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Admissibility of C*-Covers and Crossed Products of Operator Algebras

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ADMISSIBILITY OF C*-COVERS AND CROSSED PRODUCTS OF OPERATOR ALGEBRAS

by

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In 2015, E. Katsoulis and C. Ramsey introduced the construction of a non-self-adjoint crossed product that encodes the action of a group of automorphisms on an operator algebra. They did so by realizing a non-self-adjoint crossed product as the subalgebra of a C*-crossed product when dynamics of a group acting on an operator algebra by completely isometric automorphisms can be extended to self-adjoint dynamics of the group acting on a C*-algebra by $*$-automorphisms. We show that this extension of dynamics is highly dependent on the representation of the given algebra and we define a lattice structure for an operator algebra's completely isometric representation theory. We characterize when a self-adjoint extension of dynamics exists in terms of the boundary ideal structure for the given operator algebra in its maximal representation. We use this characterization to produce the first example of dynamics on a finite dimensional non-self-adjoint operator algebra that are not extendable in a given representation and the first examples of always extendable dynamics for a family of operator algebras in a non-extremal representation. We give a partial crossed product construction to extend dynamics on a family of operator algebras, even when the operator algebra is represented degenerately. We connect W. Arveson's crossed product with that of E. Katsoulis and C. Ramsey by giving a partial answer to a recent open problem.

DEDICATION

This work was made possible by the love, support, and understanding of many beings – some human, some feline, and some that remain only in treasured memory.

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Introduction

Operator algebras have applications to a wide range of mathematical disciplines. A common strategy in the theory is to build an operator algebra that encodes information about an object of interest. Be the object a group, directed graph, or dynamical system, operator algebras can be constructed in a way that preserves algebraic, combinatorial, and topological data in its structure, and often, the operator algebra is a more tractable object to study. This strategy leads back to the 1930s as F. J. Murray and J. von Neumann developed what is now called the group-measure space construction.

Our work is in the context non-self-adjoint operator algebras arising from dynamical systems. This began with the work of W. Arveson in 1967 [4] and later J. Peters in 1984 [28] as they constructed non-self-adjoint operator algebras from semigroups of endomorphisms acting on operator algebras. These objects are called *semicrossed products* and have a strong connection with analytic function theory.

Unlike the construction of J. Peters, et. al, we construct non-self-adjoint operator algebras from groups of (categorically appropriate) automorphisms acting on non-self-adjoint operator algebras. We call these objects *crossed products*. E. Katsoulis and C. Ramsey introduced the construction of a non-self-adjoint crossed product in 2015 [22] as a tool for building examples of semi-Dirichlet algebras [12] beyond the class of Dirichlet algebras and tensor algebras for C*-correspondences. E. Katsoulis and C. Ramsey established a robust theory for non-self-adjoint crossed products in [22], [20], [21], and showed their utility through many applications. In particular, E. Katsoulis and C. Ramsey were able to leverage non-self-adjoint crossed product theory to achieve a positive solution to the *Hao-Ng Isomorphism Problem* from 2008 [17] in increasing generality, which is remarkable as the problem was viewed as a self-adjoint problem prior to their work.

The crossed product of a non-self-adjoint operator algebra is constructed by realizing it as the subalgebra of crossed product C*-algebra when the action of a group on the non-self-adjoint algebra can be extended to a categorically appropriate action of the group on an enveloping C*-algebra. It is natural to ask: *when does this extension exist?* E. Katsoulis and C. Ramsey develop their theory by leveraging the existence of extensions for universal representations of the non-self-adjoint operator algebra. Not much is known beyond these two representations. The primary focus of this dissertation is to address the existence of an extension in an arbitrary representation. In the last chapter, we extend work of W. Arveson, J. Peters, et. al to the context of non-self-adjoint crossed products.

In Chapter 1, we review the background for operator algebra theory and establish notation. We introduce fundamental examples of non-self-adjoint operator algebras that will appear throughout this dissertation. The notion of a representation of an operator algebra $\mathcal{A}$ is introduced as a pair $(\mathcal{C}, j)$ including a homomorphism j along with an enveloping C*-algebra $\mathcal{C}$ for the image $j(\mathcal{A})$. These representation pairs are called C*-covers. Our motivating question is to determine when actions of a group on an operator algebra can be lifted via C*-covers. When such a lift exists, the C*-cover is called *admissible*.

In Chapter 2, we give structure to the collection of all C*-covers. We construct meet and join operations on the collection of all C*-covers equipped with a natural partial ordering. This endows the collection of all C*-covers with a complete lattice structure. We then characterize all admissible C*-covers for a fixed operator algebra in terms of the ideal structure of the algebra's maximal C*-cover. We use this characterization to show that the collection of all admissible C*-covers forms a complete sublattice and to produce the first finite-dimensional example of a non-admissible C*-cover. We finish the chapter by showing that dynamics can still be lifted to certain non-admissible C*-covers using a partial crossed product construction.

In Chapter 3, we give the first examples of C*-covers that are admissible independent of the group action and are neither the maximal nor minimal elements in their lattice. We show that every C*-cover for the upper 2×2 triangular matrices will be admissible, regardless of the action.

In Chapter 4, we investigate conjugate dynamical systems in the context of E.Katsoulis and C. Ramsey crossed products. We give a geometric characterization of conjugate dynamical systems for the disc algebra by $\mathbb{Z}$-actions, which connects crossed products with semicrossed product theory within a particular class of examples.

Chapter 1

Background – Operator Algebras

In this chapter, we will outline the relevant background material in operator algebra theory. We omit proofs of the fundamental results in the area, but we provide proofs to many folklore and technical results, particularly in Section 1.2, that will be useful in later chapters.

1.1 Basic Definitions

In our work, all algebras will be unital and over the complex field $\mathbb{C}$ unless otherwise stated. For each $n \in \mathbb{N}$, we let $M_n(\mathcal{A})$ denote the algebra of $n \times n$ matrices with entries in the algebra $\mathcal{A}$. For each $n \in \mathbb{N}$, we let $M_n := M_n(\mathbb{C})$ denote the algebra of $n \times n$ matrices with entries in $\mathbb{C}$. We will often represent elements of $M_n(\mathcal{A})$ in terms of their matrix entries by writing $A = (a_{rc}) \in M_n(\mathcal{A})$. We let $\{E_{rc}\}_{r,c=1}^n$ be the standard matrix units in $M_n(\mathcal{A})$, i.e. E_{rc} is the matrix with $1_\mathcal{A}$ in the (r, c)-entry and 0 elsewhere. We will let A^t denote the transpose of the matrix $A \in M_n(\mathcal{A})$. When $\mathcal{A}$ is a $*$-algebra, $M_n(\mathcal{A})$ is a $*$-algebra with $* : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ given by $(a_{rc})^* = (a_{cr}^*) = (a_{rc}^*)^t$.

A *C*-algebra* $\mathcal{C}$ is a Banach $*$-algebra whose norm satisfies $\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{C}$. This norm condition is called the *C*-identity*, and it guarantees that the topological and algebraic structures of the algebra are intimately related. Examples of C*-algebras include the algebra of continuous functions on a compact, Hausdorff space and direct sums of full matrix algebras. A cornerstone result in the field, credited to I. Gelfand, M. Naimark, and I. Segal, says every C*-algebra is $*$-isomorphic to a $*$-closed subalgebra of bounded linear operators $\mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. So the study of C*-algebras can be viewed as the study of bounded operators on Hilbert space.

Our main object of study is called an *operator algebra*, which is a norm-closed subalgebra of a unital C*-algebra containing that unit. Every C*-algebra is an operator algebra, but the converse is not true in general as operator algebras may or may not be closed under the $*$-operation of the parent C*-algebra. When an operator algebra is not $*$-closed, we will call it *non-self-adjoint*. We give some basic examples of operator algebras.

Example 1.1.1 (Upper Triangular Matrices)**.** Let $n \in \mathbb{N}$ be given and define T_n to be the subalgebra of all upper triangular $n \times n$ matrices in M_n. Then T_n is a non-self-adjoint unital operator subalgebra of M_n. Indeed, T_n must be norm closed since M_n is finite-dimensional and T_n is non-self-adjoint since $E_{1n} \in T_n$ but $E_{1n}^* = E_{n1} \notin T_n$.

Example 1.1.2 (The Four Cycle Algebra)**.** Let A_4 be the subalgebra of M_4 given by matrices of the form

$$\begin{bmatrix} * & & * & * \\ & * & * & * \\ & & * & \\ & & & * \end{bmatrix}.$$

Then A_4 is a unital non-self-adjoint operator subalgebra of M_4 called the *four cycle algebra*.

Example 1.1.3. Any C*-algebra is an operator algebra.

Every operator algebra $\mathcal{A}$ with parent C*-algebra $\mathcal{C}$ comes equipped with a matricial structure in addition to its norm-algebra structure. For each $n \in \mathbb{N}$, $M_n(\mathcal{A})$ inherits a norm $\|\cdot\|_n$ from $M_n(\mathcal{C})$ by viewing $M_n(\mathcal{C})$ as bounded operators on some Hilbert space $\mathcal{H}^n$. An operator algebra $\mathcal{A}$ is determined not only by its norm-algebra structure but also by its family of matrix norms $\{\|\cdot\|_n : M_n(\mathcal{A}) \to [0,\infty)\}_{n\in\mathbb{N}}$. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be linear map between operator algebras $\mathcal{A}$ and $\mathcal{B}$. Then for each $n \in \mathbb{N}$, φ induces a linear map $\varphi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ by $\varphi_n((a_{rc})) = (\varphi(a_{rc}))$. We say φ is *completely positive (cp)* when each φ_n is a positive linear map. We say φ is *completely bounded (cb)* when its *cb-norm* given by $\|\varphi\|_{cb} := \sup_{n\in\mathbb{N}} \|\varphi_n\|$ is finite. We say φ is *completely contractive* when $\|\varphi\|_{cb} \leq 1$, or equivalently, $\|\varphi_n\| \leq 1$ for all $n \in \mathbb{N}$. In the category of operator algebras, complete positivity of an algebra homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is equivalent to φ being completely contractive (see [27] as a reference). We say $\varphi : \mathcal{A} \to \mathcal{B}$ is *completely isometric (cis)* if $\varphi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is isometric for all $n \in \mathbb{N}$. A *completely isometric isomorphism* is an algebra isomomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that φ is completely isometric. When such a map exists, we say the operator algebras $\mathcal{A}$ and $\mathcal{B}$ are *completely isometrically isomorphic*, or *isomorphic* (denoted $\mathcal{A} \cong \mathcal{B}$) when the context is unambiguous.

Example 1.1.4 (The Disc Algebra)**.** The *disc algebra* $A(\mathbb{D})$ is the subalgebra of $C(\overline{\mathbb{D}})$ consisting of functions that are analytic on the open unit disc $\mathbb{D}$ in $\mathbb{C}$. Since the restriction of functions in $A(\mathbb{D})$ to the unit circle $\mathbb{T} = \partial\mathbb{D}$ is completely isometric as a consequence of the maximum modulus principle, we can also view $A(\mathbb{D})$ as a closed subalgebra of $C(\mathbb{T})$. Note $A(\mathbb{D}) = \overline{\text{alg}}\,\{1, z\} = \overline{\mathbb{C}[z]}$ is non-self-adjoint as $z \mapsto \overline{z}$ is not analytic.

Example 1.1.5 (2.2.9 & 10 in [8])**.** We can regard any subspace X of $\mathcal{B}(\mathcal{H})$, which we call an *operator space*, as an operator algebra with trivial product via the complete isometry $x \mapsto \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$. We can construct a unital operator algebra from X in the following way: define

$$\mathcal{U}_B(X) := \begin{bmatrix} \mathbb{C}I & X \\ 0 & \mathbb{C}I \end{bmatrix} = \left\{ \begin{bmatrix} \lambda_1 I & x \\ 0 & \lambda_2 I \end{bmatrix} : x \in X, \lambda_1, \lambda_2 \in \mathbb{C} \right\} \subset \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) = M_2(\mathcal{B}(\mathcal{H})).$$

Then $\mathcal{U}_B(X)$ is a unital operator subalgebra of $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ with multiplication defined in the obvious way. It should be noted that the definition of $\mathcal{U}_B(X)$ is independent of $\mathcal{H}$ – meaning its operator space structure, its norm, and its multiplication are all independent of the representation of X on $\mathcal{H}$.

1.2 C*-Covers for Operator Algebras

A given operator algebra has many completely isometric representations, and the C*-algebra generated by each representation can be drastically different. For example, norm limits of polynomials in z and $\overline{z}$ generate the C*-algebra $C(\overline{\mathbb{D}})$ by the Stone-Weierstrass theorem. Hence, norm limits of polynomials in $A(\mathbb{D})$ and $A(\mathbb{D})^*$ generate $C(\overline{\mathbb{D}})$. We denote this by $C^*(A(\mathbb{D})) = C(\overline{\mathbb{D}})$. However, since the restriction of functions in $A(\mathbb{D})$ to $\mathbb{T}$ is completely isometric, we also have $C^*(\mathcal{A}(\mathbb{D})|_{\mathbb{T}}) = C(\mathbb{T})$. Thus, when studying the completely isometric representation theory of an operator algebra, it is important to track not only representation but also the C*-algebra generated by that representation.

Definition 1.2.1. A *C*-cover* for an operator algebra $\mathcal{A}$ is a pair $(\mathcal{C}, j)$ consisting of a C*-algebra $\mathcal{C}$ together with a completely isometric algebra homomorphism $j : \mathcal{A} \to \mathcal{C}$ such that $\mathcal{C} = C^*(j(\mathcal{A}))$.

There is a natural partial-ordering on the collection of C*-covers for a given operator algebra.

Definition 1.2.2. If $\mathcal{A}$ is an operator algebra and $(\mathcal{C}_1, j_1)$ and $(\mathcal{C}_2, j_2)$ are C*-covers of $\mathcal{A}$, then we say $(\mathcal{C}_1, j_1) \leq (\mathcal{C}_2, j_2)$ if there exists a (necessarily surjective) $*$-homomorphism $\pi : \mathcal{C}_2 \to \mathcal{C}_1$ such that $\pi \circ j_2 = j_1$, i.e. such that the following diagram commutes.

$$\begin{tikzcd} & \mathcal{C}_2 \arrow[d, "\pi"] \\ \mathcal{A} \arrow[ur, "j_2"] \arrow[r, "j_1"'] & \mathcal{C}_1 \end{tikzcd}$$

We define the equivalence of C*-covers using our partial order.

Definition 1.2.3. Two C*-covers $(\mathcal{C}_1, j_1)$ and $(\mathcal{C}_2, j_2)$ for an operator algebra $\mathcal{A}$ are said to be *equivalent*, denoted $(\mathcal{C}_1, j_1) = (\mathcal{C}_2, j_2)$, if there exists a $*$-isomorphism $\varphi : \mathcal{C}_1 \to \mathcal{C}_2$ such that $\varphi \circ j_1 = j_2$.

Example 1.2.4. Several examples of C*-covers are readily apparent.

(i) The C*-algebra M_n paired with the inclusion map $incl : T_n \to M_n$ is a C*-cover for the upper triangular $n \times n$ matrices T_n.

(ii) The C*-algebra M_4 paired with the inclusion map $incl : A_4 \to M_4$ is a C*-cover for the four cycle algebra A_4.

(iii) $(C(\overline{\mathbb{D}}), incl)$ and $(C(\mathbb{T}), (\cdot)|_{\mathbb{T}})$ are C*-covers for the disc algebra $A(\mathbb{D})$. Taking the quotient of $C(\overline{\mathbb{D}})$ by functions that vanish on $\mathbb{T}$ yields a quotient map of $C(\overline{\mathbb{D}})$ onto $C(\mathbb{T})$, which is given by restriction to $\mathbb{T}$. Hence, we have $(C(\mathbb{T}), (\cdot)|_{\mathbb{T}}) \leq (C(\overline{\mathbb{D}}), incl)$.

Every operator algebra $\mathcal{A}$ has a *maximal C*-cover*, which we denote $C^*_{\max}(\mathcal{A}) \equiv (C^*_{\max}(\mathcal{A}), i_{\max})$. Up to equivalence, the maximal C*-cover $C^*_{\max}(\mathcal{A})$ for $\mathcal{A}$ is any C*-cover for $\mathcal{A}$ satisfying the following universal property: if $\varphi : \mathcal{A} \to \mathcal{C}$ is a unital completely contractive homomorphism into a unital C*-algebra $\mathcal{C}$, then there exists a $*$-epimorphism π of $C^*_{\max}(\mathcal{A})$ onto $\mathcal{C}$ such that $\pi \circ i_{\max} = \varphi$. The maximal C*-cover for $\mathcal{A}$ is the C*-algebra generated by the direct sum of all completely contractive representations of $\mathcal{A}$, which includes all completely isometric copies of $\mathcal{A}$.

Example 1.2.5. Let T_2 denote the subalgebra of upper triangular 2×2 matrices in M_2 and let $\sqrt{\cdot}$ denote the function $t \mapsto \sqrt{t}$ for $t \in [0, 1]$. D. P. Blecher shows in Example 2.4 of [6] that

$$\mathcal{C}_2 := \{F \in C([0, 1], M_2) : F(0) \text{ is diagonal}\}$$

is the maximal C*-cover of T_2 via the complete isometry $\begin{bmatrix} \lambda & x \\ 0 & \mu \end{bmatrix} \underset{j}{\mapsto} \begin{bmatrix} \lambda 1 & x\sqrt{\cdot} \\ 0 & \mu 1 \end{bmatrix}$ by verifying $C^*(j(T_2)) = \mathcal{C}_2$ and checking that $\mathcal{C}_2$ has the universal property.

Example 1.2.6. Since $A(\mathbb{D})$ is generated by a unitary, $C^*_{\max}(A(\mathbb{D}))$ is the C*-algebra generated by the direct sum of all completely contractive representations of the generator of $A(\mathbb{D})$. Hence, $C^*_{\max}(A(\mathbb{D}))$ is the C*-algebra generated by a universal (non-normal) contraction. (See Example 2.4.5 in [24] and Example 2.3 in [6]).

1.3 Noncommutative Boundaries and the Shilov Boundary Ideal

In his seminal 1969 paper [3], W. Arveson explored to what extent a norm-closed algebra of bounded operators determined features about the C*-algebra it generated. He showed that certain irreducible representations of the generated C*-algebra called *boundary representations* were invariant under any completely isometric representation of the given algebra of operators. A *boundary representation* for an operator algebra $\mathcal{A}$ in a C*-algebra $\mathcal{C} = C^*(\mathcal{A})$ is an irreducible representation ρ of $\mathcal{C}$ such that $\rho|_{\mathcal{A}}$ has a unique completely positive extension to all of $\mathcal{C}$. As a tool, W. Arveson defined the non-commutative analogue of a boundary and a Shilov boundary from function algebra theory.

Definition 1.3.1. Let $(\mathcal{C}, j)$ be a C*-cover for an operator algebra $\mathcal{A}$. A closed two-sided ideal J in $\mathcal{C}$ is called a *boundary ideal* for $\mathcal{A}$ in $\mathcal{C}$ if the canonical quotient map $q : \mathcal{C} \to \mathcal{C}/J$ is completely isometric on $j(\mathcal{A})$. The *Shilov boundary ideal* for $\mathcal{A}$ is the largest boundary ideal for $\mathcal{A}$ in $\mathcal{C}$.

When $\mathcal{A}$ is a function algebra that generates a C*-algebra $C(X)$ of continuous functions on a compact, Hausdorff space X, boundary ideals for $\mathcal{A}$ in $C(X)$ correspond to closed subsets K of X where functions in $\mathcal{A}$ achieve their maximum modulus, i.e. K is a boundary for $\mathcal{A}$. The Shilov boundary for $\mathcal{A}$ is the smallest closed subset of X where functions in $\mathcal{A}$ achieve their maximum modulus, which indeed corresponds to the largest boundary ideal for $\mathcal{A}$ in $C(X)$.

It is not obvious that the Shilov boundary ideal should always exist. However, it does, and it is the intersection of all kernels of boundary representations for $\mathcal{A}$, of which there are sufficiently many. Proving the existence of the Shilov boundary ideal in general was the result of over 40 years of work from many prominent operator algebraists. (See [3], [16], [14], [2], and [13].)

We outline several (certainly well-known) observations about the intimate relationship between C*-covers the study of boundary ideals.

Proposition 1.3.2. *Suppose* $(\mathcal{C}, j)$ *is a C*-cover for* $\mathcal{A}$*. Let* J *be a closed two-sided ideal in* $\mathcal{C}$*. The following are equivalent:*

(i) J *is a boundary ideal for* $\mathcal{A}$ *in* $\mathcal{C}$*.*

(ii) J *is contained in the Shilov boundary ideal* $\mathcal{J}$ *for* $\mathcal{A}$ *in* $\mathcal{C}$*.*

Proof. Since $\mathcal{J}$ is the largest boundary ideal for $\mathcal{A}$ in $\mathcal{C}$, it contains all other boundary ideals. Conversely, suppose J is contained in $\mathcal{J}$ and let $q : \mathcal{C} \to \mathcal{C}/J$ be the quotient map. Since $J \subseteq \mathcal{J}$, $\mathcal{J}/J$ is a closed two-sided ideal in $\mathcal{C}/J$. Hence, we can observe that

$$\frac{\mathcal{C}/J}{\mathcal{J}/J} \cong \mathcal{C}/\mathcal{J}.$$

Thus, if $q : \mathcal{C} \to \mathcal{C}/J$ failed to be completely isometric on $j(\mathcal{A})$, the quotient map of $\mathcal{C}$ onto $\mathcal{C}/\mathcal{J}$ would fail to be completely isometric on $j(\mathcal{A})$ as well, which contradicts that $\mathcal{J}$ is a boundary ideal for $\mathcal{A}$ in $\mathcal{C}$. Therefore, q must be completely isometric on $j(\mathcal{A})$, and it follows that J is a boundary ideal for $\mathcal{A}$ in $\mathcal{C}$. $\square$

Proposition 1.3.3. *Let $(\mathcal{C}, j)$ be a C^*-cover for $\mathcal{A}$ and suppose J is a boundary ideal for $\mathcal{A}$ in $\mathcal{C}$. Then $(\mathcal{C}/J, q \circ j)$ is a C^*-cover for $\mathcal{A}$ where $q : \mathcal{C} \to \mathcal{C}/J$ is the quotient map.*

Proof. Since J is a boundary ideal for $\mathcal{A}$, the quotient map $q : \mathcal{C} \to \mathcal{C}/J$ is completely isometric on $j(\mathcal{A})$, and it follows that the composition $q \circ j$ is completely isometric on $\mathcal{A}$. Since q is a $*$-homomorphism and $\mathcal{C} = C^*(j(\mathcal{A}))$, we can conclude that $\mathcal{C}/J = q(C^*(j(\mathcal{A}))) = C^*(q \circ j(\mathcal{A}))$, which proves the desired result. □

Corollary 1.3.4. *Let $\mathcal{C}$ be a C^*-algebra and suppose $j : \mathcal{A} \to \mathcal{C}$ is an algebra homomorphism such that $C^*(j(\mathcal{A})) = \mathcal{C}$. If there exists a $*$-homomorphism $\pi : C^*_{max}(\mathcal{A}) \to \mathcal{C}$ such that $\pi \circ i_{max} = j$, then $(\mathcal{C}, j)$ is a C^*-cover for $\mathcal{A}$ if and only if $\ker \pi$ is a boundary ideal for $\mathcal{A}$ in $C^*_{max}(\mathcal{A})$.*

Proof. If $(\mathcal{C}, j)$ is a C*-cover for $\mathcal{A}$, the universal property of $C^*_{\max}(\mathcal{A})$ says $\mathcal{C} \cong C^*_{\max}(\mathcal{A})/\ker \pi$ and $j = \pi \circ i_{\max}$ is a complete isometry. Hence, $\pi|_{i_{\max}(\mathcal{A})}$ is completely isometric, i.e., $\ker \pi$ is a boundary ideal for $\mathcal{A}$ in $C^*_{\max}(\mathcal{A})$.

The converse follows from Proposition 1.3.3 since $(C^*_{\max}(\mathcal{A})/\ker \pi, \pi \circ i_{\max})$ is a C*-cover and must be equivalent to $(\mathcal{C}, j)$. □

It follows from the existence of the Shilov boundary ideal that every operator algebra $\mathcal{A}$ must have a minimal C*-cover with respect to the natural partial ordering on C*-covers. This minimal C*-cover for $\mathcal{A}$ is called the *C*-envelope for* $\mathcal{A}$, and we denote it by $C^*_e(\mathcal{A}) \equiv (C^*_e(\mathcal{A}), i_{\min})$. Up to equivalence, the C*-envelope for $\mathcal{A}$ is any C*-cover for $\mathcal{A}$ satisfying the following universal property: for any other C*-cover $(\mathcal{C}, j)$ for $\mathcal{A}$, there exists a $*$-epimorphism π of $\mathcal{C}$ onto $C^*_e(\mathcal{A})$ such that $\pi \circ j = i_{\min}$. The C*-envelope for $\mathcal{A}$ can be constructed by fixing a C*-cover $(\mathcal{C}, j)$ for $\mathcal{A}$, then taking the quotient of $\mathcal{C}$ by the Shilov boundary ideal $\mathcal{J}$ of $\mathcal{A}$ in $\mathcal{C}$, i.e. $C^*_e(\mathcal{A}) = (\mathcal{C}/\mathcal{J}, q \circ j)$ where $q : \mathcal{C} \to \mathcal{C}/\mathcal{J}$ is the quotient map.

Example 1.3.5. We compute examples of C*-envelopes.

(i) $(M_n, incl)$ is the C*-envelope for T_n. Since $(M_n, incl)$ is a C*-cover for T_n and M_n is simple, the universal property yields a quotient map of M_n onto $C_e^*(\mathcal{A})$ that must be injective.

(ii) Similarly, $(M_4, incl)$ is the C*-envelope for A_4.

(iii) $(C(\mathbb{T}), (\cdot)|_{\mathbb{T}})$ is the C*-envelope for $A(\mathbb{D})$ since $\mathbb{T}$ is the smallest closed subset where $A(\mathbb{D})$ achieves its maximum modulus.

1.4 Crossed Products of Operator Algebras

Let $\mathcal{A}$ be an operator algebra contained in some C*-algebra $\mathcal{C} = C^*(\mathcal{A})$. We denote the group of all *completely isometric automorphisms* of $\mathcal{A}$ by $\mathrm{Aut}(\mathcal{A})$. When $\mathcal{A}$ is a C*-algebra, $\mathrm{Aut}(\mathcal{A})$ is the group of $*$-automorphisms since completely isometric homomorphisms of C*-algebras are necessarily $*$-homomorphisms. In [22], E. Katsoulis and C. Ramsey generalize C*-dynamical systems to include non-self-adjoint algebras.

Definition 1.4.1. A *dynamical system* $(\mathcal{A}, G, \alpha)$ consists of an operator algebra $\mathcal{A}$ together with a strongly-continuous group homomorphism α of a locally compact group G into $\mathrm{Aut}(\mathcal{A})$.

Given a dynamical system $(\mathcal{A}, G, \alpha)$ and a C*-cover $(\mathcal{C}, j)$ for $\mathcal{A}$, we'd like to determine when the action of G on $\mathcal{A}$ extends to an action of G on $\mathcal{C}$ in a way that preserves the action of G on $j(\mathcal{A})$. When such an extension exists, we say $(\mathcal{C}, j)$ is *α-admissible*.

Definition 1.4.2. Let $(\mathcal{A}, G, \alpha)$ be a dynamical system. A C*-cover $(\mathcal{C}, j)$ for $\mathcal{A}$ is *α-admissible* if there exists a strongly continuous group homomorphism $\tilde{\alpha} : G \to \mathrm{Aut}(\mathcal{C})$ such that for all $s \in G$, the following diagram commutes.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\tilde{\alpha}_s"] & \mathcal{C} \\ \mathcal{A} \arrow[u, "j"] \arrow[r, "\alpha_s"'] & \mathcal{A} \arrow[u, "j"'] \end{tikzcd}$$

In the definition of α-admissibility, the extended action $\tilde{\alpha}$ of G on $\mathcal{C}$ must be the *unique* extension of α since $\mathcal{C} = C^*(j(\mathcal{A}))$, i.e. $\tilde{\alpha}$ is determined completely by its action on $j(\mathcal{A})$. It is not obvious that there should exist an α-admissible C*-cover for a given dynamical system $(\mathcal{A}, G, \alpha)$. However, the maximal C*-cover $C^*_{\max}(\mathcal{A})$ for $\mathcal{A}$ and the minimal C*-cover $C^*_e(\mathcal{A})$ for $\mathcal{A}$ must be α-admissible by their respective universal properties. (See Lemma 3.4 of [22].)

When $(\mathcal{C}, j)$ is an α-admissible cover for $\mathcal{A}$, $(\mathcal{C}, G, \tilde{\alpha})$ is a C*-dynamical system. Thus, we can build the full and reduced C*-crossed products $\mathcal{C} \rtimes_{\tilde{\alpha}} G$ and $\mathcal{C} \rtimes^r_{\tilde{\alpha}} G$. We define the full and reduced crossed products of $\mathcal{A}$ relative to the α-admissible C*-cover $(\mathcal{C}, j)$ as an appropriate generating operator subalgebra of the full and reduced crossed products of $\mathcal{C} \rtimes_{\tilde{\alpha}} G$ and $\mathcal{C} \rtimes^r_{\tilde{\alpha}} G$, respectively.

Definition 1.4.3. Let $(\mathcal{A}, G, \alpha)$ be a dynamical system and let $(\mathcal{C}, j)$ be an α-admissible C*-cover for $\mathcal{A}$. The *relative crossed product of $\mathcal{A}$ by G*, denoted $\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} G$, is the norm-closed subalgebra of $\mathcal{C} \rtimes_{\tilde{\alpha}} G$ generated by $C_c(G, j(\mathcal{A}))$. The *relative reduced crossed product of $\mathcal{A}$ by G*, denoted $\mathcal{A} \rtimes^r_{\mathcal{C},j,\alpha} G$, is the norm-closed subalgebra of $\mathcal{C} \rtimes^r_{\tilde{\alpha}} G$ generated by $C_c(G, j(\mathcal{A}))$.

In [22], E. Katsoulis and C. Ramsey define and develop the fundamental theory for crossed products of non-self-adjoint operator algebras. In particular, they show that given a dynamical system $(\mathcal{A}, G, \alpha)$ the relative crossed products $\mathcal{A} \rtimes_{C^*_{\max}(\mathcal{A}),\alpha} G \subseteq C^*_{\max}(\mathcal{A}) \rtimes_{\tilde{\alpha}} G$ and $\mathcal{A} \rtimes^r_{C^*_e(\mathcal{A}),\alpha} G \subseteq C^*_e(\mathcal{A}) \rtimes^r_{\tilde{\alpha}} G$ are universal objects for covariant representations and regular covariant representations of $(\mathcal{A}, G, \alpha)$, respectively. (See Theorem 3.10 and Corollary 3.16 in [22].) This observation justifies the definitions for the full and reduced crossed products below.

Definition 1.4.4. Let $(\mathcal{A}, G, \alpha)$ be a dynamical system. Define the *full crossed product of* $(\mathcal{A}, G, \alpha)$ to be the operator algebra

$$\mathcal{A} \rtimes_\alpha G := \mathcal{A} \rtimes_{C^*_{\max}(\mathcal{A}),\alpha} G,$$

and define the *reduced crossed product of* $(\mathcal{A}, G, \alpha)$ to be the operator algebra

$$\mathcal{A} \rtimes^r_\alpha G := \mathcal{A} \rtimes^r_{C^*_e(\mathcal{A}),\alpha} G.$$

Both [10] and [30] are good resources for the fundamental theory of crossed products of C*-algebras. We give examples of non-self-adjoint crossed products in Section 2.3.

Chapter 2

Admissibility of C*-Covers

Given a dynamical system $(\mathcal{A}, G, \alpha)$, the α-admissiblity of an arbitrary C*-cover $(\mathcal{C}, j)$ for $\mathcal{A}$ is unclear. The goal of the present chapter is to understand the structure of the collection of all C*-covers for a given operator algebra, and then to analyze how that structure is affected when a group action is introduced. In particular, we define meet and join operations to endow the collection of all C*-covers for $\mathcal{A}$ with a complete lattice structure in the natural partial ordering of C*-covers. We conclude by showing that the collection of all α-admissible C*-covers forms a complete sublattice in Theorem 2.2.9. In Section 2.4, we construct the first example of a finite dimensional non-admissible C*-cover, and in Section 2.5, we discuss how to recover the original dynamical system in a potentially non-admissible C*-cover by introducing a natural partial action.

We will use the following notation conventions throughout our work: $\mathcal{A}, \mathcal{B}$ will be operator algebras, $\mathcal{C}, \mathcal{D}$ will be C*-algebras, and i, j will be completely isometric algebra homomorphisms. The C*-algebra of compact operators on a separable Hilbert space will be denoted $\mathbb{K}$. All ideals will be closed and two-sided unless otherwise stated. All groups G are locally compact and Hausdorff.

2.1 The Complete Lattice of C*-Covers

Recall the natural ordering on the collection of C*-covers for an operator algebra.

Definition 1.2.2. If $\mathcal{A}$ is an operator algebra and $(\mathcal{C}_1, j_1), (\mathcal{C}_2, j_2)$ are C*-covers of $\mathcal{A}$, then we say $(\mathcal{C}_1, j_1) \leq (\mathcal{C}_2, j_2)$ if there exists a $*$-homomorphism $\pi : \mathcal{C}_2 \to \mathcal{C}_1$ such that $\pi \circ j_2 = j_1$.

As we have been claiming, $\leq$ is a partial ordering on the collection of all C*-covers for an operator algebra.

Proposition 2.1.1. *Let $\mathfrak{C}$ be the collection of all C*-covers of a unital operator algebra $\mathcal{A}$. Then $\leq$ is a partial order on $\mathfrak{C}$.*

Proof. It is routine to check that the ordering is (i) reflexive, (ii) antisymmetric, and (iii) transitive, but we include the proof for completeness.

(i) If $(\mathcal{C}, j)$ is a C*-cover for $\mathcal{A}$, then $(\mathcal{C}, j) \leq (\mathcal{C}, j)$ since the identity map $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ is a $*$-homomorphism such that $\mathrm{id} \circ j = j$.

(ii) If $(\mathcal{C}_1, j_1)$ and $(\mathcal{C}_2, j_2)$ are C*-covers for $\mathcal{A}$ such that $(\mathcal{C}_1, j_1) \leq (\mathcal{C}_2, j_2)$ and $(\mathcal{C}_2, j_2) \leq (\mathcal{C}_1, j_1)$, then there exist $*$-homomorphisms $\pi : \mathcal{C}_2 \to \mathcal{C}_1$ and $\rho : \mathcal{C}_1 \to \mathcal{C}_2$ such that the following diagram commutes.

$$\begin{tikzcd} & \mathcal{C}_1 \arrow[d, dashed, "\rho"] \\ \mathcal{A} \arrow[ur, "j_1"] \arrow[r, "j_2"'] \arrow[dr, "j_1"'] & \mathcal{C}_2 \arrow[d, dashed, "\pi"] \\ & \mathcal{C}_1 \end{tikzcd}$$

Hence, $\rho \circ \pi \circ j_2 = j_2$ and $\pi \circ \rho \circ j_1 = j_1$, which implies that ρ and π are inverses since $\mathcal{C}_1 = C^*(j_1(\mathcal{A}))$ and $\mathcal{C}_2 = C^*(j_2(\mathcal{A}))$. Thus, $\rho : \mathcal{C}_1 \to \mathcal{C}_2$ is a $*$-isometric isomorphism such that $\rho \circ j_1 = j_2$, and it follows that $(\mathcal{C}_1, j_1)$ and $(\mathcal{C}_2, j_2)$ are equivalent C*-covers.

(iii) For $i = 1, 2, 3$, suppose $(\mathcal{C}_i, j_i)$ are C*-covers such that $(\mathcal{C}_1, j_1) \leq (\mathcal{C}_2, j_2)$ and $(\mathcal{C}_2, j_2) \leq (\mathcal{C}_3, j_3)$. Then there exist $*$-homomorphisms $\pi : \mathcal{C}_2 \to \mathcal{C}_1$ and $\rho : \mathcal{C}_3 \to \mathcal{C}_2$ such that the following diagram commutes.

$$\begin{tikzcd} & \mathcal{C}_3 \arrow[d, dashed, "\rho"] \\ \mathcal{A} \arrow[ur, "j_3"] \arrow[r, "j_2"'] \arrow[dr, "j_1"'] & \mathcal{C}_2 \arrow[d, dashed, "\pi"] \\ & \mathcal{C}_1 \end{tikzcd}$$

Composing maps yields that $\pi \circ \rho : \mathcal{C}_3 \to \mathcal{C}_1$ is a $*$-homomorphism satisfying $\pi \circ \rho \circ j_3 = \pi \circ j_2 = j_1$. Hence, $(\mathcal{C}_1, j_1) \leq (\mathcal{C}_3, j_3)$. $\square$

Our goal is to show that $(\mathfrak{C}, \leq)$ forms a complete lattice. We will suppress notation by writing $\mathfrak{C}$ when we refer to the partially ordered collection $(\mathfrak{C}, \leq)$. We continue by reviewing the direct sum of operator spaces.

Example 2.1.2. Let $\{X_\lambda : \lambda \in I\}$ be a family of operator spaces such that each X_λ is contained in some corresponding C*-algebra $\mathcal{C}_\lambda$. We identify $\bigoplus_\lambda X_\lambda$ via its canonical inclusion in the C*-algebra $\bigoplus_\lambda \mathcal{C}_\lambda$. Hence, $\bigoplus_\lambda X_\lambda$ inherits its operator space structure from $\bigoplus_\lambda \mathcal{C}_\lambda$, which is determined by the $*$-isomorphisms $M_n(\bigoplus_\lambda \mathcal{C}_\lambda) \cong \bigoplus_\lambda M_n(\mathcal{C}_\lambda)$ for all $n \in \mathbb{N}$ as described in the following proposition.

Proposition 2.1.3 is well-known and can be found in the literature. Example 1.2.17 in [8] mentions a strategy for a proof, for example. We give a complete proof here as the construction will be instrumental in later results.

Proposition 2.1.3. *For all $n \in \mathbb{N}$, $M_n(\bigoplus_\lambda \mathcal{C}_\lambda)$ is $*$-isomorphic to $\bigoplus_\lambda M_n(\mathcal{C}_\lambda)$.*

Proof. Let $n \in \mathbb{N}$ be given. For each $\mu \in I$, let P_μ be the projection of $\bigoplus_\lambda \mathcal{C}_\lambda$ onto $\mathcal{C}_\mu$. Define $\Phi_n : M_n(\bigoplus_\lambda \mathcal{C}_\lambda) \to \bigoplus_\lambda M_n(\mathcal{C}_\lambda)$ by $\Phi_n = \bigoplus_\lambda (P_\lambda)_n$.

We see that Φ_n is a well-defined $*$-homomorphism since each amplification $(P_\lambda)_n$ is a $*$-homomorphism. Indeed, for each $\lambda \in I$, we get $\|(P_\lambda)_n\| = 1$ since amplifications of $*$-homomorphisms are $*$-homomorphisms. Thus, $\sup_\lambda \|(P_\lambda)_n\| = 1$, and it follows that for all $(a_{rc}) \in M_n(\bigoplus_\lambda \mathcal{C}_\lambda)$ we have

$$\|\Phi_n(a_{rc})\| = \sup_{\lambda \in I} \|(P_\lambda)_n(a_{rc})\|_{M_n(\mathcal{C}_\lambda)} \leq \sup_{\lambda \in I} \|(P_\lambda)_n\| \cdot \|(a_{rc})\| = \|(a_{rc})\| < \infty.$$

To see that Φ_n is isometric, observe that $\Phi_n(a_{rc}) = 0$ if and only if $P_\lambda a_{rc} = 0$ for all $1 \leq r, c \leq n$ and each $\lambda \in I$. This happens if and only if $a_{rc} \in \ker P_\lambda$ for all $1 \leq r, c, \leq n$ and $\lambda \in I$. Hence, $\Phi_n(a_{rc}) = 0$ if and only if $a_{rc} = 0 \in \bigoplus_\lambda \mathcal{C}_\lambda$ for all $1 \leq r, c \leq n$. So Φ_n is injective, and hence, an isometric $*$-homomorphism. Since Φ_n is clearly surjective, we obtain the desired isomorphism. $\square$

We show that the direct sum of completely isometric linear maps is indeed completely isometric.

Lemma 2.1.4. *Let $\mathcal{A}$ be a unital operator algebra and let $\{\varphi_\lambda : \lambda \in I\}$ be a family of completely isometric linear maps from $\mathcal{A}$ to some corresponding C^*-algebra $\mathcal{C}_\lambda$. Then $\varphi : \mathcal{A} \to \bigoplus_\lambda \mathcal{C}_\lambda$ given by $\varphi = \bigoplus_\lambda \varphi_\lambda$ is completely isometric.*

Proof. By construction, φ is a well-defined isometry. We need only verify that φ is completely isometric. Let $n \in \mathbb{N}$ and $(a_{rc}) \in M_n(\mathcal{A})$ be given. Proposition 2.1.3 says $\Phi_n : M_n(\bigoplus_\lambda \mathcal{C}_\lambda) \to \bigoplus_\lambda M_n(\mathcal{C}_\lambda)$ given by $\Phi_n = \bigoplus_\lambda (P_\lambda)_n$ is a $*$-isomorphism. Thus, since φ_n maps $M_n(\mathcal{A})$ into $M_n\left(\bigoplus_\lambda \mathcal{C}_\lambda\right)$, we have the following equivalence of norms

$$\left\|\varphi_n((a_{rc})_{r,c=1}^n)\right\|_n = \left\|(\oplus_\lambda \varphi_\lambda(a_{rc}))_{r,c=1}^n\right\|_{M_n\left(\bigoplus_\lambda \mathcal{C}_\lambda\right)} = \left\|\Phi_n\left((\oplus_\lambda \varphi_\lambda(a_{rc}))_{r,c=1}^n\right)\right\|_{\bigoplus_\lambda M_n(\mathcal{C}_\lambda)}.$$

By definition, Φ_n acts on the matrix $(\oplus_\lambda \varphi_\lambda(a_{rc}))_{r,c=1}^n$ by projecting onto the μ-th summand of each matrix entry to produce a matrix in $M_n(\mathcal{C}_\mu)$ for all $\mu \in I$. The resulting matrices indexed by I are then direct summed. Hence, we have $\Phi_n\left((\oplus_\lambda \varphi_\lambda(a_{rc}))_{r,c=1}^n\right) = \bigoplus_{\mu\in I}(\varphi_\mu(a_{rc}))_{r,c=1}^n$. Thus, we compute

$$\begin{aligned}
\left\|\varphi_n((a_{rc})_{r,c=1}^n)\right\|_n &= \left\|\Phi_n\left((\oplus_\lambda \varphi_\lambda(a_{rc}))_{r,c=1}^n\right)\right\|_{\bigoplus_\lambda M_n(\mathcal{C}_\lambda)} \\
&= \sup_\mu \left\|(\varphi_\mu(a_{rc}))_{r,c=1}^n\right\|_{M_n(\mathcal{C}_\mu)} \\
&= \sup_\mu \left\|(\varphi_\mu)_n((a_{rc})_{r,c=1}^n)\right\|_{M_n(\mathcal{C}_\mu)}.
\end{aligned}$$

Since φ_μ is completely isometric for all $\mu \in I$, we conclude that

$$\left\|\varphi_n((a_{rc})_{r,c=1}^n)\right\|_n = \sup_\mu \left\|(\varphi_\mu)_n((a_{rc})_{r,c=1}^n)\right\| = \left\|(a_{rc})_{r,c=1}^n\right\|.$$

Therefore, φ is a complete isometry. □

We amend the previous technical lemma to observe that complete isometries can be constructed as the direct sum of a complete isometry and proper complete contraction. This observation will be used in the construction of Example 2.4.1.

Lemma 2.1.5. *Let $\mathcal{A}$ be a unital operator algebra and $\{\varphi_\lambda : \lambda \in I\}$ be a family of completely contractive linear maps from $\mathcal{A}$ to some corresponding C^*-algebra $\mathcal{C}_\lambda$. Suppose $\varphi_0 : \mathcal{A} \to \mathcal{C}_0$ is a complete isometry of $\mathcal{A}$ into a C^*-algebra $\mathcal{C}_0$. Then $\varphi : \mathcal{A} \to \mathcal{C}_0 \oplus (\bigoplus_\lambda \mathcal{C}_\lambda)$ given by $\varphi = \varphi_0 \oplus (\bigoplus_\lambda \varphi_\lambda)$ is completely isometric.*

Proof. Let $n \in \mathbb{N}$ and $(a_{rc})_{r,c=1}^n \in M_n(\mathcal{A})$ be given. Since each φ_μ is completely contractive, we have

$$\sup_{\mu\in I} \left\|(\varphi_\mu)_n\left((a_{rc})_{r,c=1}^n\right)\right\| \le \left\|(a_{rc})_{r,c=1}^n\right\| = \left\|(\varphi_0)_n\left((a_{rc})_{r,c=1}^n\right)\right\|.$$

Thus, the norm $\left\|\varphi_n((a_{rc})_{r,c=1}^n)\right\|$ will be achieved on the completely isometric summand $(\varphi_0)_n$ of φ_n. Hence, we have

$$\left\|\varphi_n((a_{rc})_{r,c=1}^n)\right\| = \sup\left\{\left\|(\varphi_\mu)_n\left((a_{rc})_{r,c=1}^n\right)\right\|_{M_n(\mathcal{C}_\mu)} : \mu \in \{0\} \cup I\right\} = \left\|(a_{rc})_{r,c=1}^n\right\|.$$

Therefore, φ is a complete isometry. □

We show that the direct sum of completely isometric representations of an operator algebra generates a C*-cover, and that C*-cover is an upper bound for each summand of the representation.

Theorem 2.1.6. *Let $S = \{(\mathcal{C}_\lambda, j_\lambda) : \lambda \in I\} \subseteq \mathfrak{C}$ be given and define $j := \bigoplus_\lambda j_\lambda$. Then $(C^*(j(\mathcal{A})), j)$ is a C*-cover for $\mathcal{A}$ that is the supremum of S.*

Proof. Lemma 2.1.4 says j is a completely isometric algebra homomorphism, and thus, $(C^*(j(\mathcal{A})), j)$ is a C*-cover for $\mathcal{A}$. Clearly, $(C^*(j(\mathcal{A})), j)$ is an upper bound for S. Suppose $(\mathcal{D}, i)$ is another upper bound for S. Then for each $\lambda \in I$, there exists a $*$-homomorphism π_λ of $\mathcal{D}$ onto $\mathcal{C}_\lambda$ such that $\pi_\lambda \circ i = j_\lambda$. Define $\pi : \mathcal{D} \to \bigoplus_\lambda \mathcal{C}_\lambda$ by $\pi = \bigoplus_\lambda \pi_\lambda$. Then π is a $*$-homomorphism of $\mathcal{D}$ into $C^*(j(\mathcal{A}))$ such that

$$\pi \circ i = \bigoplus_\lambda (\pi_\lambda \circ i) = \bigoplus_\lambda j_\lambda = j.$$

It is clear that π maps $\mathcal{D}$ onto $C^*(j(\mathcal{A}))$ since $\pi(i(\mathcal{A})) = j(\mathcal{A})$ implies $\pi(\mathcal{D}) = C^*(j(\mathcal{A}))$ since π is a $*$-homomorphism and $C^*(i(\mathcal{A})) = \mathcal{D}$. Thus, $(C^*(j(\mathcal{A})), j)$ is below $(\mathcal{D}, i)$ with respect to the partial ordering on $\mathfrak{C}$, i.e. $(C^*(j(\mathcal{A})), j) \leq (\mathcal{D}, i)$. □

A consequence of Theorem 2.1.6 is that we can equip $\mathfrak{C}$ with a join operation that makes it a complete join semi-lattice.

Definition 2.1.7. Let $S = \{(\mathcal{C}_\lambda, j_\lambda) : \lambda \in I\} \subseteq \mathcal{C}$ be given. Define $\bigvee_\lambda \mathcal{C}_\lambda := C^*\left(\bigoplus_\lambda j_\lambda(\mathcal{A})\right) \subseteq \bigoplus_{\lambda\in I} C_\lambda$ and set $\bigvee_\lambda j_\lambda := \bigoplus_\lambda j_\lambda$. Theorem 2.1.6 shows $\bigvee_\lambda j_\lambda$ is a complete isometry and $\bigvee_{\lambda\in I} C_\lambda = C^*((\bigvee_\lambda j_\lambda)(\mathcal{A}))$ is the supremum of S. We say the *join* of S is the C*-cover $(\bigvee_\lambda \mathcal{C}_\lambda, \bigvee_\lambda j_\lambda)$ for $\mathcal{A}$.

Example 2.1.8. Let $\mathcal{A}$ be a unital operator algebra and let $\mathcal{S}$ be a collection of C*-covers for $\mathcal{A}$ containing $C^*_{\max}(\mathcal{A})$. Then $\bigvee_{\mathcal{C}\in\mathcal{S}} \mathcal{C} = C^*_{\max}(\mathcal{A})$ by the universal property of $C^*_{\max}(\mathcal{A})$ and the fact that $\bigvee_{\mathcal{C}\in\mathcal{S}} \mathcal{C}$ must be a supremum.

Example 2.1.9. Let $\mathcal{A}$ be a unital operator algebra and let S be a collection of C*-covers for $\mathcal{A}$ containing $C^*_e(\mathcal{A})$ such that $S \setminus \{C^*_e(\mathcal{A})\}$ is nonempty. Then $\bigvee_{\mathcal{C}\in S} \mathcal{C} = \bigvee_{\mathcal{C}\in S\setminus\{C^*_e(\mathcal{A})\}} \mathcal{C}$.

It is tempting to view the join of two C*-covers as their direct sum. Example 2.1.10 is warning against this viewpoint. The join of C*-covers as defined in Definition 2.1.7 is *not* the same as the direct sum of two C*-algebras in general.

Example 2.1.10. Let $\mathcal{A}$ be a unital operator algebra and let $(\mathcal{C}, j)$ be a C*-cover for $\mathcal{A}$. Then for every $a \in \mathcal{A}$, we have

$$(j \vee j)(a) = j(a) \oplus j(a) = j(a)(I_\mathcal{C} \oplus I_\mathcal{C}) \in \mathcal{C} \oplus \mathcal{C}.$$

Hence, we have $(j \vee j)(\mathcal{A}) = j(\mathcal{A})(1_\mathcal{C} \oplus 1_\mathcal{C})$, and it follows that $\mathcal{C} \vee \mathcal{C} = \mathcal{C}(1_\mathcal{C} \oplus 1_\mathcal{C}) \cong \mathcal{C}$. Hence, $\mathcal{C} \vee \mathcal{C}$ is *not* isomorphic to $\mathcal{C} \oplus \mathcal{C}$.

We define a meet operation for $\mathcal{C}$ using the ideal structure of $C^*_{\max}(\mathcal{A})$.

Theorem 2.1.11. *Let $S = \{(\mathcal{C}_\lambda, j_\lambda) : \lambda \in I\} \subseteq \mathcal{C}$ be given. Then S has an infimum in $\mathcal{C}$.*

Proof. Let $\mathcal{J}$ be the Shilov boundary ideal for $\mathcal{A}$ in $C^*_{\max}(\mathcal{A})$. For each $\lambda \in I$, there exists a $*$-epimorphism $q_\lambda : C^*_{\max}(\mathcal{A}) \to \mathcal{C}_\lambda$ such that $\ker q_\lambda \subseteq \mathcal{J}$ is a boundary ideal for $\mathcal{A}$ in $C^*_{\max}(\mathcal{A})$ and the following diagram commutes.

$$\begin{tikzcd} & C^*_{\max}(\mathcal{A}) \arrow[d, "q_\lambda"] \\ \mathcal{A} \arrow[ur, "i_{\max}"] \arrow[r, "j_\lambda"'] & \mathcal{C}_\lambda \end{tikzcd}$$

Let J be the norm-closure of the two-sided ideal generated by $\bigcup_\lambda \ker q_\lambda$ in $C^*_{\max}(\mathcal{A})$, i.e. $J = \overline{\sum_\lambda \ker q_\lambda}^{\|\cdot\|}$. Then J is the smallest closed two-sided in $C^*_{\max}(\mathcal{A})$ containing $\bigcup_\lambda \ker q_\lambda$ and so $J \subseteq \mathcal{J}$. Hence, Proposition 1.3.2 says J is a boundary ideal for $\mathcal{A}$ in $C^*_{\max}(\mathcal{A})$. It follows that $(C^*_{\max}(\mathcal{A})/J, q_J \circ i_{\max})$ is a C*-cover for $\mathcal{A}$, where q_J is the quotient map of $C^*_{\max}(\mathcal{A})$ onto $C^*_{\max}(\mathcal{A})/J$.

Clearly, $(C^*_{\max}(\mathcal{A})/J, q_J \circ i_{\max})$ is a lower bound for S. We claim that $(C^*_{\max}(\mathcal{A})/J, q_J \circ i_{\max})$ is the infimum of S. Suppose $(\mathcal{D}, i)$ is a C*-cover for $\mathcal{A}$ such that $(\mathcal{D}, i) \leq (\mathcal{C}_\lambda, j_\lambda)$ for all $\lambda \in I$. Then there exists a $*$-epimorphism π of $C^*_{\max}(\mathcal{A})$ onto $\mathcal{D}$ such that $\pi \circ i_{\max} = j$. Hence, $\ker \pi$ is a boundary ideal for $\mathcal{A}$ in $C^*_{\max}(\mathcal{A})$ such that $C^*_{\max}(\mathcal{A})/\ker \pi \cong \mathcal{D}$. Note that $\bigcup_\lambda \ker q_\lambda$ is contained in $\ker \pi$ since $(\mathcal{D}, i) \leq (\mathcal{C}_\lambda, j_\lambda)$ for all $\lambda \in I$. Indeed, for each $\lambda \in I$, $(\mathcal{D}, i) \leq (\mathcal{C}_\lambda, j_\lambda)$ implies there exists a $*$-epimorphism $Q_\lambda : \mathcal{C}_\lambda \to \mathcal{D}$ such that the following diagram commutes.

$$\begin{tikzcd} & C^*_{\max}(\mathcal{A}) \arrow[d, "q_\lambda"] \\ \mathcal{A} \arrow[ur, "i_{\max}"] \arrow[r, "j_\lambda"'] \arrow[dr, "i"'] & \mathcal{C}_\lambda \arrow[d, dashed, "Q_\lambda"] \\ & \mathcal{D} \end{tikzcd}$$

Thus, for each $\lambda \in I$, we have

$$(Q_\lambda \circ q_\lambda) \circ i_{\max} = Q_\lambda \circ (q_\lambda \circ i_{\max}) = Q_\lambda \circ j_\lambda = i = \pi \circ i_{\max},$$

which implies $\pi = Q_\lambda \circ q_\lambda$. It follows that $\ker q_\lambda \subseteq \ker \pi$ for all $\lambda \in I$.

Since J is the smallest ideal containing $\bigcup_\lambda \ker q_\lambda$, we must have $J \subseteq \ker \pi$. Thus, $\ker \pi / J$ is an ideal in $C^*_{\max}(\mathcal{A})/J$ such that $(C^*_{\max}(\mathcal{A})/J)/(\ker \pi / J) \cong \mathcal{D}$, i.e. $\mathcal{D}$ is a quotient of

$C^*_{\max}(\mathcal{A})/J$. Hence, there exists a $*$-homomorphism $\tilde{\pi}$ of $C^*_{\max}(\mathcal{A})/J$ onto $\mathcal{D}$ such that $\tilde{\pi} \circ q_J = \pi$, which yields

$$\tilde{\pi} \circ (q_J \circ i_{\max}) = (\tilde{\pi} \circ q_J) \circ i_{\max} = \pi \circ i_{\max} = i.$$

Therefore, we have $(\mathcal{D}, i) \leq (C^*_{\max}(\mathcal{A})/J, q_J \circ i_{\max})$. $\square$

Definition 2.1.12. Let $S = \{(\mathcal{C}_\lambda, j_\lambda) : \lambda \in I\} \subseteq \mathfrak{C}$ be given. Using the notation of Proposition 2.1.11, set $\bigwedge_\lambda j_\lambda := q_J \circ i_{\max}$ for $J = \overline{\sum_\lambda \ker q_\lambda} \subseteq C^*_{\max}(\mathcal{A})$, and put $\bigwedge_\lambda \mathcal{C}_\lambda := C^*(\bigwedge_\lambda j_\lambda(\mathcal{A}))$. The *meet* of S is the C*-cover $(\bigwedge_\lambda \mathcal{C}_\lambda, \bigwedge_\lambda j_\lambda)$ for $\mathcal{A}$.

Example 2.1.13. Let $\mathcal{A}$ be an operator algebra and suppose S is a collection of C*-covers for $\mathcal{A}$ containing $C^*_e(\mathcal{A})$. Then $\bigwedge_{\mathcal{C} \in S} \mathcal{C} = C^*_e(\mathcal{A})$ since $C^*_e(\mathcal{A})$ is the smallest C*-cover of $\mathcal{A}$.

We can define the join of C*-covers in terms of boundary ideals in the maximal C*-cover as we did with the meet. We give this alternate definition and prove its equivalence in Proposition 2.1.14.

Proposition 2.1.14. *Let $S = \{(\mathcal{C}_\lambda, j_\lambda) : \lambda \in I\} \subseteq \mathfrak{C}$ be given. Define $J := \bigcap_\lambda \ker q_\lambda \subseteq C^*_{max}(\mathcal{A})$, where each q_λ is the quotient map of $C^*_{max}(\mathcal{A})$ onto $\mathcal{C}_\lambda$. Then $(\bigvee_\lambda \mathcal{C}_\lambda, \bigvee_\lambda j_\lambda)$ is equivalent to $(C^*_{max}(\mathcal{A})/J, q_J \circ i_{max})$, where q_J is the quotient map of $C^*_{max}(\mathcal{A})$ onto $C^*_{max}(\mathcal{A})/J$.*

Proof. Observe that J is a closed two-sided ideal contained in each $\ker q_\lambda$. Hence, we must have $(\mathcal{C}_\lambda, j_\lambda) = (C^*_{\max}(\mathcal{A})/\ker q_\lambda, q_\lambda \circ i_{\max}) \leq (C^*_{\max}(\mathcal{A})/J, q_J \circ i_{\max})$ for all $\lambda \in I$ by passing to quotients. Since $(\bigvee_\lambda \mathcal{C}_\lambda, \bigvee_\lambda j_\lambda)$ is the supremum for S, we obtain $(\bigvee_\lambda \mathcal{C}_\lambda, \bigvee_\lambda j_\lambda) \leq (C^*_{\max}(\mathcal{A})/J, q_J \circ i_{\max})$.

The universal property of $C^*_{\max}(\mathcal{A})$ guarantees a $*$-epimorphism Q of $C^*_{\max}(\mathcal{A})$ onto $\bigvee_\lambda \mathcal{C}_\lambda$ such that $Q \circ i_{\max} = \bigvee_\lambda j_\lambda$. Since $(\bigvee_\lambda \mathcal{C}_\lambda, \bigvee_\lambda j_\lambda)$ is an upper bound for S, $\ker Q$ is contained in $\ker q_\lambda$ for all $\lambda \in I$. But J is the largest closed two-sided ideal contained in each $\ker q_\lambda$. So we must have $\ker Q \subseteq J$, which implies $J/\ker Q$ is a closed two-sided ideal in $C^*_{\max}(\mathcal{A})/\ker Q \cong \bigvee_\lambda \mathcal{C}_\lambda$. Taking the quotient of $C^*_{\max}(\mathcal{A})/\ker Q$ by $J/\ker Q$ yields a C*-cover equivalent to $(C^*_{\max}(\mathcal{A})/J, q_J \circ i_{\max})$. Therefore, we have $(C^*_{\max}(\mathcal{A})/J, q_J \circ i_{\max}) \leq (\bigvee_\lambda \mathcal{C}_\lambda, \bigvee_\lambda j_\lambda)$ by chasing the diagram of quotients. This completes the proof as $(\leq)$ is a partial ordering on $\mathfrak{C}$. $\square$

The following corollary is a consequence of Theorems 2.1.6 and 2.1.11.

Corollary 2.1.15. *Let $\mathfrak{C}$ be the collection of all C*-covers for a unital operator algebra $\mathcal{A}$. Then $\mathfrak{C} \equiv (\mathfrak{C}, \leq)$ forms a complete lattice.*

2.2 α-Admissibility of C*-Covers and α-Invariant Boundary Ideals

Given a dynamical system $(\mathcal{A}, G, \alpha)$, E. Katsoulis and C. Ramsey prove in [22] that the maximal and minimal C*-covers for $\mathcal{A}$ are always α-admissible by their respective universal properties. So determining the α-admissibility of a C*-cover $(\mathcal{C}, j)$ for $\mathcal{A}$ can be thought of as an automorphism lifting problem from the quotient $C^*_e(\mathcal{A})$ to $\mathcal{C}$ or an automorphism factoring problem from $C^*_{\max}(\mathcal{A})$ to $\mathcal{C}$. In this section, we approach α-admissibility from the latter perspective.

Definition 2.2.1. Let $(\mathcal{A}, G, \alpha)$ be a dynamical system and let $(\mathcal{C}, j)$ be an α-admissible C*-cover for $\mathcal{A}$. A closed two-sided ideal J in $\mathcal{C}$ is called an *α-invariant ideal* in $\mathcal{C}$ if J is invariant under the extended automorphism $\tilde{\alpha}_s \in \mathrm{Aut}(\mathcal{C})$ for all $s \in G$, i.e. $\tilde{\alpha}_s(J) \subseteq J$ for all $s \in G$. A closed two-sided ideal J in $\mathcal{C}$ is called an *$\mathcal{A}$-invariant ideal* in $\mathcal{C}$ if J is invariant under each $*$-automorphism of $\mathcal{C}$ that leaves $j(\mathcal{A})$ invariant in $\mathcal{C}$.

Example 2.2.2. Recall that the Toeplitz algebra $\mathcal{T}$ is an extension of $C(\mathbb{T})$ by the compact operators $\mathbb{K}$ on the Hardy space $H^2(\mathbb{T})$. This yields the following short exact sequence, where $T : C(\mathbb{T}) \to \mathcal{T}$ is the symbol map $T(f) = T_f$ for all $f \in C(\mathbb{T})$.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{K} \arrow[r, "incl"] & \mathcal{T} \arrow[r, "\pi"] & C(\mathbb{T}) \arrow[r] & 0 \\ & & A(\mathbb{D}) \arrow[u, "T|_{(A(\mathbb{D}))|_{\mathbb{T}}}"] \arrow[ur, "(\cdot)|_{\mathbb{T}}"'] & & \end{tikzcd}$$

We will see in Section 3.2.2 that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is a C*-cover for $A(\mathbb{D})$. Suppose $(A(\mathbb{D}), G, \alpha)$ is a dynamical system such that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible. As the automorphisms of the Toeplitz algebra are inner in $\mathcal{B}(H^2(\mathbb{T}))$, the compact operators $\mathbb{K}$ in $\mathcal{T}$ are invariant under the automorphisms of $\mathcal{T}$. Hence, $\mathbb{K}$ must be an α-invariant ideal.

This example is important as we will see in Section 3.2.5 that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible under any dynamical system $(A(\mathbb{D}), G, \alpha)$ where G is cyclic.

We observe that the $\mathcal{A}$-invariant ideals of $C^*_{\max}(\mathcal{A})$ characterize the C*-covers for $\mathcal{A}$ that are α-admissible for any dynamical system $(\mathcal{A}, G, \alpha)$. We will call these C*-covers *always admissible.*

Theorem 2.2.3. *Let $(\mathcal{A}, G, \alpha)$ be a dynamical system and let $(\mathcal{D}, i)$ be an α-admissible C*-cover for $\mathcal{A}$. If $(\mathcal{C}, j)$ is any C*-cover for $\mathcal{A}$ such that there exists a $*$-homomorphism π of $\mathcal{D}$ onto $\mathcal{C}$ satisfying $\pi \circ i = j$, i.e. $(\mathcal{C}, j) \leq (\mathcal{D}, i)$, then the following are equivalent:*

(i) $(\mathcal{C}, j)$ *is α-admissible.*

(ii) $\ker \pi$ *is an α-invariant ideal in $\mathcal{D}$.*

Proof. Since $(\mathcal{D}, i)$ is α-admissible, there exists a strongly continuous group representation $\beta : G \to \mathrm{Aut}(\mathcal{D})$ such that $\beta_s \circ i = i \circ \alpha_s$ for all $s \in G$. Moreover, we have the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & \ker\pi \arrow[r] & \mathcal{D} \arrow[r, "\pi"] & \mathcal{C} \arrow[r] & 0 \\ & & \mathcal{A} \arrow[u, "i"] \arrow[ur, "j"'] & & \end{tikzcd}$$

If $(\mathcal{C}, j)$ is α-admissible, there exists another strongly continuous group representation $\tilde{\alpha} : G \to \mathrm{Aut}(\mathcal{C})$ such that $\tilde{\alpha}_s \circ j = j \circ \alpha_s$ for all $s \in G$. Since the above diagram commutes, π intertwines the actions β and $\tilde{\alpha}$ of G on $i(\mathcal{A})$ and $j(\mathcal{A})$, respectively. That is, for all $s \in G$, we have

$$(\pi \circ \beta_s) \circ i = \pi \circ (\beta_s \circ i) = \pi \circ (i \circ \alpha_s) = (\pi \circ i) \circ \alpha_s = j \circ \alpha_s = \tilde{\alpha}_s \circ j.$$

This intertwining corresponds to chasing the following commutative diagram.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\beta_s"] & \mathcal{D} \arrow[r, "\pi"] & \mathcal{C} \arrow[r, "\tilde{\alpha}_s"] & \mathcal{C} \\
\mathcal{A} \arrow[u, "i"] \arrow[r, "\alpha_s"] & \mathcal{A} \arrow[u, "i"] \arrow[ur, "j"'] \arrow[rr, "\alpha_s"] & & \mathcal{A} \arrow[u, "j"']
\end{tikzcd}$$

Let $s \in G$, $a, b \in \mathcal{A}$ be given and suppose $i(a)i(b)^* \in \ker \pi$. Since π is $*$-homomorphism that intertwines the $*$-automorphisms β_s and $\tilde{\alpha}_s$, we can see that $\beta_s(i(a)i(b)^*) \in \ker \pi$. Indeed, observe that

$$\begin{aligned}
\pi\beta_s\left(i(a)i(b)^*\right) &= \pi\beta_s i(a)\left(\pi\beta_s i(b)\right)^* \\
&= \tilde{\alpha}_s j(a)\left(\tilde{\alpha}_s j(b)\right)^* \\
&= \tilde{\alpha}_s\left(j(a)j(b)^*\right) \\
&= \tilde{\alpha}_s\left(\pi i(a)(\pi i(b))^*\right) \\
&= \tilde{\alpha}_s(\pi\left(i(a)i(b)^*\right)) \\
&= 0.
\end{aligned}$$

Any monomial in $i(\mathcal{A})$ and $i(\mathcal{A})^*$ can be written as the product of elements of the form $i(a)i(b)^*$ since $\mathcal{A}$ is unital. Thus, the above computation shows that π intertwines β_s and $\tilde{\alpha}_s$ on monomials in $i(\mathcal{A})$ and $i(\mathcal{A})^*$ since $\pi\beta_s$ is a $*$-homomorphism. Hence, if π vanishes on a monomial m in $i(\mathcal{A})$ and $i(\mathcal{A})^*$, then $\pi\beta_s(m) = \tilde{\alpha}_s\pi(m) = 0$. Therefore, if p is a polynomial in $i(\mathcal{A})$ and $i(\mathcal{A})^*$ such that $p \in \ker \pi$, the linearity of $\pi\beta_s$ implies that $\pi\beta_s(p) = \tilde{\alpha}_s\pi(p) = 0$. Since the polynomials in $i(\mathcal{A})$ and $i(\mathcal{A})^*$ are dense in $\mathcal{D}$, all such polynomials contained in $\ker \pi$ are dense in the closed two-sided ideal $\ker \pi$. Hence, it follows that $\beta_s(\ker \pi) \subseteq \ker \pi$ by the continuity of β_s. Therefore, $\ker \pi$ is an α-invariant ideal in $\mathcal{D}$.

Conversely, if $\ker \pi$ is α-invariant, $\ker \pi$ is invariant under β_s for each $s \in G$ so each β_s factors through the quotient to make the below diagram commute.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\beta_s"] \arrow[d, "\pi"'] & \mathcal{D} \arrow[d, "\pi"] \\
\mathcal{C} \cong \mathcal{D}/\ker \pi \arrow[r, dashed, "\tilde{\alpha}_s"] & \mathcal{D}/\ker \pi \cong \mathcal{C}
\end{tikzcd}$$

Explicitly, let $s \in G$ be given and define $\tilde{\alpha}_s : \mathcal{C} \to \mathcal{C}$ by

$$\tilde{\alpha}_s(a + \ker \pi) := \beta_s(a) + \ker \pi = \pi\beta_s(a).$$

To see that $\tilde{\alpha}_s$ is well-defined, suppose $a+\ker \pi = b+\ker \pi$. Since $\ker \pi$ is invariant under β_s, we have $\beta_s(a-b) \in \ker \pi$, and it follows that $\tilde{\alpha}_s(a+\ker \pi) = \pi(\beta_s(a)) = \pi(\beta_s(b)) = \tilde{\alpha}_s(b+\ker \pi)$ since $\pi\beta_s$ is a $*$-homomorphism. Hence, $\tilde{\alpha}_s$ is well-defined.

It is clear that $\tilde{\alpha}_s$ is a $*$-epimorphism as π and β_s are both $*$-epimorphisms. Moreover, $a + \ker \pi \in \ker \tilde{\alpha}_s$ if and only if $\beta_s(a) \in \ker \pi$, which happens if and only if $a \in \ker \pi$ since $\beta_s^{-1}(\ker \pi) = \beta_{s^{-1}}(\ker \pi) \subseteq \ker \pi$ by assumption. Thus, $\ker \tilde{\alpha}_s$ is trivial, and it follows that $\tilde{\alpha}_s$ is injective. Hence, $\tilde{\alpha}_s$ is a $*$-automorphism of $\mathcal{C}$.

Define $\tilde{\alpha} : G \to \mathrm{Aut}(\mathcal{C})$ by $s \mapsto \tilde{\alpha}_s$. It is clear that $\tilde{\alpha}$ is a strongly continuous group representation since $\beta : G \to \mathrm{Aut}(\mathcal{D})$ is a strongly continuous group representation. Indeed, let $s, t \in G$ be given. For all $a + \ker \pi \in \mathcal{C}$, observe that

$$\begin{aligned}\tilde{\alpha}_{st}(a + \ker \pi) &= \pi(\beta_{st}(a)) \\ &= \pi(\beta_s(\beta_t(a))) \\ &= \tilde{\alpha}_s(\beta_t(a) + \ker \pi) \\ &= \tilde{\alpha}_s(\pi(\beta_t(a))) \\ &= \tilde{\alpha}_s\tilde{\alpha}_t(a + \ker \pi).\end{aligned}$$

Hence, $\tilde{\alpha}$ is a group representation. Suppose $\{s_\lambda\}$ is a net in G converging to s. As β is strongly continuous and π is contractive, letting $\lambda \to \infty$ we obtain

$$\|\tilde{\alpha}_{s_\lambda}(a + \ker \pi) - \tilde{\alpha}_s(a + \ker \pi)\| = \|\pi(\beta_{s_\lambda}(a) - \beta_s(a))\| \leq \|\beta_{s_\lambda}(a) - \beta_s(a)\| \to 0.$$

Thus, $\tilde{\alpha}$ is a strongly continuous group representation, which implies $(\mathcal{C}, G, \tilde{\alpha})$ is a C*-dynamical system.

It remains to show that $\tilde{\alpha}|_{j(\mathcal{A})}$ extends α from $\mathcal{A}$ to $j(\mathcal{A})$. Let $s \in G$ and $a \in \mathcal{A}$ be given. Consider

$$\begin{aligned}\tilde{\alpha}_s j(a) &= \tilde{\alpha}_s(\pi i(a)) && \text{since } (\mathcal{C}, j) \leq (\mathcal{D}, i) \\ &= \pi(\beta_s(i(a))) && \text{by definition of } \tilde{\alpha}_s \\ &= \pi(i\alpha_s(a)) && \text{since } (\mathcal{D}, i) \text{ is } \alpha\text{-admissible} \\ &= j\alpha_s(a).\end{aligned}$$

Thus, $\tilde{\alpha}_s \circ j = j \circ \alpha_s$ for all $s \in G$, and it follows that $(\mathcal{C}, j)$ is α-admissible. □

Example 2.2.4. Suppose $(A(\mathbb{D}), G, \alpha)$ is a dynamical system such that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible. In Example 2.2.2, we saw that the compact operators $\mathbb{K}$ on $H^2(\mathbb{T})$ is an α-invariant ideal for $A(\mathbb{D})$ in $\mathcal{T}$. Thus, Theorem 2.2.3 says $\mathcal{T}/\mathbb{K} \cong C(\mathbb{T})$ (with the appropriate complete isometry) is α-admissible, which comes as no surprise since $C(\mathbb{T})$ is the C*-envelope for $A(\mathbb{D})$.

Example 2.2.5. Let $(\mathcal{A}, G, \alpha)$ be a dynamical system and suppose $(\mathcal{C}, j)$ is an α-admissible C*-cover for $\mathcal{A}$. Then there exists a $*$-epimorphism $\pi : \mathcal{C} \to C_e^*(\mathcal{A})$ by the universal property of $C_e^*(\mathcal{A})$. Hence, Theorem 2.2.3 says $\ker \pi$ is an α-invariant ideal in $\mathcal{C}$ since $C_e^*(\mathcal{A})$ is always admissible.

Example 2.2.5 demonstrates that every α-admissible C*-cover $(\mathcal{C}, j)$ for an operator algebra $\mathcal{A}$ must contain a nontrivial $\mathcal{A}$-invariant ideal when $(\mathcal{C}, j) \neq C_e^*(\mathcal{A})$, namely the Shilov boundary ideal for $\mathcal{A}$ in $\mathcal{C}$. The $\mathcal{A}$-invariance property of the Shilov boundary ideal can be found in the literature (see Lemma 3.11 in [22], for example). However, its formulation in terms of the α-admissibility/α-invariance correspondence given in Theorem 2.2.3 is new, and it yields the characterization in Corollary 2.2.6, which is a special case of Theorem 2.2.3.

Corollary 2.2.6. *Let $(\mathcal{A}, G, \alpha)$ be a dynamical system and suppose $(\mathcal{C}, j)$ is a C*-cover for $\mathcal{A}$. If $\pi : C_{max}^*(\mathcal{A}) \to \mathcal{C}$ is the quotient map of $C_{max}^*(\mathcal{A})$ onto $\mathcal{C}$ such that $\pi \circ i_{max} = j$, then the following are equivalent:*

(i) $(\mathcal{C}, j)$ is α-admissible.

(ii) $\ker \pi$ is an α-invariant ideal in $C_{max}^*(\mathcal{A})$.

Moreover, $(\mathcal{C}, j)$ is always admissible for $\mathcal{A}$ if and only if $\ker \pi$ is $\mathcal{A}$-invariant in $C_{max}^*(\mathcal{A})$.

The following corollary is a restatement of Corollary 2.2.6 in terms of boundary ideals, explicitly. It has a useful perspective to keep in mind as we proceed.

Corollary 2.2.7. *Suppose J is a boundary ideal for $\mathcal{A}$ in $C_{max}^*(\mathcal{A})$ and let $\pi : C_{max}^*(\mathcal{A}) \to C_{max}^*(\mathcal{A})/J$ be the natural quotient map. The following are equivalent:*

(i) J is an α-invariant ideal.

(ii) $(C^*_{max}(\mathcal{A})/J, \pi \circ i_{max})$ *is an* α*-admissible C*-cover for* $\mathcal{A}$*.*

Proof. Suppose J is α-invariant. Since J is a boundary ideal for $\mathcal{A}$ in $C^*_{\max}(\mathcal{A})$, Proposition 1.3.3 says $(C^*_{\max}(\mathcal{A})/J, \pi \circ i_{\max})$ is a C*-cover for $\mathcal{A}$. Since $J = \ker \pi$ is an α-invariant ideal in $C^*_{\max}(\mathcal{A})$, Corollary 2.2.6 says $(C^*_{\max}(\mathcal{A})/J, \pi \circ i_{\max})$ is α-admissible.

The converse follows directly from Theorem 2.2.3. □

Example 2.2.8. We saw in Example 1.2.5 that the maximal C*-cover for the upper triangular 2×2 matrices T_2 is

$$\mathcal{C}_2 = \{F \in C([0,1], M_2) : F(0) \text{ is diagonal}\}$$

via the complete isometry $\begin{bmatrix} \lambda & x \\ 0 & \mu \end{bmatrix} \underset{j}{\mapsto} \begin{bmatrix} \lambda 1 & x\sqrt{\cdot} \\ 0 & \mu 1 \end{bmatrix}$. Let $\rho_1 : \mathcal{C}_2 \to M_2$ be evaluation at $1 \in [0,1]$, i.e. $\rho_1(F) = F(1)$ for all $F \in \mathcal{C}_2$. It is easy to see that ρ_1 is a $*$-representation of $\mathcal{C}_2$ and $\ker \rho_1$ is a boundary ideal for T_2 in $\mathcal{C}_2$. Thus, Proposition 1.3.3 says $(\mathcal{C}_2/\ker \rho_1, \rho_1 \circ i)$ is a C*-cover for T_2, which comes as no surprise since $\mathcal{C}_2/\ker \rho_1 \cong M_2 = C^*_e(T_2)$. Since M_2 is always admissible for T_2, we get $\ker \rho_1$ is a T_2-invariant boundary ideal for T_2 in $\mathcal{C}_2$ by Corollary 2.2.6.

Theorem 2.2.3 and its corollaries are interesting as they give us a strategy for producing examples of always admissible C*-covers, as we will see in Section 3. For now, we use Corollary 2.2.7 to study the collection of C*-covers for $\mathcal{A}$ that are always admissible for $\mathcal{A}$. We will see that the collection of all such C*-covers forms a complete sublattice of $\mathfrak{C}$.

Theorem 2.2.9. *Let* $(\mathcal{A}, G, \alpha)$ *be a dynamical system. Denote the collection of all* α*-admissible C*-covers for* $\mathcal{A}$ *by*

$$\mathfrak{C}^\alpha = \{(\mathcal{C}, j) \in \mathfrak{C} : (\mathcal{C}, j) \textit{ is } \alpha\textit{-admissible}\}.$$

Then $\mathfrak{C}^\alpha \equiv (\mathfrak{C}^\alpha, \leq)$ *forms a complete sublattice of* $\mathfrak{C}$*.*

Proof. E. Katsoulis and C. Ramsey showed in Lemma 3.4 of [22] that $C^*_{\max}(\mathcal{A})$ and $C^*_e(\mathcal{A})$ are always admissible via their universal properties. So $\mathcal{C}^\alpha$ is non-empty and contains a maximal and minimal element with respect to the partial order on $\mathcal{C}$. It remains to show that the meet and join of a collection of α-admissible C*-covers remains α-admissible.

Let $S = \{(\mathcal{C}_\lambda, j_\lambda) : \lambda \in I\} \subseteq \mathcal{C}^\alpha$ be given. For each $\lambda \in I$, there exists a strongly continuous group representation $\beta^{(\lambda)} : G \to \mathrm{Aut}(\mathcal{C}_\lambda)$ such that $\beta^{(\lambda)}_s \circ j_\lambda = j_\lambda \circ \alpha_s$ for all $s \in G$. Define $\beta^\vee : G \to \mathrm{Aut}(\bigoplus_\lambda \mathcal{C}_\lambda)$ by

$$\beta^\vee_s = \bigoplus_\lambda \beta^{(\lambda)}_s.$$

We claim that $\beta^\vee$ can be viewed as a representation of G into $\mathrm{Aut}(\bigvee_\lambda C_\lambda)$. To see this, first note that for all $s \in G$, we have

$$\beta^\vee_s \circ \bigvee_\lambda j_\lambda = \left(\bigoplus_\lambda \beta^{(\lambda)}_s\right) \circ \left(\bigoplus_\lambda j_\lambda\right) = \bigoplus_\lambda \left(\beta^{(\lambda)}_s \circ j_\lambda\right) = \bigoplus_\lambda \left(j_\lambda \circ \alpha_s\right) = \left(\bigvee_\lambda j_\lambda\right) \circ \alpha_s.$$

Hence, $\bigvee_\lambda j_\lambda$ intertwines $\beta^\vee$ and α, and it follows that $\beta^\vee_s\left(\bigvee_\lambda j_\lambda\right)(\mathcal{A}) \subseteq \left(\bigvee_\lambda j_\lambda\right)(\mathcal{A})$ for all $s \in G$. Note that for any operator subalgebra $\mathcal{B}$ of a unital C*-algebra $\mathcal{D} = C^*(\mathcal{B})$, if φ is a $*$-homomorphism of $\mathcal{D}$, then the image of φ is $\varphi(\mathcal{D}) = \varphi(C^*(\mathcal{B})) = C^*(\varphi(\mathcal{B}))$. Thus, as each $\beta^\vee_s$ is a $*$-homomorphism, for all $s \in G$ we have

$$\beta^\vee_s\left(\bigvee_\lambda C_\lambda\right) = \beta^\vee_s\left(C^*\left(\bigvee_\lambda j_\lambda(\mathcal{A})\right)\right) = C^*\left(\beta^\vee_s \circ \bigvee_\lambda j_\lambda(\mathcal{A})\right) = C^*\left(\bigvee_\lambda j_\lambda \circ \alpha_s(\mathcal{A})\right) = \bigvee_\lambda C_\lambda.$$

Hence, we have $\beta^\vee_s|_{\bigvee_\lambda \mathcal{C}_\lambda} \in \mathrm{Aut}\left(\bigvee_\lambda \mathcal{C}_\lambda\right)$ for all $s \in G$, which proves our claim. Therefore, $\beta^\vee : G \to \mathrm{Aut}(\bigvee_\lambda \mathcal{C}_\lambda)$ given by $s \mapsto \beta^\vee_s|_{\bigvee_\lambda \mathcal{C}_\lambda}$ is a well-defined group representation. It only remains to show that $\beta^\vee$ is strongly continuous.

Suppose (s_μ) is a net in G converging to some $s \in G$. Then for all $a \in \mathcal{A}$, observe that

$$\begin{aligned}\left\|\beta^{\vee}_{s_\mu}\left(\left(\bigvee_\lambda j_\lambda\right)(a)\right) - \beta^{\vee}_{s}\left(\left(\bigvee_\lambda j_\lambda\right)(a)\right)\right\| &= \left\|\left(\bigvee_\lambda j_\lambda\right)(\alpha_{s_\mu}(a)) - \left(\bigvee_\lambda j_\lambda\right)(\alpha_s(a))\right\| \\ &= \left\|\left(\bigvee_\lambda j_\lambda\right)(\alpha_{s_\mu}(a) - \alpha_s(a))\right\| \\ &\leq \left\|\bigvee_\lambda j_\lambda\right\| \cdot \left\|\alpha_{s_\mu}(a) - \alpha_s(a)\right\| \\ &= \left\|\alpha_{s_\mu}(a) - \alpha_s(a)\right\|.\end{aligned}$$

Hence, for all $a \in \mathcal{A}$, the strong continuity of α yields

$$\left\|\beta^{\vee}_{s_\mu}\left(\left(\bigvee_\lambda j_\lambda\right)(a)\right) - \beta^{\vee}_{s}\left(\left(\bigvee_\lambda j_\lambda\right)(a)\right)\right\| \leq \left\|\alpha_{s_\mu}(a) - \alpha_s(a)\right\| \to 0$$

as $s_\mu \to s$. Since $\bigvee_\lambda \mathcal{C}_\lambda = C^*\left(\left(\bigvee_\lambda j_\lambda\right)(\mathcal{A})\right)$ and β_t is a $*$-automorphism for all $t \in G$, it follows that, for all x in the $*$-algebra generated by $\left(\bigvee_\lambda j_\lambda\right)(\mathcal{A})$, we have $\left\|\beta^{\vee}_{s_\mu}(x) - \beta^{\vee}_{s}(x)\right\| \to 0$ as $s_\mu \to s$. From here, an easy $\varepsilon/3$ argument confirms that for all $x \in \bigvee_\lambda \mathcal{C}_\lambda$, we have $\left\|\beta^{\vee}_{s_\mu}(x) - \beta^{\vee}_{s}(x)\right\| \to 0$ as $s_\mu \to s$.

Indeed, let $x \in \bigvee_\lambda \mathcal{C}_\lambda$ be given. Since the $*$-algebra generated by $\left(\bigvee_\lambda j_\lambda\right)(\mathcal{A})$ is norm-dense in $\bigvee_\lambda \mathcal{C}_\lambda$, there exists a net (x_ν) in the $*$-algebra generated by $\left(\bigvee_\lambda j_\lambda\right)(\mathcal{A})$ such that $\|x_\nu - x\| \to 0$ as $\nu \to \infty$. Consider the estimate

$$\begin{aligned}\left\|\beta^{\vee}_{s_\mu}(x) - \beta^{\vee}_{s}(x)\right\| &\leq \left\|\beta^{\vee}_{s_\mu}(x) - \beta^{\vee}_{s_\mu}(x_\nu)\right\| + \left\|\beta^{\vee}_{s_\mu}(x_\nu) - \beta^{\vee}_{s}(x_\nu)\right\| + \left\|\beta^{\vee}_{s}(x_\nu) - \beta^{\vee}_{s}(x)\right\| \\ &\leq \left\|\beta^{\vee}_{s_\mu}(x_\nu) - \beta^{\vee}_{s}(x_\nu)\right\| + 2\left\|x_\nu - x\right\|.\end{aligned}$$

Fix $\varepsilon > 0$. Choose ν sufficiently large so that $\|x_\nu - x\| < \frac{\varepsilon}{3}$ and pick $\tilde{\mu}$ sufficiently large so that $\left\|\beta^\vee_{s_\mu}(x_\nu) - \beta^\vee_s(x_\nu)\right\| < \frac{\varepsilon}{3}$ for all $\mu \geq \tilde{\mu}$. Then we must have

$$\left\|\beta^\vee_{s_\mu}(x) - \beta^\vee_s(x)\right\| \leq \left\|\beta^\vee_{s_\mu}(x_\nu) - \beta^\vee_s(x_\nu)\right\| + 2\left\|x_\nu - x\right\| < \varepsilon$$

for all $\mu \geq \tilde{\mu}$. Thus, $(\bigvee_\lambda \mathcal{C}_\lambda, \bigvee_\lambda j_\lambda)$ is α-admissible and is a supremum for S.

Since $C^*_{\max}(\mathcal{A})$ is α-admissible, there exists a strongly continuous group representation $\tilde{\alpha} : G \to \operatorname{Aut}(C^*_{\max}(\mathcal{A}))$ such that $\tilde{\alpha}_s \circ i_{\max} = j \circ \alpha_s$ for all $s \in G$. Recall that $\bigwedge_\lambda \mathcal{C}_\lambda = C^*_{\max}(\mathcal{A})/J$, where J is the norm-closure of $\sum_\lambda \ker q_\lambda$ in $C^*_{\max}(\mathcal{A})$ and each q_λ is the quotient map of $C^*_{\max}(\mathcal{A})$ onto $\mathcal{C}_\lambda$. For each $\lambda \in I$, the α-admissibility of $(\mathcal{C}_\lambda, j_\lambda)$ implies that $\ker q_\lambda$ is $\tilde{\alpha}$-invariant in $C^*_{\max}(\mathcal{A})$ by Corollary 2.2.6. Thus, $\sum_\lambda \ker q_\lambda$ must be $\tilde{\alpha}$-invariant, and it follows that J is $\tilde{\alpha}$-invariant in $C^*_{\max}(\mathcal{A})$ since $\tilde{\alpha}_s$ is norm-continuous for all $s \in G$. Therefore, our characterization of α-admissibility in terms of α-invariant ideals in Corollary 2.2.7 yields $(\bigwedge_\lambda \mathcal{C}_\lambda, \bigwedge_\lambda j_\lambda)$ is α-admissible and must be an infimum for S. □

Remark 2.2.10. It should be noted that an argument similar to the meet case in Theorem 2.2.9 can be applied to the join case by viewing $\bigvee_\lambda C_\lambda$ as $C^*_{\max}(\mathcal{A})/J$, where $J = \bigcap_\lambda \ker q_\lambda$ in $C^*_{\max}(\mathcal{A})$ as in Proposition 2.1.14. The proof is shorter than the argument provided, however, the given proof is more constructive.

2.3 Inner Dynamical Systems

In this section, we restrict our view to dynamical systems where the action of the group is implemented by inner automorphisms of a C*-algebra. We define an *inner dynamical system* of an operator algebra relative to a C*-cover and show that admissibility of a C*-cover is always guaranteed when the action is implemented by inner automorphisms of the original operator algebra.

Definition 2.3.1. Let $(\mathcal{A}, G, \alpha)$ be a dynamical system and let $(\mathcal{C}, j)$ be an α-admissible C*-cover for $\mathcal{A}$. Then $(\mathcal{A}, G, \alpha)$ is *inner in* $(\mathcal{C}, j)$ if there exists a norm continuous unitary group representation $U : G \to \mathcal{U}(\mathcal{C})$ such that the extended action $\tilde{\alpha} : G \to \mathrm{Aut}(\mathcal{C})$ is implemented by U, i.e. $\tilde{\alpha}_s = \mathrm{ad}(U_s)$ for all $s \in G$.

In the C*-algebra context, it is standard in the literature to require the unitary group representation in Definition 2.3.1 to be *strictly continuous* in the *multiplier algebra* for $\mathcal{C}$. See Definitions 2.70 and 1.93 in [30]. Since our algebras are unital, the multiplier algebra for $\mathcal{C}$ is $\mathcal{C}$ itself and strict convergence is equivalent to norm convergence. Thus, we will only work in the context of norm continuous unitary group representations.

Example 2.3.2. Let T_2 be the upper 2×2 triangular matrices in M_2 and suppose $(T_2, \mathbb{Z}, \alpha)$ is a dynamical system. Since (M_2, incl) is α-admissible, each α_n extends to a $*$-automorphism of M_2. Hence, for all $n \in \mathbb{Z}$, there exist unitary matrices $U_n \in M_2$ such that $\alpha_n = \mathrm{ad}(U_n)$. Since each α_n is an automorphism of T_2, conjugation by each U_n must leave T_2 invariant. Thus, an easy computation shows that each U_n must be a diagonal unitary matrix. So $\mathbb{Z}$ acts on T_2 via conjugation by diagonal unitary matrices. Since $\mathbb{Z}$ is cyclic and discrete, it must follow that $U : \mathbb{Z} \to \mathcal{U}(M_2)$ is a norm continuous group representation as α is a group representation. Therefore, $(T_2, \mathbb{Z}, \alpha)$ is inner in (M_2, incl).

When a dynamical system is inner in an α-admissible C*-cover, an inner C*-dynamical arises in the classical sense. An *inner C*-dynamical system* is a C*-dynamical system $(\mathcal{C}, G, \alpha)$ such that there exists a norm continuous unitary group representation $U : G \to \mathcal{U}(\mathcal{C})$ satisfying $\alpha_s = \mathrm{ad}(U_s)$ for all $s \in G$. It is well-known that the group action of an inner C*-dynamical system looks trivial in the crossed product as stated in the following proposition. We provide a proof of this fact as we will use its machinery later. See Example 2.53 in [30] as a reference.

Proposition 2.3.3 (Example 2.53 in [30]). *Let $(\mathcal{C}, G, \alpha)$ be a C^*-dynamical system and let $(\mathcal{C}, G, \iota)$ be the trivial C^*-dynamical system, i.e. ι_s is the identity on $\mathcal{C}$ for all $s \in G$. If there exists a norm continuous unitary group representation $U : G \to \mathcal{U}(\mathcal{C})$ such that $\alpha_s = ad(U_s)$ for all $s \in G$, then $\mathcal{C} \rtimes_\alpha G \cong \mathcal{C} \rtimes_\iota G$.*

Proof. Define $\varphi : C_c(G, \mathcal{C})_\alpha \to C_c(G, \mathcal{C})_\iota$ by $\varphi(f)(s) = f(s)U_s$. We see that φ is well-defined since U is continuous. Moreover, φ is clearly linear. To see that φ is multiplicative, let $f, g \in C_c(G, \mathcal{C})_\alpha$ be given. Recall that the multiplication on $C_c(G, \mathcal{C})_\alpha$ is given by twisted convolution and multiplication on $C_c(G, \mathcal{C})_\iota$ is convolution without a twist. So for all $s \in G$, we have

$$\varphi(f) * \varphi(g)(s) = \int_G \varphi(f)(t)\iota_t(\varphi(g)(t^{-1}s))\, d\mu(t) = \int_G f(t)U_t g(t^{-1}s)U_{t^{-1}s}\, d\mu(t).$$

Since U is a group representation, we have $U_{t^{-1}s} = U_{t^{-1}s}U_s$ for all $s, t \in G$. Hence, for each $s \in G$ we have

$$\varphi(f)*\varphi(g)(s) = \int_G f(t)(U_t g(t^{-1}s)U_{t^{-1}})U_s\, d\mu(t) = \left(\int_G f(t)\alpha_t(g(t^{-1}s))\, d\mu(t)\right) U_s = \varphi(f*g)(s).$$

Hence, φ is an algebra homomorphism. To see that φ is a $*$-homomorphism, let $f \in C_c(G, \mathcal{C})_\alpha$ be given and for each $s \in G$ consider

$$\varphi(f)^*(s) = \Delta(s^{-1})\iota_s(\varphi(f)(s^{-1})^*) = \Delta(s^{-1})(f(s^{-1})U_{s^{-1}})^*.$$

Since U is a unitary group representation, we must have $U_{s^{-1}} = U_s^{-1} = U_s^*$ for all $s \in G$. Hence, for all $s \in G$ we have

$$\varphi(f)^*(s) = \Delta(s^{-1})U_s f(s^{-1})^* = \Delta(s^{-1})(U_s f(s^{-1})^* U_{s^{-1}})U_s = \Delta(s^{-1})\alpha_s(f(s^{-1})^*)U_s = \varphi(f^*)(s).$$

Therefore, φ is a $*$-homomorphism and extends continuously to a $*$-homomorphism $\varphi : \mathcal{C} \rtimes_\alpha G \to \mathcal{C} \rtimes_\iota G$. One can see that φ is a $*$-isomorphism by verifying that $\phi : C_c(G, \mathcal{C})_\iota \to C_c(G, \mathcal{C})_\alpha$ given by $\varphi(f)(s) = f(s)U_s^*$ is a $*$-homomorphism that extends to the inverse of φ. $\square$

The previous proposition allows us to untwist the group action if it is implemented by unitaries in the operator algebra upon which the group acts. We mention this since trivial crossed products are well-understood for C*-algebras.

Proposition 2.3.4 (Lemma 2.73 and Corollary 7.17 in [30])**.** *Let $\mathcal{C}$ be a C*-algebra. If $\iota : G \to Aut(\mathcal{C})$ is the trivial action, then $\mathcal{C} \rtimes_\iota G \cong \mathcal{C} \otimes_{max} C^*(G)$ and $\mathcal{C} \rtimes_\iota^r G \cong \mathcal{C} \otimes_{min} C_r^*(G)$.*

Even in non-self-adjoint dynamical systems, an action implemented by unitaries in the operator algebra looks trivial in the crossed product, as we will see in Theorem 2.3.6. We define what it means for a dynamical system to be inner with respect to the original operator algebra.

Definition 2.3.5. Let $(\mathcal{A}, G, \alpha)$ be a dynamical system. $(\mathcal{A}, G, \alpha)$ is *inner in itself* if there exists a norm continuous unitary group representation $U : G \to \mathcal{U}(\mathcal{A})$ such that $\alpha_s = \mathrm{ad}(U_s)$ for all $s \in G$.

Every inner C*-dynamical system is inner in itself. When a dynamical system $(\mathcal{A}, G, \alpha)$ is inner in itself, there are enough unitaries in the unital C*-subalgebra $\mathcal{A} \cap \mathcal{A}^* \subseteq \mathcal{A}$ to implement the action of G on $\mathcal{A}$, which is quite restrictive. We call $\mathcal{A} \cap \mathcal{A}^*$ the *diagonal of* $\mathcal{A}$. The diagonal $\mathcal{A} \cap \mathcal{A}^*$ is well-defined and independent of C*-cover for $\mathcal{A}$ since any completely isometric representation of $\mathcal{A}$ is $*$-isometric when restricted to $\mathcal{A} \cap \mathcal{A}^*$. See 2.1.2 in [8]. We show that $(\mathcal{A}, G, \alpha)$ being inner in itself is so restrictive that every C*-cover for $\mathcal{A}$ is α-admissible and every relative crossed product coincides with its trivial counterpart.

Theorem 2.3.6. *Suppose $(\mathcal{A}, G, \alpha)$ is inner in itself and let $(\mathcal{C}, j)$ be a C*-cover for $\mathcal{A}$. Then $(\mathcal{C}, j)$ is α-admissible and the following operator algebras are completely isometrically isomorphic*

$$\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} G \cong \mathcal{A} \rtimes_{\mathcal{C},j,\iota} G,$$

where ι is the trivial action.

Proof. Let $U : G \to \mathcal{U}(\mathcal{A})$ be the norm continuous unitary group representation such that $\alpha_s = \mathrm{ad}(U_s)$ for all $s \in G$. Note that the restriction $j|_{\mathcal{A}\cap\mathcal{A}^*} : \mathcal{A} \cap \mathcal{A}^* \to \mathcal{C}$ to the diagonal $\mathcal{A} \cap \mathcal{A}^*$ is a completely isometric representation between C*-algebras. Hence, $j|_{\mathcal{A}\cap\mathcal{A}^*}$ must be a $*$-isometric homomorphism on $\mathcal{A} \cap \mathcal{A}^*$. Since $U(G)$ is contained in $\mathcal{A} \cap \mathcal{A}^*$, it follows that $j(U_s^*) = j(U_s)^*$ for all $s \in G$. Therefore, each $j(U_s)$ is unitary in $\mathcal{C}$ since U_s is unitary.

Define $\tilde{U} : G \to \mathcal{U}(\mathcal{C})$ by $\tilde{U}_s = j(U_s)$. Then $\tilde{U}$ is a norm continuous unitary group representation into $\mathcal{C}$ since U is a norm continuous unitary group representation and j is an isometric homomorphism. Hence, $\beta : G \to \mathrm{Aut}(\mathcal{C})$ given by $\beta_s = \mathrm{ad}(\tilde{U}_s)$ is a strongly continuous group representation. Moreover, for all $s \in G$ and $a \in \mathcal{A}$, we have

$$\beta_s(j(a)) = \tilde{U}_s j(a) \tilde{U}_s^* = j(U_s) j(a) j(U_s)^* = j(U_s a U_s^*) = j(\alpha_s(a)).$$

Thus, $(\mathcal{C}, j)$ is α-admissible.

Notice that $(\mathcal{C}, G, \beta)$ is an inner C*-dynamical system via the unitary group representation $\tilde{U} : G \to \mathcal{U}(\mathcal{C})$. Hence, Proposition 2.3.3 says $\mathcal{C} \rtimes_\beta G \cong \mathcal{C} \rtimes_\iota G$ via the $*$-isomorphism $\varphi : \mathcal{C} \rtimes_\beta G \to \mathcal{C} \rtimes_\iota G$ given by

$$\varphi(f)(s) = f(s)\tilde{U}_s \quad \text{for all } f \in C_c(G, \mathcal{C})_\beta.$$

Thus, the restriction $\phi := \varphi|_{\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} G}$ is a completely isometric homomorphism. It remains to see that $\phi(\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} G) = \mathcal{A} \rtimes_{\mathcal{C},j,\iota} G \subseteq \mathcal{C} \rtimes_\iota G$.

Clearly, $\phi(\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} G)$ is contained in $\mathcal{A} \rtimes_{\mathcal{C},j,\iota} G$ since $\tilde{U}$ is continuous and each $\tilde{U}_s$ lies in $j(\mathcal{A})$. Let $f \in C_c(G, j(\mathcal{A}))_\iota$ be given. Since $\tilde{U}$ is a continuous representation, $g(s) = f(s)\tilde{U}_{s^{-1}} = f(s)j(U_{s^{-1}})$ is a compactly supported continuous function from G to $j(\mathcal{A})$, i.e. $g \in C_c(G, j(\mathcal{A}))_\alpha$. Hence, for each $s \in G$, we have $\phi(g)(s) = g(s)U_s = f(s)U_{s^{-1}}U_s = f(s)$. Therefore, ϕ maps a dense subset of $\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} G$ onto a dense subset of $\mathcal{A} \rtimes_{\mathcal{C},j,\iota} G$, and thus, ϕ maps onto $\mathcal{A} \rtimes_{\mathcal{C},j,\iota} G$ by continuity. $\square$

We give an example of a dynamical system that is not inner in itself but is inner in its C*-envelope. We show that the corresponding crossed product is non-trivial.

Example 2.3.7. Let A_4 be the four-cycle algebra in M_4. Theorem 2.3.6 yields $A_4 \rtimes_\iota \mathbb{Z}/2\mathbb{Z} \cong A_4 \oplus A_4$ as operator algebras. Consider the dynamical system $(A_4, \mathbb{Z}/2\mathbb{Z}, \mathrm{ad}(U \oplus U))$, where $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in M_2$. Then $(A_4, \mathbb{Z}/2\mathbb{Z}, \mathrm{ad}(U \oplus U))$ is not inner in itself but is inner in $C^*_e(A_4) = M_4$.

Set $v = E_{11} - E_{22} \in M_2$ and define $\varphi : M_4 \rtimes_{\mathrm{ad}(U \oplus U)} \mathbb{Z}/2\mathbb{Z} \to M_4 \oplus M_4$ by

$$\varphi(a\delta_0 + b\delta_1) = \begin{bmatrix} a + bU & 0 \\ 0 & a - bU \end{bmatrix} = a \otimes I_2 + bU \otimes v.$$

Since $U = U^*$, an easy computation shows that φ is a $*$-homomorphism. Moreover, φ is clearly injective and surjects onto $M_4 \oplus M_4$. Hence, φ is a $*$-isomorphism.

Since φ is an isometric $*$-homomorphism, its restriction $\varphi|_{A_4 \rtimes_{\mathrm{ad}(U \oplus U)} \mathbb{Z}/2\mathbb{Z}}$ is a completely isometric representation of the non-self-adjoint crossed product $A_4 \rtimes_{\mathrm{ad}(U \oplus U)} \mathbb{Z}/2\mathbb{Z}$. Hence, we can identify $A_4 \rtimes_{\mathrm{ad}(U \oplus U)} \mathbb{Z}/2\mathbb{Z}$ with its image $\mathcal{A} := \varphi(A_4 \rtimes_{\mathrm{ad}(U \oplus U)} \mathbb{Z}/2\mathbb{Z})$ in $M_4 \oplus M_4$. Recall

$v = E_{11} - E_{22}$. An easy computation yields

$$\mathcal{A} = \varphi(A_4 \rtimes_{\mathrm{ad}(U \oplus U)} \mathbb{Z}/2\mathbb{Z}) = \left\{ \begin{bmatrix} A & B & & \\ & C & & \\ & & vAv & D \\ & & & vCv \end{bmatrix} : A, B, C, D \in M_2 \right\}.$$

That is, every element of $\mathcal{A}$ has the form

$$\left[\begin{array}{c|c|c|c} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} & B & 0 & 0 \\ \hline 0 & \begin{matrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{matrix} & 0 & 0 \\ \hline 0 & 0 & \begin{matrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{matrix} & D \\ \hline 0 & 0 & 0 & \begin{matrix} c_{11} & -c_{12} \\ -c_{21} & c_{22} \end{matrix} \end{array}\right] \in M_8.$$

Hence, the diagonal of $\mathcal{A}$ is $\mathcal{A} \cap \mathcal{A}^* \cong M_2 \oplus M_2$ while the diagonal of $A_4 \oplus A_4$ is isomorphic to $\mathbb{C}^8$. Therefore, $A_4 \rtimes_{\mathrm{ad}(U \oplus U)} \mathbb{Z}/2\mathbb{Z}$ is not isomorphic to $A_4 \oplus A_4 \cong A_4 \rtimes_\iota \mathbb{Z}/2\mathbb{Z}$.

Even if the action $\alpha : G \curvearrowright \mathcal{A}$ is implemented by inner automorphisms in some C*-cover $(\mathcal{C}, j)$, the existence of a unitary group representation is not guaranteed. We will call dynamical systems whose action is implemented by inner automorphisms in a C*-cover *locally inner*.

Definition 2.3.8. Let $(\mathcal{A}, G, \alpha)$ be a dynamical system and suppose $(\mathcal{C}, j)$ is a C*-cover for $\mathcal{A}$.

(i) $(\mathcal{A}, G, \alpha)$ is *locally inner in* $(\mathcal{C}, j)$ if for each $s \in G$ there exists a unitary $U_s \in \mathcal{C}$ such that $\alpha_s = j^{-1} \circ \mathrm{ad}(U_s) \circ j$.

(ii) $(\mathcal{A}, G, \alpha)$ is *locally inner in itself* if for each $s \in G$ there exists a unitary $U_s \in \mathcal{A} \cap \mathcal{A}^*$ such that $\alpha_s = \mathrm{ad}(U_s)$.

In this context, being locally inner says that the map $s \mapsto U_s$ is a projective representation as the following example illustrates. See Appendix D.3 in [30] as a reference for projective representations.

Example 2.3.9. Consider the dynamical system $(M_2, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \alpha)$ where

$$\alpha_{(0,0)} = \mathrm{id}_{M_2}, \quad \alpha_{(1,0)} = \mathrm{ad} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{U_{(1,0)}}, \quad \alpha_{(0,1)} = \mathrm{ad} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{U_{(0,1)}}, \quad \text{and} \quad \alpha_{(1,1)} = \mathrm{ad} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{U_{(1,1)}}.$$

Each α_s is an inner automorphism of M_2 so $(M_2, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \alpha)$ is locally inner in itself, however $U_{(1,0)}U_{(0,1)} = -U_{(1,1)}$ guarantees that $(M_2, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \alpha)$ is not an inner dynamical system.

We give an example of a dynamical system for a non-self-adjoint operator algebra that is locally inner in itself.

Example 2.3.10. Suppose (T_2, G, α) is a dynamical system. We saw in Example 2.3.2 that each automorphism α_s of T_2 must be implemented by an inner automorphism of M_2, i.e. there exists a unitary matrix $U_s \in M_2$ such that $\alpha_s = \mathrm{ad}(U_s)$. An easy computation shows that each U_s must be a diagonal unitary matrix in order for T_2 to be invariant under its conjugation. Hence, each U_s is a unitary matrix in T_2. Therefore, (T_2, G, α) is inner in itself. Furthermore, since (T_2, G, α) was arbitrary, we have every dynamical system for T_2 is locally inner in itself.

Locally inner dynamical systems respect the lattice of C*-covers nicely. In particular, the existence of inner automorphisms in a C*-cover that implement the action is sufficient for that C*-cover (and any C*-cover below it) to be admissible.

Theorem 2.3.11. *Let $(\mathcal{A}, G, \alpha)$ be a dynamical system that is locally inner in some C*-cover $(\mathcal{C}, j)$ for $\mathcal{A}$. Then the following hold:*

(i) $(\mathcal{C}, j)$ *is α-admissible.*

(ii) If $(\mathcal{D}, i)$ is any C-cover such that $(\mathcal{D}, i) \leq (\mathcal{C}, j)$, then $(\mathcal{D}, i)$ is α-admissible. In particular, $(\mathcal{A}, G, \alpha)$ is locally inner in $(\mathcal{D}, i)$.*

Proof. Since $(\mathcal{A}, G, \alpha)$ is locally inner in $(\mathcal{C}, j)$, there exist unitaries $\{U_s\}_{s\in G} \subseteq \mathcal{C}$ such that $\alpha_s = j^{-1} \circ \mathrm{ad}(U_s) \circ j$ for all $s \in G$. Define $\beta : G \to \mathrm{Aut}(\mathcal{C})$ by $\beta_s = \mathrm{ad}(U_s)$. Then β is a strongly continuous group representation since it extends α and $\mathcal{C}$ is generated by $j(\mathcal{A})$. By definition, we have $\beta_s \circ j = j \circ \alpha_s$ for all $s \in G$. Thus, $(\mathcal{C}, j)$ is α-admissible.

Suppose $(\mathcal{D}, i)$ is a C*-cover for $\mathcal{A}$ such that $(\mathcal{D}, i) \leq (\mathcal{C}, j)$. Then there exists a $*$-epimorphism π of $\mathcal{C}$ onto $\mathcal{D}$ such that $\pi \circ j = i$. Hence, we have $\pi(U_s)$ is unitary in $\mathcal{D}$ for all $s \in G$. Thus, $\gamma : G \to \mathrm{Aut}(\mathcal{D})$ given by $\gamma_s = \mathrm{ad}(\pi(U_s))$ is a strongly continuous group representation. Let $s \in G$ and $a \in \mathcal{A}$ be given. Since $\pi \circ j = i$, we have

$$\gamma_s(i(a)) = \pi(U_s)i(a)\pi(U_s)^* = \pi(U_s)\pi(j(a))\pi(U_s^*) = \pi(U_s j(a) U_s^*).$$

As $\beta_s = \mathrm{ad}(U_s)$ and $\beta_s \circ j = j \circ \alpha_s$, it immediately follows that

$$\gamma_s(i(a)) = \pi(\beta_s(j(a))) = \pi(j(\alpha_s(a))) = i(\alpha_s(a)).$$

Therefore, $(\mathcal{D}, i)$ is α-admissible. $\square$

A quick corollary of Theorem 2.3.11 says admissibility of any C*-cover fails to be an obstacle when the original dynamical system is locally inner in itself.

Corollary 2.3.12. *Let $(\mathcal{A}, G, \alpha)$ be a dynamical system that is locally inner in itself. Then every C*-cover for $\mathcal{A}$ is α-admissible.*

Proof. Let $(\mathcal{C}, j)$ be a C*-cover for $\mathcal{A}$. Since $(\mathcal{A}, G, \alpha)$ is inner in itself, there exist unitaries $\{U_s\}_{s\in G} \subseteq \mathcal{A}$ such that $\alpha_s = \mathrm{ad}(U_s)$. As in Theorem 2.3.6, we note that j must be a $*$-homomorphism on the diagonal $\mathcal{A} \cap \mathcal{A}^*$ for $\mathcal{A}$. Hence, $\{j(U_s)\}_{s\in G}$ are unitaries in $\mathcal{C}$ such that for all $s \in G, a \in \mathcal{A}$ we have

$$j^{-1} \circ \mathrm{ad}(j(U_s)) \circ j(a) = j^{-1}(j(U_s)j(a)j(U_s)^*) = j^{-1}(j(U_s a U_s^*)) = U_s a U_s^* = \alpha_s(a).$$

Hence, $(\mathcal{A}, G, \alpha)$ is locally inner in $(\mathcal{C}, j)$, and it follows that $(\mathcal{C}, j)$ is α-admissible by Theorem 2.3.11. $\square$

We saw that every dynamical system for T_2 was locally inner in itself in Example 2.3.10. This gives us another immediate corollary to Theorem 2.3.11.

Corollary 2.3.13. *Let (T_2, G, α) be a dynamical system. If $(\mathcal{C}, j)$ is a C*-cover for T_2, then $(\mathcal{C}, j)$ is α-admissible.*

The universal properties for the C*-envelope and maximal C*-cover yield a final corollary for this section.

Corollary 2.3.14. *Let $(\mathcal{A}, G, \alpha)$ be a dynamical system. Then the following hold:*

(i) If $(\mathcal{A}, G, \alpha)$ is locally inner in $C^*_{max}(\mathcal{A})$, then $(\mathcal{A}, G, \alpha)$ is locally inner in every C-cover for $\mathcal{A}$. Hence, every C*-cover for $\mathcal{A}$ is α-admissible.*

(ii) If $(\mathcal{A}, G, \alpha)$ is not locally inner in $C^*_e(\mathcal{A})$, then $(\mathcal{A}, G, \alpha)$ is not locally inner in any C-cover for $\mathcal{A}$.*

2.4 A New Example of a Non-Admissible C*-Cover

E. Katsoulis and C. Ramsey give an example of a C*-cover for $A(\mathbb{D})$ in [21] that is non-admissible for an action of $\mathbb{Z}$. We construct a new example of a non-admissible C*-cover in Example 2.4.1, which is the first known finite dimensional example.

Example 2.4.1. Recall that the *four cycle algebra* A_4 is the subalgebra of M_4 given by

$$A_4 = \begin{bmatrix} * & & * & * \\ & * & * & * \\ & & * & \\ & & & * \end{bmatrix}.$$

Let $\{E_{rc}\}_{r,c=1}^4$ be the standard matrix units for M_4 and let I_4 be the identity matrix in M_4. Consider the dynamical system $(A_4, \mathbb{Z}/2\mathbb{Z}, \mathrm{ad}(U \oplus U))$, where $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in M_2$. Set $p = I_4 + E_{14} + E_{41} \in M_4$. Since p is a positive matrix, the "schur product by p" map $S_p : M_4 \to M_4$ is unital and completely positive by Theorem 3.7 in [27]. Thus, Proposition 3.6 in [27] says S_p is completely contractive since S_p is unital.

Let $a, b \in A_4$ be given. Consider

$$\begin{aligned}
S_p(ab) &= S_p\left(\begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}\begin{bmatrix} b_{11} & 0 & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix}\right) \\
&= S_p\left(\begin{bmatrix} a_{11}b_{11} & 0 & a_{11}b_{13} + a_{13}b_{33} & a_{11}b_{14} + a_{14}b_{44} \\ 0 & a_{22}b_{22} & a_{22}b_{23} + a_{23}b_{33} & a_{22}b_{24} + a_{24}b_{44} \\ 0 & 0 & a_{33}b_{33} & 0 \\ 0 & 0 & 0 & a_{44}b_{44} \end{bmatrix}\right) \\
&= \begin{bmatrix} a_{11}b_{11} & 0 & 0 & a_{11}b_{14} + a_{14}b_{44} \\ 0 & a_{22}b_{22} & 0 & 0 \\ 0 & 0 & a_{33}b_{33} & 0 \\ 0 & 0 & 0 & a_{44}b_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}\begin{bmatrix} b_{11} & 0 & 0 & b_{14} \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix} \\
&= S_p(a)S_p(b).
\end{aligned}$$

Hence, $S_p|_{A_4}$ is a ucc representation of A_4 into M_4.

Define $j : A_4 \to M_4 \oplus M_4$ by

$$j(a) = \begin{bmatrix} a & 0 \\ 0 & S_p(a) \end{bmatrix}.$$

Since j is the direct sum of the ucc homomorphism S_p and the ucis inclusion $A_4 \hookrightarrow M_4$, we have that j is a ucis homomorphism by Lemma 2.1.5. It follows that $(C^*(j(A_4)), j)$ is a C*-cover for A_4, where $C^*(j(A_4))$ is a C*-subalgebra of $M_4 \oplus M_4$. We claim $(C^*(j(A_4)), j)$ is *not* ad$(U \oplus U)$-admissible.

Note that $E_{13} \oplus 0 = j(E_{13}) \in C^*(j(A_4))$. Similarly, we have $E_{23} \oplus 0$ and $E_{24} \oplus 0$ are elements in $C^*(j(A_4))$. Taking adjoints and products, we see that $M_4 \oplus 0 \subseteq C^*(j(A_4)) \subseteq$

$M_4 \oplus M_4$. Hence, we must have $C^*(j(A_4)) = M_4 \oplus C^*(S_p(A_4))$, where $C^*(S_p(A_4)) = \mathcal{D}_4 + \operatorname{span}\{E_{14}, E_{41}\} \cong M_2 \oplus \mathbb{C}^2$.

Towards a contradiction, assume that $(C^*(j(A_4)), j)$ is $\operatorname{ad}(U \oplus U)$-admissible. Then there exists a $*$-automorphism $\beta : C^*(j(A_4)) \to C^*(j(A_4))$ such that $\beta \circ j = j \circ \operatorname{ad}(U \oplus U)$. Observe that we can write $E_{14} \oplus 0 \in C^*(j(A_4))$ as

$$E_{14} \oplus 0 = (E_{13} \oplus 0)(E_{32} \oplus 0)(E_{24} \oplus 0) = j(E_{13})j(E_{23})^* j(E_{24}).$$

Hence, we can write $0 \oplus E_{14}$ as

$$0 \oplus E_{14} = E_{14} \oplus E_{14} - E_{14} \oplus 0 = j(E_{14}) - j(E_{13})j(E_{23})^* j(E_{24}).$$

Since we assumed $\beta \circ j = j \circ \operatorname{ad}(U \oplus U)$, we must have

$$\begin{aligned}
\beta(0 \oplus E_{14}) &= \beta(j(E_{14}) - j(E_{13})j(E_{23})^* j(E_{24})) \\
&= \beta(j(E_{14})) - \beta(j(E_{13}))\beta(j(E_{23}))^* \beta(j(E_{24})) \\
&= j(\operatorname{ad}(U \oplus U)E_{14}) - j(\operatorname{ad}(U \oplus U)E_{13})j(\operatorname{ad}(U \oplus U)E_{23})^* j(\operatorname{ad}(U \oplus U)E_{24}) \\
&= j(E_{23}) - j(E_{24})j(E_{14})^* j(E_{13}) \\
&= (E_{23} \oplus 0) - (E_{24} \oplus 0)(E_{41} \oplus E_{41})(E_{13} \oplus 0) \\
&= E_{23} \oplus 0 - E_{23} \oplus 0 \\
&= 0.
\end{aligned}$$

But β is injective so we have a contradiction. Therefore, $(C^*(j(A_4)), j)$ is *not* $\operatorname{ad}(U \oplus U)$-admissible.

In Example 2.4.1, the algebra is finite-dimensional and being acted upon by the smallest nontrivial group. Moreover, the C*-envelope is simple so the action of the group on the given algebra is inner in its C*-envelope. Since admissibility fails for such an "uncomplicated" dynamical system, we really shouldn't expect admissibility of a C*-cover in general.

2.5 Decomposition of Complete Isometries and Recovering Dynamics Using Partial Actions

Admissibility failed for the C*-cover in Example 2.4.1 since the generating complete isometry decomposed into the direct sum of a complete isometry and a proper complete contraction. We will see in this section that this decomposition *always* happens when the Shilov boundary ideal is maximal and not essential. An ideal I in a C*-algebra $\mathcal{C}$ is *essential* when I intersects each nonzero ideal of $\mathcal{C}$ nontrivially. Equivalently, I is essential in $\mathcal{C}$ if the annihilator ideal for I in $\mathcal{C}$ given by $I^\perp = \{x \in \mathcal{C} : xI = 0\}$ is trivial, i.e. $I^\perp = \{0\}$.

Though the decomposition of the generating representation is not optimal for admissibility (as seen in Example 2.4.1), we will see that there is a natural partial action that allows us to recover dynamics, even in the case that our C*-cover is not admissible. We begin with a lemma.

Lemma 2.5.1. *Let $(\mathcal{C}, j)$ be a C*-cover for $\mathcal{A}$ such that the Shilov boundary ideal $\mathcal{J}$ for $\mathcal{A}$ in $\mathcal{C}$ is maximal and not essential. Then the annihilator ideal $\mathcal{J}^\perp$ for $\mathcal{J}$ is $*$-isomorphic to $C_e^*(\mathcal{A})$.*

Proof. Since $\mathcal{J}$ is not essential, $\mathcal{J}^\perp$ is a nontrivial closed 2-sided ideal in $\mathcal{C}$. As $\mathcal{J}^\perp \cap \mathcal{J} = 0$, we have $\mathcal{J}^\perp$ is $*$-isomorphic to a closed 2-sided ideal of $C_e^*(\mathcal{A}) \cong \mathcal{C}/\mathcal{J}$ since

$$\frac{\mathcal{J}^\perp + \mathcal{J}}{\mathcal{J}} \cong \frac{\mathcal{J}^\perp}{\mathcal{J}^\perp \cap \mathcal{J}} \cong \mathcal{J}^\perp.$$

But $C_e^*(\mathcal{A})$ is simple since $\mathcal{J}$ is maximal. Hence, we must have $\mathcal{J}^\perp \cong C_e^*(\mathcal{A})$ since $\mathcal{J}^\perp$ is nontrivial. $\square$

We give sufficient conditions for a C*-cover decomposing as the direct sum of an always admissible C*-subcover and an ideal that vanishes in the quotient.

Theorem 2.5.2. *Let $(\mathcal{C}, j)$ be a C*-cover for $\mathcal{A}$ such that the Shilov boundary ideal $\mathcal{J}$ for $\mathcal{A}$ in $\mathcal{C}$ is a maximal and not essential. Then*

(i) $\mathcal{C} \cong C_e^*(\mathcal{A}) \oplus \mathcal{J}$, *and*

(ii) $j = j_1 + j_2$, *where $(C_e^*(\mathcal{A}) \oplus 0, j_1)$ is a C*-cover for $\mathcal{A}$ and $j_2 : \mathcal{A} \to 0 \oplus \mathcal{J}$ is a proper completely contractive homomorphism.*

Proof. Since $\mathcal{J}$ is maximal and not essential, Lemma 2.5.1 says $\mathcal{J}^\perp \cong C_e^*(\mathcal{A})$. Thus, there exists a unital completely isometric homomorphism $\varphi : \mathcal{A} \to \mathcal{J}^\perp$ such that $(\mathcal{J}^\perp, \varphi)$ is a C*-cover for $\mathcal{A}$. By the GNS theorem, there exists a non-degenerate faithful $*$-representation π of $\mathcal{J}^\perp$ on some Hilbert space $\mathcal{H}$. By Theorem II.7.3.9 in [5], π extends uniquely to a $*$-homomorphism $\tilde{\pi}$ from $\mathcal{C}$ to $\mathcal{B}(\mathcal{H})$, which yields the following diagram.

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dr, dashed, "\tilde{\pi}"] & \\
\mathcal{A} \arrow[ur, "j"] \arrow[r, "\varphi"'] & \mathcal{J}^\perp \arrow[u] \arrow[r, "\pi"'] & \mathcal{B}(\mathcal{H})
\end{tikzcd}$$

Moreover, Theorem II.7.3.9 in [5] says $\ker \tilde{\pi} = \left(\mathcal{J}^\perp\right)^\perp$ since π is faithful. Hence, $\mathcal{J}$ is contained in $\ker \tilde{\pi} = \left(\mathcal{J}^\perp\right)^\perp$ since $\mathcal{J}$ annihilates $\mathcal{J}^\perp$. Thus, $\ker \tilde{\pi}/\mathcal{J}$ is a closed two-sided ideal in the quotient $\mathcal{C}/\mathcal{J} \cong C_e^*(\mathcal{A})$. As $\mathcal{J}$ is a maximal ideal in $\mathcal{C}$, we have $\mathcal{C}/\mathcal{J}$ is simple, and so it follows that $\ker \tilde{\pi}/\mathcal{J}$ is a trivial ideal, i.e. $\ker \tilde{\pi} = \mathcal{J}$ or $\ker \tilde{\pi} = \mathcal{C}$. Since $\tilde{\pi}|_{\mathcal{J}^\perp} = \pi$ is a faithful $*$-homomorphism, $\tilde{\pi}$ is completely isometric on $\varphi(\mathcal{A})$ in $\mathcal{J}^\perp$. Hence, we must have $\left(\mathcal{J}^\perp\right)^\perp = \ker \tilde{\pi} = \mathcal{J}$.

Notice that $\varphi(1_{\mathcal{A}})$ is a unit for $\mathcal{J}^\perp$ since φ is a homomorphism and $\mathcal{A}$ is unital. Hence, $\varphi(1_{\mathcal{A}})$ is a central idempotent in $\mathcal{C}$. Indeed, for all $x \in \mathcal{C}$, we have

$$\varphi(1_{\mathcal{A}})x = (\varphi(1_{\mathcal{A}})x)\varphi(1_{\mathcal{A}}) = \varphi(1_{\mathcal{A}})(x\varphi(1_{\mathcal{A}})) = x\varphi(1_{\mathcal{A}}).$$

Moreover, since $\varphi : \mathcal{A} \to \mathcal{J}^\perp$ is completely isometric, we have that $\|\varphi(1_{\mathcal{A}})\|_{\mathcal{C}} = \|1_{\mathcal{A}}\|_{\mathcal{A}} = 1$. Hence, $p := \varphi(1_{\mathcal{A}})$ is a central projection in $\mathcal{C}$. Similarly, $p^\perp := 1_{\mathcal{C}} - \varphi(1_{\mathcal{A}})$ is a central projection in $\mathcal{C}$, which is a unit for $\mathcal{J}$. Thus, we obtain the decomposition

$$\mathcal{C} = p\mathcal{C}p \oplus p^\perp \mathcal{C} p^\perp = \mathcal{J}^\perp \oplus \mathcal{J}.$$

By identifying $\mathcal{J}^\perp$ with $C_e^*(\mathcal{A})$, we obtain the decomposition in (i) of the proposition.

Our goal for (ii) is to decompose $j : \mathcal{A} \to C_e^*(\mathcal{A}) \oplus \mathcal{J}$ as the sum of a completely isometric homomorphism and proper completely contractive homomorphism. Define $L_p : \mathcal{C} \to C_e^*(\mathcal{A}) \oplus 0$ by

$$L_p(c) = pc.$$

Since p is a central projection in $\mathcal{C}$, L_p is a $*$-homomorphism of $\mathcal{C}$ onto $C_e^*(\mathcal{A})$. We claim that $L_p \circ j : \mathcal{A} \to C_e^*(\mathcal{A}) \oplus 0$ is completely isometric.

To see this, fix $n \in \mathbb{N}$ and note that $C_e^*(M_n(\mathcal{A})) \cong M_n(C_e^*(\mathcal{A}))$. Indeed, $\varphi : \mathcal{A} \to C_e^*(\mathcal{A})$ being completely isometric implies its amplification $\varphi_n : M_n(\mathcal{A}) \to M_n(C_e^*(\mathcal{A}))$ is completely isometric. Moreover, we have $C^*(\varphi_n(M_n(\mathcal{A}))) = C^*(M_n(\varphi(\mathcal{A}))) = M_n(C_e^*(\mathcal{A}))$ since $C^*(\varphi(\mathcal{A})) = C_e^*(\mathcal{A})$. Hence, it follows that $(M_n(C_e^*(\mathcal{A})), \varphi_n)$ is a C*-cover for $M_n(\mathcal{A})$. Thus, by the universal property of $C_e^*(M_n(\mathcal{A}))$, there exists a surjective $*$-homomorphism q_φ of $M_n(C_e^*(\mathcal{A}))$ onto $C_e^*(M_n(\mathcal{A}))$. Moreover, the correspondence of ideals between $C_e^*(\mathcal{A})$ and $M_n(C_e^*(\mathcal{A}))$ guarantees that q_φ is injective since $C_e^*(\mathcal{A})$ is simple. Thus, we obtain $C_e^*(M_n(\mathcal{A})) \cong M_n(C_e^*(\mathcal{A}))$.

Let $(a_{rc})_{r,c=1}^n \in M_n(\mathcal{A})$ be given. Observe that $(L_p)_n$ is contractive since it is a $*$-homomorphism. Thus, we have

$$\left\|(L_p \circ j)_n(a_{rc})_{r,c=1}^n\right\| = \left\|((L_p)_n \circ j_n)(a_{rc})_{r,c=1}^n\right\| \leq \left\|j_n(a_{rc})_{r,c=1}^n\right\| = \left\|(a_{rc})_{r,c=1}^n\right\|.$$

Notice that $M_n(\mathcal{J})$ is the Shilov boundary ideal for $M_n(\mathcal{A})$ in $M_n(\mathcal{C})$ since $M_n(\mathcal{C})/M_n(\mathcal{J}) \cong M_n(\mathcal{C}/\mathcal{J}) \cong C_e^*(M_n(\mathcal{A}))$. Hence, there exists a $*$-homomorphism $q_j : M_n(\mathcal{C}) \to C_e^*(M_n(\mathcal{A}))$ that is completely isometric on $j_n(M_n(\mathcal{A}))$ and $\ker q_j = M_n(\mathcal{J})$. Therefore, using that $p^\perp j(a) \in \mathcal{J}$ for all $a \in \mathcal{A}$ and $\ker q_j = M_n(\mathcal{J})$, we see that

$$\begin{aligned}\left\|j_n(a_{rc})_{r,c=1}^n\right\| &= \left\|q_j(j_n(a_{rc})_{r,c=1}^n)\right\| = \left\|q_j(pj(a_{rc}))_{r,c=1}^n + q_j(p^\perp j(a_{rc}))_{r,c=1}^n\right\| \\ &= \left\|q_j(pj(a_{rc}))_{r,c=1}^n\right\| \leq \left\|(pj(a_{rc}))_{r,c=1}^n\right\| = \left\|(L_p \circ j)_n(a_{rc})_{r,c=1}^n\right\|.\end{aligned}$$

Thus, $L_p \circ j : \mathcal{A} \to C_e^*(\mathcal{A}) \oplus 0$ is completely isometric. Set $j_1 := L_p \circ j$ and define $j_2 : \mathcal{A} \to 0 \oplus \mathcal{J}$ by $j_2(a) = p^\perp j(a)$ to obtain the decomposition in (ii). Since $\mathcal{J}$ is maximal (and thus, is nonzero), j_2 must be a proper complete contraction, or else $\mathcal{J}$ contains a completely isometric copy of $\mathcal{A}$, which is absurd. □

To extend dynamics in the context of Theorem 2.5.2, we introduce a partial action on the C*-algebra generated by the decomposed representation of $\mathcal{A}$. R. Exel's textbook [15] is a comprehensive resource for partial actions.

Theorem 2.5.3. *Let G be a discrete amenable group and let $(\mathcal{A}, G, \alpha)$ be a dynamical system. Suppose $(\mathcal{C}, j)$ is a C*-cover for $\mathcal{A}$ such that the Shilov boundary ideal $\mathcal{J}$ for $\mathcal{A}$ in $\mathcal{C}$ is maximal and not essential. Then there exists a partial action $\theta : G \curvearrowright \mathcal{C}$ and a norm closed subalgebra $\mathcal{B} \subseteq \mathcal{C} \rtimes_\theta G$ such that $\mathcal{B}$ is completely isometrically isomorphic to $\mathcal{A} \rtimes_\alpha G$.*

Proof. By Theorem 2.5.2, we have $\mathcal{C} = C_e^*(\mathcal{A}) \oplus \mathcal{J}$. Moreover, there exists a central projection

$p \in \mathcal{C}$ such that $j = j_1 + j_2$, where $j_1 : \mathcal{A} \to C_e^*(\mathcal{A}) \oplus 0$ is a completely isometric homomorphism given by $j_1(a) = pj(a)$ for all $a \in \mathcal{A}$.

For each $s \in G$, define

$$D_s = \begin{cases} \mathcal{C} & \text{if } s = e \\ C_e^*(\mathcal{A}) \oplus 0 & \text{if } s \neq e \end{cases}.$$

As $(C_e^*(\mathcal{A}) \oplus 0, j_1) = (C_e^*(\mathcal{A}), i_{\min})$ is an α-admissible C*-cover for $\mathcal{A}$, there exists an action $\beta : G \to \mathrm{Aut}(C_e^*(\mathcal{A}) \oplus 0)$ such that $\beta_s \circ j_1 = j_1 \circ \alpha_s$ for all $s \in G$. Hence, for each $s \in G$, we can define $\theta_s : D_{s^{-1}} \to D_s$ by

$$\theta_s = \begin{cases} \mathrm{id}_{\mathcal{C}} & \text{if } s = e \\ \beta_s & \text{if } s \neq e \end{cases}.$$

Thus, $\theta = \left\{ \{\theta_s\}_{s \in G}, \{D_s\}_{s \in G} \right\}$ is a C*-partial action of G on $\mathcal{C}$.

Our construction yields two crossed products of C*-algebras, namely the global C*-crossed product $(C_e^*(\mathcal{A}) \oplus 0) \rtimes_\beta G$ and the partial C*-crossed product $\mathcal{C} \rtimes_\theta G$. Since G is amenable, Theorem 3.14 in [22] says all relative crossed products of $\mathcal{A}$ by G coincide by a complete isometry that maps generators to generators. In particular, this implies that the full crossed product $\mathcal{A} \rtimes_\alpha G$ is completely isometrically isomorphic to the closure of $C_c(G, j(\mathcal{A}))$ in $(C_e^*(\mathcal{A}) \oplus 0) \rtimes_\beta G$. Moving forward, we will identify $\mathcal{A} \rtimes_\alpha G$ with this subalgebra of $(C_e^*(\mathcal{A}) \oplus 0) \rtimes_\beta G$.

Recall that there exists a canonical unitary representation $U : G \to (C_e^*(\mathcal{A}) \oplus 0) \rtimes_\beta G$ given by $U(s) = U_s$ and observe that the map $\pi : \mathcal{C} \to (C_e^*(\mathcal{A}) \oplus 0) \rtimes_\beta G$ given by $\pi(x) = (px)U_e$ is a $*$-homomorphism. To see that (π, U) is a covariant representation, fix $e \neq s \in G$ and let $j_1(a) \in D_{s^{-1}} = C_e^*(\mathcal{A}) \oplus 0$ be given. Since p is the unit for $C_e^*(\mathcal{A}) \oplus 0$ as in Theorem 2.5.2, we have $px = x$ for all $x \in C_e^*(\mathcal{A}) \oplus 0$, and it follows that

$$U_s \pi(j_1(a)) U_s^* = U_s((pj_1(a))U_e)U_s^* = \beta_s(pj_1(a))U_e = p\beta_s(j_1(a))U_e = \pi(\theta_s(j_1(a))).$$

Since $j_1(\mathcal{A})$ generates $D_{s^{-1}} = C_e^*(\mathcal{A}) \oplus 0$ as a C*-algebra, we get $U_s\pi(x)U_s^* = \pi(\theta_s(x))$ for all $x \in D_{s^{-1}}$. Also, for all $x \in D_e^{-1} = \mathcal{C}$, we have

$$U_e\pi(x)U_{e^{-1}} = pxU_e = p(\mathrm{id}_{\mathcal{C}}(x))U_e = \pi(\theta_e(x)).$$

Hence, (π, U) is a covariant representation of θ in $(C_e^*(\mathcal{A}) \oplus 0) \rtimes_\beta G$. Thus, there exists an integrated form $\pi \rtimes U : \mathcal{C} \rtimes_\theta G \to (C_e^*(\mathcal{A}) \oplus 0) \rtimes_\beta G$ given by $(\pi \rtimes U)(x\delta_s) = \pi(x)U_s$ for all $x \in \mathcal{C}, s \in G$.

Define $\varphi : j_1(\mathcal{A}) \to \mathcal{C} \rtimes_\theta G$ by $\varphi(j_1(a)) = j_1(a)\delta_e$ for all $a \in \mathcal{A}$ and define $V : G \to \mathcal{C} \rtimes_\theta G$ by $V_s = V(s) = 1_s\delta_s$, where $1_s = p$ when $s \neq e$ and $1_e = 1_{\mathcal{C}}$. Fix $e \neq s \in G$ and let $a \in \mathcal{A}$ be given. Recall that the α-admissibility of $(C_e^*(\mathcal{A}) \oplus 0, j_1)$ yields $\beta_s \circ j_1 = j_1 \circ \alpha_s$. Thus, for each $a \in \mathcal{A}$, we have

$$\begin{aligned} V_s\varphi(j_1(a))V_{s^{-1}} &= (p\delta_s)(j_1(a)\delta_e)(p\delta_{s^{-1}}) = (\beta_s(\beta_{s^{-1}}(p)j_1(a))\delta_s)(p\delta_{s^{-1}}) \\ &= \beta_s(\beta_{s^{-1}}(p)j_1(a)p)\delta_e = \beta_s(pj_1(a)p)\delta_e = \beta_s(j_1(a))\delta_e \\ &= j_1(\alpha_s(a))\delta_e = \varphi(j_1(\alpha_s(a))) = (\varphi \circ j_1)(\alpha_s(a)). \end{aligned}$$

Since $\varphi \circ j_1$ is completely contractive, $(\varphi \circ j_1, V)$ is a non-degenerate covariant representation of $(\mathcal{A}, G, \alpha)$ in $\mathcal{C} \rtimes_\theta G$. Thus, Proposition 3.7 in [22] yields a completely contractive integrated form $(\varphi \circ j_1) \rtimes V$ of the full crossed product $\mathcal{A} \rtimes_\alpha G$ to $\mathcal{C} \rtimes_\theta G$ given by

$$((\varphi \circ j_1) \rtimes V)(j_1(a)U_s) = \varphi(j_1(a))V_s = j_1(a)\delta_s.$$

Set $\mathcal{B} = \overline{\bigoplus_{s \in G} D_s \cap j_1(\mathcal{A})} \subseteq \mathcal{C} \rtimes_\theta G$ and note that $((\varphi \circ j_1) \rtimes V)(\mathcal{A} \rtimes_\alpha G) \subseteq \mathcal{B}$ and $(\pi \rtimes U)(\mathcal{B}) \subseteq \mathcal{A} \rtimes_\alpha G$. We claim that $(\varphi \circ j_1) \rtimes V$ and $\pi \rtimes U|_{\mathcal{B}}$ are completely contractive inverses.

Let $B = \sum_{s \in G} j_1(a_s)\delta_s \in \mathcal{B}$ be given. Observe that

$$\begin{aligned}
((\varphi \circ j_1) \rtimes V)\left((\pi \rtimes U)(B)\right) &= ((\varphi \circ j_1) \rtimes V)\left(\sum_{s \in G} \pi(j_1(a_s))U_s\right) \\
&= ((\varphi \circ j_1) \rtimes V)\left(\sum_{s \in G} p j_1(a_s)U_s\right) \\
&= ((\varphi \circ j_1) \rtimes V)\left(\sum_{s \in G} j_1(a_s)U_s\right) \\
&= \sum_{s \in G} \varphi(j_1(a_s))V_s \\
&= \sum_{s \in G} j_1(a_s)\delta_s \\
&= B.
\end{aligned}$$

Similarly, if $A = \sum_{s \in G} j_1(a_s)U_s \in \mathcal{A} \rtimes_\alpha G$, then we have

$$\begin{aligned}
(\pi \rtimes U)\left(((\varphi \circ j_1) \rtimes V)(A)\right) &= (\pi \rtimes U)\left(\sum_{s \in G} j_1(a_s)\delta_s\right) \\
&= \sum_{s \in G} \pi(j_1(a_s))U_s \\
&= \sum_{s \in G} p j_1(a_s)U_s \\
&= \sum_{s \in G} j_1(a_s)U_s \\
&= A.
\end{aligned}$$

Thus, the integrated forms $\varphi \rtimes V$ and $\pi \rtimes U|_{\mathcal{B}}$ are completely contractive inverses on dense subsets of $\mathcal{A} \rtimes_\alpha G$ and $\mathcal{B}$. Hence, by continuity, $\varphi \rtimes V$ and $\pi \rtimes U|_{\mathcal{B}}$ are completely contractive inverses, and it follows that $\mathcal{A} \rtimes_\alpha G \cong \mathcal{B}$. $\square$

Even when G is non-amenable, the dynamical system can still be recovered in the form of the reduced crossed product.

Corollary 2.5.4. *Let $(\mathcal{A}, G, \alpha)$ be a dynamical system. Suppose $(\mathcal{C}, j)$ is a C^*-cover for $\mathcal{A}$ such that the Shilov boundary ideal $\mathcal{J}$ for $\mathcal{A}$ in $\mathcal{C}$ is maximal and not essential. Then there exists a partial action $\theta : G \curvearrowright \mathcal{C}$ and a norm closed subalgebra $\mathcal{B} \subseteq \mathcal{C} \rtimes_\theta G$ such that $\mathcal{B}$ is completely isometrically isomorphic to $\mathcal{A} \rtimes_\alpha^r G$.*

Though the assumptions on the Shilov boundary ideal in Theorem 2.5.3 are strong, the theorem can still be applied to many important examples in the literature. Indeed, Theorem 2.5.3 applies to any generating operator subalgebra whose C*-envelope is simple, like generating subalgebras of M_n, the Cuntz algebras $\mathcal{O}_k$, UHF algebras, irrational rotation algebras, the reduced group C*-algebra of the free group on 2 generators $C^*(\mathbb{F}_2)$, etc. In particular, we can revisit Example 2.4.1.

Example 2.5.5. Let $(A_4, \mathbb{Z}/2\mathbb{Z}, \mathrm{ad}(U \oplus U))$ be the dynamical system from Example 2.4.1. We showed that $(M_4 \oplus (M_2 \oplus \mathbb{C}), j)$ was not an $\mathrm{ad}(U \oplus U)$-admissible C*-cover for the four cycle algebra A_4, where $j : A_4 \to M_4 \oplus (M_2 \oplus \mathbb{C})$ was given by the ucis representation $j(a) = a \oplus S_p(a)$ for all $a \in A_4$. Recall that $S_p : A_4 \to M_2 \oplus \mathbb{C}$ is Schur multiplication by some positive matrix $p \in M_4$, which is a completely contractive homomorphism.

The Shilov boundary ideal for A_4 in $C^*(j(A_4)) = M_4 \oplus (M_2 \oplus \mathbb{C}^2)$ is $\mathcal{J} = 0 \oplus (M_2 \oplus \mathbb{C}^2)$, which is maximal and not essential since $M_4 \oplus 0$ annihilates $\mathcal{J}$. Thus, Theorem 2.5.3 applies and says there exists a partial action $\theta : \mathbb{Z}/2\mathbb{Z} \curvearrowright M_4 \oplus (M_2 \oplus \mathbb{C})$ and a subalgebra $\mathcal{B}$ of the partial C*-crossed product $(M_4 \oplus (M_2 \oplus \mathbb{C})) \rtimes_\theta \mathbb{Z}/2\mathbb{Z}$ that is completely isometrically isomorphic to $\mathcal{A}_4 \rtimes_{\mathrm{ad}(U \oplus U)} \mathbb{Z}/2\mathbb{Z}$. Working through the construction of Theorem 2.5.3, we can see that

$$(M_4 \oplus (M_2 \oplus \mathbb{C})) \rtimes_\theta \mathbb{Z}/2\mathbb{Z} \cong (M_4 \rtimes_{\mathrm{ad}(U \oplus U)} \mathbb{Z}/2\mathbb{Z}) \oplus (M_2 \oplus \mathbb{C}) \cong (M_4 \oplus M_4) \oplus (M_2 \oplus \mathbb{C}).$$

Moreover, we have $\mathcal{B} = (A_4 \rtimes_{\mathrm{ad}(U\oplus U)} \mathbb{Z}/2\mathbb{Z}) \oplus 0$. Therefore, despite the C*-cover being non-admissible, we can extend the original dynamics to the C*-cover so long as we allow partial actions to come into play.

Example 2.4.1 and the example presented in Proposition 2.1 of [21] are the only known examples of non-admissible C*-covers in the literature. The partial action construction in Theorem 2.5.3 applies to both of these examples.

Example 2.5.6. Consider the Möbius transformation $\tau(w) = \dfrac{w - \frac{1}{2}}{1 - \frac{w}{2}}$, which is a conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$, $\mathbb{T}$ onto $\mathbb{T}$, and preserves orientation. Hence, $\alpha_1 : f \mapsto f \circ \tau$ defines a completely isometric automorphism of $A(\mathbb{D})$, which yields an action of $\mathbb{Z}$ on $A(\mathbb{D})$ in the usual way. Hence, $(A(\mathbb{D}), \mathbb{Z}, \alpha)$ is a dynamical system.

In Proposition 2.1 of [21], E. Katsoulis and C. Ramsey show that $(C(\mathbb{T}) \oplus M_2, i)$, where $i : z \mapsto z \oplus \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, is a C*-cover for $A(\mathbb{D})$ that is not α-admissible. However, observe that the Shilov boundary ideal for $A(\mathbb{D})$ in $C(\mathbb{T}) \oplus M_2$ is $\mathcal{J} = 0 \oplus M_2$. $\mathcal{J}$ is not a maximal ideal, but $\mathcal{J}$ is *not* essential since $C(\mathbb{T}) \oplus 0 \cong C^*_e(A(\mathbb{D}))$ annihilates $\mathcal{J}$. Using the partial action constructed in Theorem 2.5.3, we can compute the partial C*-crossed product

$$(C(\mathbb{T}) \oplus M_2) \rtimes_\theta \mathbb{Z} \cong (C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}) \oplus M_2$$

and see that $\mathcal{B} = (\mathcal{A}(\mathbb{D}) \rtimes_\alpha \mathbb{Z}) \oplus 0$, which is clearly completely isometrically isomorphic to $A(\mathbb{D}) \rtimes_\alpha \mathbb{Z}$.

Given that the Shilov boundary ideal is not essential in every known example of a non-admissible C*-cover, we might ask the following question: does the Shilov boundary ideal being essential in a C*-cover determine if that C*-cover is always admissible? We show that the Shilov boundary ideal being essential is *not* necessary for the C*-cover to be always admissible in Proposition 3.1.9.

Chapter 3

Examples of Non-Extremal Always Admissible C*-Covers

We refer to any C*-cover that is neither maximal nor minimal in its lattice as *non-extremal*. In this section, we present the first non-extremal examples of always admissible C*-covers for various operator algebras. In Section 3.1, we construct a chain of always admissible C*-covers for a family of finite dimensional operator algebras. The construction allows us to conclude that every C*-cover for the upper 2×2 triangular matrices T_2 is always admissible for T_2. In Section 3.2, we show that the Toeplitz algebra is admissible for any action of a cyclic group on $A(\mathbb{D})$.

3.1 A Chain of Always Admissible C*-covers for a Family of Finite Dimensional Operator Algebras

Let $n \in \mathbb{N}$ be given. Using the notation of Example 1.1.5, consider the non-self-adjoint operator subalgebra of M_{2n} given by

$$\mathcal{U}_B(M_n) = \left\{ \begin{bmatrix} \lambda I_n & a \\ 0 & \mu I_n \end{bmatrix} : \lambda, \mu \in \mathbb{C}, a \in M_n \right\}.$$

Define $\mathcal{C}_{2n}$ to be the C*-subalgebra of $C([0,1],M_{2n}) \cong M_{2n}(C([0,1]))$ given by

$$\mathcal{C}_{2n} = \{F \in C([0,1],M_{2n}) : F(0) \in D_2 \otimes I_n\},$$

where D_2 are the 2×2 diagonal matrices in M_2 and I_n is the $n \times n$ identity matrix in M_n. Define $j : \mathcal{U}_B(M_n) \to \mathcal{C}_{2n}$ by

$$j\left(\begin{bmatrix} \lambda I_n & a \\ 0 & \mu I_n \end{bmatrix}\right) = \begin{bmatrix} \lambda 1_n & a\sqrt{\cdot} \\ 0 & \mu 1_n \end{bmatrix},$$

where 1_n is the identity in $M_n(C([0,1]))$, i.e. 1_n is the $n\times n$ diagonal matrix with the constant function 1 down the diagonal.

We claim that $(\mathcal{C}_{2n}, j)$ is a C*-cover for $\mathcal{U}_B(M_n)$.

Proposition 3.1.1. *For each $n \in \mathbb{N}$, $(\mathcal{C}_{2n}, j)$, as defined above, is a C*-cover for $\mathcal{U}_B(M_n)$ and $C^*_e(\mathcal{U}_B(M_n))) = M_{2n}$.*

Proof. It is routine to check that j is an algebra homomorphism. Since M_n is completely isometric to $M_n\sqrt{\cdot}$ (as the square root function achieves its maximum at one on the unit interval), Corollary 2.2.12 in [8] says $\mathcal{U}_B(M_n)$ is completely isometric to $\mathcal{U}_B(M_n\sqrt{\cdot}) = j(\mathcal{U}_B(M_n))$. It follows that j is a completely isometric algebra homomorphism.

To see that (M_{2n}, incl) and $(\mathcal{C}_{2n}, j)$ are C*-covers for $\mathcal{U}_B(M_n)$, we need only show that M_{2n} is generated by $\mathcal{U}_B(M_n)$ and $\mathcal{C}_{2n}$ is generated by $j(\mathcal{U}_B(M_n))$. To see $M_{2n} = C^*(\mathcal{U}_B(M_n))$, we'll show that the standard matrix units for M_{2n} are contained in $C^*(\mathcal{U}_B(M_n))$.

Let $\{E_{rc}\}_{r,c=1}^{2n}$ be the standard matrix units for M_n. Then $E_{rc}, E_{cr} = E_{rc}^* \in C^*(\mathcal{U}_B(M_n))$ for $r \in \{1,\ldots,n\}, c \in \{n+1,\ldots,2n\}$. Observe that for $r,c \in \{1,2,\ldots,n\}$, we have $E_{rc} = E_{r,n+1}E_{c,n+1}^* \in C^*(\mathcal{U}_B(M_n))$. A similar argument shows that $E_{rc} \in C^*(\mathcal{U}_B(M_n))$ for $r,c \in \{n+1,\ldots,2n\}$. Since $C^*(\mathcal{U}_B(M_n))$ is a $*$-subalgebra of M_{2n} that contains all the matrix units for M_{2n}, we conclude that $C^*(\mathcal{U}_B(M_n)) = M_{2n}$. The simplicity of M_{2n} yields that $C_e^*(\mathcal{U}_B(M_n)) = M_{2n}$ by the universal property of the C*-envelope.

To see that $\mathcal{C}_{2n} = C^*(j(\mathcal{U}_B(M_n)))$, we make a Stone-Weierstrass argument (adapted from Example 2.4.5 in [8]) that shows the $*$-algebra generated by $j(\mathcal{U}_B(M_n))$ is dense in $\mathcal{C}_{2n}$.

Note that $C^*(j(\mathcal{U}_B(M_n))$ is contained in $\mathcal{C}_{2n}$ since $\mathcal{C}_{2n}$ is a C*-algebra containing $j(\mathcal{U}_B(M_n))$. Let $\mathcal{E}_0$ be the $*$-algebra generated by $j(\mathcal{U}_B(M_n))$ in $\mathcal{C}_{2n}$, and let $\{E_{rc}\}_{r,c=1}^{2n}$ be the standard matrix units for $M_{2n}(C([0,1]))$. Then $\sqrt{\cdot}E_{rc}, \sqrt{\cdot}E_{cr} = (\sqrt{\cdot}E_{rc})^* \in \mathcal{E}_0$ for $r \in \{1, 2, \ldots, n\}, c \in \{n+1, \ldots, 2n\}$. As $\mathcal{E}_0$ is a $*$-algebra, products of these elements and their adjoints will yield that $zE_{rc} \in \mathcal{E}_0$ for all $r, c \in \{1, \ldots, 2n\}$, where z is the identity function on $[0,1]$. Hence, for all $k \in \mathbb{N}$, $r, c \in \{1, \ldots, 2n\}$, we get that $z^k E_{rc} \in \mathcal{E}_0$. Taking linear combinations, we can conclude that $pE_{rc} \in \mathcal{E}_0$ for all $i, j \in \{1, \ldots, 2n\}$, where p is any polynomial of degree at least 1.

We claim that $\mathcal{C}_{2n}$ is contained in the closure of $\mathcal{E}_0$ with respect to the sup norm on $C([0,1], M_{2n})$. To see this, we'll show that the closure of $\mathcal{E}_0$ contains multiples of each matrix unit by continuous functions that have roots at zero, then argue that norm-limits of the diagonal entires in $\mathcal{E}_0$ must be in $D_2 \otimes I_n$ at zero.

Fix $r, c \in \{1, \ldots, 2n\}$, and let $f \in C([0,1])$ such that $f(0) = 0$ be given. By Stone-Weierstrass (or rather, the Weierstrass Approximation theorem), there exists a sequence of polynomials $\{p_k\}_{k=1}^\infty$ converging to f uniformly on $[0,1]$. We can assume that $p_k(0) = 0$ for all $k \in \mathbb{N}$ by redefining the sequence to be $\{p_k(x) - p_k(0)\}_{k=1}^\infty$ if necessary. Thus, we have $\{p_k E_{rc}\}_{k=1}^\infty$ is a sequence in $\mathcal{E}_0$ converging to fE_{rc} in norm. Indeed, consider the estimates

$$\begin{aligned} \|p_k E_{rc} - fE_{rc}\| = \|(p_k - f)E_{rc}\| &= \sup_{x \in [0,1]} \|(p_k(x) - f(x))E_{rc}\| \\ &= \sup_{x \in [0,1]} |p_k(x) - f(x)| \, \|E_{rc}\| = \|p_k - f\|_\infty . \end{aligned}$$

Hence, we have $\|p_k E_{rc} - fE_{rc}\| \to 0$ as $k \to \infty$, and it follows that $fE_{rc} \in \overline{\mathcal{E}_0}^{\|\cdot\|} = C^*(j(\mathcal{U}_B(M_n)))$.

Let $F \in \mathcal{C}_{2n}$ be given. Since $F(0) \in D_2 \otimes I_n$, there exist $\lambda, \mu \in \mathbb{C}$ such that $F(0) = \lambda I_n \oplus \mu I_n$, which implies $(F - (\lambda 1_n \oplus \mu 1_n))(0) = 0$. Hence, the previous argument says each entry of $F - (\lambda 1_n \oplus \mu 1_n)$ is contained in $C^*(j(\mathcal{U}_B(M_n)))$. Since $j(\lambda I_n \oplus \mu I_n) = \lambda 1_n \oplus \mu 1_n \in C^*(j(\mathcal{U}_B(M_n)))$, we conclude that $F = F - (\lambda 1_n \oplus \mu 1_n) + (\lambda 1_n \oplus \mu 1_n) \in C^*(j(\mathcal{U}_B(M_n)))$ since F is the sum of $(2n)^2 + 1$ elements of $C^*(j(\mathcal{U}_B(M_n)))$. Therefore, we have $\mathcal{C}_{2n} = C^*(j(\mathcal{U}_B(M_n)))$. $\square$

Given a dynamical system $(\mathcal{U}_B(M_n), G, \alpha)$, we are interested in determining the α-admissibility of $(\mathcal{C}_{2n}, j)$. So we need to understand the action $\alpha : G \curvearrowright \mathcal{U}_B(M_n)$ more deeply. The following proposition characterizes all completely isometric automorphisms of $\mathcal{U}_B(M_n)$.

Proposition 3.1.2. *For all $n \in \mathbb{N}$, σ is a completely isometric automorphism of $\mathcal{U}_B(M_n)$ if and only if σ is implemented by a direct sum of unitary matrices in M_n.*

Proof. Suppose σ is a completely isometric automorphism of $\mathcal{U}_B(M_n)$. Then σ extends to a $*$-automorphism of $C^*_e(\mathcal{U}_B(M_n)) = M_{2n}$ that leaves $\mathcal{U}_B(M_n)$ invariant by the C*-envelope's universal property. The $*$-automorphisms of full matrix algebras are well-studied. In particular, Skolem-Noether theorem says σ must be an inner automorphism of M_{2n}. Hence, there exists a unitary matrix $U \in M_{2n}$ such that $\sigma = \mathrm{ad}(U)$ and conjugation by U leaves $\mathcal{U}_B(M_n)$ invariant in M_{2n}.

Write U as a 2×2 block matrix, i.e. write

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \text{where } a, b, c, d \in M_n.$$

Since $\mathrm{ad}(U)$ leaves $\mathcal{U}_B(M_n)$ invariant, we claim that this implies $b = c = 0$, i.e. U is block diagonal. Let $X = \begin{bmatrix} \lambda I_n & x \\ 0 & \mu I_n \end{bmatrix} \in \mathcal{U}_B(M_n)$ be given, where $x \in M_n$ and $\lambda, \mu \in \mathbb{C}$. Computing the products UXU^* and $U^*U = UU^* = I_{2n}$ yields the necessary information to verify our claim. We outline the appropriate computations below.

Since $\mathrm{ad}(U)$ leaves $\mathcal{U}_B(M_n)$ invariant, the $(2,1)$-entry of UXU^* must be 0. Since X was arbitrary, for all $x \in M_n$ and $\lambda, \mu \in \mathbb{C}$, we have

$$\lambda ca^* + cxb^* + \mu db^* = 0.$$

Thus, $ca^* = 0$ by choosing $x = 0, \mu = 0$, and $\lambda = 1$. Similarly, we have $db^* = 0$ by choosing $x = 0, \lambda = 0$, and $\mu = 1$, and $cb^* = 0$ by choosing $x = I_n$ and $\lambda = \mu = 0$.

The invariance of $\mathcal{U}_B(M_n)$ under $\mathrm{ad}(U)$ also yields that the $(1,1)$-entry of UXU^* must be a constant multiple of I_n. That is, there exists some constant $\eta(\lambda, \mu, x) \in \mathbb{C}$, which is dependent on λ, μ, and x, such that

$$\lambda aa^* + axb^* + \mu bb^* = \eta(\lambda, \mu, x) I_n.$$

Since the $(1,1)$-entry of UU^* yields $aa^* + bb^* = I_n$, we obtain that $ab^* + I_n = \eta(1,1,I_n)I_n$ when $\lambda = \mu = 1$ and $x = I_n$. Thus, ab^* is a constant multiple of I_n. In particular, we have $ab^* = \eta I_n$, where $\eta = \eta(1,1,I_n) - 1$. By right multiplying the $(1,1)$-entry of U^*U, namely $a^*a + c^*c = I_n$, by b^*, we get b^* is a constant multiply of a^*. That is, we have

$$b^* = a^*ab^* + c^*cb^* = a^*(\eta I_n) + c^*(0) = \eta a^*.$$

Reconsidering the $(1,1)$-entry of UU^*, we obtain $I_n = aa^* + bb^* = (|\eta|^2 + 1)aa^*$. Hence, a^* is invertible with $(a^*)^{-1} = (|\eta|^2 + 1)a$. Thus, we must have $c = 0$ since $ca^* = 0$. Moreover, observing that $db^* = 0$ and $b^* = \eta a^*$, we see that $\eta d = d(\eta a^*)(a^*)^{-1} = db^*(a^*)^{-1} = 0$, which implies $\eta = 0$ or $d = 0$. However, if d was the zero matrix, then the $(2,2)$-entry of UU^* yields the contradiction $I_n = cc^* + dd^* = 0$. Therefore, we must have $\eta = 0$, and thus, $b = \overline{\eta} a^* = 0$, which proves our claim. In particular, we obtain that $U = a \oplus d \in M_{2n}$, where a and d are $n \times n$ unitary matrices since the $(1,1)$ and $(2,2)$-entries of UU^* are both I_n.

Conversely, let a and d be unitary matrices in M_n. Then $U := a \oplus d$ is a unitary matrix in M_{2n}, and thus, $\sigma := \mathrm{ad}(U)$ is an inner $*$-automorphism of M_{2n}. Hence, its restriction to $\mathcal{U}_B(M_n)$ is completely isometric. Moreover, σ leaves $\mathcal{U}_B(M_n)$ invariant since for each $\lambda, \mu \in \mathbb{C}$ and $x \in M_n$,

$$\sigma\left(\begin{bmatrix} \lambda I_n & x \\ 0 & \mu I_n \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} \lambda I_n & x \\ 0 & \mu I_n \end{bmatrix} \begin{bmatrix} a^* & 0 \\ 0 & d^* \end{bmatrix} = \begin{bmatrix} \lambda I_n & axd^* \\ 0 & \mu I_n \end{bmatrix} \in \mathcal{U}_B(M_n).$$

Therefore, σ is a completely isometric automorphism of $\mathcal{U}_B(M_n)$. □

Proposition 3.1.3. *For each $n \in \mathbb{N}$, if $(\mathcal{U}_B(M_n), G, \alpha)$ is a dynamical system, then $(\mathcal{C}_{2n}, j)$ is α-admissible. Thus, $(\mathcal{C}_{2n}, j)$ is always admissible for $\mathcal{U}_B(M_n)$.*

Proof. Let $\gamma : M_{2n} \to C([0,1], M_{2n})$ be the inclusion of M_{2n} into $C([0,1], M_{2n})$ as the constant matrices, which is a $*$-homomorphism. Suppose $(\mathcal{U}_B(M_n), G, \alpha)$ is a dynamical system. Since $(\mathcal{C}_{2n}, j)$ is a C*-cover for $\mathcal{U}_B(M_n)$ and $C_e^*(\mathcal{U}_B(M_n)) = M_{2n}$, there exists a $*$-epimorphism π of $\mathcal{C}_{2n}$ onto M_{2n} such that $\pi \circ j = i_{\min}$.

$$\begin{tikzcd}
 & & C([0,1], M_{2n}) & & \\
0 \arrow[r] & \ker \pi \arrow[r, "incl"] & \mathcal{C}_{2n} \arrow[u] \arrow[r, "\pi"] & M_{2n} \arrow[r] \arrow[ul, "\gamma"'] & 0 \\
 & & & \mathcal{U}_B(M_n) \arrow[u, "i_{\min}"'] \arrow[ul, "j"] &
\end{tikzcd}$$

By Proposition 3.1.2, for all $s \in G$ there exist unitaries $a_s, d_s \in M_n$ such that $\alpha_s = \mathrm{ad}(U_s)$, where $U_s = a_s \oplus d_s$. Since γ is a $*$-homomorphism, $\gamma(U_s)$ is unitary constant matrix in $C([0,1], M_{2n})$ for all $s \in G$. For each $s \in G$, set $\beta_s := \mathrm{ad}(\gamma(U_s)) \in \mathrm{Aut}(C([0,1], M_{2n})$. We claim that $\mathcal{C}_{2n}$ is invariant for each β_s, which implies $\beta_s \in \mathrm{Aut}(\mathcal{C}_{2n})$ for all $s \in G$.

Let $s \in G$ and $F \in \mathcal{C}_{2n}$ be given. We need only show that $\beta_s(F)(0) \in D_2 \otimes I_n$. Since $\gamma(U_s)$ is a constant matrix, $\gamma(U_s)(x) = U_s$ for all $x \in [0,1]$. Hence, we have

$$\beta_s(F)(0) = \left(\gamma(U_s)F\gamma(U_s)^*\right)(0) = \gamma(U_s)(0)F(0)\gamma(U_s)^*(0) = U_sF(0)U_s^*.$$

Since $F(0)$ is diagonal and U_s is the direct sum of unitary matrices in M_n, U_s must commute with $F(0)$, which implies $\beta_s(F)(0) = U_sF(0)U_s^* = F(0) \in D_2 \otimes I_n$. Thus, we have $\tilde{\alpha}_s := \beta_s|_{\mathcal{C}_{2n}} \in \mathrm{Aut}(\mathcal{C}_{2n})$.

It remains to show that $(\mathcal{C}_{2n}, G, \tilde{\alpha})$ is a C*-dynamical system and that j intertwines α and $\tilde{\alpha}$. We begin by showing the intertwining property. Let $s \in G$ and $X \in \mathcal{U}_B(M_n)$ be given. Then there exist $\lambda, \mu \in \mathbb{C}$ and $x \in M_n$ such that $X = \begin{bmatrix} \lambda I_n & x \\ 0 & \mu I_n \end{bmatrix}$.

Consider

$$j(\alpha_s(X)) = j(\mathrm{ad}(a_s \oplus d_s)(X)) = j\left(\begin{bmatrix} \lambda I_n & a_s x d_s^* \\ 0 & \mu I_n \end{bmatrix}\right) = \begin{bmatrix} \lambda I_n & a_s x d_s^* \sqrt{\cdot} \\ 0 & \mu I_n \end{bmatrix}.$$

As $\gamma(U_s) = \gamma(a_s) \oplus \gamma(d_s)$ is a constant matrix, we also have

$$\begin{aligned}\tilde{\alpha}_s(j(X)) &= \mathrm{ad}(\gamma(a_s) \oplus \gamma(d_s)) \left(\begin{bmatrix} \lambda I_n & x\sqrt{\cdot} \\ 0 & \mu I_n \end{bmatrix} \right) \\ &= \begin{bmatrix} \lambda I_n & \gamma(a_s)(\cdot) x \gamma(d_s)^*(\cdot)\sqrt{\cdot} \\ 0 & \mu I_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda I_n & a_s x d_s^* \sqrt{\cdot} \\ 0 & \mu I_n \end{bmatrix}.\end{aligned}$$

Thus, we have $\tilde{\alpha}_s \circ j = j \circ \alpha_s$ for all $s \in G$. Hence, $(\mathcal{C}_{2n}, j)$ is α-admissible so long as we can verify that $\tilde{\alpha} : G \to \mathrm{Aut}(\mathcal{C}_{2n})$ is a strongly continuous group representation.

Let $s, t \in G$ be given. Since α is a group representation by assumption and j intertwines $\tilde{\alpha}$ and α, we obtain that for all $X \in \mathcal{U}_B(M_n)$

$$\tilde{\alpha}_{st}(j(X)) = j(\alpha_{st}(X)) = j(\alpha_s \alpha_t(X)) = \tilde{\alpha}_s(j(\alpha_t(X))) = (\tilde{\alpha}_s \tilde{\alpha}_t)(j(X)).$$

Since $\tilde{\alpha}_s t$ and $\tilde{\alpha}_s \tilde{\alpha}_t$ are $*$-automorphisms of $\mathcal{C}_{2n} = C^*(j(\mathcal{U}_B(M_n))$ that agree on generators, we conclude that $\tilde{\alpha}_s t = \tilde{\alpha}_s \tilde{\alpha}_t$. Hence, $\tilde{\alpha} : G \to \mathrm{Aut}(\mathcal{C}_{2n})$ is a group representation.

Similarly, the intertwining property yields $\tilde{\alpha}$ inherits its strong continuity from α. Indeed, fix $X \in \mathcal{U}_B(M_n)$. If a net $\{s_\lambda\}_{\lambda \in \Lambda}$ in G converges to an element s in G, as $\lambda \to \infty$ we must have

$$\|\tilde{\alpha}_{s_\lambda}(j(X)) - \tilde{\alpha}_s(j(X))\| = \|j(\alpha_{s_\lambda}(X) - \alpha_s(X))\| = \|\alpha_{s_\lambda}(X) - \alpha_s(X)\| \to 0.$$

Since $\tilde{\alpha}$ is a $*$-automorphism, the strong continuity extends to the $*$-algebra generated by $j(\mathcal{U}_B(M_n))$ and then to all of $\mathcal{C}_{2n}$ since the aforementioned $*$-algebra is dense in $\mathcal{C}_{2n}$. $\square$

Since $\mathcal{C}_{2n}$ is an always admissible C*-cover of $\mathcal{U}_B(M_n)$, the ideal structure of $\mathcal{C}_{2n}$ will determine the admissibility of each cover below $(\mathcal{C}_{2n}, j)$. Recall that the ideal structure of matrix rings and the closed ideal structure of $C([0,1])$ implies that there is a bijective correspondence between closed two-sided ideals of $C([0,1], M_{2n}) \cong M_{2n}(C([0,1]))$ and closed subsets of the unit interval. But $\mathcal{C}_{2n}$ is not a hereditary C*-subalgebra of $M_{2n}(C([0,1]))$ so we should not expect the ideal structure of $\mathcal{C}_{2n}$ to be inherited completely from $M_{2n}(C([0,1]))$. To gain insight, we compute the irreducible representations of $\mathcal{C}_{2n}$ and the Shilov boundary ideal for $\mathcal{U}_B(M_n)$ in $\mathcal{C}_{2n}$. We use $\{\omega_1, \omega_2\} \cup (0,1]$ to denote the quotient space $([0,1] \times \{\omega_1\} \sqcup [0,1] \times \{\omega_2\})/\sim$ where $(x, \omega_1) \sim (x, \omega_2)$ if and only if $x \neq 0$. This space is called the *unit interval with two origins.*

Lemma 3.1.4. *The irreducible representations of $\mathcal{C}_{2n}$ (up to unitary equivalence) correspond to point evaluation on the unit interval with two origins $\{\omega_1, \omega_2\} \cup (0,1]$.*

Proof. It is easy to see that the irreducible representations for $M_{2n}(C[0,1])$ are point evaluations at each $x \in [0,1]$. For each $x \in (0,1]$, let $\rho_x : \mathcal{C}_{2n} \to M_{2n}$ be point evaluation at x, i.e. $\rho_x(F) = F(x)$. We must have each ρ_x is irreducible since $\rho_x(\mathcal{C}_{2n}) = M_{2n}$ for all $x \in (0,1]$. Evaluation at 0 is *not* irreducible, however, as the image of evaluation at 0 is $\mathbb{C}I_n \oplus \mathbb{C}I_n$ in M_{2n}, which has nontrivial invariant subspaces. Thus, for $k = 1, 2$, define

$$\rho_{\omega_k}(F) = \begin{cases} F(0)(I_n \oplus 0), & \text{for } k = 1 \\ F(0)(0 \oplus I_n), & \text{for } k = 2 \end{cases}.$$

Since both $\rho_{\omega_1}(\mathcal{C}_{2n})$ and $\rho_{\omega_2}(\mathcal{C}_{2n})$ are isomorphic to the complex numbers $\mathbb{C}$, we have ρ_{ω_1} and ρ_{ω_2} are irreducible. We claim these are the only irreducible representations of $\mathcal{C}_{2n}$ up to unitary equivalence.

Let $\pi : \mathcal{C}_{2n} \to \mathcal{B}(\mathcal{H})$ be an irreducible representation of $\mathcal{C}_{2n}$ on some Hilbert space $\mathcal{H}$.

Then Theorem 5.5.1 in [25] says π is unitarily equivalent to the restriction of an irreducible representation for $M_{2n}(C([0,1]))$. This means there exists an irreducible representation φ of $M_{2n}(C([0,1]))$ on a Hilbert space $\mathcal{K}$ and a closed subspace $\mathcal{H}'$ of $\mathcal{K}$ invariant for $\varphi(\mathcal{C}_{2n})$ such that π is unitarily equivalent to the $*$-representation $\varphi' : \mathcal{C}_{2n} \to \mathcal{B}(\mathcal{H}')$ given by $\varphi'(F) = \varphi(F)|_{\mathcal{H}'}$. Since φ is an irreducible representation of $M_{2n}(C([0,1]))$, it is unitarily equivalent to evaluation at some $x \in [0,1]$. If $x \neq 0$, then π is unitarily equivalent to evaluation at x. If $x = 0$, then $\varphi'(\mathcal{C}_{2n})$ is unitarily equivalent to $\mathcal{C}_{2n}(0) = \mathbb{C}I_n \oplus \mathbb{C}I_n \cong \mathbb{C}^2$ restricted to an invariant subspace, i.e. φ' is unitarily equivalent to ρ_{ω_k} for either $k = 1$ or 2. Therefore, if $x = 0$, π is unitarily equivalent to either ρ_{ω_1} or ρ_{ω_2}. □

Lemma 3.1.5. *The Shilov boundary ideal for $j(\mathcal{U}_B(M_n))$ is $\ker \rho_1$, where $\rho_1 : \mathcal{C}_{2n} \to M_{2n}$ is evaluation at $1 \in [0,1]$.*

Proof. Observe that $\mathcal{C}_{2n}/\ker \rho_1 \cong M_{2n} = C_e^*(\mathcal{U}_B(M_n))$. The quotient map q of $\mathcal{C}_{2n}$ onto $\mathcal{C}_{2n}/\ker \rho_1 \cong M_{2n}$ is clearly completely isometric on $j(\mathcal{U}_B(M_n))$ since $q \circ j$ is the inclusion of $\mathcal{U}_B(M_n)$ into M_{2n}. Hence, $\ker \rho_1$ is a boundary ideal, and thus, it is contained in the Shilov boundary ideal for $j(\mathcal{U}_B(M_n))$ in $\mathcal{C}_{2n}$. However, $\ker \rho_1$ is a maximal ideal so it must be the Shilov boundary ideal. □

Alternatively, one can see $\ker \rho_1$ is the Shilov boundary ideal for $\mathcal{U}_B(M_n)$ in $\mathcal{C}_{2n}$ by computing the boundary representation(s) of $\mathcal{C}_{2n}$, which turns out to be only ρ_1. See [19] for a strategy using the matrix units of $\mathcal{C}_{2n}$.

We proceed by building a chain of always admissible C*-covers for $\mathcal{U}_B(M_n)$.

Theorem 3.1.6. *Let $n \in \mathbb{N}$ be given and suppose $(\mathcal{U}_B(M_n), G, \alpha)$ is a dynamical system. If $K_1 \supseteq K_2 \supseteq \ldots$ is a nested sequence of closed subsets of $[0,1]$ such that $1 \in \bigcap_{i\in\mathbb{N}} K_i$, then $(\mathcal{C}_{2n}, j) \geq (\mathcal{C}_{2n}|_{K_1}, |_{K_1} \circ j) \geq (\mathcal{C}_{2n}|_{K_2}, |_{K_2} \circ j) \geq \ldots \geq M_{2n}$ is a chain of α-admissible C*-covers for $\mathcal{U}_B(M_n)$.*

Proof. Fix $k \in \mathbb{N}$. Then $J = \{F \in \mathcal{C}_{2n} : F|_{K_k} = 0\}$ is a closed two-sided ideal in $\mathcal{C}_{2n}$ contained in the Shilov boundary ideal $\ker \rho_1 = \{F \in \mathcal{C}_{2n} : F(1) = 0\}$. Hence, Proposition 1.3.2 says J is a boundary ideal for $\mathcal{U}_B(M_n)$ in $\mathcal{C}_{2n}$. Therefore, Proposition 1.3.3 says $(\mathcal{C}_{2n}/J, q \circ j)$ is a C*-cover for $\mathcal{U}_B(M_n)$, where $q : \mathcal{C}_{2n} \to \mathcal{C}_{2n}/J$ is the quotient map. Proposition 3.1.3 showed $(\mathcal{C}_{2n}, j)$ is α-admissible via a strongly continuous group representation $\tilde{\alpha} : G \to \mathrm{Aut}(\mathcal{C}_{2n})$ given by $s \mapsto \tilde{\alpha}_s = \mathrm{ad}(U_s)$ for some constant unitary matrix $U_s \in M_{2n}(C([0,1]))$.

Let $s \in G$ and $R \in J$ be given. Observe $\tilde{\alpha}_s(R) = U_s R U_s^*$ is a matrix in $\mathcal{C}_{2n}$ whose entries are linear combinations of continuous functions that vanish on K_k as U_s has constant entries. This implies $\tilde{\alpha}_s(R) \in J$ since linear combinations of continuous functions that vanish on K_k must be continuous and vanish on K_k. Thus, $\tilde{\alpha}_s(J) \subseteq J$ for all $s \in J$, i.e. J is α-invariant in $\mathcal{C}_{2n}$. Therefore, Theorem 2.2.3 says $(\mathcal{C}_{2n}/J, q \circ j)$ is α-admissible.

But J is the kernel of the $*$-homomorphism $\pi_k : \mathcal{C}_{2n} \to M_{2n}(C(K_k))$ given by $\pi_i(F) = F|_{K_k}$ and $q(\mathcal{C}_{2n}) \cong \mathcal{C}_{2n}|_{K_k}$. So $\mathcal{C}_{2n}/J$ and $\mathcal{C}_{2n}|_{K_k}$ are $*$-isomorphic, and it follows that $(\mathcal{C}_{2n}|_{K_i}, |_{K_i} \circ j) = (\mathcal{C}_{2n}/J, q \circ j)$ must be an α-admissible C*-cover for $\mathcal{U}_B(M_n)$.

Suppose $\ell < m$. Then $K_\ell \supseteq K_m$ and so there exists a natural $*$-epimorphism $\pi_{\ell m}$ of $\mathcal{C}_{2n}|_{K_\ell}$ onto $\mathcal{C}_{2n}|_{K_m}$ given by $\pi_{\ell m}(F) = F|_{K_m}$, which yields the following commutative diagram.

$$\begin{tikzcd} & \mathcal{C}_{2n}|_{K_\ell} \arrow[d, dashed, "\pi_{\ell m}"] \\ \mathcal{U}_B(M_n) \arrow[ur, "|_{K_\ell} \circ j"] \arrow[r, "|_{K_m} \circ j"'] & \mathcal{C}_{2n}|_{K_m} \end{tikzcd}$$

Hence, we have $(\mathcal{C}_{2n}|_{K_\ell}, |_{K_\ell} \circ j) \geq (\mathcal{C}_{2n}|_{K_m}, |_{K_m} \circ j)$. $\square$

Example 3.1.7. Fix $n \in \mathbb{N}$ and suppose $(\mathcal{U}_B(M_n), G, \alpha)$ is dynamical system. Let $K \subseteq [0,1]$ be the middle-thirds construction of the Cantor set. For each $i \in \mathbb{N}$, set $K_i := K \cap [2^{-i}, 1]$. Then $(\mathcal{C}_{2n}, j) \geq (\mathcal{C}_{2n}|_{K_1}, |_{K_1} \circ j) \geq (\mathcal{C}_{2n}|_{K_2}, |_{K_2} \circ j) \geq \ldots \geq M_{2n}$ is a lattice of α-admissible C*-covers for $\mathcal{U}_B(M_n)$

It remains to see that every C*-cover below $(\mathcal{C}_{2n}, j)$ is always admissible for $\mathcal{U}_B(M_n)$. We appeal to Theorem 3.1.6 given that we understand the irreducible representations of $\mathcal{C}_{2n}$.

Theorem 3.1.8. *Suppose $(\mathcal{U}_B(M_n), G, \alpha)$ is a dynamical system. If $(\mathcal{D}, i)$ is any C*-cover for $\mathcal{U}_B(M_n)$ such that $(\mathcal{D}, i) \leq (\mathcal{C}_{2n}, j)$, then $(\mathcal{D}, i)$ is α-admissible.*

Proof. Each irreducible representation ρ_x of $\mathcal{C}_{2n}$ gives rise to an α-invariant ideal. Indeed, Theorem 3.1.6 says that for each $x \in (0,1]$, the boundary ideal $\ker \rho_x \cap \ker \rho_1$ is an α-invariant ideal. It is easy to see that $\ker \rho_{\omega_k} \cap \ker \rho_1$ (for $k = 1, 2$) must also be α-invariant since G must act trivially on diagonal matrices via $\tilde{\alpha} : G \to \mathrm{Aut}(\mathcal{C}_{2n})$.

Since $(\mathcal{D}, i) \leq (\mathcal{C}_{2n}, j)$, there exists a $*$-epimorphism $q : \mathcal{C}_{2n} \to \mathcal{D}$, which implies $\mathcal{D}$ is separable as $\mathcal{C}_{2n}$ is separable. Thus, the GNS Theorem says that $\mathcal{D}$ can be represented $*$-isometrically on a separable Hilbert space $\mathcal{H}$ via a $*$-representation $\varphi : \mathcal{D} \to \mathcal{B}(\mathcal{H})$.

Define $\Phi : \mathcal{C}_{2n} \to \mathcal{B}(\mathcal{H})$ by $\Phi = \varphi \circ q$. As $\mathcal{H}$ and $\mathcal{D}$ are separable, a generalization of the Weyl-von Neumann Theorem (see Corollary II.5.9 in [10]) says that Φ is approximately unitarily equivalent to a direct sum of irreducible representations of $\mathcal{C}_{2n}$. Hence, there exists a sequence of unitaries $\{V_k\}$ on $\mathcal{H}$ such that for all $F \in \mathcal{C}_{2n}$, we have

$$\Phi(F) = \lim_{k \to \infty} V_k \pi(F) V_k^*,$$

where π is a direct sum of irreducible representations of $\mathcal{C}_{2n}$. So $F \in \ker \Phi$ if and only if $\|V_k \pi(F) V_k^*\| \to 0$, which happens if and only if $\pi(F) = 0$. Hence, $\ker \Phi = \ker \pi$ and so $\ker \Phi$ is the intersection of the kernels of irreducible representations for $\mathcal{C}_{2n}$. Since $\ker \Phi = \ker \varphi \circ q = \ker q$ is also a boundary ideal, we have $\ker \Phi \cap \ker \rho_1 = \ker \Phi$ as $\ker \rho_1$ is the Shilov boundary ideal for $\mathcal{U}_B(M_n)$ in $\mathcal{C}_{2n}$. Thus, $\ker \Phi = \ker q$ must be α-invariant since $\ker \rho \cap \ker \rho_1$ is α-invariant for every irreducible representation ρ for $\mathcal{C}_{2n}$. Hence, we get $(\mathcal{D}, i)$ is α-admissible by Theorem 2.2.3. $\square$

Theorem 3.1.8 says every C*-cover for $\mathcal{U}_B(M_n)$ that is below $(\mathcal{C}_{2n}, j)$ must be always admissible for $\mathcal{U}_B(M_n)$. As $\mathcal{U}_B(M_1) = T_2$ and $\mathcal{C}_2 = C^*_{\max}(T_2)$ (see Example 1.2.5), we immediately obtain another proof to Corollary 2.3.13, which says every C*-cover for T_2 is always admissible. Since this is true, we can show that the Shilov boundary ideal being essential is not a necessary condition for a C*-cover to be always admissible.

Proposition 3.1.9. *There exist C*-covers for T_2 where the Shilov boundary ideal is not essential.*

Proof. Define $j : T_2 \to M_2 \oplus M_2$ by

$$j\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \oplus \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}.$$

It is easy to see that j is a completely isometric representation of T_2 and $C^*(j(T_2)) \cong M_2 \oplus \mathbb{C}^2$ by a matrix unit argument. Hence, $(M_2 \oplus \mathbb{C}^2, j)$ is a C*-cover for T_2. It follows that $(M_2 \oplus \mathbb{C}^2, j)$ is always admissible for T_2 by Corollary 2.3.13, yet the Shilov boundary ideal $\mathcal{J} = 0 \oplus \mathbb{C}^2$ in $M_2 \oplus \mathbb{C}^2$ is not essential since it has trivial intersection with the ideal $M_2 \oplus 0$. $\square$

We can generalize Theorem 3.1.8 further by considering any generating subalgebra of a full matrix algebra.

Corollary 3.1.10. *Suppose that $\mathcal{A}$ is an operator algebra such that $C^*_e(\mathcal{A}) = M_n$ for some $n \in \mathbb{N}$. If $(\mathcal{A}, G, \alpha)$ is a dynamical system, then*

(i) α extends to an action of G on $\mathcal{U}_B(\mathcal{A})$ such that $(\mathcal{U}_B(\mathcal{A}), G, \alpha)$ is a dynamical system,

(ii) $(\mathcal{C}_{2n}, j)$ is a C-cover for $\mathcal{U}_B(\mathcal{A})$, and*

(iii) *if* $K_1 \supseteq K_2 \supseteq \ldots$ *is a nested sequence of closed subsets of* $[0,1]$ *such that* $1 \in \bigcap_{i\in\mathbb{N}} K_i$, *then* $(\mathcal{C}_{2n}, j) \geq (\mathcal{C}_{2n}|_{K_1}, |_{K_1} \circ j) \geq (\mathcal{C}_{2n}|_{K_2}, |_{K_2} \circ j) \geq \ldots \geq M_{2n}$ *is a chain of* α*-admissible* C^**-covers for* $\mathcal{U}_B(\mathcal{A})$.

(iv) *if* $(\mathcal{D}, i)$ *is a* C^**-cover for* $\mathcal{U}_B(\mathcal{A})$ *such that* $(\mathcal{D}, i) \leq (\mathcal{C}_{2n}, j)$, *then* $(\mathcal{D}, i)$ *is* α*-admissible.*

Proof. Since $C^*_e(\mathcal{A}) = M_n$, there is a completely isometric representation $i_{\min} : \mathcal{A} \to M_n$ such that $C^*(i_{\min}(\mathcal{A})) = M_n$. Moreover, $\mathcal{U}_B(\mathcal{A})$ is completely isometrically isomorphic to $\mathcal{U}_B(i_{\min}(\mathcal{A}))$ by Corollary 2.2.12 in [8] since $\mathcal{A}$ and $i_{\min}(\mathcal{A})$ are isomorphic. Thus, we can identify $\mathcal{A}$ with $i_{\min}(\mathcal{A})$ in M_n and $\mathcal{U}_B(\mathcal{A})$ with its isomorphic copy $\mathcal{U}_B(i_{\min}(\mathcal{A}))$ in $\mathcal{U}_B(M_n)$. Since $C^*_e(\mathcal{A}) = M_n$ must be α-admissible for $\mathcal{A}$, for each $s \in G$ there exist unitaries $U_s \in M_n$ such that $\alpha_s = \mathrm{ad}(U_s)$. Hence, the strongly continuous group representation $\beta : G \to \mathrm{Aut}(\mathcal{U}_B(\mathcal{A}))$ given by $\beta_s = \mathrm{ad}(U_s \oplus U_s)$ gives rise to a dynamical system $(\mathcal{U}_B(\mathcal{A}), G, \beta)$. Note that β extends the action of G on $\mathcal{A}$ in the sense that for all $s \in G$ we have $\beta_s(\mathcal{U}_B(\mathcal{A})) = \mathcal{U}_B(\alpha_s(\mathcal{A}))$.

Since $(\mathcal{C}_{2n}, j)$ is a C*-cover for $\mathcal{U}_B(M_n)$, $j|_{\mathcal{U}_B(\mathcal{A})}$ is a completely isometric homomorphism. Since $\mathcal{A}$ is unital, it is easy to see that each of the four $n \times n$ blocks in $\mathcal{C}_{2n}$ contain a copy of $\mathcal{A}$ and $\mathcal{A}^*$. Therefore, we must have $C^*(j(\mathcal{U}_B(\mathcal{A}))) = C^*(j(\mathcal{U}_B(M_n))) = \mathcal{C}_{2n}$ since $\mathcal{A}$ generates M_n. Thus, $\mathcal{C}_{2n}$ is a C*-cover for $\mathcal{U}_B(\mathcal{A})$. Observe that we can extend the action β from $\mathcal{U}_B(\mathcal{A})$ to $\mathcal{U}_B(M_n)$ then to $\mathcal{C}_{2n}$ as conjugation by constant unitary matrices. The remainder of the result follows from Theorems 3.1.6 and 3.1.8. $\square$

3.2 The Toeplitz Algebra $\mathcal{T}$ for the Disc Algebra $A(\mathbb{D})$

We begin by describing the automorphisms of the disc algebra $A(\mathbb{D})$, which are well studied. If α is a completely isometric automorphism of $A(\mathbb{D})$, there exists a conformal map τ such that $\tau(\mathbb{D}) = \mathbb{D}$ and $\alpha(f)(w) = f(\tau(w))$ for all $f \in A(\mathbb{D}), w \in \mathbb{D}$ (see page 143 in [18]). Hence, if z is the identity function, we have $\alpha(z)(w) = z(\tau(w)) = \tau(w)$ for all $w \in \mathbb{D}$. This implies τ itself is conformal and analytic on $\mathbb{D}$.

We can say more. As a consequence of Schwarz's lemma (see Theorem 12.6 in [29]), there exists $\lambda \in \mathbb{T}, \mu \in \mathbb{D}$ such that

$$\tau(w) = \lambda \left(\frac{w - \mu}{1 - \overline{\mu} w} \right) \qquad \text{for all } w \in \mathbb{D}.$$

Hence, τ is a Möbius transformation that maps $\mathbb{T}$ onto $\mathbb{T}$, $\mathbb{D}$ onto $\mathbb{D}$, and preserves orientations. Furthermore, α must lift to a $*$-automorphism of $C(\mathbb{T})$ via $\tau|_{\mathbb{T}}$ since $\tau|_{\mathbb{T}}$ is a homeomorphism of $\mathbb{T}$ that preserves orientations.

Our goal is to show that the Toeplitz algebra is an admissible C*-cover for any action of $\mathbb{Z}$ on $A(\mathbb{D})$. We continue with a review of the construction of the Toeplitz algebra as operators on Hardy space.

3.2.1 A Review of Hardy Space

Let $\mathcal{H}(\mathbb{D})$ be the set of analytic (or holomorphic) functions on the open unit disc $\mathbb{D}$ in the complex plane. The *Hardy Space on* $\mathbb{D}$, denoted $H^2(\mathbb{D})$, is defined to be the space of analytic functions on $\mathbb{D}$ with square-summable Taylor coefficients centered at the origin, i.e.

$$H^2(\mathbb{D}) = \left\{ f \in \mathcal{H}(\mathbb{D}) : \sum_{n=0}^{\infty} \left| \hat{f}(n) \right|^2 < \infty, \text{ where } \hat{f}(n) = \frac{f^{(n)}(0)}{n!} \right\}.$$

Then $H^2(\mathbb{D})$ is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)},$$

and it follows that $H^2(\mathbb{D})$ is isomorphic as a Hilbert space to $\ell^2(\mathbb{N})$ via the unitary mapping $f \mapsto (\hat{f}(n))_{n=0}^{\infty}$. The following theorem allows us to identify $H^2(\mathbb{D})$ as a Hilbert subspace of

$L^2(\mathbb{T})$.

Proposition 3.2.1 (Theorem 3.8 in [23]). *If $f \in H^2(\mathbb{D})$, then there exists $\tilde{f} \in L^2(\mathbb{T})$ such that*

(i) $\tilde{f}(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ *for a.e.* $\theta \in [0, 2\pi]$, *and*

(ii) the mapping $f \mapsto \tilde{f}$ *is isometric.*

Thus, if $\{e_n := z^n : n \in \mathbb{Z}\}$ is the standard orthonormal basis for $L^2(\mathbb{T})$, where $z : \mathbb{T} \to \mathbb{T}$ is the identity map, we can (and will) view $H^2(\mathbb{D})$ as the closed subspace of $L^2(\mathbb{T})$ spanned $\{e_n : n \in \mathbb{N}_0\}$. Stated another way, $H^2(\mathbb{D})$ can be viewed as $L^2(\mathbb{T})$-functions whose Fourier coefficients vanish for negative powers of n. We will often blur the lines between these representations of Hardy space. When necessary, we will denote $H^2(\mathbb{T})$ as the closed subspace of $L^2(\mathbb{T})$ spanned $\{e_n : n \in \mathbb{N}_0\}$ to distinguish it from our original definition of $H^2(\mathbb{D})$.

An important feature of $H^2(\mathbb{D})$ is that it is a *reproducing kernel Hilbert space* (*RKHS*), i.e. point evaluations on $\mathbb{D}$ are bounded linear functionals. Indeed, let $\kappa_x : H^2(\mathbb{D}) \to \mathbb{C}$ be evaluation at $x \in \mathbb{D}$. Then for each $f \in H^2(\mathbb{D})$, we have

$$\begin{aligned}
|\kappa_x(f)| &= |f(x)| \\
&= \left| \sum_{n=0}^{\infty} \hat{f}(n) x^n \right| && \text{using the power series expansion for } f \text{ at the origin} \\
&\leq \sum_{n=0}^{\infty} \left| \hat{f}(n) \right| \cdot |x|^n \\
&= \left\langle f, \frac{1}{1 - |x|\, z} \right\rangle \\
&\leq \left(\sum_{n=0}^{\infty} \left| \hat{f}(n) \right|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{n=0}^{\infty} |x|^{2n} \right)^{\frac{1}{2}} && \text{by the Cauchy-Schwarz inequality} \\
&= \sqrt{\frac{1}{1 - |x|^2}} \cdot \|f\| .
\end{aligned}$$

Thus, κ_x is a bounded linear functional on $H^2(\mathbb{D})$, and it follows that $\{\kappa_x : x \in \mathbb{D}\} \subseteq H^2(\mathbb{D})^*$.

Since $H^2(\mathbb{D})$ is a Hilbert space, the Riesz Representation theorem allows us to identify each κ_x as a vector in $H^2(\mathbb{D})$ that implements the point evaluation functional in the inner product, i.e. for each $x \in \mathbb{D}$, there exists a unique vector $K_x \in H^2(\mathbb{D})$ such that $\kappa_x(f) = f(x) = \langle f, K_x\rangle$. We compute this kernel function K_x at $x \in \mathbb{D}$. Fix $x \in \mathbb{D}$. Since $K_x \in H^2(\mathbb{D})$, it has a power series expansion at the origin of the form $K_x(w) = \sum_{n=0}^{\infty} a_n w^n$. But for each $f \in H^2(\mathbb{D})$, we have

$$\sum_{n=0}^{\infty} \hat{f}(n)x^n = f(x) = \langle f, K_x\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{a_n}.$$

Hence, $a_n = \overline{x}^n$ for each $n \in \mathbb{N}_0$ by an inner product computation. Thus, we have

$$K_x(w) = \sum_{n=0}^{\infty} \overline{x}^n w^n = \sum_{n=0}^{\infty} (\overline{x}w)^n = \frac{1}{1-\overline{x}w}.$$

Furthermore, we have

$$\|K_x\| = \langle K_x, K_x\rangle^{\frac{1}{2}} = \sqrt{K_x(x)} = \sqrt{\frac{1}{1-|x|^2}}.$$

The function $K : \mathbb{D} \times \mathbb{D} \to H^2(\mathbb{D})$ by $K(w, x) = K_x(w)$ is called the *Szego kernel on* $\mathbb{D}$.

3.2.2 The Toeplitz Algebra as a C*-cover for $A(\mathbb{D})$

The C*-algebra generated by the Toeplitz operator T_z on $H^2(\mathbb{T})$ is called the *Toeplitz algebra*, and it is well known that the Toeplitz algebra $\mathcal{T} = C^*(T_z)$ has the form

$$\mathcal{T} = \{T_f + K : f \in C(\mathbb{T}) \text{ and } K \in \mathbb{K}\},$$

where $\mathbb{K}$ is the compact operators on $H^2(\mathbb{T})$. This yields the following short exact sequence that splits via the continuous, linear section $T(f) = T_f$.

$$0 \longrightarrow \mathbb{K} \xrightarrow{\text{incl}} \mathcal{T} \underset{T}{\overset{\pi}{\rightleftarrows}} C(\mathbb{T}) \longrightarrow 0$$

It is mentioned in the literature that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is a C*-cover for $A(\mathbb{D})$, however there is no clear reference for a proof of this fact. As such, we include a proof here.

Proposition 3.2.2. *$(\mathcal{T}, T|_{A(\mathbb{D})})$ is a C*-cover for $A(\mathbb{D})$.*

Proof. Since the disc algebra is contained in $H^\infty(\mathbb{T}) = H^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$, the symbol map T is multiplicative on $A(\mathbb{D})$, i.e. $T|_{A(\mathbb{D})}$ is an algebra homomorphism. It is clear that $C^*(T(A(\mathbb{D}))) = \mathcal{T} = C^*(T_z)$ as T is a continuous algebra homomorphism mapping the generator of $A(\mathbb{D})$ to the C*-generator of $\mathcal{T}$. It remains to see that $T|_{A(\mathbb{D})}$ is completely isometric.

The symbol map $T : C(\mathbb{T}) \to \mathcal{T}$ is positive. Indeed, if $f \in C(\mathbb{T})$ be positive, then $T_f^* = T_{\overline{f}} = T_f$ and the image of $\mathbb{T}$ under f is contained in $[0, \infty)$. Corollary V.1.8 in [10] says that the Fredholm theory of T_f yields $\sigma(T_f) = f(\mathbb{T}) \cup \{\lambda \in \mathbb{C} : \text{wind}(f - \lambda) \neq 0\}$, where $\text{wind}(f-\lambda)$ is the winding number of the oriented curve $(f-\lambda)(\mathbb{T})$ about 0. As $f(\mathbb{T}) \subseteq [0, \infty)$, the image of any translation of f will sit on a bounded line segment in $\mathbb{C}$. Hence, we have $\text{wind}(f - \lambda) = 0$ for all $\lambda \in \mathbb{C}$, and it follows that $\sigma(T_f) = f(\mathbb{T}) \subseteq [0, \infty)$, i.e. T_f is positive.

Since $T : C(\mathbb{T}) \to \mathcal{T}$ is a positive map from a uniform algebra into a C*-algebra, a result of Stinespring (Theorem 3.11 in [27]) says T must be completely positive. Thus, T is completely bounded with $\|T\|_{cb} = \|T\| = \|T(1)\| = 1$, and it follows that $T|_{A(\mathbb{D})}$ is completely contractive.

To see that $T|_{A(\mathbb{D})}$ completely isometric, assume that it is not. Then there exists an $n \in \mathbb{N}$ and an element $F = (f_{rc}) \in M_n(A(\mathbb{D}))$ such that $\|T_n(F)\|_{M_n(\mathcal{T})} < \|F\|_{M_n(A(\mathbb{D}))}$. Since any amplification of the $*$-homomorphism π is a $*$-homomorphism and $\pi_n \circ T_n$ is the identity on

$M_n(A(\mathbb{D}))$, we get the contradiction

$$\|T_n(F)\|_{M_n(\mathcal{T})} < \|F\|_{M_n(A(\mathbb{D}))} = \|\pi_n \circ T_n(F)\|_{M_n(A(\mathbb{D}))} \leq \|T_n(F)\|_{M_n(\mathcal{T})}.$$

Thus, $T|_{A(\mathbb{D})}$ is a complete isometry, and we conclude that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is a C*-cover for $A(\mathbb{D})$. □

It should be noted that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is a C*-cover for $A(\mathbb{D})$ that is neither maximal nor minimal in the lattice of C*-covers for $A(\mathbb{D})$. Indeed, the C*-envelope for $A(\mathbb{D})$ is $C_e^*(A(\mathbb{D})) = C(\mathbb{T})$ and the maximal C*-cover for $A(\mathbb{D})$ is the universal C*-algebra generated by a non-normal contraction as discussed in Example 1.2.6. Thus, $(\mathcal{T}, T|_{A(\mathbb{D})})$ is a *non-extremal* C*-cover for $A(\mathbb{D})$, i.e. $C(\mathbb{T}) < (\mathcal{T}, T|_{A(\mathbb{D})}) < C^*_{\max}(A(\mathbb{D}))$.

3.2.3 Dynamical Systems of the Disc Algebra by Rotation

Let G be a (possibly infinite) cyclic group and let $(A(\mathbb{D}), G, \alpha)$ be a dynamical system. We will identify G with $\mathbb{Z} = \langle 1 \rangle$ when G is infinite and G with $\mathbb{Z}/k\mathbb{Z} = \langle 1 \rangle$ when G has finite order k. Our ultimate goal is to show that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible. We start by considering the action of G on $A(\mathbb{D})$ by rotation.

Proposition 3.2.3. *Suppose $G = \langle 1 \rangle$ is a cyclic group and let $(A(\mathbb{D}), G, \alpha)$ be a dynamical system given by rotation by some $\lambda \in \mathbb{T}$, i.e. $\alpha_n(f)(w) = f(\lambda^n w)$ for all $f \in A(\mathbb{D}), w \in \mathbb{D}$. Then $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible.*

Proof. Recall that $\alpha : G \to \operatorname{Aut}(A(\mathbb{D}))$ extends to a representation of G as automorphisms on $C(\mathbb{T})$, which we will also denote by α. Let $U : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the unitary given by

$$Uf(z) = f(\lambda z).$$

By the left-invariance of Lebesgue measure on $\mathbb{T}$, for all $f \in C(\mathbb{T})$ and $g, h \in H^2(\mathbb{T})$, a change of variables computation yields

$$\langle UT_f U^* g, h\rangle = \left\langle T_{\alpha_1(f)} g, h \right\rangle .$$

Thus, $\mathrm{ad}(U)$ must be an automorphism of the Toeplitz algebra since $\mathbb{K}$ is an ideal and $\mathrm{ad}(U)$ preserves $T(C(\mathbb{T}))$.

Define $\beta : G \to \mathrm{Aut}(\mathcal{T})$ by $n \mapsto \mathrm{ad}(U^n)$. Then $(\mathcal{T}, G, \beta)$ is an inner C*-dynamical system. Moreover, for all $n \in G, f \in A(\mathbb{D})$, we have

$$(\beta_n \circ T)(f) = \beta_n(T_f) = \mathrm{ad}(U^n)(T_f) = T_{\alpha_n(f)} = (T \circ \alpha_n)(f).$$

Hence, $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible. □

Proposition 3.2.4. *Suppose $(A(\mathbb{D}), G, \alpha)$ is a dynamical system, where G is a finite cyclic group of order $n > 1$. Then $(A(\mathbb{D}), G, \alpha)$ is given by rotation by a primitive n^{th} root of unity, and hence, $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible.*

Proof. We identify G with the additive group $\mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$. By a previous observation, there exists a Möbius transformation of the form

$$\tau(w) = \lambda \left(\frac{w - \mu}{1 - \overline{\mu} w} \right), \qquad \text{where } \lambda \in \mathbb{T}, \mu \in \mathbb{D}$$

such that $\alpha_1(f)(w) = f(\tau(w))$ for all $f \in A(\mathbb{D}), w \in \mathbb{D}$. Thus, for all $k \in \mathbb{Z}/n\mathbb{Z}$, we must have $\alpha_k(f)(z) = f(\tau^k(z))$ since $\mathbb{Z}/n\mathbb{Z}$ is cyclic. In particular, we have

$$w = \alpha_n(z)(w) = \tau^n(w) \qquad \text{for all } w \in \mathbb{D}. \tag{$*$}$$

We can associate to τ an invertible 2×2 matrix of its coefficients given by

$$A := \begin{bmatrix} \lambda & -\lambda\mu \\ -\overline{\mu} & 1 \end{bmatrix}.$$

Then the matrix associated to τ^n is given by A^n. But $(*)$ says τ^n is the identity so $A^n = I_2$. Observe that $\det A = \lambda(1 - |\mu|^2)$, which implies

$$1 = |\det(A^n)| = |\det A|^n = \left|1 - |\mu|^2\right|^n = \left(1 - |\mu|^2\right)^n.$$

Hence, we must have $1 - |\mu|^2 = 1$, which happens if and only if $\mu = 0$. Therefore, $\tau(w) = \lambda w$ is a rotation transformation, where λ is a primitive n^{th} root of unity. By Proposition 3.2.3, it follows that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible. $\square$

3.2.4 A Review of Composition Operators on $H^2(\mathbb{D})$

Suppose X is a Banach space of complex-valued functions on a set Ω. Given $\varphi : \Omega \to \Omega$, we can formally define a linear mapping C_φ on X called a *composition operator* with symbol φ by

$$(C_\varphi f)(\omega) = f(\varphi(\omega)) \qquad \text{for all } f \in X, \omega \in \Omega,$$

i.e. $C_\varphi(f) = f \circ \varphi$. But it is not clear that C_φ should even be well-defined. We give some familiar examples where the mapping is well-defined.

Example 3.2.5. Let $X = \ell^2(\mathbb{N})$ and define $\varphi : \mathbb{N} \to \mathbb{N}$ by $\varphi(n) = n + 1$. Then for each $x \in \ell^2(\mathbb{N}), n \in \mathbb{N}$, we have

$$C_\varphi(x)(n) = x(\varphi(n)) = x(n + 1).$$

Thus, C_φ is the left shift on $\ell^2(\mathbb{N})$.

Example 3.2.6. Let $X = L^2(\mathbb{T})$. Choose $\theta \in \mathbb{R}$ and define $\varphi : \mathbb{T} \to \mathbb{T}$ by $\varphi(w) = e^{-2\pi i\theta}w$. Then for each $f \in C(\mathbb{T}) \subseteq L^2(\mathbb{T}), w \in \mathbb{T}$, we have

$$C_\varphi(f)(w) = f(\varphi(w)) = f(e^{-2\pi i\theta}w).$$

Thus, the extension of C_φ to $L^2(\mathbb{T})$ is a well-known unitary and $C^*(C_\varphi, M_z)$ is a rotation algebra.

We are interested in the case when $X = H^2(\mathbb{D})$. We appeal to a result of J. E. Littlewood (1925) to identify when properties of the symbol φ determines C_φ is well-defined and bounded on $H^2(\mathbb{D})$.

Proposition 3.2.7 (Littlewood Subordination Theorem (Theorem 2.22 in [9]))**.** *Let φ be an analytic mapping of the unit disk into itself such that $\varphi(0) = 0$. Then*

(i) $C_\varphi H^2(\mathbb{D}) \subseteq H^2(\mathbb{D})$, and

(ii) $\|C_\varphi f\| \leq \|f\|$ for all $f \in H^2(\mathbb{D})$.

Corollary 3.2.8. *If φ is an analytic mapping of the unit disk into itself, then C_φ is well-defined and bounded with*

$$\|C_\varphi\|^2 \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}.$$

The following result of J. Ryff (1966) allows us to continue identifying $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ when working with composition operators on Hardy space.

Theorem 3.2.9 (Proposition 2.25 in [9])**.** *If $f \in H^2(\mathbb{D})$ and φ is an analytic mapping of the unit disk into itself, then $(f \circ \varphi)^\sim = \tilde{f} \circ \tilde{\varphi}$ almost everywhere on $\mathbb{T}$, where $\sim$ is defined as in Proposition 3.2.1.*

It turns out that bounded composition operators on $H^2(\mathbb{D})$ are completely determined by how their adjoints act on the kernel functions $\{K_x : x \in \mathbb{D}\}$.

Theorem 3.2.10 (Theorem 1.4 in [9])**.** *If A is a bounded operator on $H^2(\mathbb{D})$, then A is a composition operator if and only if the set of kernel functions $\{K_x : x \in \mathbb{D}\}$ is invariant under A^*. In this case, $A = C_\varphi$, where φ and A are related by $A^* K_x = K_{\varphi(x)}$.*

In many cases, the study of composition operators is focused on relating the properties of C_φ on $H^2(\mathbb{D})$ to the geometric properties of the symbol φ on $\mathbb{D}$. In [26], E. A. Nordgren characterizes when a composition operator is isometric. Recall that a map φ is *inner* if φ is a bounded analytic function on $\mathbb{D}$ such that φ has radial limits of modulus 1 almost everywhere, i.e. $\varphi \in H^\infty(\mathbb{D})$ and $\lim_{r \to 1^-} \left|\varphi(re^{i\theta})\right| = 1$ for a.e. $\theta \in [0, 2\pi]$.

Theorem 3.2.11 (Corollary of Theorem 1 in [26])**.** *C_φ on $H^2(\mathbb{D})$ is isometric if and only if φ is inner and $\varphi(0) = 0$.*

The adjoint property of C_φ leads to a geometrically satisfying characterization of invertible composition operators.

Theorem 3.2.12 (Theorem 1.6 in [9])**.** *C_φ on $H^2(\mathbb{D})$ is invertible if and only if φ is an (analytic) automorphism of $\mathbb{D}$.*

A characterization of unitary composition operators follows from the previous theorems.

Corollary 3.2.13. *C_φ on $H^2(\mathbb{D})$ is unitary if and only if φ is a rotation of $\mathbb{D}$ by some $\lambda \in \mathbb{T}$, i.e. $\varphi(w) = \lambda w$ for all $w \in \mathbb{D}$.*

3.2.5 Admissibility of the Toeplitz Algebra by Actions of $\mathbb{Z}$

In this section, we show that the Toeplitz algebra is admissible for any action of $\mathbb{Z}$ on $A(\mathbb{D})$. Moreover, we completely characterize the unitaries that implement the extended action of $\mathbb{Z}$

on $\mathcal{T}$. In this section, let $\{e_n\}_{n\geq 0}$ be the standard orthonormal basis for $H^2(\mathbb{T})$ and let z be the identity function on $\mathbb{T}$, i.e. $z(w) = w$ for all $w \in \mathbb{T}$.

Proposition 3.2.14. *Let $(A(\mathbb{D}), \mathbb{Z}, \alpha)$ be a dynamical system with implementing Möbius transformation*

$$\tau(w) = \frac{w-\mu}{1-\overline{\mu}w} \text{ for some } \mu \in \mathbb{D},$$

and define $U : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ by

$$Ue_n = T_\tau^n \frac{K_\mu}{\|K_\mu\|}$$

extending linearly. Then U is unitary and the group representation $\beta : \mathbb{Z} \to Aut(\mathcal{T})$ given by $\beta_n = ad(U^n)$ satisfies $\beta_n \circ T|_{A(\mathbb{D})} = T|_{A(\mathbb{D})} \circ \alpha_n$ for all $n \in \mathbb{Z}$, i.e. $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible.

Proof. Observe that $\{Ue_n : n \geq 0\}$ is an orthonormal set. To see that each Ue_k is a unit vector, note that T_τ is an isometry since $\tau \in H^\infty(\mathbb{T})$ and $|\tau| \equiv 1$. Let $n \geq 0$ and consider

$$\begin{aligned}
\|Ue_n\|^2 &= \frac{1}{\|K_\mu\|^2} \langle T_\tau^n K_\mu, T_\tau^n K_\mu \rangle \\
&= \frac{1}{\|K_\mu\|^2} \langle (T_\tau^* T_\tau)^n K_\mu, K_\mu \rangle \qquad \text{since } T_\tau \text{ is an isometry} \\
&= \frac{1}{\|K_\mu\|^2} \langle K_\mu, K_\mu \rangle \\
&= 1.
\end{aligned}$$

It remains to see that $\{Ue_k\}_{k=0}^\infty$ is an orthogonal set. Let $n, m \in \mathbb{N}_0$ be given. Our goal is to verify that $\langle Ue_n, Ue_m \rangle = 0$ when $n \neq m$. Since $\langle Ue_m, Ue_n \rangle = \overline{\langle Ue_n, Ue_m \rangle}$, it is enough to assume $n \geq m \geq 0$ and compute $\langle Ue_n, Ue_m \rangle$. Consider

$$\begin{aligned}
\langle Ue_n, Ue_m \rangle &= \frac{1}{\|K_\mu\|^2} \langle T_\tau^n K_\mu, T_\tau^m K_\mu \rangle \\
&= \frac{1}{\|K_\mu\|^2} \left\langle T_\tau^{n-m} K_\mu, K_\mu \right\rangle \\
&= \frac{1}{\|K_\mu\|^2} \left\langle M_\tau^{n-m} K_\mu, K_\mu \right\rangle \qquad \text{since } \tau \text{ is analytic and } n-m \geq 0.
\end{aligned}$$

Each of the previous computations is in the $L^2(\mathbb{T})$ inner product. In fact, the above is identifying $\langle M_\tau^{n-m} K_\mu, K_\mu \rangle$ with $\left\langle \widetilde{M_\tau^{n-m} K_\mu}, \widetilde{K_\mu} \right\rangle_{L^2(\mathbb{T})}$, where $f \mapsto \tilde{f}$ is the isometric embedding of $H^2(\mathbb{D})$ into $L^2(\mathbb{T})$ from Proposition 3.2.1. Thus, we have $\left\langle \widetilde{M_\tau^{n-m} K_\mu}, \widetilde{K_\mu} \right\rangle_{L^2(\mathbb{T})} = \langle M_\tau^{n-m} K_\mu, K_\mu \rangle_{H^2(\mathbb{D})}$. As K_μ is a kernel function for $H^2(\mathbb{D})$ at $\mu \in \mathbb{D}$, it follows that $\left\langle \widetilde{M_\tau^{n-m} K_\mu}, \widetilde{K_\mu} \right\rangle_{L^2(\mathbb{T})} = \tau(\mu)^{n-m} K_\mu(\mu)$. But $\tau(\mu) = 0$ so $\tau(\mu)^{n-m}$ is 0 if $n \neq m$ and $\|K_\mu\|^2$ when $n = m$. Thus, we have

$$\langle Ue_n, Ue_m \rangle = \frac{1}{\|K_\mu\|^2} \left(\tau(\mu)^{n-m} K_\mu(\mu) \right) = \begin{cases} 1, & n = m \\ 0, & n > m \end{cases}.$$

Hence, U is isometric on linear combinations of $\{e_n : n \geq 0\}$. Extending by continuity yields that U is isometric on $H^2(\mathbb{T})$ since $\operatorname{Span}\{e_n : n \geq 0\}$ is dense in $H^2(\mathbb{T})$.

We show that U satisfies the commutation relation $UT_p = T_{p \circ \tau} U$ for all $p \in \mathbb{C}[z]$. Fix $p \in \mathbb{C}[z]$. The linearity of U and the symbol map $T : C(\mathbb{T}) \to \mathcal{T}$ yields

$$UT_p e_0 = Up = U \left(\sum_{n \geq 0} \hat{p}(n) e_n \right) = \sum_{n \geq 0} \hat{p}(n) U e_n = \sum_{n \geq 0} \hat{p}(n) T_\tau^n \frac{K_\mu}{\|K_\mu\|} = \left(\sum_{n \geq 0} \hat{p}(n) T_\tau^n \right) \left(\frac{K_\mu}{\|K_\mu\|} \right).$$

But T_τ can be identified with $T_{z\circ\tau}$, where $z:\mathbb{T}\to\mathbb{T}$ is the identity map. Thus, we obtain

$$\begin{aligned} UT_pe_0 &= \left(\sum_{n\geq 0}\hat{p}(n)T_{z^n\circ\tau}\right)\left(\frac{K_\mu}{\|K_\mu\|}\right) && \text{since } \tau\in A(\mathbb{D})\subseteq H^\infty(\mathbb{T}) \\ &= T_{p\circ\tau}\frac{K_\mu}{\|K_\mu\|} && \text{by linearity of } T \\ &= T_{p\circ\tau}Ue_0. \end{aligned}$$

Hence, for all $n\geq 0$, we have

$$\begin{aligned} UT_pe_n &= UT_pT_{z^n}e_0 \\ &= UT_{p\cdot z^n}e_0 && \text{since } p, z^n\in H^\infty(\mathbb{T}) \\ &= T_{(p\circ\tau)\cdot\tau^n}Ue_0 && \text{since } p\cdot z^n\in\mathbb{C}[z] \\ &= T_{p\circ\tau}T_\tau^nUe_0 && \text{since } p\circ\tau,\tau\in H^\infty(\mathbb{T}) \\ &= T_{p\circ\tau}Ue_n. \end{aligned}$$

Thus, we conclude that $UT_p = T_{p\circ\tau}U$ for all $p\in\mathbb{C}[z]$ by the linearity and continuity of U and T_p on $H^2(\mathbb{T})$.

We can extend this commutation relation from $T(\mathbb{C}[z])$ to $T(A(\mathbb{D}))$ using the density of $\mathbb{C}[z]$ in $A(\mathbb{D})$ with respect to the supremum norm. Indeed, let $f\in A(\mathbb{D})$ be given. Then there exists a sequence of polynomials $\{p_k\}_{k=1}^\infty$ in $\mathbb{C}[z]$ such that $\|p_k-f\|_\infty\to 0$ as $k\to\infty$. Fix $\varepsilon>0$. Choose $N\in\mathbb{N}$ so that $\|p_k-f\|_\infty<\frac{\varepsilon}{2}$ for all $k\geq N$. As τ is an automorphism of $\overline{\mathbb{D}}$, observe that for all $k\geq N$, we have

$$\|p_k\circ\tau-f\circ\tau\|_\infty = \sup_{w\in\mathbb{T}}|p_k(\tau(w))-f(\tau(w))| = \sup_{w\in\mathbb{T}}|p_k(w)-f(w)| = \|p_k-f\|_\infty < \frac{\varepsilon}{2}.$$

Hence, for all $k\geq N$, the commutation relation on polynomials and properties of Toeplitz operators with $H^\infty(\mathbb{T})$ symbol yields

$$\begin{aligned}
\|UT_f - T_{f\circ\tau}U\| &\leq \|UT_f - UT_{p_k}\| + \|UT_{p_k} - T_{p_k\circ\tau}U\| + \|T_{p_k\circ\tau}U - T_{f\circ\tau}U\| \\
&= \|UT_f - UT_{p_k}\| + \|T_{p_k\circ\tau}U - T_{f\circ\tau}U\| \\
&\leq \|U\|\,\|T_f - T_{p_k}\| + \|T_{p_k\circ\tau} - T_{f\circ\tau}\|\,\|U\| \\
&= \|T_{f-p_k}\| + \|T_{p_k\circ\tau - f\circ\tau}\| \\
&= \|f - p_k\|_\infty + \|p_k\circ\tau - f\circ\tau\|_\infty \\
&< \varepsilon.
\end{aligned}$$

Thus, we have $UT_f = T_{f\circ\tau}U$ for all $f \in A(\mathbb{D})$.

From here, we can observe that U has dense range in $H^2(\mathbb{T})$, which implies that U is unitary. Let $p \in \mathbb{C}[z]$ be given. Choose $\tilde{p} = \|K_\mu\|\,T_{1-\overline{\mu}\tau^{-1}}C_{\tau^{-1}}p \in H^2(\mathbb{T})$ and note that $1-\overline{\mu}\tau^{-1}, p\circ\tau^{-1} \in A(\mathbb{D})$. Consider

$$\begin{aligned}
U\tilde{p} &= \|K_\mu\|\,UT_{1-\overline{\mu}\tau^{-1}}C_{\tau^{-1}}p \\
&= \|K_\mu\|\,UT_{1-\overline{\mu}\tau^{-1}}T_{p\circ\tau^{-1}}e_0 \\
&= \|K_\mu\|\,T_{1-\overline{\mu}z}T_pUe_0 \qquad \text{by the commutation relation} \\
&= T_{1-\overline{\mu}z}T_pK_\mu.
\end{aligned}$$

But recall that $K_\mu(w) = \frac{1}{1-\overline{\mu}w}$ for a.e. $w \in \mathbb{T}$ so $(T_pK_\mu)(w) = \frac{p(w)}{1-\overline{\mu}w}$ for a.e. $w \in \mathbb{T}$. Hence, we must have

$$U\tilde{p} = T_{1-\overline{\mu}z}T_pK_\mu = p.$$

As the polynomials in z are dense in $H^2(\mathbb{T})$, U is an isometry with dense range, and it follows that U is unitary.

To see that conjugation by U is a $*$-automorphism on $\mathcal{T}$, we need to verify that $\mathrm{ad}(U)$

leaves the Toeplitz algebra invariant. This is clear since T_z generates $\mathcal{T}$ as a C*-algebra, $\mathrm{ad}(U)T_z = T_\tau$ by the commutation relation on $T(A(\mathbb{D}))$, and $\mathrm{ad}(U)$ is a $*$-homomorphism. Thus, we have $\mathrm{ad}(U) \in \mathrm{Aut}(\mathcal{T})$.

Define $\beta : \mathbb{Z} \to \mathrm{Aut}(\mathcal{T})$ by $n \mapsto \mathrm{ad}(U^n)$. Then β is a group representation and the commutation relation $T_f U = T_{f\circ\tau}U$ for all $f \in A(\mathbb{D})$ says the following diagram commutes.

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\beta_n"] & \mathcal{T} \\ A(\mathbb{D}) \arrow[u, "T"] \arrow[r, "\alpha_n"'] & A(\mathbb{D}) \arrow[u, "T"'] \end{tikzcd}$$

Therefore, we have $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible. $\square$

Theorem 3.2.15. *Let $(A(\mathbb{D}), \mathbb{Z}, \alpha)$ be a dynamical system with implementing Möbius transformation*

$$\tau(w) = \lambda\left(\frac{w-\mu}{1-\overline{\mu}w}\right) \quad \text{for some } \mu \in \mathbb{D}, \lambda \in \mathbb{T}.$$

Then $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible.

Proof. By Proposition 3.2.14, there exists a unitary U_1 operator on $H^2(\mathbb{T})$ such that $U_1 T_f = T_{f\circ(\overline{\lambda}\tau)}U_1$ for all $f \in A(\mathbb{D})$. Similarly, Proposition 3.2.3 says there exists a unitary U_2 on $H^2(\mathbb{T})$ such that $\mathrm{ad}(U_2)T_f = T_{f\circ\gamma}$, where $\gamma(w) = \lambda w$ for all $w \in \mathbb{T}$. Set $U := U_1U_2$. Then for all $f \in A(\mathbb{D})$, we have

$$UT_f = U_1U_2T_f = U_1T_{f\circ\gamma} = T_{f\circ\gamma\circ(\overline{\lambda}\tau)}U_1U_2 = T_{f\circ\gamma\circ(\overline{\lambda}\tau)}U.$$

But we can see that $\gamma \circ (\overline{\lambda}\tau) = \tau$ since $\lambda \in \mathbb{T}$. Thus, we have $UT_f = T_{f\circ\tau}U$ for all $f \in A(\mathbb{D})$. Since $\mathrm{ad}(U)$ is a $*$-homomorphism and $\mathcal{T}$ is generated by $T(A(\mathbb{D}))$, we see that $\mathrm{ad}(U)$ leaves $\mathcal{T}$ invariant. Therefore, $\beta : \mathbb{Z} \to \mathrm{Aut}(\mathcal{T})$ given by $n \mapsto \mathrm{ad}(U^n)$ is a well-defined group representation, and it follows that $(\mathcal{T}, T|_{A(\mathbb{D})})$ is α-admissible. $\square$

Chapter 4

Isomorphic Dynamical Systems

We begin by defining what it means to have isomorphic, or conjugate, dynamical systems. We show that conjugacy of dynamical systems yields completely isometrically isomorphic crossed products in Theorem 4.1.4. We conclude the chapter by giving a geometric characterization of conjugacy for two dynamical systems of the form $(A(\mathbb{D}), \mathbb{Z}, \alpha)$, which is progress towards answering a question of E. Katsoulis and C. Ramsey in [22].

4.1 Conjugacy of Dynamical Systems

Definition 4.1.1. Let $(\mathcal{A}, G, \alpha)$ and $(\mathcal{B}, G, \beta)$ be dynamical systems where $\mathcal{A}, \mathcal{B}$ are unital operator algebras. We say $(\mathcal{A}, G, \alpha)$ and $(\mathcal{B}, G, \beta)$ are *conjugate* if there is a completely isometric isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(\alpha_s(a)) = \beta_s(\varphi(a))$ for all $s \in G$ and $a \in \mathcal{A}$. We call $\varphi : \mathcal{A} \to \mathcal{B}$ an *equivariant completely isometric isomorphism.*

Conjugate dynamical systems, as we've defined it, are sometimes called *equivariantly isomorphic* dynamical systems in literature when the algebras are C*-algebras. See definition 2.64 in [30]. We show that conjugacy of dynamical systems lifts to conjugacy of the associated C*-envelope dynamical systems.

Lemma 4.1.2. *Suppose $(\mathcal{A}, G, \alpha)$ and $(\mathcal{B}, G, \beta)$ are conjugate dynamical systems via a completely isometric isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ that intertwines α and β. Then $(C_e^*(\mathcal{A}), G, \tilde{\alpha})$ and $(C_e^*(\mathcal{B}), G, \tilde{\beta})$ are conjugate C*-dynamical systems.*

Proof. Let $(C_e^*(\mathcal{A}), G, \tilde{\alpha})$ and $(C_e^*(\mathcal{B}), G, \tilde{\beta})$ be the C*-dynamical systems that arise from the α and β-admissibility of the C*-covers $C_e^*(\mathcal{A}) \equiv (C_e^*(\mathcal{A}), i_{\mathcal{A}})$ and $C_e^*(\mathcal{B}) \equiv (C_e^*(\mathcal{B}), i_{\mathcal{B}})$, respectively. Note $i_{\mathcal{B}} \circ \varphi : \mathcal{A} \to C_e^*(\mathcal{B})$ is a completely isometric homomorphism and $C^*(i_{\mathcal{B}}(\varphi(\mathcal{A}))) = C^*(i_{\mathcal{B}}(\mathcal{B})) = C_e^*(\mathcal{B})$. Hence, $C_e^*(\mathcal{B})$ is a C*-cover for $\mathcal{A}$, and similarly, $C_e^*(\mathcal{A})$ is a C*-cover for $\mathcal{B}$. Appealing to the universal property for both C*-envelopes yields an isomorphism $\Phi : C_e^*(\mathcal{A}) \to C_e^*(\mathcal{B})$ such that $\Phi \circ i_{\mathcal{A}} = i_{\mathcal{B}} \circ \varphi$. Thus, for all $s \in G$, α-admissibility of $C_e^*(\mathcal{A})$ yields

$$(\Phi \circ \tilde{\alpha}_s) \circ i_{\mathcal{A}} = \Phi \circ (\tilde{\alpha}_s \circ i_{\mathcal{A}}) = \Phi \circ (i_{\mathcal{A}} \circ \alpha_s) = (\Phi \circ i_{\mathcal{A}}) \circ \alpha_s = (i_{\mathcal{B}} \circ \varphi) \circ \alpha_s.$$

Recall that φ intertwines the actions of G on $\mathcal{A}$ and $\mathcal{B}$, i.e. $\varphi \circ \alpha_s = \beta_s \circ \varphi$ for all $s \in G$. Hence, we have $(i_{\mathcal{B}} \circ \varphi) \circ \alpha_s = i_{\mathcal{B}} \circ (\beta_s \circ \varphi) = (i_{\mathcal{B}} \circ \beta_s) \circ \varphi$ for all $s \in G$. But $C_e^*(\mathcal{B})$ is β-admissible so for each $s \in G$ we have $(i_{\mathcal{B}} \circ \beta_s) \circ \varphi = (\tilde{\beta}_s \circ i_{\mathcal{B}}) \circ \varphi$. Therefore, for all $s \in G$, we have

$$(\Phi \circ \tilde{\alpha}_s) \circ i_{\mathcal{A}} = (i_{\beta} \circ \varphi) \circ \alpha_s = \tilde{\beta}_s \circ (i_{\mathcal{B}} \circ \varphi) = \tilde{\beta}_s \circ (\Phi \circ i_{\mathcal{A}}) = (\tilde{\beta}_s \circ \Phi) \circ i_{\mathcal{A}}.$$

Thus, $\Phi \circ \tilde{\alpha}_s$ and $\tilde{\beta}_s \circ \Phi$ are $*$-isomorphisms that agree on the generating subalgebra $i_{\mathcal{A}}(\mathcal{A})$ for $C_e^*(\mathcal{A})$. Hence, we have $\Phi \circ \tilde{\alpha}_s = \tilde{\beta}_s \circ \Phi$ on all of $C_e^*(\mathcal{A})$, and it follows that $(C_e^*(\mathcal{A}), G, \tilde{\alpha})$ and $(C_e^*(\mathcal{B}), G, \tilde{\beta})$ are conjugate C*-dynamical systems. $\square$

By a nearly identical argument to Lemma 4.1.2, conjugacy of dynamical systems also lifts to conjugacy of the associated maximal C*-dynamical systems.

Corollary 4.1.3. *Suppose $(\mathcal{A}, G, \alpha)$ and $(\mathcal{B}, G, \beta)$ are conjugate dynamical systems via a completely isometric isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ that intertwines α and β. Then $(C^*_{max}(\mathcal{A}), G, \tilde{\alpha})$ and $(C^*_{max}(\mathcal{B}), G, \tilde{\beta})$ are conjugate C*-dynamical systems.*

By lifting conjugacy to admissible C*-covers, we are able to establish an isomorphism between the associated crossed products.

Theorem 4.1.4. *Suppose $(\mathcal{A}, G, \alpha)$ and $(\mathcal{B}, G, \beta)$ are conjugate dynamical systems via a completely isometric isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ that intertwines α and β. Let $(\mathcal{C}, j)$ and $(\mathcal{D}, i)$ be C*-covers for $\mathcal{A}$ and $\mathcal{B}$, respectively, such that either $(\mathcal{C}, j) = C^*_e(\mathcal{A})$ and $(\mathcal{D}, i) = C^*_e(\mathcal{B})$ or $(\mathcal{C}, j) = C^*_{max}(\mathcal{A})$ and $(\mathcal{D}, i) = C^*_{max}(\mathcal{B})$. Then the map $\varphi \otimes id : C_c(G, j(\mathcal{A})) \to C_c(G, i(\mathcal{B}))$ given by*

$$\varphi \otimes id(f)(s) = i \circ \varphi \circ j^{-1}(f(s))$$

extends to a completely isometric isomorphism of $\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} G$ onto $\mathcal{B} \rtimes_{\mathcal{D},i,\beta} G$.

Proof. By Lemma 4.1.2 and its subsequent corollary, $(\mathcal{C}, G, \tilde{\alpha})$ and $(\mathcal{D}, G, \tilde{\beta})$ are conjugate C*-dynamical systems via a $*$-isomorphism $\Phi : \mathcal{C} \to \mathcal{D}$ that satisfies $\Phi \circ j = i \circ \varphi$. Thus, Lemma 2.65 in [30] says the map $\Phi \otimes \mathrm{id} : C_c(G, \mathcal{C}) \to C_c(G, \mathcal{D})$ given by $\Phi \otimes \mathrm{id}(f)(s) = \Phi(f(s))$ extends to a $*$-isomorphism of the C*-crossed products $\mathcal{C} \rtimes_{\tilde{\alpha}} G$ onto $\mathcal{D} \rtimes_{\tilde{\beta}} G$. Hence, the restriction $\Phi \otimes \mathrm{id}|_{\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} G}$ is completely isometric. We claim that $\Phi \otimes \mathrm{id}|_{C_c(G, j(\mathcal{A}))} = \varphi \otimes \mathrm{id}$.

Let $f \in C_c(G, j(\mathcal{A}))$ be given. Then for all $s \in G$, we have $f(s) = j(a_s)$ for some $a_s \in \mathcal{A}$. As $\Phi \circ j = i \circ \varphi$, for each $s \in G$ we can see that

$$\Phi \otimes \mathrm{id}(f)(s) = \Phi(f(s)) = \Phi(j(a_s)) = i(\varphi(a_s)) = (i \circ \varphi \circ j^{-1})(j(a_s)) = \varphi \otimes \mathrm{id}(f)(s).$$

Since $\Phi \otimes \mathrm{id}$ is completely isometric, we have $\varphi \otimes \mathrm{id}$ is a completely isometric homomorphism of $C_c(G, j(\mathcal{A}))$ onto $C_c(G, i(\mathcal{B}))$. By continuity, $\varphi \otimes \mathrm{id}$ extends to a completely isometric isomorphism of $\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} G$ onto $\mathcal{B} \rtimes_{\mathcal{D},i,\beta} G$. □

4.2 Classifying Conjugate Dynamical Systems of the Disc Algebra

E. Katsoulis and C. Ramsey ask in [22] the following question: *when are two algebras of the form $A(\mathbb{D}) \rtimes_\alpha \mathbb{Z}$ isomorphic as algebras?* We give a geometric classification for conjugate $\mathbb{Z}$-dynamical systems of the disc algebra.

Theorem 4.2.1. *Suppose $(A(\mathbb{D}), \mathbb{Z}, \alpha)$ and $(A(\mathbb{D}), \mathbb{Z}, \beta)$ are dynamical systems with implementing Möbius transformations τ_α and τ_β given by*

$$\tau_\alpha(w) = \lambda_\alpha \left(\frac{w - \mu_\alpha}{1 - \overline{\mu_\alpha} w} \right) \qquad \textit{and} \qquad \tau_\beta(w) = \lambda_\beta \left(\frac{w - \mu_\beta}{1 - \overline{\mu_\beta} w} \right).$$

Then the following are equivalent:

(i) $(A(\mathbb{D}), \mathbb{Z}, \alpha)$ *and* $(A(\mathbb{D}), \mathbb{Z}, \beta)$ *are conjugate.*

(ii) $\lambda_\alpha = \lambda_\beta$ *and* $|\mu_\alpha| = |\mu_\beta|$.

Proof. If $(A(\mathbb{D}), \mathbb{Z}, \alpha)$ and $(A(\mathbb{D}), \mathbb{Z}, \beta)$ are conjugate, there exists an equivariant completely isometric isomorphism $\varphi : A(\mathbb{D}) \to A(\mathbb{D})$ that intertwines α and β. As φ is a completely isometric automorphism of $A(\mathbb{D})$, there exists an implementing Möbius transformation τ_φ. Using the equivariance of φ, we have

$$\tau_\alpha \circ \tau_\varphi = \varphi(\alpha_1(z)) = \beta_1(\varphi(z)) = \tau_\varphi \circ \tau_\beta,$$

where $z : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is the identity map.

By associating invertible 2×2 matrices of coefficients A, B, and F to each Möbius transformation τ_α, τ_β, and τ_φ, respectively, we can express $\tau_\alpha \circ \tau_\varphi = \tau_\varphi \circ \tau_\beta$ as the matrix product

$$\begin{pmatrix} \lambda_\alpha & -\lambda_\alpha \mu_\alpha \\ -\overline{\mu_\alpha} & 1 \end{pmatrix} F = AF = FB = F \begin{pmatrix} \lambda_\beta & -\lambda_\beta \mu_\beta \\ -\overline{\mu_\beta} & 1 \end{pmatrix}.$$

Hence, A and B are similar matrices since F is invertible. In particular, this implies that A and B have the same characteristic polynomial. Therefore, we have

$$(\lambda_\alpha - w)(1-w) - \lambda_\alpha \left|\mu_\alpha\right|^2 = \det(A - wI) = \det(B - wI) = (\lambda_\beta - w)(1-w) - \lambda_\beta \left|\mu_\beta\right|^2.$$

Rearranging terms, we obtain

$$(\lambda_\alpha - w)(1-w) - (\lambda_\beta - w)(1-w) = \lambda_\alpha \left|\mu_\alpha\right|^2 - \lambda_\beta \left|\mu_\beta\right|^2.$$

Simplifying the left hand side, we get

$$(1-w)(\lambda_\alpha - \lambda_\beta) = \lambda_\alpha \left|\mu_\alpha\right|^2 - \lambda_\beta \left|\mu_\beta\right|^2,$$

which must be true for all $w \in \mathbb{C}$. When $w = 1$, we get $\lambda_\alpha \left|\mu_\alpha\right|^2 = \lambda_\beta \left|\mu_\beta\right|^2$. As $\lambda_\alpha, \lambda_\beta \in \mathbb{T}$, it follows that

$$\left|\mu_\alpha\right|^2 = \left|\lambda_\alpha \left|\mu\right|^2\right| = \left|\lambda_\beta \left|\mu_\beta\right|^2\right| = \left|\mu_\beta\right|^2.$$

Therefore, we have $|\mu_\alpha| = |\mu_\beta|$. Hence, using $w = 0$ we get $\lambda_\alpha - \lambda_\beta = (\lambda_\alpha - \lambda_\beta)\left|\mu_\alpha\right|^2$. Thus, we must have $\lambda_\alpha = \lambda_\beta$, or else $\left|\mu_\alpha\right|^2 = 1$, which is absurd. Therefore, we have $|\mu_\alpha| = |\mu_\beta|$ and $\lambda_\alpha = \lambda_\beta$.

Conversely, suppose $|\mu_\alpha| = |\mu_\beta|$ and $\lambda_\alpha = \lambda_\beta$. Then μ_α and μ_β lie on the same circle inside $\mathbb{D}$. Hence, we can choose $\theta \in (-\pi, \pi]$ such that $e^{i\theta}\mu_\alpha = \mu_\beta$. Define $\tau_\theta : \mathbb{C} \to \mathbb{C}$ to be the rotation transformation given by $\tau_\theta(w) = e^{i\theta}w$. Since $\mu_\beta = e^{i\theta}\mu_\alpha$ and $\lambda_\alpha = \lambda_\beta$, for all $w \in \overline{\mathbb{D}}$ we have

$$\begin{aligned}(\tau_\beta \circ \tau_\theta)(w) = \lambda_\beta \left(\frac{e^{i\theta}w - \mu_\beta}{1 - \overline{\mu_\beta}(e^{i\theta}w)}\right) &= \lambda_\alpha \left(\frac{e^{i\theta}w - (e^{i\theta}\mu_\alpha)}{1 - (e^{-i\theta}\overline{\mu_\alpha})(e^{i\theta}w)}\right) \\ &= e^{i\theta}\left(\lambda_\alpha \left(\frac{w - \mu_\alpha}{1 - \overline{\mu_\alpha}w}\right)\right) = (\tau_\theta \circ \tau_\alpha)(w).\end{aligned}$$

Thus, we have $\tau_\beta \circ \tau_\theta$ and $\tau_\theta \circ \tau_\alpha$ agree on $\overline{\mathbb{D}}$. Define $\varphi : A(\mathbb{D}) \to A(\mathbb{D})$ by $\varphi(f) = f \circ \tau_\theta$ and note that φ is a a completely isometric automorphism of $A(\mathbb{D})$. Moreover, for each $n \in \mathbb{Z}$ and for all $f \in A(\mathbb{D})$, we must have

$$\varphi(\beta_n(f)) = \varphi(f \circ \tau_\beta^n) = f \circ \tau_\beta^n \circ \tau_\theta = f \circ \tau_\theta \circ \tau_\alpha^n = \alpha_n(f \circ \tau_\theta) = \alpha_n(\varphi(f)).$$

Therefore, $(A(\mathbb{D}), \mathbb{Z}, \alpha)$ and $(A(\mathbb{D}), \mathbb{Z}, \beta)$ are conjugate. $\square$

This gives a sufficient geometric condition for two crossed products of the disc algebra to be completely isometrically isomorphic by Theorem 4.1.4.

Corollary 4.2.2. *Suppose $(A(\mathbb{D}), \mathbb{Z}, \alpha)$ and $(A(\mathbb{D}), \mathbb{Z}, \beta)$ are dynamical systems with implementing Möbius transformations τ_α and τ_β given by*

$$\tau_\alpha(w) = \lambda_\alpha \left(\frac{w - \mu_\alpha}{1 - \overline{\mu_\alpha} w} \right) \qquad \textit{and} \qquad \tau_\beta(w) = \lambda_\beta \left(\frac{w - \mu_\beta}{1 - \overline{\mu_\beta} w} \right).$$

If $\lambda_\alpha = \lambda_\beta$ and $|\mu_\alpha| = |\mu_\beta|$, then $A(\mathbb{D}) \rtimes_\alpha \mathbb{Z}$ is isometrically isomorphic to $A(\mathbb{D}) \rtimes_\beta \mathbb{Z}$.

We are still interested if the algebraic structure alone on the crossed product is rigid enough to determine such geometric information about the original dynamical system. In other words, we are still interested in whether or not an algebra isomorphism of the crossed products $A(\mathbb{D}) \rtimes_\alpha G$ and $A(\mathbb{D}) \rtimes_\beta G$ implies conjugacy of the dynamical systems. In the case that τ_α and τ_β are elliptic Möbius transformations, E. Katsoulis and C. Ramsey mention in [22] that the crossed products will be isomorphic to semicrossed products of the form $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}^+$. So when τ_α and τ_β are elliptic, an algebra isomorphism of crossed products will yield conjugacy of the dynamical systems by the work of K. Davidson and E. Katsoulis. See Theorem 4.6 and Corollary 4.7 in [11].

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