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OPERATOR ALGEBRAS GENERATED BY LEFT INVERTIBLES

by

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OPERATOR ALGEBRAS GENERATED BY LEFT INVERTIBLES

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Operator algebras generated by partial isometries and their adjoints form the basis for some of the most well studied classes of C*-algebras. Representations of such algebras encode the dynamics of orthonormal sets in a Hilbert space. We instigate a research program on concrete operator algebras that model the dynamics of Hilbert space frames.

The primary object of this thesis is the norm-closed operator algebra generated by a left invertible T together with its Moore-Penrose inverse $T^\dagger$. We denote this algebra by $\mathfrak{A}_T$. In the isometric case, $T^\dagger = T^*$ and $\mathfrak{A}_T$ is a representation of the Toeplitz algebra. Of particular interest is the case when T satisfies a non-degeneracy condition called analytic. We show that T is analytic if and only if T^* is Cowen-Douglas. When T is analytic with Fredholm index -1, the algebra $\mathfrak{A}_T$ contains the compact operators, and any two such algebras are boundedly isomorphic if and only if they are similar.

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Chapter 1

Introduction

Mathematical objects are frequently defined with the intent to encode interesting dynamics. For example, groups are reflect the rigid symmetries of geometrical objects. Operator algebras and the tools surrounding them have proven to be powerful at analyzing complicated phenomena. Indeed, many operator algebras reflect the structures of algebraic and combinatorial objects, such as groups and directed graphs.

Representations of operator algebras are often formed by choosing sufficiently nice linear maps on a Hilbert space that encapsulate the features of the underlying algebraic object. Often, these maps are rigid in the sense that they will preserve Hilbert space structure from the domain into their range. For example, if $\mathscr{H}$ is a Hilbert space, $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathscr{H}$, and $U \in \mathscr{B}(\mathscr{H})$ is a unitary, then $\{Ue_n\}_{n=1}^{\infty}$ is once again an orthonormal basis. Similarly, an isometry $S \in \mathscr{B}(\mathscr{H})$ moves an orthonormal basis for $\mathscr{H}$ to its range space. More generally, if S is a partial isometry, then S preserves orthonormality on $\ker(S)^{\perp}$. In each case, the operator models the movement from one orthonormal set (on the domain space) to another (on the range space). The adjoint models walking backwards between these two subspaces.

One can take a collection of partial isometries $\{S_\alpha\}_{\alpha\in A}$. Each S_α and S_α^* encodes this "single step" dynamics discussed above - moving one orthogonal set to another. To codify all possible finite walks, one would need to consider the algebra generated by the collection

$\{S_\alpha, S_\alpha^*\}_{\alpha \in A}$. Closing this algebra with respect to some topology, such as the operator norm, describes the infinite walks as well.

An important class of operator algebras generated by partial isometries are graph C*-algebras. Representations that reflect the directed graph structure are described as follows. Given a directed graph, a Hilbert space is chosen for each vertex of the graph. Let $\mathscr{H}$ denote the direct sum of these spaces. By choosing orthonormal sequences for each of these closed, orthogonal spaces of $\mathscr{H}$, one chooses partial isometries that map one summand to another subject to the Cuntz-Krieger relations coming from the graph [36]. Specifically, let E^0 be the set of vertices and E^1 is the set of edges for a graph. Let $s(e)$ and $r(e)$ denote the range and source of an edge respectively. Given a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{S_e : e \in E^1\}$ of partial isometries, the Cuntz-Krieger relations are given by

1. $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$
2. $P_v = \sum_{e \in E^1 : r(s) = v} S_e S_e^*$ whenever v is not a source.

This representation of the graph C*-algebra can be viewed as encoding walks on the graph.

Orthonormal bases are rigid structures. The requirement that each element within the set be orthogonal to one another is strict and has precluded them from finding applications in some realms of applied harmonic analysis. This naturally led to the definition of a frame for a Hilbert space. A sequence $\{f_n\}$ of points in a Hilbert space $\mathscr{H}$ is said to be a *frame* if there exists constants $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_n |\langle x, f_n \rangle|^2 \leq B\|x\|^2$$

for all $x \in \mathscr{H}$. Associated to each Hilbert space frame $\{f_n\}$ is a (canonical) dual frame $\{g_n\}$. Using this dual frame, one can reconstruct elements f of the Hilbert space $\mathscr{H}$ in an analogous way to orthonormal basis:

$$f = \sum_{n \geq 1} \langle f, g_n \rangle f_n$$

It is easy to see that orthonormal bases are frames, but not all frames need be orthogonal, norm one, or even contain a unique set of elements. A frame does not enforce the rigidity of inner products that an orthonormal basis does - allowing for variation between individual frame elements (rather than just 0 or 1). The flexibility of the definition has found applications across signal processing and harmonic analysis. Frames may be constructed for particular features of a problem, allowing one choose linear dependent sets, or even add multiple copies of a single element. This extra redundancy helps to protect signals from degradation, ensuring that the effects of erasures are minimized. The looseness of the structure allows one to construct the analog of frames for structures that don't necessarily come equipped with suitable generalization of an orthonormal basis. Indeed, certain classes of Hilbert C*-Modules and Banach spaces posses frames [20], [6]. For more on basics of frame theory, see [5], [9], [8].

As discussed, partial isometries between closed subspaces of $\mathscr{H}$ preserve orthonormal sets. The adjoint of a partial isometry also preserves orthonormality, and acts as an inverse wherever it makes sense. More generally, if $\{f_n\}_{n=1}^{\infty}$ is a frame, and $T \in \mathscr{B}(\mathscr{H})$ is invertible, then $\{Tf_n\}_{n=1}^{\infty}$ is a new frame for the Hilbert space. Hence, a left invertible operator moves a frame to its range space. Generalizing this one last step, closed range operators preserve the property of a frame on $\ker(T)^{\perp}$. See Proposition 3.1.5.

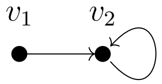
If T has closed range, T has a pseudo inverse $T^{\dagger}$ that acts like an inverse wherever it makes sense. This operator, called the *Moore-Penrose inverse* encodes the dynamics of walking backward from the range subspace to the source subspace. When T is isometric, $T^{\dagger} = T^{*}$. See Proposition 3.1.1.

The previous discussion lays the groundwork for a natural extension of C*-algebras of isometries, one that codifies frames over orthonormal bases. One arrives at a such an extension by replacing *partial isometries and their adjoints* with *closed range operators and their Moore-Penrose inverses.* As discussed above, the closed range operators preserve frame theoretic quantities. Therefore, by replacing all instances of “unitary” with “invertible”, we arrive at a natural generalization of concrete C*-algebras, integrating frame theory over orthonormal bases.

One cannot hope to fully understand the C*-algebra generated by arbitrary set of partial isometries. For this reason, algebraic conditions, such as the Cuntz Krieger relations (constraints that arise from a directed graph), are imposed. This leads us to the following general program:

Program. *Given a set of operators with closed range and their Moore-Penrose inverses, construct the norm-closed algebra subject to the constraints of a directed graph. What is the structure of these algebras?*

The focus of this paper is on one particular class of examples within this program. Consider the following directed graph Γ:



It is well known that the graph C*-algebra associated to Γ is isomorphic to the Toeplitz algebra $\mathcal{T}$ [36]. As a concrete operator algebra, $\mathcal{T}$ may be represented as the C*-algebra generated by $T = M_z$ on the Hardy space $H^2(\mathbb{T})$. The graph C*-algebra representations associated to Γ can be described as follows. Let $\mathscr{H}_i$ represent the Hilbert space associated to vertex v_i, and $T_1 : \mathscr{H}_1 \to \mathscr{H}_2$, $T_2 : \mathscr{H}_2 \to \mathscr{H}_2$ be chosen (partial) isometries. Since $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, and $\operatorname{ran}(T_1) \oplus \operatorname{ran}(T_2) = \mathscr{H}_2$, we have that $T := T_1 \oplus T_2$ defines an isometry with Fredholm index equal to $-\dim(\mathscr{H}_1)$. Thus, the representations can be succinctly written as $C^*(T)$ for some isometry T.

The same argument can be applied to the operator algebras described above. Concretely, choose $T_1 : \mathscr{H}_1 \to \mathscr{H}_2$, $T_2 : \mathscr{H}_2 \to \mathscr{H}_2$ closed range operators with orthogonal ranges summing to $\mathscr{H}_2$. Then $T := T_1 \oplus T_2$ is left invertible. The associated operator algebra can be expressed as

$$\mathfrak{A}_T := \overline{\mathrm{Alg}}(T, T^\dagger)$$

where the closure is in the operator norm. **The goal of this paper is to analyze the structure of the operator algebras $\mathfrak{A}_T$.**

If T is an isometry, then its Moore-Penrose inverse $T^\dagger$ is T^*. If T is purely isometric (no unitary summand) with Fredholm index -1, then T is unitarily equivalent to M_z on $H^2(\mathbb{T})$. Hence, $\mathfrak{A}_T$ is the Toeplitz algebra $\mathcal{T}$. This representation is particularly nice, as every operator $A \in \mathcal{T}$ can be uniquely represented as a compact perturbation of a Toeplitz operator with continuous symbol. The purpose of this thesis is to understand the following question:

Question. *To what extent do the elements of $\mathfrak{A}_T$ have the form "compact perturbation of a continuous function"?*

The paper is organized as follows. In the second chapter, we review the background material needed for this thesis. This includes an explicit construction of the Moore-Penrose inverse, and a more detailed analysis of the Toeplitz algebra.

Chapter Three is devoted to operator theoretic properties of left invertible operators, and elementary observations about $\mathfrak{A}_T$. We discover that if the Fredholm index of T is finite, $\mathfrak{A}_T$ has the following description:

Heuristic 1.0.1. *If T has finite Fredholm index, then the operators in $\mathfrak{A}_T$ are compact* *perturbations of Laurent series.*

In its construction, $\mathfrak{A}_T$ is built by replacing instances of "unitary" with "invertible". Hence, this heuristic is intuitive. Therefore our goal is to explore the extent to which this description is true. We justify that in order to make any serious progress understanding the rich structure of $\mathfrak{A}_T$, we need to restrict ourselves to a subclass of left invertible operators, known as analytic operators.

In the fourth chapter, we discuss Cowen-Douglas operators, a class of operators that have rich analytic structure. In that chapter, we connect analyticity of T to the class of Cowen-Douglas operators. Given an open set $\Omega \subset \mathbb{C}$ and a positive integer n, the operators in the Cowen-Douglas class $B_n(\Omega)$ are defined in Definition 4.0.1. We prove the following connection:

Theorem A. *Let $T \in \mathscr{B}(\mathscr{H})$ be left invertible operator with Fredholm index equal to $-n$, for a positive integer $n \in \mathbb{N}$. Then the following are equivalent:*

i. T is analytic

ii. $T^{\dagger^*}$ (the Cauchy Dual of T) is analytic

iii. There exists $\epsilon > 0$ such that $T^* \in B_n(\Omega)$ for $\Omega = \{z : |z| < \epsilon\}$

iv. There exists $\epsilon > 0$ such that $T^\dagger \in B_n(\Omega)$ for $\Omega = \{z : |z| < \epsilon\}$.

This result has several applications. First, it gives an analytic model for representing T in the sense that T is unitarily equivalent to M_z on a reproducing kernel Hilbert space of analytic functions. This further furnishes our description of $\mathfrak{A}_T$ as "compacts plus Laurent series". It also provides us with a decomposition theorem. If T is an isometry, the Wold decomposition lets us decompose T into a direct sum of Fredholm index -1 isometries (and a unitary). A corollary of Theorem A is that we cannot reduce our study to the case where the Fredholm index of T is -1. Rather, $T \sim \oplus T_j$ where each T_j are strongly irreducible operators - operators that are analogous to Jordan blocks in $\mathscr{B}(\mathscr{H})$.

Theorem A also allows us to analyze the isomorphism classes of $\mathfrak{A}_T$ in the case when the Fredholm index of T is -1. In Chapter Five, we focus on the case when the index of T is -1. Here, we determine the conditions for two such algebras to be isomorphic, establishing our main theorem. It gives a rather rigid structure on bounded isomorphisms between the algebras $\mathfrak{A}_T$:

Theorem B. *Let T_i, $i = 1, 2$ be left invertibles (analytic with Fredholm index -1) and $\mathfrak{A}_i = \mathfrak{A}_{T_i}$. Suppose that $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ a bounded isomorphism. Then there exists some invertible $V \in \mathscr{B}(\mathscr{H})$ such that $\phi(A) = VAV^{-1}$ for all $A \in \mathfrak{A}_1$.*

In particular, this theorem shows that all bounded isomorphisms are completely bounded, and reduces the isomorphism problem to a similarity orbit problem. We remark that the problem of finding the similarity orbit of Cowen-Douglas operators is classic. Using the results of Jiang and others on K_0 groups of strongly irreducible operators, we complete the classification in this case. We also analyze the similarity orbit via associated reproducing kernel Hilbert spaces.

In Chapter Five, we investigate a class of illustrative examples arising from the theory of subnormal operators. If S is a subnormal operator, we let $N = mne(S)$ denote the minimal normal extension of S, and $\sigma_{ap}(S)$ denote the approximate point spectrum of S. We show that this class, $\mathfrak{A}_S$ can be described by the heuristic of compact perturbations of Toeplitz operators with Laurent series:

Theorem C. *Let S be an analytic left invertible, Fredholm index -1, essentially normal, subnormal operator with $N := mne(S)$ such that $\sigma(N) = \sigma_{ap}(S)$. Let $\mathscr{B}$ be the uniform* *algebra generated by the functions* z *and* z^{-1} *on* $\sigma_e(S)$*. Then*

$$\mathfrak{A}_S = \{T_f + K : f \in \mathscr{B}, K \in \mathscr{K}(\mathscr{H})\}.$$

Moreover, the representation of each element as $T_f + K$ *is unique.*

Chapter 2

Background

This chapter is dedicated to reviewing the background material necessary for this thesis. We will begin by reviewing some of the basics of functional analysis to establish some notation. We also spend time reviewing the Toeplitz algebra and its associated reproducing kernel Hilbert space $H^2(\mathbb{D})$, as it forms the classic model this theory hopes to generalize. We also cover the basics of subnormal operators and K-theory required for examples and theory used later in the paper.

2.1 Basic Definitions and Notation

In this section, we briefly recall some definitions and establish some notation. Throughout, $\mathscr{H}$ and $\mathscr{K}$ will denote Hilbert spaces over the complex numbers. We denote the collection of all bounded operators over $\mathscr{H}$ by $\mathscr{B}(\mathscr{H})$.

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be a *compact operator* if the image of the unit ball under T has compact closure in $\mathscr{H}$. We let $\mathscr{K}(\mathscr{H})$ denote the set of all compact operators over $\mathscr{H}$. It is well known that $\mathscr{K}(\mathscr{H})$ is the closure of the finite rank operators in the operator norm. The compact operators are a norm-closed two sided ideal of $\mathscr{B}(\mathscr{H})$. Hence, one can form the quotient algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ known as the *Calkin algebra*.

Recall that for $T \in \mathscr{B}(\mathscr{H})$, the *spectrum of* T is defined as

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\}.$$

The point spectrum of T is the collection of eigenvalues of T, denoted $\sigma_p(T)$. The *approximate point spectrum* consists of the collection of approximate eigenvalues:

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \text{ there exists } x_n \in \mathscr{H}, \|x_n\| \leq 1 \text{ such that } \|(T - \lambda)x_n\| \to 0\}.$$

If π is the canonical map from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, the *essential spectrum* is

$$\sigma_e(T) := \sigma(\pi(T)).$$

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *Fredholm* if

1. T has closed range
2. $\dim \ker(T) < \infty$
3. $\dim(\mathscr{H}/\operatorname{ran}(T)) = \dim(\ker(T^*)) < \infty$.

The well known theorem of Atkinson classifies Fredholm operators via invertibility in the Calkin algebra. Namely, $T \in \mathscr{B}(\mathscr{H})$ is Fredholm if and only if $\pi(T)$ is invertible. The *Fredholm domain of* T is

$$\rho_F(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is Fredholm}\} = \mathbb{C} \setminus \sigma_e(T).$$

For each $\lambda \in \rho_F(T)$, the function

$$\operatorname{ind}(T - \lambda) := \dim(\ker(T)) - \dim(\ker(T^*))$$

is a well defined integer, called the *index*. It is well known that the index is constant on each component of $\rho_F(T)$.

2.2 The Toeplitz Algebra

If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for $\mathscr{H}$, the *unilateral shift on* $\{e_n\}_{n\geq 0}$ is the bounded linear operator S defined by $Se_n = e_{n+1}$. The unilateral shift is isometric ($S^*S = 1$) and is Fredholm with $\mathrm{ind}(S) = -1$. In this section, we review a particularly nice representation of $C^*(S)$, the C*-algebra generated by the unilateral shift. This representation forms the model our own analysis of $\mathfrak{A}_T$.

Let $L^2(\mathbb{T})$ denote the square integrable functions over $\mathbb{T}$ with respect to the normalized Lebesgue measure. It is well known that the functions $e_n(z) = z^n$ for $n \in \mathbb{Z}$ form an orthonormal basis for $L^2(\mathbb{T})$. The *Hardy space* $H^2(\mathbb{T})$ is the subspace functions defined by $H^2(\mathbb{T}) = \overline{\mathrm{span}}\{e_n : n \geq 0\}$. We let $H^\infty(\mathbb{T})$ denote the space $H^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$ equipped with the norm coming from $L^\infty(\mathbb{T})$.

If $f \in L^\infty(\mathbb{T})$, is a bounded measurable function on $\mathbb{T}$, define the multiplication operator $M_f \in \mathscr{B}(L^2(\mathbb{T}))$ via $M_f(g) = fg$ for each $g \in L^2(\mathbb{T})$. Let P denote the projection of $L^2(\mathbb{T})$ onto the closed subspace $H^2(\mathbb{T})$. Then the *Toeplitz operator* T_f is the bounded operator on $H^2(\mathbb{T})$ defined by

$$T_f := PM_f \mid_{H^2(\mathbb{T})} .$$

The function f is called the *symbol* of the Toeplitz operator T_f. The following result concerning the norm of Toeplitz operators is well known:

Proposition 2.2.1 ([17] Prop. V.1.1.)**.** *If* $f \in L^\infty(\mathbb{T})$*, then* $\|T_g\| = \|g\|_\infty$.

The *Toeplitz algebra* $\mathcal{T}$ is the C*-algebra generated by all the Toeplitz operators T_f with continuous symbols $f \in C(\mathbb{T})$:

$$\mathcal{T} = C^*(\{T_f : f \in C(\mathbb{T})\}).$$

Using Proposition 2.2.1 and a Stone-Weierstrass argument, one finds that $\mathcal{T}$ is generated by T_p where p is a trigonometric polynomial. It then follows that $\mathcal{T}$ is the C*-algebra generated by T_z. Now, it is easy to see that the operator T_z is unitarily equivalent to the unilateral shift S. Therefore, the C*-algebra $C^*(S)$ is unitarily equivalent to $\mathcal{T}$. However, more can be said. We begin with a definition:

Definition 2.2.2. Let $\mathfrak{A}$ be an operator algebra. Given $a, b \in \mathfrak{A}$, the **commutator of** a **and** b is the element $ab - ba$. The **commutator ideal** $\mathscr{C}$ of $\mathfrak{A}$ is the two sided ideal generated by the commutators of $\mathfrak{A}$. In other words, the commutator ideal is the smallest ideal of $\mathfrak{A}$ such that $\mathfrak{A}/\mathscr{C}$ is commutative.

The commutator ideal plays a central role in understanding the structure of $\mathcal{T}$. The following results provide us with the principal characterization of $\mathcal{T}$.

Proposition 2.2.3 ([18], Prop. 7.4 [17] Cor. V.1.4)**.** *If $T_f, T_g \in \mathcal{T}$, then $T_fT_g - T_gT_f$ is compact. That is, the commutator ideal $\mathscr{C}$ of $\mathcal{T}$ is $\mathscr{K}(H^2(\mathbb{T}))$. Moreover, the semi-commutators $T_fT_g - T_{fg}$ are compact.*

Theorem 2.2.4 ([17] Thm. V.1.5.)**.** *Each element of $\mathcal{T}$ can be written uniquely as a Toeplitz operator plus compact. Namely,*

$$\mathcal{T} = \{T_f + K : f \in C(\mathbb{T}), K \in \mathscr{K}(H^2(\mathbb{T}))\}$$

and if $A \in \mathcal{T}$, then $A = T_f + K$ for exactly one $f \in C(\mathbb{T})$ and $K \in \mathscr{K}(H^2(\mathbb{T}))$. Furthermore, $\mathcal{T}$ is irreducible, $\mathscr{K}(H^2(\mathbb{T}))$ is the unique minimal ideal of $\mathcal{T}$, and we have the following exact *sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{K}(H^2(\mathbb{T})) \arrow[r, "\iota"] & \mathcal{T} \arrow[r, "\pi"] & C(\mathbb{T}) \arrow[r] & 0 \end{tikzcd}$$

2.3 Reproducing Kernel Hilbert Spaces

In this section, we provide a brief primer on reproducing kernel Hilbert spaces. We begin with a definition:

Definition 2.3.1. Given a set X, we say that $\mathscr{H}$ is a **reproducing kernel Hilbert space (RKHS) over** X if

i. $\mathscr{H}$ is a Hilbert space of functions over X

ii. For every $x \in X$, the linear maps $E_x : \mathscr{H} \to \mathbb{C}$ defined by

$$E_x(f) = f(x)$$

are bounded.

Unless otherwise stated, throughout the remainder of this subsection, $\mathscr{H}$ will denote a reproducing kernel space over some set X. As $\mathscr{H}$ is a Hilbert space, every bounded linear functional is arises as the inner product with a unique element in $\mathscr{H}$. In particular, for every $f \in \mathscr{H}$, and $x \in X$, there exists a unique $k_x \in \mathscr{H}$ such that

$$f(x) = E_x(f) = \langle f, k_x \rangle.$$

Definition 2.3.2. The function k_x is called the **reproducing kernel for the point x**.

The function $K : X \times X \to \mathbb{C}$ defined by

$$K(x, y) := k_y(x) = \langle k_y, k_x \rangle$$

is called the **reproducing kernel for $\mathscr{H}$**.

As the name suggests, the reproducing kernel is a classifying feature of reproducing kernel Hilbert spaces. We have the following result:

Theorem 2.3.3 ([2])**.** *Let $\mathscr{H}$ be a RKHS with kernel K. Then*

$$\mathscr{H} = \overline{\operatorname{span}}\{k_x : x \in X\}$$

If $\mathscr{H}'$ is another RKHS with kernel K' such that $K(x, y) = K'(x, y)$ for all $x, y \in X$, then $\mathscr{H} = \mathscr{H}'$ and $\|f\|_{\mathscr{H}} = \|f\|_{\mathscr{H}'}$.

As reproducing kernel Hilbert spaces are classified by their kernels, and the span of the reproducing kernels are dense in $\mathscr{H}$, a natural question that arises is the following: Which functions $K : X \times X \to \mathbb{C}$ are reproducing kernels for some Hilbert space? This brings us to the following definition:

Definition 2.3.4. A function $K : X \times X \to \mathbb{C}$ is said to be a **kernel function** if for every finite subset $\{x_1, \dots x_n\} \subset X$, the matrix $(K(x_i, x_j))_{i,j=1}^n$ is a positive matrix. That is,

$$\langle (K(x_i, x_j))y, y \rangle > 0$$

for each $y \in \mathbb{C}^n$.

It is easy to see that given a RKHS $\mathscr{H}$ with reproducing kernel K, that K is a kernel function. A well known theorem of Moore gives us the converse:

Theorem 2.3.5 ([2]). *Let $K : X \times X \to \mathbb{C}$ be a kernel function. Then there exists a RKHS $\mathscr{H}$ of functions over X such that K is the kernel of $\mathscr{H}$. Indeed, if for each $x, y \in X$, we define $k_y(x) := K(x, y)$ and*

$$\mathscr{H}_0 = \operatorname{span}\{k_y : y \in X\}$$

then $\mathscr{H}$ is the closure of the $\mathscr{H}_0$ with respect to the norm induced by the inner product

$$\langle k_x, k_y \rangle := K(y, x).$$

An important example of a RKHS comes from the Hardy space $H^2(\mathbb{T})$. We let $H^2(\mathbb{D})$ denote the space of holomorphic functions on $\mathbb{D}$ that satisfy the following growth condition:

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right)^{\frac{1}{2}} < \infty.$$

The above quantity defines a norm. The norm on $H^2(\mathbb{D})$ is induced by the following inner product.

$$\langle f, g \rangle = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} \, d\theta.$$

If $f \in H^2(\mathbb{D})$, then it is holomorphic. Hence, f has a Taylor series expansion $f(z) = \sum_{n \geq 0} a_n z^n$ for each $z \in \mathbb{D}$. Therefore,

$$\|f\|^2_{H^2(\mathbb{D})} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n \geq 0} a_n r^n e^{in\theta} \right) \left(\sum_{m \geq 0} \overline{a_m} r^m e^{-im\theta} \right) \, d\theta = \sum_n |a_n|^2.$$

If $f(z) = \sum_{n \geq 0} a_n z^n$ and $g(z) = \sum_{n \geq 0} b_n z^n$, then then inner product of may be more simply computed as

$$\langle f, g \rangle = \sum a_n \overline{b_m}.$$

It is well known that there is an isometric isomorphism of $H^2(\mathbb{T})$ with $H^2(\mathbb{D})$ via the *Cauchy*

Transform [32]. Namely, if $f \in H^2(\mathbb{T})$, then the function $\hat{f}$ defined by

$$\hat{f}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - ze^{-it}} \, dt$$

is in $H^2(\mathbb{D})$.

To see that $H^2(\mathbb{D})$ is a RKHS, first note that it is a Hilbert space of (analytic) functions over the set $X = \mathbb{D}$. Furthermore, one has

$$\begin{aligned} |E_z(f)| = |f(z)| &= \left|\textstyle\sum_{n\geq 0} a_n z^n\right| \leq \textstyle\sum_{n\geq 0} |a_n||z^n| \\ &\leq \left(\textstyle\sum_{n\geq 0} |a_n|^2\right)^{\frac{1}{2}} \left(\textstyle\sum_{n\geq 0} |z^n|^2\right)^{\frac{1}{2}} \\ &= \|f\|_{H^2(\mathbb{T})} \tfrac{1}{\sqrt{1-|z|^2}}. \end{aligned}$$

The reproducing kernel at $w \in \mathbb{D}$ is given by

$$k_w(z) = \sum_{n\geq 0} \overline{w}^n z^n.$$

Hence, the reproducing kernel for $H^2(\mathbb{D})$ is

$$K(z, w) = k_w(z) = \sum \overline{w}^n z^n = \frac{1}{1 - \overline{w}z}.$$

Of common interest are the linear operators on a reproducing kernel Hilbert space over a set X. If $f : X \to \mathbb{C}$, then one can perform pointwise multiplication of f by any function g in the reproducing kernel Hilbert space. Naturally, one would want to understand when this type of operator is a bounded operator. This leads us to the following definition.

Definition 2.3.6. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be reproducing kernel Hilbert spaces over a set X. A function $f : X \to \mathbb{C}$ is called a **multiplier of $\mathscr{H}_1$ into $\mathscr{H}_2$** if $f\mathscr{H}_1 \subset \mathscr{H}_2$. We denote the set of multipliers by $\mathcal{M}(\mathscr{H}_1, \mathscr{H}_2)$. If $\mathscr{H}_1 = \mathscr{H}_2$, we write $\mathcal{M}(\mathscr{H}_1) := \mathcal{M}(\mathscr{H}_1, \mathscr{H}_1)$.

Given a multiplier $f \in \mathcal{M}(\mathscr{H}_1, \mathscr{H}_2)$, we let $M_f : \mathscr{H}_1 \to \mathscr{H}_2$ denote the linear map given by $M_f(g) = fg$. One has the following result relating the boundedness of M_f and the kernels K_i of $\mathscr{H}_i$.

Proposition 2.3.7 ([32])**.** *Let $\mathscr{H}_i$ be RKHS over X with kernels K_i, $i = 1, 2$. Let $f : X \to \mathbb{C}$. Then the following are equivalent:*

i. $f \in \mathcal{M}(\mathscr{H}_1, \mathscr{H}_2)$

ii. $M_f \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$

iii. there exists a $c > 0$ *such that* $f(x)K_1(x, y)\overline{f(y)} \leq cK_2(x, y)$ *for each* $x, y \in X$.

Furthermore, the least constant c in iii. above is $\|M_f\|^2$.

2.4 Subnormal Operators

In this section, we discuss some of the basics of subnormal operators. We begin with a definition:

Definition 2.4.1. An operator $S \in \mathscr{B}(\mathscr{H})$ is called **subnormal** if there exists a Hilbert space $\mathscr{K}$ such that $\mathscr{K} \supset \mathscr{H}$ and a normal operator $N \in \mathscr{B}(\mathscr{K})$ such that

i. $N\mathscr{H} \subset \mathscr{H}$

ii. $S = N \mid_{\mathscr{H}}$

Such a normal operator N is called a **normal extension** of S. The operator N is said to be a **minimal normal extension** if $\mathscr{K}$ has no proper subspace reducing N and containing $\mathscr{H}$.

It can be shown that any two minimal normal extensions of a subnormal operator S are unitarily equivalent [12]. Thus, we usually refer to the minimal normal extension, and denote it by $N := mne(S)$.

Classic examples of a subnormal operators are the Toeplitz operators T_f on $H^2(\mathbb{T})$ for $f \in L^\infty(\mathbb{T})$. The minimal normal extension is given by M_f on $L^2(\mathbb{T})$ (for f non-constant). It is not hard to see that all subnormal operators have this form. We make the following definition:

Definition 2.4.2. Let $S \in \mathscr{B}(\mathscr{H})$ be a subnormal operator, and $N = mne(S) \in \mathscr{B}(\mathscr{K})$. If μ is a scalar-valued spectral measure associated to N, and $f \in L^\infty(\sigma(N), \mu)$, we define the **Toeplitz operator** $T_f \in \mathscr{B}(\mathscr{H})$ via

$$T_f := P(f(N)) \mid_{\mathscr{H}}$$

where P is the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$.

In the case when S is the unilateral shift, the above are the Toeplitz operators on $H^2(\mathbb{T})$. For any subnormal operator S, we have that $T_z = S$, and that $T_{\overline{z}^n} T_{z^m} = T_{\overline{z}^n z^m}$. Consequently, $\{T_f : f \in C(\sigma(N))\} \subset C^*(S)$. We remark that, while the map from $L^\infty(\sigma(N), \mu)$ to $\mathscr{B}(\mathscr{H})$ via $f \mapsto T_f$ is positive and norm decreasing, it is not multiplicative.

Ultimately, we are interested in algebras of operators generated by left invertible operators. Salient examples will arise from the subnormal operators, due in large part to their rich spectral theory. The following is the first useful result in that direction.

Proposition 2.4.3 ([12]). *Let S be a subnormal operator with $N = mne(S)$. Then the following inclusions hold:*

$$\partial\sigma(S) \subseteq \sigma_{ap}(S) \subseteq \sigma_{ap}(N) = \sigma(N) \subseteq \sigma(S)$$

where $\sigma_{ap}(S)$ *is the approximate point spectrum of* S*.*

Next we highlight some C*-algebraic results about subnormal operators due to Olin, Thomson, Keough and McGuire. If N is a normal operator, there is a natural identification of $C^*(N)$ with $C(\sigma(N))$ given by the Gelfand transform. There is also an intimate connection between the C*-algebra generated by a subnormal operator S and its minimal normal extension N.

When S is the unilateral shift, its minimal normal extension N is a unitary. The commutative C*-algebra $C^*(N) \cong C^*(\sigma(N)) \cong C(\mathbb{T})$ appears in the symbols of the Toeplitz operators. Being a subnormal operator, by definition S dilates to a normal operator. The unilateral shift also has the additional property the image of S in the Calkin algebra is normal (in fact, unitary). This later property is known as essentially normal.

Definition 2.4.4. An operator $S \in \mathscr{B}(\mathscr{H})$ is called **essentially normal** if its image $\pi(S)$ is normal in the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$.

In summary, three key properties that the unilateral shift possesses are irreducibility, sub-normality and essential normality. If S is any operator with these three properties, one obtains a construction similar to the Toeplitz algebra. It is helpful to view the following theorem with Proposition 2.4.3 in mind.

Theorem 2.4.5 ([27] [29] [31])**.** *If* S *is an irreducible, subnormal, essentially normal operator, then*

i. $\sigma_{ap}(S) = \sigma_e(S)$

ii. For each $f, g \in C(\sigma(N))$*, we have*

  a. $T_f \in \mathscr{K}(\mathscr{H})$ *if and only if* f *vanishes on* $\sigma_e(S)$

  b. $\|T_f + \mathscr{K}(\mathscr{H})\| = \|f\|_{\sigma_e(S)}$

c. $T_{fg} - T_f T_g \in \mathscr{K}(\mathscr{H})$

d. $\sigma_e(T_f) = f(\sigma_e(S))$

iii. Every element of $C^*(S)$ *can be written as a sum of a Toeplitz operator and compact:*

$$C^*(S) = \{T_f + K : f \in C(\sigma(N)), K \in \mathscr{K}(\mathscr{H})\}.$$

Moreover, if $\sigma(N) = \sigma_{ap}(S)$*, then each element has* $A \in C^*(S)$ *has a unique representation of the form* $T_f + K$*. If* $\sigma(N) \neq \sigma_{ap}(S)$*,* A *may be expressed as* $A = T_{f_1} + K_1 = T_{f_2} + K_2$*, where* $f_1 \mid_{\sigma_e(S)} = f_2 \mid_{\sigma_e(S)}$*.*

2.5 K-Theory

To any (not necessarily self-adjoint) operator algebra $\mathfrak{A}$, one can define groups $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ that encode homological and non-commutative topological aspects of $\mathfrak{A}$. If $\mathfrak{A} \cong \mathscr{B}$, then the K-groups are naturally isomorphic. Miraculously, the K-theory of several classes of C*-algebras are a complete invariant [38].

K-theory will play an interesting role in the classification of the algebras $\mathfrak{A}_T$ for certain classes of T. Specifically, the K_0 group of a certain commutative algebra will be key in our analysis. Throughout this section, $\mathfrak{A}$ will denote a unital operator algebra. In this subsection, we review the definition of the K_0 group. We begin by defining some relationships on idempotents of the algebra $\mathfrak{A}$.

Definition 2.5.1. Two idempotents $p, q \in \mathfrak{A}$ are **algebraically equivalent** if there exists $x, y \in \mathfrak{A}$ such that $xy = p$ and $yx = q$. The two idempotents are **similar** if there exists an invertible $u \in \mathfrak{A}$ such that $up = qu$. Lastly, they are **homotopic** if there exists a continuous path of idempotents starting at p and ending at q.

THe definitions in Definition 2.5.1 are equivalence relations on the set of idempotents of $\mathfrak{A}$. Each relation encodes algebraic and topological information within them. These relations are not the same in general - however they “stably equivalent” in the following sense.

Let $M_n(\mathfrak{A}) = M_n \otimes \mathfrak{A}$, the algebra of $n \times n$ matrices with entries in $\mathfrak{A}$. If $a \in M_n(\mathfrak{A})$ and $b \in M_m(\mathfrak{A})$, define

$$\operatorname{diag}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+m}(\mathfrak{A}).$$

The algebra $M_n(\mathfrak{A})$ embeds into $M_{n+1}(\mathfrak{A})$ via $a \mapsto \operatorname{diag}(a, 0)$. Using this inclusion, we define $M_\infty(\mathfrak{A})$ as the inductive limit of the $\{M_n(\mathfrak{A})\}_{n\geq 1}$. We then have the following lemma.

Lemma 2.5.2 ([39])**.** *In $M_\infty(\mathfrak{A})$, algebraic, similarity, and homotopy equivalence coincide.*

Let $\sim$ denote any of the identical equivalence relations in $M_\infty(\mathfrak{A})$. We define

$$V(\mathfrak{A}) := \{p \in M_\infty(\mathfrak{A}) : p = p^2\}/\sim .$$

If $[p], [q] \in V(\mathfrak{A})$, we can always find $p' \in [p]$ and $q' \in [q]$ such that p is orthogonal to q in the sense that $pq = qp = 0$. This allows one to build a well-defined binary operation $+$ on $V(\mathfrak{A})$ via $[p] + [q] := [p' + q']$. This operation turns $V(\mathfrak{A})$ into an abelian semigroup with a zero element $[0]$.

It follows by definition that $V(\cdot)$ is functorial. Namely, if $\phi : A \to B$ is a homomorphism, then ϕ induces a map $\phi_* : V(\mathfrak{A}) \to V(\mathscr{B})$. Further, $V(\mathfrak{A} \oplus \mathscr{B}) \cong V(\mathfrak{A}) \oplus V(\mathscr{B})$ and is continuous with respect to direct limits.

As $V(\mathfrak{A})$ is an abelian semigroup, it can be made into a group via the Grothendieck construction, which we describe here. Let $\mathcal{S}$ be a semigroup. Consider pairs $(m_1, m_2) \in \mathcal{S} \times \mathcal{S}$ representing formal differences $m_1 - m_2$. Define addition on $\mathcal{S} \times \mathcal{S}$ coordinate-wise: $(m_1, m_2) + (n_1, n_2) = (m_1 + n_1, m_2 + n_2)$. We then define an equivalence relation on $(\mathcal{S} \times \mathcal{S}, +)$

via $(m_1, m_2) \sim (n_1, n_2)$ if there exists some element $d \in \mathcal{S}$ such that $m_1+n_2+d = m_2+n_1+d$.

Definition 2.5.3. If $\mathfrak{A}$ is a unital operator algebra, then $K_0(\mathfrak{A})$ is the Grothendieck group of $V(\mathfrak{A})$.

Chapter 3

Properties of Left Invertible Operators and $\mathfrak{A}_T$

The focus of this chapter is elementary properties of left invertible operators and the algebra $\mathfrak{A}_T$. We will begin by discussing the Moore-Penrose inverse of a closed range operator formally, and then move on to prove some basic facts about left invertible operators. In order to make meaningful headway, we impose a Fredholm condition on our left invertibles. We then discover some coarse properties of the algebra $\mathfrak{A}_T$, noting that a dense set may be written as finite rank operator plus polynomials in T and $T^\dagger$. This initiates our description of $\mathfrak{A}_T$ as compact perturbations of Laurent series. Drawing on analogies with isometric operators, we describe a non-degeneracy condition of left invertible operators called analytic. This allows one to build a type of basis on which T acts like a shift operator. We conclude this chapter by demonstrating that one cannot hope to recover a decomposition exactly like the Wold decomposition for left invertible operators.

3.1 Basics of Closed Range and Left Invertible Operators

We begin this chapter by providing a more rigorous definition of the Moore-Penrose inverse, and prove its existence. We then shift our focus towards to, left invertible operators. After proving some equivalent definitions for an operator to be left invertible, we move towards proving general results that will be required throughout the text.

Proposition 3.1.1. *Let* $T \in \mathscr{B}(\mathscr{H})$ *be an operator with* $\operatorname{ran}(T)$ *closed. Then there exists a unique operator* $T^\dagger \in \mathscr{B}(\mathscr{H})$ *such that*

i. $\ker(T^\dagger) = \operatorname{ran}(T)^\perp = \ker(T^*)$

ii. $T^\dagger T x = x$ *for each* $x \in \ker(T)^\perp$*.*

Proof. Consider the operator $\tilde{T} : \ker(T)^\perp \to \operatorname{ran}(T)$ obtained by restricting the domain of T to $\ker(T)^\perp$ and the range of T to $\operatorname{ran}(T)$. Since T has closed range, $\tilde{T}$ is a bijective operator between two Hilbert spaces, and therefore boundedly invertible by the open mapping theorem. Define $T^\dagger \in \mathscr{B}(\mathscr{H})$ via

$$T^\dagger x = \begin{cases} \tilde{T}^{-1} x & x \in \operatorname{ran}(T) \\ 0 & x \in \operatorname{ran}(T)^\perp. \end{cases}$$

By construction, $T^\dagger$ satisfies properties *i.* and *ii.*.

For uniqueness, suppose that L was another such operator. Then for all $x \in \operatorname{ran}(T)^\perp$, $Lx = 0 = T^\dagger x$. Moreover, if $x \in \operatorname{ran}(T)$, $x = Ty$ for some y. Using the second property, we have

$$Lx = LTy = y = T^\dagger T y = T^\dagger x$$

So L agrees with $T^\dagger$ on all of $\mathscr{H}$. □

Definition 3.1.2. The operator $T^\dagger$ that appears in Proposition 3.1.1 is called the **Moore-Penrose Inverse of T**.

The Moore-Penrose inverse behaves like a left inverse for an operator only where it makes sense. The focus of this thesis is on left invertible operators. Our present goal is to show that left invertible operators are a subclass of the closed range operators. To demonstrate this property, as well as other equivalent characterizations of left invertible operators, we will require the following definition.

Definition 3.1.3. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be **bounded below** if there exists a $c > 0$ such that for each $x \in \mathscr{H}$, $\|Tx\| \geq c\|x\|$.

Proposition 3.1.4. *For $T \in \mathscr{B}(\mathscr{H})$, the following are equivalent:*

i. T is left-invertible

ii. T^* is right-invertible

iii. T is bounded below

iv. T is injective and has closed range

v. T^*T is invertible.

Proof. If L is a left inverse of T, then L^* is a right inverse of T^*. The equivalence of *i.* and *ii.* follows immediately. To see that *i.* implies *iii.*, let T be left invertible. Then for each $x \in \mathscr{H}$, we have

$$\|x\| = \|LTx\| \leq \|L\|\|Tx\|.$$

Hence, $\|Tx\| \geq c\|x\|$ where $c = \|L\|^{-1}$.

Next we demonstrate the equivalence of *iii.* and *iv.*. If T is bounded below, then T is certainly injective. To see that T has closed range, suppose that $Tx_n \to y$. Then $\{x_n\}$ form a Cauchy sequence because

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \geq c\|x_n - x_m\|.$$

Since $\{x_n\}$ are Cauchy, they must converge to some $x \in \mathscr{H}$. Hence, $y = Tx$ so the range of T is closed. Conversely, suppose that T is injective and has closed range. Let $\hat{T} : \mathscr{H} \to \operatorname{ran}(T)$

be the restriction of T to the range of T. Then $\hat{T}$ is bijective, and therefore boundedly invertible by the open mapping theorem. Hence, for each $x \in \mathscr{H}$,

$$\|x\| = \|\hat{T}^{-1}\hat{T}x\| \leq \|\hat{T}^{-1}\|\|\hat{T}x\| = \|\hat{T}^{-1}\|\|Tx\|$$

Therefore, $c\|x\| \leq \|Tx\|$ for $c = \|\hat{T}^{-1}\|^{-1}$.

Finally, we show *iii.* implies *v.* implies *i.*. Notice that if T is bounded below, that for each $x \in \mathscr{H}$ we have

$$\langle T^*Tx, x\rangle = \|Tx\|^2 \geq c^2\|x\|^2.$$

It follows from Cauchy-Schwartz that

$$\langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\|.$$

Hence, $\|T^*Tx\| \geq c^2\|x\|$. Therefore, T^*T is a self-adjoint operator that is bounded below. In particular, T^*T is injective, and since $\mathrm{ran}(T^*T)^{\perp} = \ker(T^*T) = 0$, it is also bijective. Hence, T^*T is invertible by the open mapping theorem. To see that *v.* implies *i.*, notice that $[(T^*T)^{-1}T^*]$ is a left inverse of T. □

In the introduction we stated our interest in operator algebras that model dynamics of Hilbert space frames. Recall, a sequence $\{f_n\}$ of points in a Hilbert space $\mathscr{H}$ is a *frame* if there exists constants $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_n |\langle x, f_n\rangle|^2 \leq B\|x\|^2$$

for all $x \in \mathscr{H}$. We have the following result about relating Hilbert space frames and left invertible operators.

Proposition 3.1.5. *If* $\{f_n\}$ *is a frame for* $\mathscr{H}$*, and* $T \in \mathscr{B}(\mathscr{H})$ *is left invertible, then* $\{Tf_n\}$ *is a frame for* $\operatorname{ran}(T)$*.*

Proof. Let $x \in \operatorname{ran}(T)$. The upper bound follows from the fact that T is bounded:

$$\sum_n |\langle x, Tf_n\rangle|^2 = \sum_n |\langle T^*x, f_n\rangle|^2 \leq B\|T^*x\|^2 \leq B\|T\|^2\|x\|^2.$$

By Proposition 3.1.4, if we regard $T \in \mathscr{B}(\mathscr{H}, \operatorname{ran}(T))$, then T is invertible. Consequently, $T^* \in \mathscr{B}(\mathscr{H}, \operatorname{ran}(T))$ is also invertible, and in particular, is left invertible. Again by Proposition 3.1.4, T^* is bounded below by some constant $c > 0$. Hence,

$$\sum_n |\langle x, Tf_n\rangle|^2 = \sum_n |\langle T^*x, f_n\rangle|^2 \geq A\|T^*x\|^2 \geq Ac^2\|x\|^2.$$

□

In the case of left invertible operators, the Moore-Penrose inverse is a left inverse. It is a special left inverse that takes on a particular form as the following propositions demonstrate.

Proposition 3.1.6. *Let* $T \in \mathscr{B}(\mathscr{H})$ *be left invertible. Then* $T^\dagger = (T^*T)^{-1}T^*$*.*

Proof. By Proposition 3.1.4, T^*T is invertible. Let $L = (T^*T)^{-1}T^*$. Clearly L is a left inverse of T, and since $\ker(L) = \ker(T^*)$, it follows from Proposition 3.1.1 that $L = T^\dagger$. □

Proposition 3.1.7. *Given any left invertible* $T \in \mathscr{B}(\mathscr{H})$*, the following hold:*

i. $TT^\dagger$ *is the (orthogonal) projection onto ran*(T)

ii. $I - TT^\dagger$ *is the (orthogonal) projection onto ran*$(T)^\perp$

iii. $\ker(T^\dagger) = \operatorname{ran}(T)^\perp = \ker(T^*)$

iv. $\operatorname{ran}(T^\dagger) = \operatorname{ran}(T^*)$.

Proof. By Proposition 3.1.6, we know $T^\dagger = (T^*T)^{-1}T^*$. So, $TT^\dagger = T(T^*T)^{-1}T^*$. Therefore, $TT^\dagger$ is a self-adjoint idempotent. Also, since $T^\dagger$ is onto, we have $\operatorname{ran}(TT^\dagger) = \operatorname{ran}(T)$, so $TT^\dagger$ is a projection onto $\operatorname{ran}(T)$. The rest follows from previous observations. $\square$

Proposition 3.1.8. *Let $T \in \mathscr{B}(\mathscr{H})$ be left invertible. Then every left inverse is of the form*

$$L = T^\dagger + A(I - TT^\dagger).$$

for some $A \in \mathscr{B}(\mathscr{H})$.

Proof. Let $A \in \mathscr{B}(\mathscr{H})$. Then it follows that $L = T^\dagger + A(I - TT^\dagger)$ is a left inverse of T:

$$LT = T^\dagger T + A(I - TT^\dagger)T = I + A(T - T) = I$$

Conversely, suppose that L is a left inverse of T. Then if $x \in \operatorname{ran}(T)$, $x = Ty$ for some $y \in \mathscr{B}(\mathscr{H})$ so that

$$Lx = LTy = y = T^\dagger Ty = T^\dagger x$$

Hence, L agrees with $T^\dagger$ on $\operatorname{ran}(T)$. It may be the case that L is non-zero on $\operatorname{ran}(T)^\perp$. Let A denote the action of L on $\operatorname{ran}(T)^\perp$. By Proposition 3.1.7, $I - TT^\dagger$ is the projection onto $\operatorname{ran}T^\perp$, so that

$$L = T^\dagger(TT^\dagger) + A(I - TT^\dagger) = T^\dagger + A(I - TT^\dagger)$$

$\square$

Lemma 3.1.9. *Let $T \in \mathscr{B}(\mathscr{H})$ be left invertible. If $S \in \mathscr{B}(\mathscr{H})$ satisfies $\|T - S\| < \|T^\dagger\|^{-1}$, then S is also left invertible. The operator $(T^\dagger S)^{-1}T^\dagger$ is a left inverse of S.*

Proof. Notice that

$$\|T^\dagger S - I\| = \|T^\dagger(S - T)\| \leq \|T^\dagger\|\|S - T\| < 1.$$

Therefore, $T^{\dagger}S$ is invertible. Hence, $(T^{\dagger}S)^{-1}T^{\dagger}$ is a left inverse of S. $\square$

This paper will largely be concerned with the case when $\dim(\operatorname{ran}(T)^{\perp}) < \infty$. This can be viewed as a Fredholm assumption on T, which will make the theory more interesting. Furthermore, our interest is in left invertible operators which are not invertible. We make the following definition:

Definition 3.1.10. An left invertible operator $T \in \mathscr{B}(\mathscr{H})$ is said to be **natural** if the $\dim(\ker(T^*))$ is a natural number. Specifically,

$$0 < \dim(\ker(T^*)) = \dim(\operatorname{ran}(T)^{\perp}) < \infty$$

Note that if T is a natural left invertible, then $\ker(T^*)$ is a positive integer. Hence, T^* is not invertible, so neither is T. Moreover, natural left invertibles are Fredholm:

Proposition 3.1.11. *Let T be a natural left invertible. Then $0 \in \sigma(T)$, and $0 \notin \sigma_e(T)$. Indeed, T is Fredholm with $\operatorname{ind}(T) = -dim(\ker(T^{\dagger})) = -\operatorname{ind}(T^{\dagger})$.*

Proof. Since T is not invertible, $0 \in \sigma(T)$. As $\dim(\operatorname{ran}(T)^{\perp}) < \infty$, and $I - TT^{\dagger}$ is the projection onto the $\operatorname{ran}(T)^{\perp}$, T is invertible in $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$. Therefore, T is Fredholm. Because T is injective, the Fredholm index of T is

$$\operatorname{ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*)) = -\dim(\ker(T^{\dagger})).$$

Note that $(T^{\dagger})^* = T(T^*T)^{-1}$. Hence $(T^{\dagger})^*$ is injective, so that

$$\operatorname{ind}(T^{\dagger}) = \dim(\ker(T^{\dagger})) - \dim(\ker((T^{\dagger})^*)) = \dim(\ker(T^{\dagger})) = -\operatorname{ind}(T).$$

$\square$

Corollary 3.1.12. *If T is a natural left invertible, then all left inverses L of T are finite rank perturbations of $T^\dagger$. Hence, all left inverses L of T are Fredholm with index* $\operatorname{ind}(L) = \mathit{dim}(\ker(T^\dagger)) = \operatorname{ind}(T^\dagger)$.

This type of result makes our study more interesting. Hence, going forward all left invertible operators will assumed to be natural, unless otherwise specified.

Proposition 3.1.13. *If $T, S \in \mathscr{B}(\mathscr{H})$, T is a natural left invertible and $\|T - S\| < \|T^\dagger\|^{-1}$, then S is Fredholm with* $\operatorname{ind}(S) = \operatorname{ind}(T)$.

Proof. Let $\tilde{T} : \mathscr{H} \to \operatorname{ran}(T)$ be the restriction of T. Then $\tilde{T}$ is invertible, with $\|\tilde{T}\| = \|T^\dagger\|$. Therefore, if $A \in \mathscr{B}(\mathscr{H}, \operatorname{ran}(T))$ with

$$\|A - \tilde{T}\| < \|\tilde{T}\|^{-1} = \|T^\dagger\|^{-1}$$

then A is invertible as well.

By assumption, T has closed range. So, $\mathscr{H} = \operatorname{ran}(T) \oplus \operatorname{ran}(T)^\perp$. Write $S = S_1 + S_2$ where $S_1 = P_{\operatorname{ran}(T)} S$ and $S_2 = P_{\operatorname{ran}(T)^\perp} S$. Then,

$$\|S_1 - \tilde{T}\| = \|P_{\operatorname{ran}(T)}(S - T)\| < \|T^\dagger\|^{-1}.$$

Hence, S_1 is invertible. Moreover since $\dim(\operatorname{ran}(T)^\perp)$ is finite, $S_2 \in \mathscr{K}(\mathscr{H})$. Therefore, S is a compact perturbation of an invertible operator, and thus Fredholm.

By Lemma 3.1.9, S is left invertible with left inverse $L = (T^\dagger S)^{-1} T^\dagger$. By Proposition 3.1.8, $S^\dagger = K + L$ for some compact $K \in \mathscr{K}(\mathscr{H})$. Therefore,

$$\operatorname{ind}(S^\dagger) = \operatorname{ind}((T^\dagger S)^{-1} T^\dagger) = \operatorname{ind}((T^\dagger S)^{-1}) = \operatorname{ind}(T^\dagger) = \operatorname{ind}(T^\dagger)$$

By Proposition 3.1.11, $\operatorname{ind}(S) = \operatorname{ind}(T)$. $\square$

We will always use $\mathscr{E}_T := \operatorname{ran}(T)^\perp$. If T is understood, we simply write $\mathscr{E}$. That is,

$$\mathscr{E} := \operatorname{ran}(T)^\perp = \ker(T^\dagger) = \ker(T^*).$$

For isometric operators, $T^n\mathscr{E} \perp T^m\mathscr{E}$ for all $n \neq m$. This is not true for general left invertible operators, even though $\mathscr{E}$ is perpendicular to the range of T. However, it is true that $\ker((T^\dagger)^n) = \bigvee_{k=0}^{n-1} T^k\mathscr{E}$:

Proposition 3.1.14. *Let T be a natural left invertible, and $P = I - TT^\dagger$ be the projection onto $\mathscr{E}$. Then for each $n \geq 1$, we have*

$$I - T^n {T^\dagger}^n = \sum_{k=0}^{n-1} T^k P {T^\dagger}^k. \tag{3.1}$$

Consequently,

$$\ker((T^\dagger)^n) = \bigvee_{k=0}^{n-1} T^k\mathscr{E}.$$

Proof. By a telescopic sum, $I - T^n {T^\dagger}^n = \sum_{k=0}^{n-1} T^k P {T^\dagger}^k$. To prove the set equality, suppose $x \in \bigvee_{k=0}^{n-1} T^k\mathscr{E}$. Then it follows immediately that ${T^\dagger}^n x = 0$. On the other hand, if $x \in \ker((T^\dagger)^n)$, then by Equation (3.1),

$$x = (I - T^n {T^\dagger}^n)x = \sum_{k=0}^{n-1} T^k P {T^\dagger}^k x.$$

Since $P{T^\dagger}^k x \in \mathscr{E}$ for all k, it follows that $x \in \bigvee_{k=0}^{n-1} T^k\mathscr{E}$. □

3.2 Basic Properties of $\mathfrak{A}_T$

In this section, we begin to analyze the basics of the algebra $\mathfrak{A}_T$. We note two ways in which left invertible operators are close to invertible. In Section 2.4, we remarked how the shift S

is almost unitary. If S is an isometry, it dilates to a unitary. Moreover, $\pi(S)$ is a unitary in $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$. Similar statements are true for general left invertibles. This is done first by taking a particular quotient of $\mathfrak{A}_T$, and then by looking at a dilation. This allows us to describe the algebra $\mathfrak{A}_T$ as "Laurent series plus compacts".

Throughout, let $\mathscr{C}$ denote the commutator ideal of $\mathfrak{A}_T$. We make the following trivial but important observation.

Lemma 3.2.1. *The projection* $P = I - TT^\dagger = T^\dagger T - TT^\dagger \in \mathscr{C}$

We prove that when the dimension of $\ker(T^*)$ is finite, $\mathscr{C} \subset \mathscr{K}(\mathscr{H})$. We then show that $\mathfrak{A}_T/\mathscr{C}$ consists of formal Laurent polynomials, namely polynomials in z and z^{-1}. Moreover T may also be dilated to an invertible, allowing us to identify $\mathfrak{A}_T$ as the corner of the algebra generated by this invertible. Combining these results allows one to heuristically describe $\mathfrak{A}_T$ as sums of compact operators and Laurent series. We begin this section with a simple observation that will be used throughout the paper:

Lemma 3.2.2. *Let T be a left invertible operator. Then* $\mathfrak{A}_T \subset C^*(T)$.

Proof. Since $T^\dagger = (T^*T)^{-1}T^*$, $T^\dagger \in C^*(T)$. □

This paper is concerned with the case when $\dim(\mathscr{E}) < \infty$, the natural left invertible operators. In particular, we will have much to say when the Fredholm index of T is -1. We have the following result about the commutator ideal of $\mathfrak{A}_T$.

Lemma 3.2.3. *Let T be left invertible. If T is natural, then* $\mathscr{C} \subset \mathscr{K}(\mathscr{H})$.

Proof. Let $X = T^n {T^\dagger}^m$ and $Y = T^k {T^\dagger}^l$. If we can show that $XY - YX$ is finite rank, then it will follow from taking linear combinations and limits that $\mathscr{C} \subset \mathscr{K}(\mathscr{H})$. To this end, notice that

$$XY - YX = T^n {T^\dagger}^m T^k {T^\dagger}^l - T^k {T^\dagger}^l T^n {T^\dagger}^m$$

Now if $m \leq k$, $T^n {T^\dagger}^m T^k {T^\dagger}^l = T^{n+k-m} {T^\dagger}^l$. On the other hand, if $m \geq k$, then $T^n {T^\dagger}^m T^k {T^\dagger}^l = T^n {T^\dagger}^{l+m-k}$. Likewise, $T^k {T^\dagger}^l T^n {T^\dagger}^m = T^{n+k-l} {T^\dagger}^m$ if $l \leq n$ and $T^k {T^\dagger}^{l+m-n}$ otherwise. Therefore, the expression $T^n {T^\dagger}^m T^k {T^\dagger}^l - T^k {T^\dagger}^l T^n {T^\dagger}^m$ can be simplified depending on the values of n, m, k and l. This leaves us with eight total cases to check. For example, two cases arise from $m \geq k$ and $l \geq n$. By above, if $m \geq k$ and $l \geq n$, then

$$T^n {T^\dagger}^m T^k {T^\dagger}^l - T^k {T^\dagger}^l T^n {T^\dagger}^m = T^n {T^\dagger}^{l+m-k} - T^k {T^\dagger}^{l+m-n}.$$

This leaves us with two sub-cases: either $n \leq k$ or $k \leq n$. If $n \leq k$, we have

$$T^n {T^\dagger}^{l+m-k} - T^k {T^\dagger}^{l+m-n} = T^n (I - T^{k-n} {T^\dagger}^{k-n}) {T^\dagger}^{l+m-k}.$$

By Proposition 3.1.14, $I - T^{k-n} {T^\dagger}^{k-n}$ is a sum of finite rank operators, and thus, $T^n {T^\dagger}^m T^k {T^\dagger}^l - T^k {T^\dagger}^l T^n {T^\dagger}^m$ is finite rank. The case when $k \leq n$ is the same. The other six cases are similar. ☐

We now investigate the quotient of $\mathfrak{A}_T$ by the commutator ideal $\mathscr{C}$. Let π denote the canonical map $\pi : \mathfrak{A}_T \to \mathfrak{A}_T/\mathscr{C}$. As $P = I - TT^\dagger$ is in $\mathscr{C}$, it follows that $\pi(T)$ is invertible with inverse $\pi(T^\dagger)$. Hence, $\mathfrak{A}_T/\mathscr{C}$ is a commutative Banach algebra (in fact, operator algebra [3]) generated by the invertible $\pi(T)$ and its inverse $\pi(T^\dagger)$. We have the following:

Lemma 3.2.4. *Let $\mathfrak{A}$ be a commutative unital Banach algebra generated by an invertible a and its inverse a^{-1}. Then the character space $\Omega(\mathfrak{A})$ is homeomorphic to $\sigma(a)$.*

Proof. Let $\phi \in \Omega(\mathfrak{A})$. Since $a - \phi(a) \in \ker(\phi)$, a proper ideal of $\mathfrak{A}$, we must have that $a - \phi(a)$ is not invertible. Hence, $\phi(a) \in \sigma(a)$. Let $\theta : \Omega(\mathfrak{A}) \to \sigma(a)$ via $\theta(\phi) = \phi(a)$.

By definition, θ is continuous. Moreover, θ is injective. Indeed, $\theta(\phi) = \theta(\psi)$ if and only if $\phi(a) = \psi(a)$. This happens if and only if ϕ and ψ agree on the dense subset $\sum_{-N}^{M} \alpha_n a^n$.

By continuity, ψ agrees with ϕ on all of $\mathfrak{A}$.

Next, we show that θ is onto. If $\lambda \in \sigma(a)$, then $a - \lambda$ is not invertible. Thus, there exists some maximal ideal $\mathscr{J}$ of $\mathfrak{A}$ that contains $a - \lambda$. Let $\phi \in \Omega(\mathfrak{A})$ be the character that corresponds to $\mathscr{J}$. Then $\phi(a - \lambda) = \phi(a) - \lambda = 0$, so that $\phi(a) = \lambda$. As θ is a bijective continuous map between compact Hausdorff spaces, θ is a homeomorphism. $\square$

By the previous lemma, the Gelfand map provides a norm decreasing homomorphism of

$$\Gamma : \mathfrak{A}_T/\mathscr{C} \to C(\sigma(\pi(T))).$$

For each $\lambda \in \sigma(\pi(T))$, let $z : \sigma(\pi(T)) \hookrightarrow \mathbb{C}$ represent the inclusion function. Namely, $z(\lambda) = \lambda$ for all $\lambda \in \sigma(\pi(T))$. Then z is invertible by construction, with inverse $z^{-1}(\lambda) := \lambda^{-1}$ for all $\lambda \in \sigma(\pi(T))$. Under the Gelfand identification, $\pi(T) \mapsto z$ and $\pi(T^\dagger) \mapsto z^{-1}$ on $\sigma(\pi(T))$. Consequently, z and z^{-1} generate the image of $\mathfrak{A}_T/\mathscr{C}$ under Γ. In this sense, $\mathfrak{A}_T/\mathscr{C}$ consists of Laurent polynomials centered at zero.

A few comments are necessary at this point. First, the Gelfand map need not have closed range, and thus, $\Gamma(\mathfrak{A}_T/\mathscr{C})$ may not be complete. Moreover, Γ may not even be injective in general. If $\mathfrak{A}$ is a commutative Banach algebra, and $a \in \mathfrak{A}$ has $\sigma(a) = 0$, then $\Gamma(a) = 0$. However, since $\mathfrak{A}_T/\mathscr{C}$ is generated by $\pi(T)$ and $\pi(T^\dagger) = \pi(T)^{-1}$, it follows that z (and therefore z^{-1}) are non-zero. As Γ is norm decreasing, we do have that every function in the range of Γ is a Laurent series in z and z^{-1}.

It will be shown in Section 5.1 that when the Fredholm index of T is -1, $\mathscr{C} = \mathscr{K}(\mathscr{H})$. In some cases, this furnishes a rather detailed analysis of the quotient. In particular, the case of essentially normal subnormal operators will be studied in Section 5.4. However, presently we will concern ourselves with an algebraic characterization of the commutator ideal. To do this, we will first get a description of the algebra generated by T and $T^\dagger$ pre-closure.

We just analyzed how quotienting by the commutator ideal results in T becoming invertible. As a consequence, "Laurent polynomials" in z and z^{-1} over $\sigma_e(T)$ are dense in the quotient. Next, we observe that if $T \in \mathscr{B}(\mathscr{H})$ is left invertible, then it dilates to an invertible. This will allow us to succinctly describe $\mathrm{Alg}(T, T^\dagger)$.

Let $P = I - TT^\dagger$. Then the operator $W \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ given by

$$W = \begin{array}{c} \begin{array}{cc} \mathscr{H} & \mathscr{H} \end{array} \\ \begin{pmatrix} T^\dagger & 0 \\ P & T \end{pmatrix} \end{array}$$

is invertible, with inverse given by

$$W^{-1} = \begin{pmatrix} T & P \\ 0 & T^\dagger \end{pmatrix}.$$

Let Q_1 and Q_2 denote the projections onto $\mathscr{H}_1 := \mathscr{H} \oplus 0$ and $\mathscr{H}_2 := 0 \oplus \mathscr{H}$ respectively. By construction $T = Q_2 W \mid_{\mathscr{H}_2}$ and $T^\dagger = Q_2 W^{-1} \mid_{\mathscr{H}_2}$. Furthermore, for each n,

$$W^n = \begin{pmatrix} {T^\dagger}^n & 0 \\ D_n & T^n \end{pmatrix} \qquad W^{-n} = \begin{pmatrix} T^n & D_n \\ 0 & {T^\dagger}^n \end{pmatrix}$$

where $D_n := \sum_{k=0}^{n-1} T^k P {T^\dagger}^{n-1-k}$. Since $\dim(\mathscr{E}) < \infty$ by assumption, D_n is a finite rank operator for each n. Furthermore, for every n, $T^n = Q_2 W^n \mid_{\mathscr{H}_2}$ and ${T^\dagger}^n = Q_2 W^{-n} \mid_{\mathscr{H}_2}$. It therefore follows that $\mathrm{Alg}(T, T^\dagger) = Q_2 \mathrm{Alg}(W \mid_{\mathscr{H}_2}, W^{-1} \mid_{\mathscr{H}_2})$. Now, a straightforward calculation reveals the following:

$$\begin{aligned}
Q_2W^{-n}Q_1W^m \mid_{\mathscr{H}_2} &= 0 \\
Q_2W^{-n}Q_2W^m \mid_{\mathscr{H}_2} &= {T^\dagger}^n T^m \\
Q_2W^mQ_1W^{-n} \mid_{\mathscr{H}_2} &= D_mD_n \\
Q_2W^mQ_2W^{-n} \mid_{\mathscr{H}_2} &= T^m{T^\dagger}^n.
\end{aligned} \tag{3.2}$$

Since $\mathrm{Alg}(T, T^\dagger) = Q_2\mathrm{Alg}(W \mid_{\mathscr{H}_2}, W^{-1} \mid_{\mathscr{H}_2})$, the operators appearing in Equation (3.2) span $\mathrm{Alg}(T, T^\dagger)$. Namely, using Equation (3.2) we have

$$D_mD_n + T^m{T^\dagger}^n = Q_2W^mW^{-n} \mid_{\mathscr{H}_2} = Q_2W^{m-n} \mid_{\mathscr{H}_2} = \begin{cases} T^{m-n} & \text{if } m > n \\ {T^\dagger}^{m-n} & \text{else.} \end{cases} \tag{3.3}$$

Also,

$$ {T^\dagger}^n T^m = Q_2W^{-n}W^m \mid_{\mathscr{H}_2} = \begin{cases} T^{m-n} & \text{if } m > n \\ {T^\dagger}^{n-m} & \text{else.} \end{cases} \tag{3.4}$$

Thus, $T^m{T^\dagger}^n$ is equal to some power of a generator, up to the finite rank perturbation D_mD_n. Consequently, every operator A in $\mathrm{Alg}(T, T^\dagger)$ may be "simplified" to an operator of the form

$$F + \sum_{k=0}^{N} a_kT^k + \sum_{l=1}^{M} b_l{T^\dagger}^l,$$

where F is some finite rank operator. Hence, the dense subalgebra $\mathrm{Alg}(T, T^\dagger)$ are finite rank operators plus Laurent polynomials in T and $T^\dagger$. We record this result here for future reference:

Proposition 3.2.5. *Let T be a natural left invertible operator with* $\mathrm{ind}(T) = -n$ *for some* *positive integer n. If $A \in \mathrm{Alg}(T, T^\dagger)$ (pre-closure of $\mathfrak{A}_T$), is the operator*

$$A = \sum_{n,m=0}^{N} \alpha_{n,m} T^m {T^\dagger}^n$$

then A may be rewritten as

$$A = F + \sum_{N \geq m \geq n \geq 0} \alpha_{n,m} T^{m-n} + \sum_{N \geq n \geq m \geq 1} \alpha_{n,m} {T^\dagger}^{n-m}$$

where F is the finite rank operator given by $F = -\sum_{n,m=0}^{N} \alpha_{n,m} D_m D_n$, and $D_n = \sum_{k=0}^{n-1} T^k P {T^\dagger}^{n-1-k}$.

Combining these two coarse descriptions of $\mathfrak{A}_T$ - one via the quotient and one via dilation, we arrive at our heuristic for $\mathfrak{A}_T$:

Heuristic 3.2.6. *The algebra $\mathfrak{A}_T$ is compact perturbations of Laurent series centered at zero.*

One further comment on the commutator ideal $\mathscr{C}$ of $\mathfrak{A}_T$. Recall that $P = I - TT^\dagger \in \mathscr{C}$. Hence by the preceding, all the finite rank operators F from this construction are in the commutator ideal $\mathscr{C}$. Combined with Proposition 3.2.5, this observation allows us to algebraically characterize a dense subset of $\mathscr{C}$.

Proposition 3.2.7. *Let $P = I - TT^\dagger$ and set*

$$\mathscr{K}_T := \overline{\mathrm{span}}\{T^n P {T^\dagger}^m : n, m \geq 0\}.$$

Then $\mathscr{K}_T = \mathscr{C}$.

Proof. First we show that $\mathscr{K}_T$ is an ideal of $\mathfrak{A}_T$. If $A \in \mathrm{Alg}(T, T^\dagger)$, then by Proposition 3.2.5,

$$A = -\sum_{n,m=0}^{N} \alpha_{n,m} D_m D_n + \sum_{N \geq m \geq n \geq 0} \alpha_{n,m} T^{m-n} + \sum_{N \geq n \geq m \geq 1} \alpha_{n,m} {T^\dagger}^{n-m}.$$

Now consider the product $A(T^kPT^{\dagger^l})$ for some k, l. Using Equations (3.3) and (3.4), it follows that $T^kPT^{\dagger^l}$ multiplied by any part in the decomposition of A above is once again in $\operatorname{span}\{T^nPT^{\dagger^m} : n, m \geq 0\}$. Similarly, $(T^kPT^{\dagger^l})A \in \operatorname{span}\{T^nPT^{\dagger^m} : n, m \geq 0\}$. It follows that all polynomials from $\operatorname{span}\{T^nPT^{\dagger^m} : n, m \geq 0\}$ multiplied by A belong to $\operatorname{span}\{T^nPT^{\dagger^m} : n, m \geq 0\}$.

If $B \in \mathscr{K}_T$, it follows from taking limits and using the closure of $\mathscr{K}_T$ that $AB, BA \in \mathscr{K}_T$. By density of $\operatorname{Alg}(T, T^\dagger)$ in $\mathfrak{A}_T$, we have that $\mathscr{K}_T$ is an ideal for $\mathfrak{A}_T$.

By definition, $P \in \mathscr{K}_T$ and so, $\mathfrak{A}_T/\mathscr{K}_T$ is commutative. Hence, $\mathscr{C} \subseteq \mathscr{K}_T$. However, notice that $\mathscr{K}_T$ is the principal ideal generated by P. Indeed, if $\mathscr{J}$ is an ideal of $\mathfrak{A}_T$, and $P \in \mathscr{J}$, then at a minimum each $T^nPT^{\dagger^m}$ must be inside of $\mathscr{J}$. Hence, $\mathscr{K}_T = \mathscr{C}$. □

Ideally, we would like a canonical representation of T as multiplication by z on some reproducing kernel Hilbert space. If we further have $T^\dagger$ represented as multiplication by z^{-1}, then $\mathfrak{A}_T$ could be further described as compact perturbations of multiplication operators with symbols Laurent series. This turns out to be the case for special class of operators, which we call analytic. We will expand on this particular topic in our discussion of Cowen-Douglas operators.

3.3 Wold-Type Decompositions

Much of the model theory and elementary properties of left invertible operators draws its inspiration from isometric operators. Isometries are a tractable class of operators due to the celebrated Wold decomposition. For future notational considerations, we state the Wold Decomposition here:

Theorem 3.3.1 (Wold Decomposition for Isometries)**.** *Let S be an isometry on $\mathscr{H}$. Define*

$$\begin{aligned} \mathscr{H}_I &:= \bigcap\nolimits_{n\geq 1} S^n \mathscr{H} \\ \mathscr{H}_A &:= \bigvee\nolimits_{n\geq 0} S^n \mathscr{E}. \end{aligned}$$

Then $\mathscr{H}_I$ and $\mathscr{H}_A$ are reducing for S, $\mathscr{H} = \mathscr{H}_I \oplus \mathscr{H}_A$, $S\mid_{\mathscr{H}_I}$ is a unitary and $S\mid_{\mathscr{H}_A}$ is a unilateral shift of rank n.

In other words, all isometries decompose the Hilbert space into two orthogonal, reducing subspaces for S. On $\mathscr{H}_I$, the isometry S is invertible, and hence, a unitary. On $\mathscr{H}_A$, the isometry is purely isometric. The isometric summand yields an analytic model. Concretely, $S\mid_{\mathscr{H}_A}$ is unitarily equivalent to $\dim(\mathscr{E})$ orthogonal copies of the unilateral shift. The unilateral shift is is unitarily equivalent to the operator of multiplication by z on a reproducing kernel Hilbert space of analytic functions. For a general left invertible operator $T \in \mathscr{B}(\mathscr{H})$, one would like to arrive at a similar type of decomposition. We make the following definition:

Definition 3.3.2. Given a left invertible $T \in \mathscr{B}(\mathscr{H})$, we define:

$$\begin{aligned} \mathscr{H}_I &:= \bigcap\nolimits_{n\geq 1} T^n \mathscr{H} \\ \mathscr{H}_A &:= \bigvee\nolimits_{n\geq 0} T^n \mathscr{E}. \end{aligned}$$

As a caution to the reader, $\mathscr{H}_I$ and $\mathscr{H}_A$ need not be reducing. However, $\mathscr{H}_I$ and $\mathscr{H}_A$ are clearly invariant subspaces for T. Moreover, $\mathscr{H}_I$ is invariant for $T^\dagger$ and $T\mid_{\mathscr{H}_I}$ is invertible, with inverse $T^\dagger\mid_{\mathscr{H}_I}$. We shall show that $T\mid_{\mathscr{H}_A}$ acts like a shift, not on a orthonormal basis, but on a more general basis. This will be discussed below.

For some isometries, the Wold-decomposition is trivial. For example, the unilateral shift on $\ell^2(\mathbb{N})$ is purely isometric since the subspace $\mathscr{H}_I = 0$. This leads us to the following definition:

Definition 3.3.3 ([40])**.** An operator $T \in \mathscr{B}(\mathscr{H})$ is **analytic** if $\mathscr{H}_I = 0$.

The terminology analytic is appropriate because we show that when a natural left invertible operator is analytic, then T is unitarily equivalent to M_z on a reproducing kernel Hilbert space of analytic functions.

In general, there is no Wold-type decomposition for T with regards to the spaces $\mathscr{H}_I$ and $\mathscr{H}_A$. See Example 3.5.1 below. However, Shimorin in [40] observed that there is almost a Wold-type decomposition. This decomposition is related to a canonical left invertible operator associated to T, called the Cauchy dual of T:

Definition 3.3.4 ([40])**.** Given a left invertible operator T, the **Cauchy dual** of T, denoted T', is the left invertible given by

$$T' := T(T^*T)^{-1} = T^{\dagger^*}.$$

Proposition 3.3.5. *Let T be a left invertible operator, and T' its Cauchy dual. The following statements hold:*

i. T' *is left invertible with Moore-Penrose inverse* $T'^{\dagger} = T^*$

ii. $\mathscr{E}' := \ker((T')^*) = \ker(T^{\dagger}) = \ker(T^*) = \mathscr{E}$

iii. $\operatorname{ind}(T') = \operatorname{ind}(T)$

Proof. It is clear from the definition that T' is left invertible with T^* a left inverse. That $T'^{\dagger} = T^*$ follows from a simple computation:

$$T'^{\dagger} = (T'^*T')^{-1}T'^* = (T^{\dagger}T')^{-1}T^{\dagger} = (T^*T)T^{\dagger} = T^*.$$

The remaining observations now follow. □

For the Cauchy dual T', we define the analogous invariant subspaces:

$$\begin{aligned} \mathscr{H}'_I &:= \bigcap_{n\geq 1} T'^n \mathscr{H} \\ \mathscr{H}'_A &:= \bigvee_{n\geq 0} T'^n \mathscr{E}. \end{aligned}$$

We now explain why the terminology of Cauchy dual is sensible. While one cannot hope to arrive at a decomposition $\mathscr{H} = \mathscr{H}_I \oplus \mathscr{H}_A$, there is a duality between the spaces $\mathscr{H}_I, \mathscr{H}'_I$ and $\mathscr{H}_A, \mathscr{H}'_A$.

Proposition 3.3.6 ([40], Prop 2.7)**.** *Let T be a left invertible operator. Then*

$$\mathscr{H} = \mathscr{H}_I \oplus \mathscr{H}'_A = \mathscr{H}'_I \oplus \mathscr{H}_A.$$

where $\oplus$ is an orthogonal direct summand of closed subspaces.

This duality is key in analyzing $\mathfrak{A}_T$. We will leverage information between T and T' (or $T^\dagger$ and T^*) in order to prove theorems about $\mathfrak{A}_T$. The first example of this is the construction of a Schauder bases used throughout the subsequent analysis.

3.4 Basis and Dual Basis

We now explore how $T\mid_{\mathscr{H}_A}$ acts as a shift on a general basis. This will be done by showing if T is a natural analytic left invertible, then it endows the Hilbert space with a type of basis analogous to that of a (Hamel) basis for a vector space, called a Schauder basis.

Definition 3.4.1. A Banach space X is said to have a **Schauder basis** if there exists a sequence $\{x_n\}$ of X such that for every element $x \in X$, there is a unique sequence of scalars α_n such that

$$x = \sum_{n\geq 0} \alpha_n x_n$$

where the above sum is converging in the norm topology of X. Alternatively, $\{x_n\}$ is a Schauder basis if and only if

i. $\overline{\text{span}}\{x_n\} = X$

ii. $\sum a_n x_n = 0$ if and only if $a_n = 0$ for all n.

Recall that a subspace $\mathscr{E}$ is said to be a *wandering subspace* for an operator $T \in \mathscr{B}(\mathscr{H})$ if for each $n \in \mathbb{N}$, $\mathscr{E} \perp T^n\mathscr{E}$ [21]. In the case of isometric operators, one further has $T^n\mathscr{E} \perp T^m\mathscr{E}$ for each $n, m \in \mathbb{N}$ with $n \neq m$.

Let T be a natural analytic left invertible operator, and L be a left inverse of T. The next result shows that $\mathscr{E} = \ker(T^*)$ is a wandering subspace for T and L^*. However, $T^n\mathscr{E}$ may not be orthogonal to $T^m\mathscr{E}$ for $n \neq m$. The invariant subspace generated in this fashion in the whole Hilbert space. Thus, the orbit of T and L^* on $\ker(T^*)$ give rise to a Schauder basis:

Theorem 3.4.2. *Let T be a natural analytic left invertible operator with* $\text{ind}(T) = -n$ *for some positive integer n. Let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for* $\ker(T^*)$, *and L be a left inverse of T. Then*

i. $x_{i,j} := T^j x_{i,0}$, $i = 1, \ldots n$, $j = 0, 1, \ldots$ is a Schauder basis for $\mathscr{H}$

ii. $x'_{i,j} := (L^*)^j x_{i,0}$, $i = 1, \ldots n$, $j = 0, 1, \ldots$ is a Schauder basis for $\mathscr{H}$.

Proof. We will only prove the case when $\text{ind}(T) = -1$. The general case is no more complicated, but simply requires extra notation for bookkeeping. In this case, $\ker(T^*) = \text{span}\{x_0\}$ for some norm one element $x_0 \in \mathscr{H}$.

The proof will proceed as follows. First we will show that the wandering space for $T' := T^{\dagger^*}$ produces a Schauder basis. Then we show that the orbit of x_0 under powers of T

will produce a Schauder basis, which will allow us to conclude that for any left inverse L, the orbit of L^* yields a Schauder basis.

Since T is analytic, by Proposition 3.3.6, we have that

$$\mathscr{H} = \mathscr{H}'_A = \bigvee_{j \geq 0} T'^j \ker(T^*). \tag{3.5}$$

Let $x'_j := T'^j x_0$ for $j = 0, 1, \dots$. Then by construction, $T'x'_j = x'_{j+1}$ and

$$T^{*m} x'_j = \begin{cases} 0 & \text{if } m > j \\ x'_{j-m} & \text{if } m \leq j \end{cases}$$

Notice that $\{x'_j\}$ is a Schauder basis. Indeed by (3.5), $\overline{\text{span}}\{x'_j\} = \mathscr{H}$. Furthermore, if $\sum_{j \geq 0} a_j x'_j = 0$, then

$$0 = (I - TT^\dagger) T^{*m} \left(\sum_{j \geq 0} a_j x'_j \right) = (I - TT^\dagger) \left(\sum_{j \geq m} a_j x'_{j-m} \right) = a_m x_0. \tag{3.6}$$

Thus, $a_j = 0$ for all j. Therefore $\{x'_j\}$ form a Schauder basis.

We now show that $x_j := T^j x_0$ is a Schauder basis. Let $\mathscr{K}$ be the closed subspace of $\mathscr{H}$ given by $\mathscr{K} := \overline{\text{span}_{j \geq 1}}\{x_j\}$. Suppose that $z \perp \mathscr{K}$. Then by above, z has a unique expansion in the Schauder basis x'_j. Say, $z = \sum_{j \geq 0} b_j x'_j$. Thus,

$$0 = \langle z, x_m \rangle = \langle T^{*m} z, x_0 \rangle = \langle T^{*m} z, (I - TT^\dagger) x_0 \rangle = \langle (I - TT^\dagger) T^{*m} z, x_0 \rangle = b_m.$$

Hence, $b_j = 0$ for all j, so $z = 0$. Therefore, $\mathscr{K}$ is dense in $\mathscr{H}$. But since $\mathscr{K}$ is closed, $\mathscr{K} = \mathscr{H}$. Now suppose that $\sum_{j \geq 0} c_j x_j = 0$. Then the exact same argument appearing in Equation (3.6) with T^{*m} replaced with $T^{\dagger^m}$ shows $c_j = 0$ for all j.

Finally, suppose L is any left inverse of T. Let $y_j = L^{*j} x_0$. Replacing the roles of x_j with

y_j and x'_j with x_j in the preceding paragraph, one concludes that y_j is a Schauder basis for $\mathscr{H}$. ☐

Corollary 3.4.3. *Let $T \in \mathscr{B}(\mathscr{H})$ be left invertible. Then T is analytic if and only if T' is analytic.*

Proof. If T is analytic, then by Theorem 3.4.2, $\mathscr{H}_A = \mathscr{H}$. Hence $\mathscr{H}'_I = 0$. The converse statement is identical. ☐

Theorem 3.4.2 illustrates how to construct Schauder bases for $\mathscr{H}$ using a natural analytic left invertible operator T and its Cauchy dual. We reserve the notation of Theorem 3.4.2 for these bases. We make the following definition:

Definition 3.4.4. Let T be a natural analytic left invertible operator and L be a left inverse of T. Fix an orthonormal basis $\{x_{i,0}\}_{i=1}^n$ for $\mathscr{E} = \ker(T^*)$. Then

$$\begin{aligned} x_{i,j} &:= T^j x_{i,0} \\ x'_{i,j} &:= L^{*j} x_{i,0}. \end{aligned} \tag{3.7}$$

We refer to the Schauder basis $\{x_{i,j}\}$ in Equation (3.7) as the **basis of T with respect to** $\{x_{i,0}\}_{i=1}^n$. Similarly, we refer to the basis $\{x'_{i,j}\}$ as the **dual basis of T with respect to** $\{x_{i,0}\}_{i=1}^n$ **and** L.

If no mention is made to the choice of left inverse L, it is assumed that $L = T^\dagger$. While the above definition depends on the choice of orthonormal basis $\{x_{i,0}\}_{i=1}^n$ for $\mathscr{E}_T$, we will usually refer to each as the *basis of* T and *dual basis of* T without reference.

By definition of a Schauder basis, for each $f \in \mathscr{H}$, there exists a unique sequences of scalars $\{\alpha_{i,j}\}$ and $\{\alpha'_{i,j}\}$ such that

$$f = \sum_{j\geq 0} \sum_{i=1}^{n} \alpha_{i,j} x_{i,j} = \sum_{j\geq 0} \sum_{i=1}^{n} \alpha'_{i,j} x'_{i,j}.$$

Naturally, one would like to have a relationship between $\{\alpha_{i,j}\}$ or $\{\alpha'_{i,j}\}$ in terms of the element $f \in \mathscr{H}$. We have the following useful characterization:

Proposition 3.4.5. *For each $f \in \mathscr{H}$, we have the following expansions:*

$$f = \sum_{j\geq 0}\sum_{i=1}^{n} \langle f, x'_{i,j}\rangle x_{i,j} = \sum_{j\geq 0}\sum_{i=1}^{n} \langle f, x_{i,j}\rangle x'_{i,j}.$$

Proof. Suppose that $f = \sum_{j\geq 0}\sum_{i=1}^{n} \alpha_{i,j} x_{i,j}$. Now, ${T^{\dagger}}^{m} x_{i,j} = 0$ if $j \leq m$ and $x_{i,j-m}$ otherwise. Also, since $\{x_{i,0}\}$ is an orthonormal basis for $\ker(T^*)$, we have for each $m \geq 0$,

$$\langle f, x'_{i,m}\rangle = \langle {T^{\dagger}}^{m} f, x_{i,0}\rangle = \alpha_{i,m}.$$

The same argument shows that if we expand f in terms of the dual basis of T as $f = \sum_{j\geq 0}\sum_{i=1}^{n} \alpha'_{i,j} x'_{i,j}$, then $\alpha'_{i,m} = \langle f, x_{i,m}\rangle$. ☐

Corollary 3.4.6. *The basis of T is bi-orthogonal to the dual basis of T. That is, $\langle x_{l,m}, x'_{i,j}\rangle = \delta_{l,i}\delta_{m,j}$*

Proof. By Proposition 3.4.5, we have that

$$x_{l,m} = \sum_{j\geq 0}\sum_{i=1}^{n} \langle x_{l,m}, x'_{i,j}\rangle x_{i,j}.$$

However by definition, Schauder bases have a unique expansion in terms of the basis. Hence, $\langle x_{l,m}, x'_{i,j}\rangle = 0$ unless $i = l$ and $j = m$. ☐

Briefly, we would like to caution the reader about the order of basis and dual basis of T. A convergent series $\sum_{n\geq 0} x_n$ in a Banach space X is said to be *unconditionally convergent* if for every permutation σ of $\mathbb{N}$, the series $\sum_{n\geq 0} x_{\sigma(n)}$ converges. Otherwise, the series is said to be *conditionally convergent.* A Schauder basis $\{x_n\}$ in a Banach space X is said to be a *unconditional basis* if the series expansion $x = \sum_{n\geq 0} \alpha_n x_n$ is unconditional for every $x \in X$. Otherwise, the basis is said to be *conditional.* Examples of unconditional bases for Hilbert spaces include orthonormal bases, and more generally, some frames.

Unfortunately, all infinite dimensional Banach spaces with a basis must have conditional bases [33]. What is worse, verifying that a basis is unconditional is, in general, a very difficult task. Explicit constructions of conditional bases exist for Hilbert spaces. Indeed, there is a class of examples for $L^2(\mathbb{T})$ of the form $\{e^{2\pi int}\phi(t)\}_{n\in\mathbb{Z}}$ for some $\phi \in L^2(\mathbb{T})$ (See [41], Example 11.2). From the author's perspective, it is not clear when the basis and dual basis of T are unconditional. Fortunately, this will not affect our analysis in any serious way. At a minimum, we have the following trivial rearrangements:

Proposition 3.4.7. *Let T be a natural analytic left invertible with* $\operatorname{ind}(T) = -n$ *for some* $1 \leq n < \infty$. *Then for any permutation σ of* $\{1, \dots, n\}$, *we have*

$$\sum_{j\geq 0}\sum_{i=1}^{n} \alpha_{i,j}x_{i,j} = \sum_{i=1}^{n}\sum_{j\geq 0} \alpha_{i,j}x_{i,j} = \sum_{i=1}^{n}\sum_{j\geq 0} \alpha_{\sigma(i),j}x_{\sigma(i),j} = \sum_{j\geq 0}\sum_{i=1}^{n} \alpha_{\sigma(i),j}x_{\sigma(i),j}$$

whenever the sum converges. Consequently, $\sum_{j\geq 0}\sum_{i=1}^{n} \alpha_{i,j}x_{i,j}$ *converges if and only if* $\sum_{j\geq 0} \alpha_{i,j}x_{i,j}$ *converges for each* $i = 1, \dots, n$.

Since we are interested in the case when $\operatorname{ind}(T)$ is a negative integer, the above proposition fits into the purview of our study. This remark is useful when we construct a canonical model for T as multiplication by z on a reproducing kernel Hilbert space of analytic functions in Chapter Four. In order to conduct a more thorough analysis of $\mathfrak{A}_T$, we will later consider the case when $\operatorname{ind}(T) = -1$. In the next section, we discuss the ways in which a left invertible can fail to have a Wold-type decomposition.

3.5 Failure of Wold Decompositions for Left Invertibles

We have mentioned that for a general left invertible operator, one cannot hope to reconstruct a exact replica of the Wold decomposition. Namely, it is not the case that $\mathscr{H} = \mathscr{H}_I \oplus \mathscr{H}_A$. Their sum can fail to be orthogonal, and hence, $\mathscr{H}_I + \mathscr{H}_A$ may not be equal to $\mathscr{H}$. We have the following example:

Example 3.5.1. Let $\mathscr{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{Z})$, and define $T \in \mathscr{B}(\mathscr{H})$ as

$$T = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where A is the unilateral shift on $\ell^2(\mathbb{N})$, C is the bilateral shift on $\ell^2(\mathbb{Z})$, and $B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{Z})$ is the inclusion map given by

$$B((a_n)_{n\geq 1}) = (\dots 0, \hat{0}, a_1, a_2, \dots)$$

where the ˆ symbol denotes the entry in the zeroth slot. Let $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=-\infty}^{\infty}$ denote the standard orthonormal basis for $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ respectively.

In order to compute the subspaces $\mathscr{H}_I$ and $\mathscr{H}_A$ above, we will first need to analyze T^n. Note that

$$T^n = \begin{pmatrix} A^n & 0 \\ D_n & C^n \end{pmatrix}$$

where $D_n := \sum_{k=0}^{n-1} C^k B A^{n-1-k}$. By construction, $D_n e_m = n f_{m+n-1}$. Therefore, $D_n = nC^{n-1}B$, so

$$T^n = \begin{pmatrix} A^n & 0 \\ nC^{n-1}B & C^n \end{pmatrix}.$$

Notice that if $x \oplus y \in \mathscr{H}$, then

$$T^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^n x \\ nC^{n-1}Bx + C^n y \end{pmatrix}. \tag{3.8}$$

We now show that

$$\mathscr{H}_I = 0 \oplus \ell^2(\mathbb{Z}).$$

Indeed, suppose that $x \oplus y \in \mathscr{H}_I$. Then for each $n \in \mathbb{N}$ there exists a sequence $x_n \in \ell^2(\mathbb{N})$ and $y_n \in \ell^2(\mathbb{Z})$ such that $T^n(x_n \oplus y_n) = x \oplus y$. By Equation (3.8), we must have $A^n x_n = x$. But since the unilateral shift A is analytic, it follows that $x = 0$ so that $\mathscr{H}_I \subseteq 0 \oplus \ell^2(\mathbb{Z})$. On the other hand, suppose $y \in \ell^2(\mathbb{Z})$. Since the bilateral shift C is invertible, C^n is invertible for all $n \in \mathbb{Z}$. Thus, for all n there exists $y_n \in \ell^2(\mathbb{Z})$ such that $C^n y_n = y$. Hence, $0 \oplus y \in \mathscr{H}_I$, demonstrating equality.

Next we compute $\mathscr{H}_A$. Notice that

$$T^* = \begin{pmatrix} A^* & B^* \\ 0 & C^* \end{pmatrix}$$

where $B^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ is the projection onto the coordinates greater than zero. Consequently, if $x \oplus y \in \mathscr{H}$,

$$T^* \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^* x + B^* y \\ C^* y \end{pmatrix}.$$

If $x \oplus y \in \ker(T^*)$, then since C^* is invertible, it follows that $y = 0$. Consequently, $x \in \ker(A^*) = \operatorname{span}\{e_1\}$. Therefore, $\mathscr{E} = \ker(T^*) = \ker(A^*) \oplus 0 = \operatorname{span}\{e_1\} \oplus 0$. Now, by

Equation (3.8),

$$T^n \begin{pmatrix} e_1 \\ 0 \end{pmatrix} = \begin{pmatrix} e_{n+1} \\ nf_n \end{pmatrix}. \tag{3.9}$$

As a result, we have that

$$\operatorname{span}_{0\leq n\leq N}\{T^n(e_1 \oplus 0)\} = \left\{ \left(\sum_{n=0}^{N} \alpha_n e_{n+1} \right) \oplus \left(\sum_{n=1}^{N} \alpha_n n f_n \right) : \alpha_0, \ldots \alpha_N \in \mathbb{C} \right\}.$$

Now because

$$\begin{aligned} \left\| \left(\textstyle\sum_{n=0}^{N} \alpha_n e_{n+1} \right) \oplus \left(\textstyle\sum_{n=1}^{N} \alpha_n n f_n \right) \right\|^2 &= \left\| \textstyle\sum_{n=0}^{N} \alpha_n e_{n+1} \right\|^2 + \left\| \textstyle\sum_{n=1}^{N} \alpha_n n f_n \right\|^2 \\ &= |\alpha_0|^2 + |\alpha_{N+1}|^2 + \textstyle\sum_{n=1}^{N} (1+n^2)|\alpha_n|^2 \end{aligned}$$

it follows that

$$\mathscr{H}_A = \left\{ \left(\sum_{n\geq 0} \alpha_n e_{n+1} \right) \oplus \left(\sum_{n\geq 1} \alpha_n n f_n \right) : \sum_{n\geq 1} (1+n^2)|\alpha_n|^2 < \infty \right\}.$$

With $\mathscr{H}_A$ computed, we now remark that $\mathscr{H}_I = 0 \oplus \ell^2(\mathbb{Z})$ is not orthogonal to $\mathscr{H}_A$. Nevertheless, $\mathscr{H}_I \cap \mathscr{H}_A = 0$. This is clear by the form of $\mathscr{H}_A$ and $\mathscr{H}_I$.

Finally, we remark that $\mathscr{H}_I + \mathscr{H}_A$ is dense in $\mathscr{H}$, but not closed. To see this, note that $0 \oplus f_n \in 0 \oplus \ell^2(\mathbb{Z}) = \mathscr{H}_I$ for all n. By Equation (3.9), it follows that $\{e_n \oplus 0\}_{n\geq 0} \subset \mathscr{H}_I + \mathscr{H}_A$. Since $\{0 \oplus f_n\}_{n\in\mathbb{Z}} \subset \mathscr{H}_I$, it follows that $\mathscr{H}_I + \mathscr{H}_A$ is dense in $\mathscr{H}$. However, $\mathscr{H}_I + \mathscr{H}_A \neq \mathscr{H}$, as $\mathscr{H}_I + \mathscr{H}_A$ is not closed. Indeed, if we let $z = ((1+n^2)^{-1}) \oplus 0$, then $z \in \mathscr{H}$ but $z \notin \mathscr{H}_I + \mathscr{H}_A$. This concludes the example.

The above example turns out to be generic. If $T \in \mathscr{B}(\mathscr{H})$ is left invertible, then $\mathscr{H}_I + \mathscr{H}_A$ is dense in $\mathscr{H}$ with $\mathscr{H}_I \cap \mathscr{H}_A = 0$. To show this, we establish a few simple results.

Proposition 3.5.2. *Let $T \in \mathscr{B}(\mathscr{H})$ be left invertible. Consider the decomposition $\mathscr{H} =$*

$\mathscr{H}_A' \oplus \mathscr{H}_I$ *afforded by Proposition 3.3.6. Then with respect to this decomposition,*

$$T = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

with A *analytic left invertible, and* C *invertible.*

Proof. Note that $\mathscr{H}_I$ is invariant for T. Therefore, T necessarily has the form above. That the operator $C = T\mid_{\mathscr{H}_I}$ is invertible is clear. Let Q be the projection onto $\mathscr{H}_A'$. To show that $A = QT\mid_{\mathscr{H}_A'}$ is left invertible, we show that A^* is right invertible. Indeed, notice that $\mathscr{H}_A'$ is invariant under T', and that

$$T^* = \begin{pmatrix} A^* & B^* \\ 0 & C^* \end{pmatrix}.$$

Thus, if $x \in \mathscr{H}_A'$, we have

$$A^*(T'x) = T^*(T'x) = x$$

since $T^*T' = I$. Therefore A^* is right invertible, so A is left invertible. That A is analytic follows from the orthogonality of the decomposition. To see this, observe

$$T^n = \begin{pmatrix} A^n & 0 \\ * & C^n \end{pmatrix}.$$

Hence, $A^n = QT^n\mid_{\mathscr{H}_A'}$. Now,

$$\bigcap A^n \mathscr{H}_A' = \bigcap QT^n \mathscr{H}_A' \subset Q\left(\bigcap T^n \mathscr{H}\right) = Q\mathscr{H}_I = 0. \qquad \square$$

Proposition 3.5.3. *Suppose that* $T \in \mathscr{B}(\mathscr{H})$, $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ *and*

$$T = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

with A *analytic left invertible, and* C *invertible. Then* T *is left invertible, with* $\mathscr{H}_I = 0 \oplus \mathscr{H}_2$, $\ker(T^*) = \ker(A^*) \oplus 0$, *and* $\mathscr{H}_I \cap \mathscr{H}_A = 0$.

Proof. Let L be the operator defined by

$$L = \begin{pmatrix} A^\dagger & 0 \\ -C^{-1}BA^\dagger & C^{-1} \end{pmatrix}.$$

Then L is a left inverse of T, so T is left invertible. Now, we remark that

$$T^n = \begin{pmatrix} A^n & 0 \\ D_n & C^n \end{pmatrix}$$

where D_n is an operator whose formula is not relevant for the remainder of the proof. If $x \oplus y \in \bigcap T^n \mathscr{H}$, then there exists x_n, y_n such that

$$T^n \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} A^n x_n \\ D_n x_n + C^n y_n \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since A is analytic, it follows that $x = 0$. Thus, $\bigcap T^n \mathscr{H} \subset 0 \oplus \mathscr{H}_2$. Conversely, given $y \in \mathscr{H}_2$, since C^n is invertible, there exists y_n such that $C^n y_n = y$. So, $T^n(0 \oplus y_n) = 0 \oplus y$. It follows that $\bigcap T^n \mathscr{H} = 0 \oplus \mathscr{H}_2$.

Concerning the intersection of $\mathscr{H}_I$ and $\mathscr{H}_A$, notice that

$$T^* = \begin{pmatrix} A^* & B^* \\ 0 & C^* \end{pmatrix}.$$

Since C^* is invertible, it follows that $x \oplus y \in \ker(T^*)$ if and only if $y = 0$ and $x \in \ker(A^*)$. Thus, $\mathscr{E} = \ker(A^*) \oplus 0$. Consequently if $x_0 \in \ker(A^*)$, $\mathscr{H}_A$ is densely spanned by elements of the form

$$T^n \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^n x_0 \\ D_n x_0 \end{pmatrix}.$$

Since A is analytic, $A^n x_0$ form a Schauder basis for $\mathscr{H}_A$ by Theorem (3.4.2). As a result, $0 \oplus y \in \mathscr{H}_A$ if and only if $y = 0$. □

Corollary 3.5.4. *Given a left invertible operator $T \in \mathscr{B}(\mathscr{H})$, $\mathscr{H}_I + \mathscr{H}_A$ is dense in $\mathscr{H}$ with $\mathscr{H}_I \cap \mathscr{H}_A = 0$.*

Proof. Proposition (3.5.3) established that $\mathscr{H}_I \cap \mathscr{H}_A = 0$. All that remains to be shown is that $\mathscr{H}_I + \mathscr{H}_A$ is dense in $\mathscr{H}$. To this end, consider the decomposition $\mathscr{H} = \mathscr{H}_A' \oplus \mathscr{H}_I$. Write,

$$T = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$

Let $x_0 \oplus 0 \in \ker(T^*) = \ker(A^*) \oplus 0$, so that

$$T^n \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^n x_0 \\ D_n x_0 \end{pmatrix}$$

as before. Given that $0 \oplus (-D_n x_0) \in 0 \oplus \mathscr{H}_I$, we have

$$T^n \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -D_n x_0 \end{pmatrix} = \begin{pmatrix} A^n x_0 \\ 0 \end{pmatrix} \in \mathscr{H}_A + \mathscr{H}_I.$$

Since A is an analytic left invertible on $\mathscr{H}_A'$, $A^n x_0$ is a Schauder basis for $\mathscr{H}_A'$. It follows that the closure of $\mathscr{H}_A + \mathscr{H}_I$ contains $\mathscr{H}_A'$ and $\mathscr{H}_I$, and therefore is dense in $\mathscr{H} = \mathscr{H}_A' \oplus \mathscr{H}_I$. $\square$

Chapter 4

Cowen-Douglas Operators - The Analytic Model

In the late 70s, Cowen and Douglas discovered that operators possessing an open set of eigenvalues can be associated with a particular Hermitian holomorphic bundle [14], [15]. These operators, now called Cowen-Douglas operators, could in some cases be completely classified by simple geometric properties. For example, when the rank of the bundle is one, the curvature serves as a complete set of unitary invariants [15].

Cowen-Douglas operators have played an important role in operator theory, servicing as a bridge between operator theory and complex geometry. The definition is rigid enough to allow for classification based on local spectral data. However, the definition is also flexible enough to allow for rich examples - including many backward weighted shifts and adjoints of some subnormal operators. The definition of Cowen-Douglas operators is as follows:

Definition 4.0.1. Given an open subset Ω of $\mathbb{C}$ and a positive integer n, we say that R is of **Cowen-Douglas class n**, and write $R \in B_n(\Omega)$ if

i. $\Omega \subset \sigma(R)$

ii. $(R-\lambda)\mathscr{H} = \mathscr{H}$ for all $\lambda \in \Omega$

iii. $\dim(\ker(R-\lambda)) = n$ for all $\lambda \in \Omega$

iv. $\bigvee_{\lambda \in \Omega} \ker(R-\lambda) = \mathscr{H}$

Thus if $R \in B_n(\Omega)$, then R contains an open set of eigenvalues such that each eigenspace has dimension n, and the span of these eigenspaces is dense in $\mathscr{H}$. Associated to Cowen-Douglas operators is a bundle structure known as a Hermitian holomorphic vector bundle.

Definition 4.0.2. A **Hermitian holomorphic vector bundle of rank** n **over** Ω consists of the following data:

i. A complex manifold E

ii. A holomorphic map $\pi : E \to \Omega$ such that each fiber $E_\lambda := \pi^{-1}(\lambda)$ is isomorphic to $\mathbb{C}^n$

iii. For each $\lambda_0 \in \Omega$, there exists a neighborhood Δ of λ_0 and functions $\{\gamma_i\}_{i=1}^n$ with $\gamma_i : \Omega \to E$ such that $\{\gamma_i(\lambda)\}_{i=1}^n$ form a basis for E_λ.

A **cross-section** E is a map $\gamma : \Omega \to E$ such that $\pi(\gamma(\lambda)) = \lambda$ for all $\lambda \in \Omega$ (namely $\gamma(\lambda) \in E_\lambda$ for each λ). The bundle is **trivial** if Δ may be taken to be Ω. The **trivial bundle of rank** n **over** Ω is $\Omega \times \mathbb{C}^n$ with $\pi(\lambda, x) = \lambda$.

If $R \in B_n(\Omega)$, then the set

$$E_R := \{(\lambda, x) \in \Omega \times \mathscr{H} : x \in \ker(R - \lambda)\}$$

with the mapping $\pi : E_R \to \Omega$ via $\pi(\lambda, x) = \lambda$ defines sub-bundle of the trivial bundle of rank n over Ω. It is known that E_R provides a complete set of unitary invariants for operators in the Cowen-Douglas class [14]. Specifically, if E_{R_1} is isomorphic to E_{R_2} as holomorphic vector bundles, then R_1 is unitarily equivalent to R_2. This approach to Cowen-Douglas theory highlights the beautiful connections that exist between complex geometry and operator theory.

The sections of the bundle E_R provide an equivalent avenue of study. Given $R \in B_n(\Omega)$, we can represent R as the adjoint of multiplication by z on a reproducing kernel Hilbert space.

The approach of this paper more closely follows this model. We will outline this construction below, and connect it to our work on bases in Chapter Three. For more information about Cowen-Douglas operators, see [14], [16], [44].

4.1 Analytic Left Invertibles and Cowen-Douglas Operators

The connection between Cowen-Douglas operators and left invertibles is found in the following:

Theorem A. *Let $T \in \mathscr{B}(\mathscr{H})$ be a left invertible operator with $\operatorname{ind}(T) = -n$, for $n \geq 1$. Then the following are equivalent:*

i. T is analytic

ii. T' is analytic

iii. There exists $\epsilon > 0$ such that $T^* \in B_n(\Omega)$ for $\Omega = \{z : |z| < \epsilon\}$

iv. There exists $\epsilon > 0$ such that $T^\dagger \in B_n(\Omega)$ for $\Omega = \{z : |z| < \epsilon\}$

Theorem A is a cornerstone result for this work. It serves two fundamental roles. First, Theorem A allows us to leverage the powerful machinery associated with Cowen-Douglas operators into classifying the algebras $\mathfrak{A}_T$. Second, it provides us with a desirable canonical model. Concretely, Theorem A allows us to represent T as multiplication by z restricted to a reproducing kernel Hilbert space of analytic functions.

To help illuminate this relationship, we will take a constructive approach to proving Theorem A. This will also connect to our results on Schauder bases from the previous chapter. We prove the implication (3) implies (1) after stating the following lemma noted in Cowen and Douglas' original work:

Lemma 4.1.1 ([14]). *Let Θ be an open subset of $\mathbb{C}$ and $S \in B_m(\Theta)$. Then for any fixed $\mu_0 \in \Theta$,*

$$\bigvee_{k\geq 1} \ker(S-\mu_0)^k = \mathscr{H}.$$

Moreover, if $\Omega \subset \mathbb{C}$ is open, $\lambda_0 \in \Omega$, n is a positive integer, and $R \in \mathscr{B}(\mathscr{H})$ satisfies

i. $\Omega \subset \sigma(S)$

ii. $(R-\lambda)\mathscr{H} = \mathscr{H}$ *for all* $\lambda \in \Omega$

iii. $\dim(\ker(R-\lambda)) = n$ *for all* $\lambda \in \Omega$

iv. $\bigvee_{k\geq 1} \ker(R-\lambda_0)^k = \mathscr{H}$.

Then $R \in B_n(\Omega)$.

Corollary 4.1.2. *Let $T \in \mathscr{B}(\mathscr{H})$, $n \in \mathbb{N}$, $\epsilon > 0$ and $\Omega = \{z : |z| < \epsilon\}$. If $T^* \in B_n(\Omega)$, then T is an analytic, left invertible operator with* $\operatorname{ind}(T) = -n$.

Proof. By assumption, $0 \in \Omega \subset \sigma(T^*)$. By condition (2) of the definition of Cowen-Douglas operators, T^* is onto. Since T^* has closed range, it follows from the closed range theorem that T also has closed range. Moreover, since T^* is onto, T must be injective. Therefore, T is left invertible.

As T is left invertible, its Cauchy dual T' is well defined. Recall that $T^* = (T')^\dagger$. Since $T^* \in B_n(\Omega)$, it follows that $\operatorname{ind}(T') = -n$. By Proposition 3.3.5 and condition (3) of Cowen-Douglas operators, we have $\operatorname{ind}(T) = \operatorname{ind}(T') = -n$. Thus, all that remains to be shown is that T is analytic. By lemma 4.1.1, $\mathscr{H} = \bigvee_{k\geq 1} \ker(T^{*k})$. Therefore,

$$0 = \left(\bigvee_{k\geq 1} \ker(T^{*k})\right)^{\perp} = \bigcap_{k\geq 1} \ker(T^{*k})^{\perp} = \bigcap_{k\geq 1} \operatorname{ran}(T^k). \qquad \square$$

Next we show that if T is a natural analytic left invertible, then $T^* \in B_n(\Omega)$. This will be done in several steps. First, we will show that T^* possess an open set Ω of eigenvalues. We establish some notation for the open set Ω that will appear in the implication (1) implies (3) of Theorem A:

Definition 4.1.3. Suppose T is a natural analytic left invertible operator. We define

$$\Omega_T := \{z \in \mathbb{C} : |z| < \|T^\dagger\|^{-1}\}.$$

Corollary 4.1.4. *If T is a natural analytic left invertible operator, and $\lambda \in \Omega_T$, then $T + \lambda$ is left invertible with* $\operatorname{ind}(T) = \operatorname{ind}(T + \lambda)$.

Proof. Notice that

$$\|(T + \lambda) - T\| = |\lambda| < \|T^\dagger\|^{-1}.$$

By Lemma 3.1.9 and Proposition 3.1.13, $T + \lambda$ is left invertible with the same Fredholm index as T. □

Lemma 4.1.5. *Let T be an analytic left invertible operator with* $\operatorname{ind}(T) = -n$ *for some $n \geq 1$. Then for all $\lambda \in \Omega_T$, the operator $I - \lambda T'$ is invertible with*

$$(I - \lambda T')^{-1} = \sum_{j \geq 0} \lambda^j T'^j.$$

Proof. As $|\lambda| < \|T^\dagger\|^{-1}$ and $T' = T^{\dagger *}$, the operator $\lambda T'$ has norm less than 1. □

Lemma 4.1.6. *Let T be an analytic left invertible operator with* $\operatorname{ind}(T) = -n$ *for some positive integer n. Let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for* $\ker(T^*)$*, and*

$$x'_{i,j} = T'^j x_{i,0} = ((T^\dagger)^*)^j x_{i,0}$$

be the dual basis of T with respect to $T^\dagger$. Then for each $i = 1, \ldots, n$, the maps $\gamma_i : \Omega_T \to \mathscr{H}$ via

$$\gamma_i(\lambda) := \sum_{j\geq 0} \lambda^j x'_{i,j}$$

are well defined. Furthermore, the maps $\gamma_i : \Omega_T \to \mathscr{H}$ are analytic.

Proof. By Lemma 4.1.5, $I - \lambda T'$ is invertible. Thus for each $i = 1, \ldots n$,

$$(I - \lambda T')^{-1} (x_{i,0}) = \sum_{j\geq 0} \lambda^j T'^j (x_{i,0}) = \sum_{j\geq 0} \lambda^j x'_{i,j} = \gamma_i(\lambda)$$

exists for each $\lambda \in \Omega_T$. Since the map $\lambda \mapsto (I - \lambda T')^{-1}$ is well defined and analytic on Ω_T, we have that the maps γ_i are analytic. □

In light of these observations, we make the following definition:

Definition 4.1.7. Given an analytic left invertible T with $\text{ind}(T) = -n$ for some positive integer n, let Ω_T be as in Definition 4.1.3. Let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$, and $x'_{i,j} = T'^j x_{i,0}$ be the dual basis of T with respect to $T^\dagger$. We define

$$\gamma_i(\lambda) := \sum_{j\geq 0} \lambda^j x'_{i,j}.$$

Lemma 4.1.8. *Let T be an analytic left invertible with* $\text{ind}(T) = -n$*, and $\{\gamma_i\}_{i=1}^n$ be as in Definition 4.1.7. Then for each i,*

$$\gamma_i(\lambda) \in \ker(T^* - \lambda).$$

Hence, $\Omega_T \subset \sigma_p(T^*)$.

Proof. Since T^* is the Moore-Penrose inverse of T', it follows from the definition of γ_i that

$$T^*\gamma_i(\lambda) = \sum_{j=0}^{\infty} \lambda^j T^* x'_{i,j} = \sum_{j=1}^{\infty} \lambda^j x'_{i,j-1} = \lambda\gamma_i(\lambda).$$

The rest of the statement follows. □

Proposition 4.1.9. *Let T be an analytic left invertible operator with* $\operatorname{ind}(T) = -n$ *for some positive integer n. Let Ω_T be be as in Definition 4.1.3. Then $T^* \in B_n(\Omega_T)$*

Proof. Pick an orthonormal basis $\{x_{i,0}\}$ for $\ker(T^*)$. By Corollary 4.1.4, if $\lambda \in \Omega_T$, then $T - \lambda$ is left invertible with Fredholm index $-n$. Therefore, each eigenspace $\ker(T^* - \lambda)$ is n-dimensional for each $\lambda \in \Omega_T$. By Lemma 4.1.8, we have $\{\gamma_i(\lambda)\}_{i=1}^n \subset \ker(T^* - \lambda)$. Moreover, since $\{x'_{i,j}\}$ form a Schauder basis, we must have that the collection $\{\gamma_i(\lambda)\}_{i=1}^n$ is linearly independent.

Indeed, suppose there exists a $\mu \in \mathbb{C}$ such that $\gamma_i(\lambda) = \mu\gamma_k(\lambda)$ for some $\lambda \in \Omega_T$ with $i \neq k$. If $x_{i,j} = T^j x_{i,0}$ is the basis associated to T, then by Lemma 3.4.6 we have for each j

$$\lambda^j = \langle \gamma_i(\lambda), x_{i,j} \rangle = \mu \langle \gamma_k(\lambda), x_{i,j} \rangle = \mu \sum_{j=0}^{\infty} \lambda^j \langle x'_{k,j}, x_{i,j} \rangle = 0.$$

This forces $\lambda = 0$. Hence, $x'_{i,0} = \gamma_i(0) = \mu\gamma_k(0) = \mu x'_{k,0}$. But since $\{x_{i,0}\}_{i=1}^n$ form an orthonormal basis for $\ker(T^*)$, this cannot happen. Hence, $\{\gamma_i(\lambda)\}_{i=1}^n$ form a (perhaps non-orthogonal) basis for $\ker(T^* - \lambda)$.

Lastly, if we choose $\lambda_0 = 0$, then

$$\ker(T^* - \lambda_0)^k = \ker((T^*)^k) = (\operatorname{ran} T^k)^\perp = \left(\bigcap_{j=0}^{k} T^j \mathscr{H} \right)^\perp.$$

Since T is analytic, it follows that $\bigvee_{k\geq 1} \ker((T^*)^k) = \mathscr{H}$. By Lemma 4.1.1, we have that $T^* \in B_n(\Omega)$. □

We highlight an important and interesting feature of the basis $\{x'_{i,j}\}$ that came up in the previous proof:

Corollary 4.1.10. *Let T be an analytic left invertible operator with* $\operatorname{ind}(T) = -n$ *for some positive integer n, and $\{\gamma_i\}_{i=1}^n$ be the analytic maps from Definition 4.1.7. Then for each $\lambda \in \Omega_T$, $\{\gamma_i(\lambda)\}_{i=1}^n$ form a spanning set for* $\ker(T^* - \lambda)$.

We have thus shown that statements (1) and (3) of Theorem A are equivalent. However, when paired with Corollary 3.4.3 we see that $T^\dagger$ must also be Cowen-Douglas. This completes the proof of Theorem A.

One consequence of Theorem A is a reformulation of the definition of $\mathfrak{A}_T$ and the operator algebra generated by a Cowen-Douglas operator and a particular right inverse. Indeed, recall that $\mathfrak{A}_T$ is defined by

$$\mathfrak{A}_T := \overline{\operatorname{Alg}}\{T, T^\dagger\}.$$

If $\epsilon > 0$, $\Omega = \{z : |z| < \epsilon\}$, and $R \in B_n(\Omega)$, then by definition R is right invertible. There exists a canonical right inverse of R, which we denote by T, such that $\operatorname{ran}(T) = \ker(R)^\perp$. By construction, T is left invertible, and $R = T^\dagger$, the Moore-Penrose inverse of T. Thus, we arrive at an equivalent viewpoint of study:

Corollary 4.1.11. *Let $\epsilon > 0$, $\Omega = \{z : |z| < \epsilon\}$, and $R \in B_n(\Omega)$. If T is the right inverse of R such that ran$(T) = \ker(R)^\perp$, then T is an analytic left invertible operator with $R = T^\dagger$. Hence,*

$$\mathfrak{A}_T = \overline{\operatorname{Alg}}\{T, R\}.$$

4.2 The Associated Reproducing Kernel Hilbert Space

As previously remarked, the general theory of Cowen-Douglas operators allows one to represent T as multiplication by z on a reproducing kernel Hilbert space of analytic functions over

Ω. This construction is highlighted here. We then connect this model to the Schauder bases associated to T and T' discussed in Chapter Three. First, let us establish some notation. Given a set $G \subset \mathbb{C}$, let $G^* := \{\overline{\lambda} : \lambda \in G\}$. Notice that $\Omega_T^* = \Omega_T$ as a set. We make the following definition:

Definition 4.2.1. Let $R \in B_n(\Omega)$. A holomorphic cross-section of $\gamma : \Omega \to E_R$ of the bundle E_R is a **spanning holomorphic cross-section** if

$$\overline{\text{span}}\{\gamma(\lambda) : \lambda \in \Omega\} = \mathscr{H}.$$

Spanning holomorphic cross-sections give rise to reproducing kernel Hilbert spaces of analytic functions. Indeed, fix a spanning holomorphic section γ. For each $f \in \mathscr{H}$, define an analytic function $\hat{f}_\gamma \in H(\Omega^*)$ as follows:

$$\hat{f}_\gamma(\lambda) = \langle f, \gamma(\overline{\lambda})\rangle \qquad\qquad \lambda \in \Omega^*. \tag{4.1}$$

Let $\widehat{\mathscr{H}_\gamma} = \{\hat{f}_\gamma : f \in \mathscr{H}\} \subset H(\Omega^*)$. Equip $\widehat{\mathscr{H}_\gamma}$ with the inner product afforded by $\mathscr{H}$. That is, for each $f, g \in \mathscr{H}$, define the inner product on $\widehat{\mathscr{H}_\gamma}$ via

$$\langle \hat{f}_\gamma, \hat{g}_\gamma\rangle_\gamma := \langle f, g\rangle.$$

Define a linear map $U_\gamma : \mathscr{H} \to \widehat{\mathscr{H}_\gamma}$ via $U_\gamma(f) = \hat{f}_\gamma$. Notice that because γ is a spanning section, U_γ is a unitary. Indeed, if $\hat{f}_\gamma = \hat{g}_\gamma$, then for each $\lambda \in \Omega^*$,

$$0 = \hat{f}_\gamma(\lambda) - \hat{g}_\gamma(\lambda) = \langle f - g, \gamma(\overline{\lambda})\rangle$$

Since the span of $\{\gamma(\lambda) : \lambda \in \Omega\}$ is dense in $\mathscr{H}$, $f - g = 0$.

Furthermore, $\widehat{\mathscr{H}_\gamma}$ is a reproducing kernel Hilbert space over the set Ω^*. Indeed, as

$\gamma(\overline{\lambda}) \in \mathscr{H}$, there exists a function $\widehat{\gamma(\overline{\lambda})}_\gamma \in \widehat{\mathscr{H}_\gamma}$. For all $f \in \mathscr{H}$ and $\lambda \in \Omega$,

$$\hat{f}_\gamma(\lambda) = \langle f, \gamma(\overline{\lambda})\rangle = \left\langle \hat{f}_\gamma, \widehat{\gamma(\overline{\lambda})}_\gamma \right\rangle_\gamma .$$

Hence, the reproducing kernel at $\lambda \in \Omega^*$ is given by $k_\lambda = \widehat{\gamma(\overline{\lambda})}_\gamma$. Therefore, given $\lambda, \mu \in \Omega^*$, the reproducing kernel may be computed as follows:

$$K(\lambda, \mu) = \langle k_\mu, k_\lambda \rangle = \left\langle \widehat{\gamma(\overline{\mu})}_\gamma, \widehat{\gamma(\overline{\lambda})}_\gamma \right\rangle_\gamma = \langle \gamma(\overline{\mu}), \gamma(\overline{\lambda})\rangle.$$

If $R \in B_n(\Omega)$, then the Hermitian holomorphic vector bundle (E_R, π) has many choices of cross sections $\gamma : \Omega \to E_R$. For example, if T is a natural analytic left invertible, the γ_i in Definition 4.1.7 are cross sections for T^*. By construction, the collection of cross-sections $\{\gamma_i\}_{i=1}^n$ satisfy $\{\gamma_i(\lambda)\}_{i=1}^n$ form a basis for E_λ. Since the fibers E_λ of E_R are $\ker(R - \lambda)$, and $\bigvee \ker(R - \lambda) = \mathscr{H}$, we have that the collection of $\gamma_i : \Omega \to \mathscr{H}$ have dense span in $\mathscr{H}$. The following theorem states that we can combine these sections to get a spanning holomorphic cross-section:

Theorem 4.2.2 ([44] - Theorem 5). *Let $\mathscr{H}$ be a Hilbert space, and $\{\gamma_i\}_{i=1}^n$ be holomorphic functions from Ω to $\mathscr{H}$ such that*

$$\bigvee_{\lambda \in \Omega} \operatorname{span}_{i=1,\ldots,n}\{\gamma_i(\lambda)\} = \mathscr{H}.$$

Then there exists holomorphic functions $\{\phi_i\}_{i=1}^n$ from $\Omega \to \mathbb{C}$ such that the map $\gamma : \Omega \to \mathscr{H}$ defined by

$$\gamma(\lambda) := \sum_{i=1}^n \phi_i(\lambda)\gamma_i(\lambda) \qquad \lambda \in \Omega$$

also spans $\mathscr{H}$.

The functions ϕ_i that appear in Theorem 4.2.2 are built as follows. Let $\mathscr{H}_1 = \bigvee_{\lambda\in\Omega} \gamma_1(\lambda)$. Then by construction, γ_1 is a holomorphic spanning cross-section for $\mathscr{H}_1$. Consider the RKHS of analytic functions built from γ_1. One can find a set of points $\{a_l\} \subset \Omega$ that is a *uniqueness set of* Ω, in the sense that the only function in this space associated to γ_1 that vanishes on $\{a_l\}$ is the zero function. Using a separation theorem due to Weierstrass, one can pick a holomorphic function ϕ_2 that vanishes exactly on $\{a_l\}$. Then $\gamma_1 + \phi_2\gamma_2$ ends up being a spanning section for the space $\mathscr{H}_2 = \bigvee_{\lambda\in\Omega} \operatorname{span}_{i=1,2}\{\gamma_i(\lambda)\}$. Iteratively, one selects holomorphic functions ϕ_i until a spanning section for the whole Hilbert space is built. In particular, one can choose ϕ_1 to be the identity function on Ω. For details, see [44].

For a concrete example of how this idea may be applied, let $\mathscr{H} = H^2(\mathbb{D}) \oplus H^2(\mathbb{D})$ and $T = T_z \oplus T_z$. Then $T^* \in B_2(\mathbb{D})$. Define $\gamma_1, \gamma_2 : \mathbb{D} \to \mathscr{H}$ via

$$\begin{array}{rcccc} \gamma_1(\lambda)(z) & = & k_\lambda(z) & \oplus & 0 \\ \gamma_2(\lambda)(z) & = & 0 & \oplus & k_\lambda(z) \end{array}$$

It is well-known that if $f \in H^2(\mathbb{D})$ is non-zero, then the zero set $Z(f) = \{a_n\}$ satisfies the Blaschke condition: $\sum 1 - |a_n| < \infty$. Therefore, if $S = \{1 - \frac{1}{n}\}_{n\geq 1}$, the only function $f \in H^2(\mathbb{D})$ that vanishes on S is the zero function. Using Blaschke products, there exists an analytic function ϕ over $\mathbb{D}$ with $Z(\phi) = S$. Now, define $\gamma : \mathbb{D} \to \mathscr{H}$ via

$$\gamma(\lambda)(z) = \gamma_1(\lambda)(z) + \phi(\lambda)\gamma_2(\lambda)(z) = k_\lambda(z) \oplus (\phi(\lambda)k_\lambda(z)).$$

The map γ is a spanning section for $\mathscr{H}$. Indeed, if $f = f_1 \oplus f_2 \in \mathscr{H}$ is orthogonal to $\gamma(\lambda)$ for each $\lambda \in \mathbb{D}$, then

$$0 = \langle f, \gamma(\lambda)\rangle = \langle f_1, k_\lambda\rangle + \overline{\phi(\lambda)}\langle f_2, k_\lambda\rangle = f_1(\lambda) + \overline{\phi(\lambda)} f_2(\lambda).$$

Since ϕ vanishes on S, we have that for each $\lambda \in S$, $0 = f_1(\lambda) + 0 = f_1(\lambda)$. Hence, f_1 vanishes on S so $f_1 = 0$. Therefore, $0 = \phi(\lambda) f_2(\lambda)$ for all $\lambda \in \mathbb{D}$. In particular, f_2 vanishes on a set with a limit point in $\mathbb{D}$, and thus is the zero function as well. Therefore, γ spans $\mathscr{H}$.

Notice that this construction is far from unique. Indeed, γ depends on a choice S and function ϕ which vanishes on S. Nevertheless, Theorem 4.2.2 provides a method for constructing spanning sections for all $R \in B_n(\Omega)$.

Corollary 4.2.3. *If $R \in B_n(\Omega)$, then (E_R, π) admits a spanning holomorphic cross-section.*

Suppose $R \in B_n(\Omega)$. A consequence of Corollary 4.2.3 is that R is unitarily equivalent to multiplication by z on a collection of analytic functions over Ω^*.

Let M_z denote the operator of multiplication by the indeterminate z. That is, for each $\lambda \in \Omega^*$, $M_z(\hat{f}_\gamma)(\lambda) = \lambda \hat{f}_\gamma(\lambda)$. Since $\overline{\lambda} \in \Omega$, it follows from the definition Cowen-Douglas operators that $\overline{\lambda}$ is an eigenvalue for R. Consequently, U_γ intertwines M_z on $\widehat{\mathscr{H}_\gamma}$ and R^* on $\mathscr{H}$. Indeed for all $f \in \mathscr{H}$,

$$\begin{aligned} (U_\gamma R^* f)(\lambda) = \widehat{(R^* f)}_\gamma(\lambda) &= \langle R^* f, \gamma(\overline{\lambda}) \rangle \\ &= \langle f, R\gamma(\overline{\lambda}) \rangle \\ &= \langle f, \overline{\lambda}\gamma(\overline{\lambda}) \rangle \\ &= (M_z U_\gamma f)(\lambda). \end{aligned} \tag{4.2}$$

Thus, we have $U_\gamma R^* = M_z U_\gamma$, so R^* is unitarily equivalent to M_z on $\widehat{\mathscr{H}_\gamma}$.

In our current study of natural analytic left invertible operators, Theorem A says that $T^* \in B_n(\Omega_T)$. Therefore, Equation (4.2) tells us that T is unitarily equivalent to M_z on $\widehat{\mathscr{H}_\gamma}$. Furthermore, $\Omega_T = \Omega_T^*$ as sets, so for ease of notation, we consider the functions in $\widehat{\mathscr{H}_\gamma}$ on Ω_T. We record this as a corollary.

Corollary 4.2.4. *Let T be an analytic, left invertible operator with* $\operatorname{ind}(T) = -n$ *for some positive integer n. Then T is unitarily equivalent to multiplication by z on a reproducing kernel Hilbert space of analytic functions on $\Omega_T^* = \Omega_T$.*

A natural question one might ask is, "What are the analytic functions in $\widehat{\mathscr{H}_\gamma}$"? The answer will depend on the choice of analytic section γ described above. We will describe a salient representation U_γ that blends together the Cowen-Douglas theory with the basis theory developed in Chapter Three.

Let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$, and $\{\gamma_i\}_{i=1}^n$ be defined as in Definition 4.1.7. By Corollary 4.1.10 and Theorem 4.2.2, there exists holomorphic functions $\{\phi_i\}_{i=1}^n$ from $\Omega \to \mathbb{C}$ such that

$$\gamma(\lambda) := \sum_{i=1}^n \phi_i(\lambda)\gamma_i(\lambda) = \sum_{i=1}^n \phi_i(\lambda) \sum_{j\geq 0} \lambda^j x'_{i,j}$$

is a holomorphic spanning cross-section for $\mathscr{H}$. By the comments following Theorem 4.2.2, ϕ_1 may be chosen to be the identity function. For each $f \in \mathscr{H}$ and $\lambda \in \Omega_T$, we have by Equation (4.1)

$$\hat{f}(\lambda) = \langle f, \gamma(\overline{\lambda})\rangle = \sum_{i=1}^n \phi_i(\lambda) \sum_{j\geq 0} \lambda^j \langle f, x'_{i,j}\rangle$$

where here we have repressed the subscript γ on $\hat{f}$. The reproducing kernel Hilbert space associated with this choice of analytic section will be simply denoted $\widehat{\mathscr{H}}$. We store this information in a definition:

Definition 4.2.5. Given a natural analytic left invertible T, let Ω_T be as in Definition 4.1.3. Let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$. Pick $\{\phi_i\}_{i=1}^n$ holomorphic functions such that the map

$$\gamma(\lambda) = \sum_{i=1}^n \phi_i(\lambda) \sum_{j\geq 0} \lambda^j x'_{i,j}$$

each $\lambda \in \Omega_T$ is a spanning holomorphic cross-section with $\phi_1 = 1$. For each each $f \in \mathscr{H}$, set

$$\hat{f}(\lambda) = \sum_{i=1}^{n} \phi_i(\lambda) \sum_{j\geq 0} \lambda^j \langle f, x'_{i,j}\rangle. \tag{4.3}$$

Let $\widehat{\mathscr{H}}$ denote the reproducing kernel Hilbert space of functions $\hat{f}$ arising from Equation (4.3) with inner product $\langle \hat{f}, \hat{g}\rangle = \langle f, g\rangle$. The representation of T as M_z on $\widehat{\mathscr{H}}$ is called the **canonical representation of T relative to $\{x_{i,0}\}_{i=1}^n$ and $\{\phi_i\}_{i=1}^n$.**

The terminology canonical is fitting for the above representation. In the canonical representation, the basis elements associated to T become the functions $\phi_k z^l$. That is, if $k = 1, \ldots n$, then $\widehat{x_{k,l}}(\lambda) = \phi_k(\lambda)\lambda^l$ for each $\lambda \in \Omega$. This follows directly by Corollary 3.4.6 and Equation (4.3):

$$\widehat{x_{k,l}}(\lambda) = \sum_{i=1}^{n} \phi_i(\lambda) \sum_{j\geq 0} \lambda^j \langle x_{k,l}, x'_{i,j}\rangle = \phi_k(\lambda)\lambda^l \tag{4.4}$$

In particular, since $\phi_1 = 1$, we have that $\widehat{\mathscr{H}}$ contains the functions of the form z^l. Furthermore, $\hat{x_{k,0}} = \phi_k \in \widehat{\mathscr{H}}$ for each $k = 1, \ldots, n$. Since $\{x_{k,0}\}_{k=1}^n$ form an orthonormal basis for $\ker(T^*)$, the functions $\{\phi_k\}_{k=1}^n$ are also orthogonal.

Recall that in general, the reproducing kernel at λ is given by $k_\lambda = \gamma(\overline{\lambda})$. Hence, for the canonical representation, the reproducing kernel $K : \Omega^2 \to \mathbb{C}$ for $\widehat{\mathscr{H}}$ takes on the following form:

$$K(\lambda, \mu) = \langle \gamma(\overline{\mu}), \gamma(\overline{\lambda})\rangle = \sum_{k=1}^{n}\sum_{i=1}^{n} \phi_i(\lambda)\overline{\phi_k(\mu)} \sum_{l\geq 0}\sum_{j\geq 0} \overline{\mu}^l \lambda^j \langle x'_{k,l}, x'_{i,j}\rangle$$

where by Proposition 3.4.7, convergence does not depend on the order of the four sums. The kernel is analytic in λ, and co-analytic in μ by construction.

Under the canonical representation, $T^\dagger$ becomes "division by z". To make this precise, we require a simple lemma:

Lemma 4.2.6. *Let T_1 and T_2 be left invertible operators with Moore-Penrose inverses $T_1^\dagger$ and ${T_2}^\dagger$. If $T_2 = UT_1U^*$ for some unitary U, then ${T_2}^\dagger = UT_1^\dagger U^* = (UT_1U^*)^\dagger$.*

Proof. Recall that ${T_2}^\dagger = ({T_2}^*T_2)^{-1}{T_2}^*$. Hence,

$${T_2}^\dagger = (UT_1^*T_1U^*)^{-1}UT_1^*U^* = U(T_1^*T_1)^{-1}U^*UT_1^*U^* = UT_1^\dagger U^*.$$

□

Corollary 4.2.7. *If T is analytic with index $-n$, and $U_\gamma : \mathscr{H} \to \widehat{\mathscr{H}_\gamma}$ is the unitary such that $M_z = U_\gamma T U_\gamma^*$, then $M_z^\dagger = (U_\gamma T U_\gamma^*)^\dagger$.*

Now, the functions inside $\ker(M_z^\dagger)$ are the span of the orthogonal functions $\{\phi_i\}_{i=1}^n$. Furthermore, $\operatorname{ran}(M_z) = \ker(M_z^\dagger)^\perp$ consists of functions of the form $z\hat{g}$. From the preceding corollary, $M_z^\dagger M_z = I$, so it follows that either $M_z^\dagger \hat{f} = 0$ (if $\hat{f}$ is linear combination of the ϕ_i) or $M_z^\dagger \hat{f} = z^{-1}\hat{f}$ otherwise.

Expanding on this computation, suppose that $\hat{f} \in \widehat{\mathscr{H}}$ is of the form $\phi_i z^j$. Consider the action of $M_z^{\dagger n}$ on $\hat{f}$. By construction, $M_z^{\dagger n}(\phi_i z^j)(\lambda)$ is equal to 0 if $n \geq j$ and $\phi_i z^{j-n}$ otherwise.

For emphasis, the operator $M_{z^{-1}}$ of division by z is not well defined on $\widehat{\mathscr{H}}$ since $0 \in \Omega$ and $\widehat{\mathscr{H}}$ contains the constant functions. Yet $M_{z^{-1}}$ is well defined as a map from $\operatorname{ran}(M_z) = \ker(M_z^\dagger)^\perp$ to $\widehat{\mathscr{H}}$. By the above computation, $M_z^\dagger$ is $M_{z^{-1}}$ on $\ker(M_z^\dagger)^\perp$. Hence, $T^\dagger$ is $M_{z^{-1}}$ wherever the operator $M_{z^{-1}}$ is well defined, and 0 otherwise. This can be succinctly written as

$$M_z^\dagger = M_{z^{-1}}Q_1$$

where Q_1 is the projection onto $\ker(M_z^\dagger)^\perp$. More generally for each n, we have that

$$M_z^{\dagger n} = M_{z^{-n}}Q_n$$

where Q_n is the projection onto $\ker(M_z^{\dagger^n})^\perp$.

This model gives intuition into the structure of $\mathfrak{A}_T$. By Proposition 3.2.5, $\mathrm{Alg}(M_z, M_z^\dagger)$ consists of operators of the form

$$F + \sum_{k=0}^{N} a_k {M_z}^k + \sum_{l=1}^{M} b_l M_z^{\dagger^l} = F + \sum_{k=0}^{N} a_k M_{z^k} + \sum_{l=1}^{M} b_l M_{z^{-l}} Q_l$$

where F is a finite rank operator. One could combine via linearity the "analytic" component of the above sum to get

$$F + M_{\sum_{k=0}^{N} a_k z^k} + \sum_{l=1}^{M} b_l M_{z^{-l}} Q_l.$$

In some sense, the "principal part" $\sum_{l=1}^{M} b_l M_{z^{-l}} Q_l$ may also be combined into a single multiplication operator. Unfortunately, this is not done as effortlessly. We do have that $Q_l \leq Q_k$ for all $k \leq l$. Therefore, for all $\hat{f} \in ker({T^\dagger}^M)^\perp$, the sum of the principal pieces combine into a single multiplication operator. That is,

$$\left(\sum_{l=1}^{M} b_l M_{z^{-l}} Q_l \right) (\hat{f})(\lambda) = \sum_{l=1}^{M} b_l \frac{\hat{f}(\lambda)}{\lambda^l} = \left(M_{\sum_{l=1}^{M} b_l z^{-l}} \hat{f} \right) (\lambda)$$

However, this fails on $\ker({T^\dagger}^M)$, as some operators in the principal part have kernels contained in $\ker({T^\dagger}^M)$. For example, if $\hat{f}$ is perpendicular to $\ker({T^\dagger}^L)$ but not perpendicular to $\ker({T^\dagger}^{L+1})$, then

$$\left(\sum_{l=1}^{M} b_l M_{z^{-l}} Q_l \right) (\hat{f})(\lambda) = \sum_{l=1}^{L} b_l \frac{\hat{f}(\lambda)}{\lambda^l} = \left(M_{\sum_{l=1}^{L} b_l z^{-l}} \hat{f} \right) (\lambda).$$

This discussion demonstrates that we have a canonical analytic model to represent $\mathfrak{A}_T$. It is the norm limit of finite rank operators plus multiplication operators that have "Laurent" polynomials as symbols.

Heuristic 4.2.8. *If T is a natural analytic left invertible operator, then the algebra $\mathfrak{A}_T$ is compact perturbations of multiplication operators whose symbols are Laurent series centered at zero.*

In this section, we have shown that $T = M_z$ on a RKHS of analytic functions. To some extent, a converse statement is true as well. In [37], Richter shows if T is M_z on a reproducing kernel Hilbert space of analytic functions, then under suitable assumptions, T is an analytic left invertible operator. We discuss this result in Section 5.3.2 on the classification of $\mathfrak{A}_T$ via the reproducing kernel Hilbert space $\widehat{\mathscr{H}}$.

4.3 Reduction of Index - Strongly Irreducible Operators

Suppose that T is an analytic (pure) isometry with Fredholm index $-n$ for $n \geq 2$. Then T can be decomposed as a direct sum of pure isometries T_i each with Fredholm index -1. This decomposition is clearly unique up to unitary equivalence. A similar, though much weaker, statement is true for general analytic left invertible operators. We require some terminology.

Definition 4.3.1 ([23])**.** An operator $R \in \mathscr{B}(\mathscr{H})$ is **strongly irreducible** if there is no non-trivial idempotent in $\{R\}'$, the commutant of R. Equivalently, R is strongly irreducible if XRX^{-1} is an irreducible operator for every invertible operator X. We denote the set of all strongly irreducible operators over $\mathscr{H}$ by (SI).

Clearly, strong irreducibility is a similarity invariant. Moreover, it follows by definition that $R \in (SI)$ if and only if $R^* \in (SI)$.

Strongly irreducible operators play an important role in single operator theory. They serve a role equivalent to the Jordan blocks in the infinite dimensional setting. To see why, we recall some facts about Jordan canonical forms.

Definition 4.3.2. For $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, let

$$J_k(\lambda) := \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix}$$

denote the **Jordan block of size k for λ**.

The next proposition lists some important facts about Jordan blocks for our current conversation. It will also be useful in characterizing the similarity orbit of Cowen-Douglas operators in Section 5.2. First, we recall a definition:

Definition 4.3.3. If $\mathfrak{A}$ is a unital Banach algebra, then **Jacobson radical** is

$$\mathrm{rad}(\mathfrak{A}) = \{b \in \mathfrak{A} : \rho(ab) = 0 \text{ for all } a \in \mathfrak{A}\}$$

where $\rho(x)$ is the spectral radius of x. Equivalently, it is the largest ideal satisfying $\sigma(b) = 0$ for all b in the ideal.

Proposition 4.3.4 ([26])**.** *For $k \in \mathbb{N}$ and all $\lambda \in \mathbb{C}$, the following hold:*

i. The commutant of the Jordan block $J_k(\lambda)$ is

$$\{J_k(\lambda)\}' = \left\{ \begin{pmatrix} a_1 & & & 0 \\ a_2 & a_1 & & \\ & \ddots & \ddots & \\ a_k & & a_2 & a_1 \end{pmatrix} \right\}.$$

ii. $J_k(\lambda)$ is strongly irreducible.

iii. If $A \in M_k$ is strongly irreducible, then A is similar to $J_k(\mu)$ for some $\mu \in \mathbb{C}$.

If $A \in M_n$, the Jordan canonical forms theorem states that A is similar to a direct sum of Jordan blocks. This decomposition is unique, up to the ordering of the blocks. If $\sigma(A) = \{\lambda_i\}_{i=1}^n$, then we write

$$A \sim \bigoplus_{i=1}^{l} J_{k_i}(\lambda_i)^{(m_i)}$$

where the superscript (m_i) denotes the orthogonal direct sum of m_i copies of the Jordan block $J_{k_i}(\lambda_i)$. In other words, the Jordan decomposition theorem states that, up to similarity, each matrix has a unique decomposition as a direct sum of strongly irreducible operators.

Our current goal is to understand how this statement translates into the infinite dimensional setting. To help make this more precise, we have the following definition:

Definition 4.3.5 ([23])**.** A sequence $\{E_j\}_{j=1}^l$, $1 \leq l \leq \infty$ of non-zero idempotents on $\mathscr{H}$ is called a **spectral family** if

i. there exists an invertible operator $X \in \mathscr{B}(\mathscr{H})$ such that $\{XE_jX^{-1}\}$ are pairwise orthogonal projections

ii. $\sum_{j=1}^l E_j = I$.

Furthermore, if $R \in \mathscr{B}(\mathscr{H})$, then the spectral family is a **strongly irreducible decomposition of** R if

iii. $E_jR = RE_j$ for all j

iv. $R \mid \mathrm{ran}(E_j) \in (SI)$.

In other words, R has a strongly irreducible decomposition if R is the topological direct sum strongly irreducible operators. Equivalently, R is similar to the orthogonal direct sum of strongly irreducible operators. We denote this by $R \sim \oplus_{j=1}^l R_j$.

In finite dimensions, Jordan canonical forms force each matrix to have a unique SI decomposition up to similarity. This is not the case for operators in $\mathscr{B}(\mathscr{H})$. Not every operator in $\mathscr{B}(\mathscr{H})$ has a strongly irreducible decomposition. Moreover, even if an operator has a strongly irreducible decomposition, it may not be unique [26]. Therefore, we make the following definition:

Definition 4.3.6. Let $R \in \mathscr{B}(\mathscr{H})$, and $\mathcal{E} = \{E_j\}_{j=1}^{l_1}$ and $\mathcal{E}' = \{E_j'\}_{j=1}^{l_2}$ be two strongly irreducible decompositions of R. We say $\mathcal{E}$ and $\mathcal{E}'$ are **similar** if

i. $l_1 = l_2 = l$

ii. there exists an invertible operator $X \in \{R\}'$, the commutant of R, such that $XE_jX^{-1} = E_j'$ for all $1 \leq j \leq l$.

If R has a strongly irreducible decomposition, we say that R has a **unique strongly irreducible decomposition up to similarity** if any two of the decompositions are similar.

There is an extensive amount of work relating strongly irreducible decompositions of operators to K-theory [4], [23], [25], [26]. We will mention some of these results in in a later section. Of particular interest to us in the present are the following deep results due to Y. Cao, J. Fang and C. Jiang:

Theorem 4.3.7 ([26] - Theorem 5.5.12)**.** *Each operator in $S \in B_1(\Omega)$ is strongly irreducible. Moreover for any n, if $R \in B_n(\Omega)$, then R has a unique SI decomposition up to similarity. Furthermore, $R \sim \oplus_{j=1}^{m} R_j$ where $R_j \in (SI) \cap B_{n_j}(\Omega)$ and $\sum_{j=1}^{m} n_j = n$.*

Corollary 4.3.8. *Let T be an analytic left invertible operator with* $\operatorname{ind}(T) = -n$ *for some* $1 \leq n < \infty$. *Then $T \sim \oplus_{j=1}^{m} T_j$ where T_j are analytic, $\sum_{j=1}^{m} \operatorname{ind}(T_j) = -n$ and $T_j \in (SI)$.*

Theorem 4.3.7 states that operators in the Cowen-Douglas class have a decomposition analogous to the Jordan canonical forms for matrices. Without loss of generality, we may assume that if $R \in B_n(\Omega)$, then $R = \oplus_{j=1}^{m} R_j$ where $R_j \in (SI) \cap B_{n_j}(\Omega)$ where $\sum_{j=1}^{m} n_j = n$. This decomposition suggests that in order to understand $\mathfrak{A}_T$, we should first study the natural analytic left invertible operators that are strongly irreducible. In particular, we should study the analytic left invertible operators with Fredholm index -1.

In the isometric case, $T^* \in B_n(\Omega)$ decomposes to a direct sum of n strongly irreducible operators in $B_1(\Omega)$. Equivalently, pure isometric operators with $\text{ind}(T) = -n$ decompose into n "Jordan blocks" of size 1. This turns out to not be the case in general. Notice that if $R \in B_n(\Omega) \cap (SI)$, then it cannot be further decomposed as a direct sum. Indeed, suppose to the contrary that $R \in B_n(\Omega) \cap (SI)$ and $R \sim \oplus_{k=1}^{n} R_k$ with $R_k \in B_1(\Omega)$. By Theorem 4.3.7, each operator in $B_1(\Omega)$ is strongly irreducible. Hence, R would have two strongly irreducible decompositions that are dissimilar. But Theorem 4.3.7 states that all Cowen-Douglas operators have a unique SI decomposition up to similarity, contradicting the assumption that $R \in B_n(\Omega) \cap (SI)$ and $R \sim \oplus_{k=1}^{n} R_k$.

Thus, if there exists left invertible operators with $T^* \in B_n(\Omega) \cap (SI)$ for $n \geq 2$, it would not be possible to decompose T as a direct sum of left invertibles with Fredholm index -1. This is unfortunately the case, as the following example outlines:

Example 4.3.9. In this example, we will construct Toeplitz operators on a subspace of a Sobolev space. These operators will be strongly irreducible, and after combining them into an operator that looks like a Jordan block, we can form strongly irreducible operators of any index. Throughout, we fix $\epsilon > 0$, and let $\Omega = \{\lambda : |\lambda| < \epsilon\}$. We begin with a definition.

Definition 4.3.10. If dm denotes the planar Lebesgue measure, then the Hilbert space $W^{2,2}(\Omega)$ consists of the $f \in L^2(\Omega,\ dm)$ such that the first and second order distributional partial derivatives of f belong to $L^2(\Omega,\ dm)$.

Let M_z be multiplication by the independent variable on Ω. Then $M_z \in W^{2,2}(\Omega)$. Let $\mathscr{R}$ denote the algebra generated by rational functions of M_z with poles off $\overline{\Omega}$. Consider the action of this algebra on the identity function 1 over Ω. We let $R(\Omega)$ be the subspace of $W^{2,2}(\Omega)$ given by

$$R(\Omega) := \mathscr{R}1.$$

Note that $R(\Omega)$ is the subspace generated by rational functions with poles off of $\overline{\Omega}$. Moreover, $R(\Omega)$ is invariant under M_f for $f \in R(\Omega)$. For $f \in R(\Omega)$, define $T_f \in \mathscr{B}(R(\Omega))$ via $T_f := M_f \mid_{R(\Omega)}$. Then we have the following:

Lemma 4.3.11 ([23] - Corollary 3.3)**.** *T_z is a left invertible operator with* $\operatorname{ind}(T_z) = -1$*. In particular, $T_z^* \in B_1(\Omega)$.*

Now for any $n \in \mathbb{N}$, define $J_n(T_z) \in \mathscr{B}(\bigoplus_{j=1}^{n} \mathscr{R})$ via

$$J_n(T_z) := \begin{pmatrix} T_z & & & 0 \\ 1 & T_z & & \\ & \ddots & \ddots & \\ 0 & & 1 & T_z \end{pmatrix}.$$

Proposition 4.3.12 ([23] - Theorem 3.5)**.** *For $J_n(T_z)$ defined above, we have*

$$\{J_n(T_z)\}' = \left\{ \begin{pmatrix} T_{f_1} & & & 0 \\ T_{f_2} & T_{f_1} & & \\ & \ddots & \ddots & \\ T_{f_n} & & T_{f_2} & T_{f_1} \end{pmatrix} : f_1, \dots f_n \in R(\Omega) \right\}.$$

From Proposition 4.3.12, it follows that $J_n(T_z)$ is strongly irreducible. Indeed, if $P \in \{J_n(T_z)\}'$ is an idempotent, then since $P^2 = P$, it follows that $f_1^2 = f_1$ on Ω. Hence, $f_1 = 1$ or $f_1 = 0$. In either case, if $P^2 = P$, then the terms on the off diagonal must all be zero. This concludes our example.

The previous example illustrates a general result about Cowen-Douglas operators. Namely, Cowen-Douglas operators of rank n take the form of triangular operators of size n:

Theorem 4.3.13 ([23] - Theorem 1.49)**.** *Let* $R \in B_n(\Omega)$ *for* $1 \leq n < \infty$*. Then there exists* n *operators* $R_1, \dots R_n$ *such that* $R_i \in B_1(\Omega)$ *and*

$$R = \begin{pmatrix} R_1 & * & * & * \\ & R_2 & * & * \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$$

with respect to some decomposition $\mathscr{H} = \oplus_{i=1}^{n} \mathscr{H}_i$*.*

Corollary 4.3.14. *If* T *is an analytic left invertible with* $\operatorname{ind}(T) = -n$ *for* $1 \leq n < \infty$*, then there exists* n *analytic left invertibles* $T_1, \dots T_n$ *such that* $\operatorname{ind}(T_i) = -1$ *and*

$$T = \begin{pmatrix} T_1 & & & \\ * & T_2 & & \\ \vdots & \vdots & \ddots & \\ * & * & \dots & T_n \end{pmatrix} \tag{4.5}$$

with respect to some decomposition $\mathscr{H} = \oplus_{i=1}^{n} \mathscr{H}_i$*.*

Corollary 4.3.14 further emphasizes the need to analyze analytic left invertible operators with $\operatorname{ind}(T) = -1$. We showed above that we can always decompose T into a direct sum of strongly irreducible pieces. The strongly irreducible blocks have the form of lower triangular operators. If T is decomposed as in Corollary 4.3.14, then $T_n = T \mid_{\mathscr{H}_n}$ and T_n is an analytic left invertible operator with $\operatorname{ind}(T_n) = -1$. If we are to gain any insight into a general $\mathfrak{A}_T$, it is mandatory to understand the index -1 case first. This analysis will be taken up next chapter.

Chapter 5

The Algebra $\mathfrak{A}_T$

As remarked earlier, given a left invertible T, we view $\mathfrak{A}_T$ as a natural generalization of the concrete C*-algebra generated by an isometry. By the Wold-decomposition, we can always reduce an isometry to its purely (analytic) isometric component. If the Fredholm index of the analytic isometry is $-n$, then this isometry is unitarily equivalent to a direct sum of n unilateral shift operators. Hence, in order to analyze the C*-algebra generated by an isometry, it is important to first understand the C*-algebra generated by an analytic isometry of Fredholm index -1.

The preceding sections showed that, in general, we cannot reduce to either of these assumptions (analytic or $\text{ind}(T) = -1$) as we could in the isometric case. Example 3.5.1 demonstrated that T cannot be decomposed as a direct sum of an analytic operator and an invertible operator. Furthermore, Example 4.3.9 shows that even if an operator is analytic, it cannot be reduced to the index -1 case. Nevertheless, there is a summand on which T will be analytic. Similar statements may be made about strong irreducibility and the Fredholm index. Under the assumption of analytic, Theorem A implies that T^* is Cowen-Douglas. Corollary 4.3.14 tells us that, in this case, T may be written as a triangular operator where each element on the diagonal is an analytic left invertible of index -1.

Although we cannot reduce to the case of analytic or index -1, the epistemological viewpoint of the author is that an important first step in understanding $\mathfrak{A}_T$ is simplifying to this case. We therefore make the following minimality assumptions on T for the remainder of this chapter:

Assumption. *Henceforth, our left invertible operators will satisfy*

i. The Fredholm index: $\mathrm{ind}(T) = -1$

ii. Analytic: $\bigcap T^n \mathscr{H} = 0$

As discussed in Section 2.2, if T is an analytic isometry with $\mathrm{ind}(T) = -1$, we can represent T as M_z on $H^2(\mathbb{T})$. This yielded an elegant representation for $C^*(T)$. The analyticity ensures that the basis associated to M_z, the orthonormal basis z^n, spans the Hilbert space. The Fredholm index guarantees that $\mathcal{T}$ will be an irreducible C*-algebra, which contains a compact $I - TT^*$, and therefore all the compacts. Furthermore, one discovers that each element of $\mathcal{T}$ may be uniquely written as $T_f + K$ for some $f \in C(\mathbb{T})$ and $K \in \mathscr{K}(\mathscr{H})$.

The general case is similar. That is, if T is an analytic, left invertible operator with Fredholm index -1, then $\mathfrak{A}_T$ contains the compact operators. As a consequence, we will determine the isomorphism classes of $\mathfrak{A}_T$.

It is worth remarking that since $\mathfrak{A}_T$ is a concrete operator algebra, it belongs to many reasonable categories. A priori, it is not clear which choice of morphism one should consider (bounded, completely bounded, etc.). Fortunately, all reasonable choices are equivalent. It will be shown that two such algebras are boundedly isomorphic if and only if the isomorphism is implemented by an invertible. This will bring us to analyze the similarity orbit of T. For Cowen-Douglas operators, the similarity orbit has been extensively studied. We will leverage these results into our analysis of the study of $\mathfrak{A}_T$.

5.1 The Compact Operators

In this section, we show that if T is analytic left invertible with $\operatorname{ind}(T) = -1$, then $\mathfrak{A}_T$ contains the compact operators. Our approach is to show that, more generally $\overline{\operatorname{Alg}}(T, L)$ contains the compact operators for any left inverse T and left inverse L. This will allow us to conclude that $\overline{\operatorname{Alg}}(T, L) = \mathfrak{A}_T$ for any left inverse L. First, let us establish some notation.

Fix a left inverse L of T. We set $F_{0,0} = I - TT^\dagger$. That is, $F_{0,0}$ is the projection onto $\ker(T^\dagger)$. We define

$$F_{n,m,L} := T^n(I - TT^\dagger)L^m$$

for each $n, m \in \mathbb{Z}_{\geq 0}$. For $x, y, z \in \mathscr{H}$ we use $\theta_{x,y}$ to denote the rank one operator $z \mapsto \langle z, y\rangle x$.

Recall the Schauder basis and dual basis associated to T and L. Notice that since $\operatorname{ind}(T) = -1$, we have a simplified notation. Concretely, let $x_0 \in \ker(T^*)$ be a unit vector. Then $\operatorname{span}\{x_0\} = \ker(T^*)$. Denote the Schauder basis of T and dual basis T (with respect to L) via $x_n := T^n x_0$ and $x'_n := (L^*)^n x_0$. Then by definition, $I - TT^\dagger$ is the projection θ_{x_0,x_0}. So for each n, m and $x \in \mathscr{H}$,

$$F_{n,m,L}(x) = T^n(I - TT^\dagger)L^m(x) = T^n(\langle L^m(x), x_0\rangle x_0) = \langle x, x'_m\rangle x_n.$$

That is, $F_{n,m,L}$ is the rank one operator θ_{x_n,x'_m}. Let

$$\mathscr{K}_L := \overline{\operatorname{span}}\{F_{n,m,L}\}_{n,m\geq 1}.$$

Recall from Proposition 3.2.7 that if $L = T^\dagger$, then $\mathscr{K}_L = \mathscr{K}_T = \mathscr{C}$, the commutator ideal. As $F_{n,m,L} \in \operatorname{Alg}(T, L)$, $\mathscr{K}_L \subset \overline{\operatorname{Alg}}(T, L)$. Furthermore, the $F_{n,m,L}$ are rank one operators for each n, m; and so $\mathscr{K}_L \subset \mathscr{K}(\mathscr{H})$. Our previous work on Schauder bases allows us to conclude that $\mathscr{K}_L = \mathscr{K}(\mathscr{H})$.

Theorem 5.1.1. *Let* $T \in \mathscr{B}(\mathscr{H})$ *be an analytic, left invertible with* $\operatorname{ind}(T) = -1$*, and* L *be a left inverse of* T*. Then* $\mathscr{K}(\mathscr{H}) = \mathscr{K}_L$*. Thus,* $\overline{\operatorname{Alg}}(T, L)$ *contains the algebra of compact operators* $\mathscr{K}(\mathscr{H})$*.*

Proof. Let $y, z \in \mathscr{H}$. Since $\overline{\operatorname{span}}\{x_n\} = \mathscr{H} = \overline{\operatorname{span}}\{x'_n\}$, there exists a sequence of sums in x_n and x'_n converging to y and z respectively. It follows that the rank one operator $\theta_{y,z}$ is a norm limit of the span of the $\{F_{n,m,L}\}$ by simple estimates. Thus, $\mathscr{K}_L$ contains all the rank one operators. Since $\mathscr{K}_L$ is norm-closed by definition, $\mathscr{K}_L \supset \mathscr{K}(\mathscr{H})$. Since $\mathscr{K}_L \subset \mathscr{K}(\mathscr{H})$, we have $\mathscr{K}_L = \mathscr{K}(\mathscr{H})$. ☐

A consequence of Theorem 5.1.1 is that the definition of $\mathfrak{A}_T$ is not dependent on the choice of left inverse.

Corollary 5.1.2. *Let* $T \in \mathscr{B}(\mathscr{H})$ *be left invertible (analytic with* $\operatorname{ind}(T) = -1$*), and* L *be a left inverse of* T*. Then* $\mathfrak{A}_T = \overline{\operatorname{Alg}}(T, L)$*.*

Proof. By Proposition 3.1.8, each left inverse L of T has the form

$$L = T^{\dagger} + A(I - TT^{\dagger})$$

for some $A \in \mathscr{B}(\mathscr{H})$. Thus, each left inverse of T differs from $T^{\dagger}$ by a compact operator. By Theorem 5.1.1, $\overline{\operatorname{Alg}}(T, L)$ contains $\mathscr{K}(\mathscr{H})$, and therefore $T^{\dagger}$. So $\overline{\operatorname{Alg}}(T, L) \subseteq \mathfrak{A}_T$. Reversing the argument, $\overline{\operatorname{Alg}}(T, L) = \mathfrak{A}_T$. ☐

Recall that an ideal $\mathscr{K}$ of a Banach Algebra $\mathfrak{A}$ is said to be *essential* if it has non-trivial intersection with all non-zero ideals of $\mathfrak{A}$. Alternatively, if $A \in \mathfrak{A}$ and $A\mathscr{K} = 0$, then $A = 0$. In the next section, we investigate the morphisms between algebras of the form $\mathfrak{A}_T$. An important result required in subsequent analysis is the following:

Proposition 5.1.3. *The compact operators $\mathscr{K}(\mathscr{H})$ are an essential ideal of $\mathfrak{A}_T$. In fact, $\mathscr{K}(\mathscr{H})$ is contained in any closed ideal of $\mathfrak{A}_T$.*

Proof. Let $\mathfrak{J}$ be a non-zero closed two sided ideal of $\mathfrak{A}_T$, and $A \in \mathfrak{J}$ be non-zero. Then there is some $x \in \mathscr{H}$ such that $\|Ax\| = 1$. Fix $y \in \mathscr{H}$, and let $B := \theta_{y,A(x)}$. Then $B(A(x)) = y$. Thus for all $h \in \mathscr{H}$, we have

$$BA\theta_{x,x}A^*B^*(h) = BA\left(\langle h, BA(x)\rangle x\right) = \langle h, y\rangle y = \theta_{y,y}(h).$$

Since $\mathscr{K}(\mathscr{H}) \subset \mathfrak{A}_T$, it follows that the rank one operators B and $\theta_{x,x}A^*B^*$ are in $\mathfrak{A}_T$. Since $A \in \mathfrak{J}$ and $\mathfrak{J}$ is an ideal, we must have that $\theta_{y,y}$ is inside of $\mathfrak{J}$. Thus for any $w, z \in \mathscr{H}$, $\theta_{w,z} = \theta_{w,y}\theta_{y,y}\theta_{y,z}$ is in $\mathfrak{J}$, so $\mathfrak{J}$ contains all the finite rank operators, and thus contains $\mathscr{K}(\mathscr{H})$. □

5.2 Isomorphisms of $\mathfrak{A}_T$

Now that we have established that the compact operators $\mathscr{K}(\mathscr{H}) \subseteq \mathfrak{A}_T$ as a minimal ideal, we may identify the isomorphism classes of $\mathfrak{A}_T$. We will show that if T_1 and T_2 are two analytic left invertible operators with Fredholm index -1, then $\mathfrak{A}_{T_1}$ is boundedly isomorphic to $\mathfrak{A}_{T_2}$ if and only if the algebras are similar. This will be done by looking at how the bounded isomorphism behaves on the compact operators.

An interesting fact about bounded homomorphisms of C*-algebras is that they necessarily have closed range. Indeed, we have the following observation due to Pitts:

Theorem 5.2.1 ([34] - Theorem 2.6)**.** *Suppose $\mathfrak{A}$ is a C*-algebra and $\phi : \mathfrak{A} \to \mathscr{B}(\mathscr{H})$ is a bounded homomorphism. Let $\mathfrak{J} = \ker\phi$. Then there exists a real number $k > 0$ such that for* *each $n \in \mathbb{N}$, and $R \in M_n(\mathfrak{A})$,*

$$k\,dist(R, M_n(\mathfrak{J})) \leq \|\phi_n(R)\|.$$

Corollary 5.2.2. *If $\phi : \mathscr{K}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ is a bounded monomorphism, then there exists a real number k such that*

$$k\|R\| \leq \|\phi(R)\|.$$

That is, ϕ has closed range.

Given an invertible operator $V \in \mathscr{B}(\mathscr{H})$, we define $\mathrm{Ad}_V : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ via $\mathrm{Ad}_V(T) = VTV^{-1}$. As previously mentioned, to fully analyze $\mathfrak{A}_T$, we need to determine which category we are working in. On the one hand, we can view $\mathfrak{A}_T$ as an operator algebra, with our morphisms being completely bounded homomorphisms. On the other hand, we may want to simply view $\mathfrak{A}_T$ as a Banach algebra, where the morphisms are bounded homomorphisms. Fortunately, Theorem 5.1.1 forces the monomorphisms of these two categories to coincide:

Theorem B. *Let T_i, $i = 1, 2$ be left invertibles (analytic with $\mathrm{ind}(T_i) = -1$) and $\mathfrak{A}_i = \mathfrak{A}_{T_i}$. Suppose that $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a bounded isomorphism. Then $\phi = Ad_V$ for some invertible $V \in \mathscr{B}(\mathscr{H})$.*

Proof. Let $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ be a bounded isomorphism. A brief outline of the proof is as follows. We first show that $\phi\,|_{\mathscr{K}(\mathscr{H})}$ is similar to a *-automorphism of $\mathscr{K}(\mathscr{H})$. It is well known that all *-automorphisms of $\mathscr{K}(\mathscr{H})$ have the form Ad_U for some unitary operator U. We then use the fact that ϕ restricted to an essential ideal has the form Ad_V to conclude that it must be equal to Ad_V on all of $\mathfrak{A}_1$. The details are as follows.

Note that $\phi\,|_{\mathscr{K}(\mathscr{H})}: \mathscr{K}(\mathscr{H}) \to \mathfrak{A}_2 \subset \mathscr{B}(\mathscr{H})$ is a bounded representation of the compact operators. It can be shown that every bounded representation of the compact operators is similar to a *-representation (more generally, every bounded representation of a nuclear C*-algebra is similar to a *-representation [7]). Let $W \in \mathscr{B}(\mathscr{H})$ be the invertible that conjugates $\phi\mid_{\mathscr{K}(\mathscr{H})}$ to a *-representation ψ. That is, $\phi(u) = W\psi(u)W^{-1}$ for every $u \in \mathscr{K}(\mathscr{H})$.

Now let us consider the $*-$representation ψ. Note that $\psi : \mathscr{K}(\mathscr{H}) \to W^{-1}\mathfrak{A}_2W$. The map $\mathrm{Ad}_{W^{-1}} : \mathfrak{A}_2 \to W^{-1}\mathfrak{A}_2W$ carries $\mathscr{K}(\mathscr{H})$ to $\mathscr{K}(\mathscr{H})$. Since every ideal of $W^{-1}\mathfrak{A}_2W$ has the form $W^{-1}\mathfrak{J}W$ for $\mathfrak{J}$ an ideal of $\mathfrak{A}_2$, it follows that $\mathscr{K}(\mathscr{H})$ is minimal in $W^{-1}\mathfrak{A}_2W$. Therefore, we must have that $\mathscr{K}(\mathscr{H}) \subseteq \psi(\mathscr{K}(\mathscr{H}))$.

Now, $\mathscr{K}(\mathscr{H})$ is equal to the closed span of the rank one projections on $\mathscr{H}$. As a result, if we can show that each rank one projection p gets sent to another rank one projection under ψ, then $\psi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$, yielding equality.

To this end, let p be a rank one projection, and $p' = \psi(p)$. If p' is not rank one, then there exists a non-zero projection q' properly contained under p'. Since $\psi(\mathscr{K}(\mathscr{H}))$ contains $\mathscr{K}(\mathscr{H})$, there exists a projection $q \in \mathscr{K}(\mathscr{H})$ such that $\psi(q) = q'$. Regarding ψ mapping from $\mathscr{K}(\mathscr{H})$ to $\psi(\mathscr{K}(\mathscr{H}))$, ψ is a *-isomorphism and hence invertible. ψ^{-1} is of course also a *-isomorphism, and therefore a positive map. Hence, if $q' < p'$, then $q < p$ by positivity of ψ^{-1}. This is absurd, since p was rank one. Thus, $\psi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$, so that $\mathscr{K}(\mathscr{H}) = \psi(\mathscr{K}(\mathscr{H}))$.

What we have just shown is that $\phi\mid_{\mathscr{K}(\mathscr{H})}$ is similar to a *-automorphism ψ of $\mathscr{K}(\mathscr{H})$. Every *-automorphism of $\mathscr{K}(\mathscr{H})$ is of the form Ad_U for some unitary operator U. Hence, we have that

$$\phi\mid_{\mathscr{K}(\mathscr{H})} = \mathrm{Ad}_W\psi = \mathrm{Ad}_W\mathrm{Ad}_U = \mathrm{Ad}_V$$

where $V = UW$. We now show that $\phi = \mathrm{Ad}_V$. To do this, first note that for all $A \in \mathfrak{A}_1$ and $K \in \mathscr{K}(\mathscr{H})$,

$$\phi(A)\phi(K) = \phi(AK) = \psi(AK) = \mathrm{Ad}_V(AK) = \mathrm{Ad}_V(A)\mathrm{Ad}_V(K) = \mathrm{Ad}_V(A)\phi(K)$$

So it follows that

$$(\phi(A) - \mathrm{Ad}_V(A))\mathrm{Ad}_V(K) = 0$$

for each $K \in \mathscr{K}(\mathscr{H})$. Cycling over all $K \in \mathscr{K}(\mathscr{H})$, we see that

$$(\phi(A) - \mathrm{Ad}_V(A))\mathscr{K}(\mathscr{H}) = 0.$$

Since $\mathscr{K}(\mathscr{H})$ is essential in $\mathfrak{A}_2$, we have that $\phi(A) = \mathrm{Ad}_V(A)$. $\square$

Theorem B is a harsh rigidity statement about classification. Indeed, $\mathfrak{A}_1$ is boundedly isomorphic to $\mathfrak{A}_2$ if and only if the algebras are similar. Consequently, if we wish to delineate these operator algebras into isomorphism classes, we need to understand the similarity orbit of left invertible operators. We define the following notation for the similarity orbit:

$$\mathcal{S}(T) := \{VTV^{-1} : V \in \mathscr{B}(\mathscr{H}) \text{ is invertible}\}.$$

In classifying the algebra $\mathfrak{A}_T$, we do not need to keep track of the similarity orbit of the Moore-Penrose inverse. Indeed, suppose T is left invertible with Moore-Penrose inverse $T^\dagger$, V is an invertible operator, and $T_2 := VTV^{-1}$. Then $L_2 := VT^\dagger V^{-1}$ is a left inverse of T_2. By Corollary 5.1.2, $\overline{\mathrm{Alg}}(T_2, L_2) = \mathfrak{A}_{T_2}$. Therefore to identify the isomorphism class of $\mathfrak{A}_T$, we may disregard $\mathcal{S}(T^\dagger)$. Hence, we pose the following question:

Question. *If T is left invertible (analytic,* $\mathrm{ind}(T) = -1$*), what is $\mathcal{S}(T)$?*

In general, it is impossible to completely classify the similarity orbit of an operator. However, analytic left invertible operators have added structure that aid in this analysis. By Theorem A, if T is analytic, $T^* \in B_n(\Omega)$ for a disc Ω centered at the origin. Clearly if we could identify $\mathcal{S}(T^*)$, then we would know $\mathcal{S}(T)$. Fortunately, similarity orbits of Cowen-Douglas operators have been extensively studied [15] [16] [25] [28] [44]. The similarity orbit of Cowen-Douglas operators can be completely described by K-theoretic means. We will highlight these results in the next section.

While the question of addressing the similarity orbit is paramount to a complete classification of our algebras $\mathfrak{A}_T$, it is not sufficient. More explicitly, suppose T_1 and T_2 are left invertible operators (analytic, $\mathrm{ind}(T) = -1$) with $\mathfrak{A}_1$ and $\mathfrak{A}_2$ isomorphic. Let V be the invertible that implements the isomorphism between $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and let $T_3 := VT_1V^{-1}$ and $L_3 := VT_1^{\dagger}V^{-1}$. Notice L_3 is a left inverse of T_3 and that $\overline{\mathrm{Alg}}(T_3, L_3) = \mathfrak{A}_2$. By Corollary 5.1.2, $\mathfrak{A}_3 = \mathfrak{A}_2$.

One would therefore be tempted to reduce to the case where $T_2 = T_3 = \mathrm{Ad}_V(T_1)$. However, it turns out that not every left invertible $S \in \mathfrak{A}_T$ will satisfy $\mathfrak{A}_S = \mathfrak{A}_T$. Consider the following example:

Example 5.2.3. We will construct a left invertible operator T inside the Toeplitz algebra $\mathcal{T}$ such that $\mathfrak{A}_T \neq \mathcal{T}$. Consider the Hardy space $H^2(\mathbb{T})$. Let $\phi_0 \in C(\mathbb{T})$ be given by

$$\phi_0(z) := \exp\left(\frac{\pi i}{2}(z-1)z\right)$$

for all $z \in \mathbb{T}$. Then $\phi_0(1) = 1$ and $\phi_0(-1) = -1$. Let

$$\epsilon_n(z) = z^n.$$

Define $\phi := M_{\epsilon_1}\phi_0$. Then ϕ satisfies $\phi(1) = \phi(-1) = 1$. Recall the following facts about invertible functions on $C(\mathbb{T})$ and their associated Toeplitz operators:

Theorem 5.2.4 ([30] Lem. 3.5.14, Thm. 3.5.15)**.** *Let $\phi \in C(\mathbb{T})$ be invertible. Then*

i. There exists a unique integer n such that $\phi = \epsilon_n e^{\psi}$ some $\psi \in C(\mathbb{T})$

ii. If $\phi = \epsilon_n e^{\psi}$, then the winding number n

iii. *We have* $\operatorname{ind}(T_\phi) =$ *negative the winding number of* ϕ

iv. T_ϕ *is invertible if and only if the winding number is zero if and only if* $\phi = e^\psi$ *some* $\psi \in C(\mathbb{T})$

By Theorem 5.2.4, the winding number of ϕ is 1, so $\operatorname{ind}(T_\phi) = -1$. Since both ϵ_1 and ϕ_0 belong to $H^\infty(\mathbb{T})$ we have that T_{ϵ_1} and T_{ϕ_0} commute, so the Toeplitz operator T_ϕ factors:

$$T_\phi = T_{\epsilon_1 \phi_0} = T_{\epsilon_1} T_{\phi_0}.$$

Also by Theorem 5.2.4, T_{ϕ_0} is invertible. The point-wise inverse of ϕ_0 is also continuous on $\mathbb{T}$. Therefore, the Toeplitz operator T_ϕ is left invertible with left inverse

$$L = {T_{\phi_0}}^{-1} T_{\epsilon_1}^* = T_{{\phi_0}^{-1}} T_{\epsilon_1}^* \in \mathcal{T}.$$

Moreover, since T_{ϵ_1} and T_{ϕ_0} commute, we have $(T_\phi)^n = T_{\epsilon_n} T_{{\phi_0}^n}$. Since $T_{{\phi_0}^n}$ is invertible, $T_{{\phi_0}^n} H^2(\mathbb{T}) = H^2(\mathbb{T})$. Consequently,

$$\bigcap {T_\phi}^n H^2(\mathbb{T}) = \bigcap T_{\epsilon_n} H^2(\mathbb{T}) = 0$$

so T_ϕ is analytic. Recall that $\mathfrak{A}_T \subset C^*(T)$ for any left invertible T. We remark that $C^*(T_\phi) \neq \mathcal{T}$. This follows from the following result due to Coburn:

Lemma 5.2.5 ([10] Cor. 6.3)**.** *If* ϕ *is in the disc algebra, then* $C^*(T_\phi) = \mathcal{T}$ *if and only if* ϕ *is injective.*

It is shown in [10] that $C^*(T_\phi)/\mathscr{K}(\mathscr{H})$ is isomorphic to continuous functions on $\mathbb{T}/\sim$, where $\sim$ is an equivalence relation identifying all points $z, w \in \mathbb{T}$ such that $\phi(z) = \phi(w)$.

Since $\phi(1) = \phi(-1)$, it follows by the above lemma that $\mathfrak{A}_T \subseteq C^*(T_\phi) \neq \mathcal{T}$. This concludes our example.

What the above example demonstrates is that not every left invertible operator in $\mathfrak{A}_T$ generates $\mathfrak{A}_T$. Therefore, determining the similarity orbit is not sufficient to delineate the isomorphism classes of $\mathfrak{A}_T$. Concretely, suppose $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are generated by T_1 and T_2 respectively. To determine if $\mathfrak{A}_1$ is isomorphic to $\mathfrak{A}_2$, it is not sufficient to verify that $\mathfrak{A}_2$ possesses an operator T_3 similar to T_1. This would demonstrate that $\mathfrak{A}_1$ is isomorphic to a subalgebra of $\mathfrak{A}_2$. If one wanted $\mathfrak{A}_1$ to be isomorphic to $\mathfrak{A}_2$, it is necessary to show that T_3 also generates $\mathfrak{A}_2$. With this caveat emphasized, we spend the next section investigating the similarity orbit of our class of left invertible operators.

5.3 The Similarity Orbit of T

If T is an analytic left invertible operator with $\operatorname{ind}(T) = -1$, then by Theorem A, $T^* \in B_1(\Omega)$ for $\Omega = \{\lambda : |\lambda| < \epsilon\}$. Therefore, classifying $\mathcal{S}(T)$ is equivalent to classifying the similarity orbit of Cowen-Douglas operators over a small disc centered at the origin. The problem of identifying when two Cowen-Douglas operators are similar is a classic one. In Cowen and Douglas' original work, they show that two operators $R_1, R_2 \in B_1(\Omega)$ are unitarily equivalent if and only if the curvature on the associated hermitian holomorphic vector bundles are equal [15]. Cowen and Douglas did not find a similarity classification however. They asked what is a complete similarity invariant of $B_1(\Omega)$, and more generally, $B_n(\Omega)$. Various authors have since worked on this problem, successfully describing the similarity orbit of Cowen-Douglas operators.

There are two approaches one could take to classification of the algebra $\mathfrak{A}_T$ for T analytic, left invertible with Fredholm index -1. One might try to parameterize the similarity orbit $\mathcal{S}(T)$ via some abstract object. Another approach is to try to find computable methods for determining when two left invertibles T_1 and T_2 are similar. In this section, we tackle both of these problems.

We begin by discussing some results of Jiang et. al. that allow us to classify $\mathcal{S}(T)$ via a K_0 group. This approach also provides a semi-computable method to determine when $T_1 \sim T_2$. We then seek a more concrete invariant that would allow one to quickly determine when two analytic left invertible operators with index -1 are not similar. We leverage the canonical reproducing kernel Hilbert space associated with T to achieve this result.

5.3.1 Similarity via K_0

In [24], Jiang describes the similarity orbit of strongly irreducible Cowen-Douglas operators using the K_0-group of the commutant algebra. Later, Jiang, Guo, and Ji gave a similarity classification of all Cowen-Douglas operators using the commutant [26]. Here we briefly outline these results, and how they connect to the discussion about strongly irreducible operators and Jordan forms from Section 4.3.

We begin by demonstrating how the classic Jordan canonical forms theorem can be phrased in terms of K-theory. Let $A \in M_n$. Then $A \sim \bigoplus_{i=1}^{l} J_{k_i}(\lambda_i)^{(m_i)}$ as in Section 4.3. We then have the following:

Proposition 5.3.1 ([26] - Theorem 2.2.6, 2.2.7)**.** *Let $A \in M_n$, with $A \sim \bigoplus_{i=1}^{l} J_{k_i}(\lambda_i)^{(m_i)}$. Then*

$$\begin{aligned} V(\{A\}') &\cong \mathbb{N}^l \\ K_0(\{A\}') &\cong \mathbb{Z}^l. \end{aligned}$$

The map that induces this isomorphism is given by

$$h([I]) = (m_1, m_2, \dots m_l)$$

where $[I]$ is the equivalence class corresponding to the identity matrix. Moreover, let $B, C \in$

M_n and $B = \bigoplus_{i=1}^{l} B_{k_i}^{(m_i)}$ where B_{k_i} are strongly irreducible (i.e. B_{k_i} are similar to Jordan block) and B_{k_i} is not similar to B_{k_j} for $i \neq j$. Then $B \sim C$ if and only if there exists an isomorphism

$$h : K_0(\{B \oplus C\}') \to \mathbb{Z}^l$$

with $h([I]) = (2m_1, 2m_2, \ldots 2m_l)$.

In other words, the K_0 group of the commutant of A contains all the information of the Jordan decomposition. Two matrices are similar if and only if they are both similar to the same Jordan decomposition $\bigoplus_{i=1}^{l} J_{k_i}(\lambda_i)^{(m_i)}$. This is equivalent to the direct sum of the matrices having Jordan decomposition $\bigoplus_{i=1}^{l} J_{k_i}(\lambda_i)^{(2m_i)}$, and this information is encoded in the K_0 group of the commutant.

This theory extends to the infinite dimensional setting. We have the following deep result due to Jiang et al.:

Theorem 5.3.2 ([26] - Theorem 4.2.1, 4.3.1)**.** *Let $R \in \mathscr{B}(\mathscr{H})$. Then the following are equivalent:*

- *i. $R \sim \bigoplus_{i=1}^{l} R_i^{m_i}$, $R_i \in (SI)$, R_i is not similar to R_j for $i \neq j$ and $R^{(n)}$ has a unique SI decomposition for all n.*
- *ii. $K_0(\{R\}') \cong \mathbb{Z}^l$ via the isomorphism*

$$h([I]) = (m_1, m_2, \ldots m_l).$$

By Theorem 4.3.7, we know that Cowen-Douglas operators have a unique SI decomposition. By definition, if $R \in B_n(\Omega)$, then $R^{(m)} \in B_{n*m}(\Omega)$, and hence has a unique SI decomposition. Combining Theorems 4.3.7 and 5.3.2, Jiang, Guo and Ji gave the complete classification of Cowen-Douglas operators up to similarity:

Theorem 5.3.3 ([24]). *Let $A, B \in B_n(\Omega)$. Suppose that $A = \bigoplus_{i=1}^{l} A_i^{(m_i)}$ where $A_i \in (SI)$ and A_i is not similar to A_j for $i \neq j$. Then $A \sim B$ if and only if $(K_0(\{A \oplus B\}'), V(\{A \oplus B\}'), I) \cong (\mathbb{Z}^l, \mathbb{N}^l, 1)$ via the isomorphism*

$$h([I]) = (2m_1, 2m_2, \ldots, 2m_l).$$

In particular, the result for Cowen-Douglas operators with Fredholm index 1 is as follows:

Theorem 5.3.4 ([26] - Proposition 5.1.7). *Let $A, B \in B_1(\Omega)$. Then A is similar to B if and only if*

$$K_0(\{A \oplus B\}') \cong \mathbb{Z}.$$

This result and its generalizations solve the question of the similarity orbit, and therefore, the isomorphism problem for $\mathfrak{A}_T$ from the last section. As stated, Theorem 5.3.4 is a rather difficult theorem to apply. Luckily in [24], Jiang provided the following theorem which concretely identifies the requirements on the isomorphism between the K_0 groups generated by A and B:

Theorem 5.3.5 ([24] - Theorem 4.4). *Two strongly irreducible Cowen-Douglas operators A and B are similar if and only if there is a group isomorphism $\alpha : K_0(\{A\}') \to K_0(\{B\}')$ satisfying the following:*

i. $\alpha(V(\{A\}')) = V(\{B\}')$

ii. $\alpha([I_{\{A\}'}]) = [I_{\{B\}'}]$, *where* $[I_{\{A\}'}]$ *is the equivalence class associated to the identity in the idempotents of* $M_\infty(\{A\}')$

iii. there exists non-zero idempotents $p \in M_\infty(\{A\}')$ *and* $q \in M_\infty(\{B\}')$ *such that* $\alpha([p]) = [q]$ *and* p *is equivalent to* q *in* $M_\infty(\{A \oplus B\}')$.

5.3.2 Similarity via $\widehat{\mathscr{H}}$

Theorems 5.3.4 and 5.3.5 establish that the obstruction to similarity is K-theoretic. As remarked, Theorem 5.3.4 is difficult to verify in practice. Theorem 5.3.5 is more intuitive, since it prescribes the similarity largely in terms of K-theory of the algebras $\{A\}'$ and $\{B\}'$ rather than $\{A \oplus B\}'$. Nevertheless, from an applications standpoint, this theorem is still a bit mysterious since we don't have an understanding of commutant $\{T^*\}'$ for $T^* \in B_1(\Omega)$. Here we attempt to remedy this by providing a description of the commutant in terms of multipliers, as well as a method to determine when two Cowen-Douglas operators fail to be similar in terms of the associated reproducing kernel Hilbert spaces.

As discussed in Chapter Four, if T is an analytic, left invertible operator with $\text{ind}(T) = -n$, then $T^* \in B_n(\Omega_T)$. Consequently, T is unitarily equivalent to M_z on a reproducing kernel Hilbert space of analytic functions over Ω_T. There, we discussed one particularly interesting representation involving the Schauder bases associated to T and T', namely, the canonical representation of T. In this chapter, we have assumed that $\text{ind}(T) = -1$. In this case, the Schauder basis representation of Definition 4.2.5 cleans up spectacularly. Fixing $x_0 \in \ker(T^*)$, we let $x_n := T^n x_0$ and $x'_n := T'^n x_0$. Then for every $\lambda \in \Omega_T$, we have

$$x_\lambda = \sum_{j \geq 0} \lambda^j x'_j$$

Then for each $f \in \mathscr{H}$, we have that $\hat{f} \in \widehat{\mathscr{H}}$ is given by

$$\hat{f}(\lambda) = \langle f, x_{\overline{\lambda}} \rangle = \sum_{j \geq 0} \lambda^j \langle f, x'_j \rangle$$

for all $\lambda \in \Omega_T$. The reproducing kernel simplifies to

$$K(\lambda, \mu) = \langle x_{\overline{\mu}}, x_{\overline{\lambda}} \rangle = \sum_{i \geq 0} \sum_{j \geq 0} \overline{\mu}^i \lambda^j \langle x_i', x_j' \rangle \tag{5.1}$$

Our goal in this section is to show that the reproducing kernel Hilbert space $\widehat{\mathscr{H}}$ is a similarity invariant of T. This will be done in two different ways - one emphasizing multipliers while the other emphasizes the positive kernel K.

The functions in $\widehat{\mathscr{H}}$ satisfy a nice factorization property studied by Richter [37]. He was interested in the invariant subspaces of well-behaved Banach spaces of analytic functions. Concretely, he investigated Banach spaces that satisfied the following axioms.

Properties 5.3.6. Given $\Omega \subset \mathbb{C}$ be open and connected, let $\mathscr{B}$ be a Banach space of analytic functions that satisfy properties:

I. The functional of evaluation at λ is continuous for all $\lambda \in \Omega$

II. If $f \in \mathscr{B}$, then $zf \in \mathscr{B}$

III. If $f \in \mathscr{B}$ and $f(\lambda) = 0$, then there exists a $g \in \mathscr{B}$ such that $(z - \lambda)g = f$.

Note that if a Hilbert space $\mathscr{H}$ satisfies the above axioms, the first condition requests $\mathscr{H}$ be a reproducing kernel Hilbert space. The second condition says that $\mathscr{H}$ is invariant under multiplication, and combined with the first says that M_z is bounded. The final condition is equivalent to asking that $M_z - \lambda$ is bounded below for every $\lambda \in \Omega$.

It is easy to see that if T is an analytic left invertible with $\text{ind}(T) = -1$, then the reproducing kernel Hilbert space $\widehat{\mathscr{H}}$ will satisfy these axioms. In [37] it is shown that a Hilbert space satisfies the axioms if and only if the Hilbert space arises from a Cowen-Douglas operator:

Proposition 5.3.7 ([37] - Theorem 2.10)**.** *Let $\Omega \subset \mathbb{C}$ be connected and open, and $R \in \mathscr{B}(\mathscr{H})$. Then $R \in B_1(\Omega^*)$ if and only if there is a Hilbert space $\widehat{\mathscr{H}}$ of analytic functions on Ω satisfying I-III in Properties 5.3.6 such that R^* is unitarily equivalent to $M_z \in \mathscr{B}(\widehat{\mathscr{H}})$.*

In the same paper, Richter showed that when a Banach space of analytic functions satisfies Properties 5.3.6, the similarity orbit of M_z can be identified in terms of multipliers. If the Banach spaces are actually Hilbert spaces, then the multipliers between them arise in a very natural way:

Proposition 5.3.8 ([37], Prop 2.4)**.** *Suppose $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces of analytic functions over Ω that satisfy I-III in Properties 5.3.6. Write M_i for multiplication by z on $\mathscr{H}_i$. Then $V \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ satisfies $VM_1 = M_2V$ if and only if there exists a multiplier $\phi \in \mathcal{M}(\mathscr{H}_1, \mathscr{H}_2)$ such that $V = M_\phi$. In particular, $\{M_1\}' = \{M_\phi : \phi \in \mathcal{M}(\mathscr{H}_1)\}$.*

Proposition 5.3.8 translates the problem of classifying the similarity orbit of T, a hard operator theoretic question, into a question about geometry of Hilbert spaces. To see how, we will require the following notation and lemma:

Definition 5.3.9 ([1])**.** Suppose $\Omega = \{z : |z| < \epsilon\}$, $K : \Omega^2 \to \mathbb{C}$ is an analytic kernel, and $0 < r < 1$. Set $\Omega_r := \{z : rz \in \Omega\}$. The **r-dilation of** K is the function $K_r : \Omega_r^2 \to \mathbb{C}$ given by $K_r(\lambda, \mu) := K(r\lambda, r\mu)$. If f is a function on Ω, let $f_r : \Omega_r \to \mathbb{C}$ be the function $f_r(\lambda) := f(r\lambda)$.

Lemma 5.3.10 ([1], Thm. 2.3)**.** *Suppose $\Omega = \{z : |z| < \epsilon\}$, $K : \Omega^2 \to \mathbb{C}$ is an analytic kernel, and $0 < r < 1$. Let $\mathscr{H}$ and $\mathscr{H}_r$ denoted the reproducing kernel space associated with K and K_r respectively. The operator $V : \mathscr{H} \to \mathscr{H}_r$ via $Vf = f_r$ is a unitary operator that preserves multipliers. That is, $\phi \in \mathcal{M}(\mathscr{H})$ if and only if $\phi_r \in \mathcal{M}(\mathscr{H}_r)$, and M_ϕ is unitarily equivalent to M_{ϕ_r}.*

Supposing $T_1, T_2 \in \mathscr{B}(\mathscr{H})$ are left invertible (analytic, $\mathrm{ind}(T_i) = -1$), represent T_i as M_i on $\widehat{\mathscr{H}_i}$. Now, it may be the case that the underlying sets $\Omega_i = \{z : |z| < \epsilon_i\}$ for $\widehat{\mathscr{H}_i}$ might not agree. However, one is certainly contained in the other. Without loss of generality, suppose that $\epsilon_1 < \epsilon_2$. We can perform an r-dilation of K_2 so that $\Omega_{2,r} = \Omega_1$. Lemma 5.3.10 says that this new reproducing kernel Hilbert space $\widehat{\mathscr{H}_{2,r}}$ will be unitarily equivalent to $\widehat{\mathscr{H}_2}$ in a way that preserves multipliers. Furthermore, the operator of M_2 will be unitarily equivalent to M_z on $\widehat{\mathscr{H}_{2,r}}$. So without loss of generality, we may assume that $\Omega_1 = \Omega_2$. Going forward, we refer to this set simply as Ω.

In light of this observation and Proposition 5.3.8, our goal is to determine if there exists an invertible multiplier between $\widehat{\mathscr{H}_1}$ and $\widehat{\mathscr{H}_2}$. This opens the following question:

Question. *Let $T_1, T_2 \in \mathscr{B}(\mathscr{H})$ with $T_i^* \in B_1(\Omega)$. Represent each as T_i as M_i on $\widehat{\mathscr{H}_i}$. Does there exist $\phi \in \mathcal{M}(\widehat{\mathscr{H}_1}, \widehat{\mathscr{H}_2})$ such that M_ϕ is invertible?*

In order to answer this question, one might first want investigate the structure of $\mathcal{M}(\widehat{\mathscr{H}_1}, \widehat{\mathscr{H}_2})$. In particular, one would like to know that $\mathcal{M}(\widehat{\mathscr{H}_1}, \widehat{\mathscr{H}_2}) \neq 0$. The theory of multipliers on reproducing kernel Hilbert spaces of analytic functions is a well explored subject [32]. Much work has been done classifying the multipliers of various well studied reproducing kernel Hilbert spaces. However, multipliers *between* reproducing kernel Hilbert spaces of analytic functions is a more sensitive subject. To understand why this is delicate problem, we make a few simple observations:

Proposition 5.3.11. *If $\phi \in \mathcal{M}(\widehat{\mathscr{H}_1}, \widehat{\mathscr{H}_2})$, and $\hat{f}_1 \in \widehat{\mathscr{H}_1}$, then*

i. $\phi \in \widehat{\mathscr{H}_2}$

ii. for each $\lambda \in \Omega$ such that $\hat{f}_1(\lambda) = 0$, $M_\phi(\hat{f}_1)(\lambda) = 0$

If in addition if M_ϕ is invertible, then *i.* $\phi(\lambda) \neq 0$ *for all* $\lambda \in \Omega$

ii. the function ψ *on* Ω *defined by* $\psi(\lambda) := \phi(\lambda)^{-1}$ *satisfies* $\psi \in \mathcal{M}(\widehat{\mathscr{H}_2}, \widehat{\mathscr{H}_1})$

iii. $(M_\phi)^{-1} = M_\psi$

iv. for each $\lambda \in \Omega$, $\hat{f}_1(\lambda) = 0$ *if and only if* $M_\phi(\hat{f}_1)(\lambda) = 0$

Proof. Note that since $\widehat{\mathscr{H}_i}$ contains the constant functions, if $\phi \in \mathcal{M}(\widehat{\mathscr{H}_1}, \widehat{\mathscr{H}_2})$ then $\phi = M_\phi(1) \in \widehat{\mathscr{H}_2}$. This proves (1). Clearly if $\hat{f}_1(\lambda) = 0$, then $M_\phi(\hat{f}_1)(\lambda) = \hat{f}_1(\lambda)\phi(\lambda) = 0$.

Now suppose that M_ϕ is invertible. Since $\phi \in \mathcal{M}(\widehat{\mathscr{H}_1}, \widehat{\mathscr{H}_2})$, Proposition 5.3.8 forces $M_\phi M_1 = M_2 M_\phi$. Since M_ϕ is invertible, clearly we have that $(M_\phi)^{-1} M_2 = M_1 (M_\phi)^{-1}$. Therefore once again by Proposition 5.3.8, $(M_\phi)^{-1} = M_\psi$ for some $\psi \in \mathcal{M}(\widehat{\mathscr{H}_2}, \widehat{\mathscr{H}_1})$.

Now, suppose to the contrary that $\phi(\lambda) = 0$ for some $\lambda \in \Omega$. Since $1 \in \widehat{\mathscr{H}_1}$, we have $M_\phi(1) = \phi \in \widehat{\mathscr{H}_2}$. Hence, $1 = (M_\phi)^{-1}(\phi) = M_\psi(\phi) = \psi\phi$. Then $1 = \psi(\lambda)\phi(\lambda) = 0$, which is absurd. Consequently, $\phi(\lambda) \neq 0$ for all $\lambda \in \Omega$. Since the constant functions are in both $\widehat{\mathscr{H}_i}$, we must have that $\psi(\lambda) = \phi(\lambda)^{-1}$ for all $\lambda \in \Omega$. Since $\phi(\lambda) \neq 0$ for all λ, it follows that $\hat{f}_2(\lambda) = 0$ if and only if $\hat{f}_1(\lambda) = 0$. ☐

It is well known that when Ω is the unit disc, $\mathcal{M}(H^2, H^2) = \mathcal{M}(A^2, A^2) = H^\infty(\Omega)$, where H^2 and A^2 denote the Hardy and Bergman space on Ω respectively. Stegenga characterized the elements of $\mathcal{M}(H^2, A^2)$ in terms of a growth condition on the boundary of the disc [42]. In particular, $\mathcal{M}(H^2, A^2) \neq 0$. However since there are $g \in A^2$ that have zeros different from all $f \in H^2$, Proposition 5.3.11 forces $\mathcal{M}(A^2, H^2) = 0$.

This example illustrates the sensitive nature of multipliers between reproducing kernel Hilbert spaces. Indeed, $\mathcal{M}(H^2, H^2)$ and $\mathcal{M}(A^2, A^2)$ are in some sense, as large as possible, filling up the entire space $H^\infty(\Omega)$. On the other extreme, $\mathcal{M}(A^2, H^2) = 0$. Yet reversing the roles of A^2 and H^2, we find that $\mathcal{M}(H^2, A^2)$ is a non-zero subspace of $H^\infty(\Omega)$. Since our class of operators contains shifts on the Hardy space and the Bergman space, we concede that a general solution to the isomorphism problem in term of multipliers is likely unobtainable.

Nevertheless, the theory in this section provides a tool by which to verify when two left invertible operators (analytic, Fredholm index -1) fail to be similar. The following proposition displays a necessarily relationship between the kernels of the spaces $\mathscr{H}_1$ and $\mathscr{H}_2$.

Proposition 5.3.12. *Given $T_1, T_2 \in \mathscr{B}(\mathscr{H})$ left invertible operators (analytic,* $\operatorname{ind}(T_i) = -1$*) represent each as $M_i = M_z$ on $\widehat{\mathscr{H}_i}$ over Ω. Suppose that $\phi \in H(\Omega)$, and define $K_{1,\phi} : \Omega^2 \to \mathbb{C}$ via $K_{1,\phi}(\lambda, \mu) := \phi(\lambda) K_1(\lambda, \mu) \overline{\phi(\mu)}$. Then the following are equivalent:*

i. $\phi \in \mathcal{M}(\widehat{\mathscr{H}_1}, \widehat{\mathscr{H}_2})$ *and M_ϕ is invertible*

ii. $K_2 = K_{1,\phi}$

Proof. Note M_ϕ is invertible if and only if every $\hat{f}_2 \in \widehat{\mathscr{H}_2}$ may be uniquely represented as $\hat{f}_2 = \phi \hat{f}_1$ for some $\hat{f}_1 \in \widehat{\mathscr{H}_1}$. Let $x_{i,\overline{\lambda}}$ denote the reproducing kernel at λ associated to $\widehat{\mathscr{H}_i}$. Then $\hat{f}_2 = \phi \hat{f}_1$ if and only if for each $\lambda \in \Omega$,

$$\langle f, x_{2,\overline{\lambda}} \rangle = \hat{f}_2(\lambda) = \phi(\lambda) \hat{f}_1(\lambda) = \langle f, \overline{\phi(\lambda)} x_{1,\overline{\lambda}} \rangle$$

Since this holds for all $f \in \mathscr{H}$, it follows that $\hat{f}_2 = \phi \hat{f}_1$ if and only if $x_{2,\overline{\lambda}} = \overline{\phi(\lambda)} x_{1,\overline{\lambda}}$, which is equivalent to

$$K_2(\lambda, \mu) = \langle x_{2,\overline{\mu}}, x_{2,\overline{\lambda}} \rangle = \phi(\lambda) \langle x_{1,\overline{\mu}}, x_{1,\overline{\lambda}} \rangle \overline{\phi(\mu)} = K_{1,\phi}(\lambda, \mu)$$

□

A consequence of this result is the following. If K_2 cannot be factored as $K_{1,\phi}$ for some ϕ, then there is no invertible multiplier between the two reproducing kernel Hilbert spaces. In this case, T_1 is not similar to T_2.

5.4 Example from Subnormal Operators

We now turn to an important class of non-trivial examples of $\mathfrak{A}_T$. These examples will involve the theory of subnormal operators. Using the work of Olin, Thomson, Keough and McGuire describing the C*-algebra generated by a subnormal, essentially normal, irreducible operator (Theorem 2.4.5), we characterize the algebras $\mathfrak{A}_S$ for S a subnormal, essentially normal left invertible operator. We begin with a simple connection between spectral data of the operators appearing in Theorem 2.4.5 and left invertibility.

Lemma 5.4.1. *Let S be a subnormal operator with $N = mne(S)$. If N is invertible, then S is left invertible with $L = T_{z^{-1}}$ a left inverse. If $\sigma(N) = \sigma_{ap}(S)$, then S is left invertible if and only if N is invertible.*

Proof. If N is invertible, then the Toeplitz operator $T_{z^{-1}} = P(N^{-1}) \mid_{\mathscr{H}}$ is well defined. Since N is a normal extension of S, we have for each $x \in \mathscr{H}$

$$T_{z^{-1}}Sx = T_{z^{-1}}(Nx) = P(N^{-1}Nx) = Px = x.$$

If $\sigma(N) = \sigma_{ap}(S)$, then S is left invertible implies $0 \notin \sigma_e(S) = \sigma(N)$. $\square$

Using the basic theory of subnormal operators, we now describe the structure of $\mathfrak{A}_S$ for a prototypical class of subnormal operators.

Theorem C. *Let S be an analytic left invertible,* $\operatorname{ind}(S) = -1$*, essentially normal, subnormal operator with $N := mne(S)$ such that $\sigma(N) = \sigma_{ap}(S)$. Let $\mathscr{B}$ be the uniform algebra generated by the functions z and z^{-1} on $\sigma_e(S)$. Then*

$$\mathfrak{A}_S = \{T_f + K : f \in \mathscr{B}, K \in \mathscr{K}(\mathscr{H})\}.$$

Moreover, the representation of each element as $T_f + K$ is unique.

Proof. By Lemma 5.4.1, $L := T_{z^{-1}}$ is a left inverse of S. By Corollary 5.1.2, $\mathfrak{A}_S$ is the norm-closed subalgebra of $C^*(S)$ generated by T_z and $T_{z^{-1}}$. Since S is analytic, it is strongly irreducible, and hence, irreducible. Therefore by Theorem 2.4.5, each element of $\mathfrak{A}_S$ has a unique representation as $T_f + K$ for some $f \in C(\sigma(N))$ and $\sigma(N) = \sigma_{ap}(S) = \sigma_e(S)$. Moreover by Theorem 2.4.5, $L^n = T_{z^{-n}} + K$ for some compact operator K. Since $\mathfrak{A}_S$ contains the compacts, it follows that $T_{z^k} \in \mathfrak{A}_S$ for each $k \in \mathbb{Z}$. Hence, for each $p \in \mathrm{Alg}(z, z^{-1})$, we have that $T_p \in \mathfrak{A}_S$. Using this information, we now show that $\mathfrak{A}_S = \{T_f + K : f \in \mathscr{B}, K \in \mathscr{K}(\mathscr{H})\}$. To do this, it suffices to show that $T_f \in \mathfrak{A}_S$ if and only if $f \in \mathscr{B}$.

First, suppose that $T_f \in \mathfrak{A}_S$ for some $f \in C(\sigma(N))$. Since $\mathrm{Alg}\{T_z, T_{z^{-1}}\}$ is dense in $\mathfrak{A}_S$, for every $\epsilon > 0$ there exists a Laurent polynomial $p \in \mathrm{Alg}(z, z^{-1})$ and compact K such that $\|T_f - (T_p + K)\| < \epsilon$. By Theorem 2.4.5,

$$\epsilon > \|T_f - (T_p + K)\| = \|T_{f-p} - K\| \geq \|T_{f-p} + \mathscr{K}(\mathscr{H})\| = \|f - p\|.$$

Hence, $f \in \mathscr{B}$. For the other inclusion, suppose to the contrary that $f \in \mathscr{B}$ but $T_f \notin \mathfrak{A}_S$. Then there exists a $\delta > 0$ such that for each $p \in \mathrm{Alg}(z, z^{-1})$ and $K \in \mathscr{K}(\mathscr{H})$, we have $\|T_f - (T_p + K)\| > \delta$. In particular, this should hold for any p such that $\|f - p\| < \frac{\delta}{2}$. Hence

$$\delta \leq \inf_{K \in \mathscr{K}(\mathscr{H})} \|T_f - (T_p + K)\| = \|T_{f-p} + \mathscr{K}(\mathscr{H})\| = \|f - p\| < \frac{\delta}{2}$$

which is absurd. Hence, T_f must be in $\mathfrak{A}_S$, completing the proof. □

Notice that in Theorem C, we can drop the requirement that $\sigma(N) = \sigma_{ap}(S)$, so long as the minimal normal extension is invertible. In this case however, one will lose the uniqueness of the representation $T_f + K$ as discussed in Theorem 2.4.5. As a corollary to Theorem C, we get a description of $\mathfrak{A}_T$ for analytic Toeplitz operators on $H^2(\mathbb{T})$ with Fredholm index -1.

Corollary 5.4.2. *Let g be an analytic function on $\mathbb{T}$ and $X = \operatorname{ran}(g)$ with winding number of g equal to 1. Then $\sigma_e(T_g) = X$, and T_g is an analytic left invertible operator with ind$(T_g) = -1$. Moreover, if $\mathscr{B}$ is the uniform algebra generated by z and z^{-1} on X, then we have the following short exact sequence*

$$0 \longrightarrow \mathscr{K}(H^2(\mathbb{T})) \xrightarrow{\iota} \mathfrak{A}_{T_g} \xrightarrow{\pi} \mathscr{B} \longrightarrow 0$$

Moreover, each element of $\mathfrak{A}_{T_g}$ has a unique representation of $T_f + K$ for f in the uniform algebra generated by g and g^{-1} and K compact.

The hypotheses of Theorem C are natural, but numerous. This is to guarantee that S remain within our current focus of study. We remark that even if S is left invertible, irreducible, subnormal, essentially normal operator, it need not be analytic.

Recall, an operator $R \in \mathscr{B}(\mathscr{H})$ is said to be *cyclic* if there exists an $x \in \mathscr{H}$ such that $\{R^n x\}_{n=0}^{\infty}$ is norm dense in $\mathscr{H}$. A result by Qing shows that every Cowen-Douglas operator is cyclic [35]. While all Cowen-Douglas operators must be cyclic, the adjoints of general subnormal operators need not be cyclic. A long-standing problem posed by Deddens and Wogen asked which subnormal operators had cyclic adjoints [12]. Feldman answered this question in [19]. A subnormal operator is said to be *pure* if it has no non-trivial normal summand. Every subnormal operator can be decomposed as $S = S_p \oplus N$, where S_p is pure and N is normal. The general cyclicity result is as follows:

Theorem 5.4.3 (Feldman [19])**.** *If $S = S_p \oplus N$ is a subnormal operator, then S^* is cyclic if and only if N is cyclic. In particular, pure subnormal operators have cyclic adjoints.*

Having a cyclic vector clearly is not sufficient for an operator to be Cowen-Douglas. However, Theorem 5.4.3 is a condition of necessity. Thomson showed in [43] that if S is a pure, cyclic subnormal operator, then S^* is Cowen-Douglas. However, as far the author is aware, there is no known elementary equivalence to guarantee S^* is Cowen-Douglas.

We remark that the similarity orbit of subnormal operators was classified by Conway [13]. He showed two subnormal operators are similar if and only if the scalar valued spectral measure associated to the minimal normal extensions were the same. In this case, there is no need to investigate the K_0 group of the commutant. Rather, the spectral data encodes all the information about the similarity orbit.

Chapter 6

Further Directions

In this dissertation, we initiated a research program on concrete operator algebras that model the dynamics of Hilbert space frames. We directed our focus on the simplest possible classes of algebras, those generated by a single left invertible operator and its Moore-Penrose inverse. We argued for the study of a non-degenerate class of left invertibles (the analytic operators) was necessary, and paid extra attention to those analytic left invertible operators with Fredholm index -1. In this special case, we concluded that the algebras $\mathfrak{A}_T$ had very similar characteristics to the Toeplitz algebra. We showed that $\mathfrak{A}_T$ contained the compact operators, and that the elements of $\mathfrak{A}_T$ could heuristically be described as "compact plus Laurent series". We also determined two such algebras are isomorphic if and only if they are similar, and characterized the isomorphism classes by K-theoretic and RKHS methods.

Concerning the algebras $\mathfrak{A}_T$, there is still a lot which is not known. A great portion of our efforts were directed at the case when $\operatorname{ind}(T) = -1$. Following the discussion in Section 4.3, the next logical class to investigate are the strongly irreducible left invertible operators of finite index. In the index -1 case, we showed that if $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a bounded isomorphism, then ϕ restricted to the compact operators (which was equal to the commutator ideal $\mathscr{C}$) mapped back into the compact operators. The essentialness and nuclearity of $\mathscr{K}(\mathscr{H})$ in $\mathfrak{A}_T$ allowed one to then conclude that ϕ was adjunction. The hope is that a similar theorem holds for the strongly irreducible operators.

Specifically, if T_i are strongly irreducible left invertibles with the same index, and $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a bounded isomorphism, we would like it to be the case that ϕ is a similarity. By Proposition 3.2.7, the commutator ideal in the strongly irreducible case is still $\mathscr{C} = \mathscr{K}_T = \overline{\text{sp}}\{T^n(I - TT^\dagger)T^{\dagger^m}\}$. Moreover, $\phi(\mathcal{C}_1) = \mathcal{C}_2$ However, it is not clear how this information could be elevated to show that the two algebras are isomorphic if and only if they are similar. We would require $\mathcal{C}$ be essential in $\mathfrak{A}_T$. This implies that such a theorem might require a new proof technique.

If this type of similarity result is true for the strongly irreducible operators, then we again we can apply Theorem 5.3.5 to classify the algebras $\mathfrak{A}_T$ by the K_0 group. For the general case, recall that each Cowen-Douglas operator has a unique strongly irreducible decomposition up to similarity. Therefore, in the general case, we can consider $T = \oplus_{j=1}^m T_j$ where each T_j are strongly irreducible. The author conjectures that this type of diagonalization result could be leveraged to state that isomorphisms are direct sums of similarities. The general classification of Cowen-Douglas operators by K_0 groups could be used to classify the algebras $\mathfrak{A}_T$ in this case as well.

It is interesting to note that in the index -1 case, the similarity orbit $\mathcal{S}(T)$ determined the isomorphism classes of $\mathfrak{A}_T$. Given $T \in \mathscr{B}(\mathscr{H})$, the *fine spectral picture of* T consists of detailed spectral, (semi)-Fredholm and algebraic (Riesz projections) data. All the information that is contained in the fine spectral picture is retained under similarity. The work of Herrero, Apostol, Voiculescu and many others showed that under most conditions, this criterion determined the closure of the similarity orbit [11]. In [22], Herrero classified the spectral pictures of Cowen-Douglas operators. Combining these results and our work above, we have a classification of $\overline{\mathcal{S}(T)}$ for T a natural analytic left invertible operator. Consequently, if $\overline{\mathcal{S}(T_1)} = \overline{\mathcal{S}(T_2)}$, then we have a sequence of invertibles that, in the limit, conjugate T_1 with T_2. If two natural analytic left invertibles are approximately similar in this sense, what can be said about their algebras? Is it the case that the algebras are similar? This would be interesting, because it would imply that spectral data is all that is required to classify the algebras (as was the case for subnormal operators).

Nearly all the operator algebraic analysis in this thesis is centered around $\mathfrak{A}_T$, the algebra generated by a single left invertible operator and its Moore-Penrose inverse. This algebra arose from considering the directed graph generated by a Γ in the introduction. There are, of course, many other directed graphs that give rise to well studied classes of C*-algebras. Most notably, the Cuntz algebra $\mathcal{O}_n$ is the graph algebra from a single vertex with n loops. This graph yields the next natural class of operator algebras to study in this program.

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