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The exact vector integral solution for all the electromagnetic field components of a general flattened Gaussian laser mode is derived by using the angular spectrum method. This solution includes the pure and annular Gaussian modes as special cases. The integrals are of the form of Gegenbauer's finite integral and are computed analytically for each case, yielding fields satisfying the Maxwell equations exactly in the form of quickly converging Fourier–Gegenbauer series. © 2006 Optical Society of America

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As laser focusing tightens to reach ever higher intensities, nonparaxial field components begin to play an important role in laser–matter interactions and must be accurately accounted for. Longitudinal laser field components were first introduced by Lax *et al.*[1] and then discussed in relation to lasers.[2] This work was quickly followed by the derivation of higher-order nonparaxial corrections for Gaussian laser beams by Davis[3] and Barton and Alexander.[4] Hora and colleagues have also reviewed and contributed to the development of laser field models and their application to electrodynamics—the accuracy principle.[2,5–7] The importance of the physical model of focused laser fields has been clearly demonstrated in direct laser–electron scattering[5,8–13] and must be considered to accurately model any high-intensity experiment.[14]

In this Letter we derive an exact analytical series solution to the full Maxwell wave equation for a laser having a flattened Gaussian transverse profile in the focal plane by using the angular spectrum method. The technique is identical to that previously used in a purely integral form for a pure Gaussian.[5,10,15]

Several formalisms have been developed to model flat-top laser profiles: super-Gaussians,[16] weighted sums of Gaussians with different focusing parameters,[17–20] $(1-r^2)^{1/\gamma}\,\mathrm{rect}(r-1/2)$ and $\{1+\exp[\gamma(r-1)]\}^{-1}$ $(\gamma\in\mathbb{Z})$ mentioned by Gori,[21] a super-Lorentzian,[17] a wavelet technique,[22] and the flattened Gaussian introduced by Gori[21] and also utilized by Santarsiero and Borghi *et al.*[23,24] All of these have been used to describe the scalar–paraxial properties of flat-top fields and are suitable in principle to generate a full vector theory for a flattened–annular focused laser beam. In this Letter, owing to the term-by-term simplicity of the Fourier transform, the general flattened Gaussian distribution is chosen,

$$E_x(x,y,z=0)=E_0\sum_{N=0}^{\infty}A_N\left(\frac{r^2}{w_0^2}\right)^N e^{-(r^2/w_0^2)}, \qquad (1)$$

where the parameters A_N are a set of arbitrary constants describing the detailed structure of the distribution. Note that this reduces to the Gaussian case for $A_0=1$ and $A_{N>0}=0$. This holds for arbitrary complex values of A_N, allowing a large amount of flexibility in specifying the field distribution.

When we impose the boundary condition from Eq. (1) and with $E_y\equiv 0$, the five remaining field components are

$$E_x^a=\sum\int_0^1 e^{-(b^2/\epsilon^2)}e^{imk_0z}J_0(k_0rb)L_N\left(\frac{b^2}{\epsilon^2}\right)b\,\mathrm{d}b, \qquad (2)$$

$$E_z^a=i\sum\partial_{\bar{x}}\int_0^1 e^{-(b^2/\epsilon^2)}\frac{e^{imk_0z}}{m}J_0(k_0rb)L_N\left(\frac{b^2}{\epsilon^2}\right)b\,\mathrm{d}b, \qquad (3)$$

$$B_x^a=\sum\partial^2_{\bar{x}\bar{y}}\int_0^1 e^{-(b^2/\epsilon^2)}\frac{e^{imk_0z}}{m}J_0(k_0rb)L_N\left(\frac{b^2}{\epsilon^2}\right)b\,\mathrm{d}b, \qquad (4)$$

$$B_y^a=\sum\left[\int_0^1 e^{-(b^2/\epsilon^2)}me^{imk_0z}J_0(k_0rb)L_N\left(\frac{b^2}{\epsilon^2}\right)b\,\mathrm{d}b - \partial^2_{\bar{x}}\int_0^1 e^{-(b^2/\epsilon^2)}\frac{e^{imk_0z}}{m}J_0(k_0rb)L_N\left(\frac{b^2}{\epsilon^2}\right)b\,\mathrm{d}b\right], \qquad (5)$$

$$B_z^a=i\sum\partial_{\bar{y}}\int_0^1 e^{-(b^2/\epsilon^2)}e^{imk_0z}J_0(k_0rb)L_N\left(\frac{b^2}{\epsilon^2}\right)b\,\mathrm{d}b, \qquad (6)$$

where $J_n(x)$ is the nth order Bessel function of the first kind, $L_N(x)$ is the Nth order unassociated Laguerre polynomial, $\partial_{\bar{a}}(\cdot)$ denotes $k_0^{-1}\partial_a(\cdot)$, Σ denotes $2\epsilon^{-2}\tilde{E}_0\Sigma_{N=0}^{\infty}A_N N!$, $\tilde{E}_0\equiv E_0\exp[-i(\omega_0 t+\phi_0)]$, and ϕ_0 is an arbitrary phase constant.

Imposing an identical boundary condition to $B_y(x,y,z=0)$ and averaging the results, the real symmetric flat-top electric field components are then

$$E_x(x,y,z)=\frac{1}{2}\left(I_1+\frac{x^2-y^2}{k_0r^3}I_2+\frac{y^2}{r^2}I_3\right), \qquad (7)$$

$$E_y(x,y,z) = \frac{1}{2}\frac{xy}{k_0 r^3}(2I_2 - k_0 r I_3), \tag{8}$$

$$E_z(x,y,z) = \frac{1}{2}\frac{x}{r}I_4. \tag{9}$$

The magnetic field is formally identical with the roles of x and y reversed, and the integrals I_n are defined as

$$I_1 = \sum \int_0^1 e^{-\zeta^2}(m + m^2)\sin(\phi_m)J_0(\Lambda)L_N(\zeta^2)\mathrm{d}m,$$

$$I_2 = \sum \int_0^1 e^{-\zeta^2}\sin(\phi_m)J_1(\Lambda)L_N(\zeta^2)\sqrt{1-m^2}\mathrm{d}m,$$

$$I_3 = \sum \int_0^1 e^{-\zeta^2}\sin(\phi_m)J_0(\Lambda)L_N(\zeta^2)(1-m^2)\mathrm{d}m,$$

$$I_4 = \sum \int_0^1 e^{-\zeta^2}\kappa(m)\cos(\phi_m)J_1(\Lambda)L_N(\zeta^2)\mathrm{d}m,$$

for $\Lambda = k_0 r\sqrt{1-m^2}$, $\zeta^2 = (1-m^2)/\epsilon^2$, $\phi_m = \omega_0 t - k_0 z m + \phi_0$, and $\kappa(m) = (1+m)\sqrt{1-m^2}$. The integration variable has been changed to $m = \sqrt{1-b^2}$ to eliminate the removable singularity in b present Eqs. (3)–(5).

Each of the field integrals in Eqs. (2)–(6) is of the form

$$\int_{\alpha_1}^{\alpha} f_d(\cos\theta, \sin\theta)\sin\theta J_n\left(\frac{v\sin\theta}{\sin\alpha}\right)\exp\left(\frac{iu\cos\theta}{\sin^2\alpha}\right)\mathrm{d}\theta,$$

where $v = k_0 r\sin\alpha$, $u = k_0 z\sin^2\alpha$, $\alpha_1 = 0$, and $\alpha = \pi/2$. This form possesses an exact analytical solution based on the Fourier–Gegenbauer expansion. Specifically, $f_d(\cos\theta, \sin\theta) = \epsilon^{-2}\exp(-\sin^2/\epsilon^2)L_N(\sin^2\theta/\epsilon^2)\cos^d\theta$ for $d \in [0,1,2]$, and using Lame's method as described in Watson[25] or Kant,[26] the function f_d is expanded as

$$f_d(\cos\theta, \sin\theta) = \sum_{s=0}^{\infty} a_{s,N}^d C_s^{1/2}(\cos\theta),$$

where $C_s^\lambda(x)$ are the Gegenbauer polynomials, and the expansion coefficients $a_{s,N}^d$ are defined by

$$a_{s,N}^d = N_s \int_0^1 e^{\xi^2/\epsilon^2}L_N\left(\frac{1-\xi^2}{\epsilon^2}\right)\xi^d C_s^{1/2}(\xi)\mathrm{d}\xi, \tag{10}$$

where $N_s = \epsilon^{-2}\exp(-1/\epsilon^2)(2s+1)/2$. The integrals in Eqs. (2)–(6) have now been reduced to Gegenbauer's finite integral and can be evaluated directly to yield

$$I_d = 2\sum_{s=0}^{\infty} a_{s,N}^d i^s C_s^{1/2}(z/\rho)j_s(k_0\rho), \tag{11}$$

where $\rho^2 = r^2 + z^2$ and $j_s(x)$ is the sth order spherical Bessel function of the first kind.[27]

The problem has now been recast from evaluating the integrals of Eqs. (2)–(6) to calculating the integrals in Eq. (10). In fact, only the case of $s=0$ needs to be computed for all $d = 0,1,2,\ldots$. Once these are known, the $s=1$ terms are automatically known from the relation $a_{1,N}^{d-1} = 3a_{0,N}^d$ following from the properties of the Gegenbauer polynomials. Similarly, the remaining terms follow from the recursion

$$a_{s,N}^d = \left(\frac{2s+1}{s}\right)\left[a_{s-1,N}^{d+1} - \left(\frac{s-1}{2s-3}\right)a_{s-2,N}^d\right],$$

where $s = 2,3,4,\ldots$ and $d = 0,1,2,\ldots$ For $s=0$, the integral of Eq. (10) can be easily evaluated:

$$a_{0,N}^d = \frac{1}{4}e^{-1/\epsilon^2}\sum_{m=0}^{N}\sum_{k=0}^{m}\binom{N}{m}\left(\frac{1}{m!}\right)\binom{m}{k} \times (-i)^\mu \epsilon^{\mu+2m-2k-2}\gamma\left(\frac{2m+\mu}{2}, \frac{1}{\epsilon^2}\right),$$

where $\mu = d+1-2k$, and $\gamma(a,x)$ is the lower incomplete gamma function.[27] For a pure Gaussian (i.e., $A_0 = 1$, $A_{N>0} = 0$), this can be simplified to

$$a_{0,0}^d = \frac{1}{4}(-i)^{d+1}\epsilon^{d-1}e^{-1/\epsilon^2}\gamma\left(\frac{d+1}{2}, -\frac{1}{\epsilon^2}\right),$$

as the Laguerre polynomial $L_0(x) \equiv 1$. Note that the inclusion of non-Gaussian terms in the boundary condition only alters the expansion coefficients and does not add any additional complexity to the problem.

The expansion coefficients of Eq. (11) are now known, and all that remains to compute the field distribution is to evaluate the derivatives of Eqs. (2)–(6), and then average this result with the analogous solution for $(x,y) \to (y,x)$. The final exact solution of the Maxwell wave equation for an arbitrarily tightly focused flattened Gaussian focal plane distribution is

$$E_x = \sum\sum_{s=0}^{\infty} i^s \Bigg\{ (a_{s,N}^1 + a_{s,N}^2)C_s^{1/2}\left(\frac{z}{\rho}\right)j_s(k_0\rho) - a_{s,N}^0\frac{x^2+z^2}{k_0\rho^3} \times \left[C_s^{1/2}\left(\frac{z}{\rho}\right)j_s'(k_0\rho) - \left(\frac{z}{k_0\rho^2}\right)C_s^{3/2}\left(\frac{z}{\rho}\right)j_s(k_0\rho)\right] + a_{s,N}^0\frac{y^2}{\rho^2}\left[C_s^{1/2}\left(\frac{z}{\rho}\right)j_s''(k_0\rho) - \frac{z}{k_0\rho^2}C_s^{3/2}\left(\frac{z}{\rho}\right)j_s'(k_0\rho)\right] - a_{s,N}^0\frac{y^2 z}{k_0\rho^4}\left[C_s^{3/2}\left(\frac{z}{\rho}\right)j_s'(k_0\rho) - \frac{3z}{k_0\rho^2}C_s^{5/2}\left(\frac{z}{\rho}\right)j_s(k_0\rho)\right]\Bigg\},$$

$$E_y = -\left(\frac{xy}{\rho^2}\right) \sum \sum_{s=0}^{\infty} i^s \left\{ a_{s,N}^0 \frac{1}{k_0\rho} \left[C_s^{1/2}\left(\frac{z}{\rho}\right) j_s'(k_0\rho) - \left(\frac{z}{k_0\rho^2}\right) C_s^{3/2}\left(\frac{z}{\rho}\right) j_s(k_0\rho) \right] - a_{s,N}^0 \left[C_s^{1/2}\left(\frac{z}{\rho}\right) j_s''(k_0\rho) - \frac{z}{k_0\rho^2} C_s^{3/2}\left(\frac{z}{\rho}\right) j_s'(k_0\rho) \right] + a_{s,N}^0 \frac{z}{k_0\rho^2} \left[C_s^{3/2}\left(\frac{z}{\rho}\right) j_s'(k_0\rho) - \frac{3z}{k_0\rho^2} C_s^{5/2}\left(\frac{z}{\rho}\right) j_s(k_0\rho) \right] \right\},$$

$$E_z = -i\left(\frac{x}{\rho}\right) \sum \sum_{s=1}^{\infty} (a_{s,N}^0 + a_{s,N}^1) i^s \left[C_s^{1/2}\left(\frac{z}{\rho}\right) j_s'(k_0\rho) - \frac{z}{k_0\rho^2} C_{s-1}^{3/2}\left(\frac{z}{\rho}\right) j_s(k_0\rho) \right],$$

for $j_s'(x) = \partial_x j_s(x) = j_{s-1}(x) - [(s+1)/x] j_s(x)$, and all Gegenbauer polynomials C_s^{λ} for $s<0$ are taken by convention to be zero. The number of terms, S, required in the Fourier–Gegenbauer series to ensure convergence of all the field components scales roughly linearly with the waist, w_0, as $S \sim 20(w_0/\lambda_0)$, where λ_0 is the laser wavelength. Provided that at least this many terms are retained, the solutions are stable for all temporal and spatial locations and quickly computed relative to the integral solution. For example, the time required to compute a 77×77 field grid in the focal plane for a $2w_0=\lambda_0$ Gaussian was decreased by a factor of 130. As $w_0 \gg \lambda_0$, the number of terms becomes large, and, in fact, the expansion coefficients become difficult to evaluate as the integrand $\exp(-\zeta^2)$ becomes sharply peaked in the forward direction. In this case, however, the field is now propagating nearly exactly along the $\hat{\mathbf{z}}$ axis, and the standard paraxial–perturbative solutions are applicable and, indeed, much simpler.[4,10,17]

The six symmetric electromagnetic field components of a focused laser with a general, flattened Gaussian transverse profile have been derived exactly. The formal prototype used gives the user an infinite set of fitting parameters allowing great flexibility in matching a realistic laser profile including nonideal Gaussians with flattened profiles, purposefully created flat-top beams, and even annular beams. The simplest case sets $A_0=1$ and $A_{N>1}=0$, reducing this to the standard Gaussian solution, and the inclusion of even one or two additional terms generates a wide array of flattened and hollow Gaussian beam profiles.

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