Gaussian transverse profile of width w, $E_x(z=0)=B_y(z=0)=\tilde{E}_0\exp(-r^2/w^2)$, is given by

$$E_x=\frac{\tilde{E}_0}{\epsilon^2}\left(I_1+\frac{x^2-y^2}{r^3}I_2+\frac{y^2}{r^2}I_3\right),$$

$$E_y=\frac{\tilde{E}_0}{\epsilon^2}\frac{xy}{r^2}\left(\frac{2}{r}I_2-I_3\right),\qquad E_z=\frac{\tilde{E}_0}{\epsilon^2}\frac{x}{r}I_4, \quad (1)$$

defined in terms of the diffraction angle $\epsilon=2/kw$ and the integrals (following the model of Ref. 12):

$$I_1=2e^{-i\omega_0 t}\sum_{s=0}^{\infty}(a_s^1+a_s^2)i^s C_s^{1/2}\left(\frac{z}{\rho}\right)j_s(\omega\bar{\rho}), \quad (2)$$

$$I_2=-2e^{-i\omega_0 t}\frac{c^2}{\omega^2}\frac{\partial}{\partial r}\sum_{s=0}^{\infty}a_s^0 i^s C_s^{1/2}\left(\frac{z}{\rho}\right)j_s(\omega\bar{\rho}), \quad (3)$$

$$I_3=2e^{-i\omega_0 t}\sum_{s=0}^{\infty}(a_s^0-a_s^2)i^s C_s^{1/2}\left(\frac{z}{\rho}\right)j_s(\omega\bar{\rho}), \quad (4)$$

$$I_4=-2ie^{-i\omega_0 t}\frac{c}{\omega}\frac{\partial}{\partial r}\sum_{s=0}^{\infty}(a_s^0+a_s^1)i^s C_s^{1/2}\left(\frac{z}{\rho}\right)j_s(\omega\bar{\rho}), \quad (5)$$

where $\tilde{E}_0=E_0\exp(i\phi_0)$, E_0 is the field amplitude, ω_0 is the laser central frequency, ϕ_0 is the carrier-envelope phase, $r^2=x^2+y^2$, $\rho^2=r^2+z^2$, $\bar{\rho}=\rho/c$, $C_s^{1/2}(x)$ are the Gegenbauer polynomials, $j_s(x)$ are the spherical Bessel functions of the first kind, and the expansion coefficients a_s^n are given in terms of the lower incomplete γ function by the recursion $a_0^n=(1/4)e^{-1/\epsilon^2}(-i\epsilon)^{n+1}\gamma([n+1]/2,-\epsilon^{-2})$, $a_1^{n-1}=3a_0^n$, and $a_s^n([2s-1]/s)[a_{s-1}^{n+1}-([s-1]/[2s-3])a_{s-2}^n]$, being only functions of the diffraction angle $\epsilon=2/kw$ where $k=\omega/c$ is the laser wavenumber.

The polychromatic, finite-pulse solution is formally identical to Eqs. (1), but now the monochromatic integrals I_{1-4} are redefined to account for the bandwidth introduced by a finite pulse length according to $I_n\rightarrow\int_{-\infty}^{\infty}\tilde{f}(\omega-\omega_0)e^{-i\omega t}I_n\mathrm{d}\omega$ for $n=1,2,3,4$, where $\tilde{f}(\omega-\omega_0)=(2\pi)^{-1}\int_{-\infty}^{\infty}f(t)\exp(i[\omega-\omega_0]t)\mathrm{d}t$, for temporal envelope $f(t)$. Employing the result of Eqs. (2)–(5),

$$I_1=\sum_{s=0}^{\infty}(a_s^1+a_s^2)C_s^{1/2}\left(\frac{z}{\rho}\right)\mathcal{I}_{s,0}, \quad (6)$$

$$I_2=\frac{cr}{\rho}\sum_{s=0}^{\infty}a_s^0\left\{\left(\frac{cz}{\rho^2}\right)C_{s-1}^{3/2}\left(\frac{z}{\rho}\right)\mathcal{I}_{s,2}-iC_s^{1/2}\left(\frac{z}{\rho}\right)\times\left[\frac{s\mathcal{I}_{s-1,1}+(s+1)\mathcal{I}_{s+1,1}}{2s+1}\right]\right\}, \quad (7)$$

$$I_3=\sum_{s=0}^{\infty}(a_s^0-a_s^2)C_s^{1/2}\left(\frac{z}{\rho}\right)\mathcal{I}_{s,0}, \quad (8)$$

$$I_4=\frac{r}{\rho}\sum_{s=0}^{\infty}(a_s^0+a_s^1)\left\{\left(i\frac{cz}{\rho^2}\right)C_{s-1}^{3/2}\left(\frac{z}{\rho}\right)\mathcal{I}_{s,1}+C_s^{1/2}\left(\frac{z}{\rho}\right)\times\left[\frac{s\mathcal{I}_{s-1,0}+(s+1)\mathcal{I}_{s+1,0}}{2s+1}\right]\right\}, \quad (9)$$

given the definition

$$\mathcal{I}_{s,\nu}=2i^s\int_{-\infty}^{\infty}\tilde{f}(\omega-\omega_0)\exp[-i(\omega-\omega_0)t]\omega^{-\nu}j_s(\omega\bar{\rho})\mathrm{d}\omega.$$

Equations (1) and (6)–(9) are the exact wave equation solution for all physically realizable, i.e., absolutely integrable, laser temporal profiles $f(t)$. To finish the model, then, only the integrals $\mathcal{I}_{s,\nu}$ need to be evaluated for $\nu=0,1,2$.

This result differs from that of Ref. 8, which implicitly assumes that all of the frequency components focus to the same spot size. Practically speaking, tightly focusing short laser pulses is generally accomplished by using an optic having a small f-number, which is defined as the ratio of the focal length to the diameter. In solving the exact diffraction formulation for an arbitrary f-number paraboloidal mirror, for example, by using the Stratton and Chu[19] integrals numerically, the product of the incident wave number, k, and the resulting beam waist, w, $kw=2\epsilon^{-1}$ is constant for a given f-number.[19,20] Physically, this means that each frequency component focuses to a spot size proportional to its wavelength, as is also anticipated from elementary optics. Therefore the diffraction angle ϵ is constant and can be removed from each $\mathcal{I}_{s,v}$. By this same argument, the Fourier–Gegenbauer coefficients, a_s^n, also do not depend on the frequency, but only on ϵ, and are hence not included in the frequency transform.

To evaluate $\mathcal{I}_{s,n}$, first consider a system where $\tilde{f}(\omega-\omega_0)=\delta(\omega-\omega_0)$. This correctly reduces to the monochromatic solution [Eqs. (2)–(5)]. To evaluate more realistic pulse shapes, note that these integrals are convolution types. Motivated by this, recast them in the time domain by utilizing the Fourier transform of the spherical Bessel function of the first kind,

$$\mathcal{F}\{j_s(\omega\bar{\rho})\}=\pi i^s\,\mathrm{rect}(T/\bar{\rho})\sum_{k=0}^{2k\leq s}C_k^{(s)}(-1)^k\bar{\rho}^{2k-s-1}T^{s-2k},$$

where the constants $C_k^{(s)}$ are defined by the recursion $(l+1)C_k^{(l+1)}=lC_{k-1}^{(l-1)}-(2l+1)C_k^{(l)}$ for $C_0^{(0)}=1$ and $C_1^{(2)}=1/2$, and $\mathrm{rect}(x)=1$ for $|x|<1$ and $\mathrm{rect}(x)=0$ otherwise.[21] The integrals $\mathcal{I}_{s,n}$ are then, in the time domain,

$$\mathcal{I}_{s,\nu}=\frac{i^s}{\pi}\int_{-\infty}^{\infty}f(t-T)G_\nu(T)e^{i\omega_0 T}\mathrm{d}T, \quad (10)$$

where $G_0(T)=\mathcal{F}\{j_s(\omega\bar{\rho})\}$, $\partial_T G_1=-iG_0(T)$, and $\partial_T^2 G_2=-G_0(T)$.

Taking the temporal envelope to be a cosine-squared function with a FWHM duration $\Delta\tau$, $f(t-T)=\cos^2(\Omega_0[t-T]/2)$, where $\Omega_0=\pi/\Delta\tau$, the integrals in Eq. (10) can be evaluated directly, yielding

$$\mathcal{I}_{s,0}=\sum_{k=0}^{2k\leq s}C_k^{(s)}(-1)^{s+k}\bar{\rho}^{2k-s-1}F_{s-2k}(-\bar{\rho},\bar{\rho}),$$

$$\mathcal{I}_{s,1}=-i\sum_{k=0}^{2k\leq s}\frac{C_k^{(s)}(-1)^{s+k}}{2-sk+1}\left(\frac{F_{s-2k+1}(-\bar{\rho},\bar{\rho})}{\bar{\rho}^{s-2k+1}}+(-1)^{s-2k}\{F_0(-\bar{\rho},\bar{\rho})+[1+(-1)^{s-2k}]F_0(\bar{\rho},\infty)\}\right),$$

$$\mathcal{I}_{s,2}=-\sum_{k=0}^{2k\leq s}C_k^{(s)}(-1)^{s+k}\left(\frac{F_{s-2k+2}(-\bar{\rho},\bar{\rho})\bar{\rho}^{2k-2-1}}{(s-2k+1)(s-2k+2)}+\frac{\bar{\rho}(-1)^{s-2k}}{s-2k+2}\{F_0(-\bar{\rho},\bar{\rho})+[1-(-1)^{s-2k}]F_0(\bar{\rho},\infty)\}+\frac{(-1)^{s-2k}}{s-2k+1}\times\{F_1(-\bar{\rho},\bar{\rho})+[1+(-1)^{s-2k}]F_1(\bar{\rho},\infty)\}\right), \quad (11)$$

where the functions $F_n(\alpha,\beta)$ are defined as

$$F_n(\alpha,\beta)=\frac{1}{4}(\{2\mathcal{C}_n(\omega_0)+\cos(\Omega_0 t)[\mathcal{C}_n(\Omega_+)+\mathcal{C}_n(\Omega_-)]+\sin(\Omega_0 t)[\mathcal{S}_0(\Omega_+)+\mathcal{S}_n(\Omega_-)]\}+i\{2\mathcal{S}_n(\omega_0)+\cos(\Omega_0 t)[\mathcal{S}_n(\Omega_+)-\mathcal{S}_n(\Omega_-)]-\sin(\Omega_0 t)\times[\mathcal{C}_n(\Omega_+)-\mathcal{C}_n(\Omega_-)]\})|_\alpha^\beta,$$

for $\Omega_\pm=\Omega_0\pm\omega_0$. The limits α and β shown in Eqs. (11) are those formally imposed by the rect(x) function. As the cosine-squared envelope is only nonzero in a finite region, the true bounds of this integration are the intersection of these regions: $T\in\{[\alpha,\beta]\cap[t-\Delta\tau,t+\Delta\tau]\}$. Finally, to complete this solution, define the function $S_p(\omega)$ as the well-known integral

$$\mathcal{S}_p(\omega) \equiv \int x^p \sin(\omega x)\mathrm{d}x = \begin{cases} (-1)^{p/2+1}(p!)\left[\cos(\omega x)\sum_{k=0}^{p/2} \frac{(-1)^k x^{2k}}{(2k)!\omega^{p-2k+1}} + \sin(\omega x)\sum_{k=1}^{p/2} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!\omega^{p-2k+2}}\right] & p \text{ even} \\ -(x^p/\omega)\cos(\omega x) + (p/\omega)\mathcal{C}_{p-1}(\omega) & p \text{ odd} \end{cases},$$

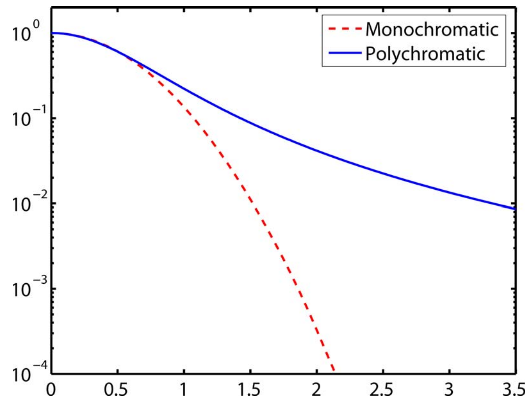

Fig. 1. (Color online) E_x evaluated for a single-cycle pulse ($\Delta\tau=\lambda/c$) at the peak of the pulse along $x=y$ in the focal plane by using the monochromatic model of Eqs. (2)–(5) (red dashed curve) and the full polychromatic model (blue solid curve).

and $\mathcal{C}_n(\omega)$ identically following the transformation $\omega x \rightarrow \omega x + \pi/2$, for all nonnegative integers p.

A key change when the polychromatic corrections are included is their tendency to broaden the beam relative to the monochromatic case. Since longer wavelengths focus to larger spot sizes for a given optic, inclusion of these frequencies spreads the laser energy over a wider radius, and hence the fraction of the energy contained within the nominal spot is significantly reduced. Figure 1 shows E_x computed in the focal plane at the peak of the pulse with $E_0=1$ and $w_0=\lambda$ by using both the monochromatic and polychromatic fields described above for $\Delta\tau$ of 1 cycle, and the broadening is readily apparent. Specifically, for a monochromatic pulse focused to $w_0=\lambda_0$, the percentage of the energy within w_0 is 86.51%. For the 5 fs Ti:sapphire pulse described by Nisoli *et al.*[1] that contains roughly 2 cycles, this is only 83.47%, and in the limit of a single cycle, this fraction is further reduced to only 72.66%. To correctly model intensity-dependent processes by using a few-cycle laser pulse, for example, field ionization, for a given laser energy and focusing optic, the bandwidth is essential, since by using a monochromatic field model the intensity in the focal plane will be systematically overestimated. Only with this polychromatic field model can such an experiment be correctly compared with theoretical predictions.

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