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Unbounded Derivations of C*-algebras and the Heisenberg Commutation Relation

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UNBOUNDED DERIVATIONS OF C^*-ALGEBRAS AND THE HEISENBERG COMMUTATION RELATION

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This dissertation investigates the properties of unbounded derivations on C^*-algebras, namely the density of their analytic vectors and a property we refer to as "kernel stabilization." We focus on a weakly-defined derivation δ_D which formalizes commutators involving unbounded self-adjoint operators on a Hilbert space. These commutators naturally arise in quantum mechanics, as we briefly describe in the introduction.

A first application of kernel stabilization for δ_D shows that a large class of abstract derivations on unbounded C^*-algebras, defined by O. Bratteli and D. Robinson, also have kernel stabilization. A second application of kernel stabilization provides a sufficient condition for when a pair of self-adjoint operators which satisfy the Heisenberg Commutation Relation on a Hilbert space must both be unbounded.

A directly related classification program is of pairs of unitary group representations which satisfy the Weyl Commutation Relation on a Hilbert space. The famous Stone-von Neumann Theorem classifies these pairs when the group is locally compact abelian. In collaboration with L. Huang, we extend the Stone-von Neumann Theorem to a uniqueness statement for representations of C^*-dynamical systems on Hilbert $\mathcal{K}(\mathcal{H})$-modules.

DEDICATION

For Grandma & Grandpa Ismert and Nanny & Papa McCurdy:

> "Now they'll walk on my arm through the distant night,
>
> and I won't let them stray from my heart.
>
> Through the wind, through the dark, through the winter light,
>
> I will read all their dreams to the stars."

— Those You've Known, *Spring Awakening*

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Chapter 1

Introduction

1.1 Quantum Mechanics and Operators on Hilbert Space

A quantum system can be represented by a Hilbert space $\mathcal{H}$ with time evolution of the system modeled by a strongly continuous one-parameter group of unitaries $\{U_t\}_{t\in\mathbb{R}}$ on $\mathcal{H}$. By time evolution, we mean that the state of the system at time t is given by $\psi_t = U_{-t}\psi_o$, where $\psi_o \in \mathcal{H}$ is the system's initial state. Stone's Theorem provides a (possibly unbounded) self-adjoint operator D whose functional calculus implements $\{U_t\}_{t\in\mathbb{R}}$; specifically, $e^{itD} = U_t$ for each $t \in \mathbb{R}$. The operator D is called the *Hamiltonian* of the system. If D is unbounded, the domain of D is only a proper dense subspace of $\mathcal{H}$. Consequently, domains of sums and compositions involving D may not be dense. Nonetheless, quantum mechanics necessitates taking such sums and compositions.

An observable of a quantum system modeled by $\mathcal{H}$ is a self-adjoint operator that represents a measurable quantity such as the position or momentum of a particle. Like the Hamiltonian, a general observable x might also be unbounded, but we restrict our attention to bounded observables. Ehrenfest's Theorem (Eqn. 6.2 of [20]) states that the commutator $[iD, x] = i(Dx - xD)$ determines the time-dependence of the observable x. Without supplemental conditions on x, however, the density of the domain of $[iD, x]$ is not guaranteed, so Ehrenfest's Theorem requires some formalization. To better understand the definedness and boundedness of $[iD, x]$, let us investigate how the commutator arises in Ehrenfest's Theorem as the descriptor of time evolution.

The *expected value* of an observable $x \in \mathcal{B}(\mathcal{H})$ at time t is given by $\langle x\psi_t, \psi_t \rangle$. Notice how

$$\langle x\psi_t, \psi_t \rangle = \left\langle xe^{-itD}\psi_0, e^{-itD}\psi_0 \right\rangle = \left\langle e^{itD}xe^{-itD}\psi_0, \psi_0 \right\rangle$$

shifts the time dependence from the vector ψ_t to the operator $e^{itD}xe^{-itD}$. These two perspectives are known as the Schrödinger picture and the Heisenberg picture, respectively. For $t \in \mathbb{R}$, define

$$\alpha_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \qquad \text{by} \qquad \alpha_t(x) := e^{itD}xe^{-itD} \text{ for all } x \in \mathcal{B}(\mathcal{H}).$$

The family $\{\alpha_t\}_{t\in\mathbb{R}}$ is a norm-continuous group of $*$-automorphisms of $\mathcal{B}(\mathcal{H})$. Informally,

$$\frac{d}{dt}\left(\alpha_t(x)\right) = \frac{d}{dt}\left(e^{itD}xe^{-itD}\right) = iD\left(e^{itD}xe^{-itD}\right) - \left(e^{itD}xe^{-itD}\right)iD = [iD, \alpha_t(x)].$$

We now interpret Ehrenfest's Theorem to mean $\frac{d}{dt}\left(\alpha_t(x)\right)|_{t=0} = [iD, x]$, but *the topology in which the derivative is taken is really the heart of the matter.* The work of E. Christensen in [6] and [5] seeks to connect the topology in which this derivative is taken to the domain of $[iD, x]$ via a *derivation* on $\mathcal{B}(\mathcal{H})$. In section 3.1, we introduce this derivation and its desirable properties.

1.2 Derivations on C^*-algebras

Given a complex $*$-algebra $\mathcal{A}$, a *derivation* on $\mathcal{A}$ is a linear map $\delta : \mathcal{A} \to \mathcal{A}$ which satisfies the Leibniz rule: $\delta(bc) = \delta(b)c + b\delta(c)$ for all $b, c \in \mathcal{A}$. We can easily construct a derivation on $\mathcal{A}$ by fixing an element $a \in \mathcal{A}$ such that $a = a^*$ and defining a map

$$\begin{aligned} \delta_a : \quad \mathcal{A} &\to \quad \mathcal{A} \\ b &\mapsto [ia, b]. \end{aligned}$$

The map δ_a is a $*$-*derivation*, that is, $\delta_a(b^*) = \delta_a(b)^*$ for all $b \in \mathcal{A}$. Conversely, for an arbitrary $*$-derivation $\delta : \mathcal{A} \to \mathcal{A}$, certain conditions on the algebra and the derivation imply $\delta = \delta_a$ for some $a \in \mathcal{A}$ satisfying $a = a^*$. The correspondence between derivations on algebras and their representation as commutators has a rich history and is deeply connected to the mathematical formulation of quantum mechanics.

We wish to define a derivation $\delta_D : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which implements the derivative informally taken in the previous section: $\delta_D(x) := [iD, x]$ for $x \in \mathcal{B}(\mathcal{H})$. However, as not every $x \in \mathcal{B}(\mathcal{H})$ makes the commutator $[iD, x]$ defined and bounded on a dense subspace of $\mathcal{H}$, the definition of the derivation "δ_D" is ambiguous. A plethora of literature is dedicated to exploring the various definitions of δ_D and their corresponding domains. In each situation, if D is unbounded then the domain of δ_D is a proper subspace of $\mathcal{B}(\mathcal{H})$. In turn, further research has been dedicated to the more general study of unbounded derivations on an abstract C^*-algebra. The unboundedness of such a derivation creates complexities that are not found with bounded derivations, i.e., derivations defined on the entire C^*-algebra. In [10], Kadison summarizes three of the many significant results pertaining to bounded derivations:

1. Every bounded derivation on a commutative C^*-algebra is 0. (This follows from the Singer-Wermer Theorem from 1955 in [23].)

2. Sakai (1959) showed in [19] that any everywhere-defined derivation of a C^*-algebra is automatically bounded, thus affirmatively settling a 1953 conjecture of Kaplansky.

3. In [12], Kaplansky showed every bounded derivation δ of a type I von Neumann algebra

M is *inner*, i.e., there exists $a \in M$ such that $\delta = \delta_a$.

We turn our attention to densely-defined derivations on C^*-algebras. In section 3.1 we give a formal definition of δ_D, its domain, domains of its higher powers, and state its desirable properties. In particular, Christensen shows in [6] that the domain of δ_D is strong operator topology (SOT)-dense in $\mathcal{B}(\mathcal{H})$.

In section 3.4 we generalize Christensen's SOT-density result for $\mathrm{Dom}(\delta_D)$ to include SOT-density of $\mathrm{Dom}(\delta_D^n)$ for all $n \in \mathbb{N}$, and we further strengthen this result by proving SOT-density of the analytic vectors for δ_D. Both of these proofs utilize the norm-density of $\mathrm{Dom}(D^n)$ and the analytic vectors for D in $\mathcal{H}$, which displays a nice parallel between the domain of a self-adjoint operator D on a Hilbert space and the domain of the derivation δ_D that D implements.

Theorem 1.1. *The set of analytic vectors for δ_D is SOT-dense in $\mathcal{B}(\mathcal{H})$.*

On the other hand, our second main result pertaining to δ_D shows that δ_D has a property which is not analogous to properties of self-adjoint operators.

Theorem 1.2. *If $\mathcal{H}$ is a Hilbert space and D is a (possibly unbounded) self-adjoint operator on $\mathcal{H}$, then $\ker \delta_D^n = \ker \delta_D$ for all $n \in \mathbb{N}$.*

The oddity of this result is illustrated by a simple example from calculus: if $f(z) = z$, then $f''(z) = 0$, but $f'(z) = 1 \neq 0$. In other words, the function f belongs to the kernel of the second-derivative, but not to the first. Notice, however, that due to unboundedness of f on $\mathbb{C}$ that an analogue of f inside of $\mathcal{B}(\mathcal{H})$ does not exist. Given $x \in \ker \delta_D^n$, the operator x is both bounded *and* analytic for δ_D. The implication of Theorem 1.2 is that x must belong to $\ker \delta_D$, or that x is a "constant." So, perhaps kernel stabilization is suggestive of a Liouville Theorem for bounded operators on a Hilbert space.

In chapter 4, we prove Theorem 1.2, and in section 4.3, we give two applications. The first application extends the property of kernel stabilization to a class of unbounded $*$-derivations on C^*-algebras described in the following theorem.

Theorem 1.3 (Bratteli-Robinson, [3]). *Let δ be a derivation of a C^*-algebra $\mathcal{A}$, and assume there exists a state ω on $\mathcal{A}$ which generates a faithful cyclic representation (π, H, f) satisfying*

$$\omega(\delta(a)) = 0 \textit{ for all } a \in \mathrm{Dom}(\delta).$$

Then δ is closable and there exists a symmetric operator S on $\mathcal{H}$ such that

$$\mathrm{Dom}(S) = \{h \in \mathcal{H} : h = \pi(a)f \ \ \textit{for some } a \in \mathcal{A}\}$$

and $\pi(\delta(a))h = [S, \pi(a)]h$ for all $a \in \mathrm{Dom}(\delta)$ and all $h \in \mathrm{Dom}(S)$. Moreover, if the set $\mathsf{A}(\delta)$ of analytic vectors for δ is dense in $\mathcal{A}$, then S is essentially self-adjoint. For $x \in \mathcal{B}(\mathcal{H})$ and $t \in \mathbb{R}$, define

$$\alpha_t(x) := e^{i\overline{S}t} x e^{-i\overline{S}t}$$

where $\overline{S}$ denotes the self-adjoint closure of S. It follows that $\alpha_t(\pi(\mathcal{A})) = \pi(\mathcal{A})$ for all $t \in \mathbb{R}$, and $\{\alpha_t\}_{t\in\mathbb{R}}$ is a strongly continuous group of $*$-automorphisms with closed infinitesimal generator $\widetilde{\delta}$ equaling the closure of $\pi \circ \delta|_{\mathsf{A}(\delta)}$.

Theorem 1.4. *Let $\mathcal{A}$ be a C^*-algebra, δ a derivation on $\mathcal{A}$, and ω a state on $\mathcal{A}$ which satisfy the hypotheses of Theorem 1.3. For every $n \in \mathbb{N}$, $\ker \delta^n = \ker \delta$.*

As a second application of kernel stabilization, we provide a sufficient condition for when a pair of self-adjoint operators which satisfy the Heisenberg Commutation Relation must both be unbounded.

Definition 1.5. Let A and B be two (possibly unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$. We say A and B *satisfy the Heisenberg Commutation Relation* (HCR) if there is a dense subspace K of $\mathcal{H}$ such that

(i) $K \subseteq \mathrm{Dom}([A, B])$ and

(ii) $[A, B]k = ik$ for all $k \in K$.

We include the condition that the HCR be satisfied on a dense subspace of $\mathcal{H}$ because of the possible unboundedness of A and B. In general,

$$\mathrm{Dom}([A, B]) = \{h \in \mathrm{Dom}(A) \cap \mathrm{Dom}(B) : Ah \in \mathrm{Dom}(B),\ Bh \in \mathrm{Dom}(A)\}.$$

Even if $\mathrm{Dom}(A) \cap \mathrm{Dom}(B)$ were dense in $\mathcal{H}$, $\mathrm{Dom}([A, B])$ may fail to be dense. If, however, K is a dense subspace of $\mathcal{H}$ such that $K \subseteq \mathrm{Dom}([A, B])$, the equality $[A, B]|_K = iI|_K$ implies $[A, B]$ continuously extends to the bounded and everywhere-defined operator iI. The condition on K that we give in Theorem 1.6 is that K be a *core* for both A and B.

Theorem 1.6. *If A and B satisfy the HCR on a common core for A and B, then both A and B must be unbounded.*

1.3 The Heisenberg and Weyl Commutation Relations

We adopt the following formal definition of a Heisenberg pair.

Definition 1.7. A pair of (possibly unbounded) self-adjoint operators (A, B) on a Hilbert space $\mathcal{H}$ form a *Heisenberg pair* if A and B satisfy the HCR.

By Stone's Theorem, A and B yield strongly-continuous one-parameter unitary groups R and S, which are families are bounded operators. Thus, one common method in the classification of Heisenberg pairs is to find sufficient conditions on A and B for when R and S form a *Heisenberg representation* of $\mathbb{R}$.

Definition 1.8. Let G be a locally compact abelian group and $\widehat{G}$ its Pontryagin dual. A pair of strongly-continuous unitary groups $R = \{R_x\}_{x\in G}$ and $S = \{S_\gamma\}_{\gamma\in\widehat{G}}$ satisfy the *Weyl Commutation Relation* (WCR) if

$$S_\gamma R_x = \gamma(x) R_x S_\gamma \text{ for all } x \in G, \gamma \in \widehat{G}.$$

The pair (R, S) is a *Heisenberg representation of* G (not to be confused with a Heisenberg *pair*).

Definition 1.9. Let μ be a Haar measure for G, and denote $L^2(G,\mu)$ by $L^2(G)$. Consider the maps $\lambda : G \to \mathcal{U}(L^2(G))$ and $V : \widehat{G} \to \mathcal{U}(L^2(G))$, where for each $x \in G$, $\gamma \in \widehat{G}$, and $f \in C_c(G)$,

$$[\lambda_x f](y) := f(x^{-1}y) \text{ and } [V_\gamma f](y) := \gamma(y) f(y) \text{ for all } y \in G.$$

The pair (λ, V) is a Heisenberg representation of G called the *Schrödinger representation.*

Theorem 1.10 (Stone-von Neumann Theorem)**.** *Every Heisenberg representation of G is unitarily equivalent to a direct sum of copies of the Schrödinger representation.*

Since Heisenberg representations of a locally compact group G are classified by the Stone-von Neumann Theorem, classification of Heisenberg pairs whose generated unitary groups form a Heisenberg representation of $\mathbb{R}$ are immediately classified.

Chapter 5 of this dissertation is joint work with Leonard Huang (University of Nevada, Reno), in which we state and prove a "Covariant Stone-von Neumann Theorem." Our result generalizes the classical Stone-von Neumann Theorem in two ways. First, we consider representations of C^**-dynamical systems* involving locally compact abelian groups as opposed to just locally compact abelian groups. We also consider representations of these dynamical systems on *Hilbert* $\mathcal{K}(\mathcal{H})$*-modules* as opposed to representations only on Hilbert spaces. Requisite background for C^*-dynamical systems and Hilbert C^*-modules is in Chapter 2.

Theorem 1.11. *Every* $(G, \mathcal{K}(\mathcal{H}), \alpha)$*-Heisenberg representation is unitarily equivalent to a direct sum of copies of the* $(G, \mathcal{K}(\mathcal{H}), \alpha)$*-Schrödinger representation.*

In Chapter 5, we define a $(G, \mathcal{A}, \alpha)$-Heisenberg representation and the $(G, \mathcal{A}, \alpha)$-Schrödinger representation for an arbitrary C^*-algebra, and we show that the $(G, \mathcal{A}, \alpha)$-Schrödinger representation is a $(G, \mathcal{A}, \alpha)$-Heisenberg representation. We then provide and prove some results about Hilbert $\mathcal{K}(\mathcal{H})$-modules that are necessary to prove Theorem 1.11.

While interesting in a purely mathematical context, our generalization of the Stone-von Neumann Theorem has a rich interpretation from the perspective of quantum mechanics. Namely, representations of dynamical systems allow for the consideration of an inherit time-dependence of the space of observables in addition to the time-dependence of the state space. This occurs when the Hamiltonian of the system is time-dependent, i.e., the energies influencing the system are not constant. Informally, we obtain a new description of the time-evolution of x:

$$\frac{dx}{dt}\Big|_{t=0} = [iD, x] + \frac{\partial x}{\partial t}\Big|_{t=0}, \qquad \text{[Eqn. 3.22 [26]]}$$

where the partial term $\frac{\partial x}{\partial t}|_{t=0}$ is the addition of time-dependence for the observable x in the presence of a time-dependent Hamiltonian. If the Hamiltonian is time-independent, this term vanishes, and we recover the time-independent version of Ehrenfest's Theorem. The time-dependence of x indicated by a nonzero partial derivative term can be modeled by an action of $\mathbb{R}$ on the C^*-algebra $\mathcal{A}$ of observables. More generally, we may consider a locally compact abelian group G acting on $\mathcal{A}$ via a continuous group homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{A})$, which we call a C^*-dynamical system $(G, \mathcal{A}, \alpha)$.

The goal of representing these dynamical systems on Hilbert $\mathcal{K}(\mathcal{H})$-modules is motivated in large part by the flexibility of modeling quantum field theory (where relativity may be in play) with representations on Hilbert C^*-modules. Tangent to this physical motivation is the goal of generalizing major theorems for operators on Hilbert spaces, such as Stone's Theorem and Stinespring's Theorem, to the setting of Hilbert C^*-modules. Works in this realm include [1] and [24]. A drawback of our work is that our main result pertains only to C^*-dynamical systems $(G, \mathcal{K}(\mathcal{H}), \alpha)$, where G is locally compact abelian, represented on Hilbert $\mathcal{K}(\mathcal{H})$-modules. Ideally our results hold in a more general context, but our current techniques rely heavily on this choice of C^*-algebra. Nonetheless, our result is a nontrivial extension of the classical Stone-von Neumann Theorem.

Chapter 2

Background

2.1 $\mathcal{B}(\mathcal{H})$ and C^*-algebras

Throughout we take $\mathcal{H}$ to be a complex Hilbert space, and we denote the continuous linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. Recall that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra with respect to the adjoint operation and the operator norm. In addition to the operator norm, there are two other topologies we consider on $\mathcal{B}(\mathcal{H})$:

Definition 2.1. The *strong operator topology* (SOT) on $\mathcal{B}(\mathcal{H})$ is the topology induced by the seminorms $\{x \mapsto \|xh\| : h \in \mathcal{H}\}$. Equivalently, a net $(x_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{B}(\mathcal{H})$ converges in the strong operator topology to $x \in \mathcal{B}(\mathcal{H})$ if and only if $\lim\limits_{\lambda\to\infty} \|x_\lambda h - xh\| = 0$ for all $h \in \mathcal{H}$.

Definition 2.2. The *weak operator topology* (WOT) on $\mathcal{B}(\mathcal{H})$ is the topology induced by the seminorms $\{x \mapsto |\langle xh, k\rangle| : h, k \in \mathcal{H}\}$. Equivalently, a net $(x_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{B}(\mathcal{H})$ converges in the weak operator topology to $x \in \mathcal{B}(\mathcal{H})$ if and only if $\lim\limits_{\lambda\to\infty} |\langle x_\lambda h, k\rangle - \langle xh, k\rangle| = 0$ for all $h, k \in \mathcal{H}$.

Remark 2.3. The norm topology on $\mathcal{B}(\mathcal{H})$ is finer than the strong operator topology, and the strong operator topology is finer than the weak operator topology.

Definition 2.4. A *von Neumann algebra* is a SOT-closed unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

2.2 Unbounded Symmetric Operators on Hilbert Space

Let $\mathcal{H}$ be a Hilbert space, K_1 and K_2 subspaces of $\mathcal{H}$, and $T : K_1 \to K_2$ a linear map. We call K_1 the *domain* of T, denoted $\mathrm{Dom}(T)$.

Definition 2.5. A linear operator T is *densely-defined* if $\mathrm{Dom}(T)$ is dense in $\mathcal{H}$.

If $\mathrm{Dom}(T) = \mathcal{H}$ and T is continuous, then T is simply an element of $\mathcal{B}(\mathcal{H})$. If $\mathrm{Dom}(T)$ is only dense in $\mathcal{H}$, but T is bounded on $\mathrm{Dom}(T)$, we may extend T by continuity to a bounded operator on all of $\mathcal{H}$. Thus, the domain of a densely-defined bounded linear operator can always be extended to all of $\mathcal{H}$, but this is not the case for densely-defined linear operators which are unbounded.

Example 2.6. For each $f \in C_c(\mathbb{R})$, the continuous compactly supported functions on $\mathbb{R}$, define

$$[Qf](x) := xf(x) \text{ for all } x \in \mathbb{R}.$$

Clearly, $Qf \in C_c(\mathbb{R})$ and Q is linear, so Q defines a linear operator on the $\|\cdot\|_2$-dense subspace $C_c(\mathbb{R})$ of the Hilbert space $L^2(\mathbb{R})$. However, Q is not extendable to an everywhere-defined operator on $L^2(\mathbb{R})$ because Q is not bounded on $C_c(\mathbb{R})$.

For each $k \in \mathbb{N}$, choose $f_k \in C_c(\mathbb{R})$ with $\mathrm{Supp}(f_k) \subseteq [k, k+1]$. Then

$$\|Qf_k\|_2 = \left(\int_{[k,k+1]} |xf_k(x)|^2 \, dm(x)\right)^{1/2} \geq k \left(\int_{[k,k+1]} |f_k(x)|^2 \, dm(x)\right)^{1/2} = k \|f_k\|_2 .$$

Thus, $\|Q\| \geq k$ for all $k \in \mathbb{N}$, which implies Q is unbounded. The largest subspace of $L^2(\mathbb{R})$ on which Q is defined is

$$\mathrm{Dom}(Q) := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |xf(x)|^2 \, dm(x) < \infty \right\}.$$

While Q is not extendable to all of $L^2(\mathbb{R})$, Q *is* continuous in a certain sense.

Definition 2.7. A linear operator T is *closed* if the *graph of* T, $\Gamma(T) := \{(h, Th) : h \in \mathrm{Dom}(T)\}$, is closed in $\mathcal{H} \oplus \mathcal{H}$.

The operator Q in Example 2.6 is closed.

Definition 2.8. Given a closed linear operator T on a Hilbert space $\mathcal{H}$, a *core* for T is a subspace $\mathscr{C} \subseteq \mathrm{Dom}(T)$ such that

$$\overline{\Gamma(T|_{\mathscr{C}})}^{\mathcal{H}\oplus\mathcal{H}} = \Gamma(T).$$

Example 2.9. For $f \in C_c^\infty(\mathbb{R})$, define $Pf := -if'$. Then P with domain

$$\mathrm{Dom}(P) := \{f \in L^2(\mathbb{R}) : f \text{ is absolutely continuous on every interval } [a,b] \text{ and } f' \in L^2(\mathbb{R})\}$$

is a closed operator.

In addition to being closed, the operators Q and P are *self-adjoint*.

Definition 2.10 (Conway, X.1.5 [7])**.** Let T be a densely-defined linear operator on $\mathcal{H}$, and let

$$\mathrm{Dom}(T^*) = \{k \in \mathcal{H} : h \mapsto \langle Th, k\rangle \text{ defines a bounded linear functional on } \mathrm{Dom}(T)\}.$$

By density of $\mathrm{Dom}(T)$ in $\mathcal{H}$, for each $k \in \mathrm{Dom}(T^*)$ the Riesz Representation Theorem provides a unique $f \in \mathcal{H}$ such that $\langle Th, k\rangle = \langle h, f\rangle$ for all $h \in \mathrm{Dom}(T)$. Let $T^*k := f$. Then,

$$\langle Th, k\rangle = \langle h, T^*k\rangle \text{ for all } h \in \mathrm{Dom}(T) \text{ and } k \in \mathrm{Dom}(T^*).$$

Definition 2.11. A densely-defined linear operator D is *self-adjoint* if

(i) $\langle Dh, k\rangle = \langle h, Dk\rangle$ for all $h, k \in \mathrm{Dom}(D)$ (i.e., D is *symmetric*)

(ii) and $\mathrm{Dom}(D) = \mathrm{Dom}(D^*)$.

Definition 2.12. A densely-defined linear operator S on $\mathcal{H}$ is *essentially self-adjoint* if the closure of the graph $\Gamma(S)$ in $\mathcal{H} \oplus \mathcal{H}$ defines the graph of a self-adjoint operator.

A symmetric operator automatically satisfies $\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D^*)$. In fact, when D is bounded, symmetry implies condition (ii). When D is unbounded, however, condition (ii) requires D to have an adequately large domain—as large as the domain of its adjoint. The domains of higher powers of a self-adjoint operator is one of the properties that make self-adjoint operators so desirable.

Notation 2.13. Let S be a linear operator on a Banach space X. For each $n \in \mathbb{N}$,

$$\mathrm{Dom}(S^n) := \{x \in \mathrm{Dom}\big(S^{n-1}\big) : S^{n-1}x \in \mathrm{Dom}(S)\}.$$

Definition 2.14. Let S be a linear operator on a Banach space X. A vector $x \in X$ is an *analytic vector* for S if

(i) $x \in \mathrm{Dom}(S^n)$ for all $n \in \mathbb{N}$ and

(ii) $\sum_{n=0}^{\infty} \frac{\|S^n x\|}{n!} t^n < \infty$ for some $t > 0$.

Denote the set of analytic vectors for S by $\mathsf{A}(S)$.

Given a *densely*-defined operator T, domains of higher powers of T may fail to be dense as

$$\mathrm{Dom}(T) \supsetneq \mathrm{Dom}\big(T^2\big) \supsetneq \mathrm{Dom}\big(T^3\big) \supsetneq \ldots$$

When T is self-adjoint, however, $\mathrm{Dom}(T^n)$ is dense in $\mathcal{H}$ for all $n \in \mathbb{N}$. In fact, the set of analytic vectors for T is dense in $\mathcal{H}$.

Theorem 2.15 (Nelson, [16]). *A densely-defined operator on a Hilbert space $\mathcal{H}$ is essentially self-adjoint if and only if its set of analytic vectors is dense in $\mathcal{H}$.*

This remarkable fact is known as "Nelson's Analytic Vector Theorem." Additionally, self-adjoint operators are the infinitesimal generators of a special type of one-parameter family.

Definition 2.16. A family $\{U_t\}_{t\in\mathbb{R}}$ of operators on a Hilbert space $\mathcal{H}$ which satisfies

(i) U_t is unitary for each $t \in \mathbb{R}$, that is, $U_t^* U_t = I = U_t U_t^*$,

(ii) $U_o = I$,

(iii) $U_s U_t = U_{s+t}$ for all $s, t \in \mathbb{R}$, and

(iv) $\lim\limits_{t\to 0} \|U_t h - h\| = 0$ for all $h \in \mathcal{H}$

is a *strongly-continuous one-parameter group of unitaries.*

Theorem 2.17 (Stone's Theorem). *Given a self-adjoint operator D, the family $\{e^{itD}\}_{t\in\mathbb{R}}$ is a strongly-continuous one-parameter group of unitaries. Conversely, given a strongly-continuous one-parameter group of unitaries $\{U_t\}_{t\in\mathbb{R}}$, there exists a self-adjoint operator D such that $U_t = e^{itD}$ for all $t \in \mathbb{R}$.*

The self-adjoint operator D is called the *infinitesimal generator* for the group $\{e^{itD}\}_{t\in\mathbb{R}}$:

$$\mathrm{Dom}(D) = \left\{ h \in \mathcal{H} : \lim_{t\to 0} \frac{e^{itD}h - h}{t} \text{ exists} \right\},$$

and for $h \in \mathrm{Dom}(D)$,

$$Dh := -i \left(\lim_{t\to 0} \frac{e^{itD}h - h}{t} \right).$$

2.3 Unitary Group Representations

Let $\mathcal{U}(\mathcal{H})$ denote the unitary group of $\mathcal{B}(\mathcal{H})$, and let G be a locally compact group. Up to a scalar, G has a unique nonzero left-invariant Radon measure, called a *Haar measure*, which we denote by μ. We may then consider the Hilbert space $L^2(G,\mu)$, which we denote by $L^2(G)$. In the case when G is abelian, μ is also right-invariant, and its Pontryagin dual is a locally compact abelian group $\widehat{G}$ whose Haar measure we denote by $\hat{\mu}$.

Definition 2.18. A *unitary group representation* of G on a Hilbert space $\mathcal{H}$ is a group homomorphism $U : G \to \mathcal{U}(\mathcal{H})$ such that for each $h \in \mathcal{H}$, the map $s \mapsto U_s h$ is continuous.

Example 2.19. Any strongly-continuous one-parameter group of unitaries $\{U_t\}_{t\in\mathbb{R}}$ on $\mathcal{H}$ defines a unitary group representation $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ by $t \mapsto U_t$.

Example 2.20. Let G be a locally compact abelian group. The *left regular representation* $\lambda : G \to \mathcal{U}(L^2(G))$ and representation $V : \widehat{G} \to \mathcal{U}(L^2(G))$ in the Schrödinger representation (λ, V) of G (recall Definition 1.9) are examples of unitary group representations.

2.4 C^*-Dynamical Systems and Crossed Products

The reader is referred to [25] for a detailed treatment of foundational material on C^*-dynamical systems and crossed product C^*-algebras. Some definitions and facts are included here for convenience. Throughout, G is a locally compact abelian group with Haar measure μ and $\mathcal{A}$ is a C^*-algebra.

Definition 2.21. A *C^*-dynamical system* is a triple $(G, \mathcal{A}, \alpha)$ where $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a continuous homomorphism.

Example 2.22. Let $C_o(G)$ be the C^*-algebra of continuous functions $f : G \to \mathbb{C}$ such that for each $\epsilon > 0$, there is a compact subset $K \subseteq G$ where $\left\| f|_{G\setminus K} \right\|_\infty < \epsilon$. Consider an action of G on $C_o(G)$ via *left translation:*

$$\begin{array}{rccc} \mathsf{lt}: & G & \to & \mathrm{Aut}(C_o(G)) \\ & x & \mapsto & \mathsf{lt}_x, \end{array}$$

where for each $f \in C_o(G)$,

$$[\mathsf{lt}_x f](y) := f(x^{-1}y) \text{ for all } y \in G.$$

Then $(G, C_o(G), \mathsf{lt})$ is a C^*-dynamical system.

Definition 2.23. A *covariant representation* of a C^*-dynamical system $(G, \mathcal{A}, \alpha)$ is a pair (π, U) consisting of a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a unitary group representation $U : G \to \mathcal{U}(\mathcal{H})$ such that

$$\pi(\alpha_x(a)) = U_x \pi(a) U_x^* \text{ for all } x \in G, a \in \mathcal{A}.$$

Example 2.24 (Williams, 2.12 [25]). Let $M : C_o(G) \to \mathcal{B}(L^2(G))$ denoted $f \mapsto M_f$ be given by pointwise multiplication, that is, for each $f \in C_o(G)$ and $h \in C_c(G)$,

$$[M_f h](x) := f(x)h(x) \text{ for all } x \in G.$$

By density of $C_c(G)$ in $L^2(G)$ and boundedness of $M_f|_{C_c(G)}$, we may extend M_f to a bounded linear operator on all of $L^2(G)$. If λ denotes the left regular representation of G, then the pair (M, λ) is a covariant representation of $(G, C_o(G), \mathsf{lt})$.

Given a C^*-dynamical system $(G, \mathcal{A}, \alpha)$, one can construct the *crossed product* C^*-algebra $\mathcal{A} \rtimes_\alpha G$ which is universal with respect to the covariant representations of $(G, \mathcal{A}, \alpha)$. Let $C_c(G, \mathcal{A})$ denote the set of continuous functions $f : G \to \mathcal{A}$ such that for each $f \in C_c(G, \mathcal{A})$, there exists a compact subset $K \subseteq G$ where $\mathrm{Supp}(f) \subseteq K$. The crossed product corresponding to a C^*-dynamical system $(G, \mathcal{A}, \alpha)$ is constructed by considering representations of $C_c(G, \mathcal{A})$ which are induced by covariant representations of $(G, \mathcal{A}, \alpha)$.

Definition 2.25. Given a covariant representation (π, U) for $(G, \mathcal{A}, \alpha)$ on $\mathcal{H}$, define the *integrated form* of (π, U) to be the $*$-representation $\pi \rtimes U : C_c(G, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ given by

$$[\pi \rtimes U](f) := \int_G \pi(f(x))U_x \, d\mu(x) \text{ for all } f \in C_c(G, \mathcal{A}).$$

The above integral is $\mathcal{B}(\mathcal{H})$-valued and converges in the WOT, i.e.,

$$\langle [\pi \rtimes U](f)h, k\rangle = \int_G \langle \pi(f(x))U_x h, k\rangle \; d\mu(x) \text{ for all } h, k \in \mathcal{H}.$$

Lemma 2.26 (Williams, 2.27 [25])**.** *For each $f \in C_c(G, \mathcal{A})$, define the* universal norm *on $C_c(G, \mathcal{A})$ by*

$$\|f\| := \sup\{\|[\pi \rtimes U](f)\| : (\pi, U) \textit{ is a covariant representation of } (G, \mathcal{A}, \alpha)\}.$$

The universal norm is dominated by the $L^1(G, \mathcal{A})$-norm and the completion of $C_c(G, \mathcal{A})$ with respect to $\|\cdot\|$ is a C^*-algebra which we denote by $\mathcal{A} \rtimes_\alpha G$.

2.5 Hilbert C^*-modules

Let G be a locally compact abelian group with Haar measure μ and $\mathcal{A}$ a C^*-algebra.

Definition 2.27. An *inner product $\mathcal{A}$-module* is a linear space X which is a right $\mathcal{A}$-module via an action $\bullet : \mathsf{X} \times \mathcal{A} \to \mathsf{X}$ denoted $(\xi, a) \mapsto \xi \bullet a$ which satisfies

$$\lambda(\xi \bullet a) = (\lambda\xi) \bullet a = \xi \bullet (\lambda a) \text{ for all } \xi \in \mathsf{X},\ a \in \mathcal{A},\ \lambda \in \mathbb{C},$$

together with a map $\langle \cdot \,|\, \cdot \rangle : \mathsf{X} \times \mathsf{X} \to \mathcal{A}$ such that for all $\xi,\ \eta,\ \nu \in \mathsf{X},\ \alpha,\ \beta \in \mathbb{C}$, and $a \in \mathcal{A}$,

(i) $\langle \xi \,|\, \alpha\eta + \beta\nu \rangle = \alpha \langle \xi \,|\, \eta \rangle + \beta \langle \xi \,|\, \nu \rangle$,

(ii) $\langle \xi \,|\, \eta \bullet a \rangle = \langle \xi \,|\, \eta \rangle\, a$,

(iii) $\langle \eta \,|\, \xi \rangle = \langle \xi \,|\, \eta \rangle^*$, and

(iv) $\langle \xi \,|\, \xi \rangle \geq 0$ as an element of $\mathcal{A}$, and if $\langle \xi \,|\, \xi \rangle = 0$, then $\xi = 0$.

We sometimes subscript $\langle \cdot \,|\, \cdot \rangle$ to avoid ambiguity when multiple algebras or modules are present.

Definition 2.28. Let X be an inner product $\mathcal{A}$-module, and define a norm on X by

$$\|\xi\| := \|\langle \xi \,|\, \xi \rangle\|_{\mathcal{A}}^{1/2} \text{ for all } \xi \in \mathsf{X}.$$

Then X is a (right) *Hilbert $\mathcal{A}$-module* if X is complete with respect to $\|\cdot\|$.

Note that when $\mathcal{A} = \mathbb{C}$, a Hilbert $\mathcal{A}$-module is simply a Hilbert space. Left Hilbert $\mathcal{A}$-modules are defined similarly.

Example 2.29. For $\phi \in C_c(G, \mathcal{A})$ and $a \in \mathcal{A}$, define

$$[\phi \bullet a](x) := \phi(x)a \text{ for all } x \in G.$$

Then $C_c(G, \mathcal{A})$ along with the action $\bullet$ by $\mathcal{A}$ is a right $\mathcal{A}$-module. For $\phi, \psi \in C_c(G, \mathcal{A})$, define

$$\langle \psi \,|\, \phi \rangle := \int_G \psi(x)^* \phi(x)\, d\mu(x),$$

where this $\mathcal{A}$-valued integral is characterized by

$$\zeta\left(\langle \psi \,|\, \phi \rangle\right) = \int_G \zeta(\psi(x)^* \phi(x))\, d\mu(x) \text{ for all } \zeta \in \mathcal{A}^*.$$

One easily checks that $\langle \cdot \mid \cdot \rangle$ satisfies the axioms in Definition 2.27, so $C_c(G, \mathcal{A})$ with $\langle \cdot \mid \cdot \rangle$ is an inner product $\mathcal{A}$-module. Denote the completion of $C_c(G, \mathcal{A})$ with respect to the induced norm $\|\cdot\| := \|\langle \cdot \mid \cdot \rangle\|_{\mathcal{A}}^{1/2}$ by $\mathsf{L}^2(G, \mathcal{A})$.

Example 2.30. Let $(G, \mathcal{A}, \alpha)$ be a dynamical system. For each $\phi \in C_c(G, \mathcal{A})$ and $a \in \mathcal{A}$, define

$$[\phi \bullet a](x) := \phi(x)\alpha_x(a) \text{ for all } x \in G.$$

Then $\bullet$ makes $C_c(G, \mathcal{A})$ into a right $\mathcal{A}$-module. For $\phi, \psi \in C_c(G, \mathcal{A})$, define

$$\langle \psi \mid \phi \rangle_\alpha := \int_G \alpha_{x^{-1}} \left(\psi(x)^* \phi(x) \right) \, d\mu(x).$$

Then $C_c(G, \mathcal{A})$ along with $\langle \cdot \mid \cdot \rangle_\alpha$ defines an inner product $\mathcal{A}$-module. Denote the completion of $C_c(G, \mathcal{A})$ with respect to the induced norm $\|\cdot\|_\alpha := \|\langle \cdot \mid \cdot \rangle_\alpha\|_{\mathcal{A}}^{1/2}$ by $\mathsf{L}^2(G, \mathcal{A}, \alpha)$.

Remark 2.31. When completing $C_c(G, \mathcal{A})$ with respect to $\|\cdot\|_\alpha$, an isomorphic copy of $C_c(G, \mathcal{A})$ exists in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ via an embedding $q : C_c(G, \mathcal{A}) \to \mathsf{L}^2(G, \mathcal{A}, \alpha)$. When considering the dense subalgebra $q(C_c(G, \mathcal{A}))$ inside $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, we will suppress the "copy" and simply identify $C_c(G, \mathcal{A})$ inside $\mathsf{L}^2(G, \mathcal{A}, \alpha)$.

Proposition 2.32. *Let $(G, \mathcal{A}, \alpha)$ be a C^*-dynamical system. A norm $\|\cdot\|_2$ can be defined on $C_c(G, \mathcal{A})$ by*

$$\|\phi\|_2 := \left(\int_G \|\phi(x)\|_{\mathcal{A}}^2 \, d\mu(x) \right)^{1/2} \text{ for each } \phi \in C_c(G, \mathcal{A}).$$

This norm has the property that $\|\phi\|_\alpha \leq \|\phi\|_2$ for all $\phi \in C_c(G, \mathcal{A})$.

Proof. Checking that $\|\cdot\|_2$ is a norm on $C_c(G, \mathcal{A})$ is a simple exercise. For $\phi \in C_c(G, \mathcal{A})$, observe

$$
\begin{aligned}
\|\phi\|_\alpha^2 &= \left\| \int_G \alpha_{x^{-1}}(\phi(x)^*\phi(x)) \, d\mu(x) \right\|_{\mathcal{A}} \\
&\leq \int_G \|\alpha_{x^{-1}}(\phi(x)^*\phi(x))\|_{\mathcal{A}} \, d\mu(x) \\
&= \int_G \|\phi(x)^*\phi(x)\|_{\mathcal{A}} \, d\mu(x) \\
&= \int_G \|\phi(x)\|_{\mathcal{A}}^2 \, d\mu(x) \\
&= \|\phi\|_2^2.
\end{aligned}
$$

□

Corollary 2.33. *Suppose* $\{\psi_\lambda\}_{\lambda\in\Lambda} \subseteq C_c(G, \mathcal{A})$ *converges uniformly to* $\psi \in C_c(G, \mathcal{A})$*, i.e.,* $\|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})} \to 0$ *as* $\lambda \to \infty$*. Then* $\|\psi_\lambda - \psi\|_\alpha \to 0$ *as* $\lambda \to \infty$*.*

Proof. By Proposition 2.32, it suffices to prove that $\|\psi_\lambda - \psi\|_2 \to 0$ as $\lambda \to \infty$. Let $\epsilon > 0$, and choose $\lambda_1 \in \Lambda$ such that $\|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})} < \frac{\epsilon}{\sqrt{2\cdot\mu(\mathrm{Supp}(\psi))+1}}$ for all $\lambda \geq \lambda_1$. Also, since $\|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})} \to 0$ as $\lambda \to \infty$, there exists $\lambda_2 \in \Lambda$ such that $\mu(\mathrm{Supp}(\psi_\lambda) \setminus \mathrm{Supp}(\psi)) < \mu(\mathrm{Supp}(\psi))$ for all $\lambda \geq \lambda_2$. Choose $\lambda_o := \max\{\lambda_1, \lambda_2\}$. Then for all $\lambda \geq \lambda_o$,

$$
\begin{aligned}
\|\psi_\lambda - \psi\|_2^2 &= \int_G \|\psi_\lambda(y) - \psi(y)\|_{\mathcal{A}}^2 \, d\mu(y) \\
&= \int_{\mathrm{Supp}(\psi)} \|\psi_\lambda(y) - \psi(y)\|_{\mathcal{A}}^2 \, d\mu(y) + \int_{\mathrm{Supp}(\psi_\lambda)\setminus\mathrm{Supp}(\psi)} \|\psi_\lambda(y) - \psi(y)\|_{\mathcal{A}}^2 \, d\mu(y) \\
&\leq \int_{\mathrm{Supp}(\psi)} \|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})}^2 \, d\mu(y) + \int_{\mathrm{Supp}(\psi_\lambda)\setminus\mathrm{Supp}(\psi)} \|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})}^2 \, d\mu(y) \\
&< \frac{\epsilon^2}{2\cdot\mu(\mathrm{Supp}(\psi))+1} \cdot \mu(\mathrm{Supp}(\psi)) + \frac{\epsilon^2}{2\cdot\mu(\mathrm{Supp}(\psi))+1} \cdot \mu(\mathrm{Supp}(\psi_\lambda)\setminus\mathrm{Supp}(\psi)) \\
&< \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2\cdot\mu(\mathrm{Supp}(\psi))+1} \cdot \mu(\mathrm{Supp}(\psi)) \\
&< \epsilon^2.
\end{aligned}
$$

By Proposition 2.32, $\|\psi_\lambda - \psi\|_\alpha \leq \|\psi_\lambda - \psi\|_2 \to 0$ as $\lambda \to \infty$. $\square$

Notation 2.34. Let X be a Hilbert $\mathcal{A}$-module, and let X_o be a closed $\mathcal{A}$-submodule of X. Denote $\mathrm{Span}\{\xi \bullet a : \xi \in \mathsf{X}_o, a \in \mathcal{A}\}$ by $\mathsf{X}_o \bullet \mathcal{A}$.

Notation 2.35. Given a Hilbert $\mathcal{A}$-submodule X_o of X, define

$$\langle \mathsf{X}_o \,|\, \mathsf{X}_o \rangle := \mathrm{Span}\{\langle \xi \,|\, \eta \rangle : \xi, \eta \in \mathsf{X}_o\}.$$

Definition 2.36. A Hilbert $\mathcal{A}$-module X is *full* if $\langle \mathsf{X} \,|\, \mathsf{X} \rangle$ is dense in $\mathcal{A}$.

Proposition 2.37. *The Hilbert $\mathcal{A}$-module $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is full.*

Fullness of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ follows from Green's Imprimitivity Theorem stated in Theorem 4.21 of [25]. We will need Green's Imprimitivity Theorem again later, so we will wait until Chapter 5 to give its statement.

Definition 2.38. Given a family $\{\mathsf{X}_j\}_{j \in J}$ of Hilbert $\mathcal{A}$-modules, define

$$\oplus_j \mathsf{X}_j := \left\{ (\xi_j)_{j \in J} : \xi_j \in \mathsf{X}_j \text{ for each } j \in J \text{ and } \sum_{j \in J} \langle \xi_j \,|\, \xi_j \rangle \text{ converges in the norm on } \mathcal{A} \right\}.$$

For $\xi = (\xi_j)_{j \in J}$ and $\eta = (\eta_j)_{j \in J}$ in $\oplus_j \mathsf{X}_j$, define

$$\langle \xi \,|\, \eta \rangle := \sum_{j \in J} \langle \xi_j \,|\, \eta_j \rangle_{\mathsf{X}_j}.$$

It is an exercise in [13] to show that $\oplus_j \mathsf{X}_j$ with this inner product forms a Hilbert $\mathcal{A}$-module.

Proposition 2.39. *Given a family of Hilbert $\mathcal{A}$-modules $\{\mathsf{X}_j\}_{j \in J}$, let $\mathsf{Y} := \oplus_j \mathsf{X}_j$. Then*

$$\mathsf{Y}_o := \{(\xi_j)_{j \in J} \in \mathsf{Y} : \xi_j = 0 \textit{ for all but finitely many } j \in J\}$$

is dense in Y.

Proof. Let $\xi = (\xi_j)_{j\in J} \in \mathsf{Y}$. Then $\sum_{j\in J} \langle \xi_j \,|\, \xi_j \rangle$ converges in $\mathcal{A}$, so in particular, given $\epsilon > 0$, there exists a finite set $F \subseteq J$ such that

$$\left\| \sum_{j\in J\setminus F} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} = \left\| \sum_{j\in J} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} - \sum_{j\in F} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} < \epsilon^2.$$

Define $(\eta_j)_{j\in J} \in \mathsf{Y}_o$ by $\eta_j = \xi_j$ whenever $j \in F$ and $\eta_j = 0$ otherwise. Then

$$\begin{aligned}
\|\xi - \eta\|_{\mathsf{Y}}^2 &= \|\langle \xi - \eta \,|\, \xi - \eta \rangle_{\mathsf{Y}}\|_{\mathcal{A}} \\
&= \left\| \sum_{j\in J} \langle \xi_j - \eta_j \,|\, \xi_j - \eta_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} \\
&= \left\| \sum_{j\in F} \langle \xi_j - \eta_j \,|\, \xi_j - \eta_j \rangle_{\mathsf{X}_j} + \sum_{j\in J\setminus F} \langle \xi_j - \eta_j \,|\, \xi_j - \eta_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} \\
&= \left\| \sum_{j\in F} \langle \xi_j - \xi_j \,|\, \xi_j - \xi_j \rangle_{\mathsf{X}_j} + \sum_{j\in J\setminus F} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} \\
&= \left\| \sum_{j\in J\setminus F} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} \\
&< \epsilon^2.
\end{aligned}$$

Therefore, Y_o is dense in Y. □

2.6 Adjointable Operators on Hilbert C^*-modules

Throughout, X and Y are (right) Hilbert $\mathcal{A}$-modules. A map $T : \mathsf{X} \to \mathsf{Y}$ which satisfies $T(\xi \bullet a) = (T\xi) \bullet a$ for all $\xi \in \mathsf{X}$ and $a \in \mathcal{A}$ is referred to as $\mathcal{A}$-*linear*.

Definition 2.40. A map $T : \mathsf{X} \to \mathsf{Y}$ is *adjointable* if there exists a map $S : \mathsf{Y} \to \mathsf{X}$ such that

$$\langle T\xi \,|\, \eta \rangle_{\mathsf{Y}} = \langle \xi \,|\, S\eta \rangle_{\mathsf{X}} \text{ for all } \xi \in \mathsf{X},\ \eta \in \mathsf{Y}.$$

If T is adjointable, its adjoint is unique and denoted by T^*. Denote the set of all adjointable maps from X to Y by $\mathcal{L}(\mathsf{X}, \mathsf{Y})$, and denote $\mathcal{L}(\mathsf{X}, \mathsf{X})$ by $\mathcal{L}(\mathsf{X})$.

It is well-known that any adjointable operator is both bounded and A-linear. A short proof of this fact is given on page 8 of [13]. Thus, the algebra $\mathcal{L}(\mathsf{X})$ is then closed under the adjoint operation and is complete with respect to the operator norm, so $\mathcal{L}(\mathsf{X})$ is in fact a C^*-algebra.

Definition 2.41. The *strict topology* on $\mathcal{L}(\mathsf{X})$ is the topology induced by the seminorms

$$\{T \mapsto \|T\xi\| : \xi \in \mathsf{X}\} \text{ and } \{T \mapsto \|T^*\eta\| : \eta \in \mathsf{X}\}.$$

Notation 2.42. Given $\xi \in \mathsf{Y}$ and $\eta \in \mathsf{X}$, define $\theta_{\xi,\eta} : \mathsf{X} \to \mathsf{Y}$ by

$$\theta_{\xi,\eta}(\nu) := \xi \bullet \langle \eta \,|\, \nu \rangle_{\mathsf{X}} \text{ for all } \nu \in \mathsf{X}.$$

Then $\theta_{\xi,\eta} \in \mathcal{L}(\mathsf{X}, \mathsf{Y})$. Let $\mathcal{K}(\mathsf{X}, \mathsf{Y})$ denote the closed span of $\{\theta_{\xi,\eta} : \xi \in \mathsf{X}, \eta \in \mathsf{Y}\}$ in $\mathcal{L}(\mathsf{X}, \mathsf{Y})$.

Definition 2.43. Let $\{\mathsf{X}_j\}_{j\in J}$ be a collection of Hilbert $\mathcal{A}$-modules, and let $\mathsf{Y} := \oplus_j \mathsf{X}_j$ be the Hilbert $\mathcal{A}$-module formed in Definition 2.38. Given $T_j \in \mathcal{L}(\mathsf{X}_j)$ for each $j \in J$ such that the family $\{T_j\}_{j\in J}$ satisfies $\sup_{j\in J} \|T_j\| < \infty$, define $\oplus_j T_j : \oplus_j \mathsf{X}_j \to \oplus_j \mathsf{X}_j$ by

$$[\oplus_j T_j](\xi_j)_{j\in J} := (T_j \xi_j)_{j\in J} \text{ for all } (\xi_j)_{j\in J} \in \oplus_j \mathsf{X}_j.$$

Then $\oplus_j T_j$ is a well-defined adjointable operator on $\oplus_j \mathsf{X}_j$ with adjoint $\oplus_j T_j^*$.

2.7 Representations on Hilbert C^*-modules

Definition 2.44. An operator $u \in \mathcal{L}(\mathsf{X})$ is *unitary* if $u^*u = I_{\mathsf{X}} = uu^*$.

Let $\mathcal{U}(\mathsf{X})$ denote the unitary group of $\mathcal{L}(\mathsf{X})$.

Definition 2.45. A *unitary group representation* of G on a Hilbert $\mathcal{A}$-module X is a strictly continuous group homomorphism $u : G \to \mathcal{U}(\mathsf{X})$, which we henceforth denote by $x \mapsto u_x$.

Note that the requirement $u : G \to \mathcal{U}(\mathsf{X})$ be strictly continuous is equivalent to requiring that the maps $x \mapsto u_x\xi$ be continuous for each fixed $\xi \in \mathsf{X}$.

Definition 2.46. Let $u : G \to \mathcal{U}(\mathsf{X})$ be a unitary group representation, and given an arbitrary index set J, let $\oplus_j\mathsf{X} = \oplus_j\mathsf{X}_j$ where $\mathsf{X}_j = \mathsf{X}$ for all $j \in J$. Define

$$\oplus_j u : G \to \mathcal{U}(\oplus_j\mathsf{X}) \quad \text{by} \quad x \mapsto [\oplus_j u]_x := \oplus_j u_x \text{ for each } x \in G,$$

where $\oplus_j u_x$ is as in Definition 2.43. Then $\oplus_j u$ defines a unitary group representation of G.

Definition 2.47. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and let X be a Hilbert $\mathcal{B}$-module. A representation $\pi : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ is *nondegenerate* if $\pi(\mathcal{A})\mathsf{X}$ is dense in X.

Definition 2.48. Let X be a Hilbert $\mathcal{B}$-module and suppose $\pi : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ is a nondegenerate $*$-representation. Let $\mathsf{Y} = \oplus_j\mathsf{X}$, and define

$$\oplus_j\pi : \mathcal{A} \to \mathcal{L}(\mathsf{Y})$$

by $[\oplus_j\pi](a) := \oplus_j\pi(a)$ for each $a \in \mathcal{A}$, as in Definition 2.43. If Y_o denotes the dense $\mathcal{B}$-submodule of Y defined in Proposition 2.39, nondegeneracy of $\oplus_j\pi$ is easily established by showing $\mathrm{Span}\{[\oplus_j\pi(a)]\xi : a \in \mathcal{A},\ \xi \in \mathsf{Y}_o\}$ approximates elements of Y_o.

Definition 2.49. Let $(G, \mathcal{A}, \alpha)$ be a C^*-dynamical system, let $\mathcal{B}$ be a C^*-algebra, and let X be a Hilbert $\mathcal{B}$-module. A *covariant homomorphism* of $(G, \mathcal{A}, \alpha)$ into $\mathcal{L}(\mathsf{X})$ is a pair (π, u) consisting of homomorphisms $\pi : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ and a unitary group representation $u : G \to \mathcal{U}(\mathsf{X})$ such that

$$\pi(\alpha_x(a)) = u_x \pi(a) u_x^* \text{ for all } x \in G,\ a \in \mathcal{A}.$$

We say (π, u) is *nondegenerate* if π is nondegenerate.

Proposition 2.50 (Williams, 2.39 [25])**.** *Let X be a Hilbert $\mathcal{B}$-module and let (π, u) be a covariant homomorphism of $(G, \mathcal{A}, \alpha)$ into $\mathcal{L}(\mathsf{X})$. Consider the integrated form $\pi \rtimes u : C_c(G, \mathcal{A}) \to \mathcal{L}(\mathsf{X})$ defined by*

$$[\pi \rtimes u](f) := \int_G \pi(f(x)) u_x \, d\mu(x) \text{ for all } f \in C_c(G, \mathcal{A}),$$

where this integral is the image of the function $x \mapsto \pi(f(x))u_x$ under the linear map described in Lemma 1.91 of [25]. Each $[\pi \rtimes u](f)$ is a well-defined operator in $\mathcal{L}(\mathsf{X})$, and $\pi \rtimes u$ extends to a homomorphism of $\mathcal{A} \rtimes_\alpha G$ which is nondegenerate whenever π is nondegenerate. We denote this extension by $\pi \rtimes u$.

Conversely, if $L : \mathcal{A} \rtimes_\alpha G \to \mathcal{L}(\mathsf{X})$ is a nondegenerate homomorphism, then there is a unique nondegenerate covariant homomorphism (π, u) of $(G, \mathcal{A}, \alpha)$ into $\mathcal{L}(\mathsf{X})$ such that $L = \pi \rtimes u$.

We can further characterize integrals involving continuous compactly supported functions from a locally compact group G into a C^*-algebra $\mathcal{A}$ by the following lemma.

Lemma 2.51 (Raeburn-Williams, C.12 [18])**.** *Let X be a Hilbert $\mathcal{A}$-module and F a compactly supported function of G into $\mathcal{L}(\mathsf{X})$ which is continuous for the strict topology. Then for each*

$\xi, \eta \in \mathsf{X}$, *the map* $x \mapsto \langle \xi \,|\, F(x)\eta \rangle$ *belongs to* $C_c(G, \mathcal{A})$ *and*

$$\left\langle \xi \,\middle|\, \left(\int_G F(x)\, d\mu(x) \right) \eta \right\rangle = \int_G \langle \xi \,|\, F(x)\eta \rangle \; d\mu(x).$$

Proposition 2.52. *Suppose* (π, u) *is a covariant homomorphism for* $(G, \mathcal{A}, \alpha)$ *into* $\mathcal{L}(\mathsf{X})$ *for some Hilbert* $\mathcal{B}$*-module* X*. Then* $(\oplus_j \pi, \oplus_j u)$ *is a covariant homomorphism for* $(G, \mathcal{A}, \alpha)$ *into* $\mathcal{L}(\oplus_j \mathsf{X})$*, and*

$$(\oplus_j \pi) \rtimes (\oplus_j u) = \oplus_j (\pi \rtimes u).$$

Proof. Covariance of $(\oplus_j \pi, \oplus_j u)$ is straightforward to check. Let $\mathsf{Y} := \oplus_j \mathsf{X}$, and recall from Proposition 2.39 that

$$\mathsf{Y}_o := \{(\xi_j)_{j \in J} \in \mathsf{Y} : \xi_j = 0 \text{ for all but finitely many } j \in J\}$$

is dense in Y. We claim

$$[(\oplus_j \pi) \rtimes (\oplus_j u)](f)|_{\mathsf{Y}_o} = [\oplus_j (\pi \rtimes u)](f)|_{\mathsf{Y}_o} \text{ for all } f \in C_c(G, \mathcal{A}).$$

Fix $f \in C_c(G, \mathcal{A})$, and observe that

$$\begin{aligned} [(\oplus_j \pi) \rtimes (\oplus_j u)]\,(f) &= \int_G [\oplus_j \pi](f(x))[\oplus_j u]_x \, d\mu(x) \\ &= \int_G [\oplus_j \pi(f(x))]\, [\oplus_j u_x] \; d\mu(x) \\ &= \int_G [\oplus_j \pi(f(x)) u_x] \; d\mu(x). \end{aligned}$$

For each $x \in G$, define $F(x) := \oplus_j \,[\pi(f(x)) u_x]$. The maps $x \mapsto [\oplus_j \pi(f(x))]|_{\mathsf{Y}_o}$ and $x \mapsto [\oplus_j u_x]|_{\mathsf{Y}_o}$ from G into $\mathcal{L}(\mathsf{Y})$ are strictly continuous, and density of Y_o in Y establishes strict continuity of $x \mapsto [\oplus_j \pi(f(x))] \circ [\oplus_j u_x]$. Therefore, $F : G \to \mathcal{L}(\mathsf{Y})$ is strictly continuous

Let $\eta \in \mathsf{Y}_o$, and let $\operatorname{Supp}(\eta) \subseteq J$ be the finite subset such that $\eta_j = 0$ for all $j \notin \operatorname{Supp}(\eta)$. Then, for any $\xi \in \mathsf{Y}$,

$$\begin{aligned}
\left\langle \xi \,\middle|\, \left(\int_G [\oplus_j \pi(f(x)) u_x] \, d\mu(x) \right) \eta \right\rangle_{\mathsf{Y}} &= \left\langle \xi \,\middle|\, \left(\int_G F(x) \, d\mu(x) \right) \eta \right\rangle_{\mathsf{Y}} \\
&= \int_G \langle \xi \,|\, F(x)\eta \rangle_{\mathsf{Y}} \, d\mu(x) \qquad [\text{ Lemma 2.51 }] \\
&= \int_G \left(\sum_{j \in J} \langle \xi_j \,|\, [\pi(f(x))u_x]\eta_j \rangle_{\mathsf{X}} \right) d\mu(x) \\
&= \int_G \left(\sum_{j \in \operatorname{Supp}(\eta)} \langle \xi_j \,|\, [\pi(f(x))u_x]\eta_j \rangle_{\mathsf{X}} \right) d\mu(x) \\
&= \sum_{j \in \operatorname{Supp}(\eta)} \int_G \langle \xi_j \,|\, [\pi(f(x))u_x]\eta_j \rangle_{\mathsf{X}} \, d\mu(x) \\
&= \sum_{j \in \operatorname{Supp}(\eta)} \left\langle \xi_j \,\middle|\, \left(\int_G \pi(f(x)) u_x \, d\mu(x) \right) \eta_j \right\rangle_{\mathsf{X}} \quad [\text{ Lemma 2.51 }] \\
&= \sum_{j \in J} \langle \xi_j \,|\, [\pi \rtimes u](f) \, \eta_j \rangle_{\mathsf{X}} \\
&= \langle \xi \,|\, [\oplus_j (\pi \rtimes u)(f)] \eta \rangle_{\mathsf{Y}} \,.
\end{aligned}$$

As $\xi \in \mathsf{Y}$ was arbitrary, we have that $[(\oplus_j \pi) \rtimes (\oplus_j u)](f)\eta = [\oplus_j (\pi \rtimes u)(f)]\eta$ for all $\eta \in \mathsf{Y}_o$. By density of Y_o in Y and continuity of both $[(\oplus_j \pi) \rtimes (\oplus_j u)](f)$ and $\oplus_j [\pi \rtimes u](f)$, we have $[(\oplus_j \pi) \rtimes (\oplus_j u)](f) = \oplus_j [\pi \rtimes u](f)$ as adjointable operators on $\mathcal{L}(\mathsf{Y})$. As $f \in C_c(G, \mathcal{A})$ was arbitrary and $C_c(G, \mathcal{A})$ is dense in $\mathcal{A} \rtimes_\alpha G$, we conclude $\oplus_j [\pi \rtimes u] = (\oplus_j \pi) \rtimes (\oplus_j u)$. $\square$

2.8 Hilbert $\mathcal{K}(\mathcal{H})$-modules

A substantial portion of the collaboration with L. Huang is in the setting of $\mathcal{A} = \mathcal{K}(\mathcal{H})$, the $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ obtained by closing the finite-rank operators on $\mathcal{H}$ in the norm topology. The following results are used later in the paper and provide some evidence of why

$\mathcal{K}(\mathcal{H})$ was desirable to work with. As a first attractive property, recall that $\mathcal{K}(\mathcal{H})$ is simple, so every nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module X is full since $\langle \mathsf{X} \,|\, \mathsf{X} \rangle$ forms a nontrivial two-sided ideal in $\mathcal{K}(\mathcal{H})$.

Lemma 2.53 (Arveson, 1.4.1 [2])**.** *Let p be a nonzero projection in $\mathcal{K}(\mathcal{H})$. Then p is rank-one if and only if $p\mathcal{K}(\mathcal{H})p = \mathbb{C}p$.*

Corollary 2.54. *If $p \in \mathcal{K}(\mathcal{H})$ is a rank-one projection, there is a linear functional $f_p : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ such that $pap = f_p(a)p$ for all $a \in \mathcal{K}(\mathcal{H})$.*

Corollary 2.55. *Let X be a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module, and let p be a rank-one projection in $\mathcal{K}(\mathcal{H})$. Then there exists $\psi \in \mathsf{X}$ such that $\langle \psi \,|\, \psi \rangle = p$.*

Proof. Let $p \in \mathcal{K}(\mathcal{H})$ be a rank-one projection. Then there exists $\psi_o \in \mathsf{X}$ such that $\psi_o \bullet p \neq 0$ (since $\mathsf{X} \bullet p$ is a full Hilbert $\mathcal{K}(\mathcal{H})$-module). Thus,

$$0 \neq \langle \psi_o \bullet p \,|\, \psi_o \bullet p \rangle = p \, \langle \psi_o \,|\, \psi_o \rangle \, p = f_p \left(\langle \psi_o \,|\, \psi_o \rangle \right) p,$$

where f_p is the linear functional corresponding to p obtained in Corollary 2.54. Let $\lambda := f_p \left(\langle \psi_o \,|\, \psi_o \rangle \right)$, and define $\psi := \lambda^{-1/2}(\psi_o \bullet p)$. Then $\langle \psi \,|\, \psi \rangle = p$. □

Lemma 2.56. *Let X be a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module and p a rank-one projection on $\mathcal{H}$. Then $\mathsf{X} \bullet p$ is a nontrivial closed subspace of X that is also a Hilbert space with inner product*

$$\langle \xi \bullet p \,|\, \eta \bullet p \rangle_{\mathsf{X} \bullet p} = f_p(\langle \xi \,|\, \eta \rangle_{\mathsf{X}}) \text{ for every } \xi, \eta \in \mathsf{X},$$

where f_p is the linear functional related to p in Corollary 2.54. Furthermore, the norm on $\mathsf{X} \bullet p$ induced by $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X} \bullet p}$ coincides with the restriction of $\|\cdot\|_{\mathsf{X}}$ to $\mathsf{X} \bullet p$.

Proof. (Huang) It is obvious that $\mathsf{X} \bullet p$ is a subspace of X. To see that it is closed in X, let $(\zeta_n)_{n \in \mathbb{N}}$ be a sequence in $\mathsf{X} \bullet p$ such that $(\zeta_n)_{n \in \mathbb{N}}$ converges to some $\eta \in \mathsf{X}$. Because $\zeta_n \bullet p = \zeta_n$

for all $n \in \mathbb{N}$, we have

$$\eta = \lim_{n \to \infty} \zeta_n = \lim_{n \to \infty} \zeta_n \bullet p = \left[\lim_{n \to \infty} \zeta_n \right] \bullet p = \eta \bullet p.$$

Hence, $\eta \in \mathsf{X} \bullet p$, which proves that $\mathsf{X} \bullet p$ is a closed subspace of X.

Clearly, $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X} \bullet p}$ is a sesquilinear form on $\mathsf{X} \bullet p$, so it remains to prove that it is positive definite and complete. Let $\zeta \in \mathsf{X} \bullet p$. Then $\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}}$ is positive in $\mathcal{K}(\mathcal{H})$, which means that

$$f_p \left(\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}} \right) p = p \langle \zeta \,|\, \zeta \rangle_{\mathsf{X}} \, p = p \langle \zeta \,|\, \zeta \rangle_{\mathsf{X}} \, p^*$$

is positive in $\mathcal{K}(\mathcal{H})$ as well. As $p(I - p) = 0$, we deduce that $I - p$ is not invertible in $\mathcal{K}(\mathcal{H})$, so $1 \in \sigma_{\mathcal{K}(\mathcal{H})}(p)$. Hence, $f_p \left(\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}} \right) \in \sigma_{\mathcal{K}(\mathcal{H})} \left(f_p \left(\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}} \right) p \right) \subseteq \mathbb{R}_{\geq 0}$, which shows that $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X} \bullet p}$ is at least positive semidefinite. Next, observe that

$$\begin{aligned}
\left| \langle \zeta \,|\, \eta \rangle_{\mathsf{X} \bullet p} \right| &= \left| f_p \left(\langle \zeta \,|\, \eta \rangle_{\mathsf{X}} \right) \right| \\
&= \left\| f_p \left(\langle \zeta \,|\, \eta \rangle_{\mathsf{X}} \right) p \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| p \langle \zeta \,|\, \eta \rangle_{\mathsf{X}} \, p \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \langle \zeta \bullet p \,|\, \eta \bullet p \rangle_{\mathsf{X}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \langle \zeta \,|\, \eta \rangle_{\mathsf{X}} \right\|_{\mathcal{K}(\mathcal{H})}. \qquad [\text{As } \zeta \bullet p = \zeta \text{ and } \eta \bullet p = \eta.]
\end{aligned}$$

Consequently, if $\langle \zeta \,|\, \zeta \rangle_{\mathsf{X} \bullet p} = 0$ for some $\zeta \in \mathsf{X} \bullet p$, then $\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}} = 0$, which yields $\zeta = 0$. This proves that $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X} \bullet p}$ is positive definite. Incidentally, this also proves that $\|\zeta\|_{\mathsf{X} \bullet p} = \|\zeta\|_{\mathsf{X}}$ for all $\zeta \in \mathsf{X} \bullet p$. As $\mathsf{X} \bullet p$ is a closed subspace of X, it is a Banach space with respect to the restriction of $\|\cdot\|_{\mathsf{X}}$ to $\mathsf{X} \bullet p$, and is thus a Banach space with respect to $\|\cdot\|_{\mathsf{X} \bullet p}$. Therefore, $\mathsf{X} \bullet p$ is a Hilbert space whose inner product is given by $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X} \bullet p}$, and the induced norm on $\mathsf{X} \bullet p$ is the restriction of $\|\cdot\|_{\mathsf{X}}$ to $\mathsf{X} \bullet p$. $\square$

Theorem 2.57 (Bakić-Guljaš, 5 & 6 [8]). *Given a rank-one projection $p \in \mathcal{K}(\mathcal{H})$, the maps*

$$\Psi : \mathcal{L}(\mathsf{X}) \to \mathcal{B}(\mathsf{X} \bullet p) \quad \text{and} \quad \Psi|_{\mathcal{K}(\mathsf{X})} : \mathcal{K}(\mathsf{X}) \to \mathcal{K}(\mathsf{X} \bullet p)$$

given by $T \mapsto T|_{\mathsf{X} \bullet p}$ are C^*-isomorphisms.

Theorem 2.58 (Magajna, 1 [14]). *Every Hilbert $\mathcal{K}(\mathcal{H})$-module X is complementable, that is, every closed $\mathcal{K}(\mathcal{H})$-submodule $\mathsf{Y} \subseteq \mathsf{X}$ has an orthogonal complement $\mathsf{Y}^\perp$ such that $\mathsf{X} = \mathsf{Y} \oplus \mathsf{Y}^\perp$.*

Proposition 2.59. *Let X be a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module, let Y be a nonzero $\mathcal{K}(\mathcal{H})$-submodule of X that is not necessarily closed, and let p be a rank-one projection on $\mathcal{H}$. Then*

$$\overline{(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})} = \overline{\mathsf{Y}}.$$

Proof. As Y is a $\mathcal{K}(\mathcal{H})$-submodule of X, we have that $\mathsf{Y} \bullet p \subseteq \mathsf{Y}$, and thus, $(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})$ is contained in Y. Hence, $\overline{(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})}$ is contained in $\overline{\mathsf{Y}}$. It thus remains to establish the reverse containment.

Note that $\{pa : a \in \mathcal{K}(\mathcal{H}) \setminus \{0\}\}$ is the set of all rank-one projections on $\mathcal{H}$. Let $\zeta \in \mathsf{Y}$ and let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for $\mathcal{K}(\mathcal{H})$. Then $\|\zeta \bullet e_\lambda - \zeta\| \to 0$ as $\lambda \to \infty$. Moreover, $\mathrm{Span}\{pa : a \in \mathcal{K}(\mathcal{H})\}$ contains all finite-rank operators on $\mathcal{H}$, so $(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})$ can approximate $\zeta \bullet e_\lambda$ for any choice of $\lambda \in \Lambda$. An $\frac{\epsilon}{2}$-argument shows that the closure of $(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})$ in X equals the closure of Y. $\square$

Chapter 3

Analytic Vectors for δ_D

3.1 Definition of Weak D-Differentiability

Throughout, $\mathcal{H}$ is a Hilbert space and D is a (possibly unbounded) self-adjoint operator on $\mathcal{H}$. For each $t \in \mathbb{R}$, both Stone's Theorem and the Spectral Theorem for Self-Adjoint Operators yields a strongly-continuous one-parameter group of unitaries $\{e^{itD}\}_{t\in\mathbb{R}}$. For each $t \in \mathbb{R}$, define a map $\alpha_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\alpha_t(x) := e^{itD}xe^{-itD} \text{ for all } x \in \mathcal{B}(\mathcal{H}).$$

Then $\{\alpha_t\}_{t\in\mathbb{R}}$ defines a flow on $\mathcal{B}(\mathcal{H})$ and forms group of $*$-automorphisms on $\mathcal{B}(\mathcal{H})$.

Definition 3.1. An operator $x \in \mathcal{B}(\mathcal{H})$ is *weakly D-differentiable* if there exists $y \in \mathcal{B}(\mathcal{H})$ such that

$$\lim_{t\to 0}\left|\left\langle\left(\frac{\alpha_t(x)-x}{t}-y\right)h,k\right\rangle\right| = 0 \text{ for all } h,k \in \mathcal{H}. \qquad (*)$$

Denote the set of all weakly D-differentiable operators by $\mathrm{Dom}(\delta_D)$, and for $x \in \mathrm{Dom}(\delta_D)$, let $\delta_D(x) := y$, where y satisfies condition $(*)$.

Theorem 3.2 (Christensen, 3.8 [6]). *Let $x \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(i) x is weakly D-differentiable.

(ii) There exists $y \in \mathcal{B}(\mathcal{H})$ *such that for every* $h \in \mathcal{H}$*,*

$$\lim_{t \to 0} \left\| \left(\frac{\alpha_t(x) - x}{t} - y \right) h \right\| = 0.$$

(iii) There exists $c > 0$ *such that* $\|\alpha_t(x) - x\| \leq c\,|t|$ *for all* $t \in \mathbb{R}$*.*

(iv) The commutator $[iD, x]$ *is defined and bounded on the domain of* D*.*

(v) The commutator $[iD, x]$ *is defined and bounded on a core for* D*.*

If any of the above conditions hold, then $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ *and* $\delta_D(x)|_{\mathrm{Dom}(D)} = [iD, x]$*.*

Theorem 3.3 (Christensen, 3.9 [6])**.** *The domain of definition* $\mathrm{Dom}(\delta_D)$ *is a SOT-dense* $*$*-subalgebra of* $\mathcal{B}(\mathcal{H})$ *and* δ_D *is a* $*$*-derivation into* $\mathcal{B}(\mathcal{H})$*. The graph of* δ_D *is WOT-closed.*

Theorem 3.3 supports Christensen's argument in [6] for considering differentiability of $x \in \mathcal{B}(\mathcal{H})$ in the weak operator topology as opposed to the norm topology on $\mathcal{B}(\mathcal{H})$. In a subsequent paper, [5], Christensen defines higher weak D-differentiability via higher powers of δ_D.

Definition 3.4. An operator $x \in \mathcal{B}(\mathcal{H})$ is *n-times weakly D-differentiable* if $x \in \mathrm{Dom}(\delta_D^n)$.

Proposition 3.5 (Christensen, 2.6 [5])**.** *An operator* $x \in \mathcal{B}(\mathcal{H})$ *is* n*-times weakly* D*-differentiable if and only if for each pair* $h, k \in \mathcal{H}$*, the function* $t \mapsto \langle \alpha_t(x)h, k \rangle$ *is* n*-times continuously differentiable. Moreover, if* x *is* n*-times weakly* D*-differentiable, then*

$$\frac{d^n}{dt^n} \langle \alpha_t(x)h, k \rangle = \langle \alpha_t[\delta_D^n(x)]h, k \rangle .$$

Analogous to Theorem 3.2, the following proposition and theorem connect higher-order weak D-differentiability of $x \in \mathcal{B}(\mathcal{H})$ to definedness and boundedness of iterated commutators $[iD, ..., [iD, x]]$.

Proposition 3.6 (Christensen, 3.3 [5])**.** *Let $x \in \mathrm{Dom}(\delta_D^n)$. Then for $k = 1, ..., n$,*

(i) $\delta_D^{k-1}(x)(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$

(ii) $x(\mathrm{Dom}\big(D^k\big)) \subseteq \mathrm{Dom}\big(D^k\big)$

(iii) $\mathrm{Dom}\left(\underbrace{[iD, ..., [iD, x]]}_{k\ times}\right) = \mathrm{Dom}\big(D^k\big)$

(iv) $\delta_D^k(x)|_{\mathrm{Dom}\left(D^k\right)} = \underbrace{[iD, ..., [iD, x]]}_{k\ times}$

(v) $\delta_D^k(x)$ *is the bounded extension of* $\underbrace{[iD, ..., [iD, x]]}_{k\ times}$ *from* $\mathrm{Dom}\big(D^k\big)$ *to all of* $\mathcal{H}$*.*

Theorem 3.7 (Christensen, 4.1 [5])**.** *Let $x \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(i) *x is n times weakly D-differentiable.*

(ii) *For all $k = 1, ..., n$, $x(\mathrm{Dom}\big(D^k\big)) \subseteq \mathrm{Dom}\big(D^k\big)$ and $\underbrace{[iD, ..., [iD, x]]}_{k\ times}$ is defined and bounded on $\mathrm{Dom}\big(D^k\big)$ with bounded extension $\delta_D^k(x)$.*

(iii) *There exists a core $\mathscr{C}$ for D such that for any $k = 1, ..., n$, the operator $\underbrace{[iD, ..., [iD, x]]}_{k\ times}$ is defined and bounded on $\mathscr{C}$.*

Notation 3.8. For notational convenience, for each $k \in \mathbb{N}$ we define

$$d^k(x) := \underbrace{[iD, ..., [iD, x]]}_{k \text{ times}}.$$

3.2 Weakly $-i\frac{d}{d\theta}$-Differentiable Multiplication Operators on $L^2(\mathbb{T})$

Consider the operator $D = -i\frac{d}{d\theta}$ on $L^2(\mathbb{T})$ with domain

$$\mathrm{Dom}(D) = \left\{ f \in L^2(\mathbb{T}) : f \text{ is absolutely continuous}, f' \in L^2(\mathbb{T}) \right\}.$$

Notation 3.9. Given a σ-finite measure space (X, μ), define

$$\begin{array}{rccc} \mathrm{diag}: & L^\infty(X,\mu) & \to & \mathcal{B}(L^2(X,\mu)) \\ & f & \mapsto & M_f \end{array}$$

where $M_f g = fg$ for each $g \in L^2(X, \mu)$.

Proposition 3.11 characterizes the n-times weakly D-differentiable multiplication operators $M_f \in \mathrm{diag}(L^\infty(\mathbb{T}))$, and Proposition 3.10 provides as the case when $n = 1$.

Proposition 3.10. *Let $f \in L^\infty(\mathbb{T})$. The following statements are equivalent:*

(i) M_f is weakly D-differentiable.

(ii) $f \in \mathrm{Dom}(D)$ and $Df \in L^\infty(\mathbb{T})$.

When either condition is satisfied, $\delta_w^D(M_f) = M_{f'}$.

Proof. ($\Rightarrow$) If $M_f \in \mathrm{Dom}(\delta_D)$, then $M_f(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ by Theorem 3.2. Let $\mathbf{1}$ denote the function which takes the value 1 for all $z \in \mathbb{T}$. Then $\mathbf{1}$ is in $\mathrm{Dom}(D)$, and so $f = M_f\mathbf{1} \in \mathrm{Dom}(D)$. In [6], Christensen remarks that in this particular setting, condition (iii) of Theorem 3.2 holds if and only if there exists $c > 0$ such that for all $z \in \mathbb{T}$ and $t \in \mathbb{R}$,

$$\left| f(ze^{it}) - f(z) \right| \leq c\,|t|\,.$$

As $f \in \mathrm{Dom}(D)$, f is absolutely continuous and thus differentiable a.e. Hence, for a.e. $z \in \mathbb{T}$,

$$|f'(z)| = \lim_{t \to 0} \left| \frac{f(ze^{it}) - f(z)}{t} \right| \leq c.$$

Therefore, $\|f'\|_\infty \leq c$, so $f' \in L^\infty(\mathbb{T})$. Hence, $Df = -if' \in L^\infty(\mathbb{T})$.

($\Leftarrow$): Suppose $f \in \mathrm{Dom}(D)$ and $Df \in L^\infty(\mathbb{T})$. We show $M_f(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ and $[iD, M_f]$ agrees with the bounded operator $M_{f'}$ on $\mathrm{Dom}(D)$. Fix $g \in \mathrm{Dom}(D)$. Then $g' \in L^2(\mathbb{T})$, so

$$\|(fg)'\|_2 = \|fg' + f'g\|_2 \leq \|fg'\|_2 + \|f'g\|_2 \leq \|f\|_\infty \|g'\|_2 + \|f'\|_\infty \|g\|_2 < \infty.$$

Also, the product of two absolutely continuous functions is absolutely continuous. Therefore, $fg \in \mathrm{Dom}(D)$. As $g \in \mathrm{Dom}(D)$ was arbitrary, we have $M_f(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$. Observe

$$[iD, M_f]g = (fg)' - fg' = f'g + fg' - fg' = f'g = M_{f'}g \text{ for all } g \in \mathrm{Dom}(D).$$

As $f' \in L^\infty(\mathbb{T})$ and $[iD, M_f]|_{\mathrm{Dom}(D)} = M_{f'} \in \mathcal{B}(L^2(\mathbb{T}))$, we have that $[iD, M_f]$ is defined and bounded on $\mathrm{Dom}(D)$. By (i) $\iff$ (iv) of Theorem 3.2, we conclude $M_f \in \mathrm{Dom}(\delta_D)$ and $\delta_D(M_f) = M_{f'}$. □

Proposition 3.11. *Let $f \in L^\infty(\mathbb{T})$. The following statements are equivalent:*

(i) M_f is n-times weakly D-differentiable.

(ii) $f \in \mathrm{Dom}(D^n)$ and $D^n f \in L^\infty(\mathbb{T})$.

When either condition is satisfied, $\delta_D^n(M_f) = M_{f^{(n)}}$.

Proof. Fix $n \in \mathbb{N}$. We proceed by induction. The base case was established in Proposition 3.10.

$(\Rightarrow)$: Suppose for all $k \leq n-1$, if $M_f \in \mathrm{Dom}\big(\delta_D^k\big)$ then $f \in \mathrm{Dom}\big(D^k\big)$ and $D^k f \in L^\infty(\mathbb{T})$. Let $M_f \in \mathrm{Dom}(\delta_D^n)$, so $M_f \in \mathrm{Dom}\big(\delta_D^k\big)$ for each $k \leq n$. The inductive hypothesis implies $f \in \mathrm{Dom}\big(D^k\big)$ and $D^k f \in L^\infty(\mathbb{T})$ for each $k \leq n-1$.

As in the proof of Proposition 3.10, let $\mathbf{1}$ the function which takes the value 1 for all $z \in \mathbb{T}$. By Proposition 3.6 (ii), $M_f(\mathrm{Dom}(D^n)) \subseteq \mathrm{Dom}(D^n)$, and so $f = M_f\mathbf{1} \in \mathrm{Dom}(D^n)$. To see that $D^n f \in L^\infty(\mathbb{T})$, note $M_f \in \mathrm{Dom}(\delta_D^n)$ implies $\delta_D^{n-1}(M_f) \in \mathrm{Dom}(\delta_D)$. By the inductive hypothesis,

$$\delta_D^{n-1}(M_f) = M_{f^{(n-1)}}.$$

By (i) $\iff$ (iii) of Theorem 3.2, $M_{f^{(n-1)}} \in \mathrm{Dom}(\delta_D)$ if and only if there exists $c > 0$ such that for all $z \in \mathbb{T}$ and $t \in \mathbb{R}$,

$$\left| f^{(n-1)}(ze^{it}) - f^{(n-1)}(z) \right| \leq c\,|t|\,.$$

Now, $f \in \mathrm{Dom}(D^n)$ by definition means $D^{n-1}f \in \mathrm{Dom}(D)$, which is equivalent to $f^{(n-1)} \in \mathrm{Dom}(D)$. In particular, $f^{(n-1)}$ is differentiable a.e., and thus, for almost every $z \in \mathbb{T}$, we have

$$\left| f^{(n)}(z) \right| = \lim_{t \to 0} \left| \frac{f^{(n-1)}(ze^{it}) - f^{(n-1)}(z)}{t} \right| \leq c.$$

Therefore, $\left\| f^{(n)} \right\|_\infty \leq c$, and hence, $f^{(n)} \in L^\infty(\mathbb{T})$. Given $D^n f = (-i)^n f^{(n)}$, we have shown $D^n f \in L^\infty(\mathbb{T})$.

$(\Leftarrow)$: Let $f \in \mathrm{Dom}(D^n)$ and suppose $D^n f \in L^\infty(\mathbb{T})$. Further, suppose for all $k \leq n-1$, if $f \in \mathrm{Dom}\big(D^k\big)$ and $D^k f \in L^\infty(\mathbb{T})$, then $M_f \in \mathrm{Dom}\big(\delta_D^k\big)$. To prove $M_f \in \mathrm{Dom}(\delta_D^n)$, by Theorem 3.7, it suffices to prove $M_f(\mathrm{Dom}(D^n)) \subseteq \mathrm{Dom}(D^n)$ and the commutator

$$d^n(M_f) = \underbrace{[iD, ..., [iD, M_f]]}_{n \text{ times}}$$

is bounded on $\mathrm{Dom}(D^n)$. Given $g \in \mathrm{Dom}(D^n)$, showing $M_f g \in \mathrm{Dom}(D^n)$ amounts to proving

(i) $fg \in \mathrm{Dom}(D^{n-1})$,

(ii) $D^{n-1}(fg)$ is absolutely continuous, and

(iii) $(D^{n-1}(fg))' \in L^2(\mathbb{T})$.

Since $M_f \in \mathrm{Dom}\big(\delta_D^{n-1}\big)$, Proposition 3.6 implies $M_f(\mathrm{Dom}(D^{n-1})) \subseteq \mathrm{Dom}(D^{n-1})$. Hence, $g \in \mathrm{Dom}(D^n) \subseteq \mathrm{Dom}(D^{n-1})$ implies $M_f g = fg \in \mathrm{Dom}(D^{n-1})$. Now,

$$D^{n-1}(fg) = (-i)^{n-1} \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(n-1-j)} g^{(j)}.$$

Each term of the above sum is the product of absolutely continuous functions because $D^{n-1-j} f \in \mathrm{Dom}(D)$ and $D^j g \in \mathrm{Dom}(D)$ for all $j = 0, ..., n-1$. The product of any two absolutely continuous functions on a bounded interval is again absolutely continuous, and thus the entire sum is as well. Therefore, (ii) is satisfied. Also,

$$\left\|(D^{n-1}(fg))'\right\|_2 = \|D^n(fg)\|_2 \le \sum_{j=0}^{n} \binom{n}{j} \left\|f^{(n-j)} g^{(j)}\right\|_2 \le \sum_{j=0}^{n} \binom{n}{j} \left\|f^{(n-j)}\right\|_\infty \left\|g^{(j)}\right\|_2.$$

As $\|f^{n-j}\|_\infty = \left\|D^{(n-j)} f\right\|_\infty < \infty$ and $g \in \mathrm{Dom}(D^n)$ ensures $g^{(j)} \in L^2(\mathbb{T})$ for all $j = 0, ..., n$, we conclude that $\|(D^{n-1}(fg))'\|_2 < \infty$. Therefore, $M_f(\mathrm{Dom}(D^n)) \subseteq \mathrm{Dom}(D^n)$.

Having established that $d^n(M_f)$ is defined on $\mathrm{Dom}(D^n)$, we now show $d^n(M_f)$ is bounded on $\mathrm{Dom}(D^n)$. In Proposition 3.10 we observed $[iD, M_f]|_{\mathrm{Dom}(D)} = M_{f'}$. Since $f' \in L^\infty(\mathbb{T})$, we concluded $\delta_D(M_f) = M_{f'}$. Following this same argument, we have $d^k(M_f) = M_{f^{(k)}}|_{\mathrm{Dom}\left(D^k\right)}$, so $\delta_D^k(M_f) = M_{f^{(k)}}$ for all $k \le n-1$. As $\mathrm{Dom}(D^n) \subseteq \mathrm{Dom}(D^{n-1})$,

$$d^n(M_f)|_{\mathrm{Dom}(D^n)} = d(d^{n-1}(M_f))|_{\mathrm{Dom}(D^n)} = d(M_{f^{(n-1)}})|_{\mathrm{Dom}(D^n)} = [iD, M_{f^{(n-1)}}]|_{\mathrm{Dom}(D^n)}.$$

Furthermore, $[iD, M_{f^{(n-1)}}]|_{\mathrm{Dom}(D^n)} = M_{f^{(n)}}$. By assumption, $D^n f \in L^\infty(\mathbb{T})$, which is equivalent to $f^{(n)} \in L^\infty(\mathbb{T})$. Therefore, the commutator $d^n(M_f)$ agrees with the bounded operrator $M_{f^{(n)}}$ on $\mathrm{Dom}(D^n)$, which establishes by Theorem 3.7 that $M_f \in \mathrm{Dom}(\delta_D^n)$ and $\delta_D^n(M_f) = M_{f^{(n)}}$. □

3.3 Domains of Higher Powers

Throughout this section, D denotes an arbitrary self-adjoint operator on a Hilbert space $\mathcal{H}$. While Theorem 3.7 extends Theorem 3.2 by connecting n-times weak D-differentiability of a bounded operator x to definedness and boundedness of an iterated commutator of x with iD, there is no analogous theorem to Theorem 3.3 stating that $\mathrm{Dom}(\delta_D^n)$ remains SOT-dense in $\mathcal{B}(\mathcal{H})$. The purpose of this section is to give a constructive proof of SOT-density of $\mathrm{Dom}(\delta_D^n)$ for all $n \in \mathbb{N}$.

Given $f, g \in \mathcal{H}$, recall the rank-one operator $f \otimes g^* : \mathcal{H} \to \mathcal{H}$ is defined as

$$(f \otimes g^*)(v) := \langle v, g \rangle\, f \text{ for all } v \in \mathcal{H}.$$

Fix $n \in \mathbb{N}$. We use the facts that $\mathrm{Span}\{f \otimes g^* : f, g \in \mathcal{H}\}$ is norm-dense in $\mathcal{K}(\mathcal{H})$ and that $\mathcal{K}(\mathcal{H})$ is SOT-dense in $\mathcal{B}(\mathcal{H})$ to prove $\mathrm{Dom}(\delta_D^n)$ is SOT-dense in $\mathcal{B}(\mathcal{H})$.

Lemma 3.12. *Let $n \in \mathbb{N}$. If $h, k \in \mathrm{Dom}(D^n)$, then $h \otimes k^* \in \mathrm{Dom}(\delta_D^n)$ and*

$$\delta_D^n(h \otimes k) = \sum_{j=0}^{n} (iD)^{n-j} h \otimes [(iD)^j k]^*.$$

Proof. Let $h, k \in \mathrm{Dom}(D^n)$. First, observe that for all $f, g \in \mathcal{H}$,

$$\begin{aligned}\langle \alpha_t(h \otimes k^*)f, g\rangle &= \left\langle e^{itD}(h \otimes k^*)e^{-itD}f, g\right\rangle \\ &= \left\langle (h \otimes k)e^{-itD}f, e^{-itD}g\right\rangle \\ &= \left\langle \left\langle e^{-itD}f, k\right\rangle h, e^{-itD}g\right\rangle \\ &= \left\langle e^{itD}h, g\right\rangle \left\langle f, e^{itD}k\right\rangle.\end{aligned}$$

Let us consider the case when $n = 1$. By Proposition 3.5, $h \otimes k^* \in \mathrm{Dom}(\delta_D)$ if and only if for every $f, g \in \mathcal{H}$ the map $t \mapsto \langle \alpha_t(h \otimes k^*)f, g\rangle$ is continuously differentiable. Thus, it suffices to prove that

$$t \mapsto \langle \alpha_t(h \otimes k^*)f, g\rangle = \left\langle f, e^{itD}k\right\rangle \left\langle e^{itD}h, g\right\rangle$$

is n-times continuously differentiable for all $f, g \in \mathcal{H}$.

Fix $f, g \in \mathcal{H}$. By Stone's Theorem,

$$\lim_{t\to 0}\left\|\frac{e^{itD}h - h}{t} - iDh\right\| = 0 \quad \text{and} \quad \lim_{t\to 0}\left\|\frac{e^{itD}k - k}{t} - iDk\right\| = 0.$$

By the Schwarz inequality, the maps $t \mapsto \left\langle f, e^{itD}k\right\rangle$ and $t \mapsto \left\langle e^{itD}h, g\right\rangle$ are continuously differentiable with derivatives $t \mapsto \left\langle f, e^{itD}(iDk)\right\rangle$ and $t \mapsto \left\langle e^{itD}(iDh), g\right\rangle$, respectively. Since the product of two continuously differentiable functions is continuously differentiable, $t \mapsto \langle \alpha_t(h \otimes k^*)f, g\rangle = \left\langle e^{itD}h, g\right\rangle \left\langle f, e^{itD}k\right\rangle$ is continuously differentiable. As $f, g \in \mathcal{H}$ were arbitrary, we conclude $h \otimes k^* \in \mathrm{Dom}(\delta_D)$.

Furthermore, Proposition 3.5 states that for all $f, g \in \mathcal{H}$,

$$\frac{d}{dt}\left\langle \alpha_t(h \otimes k^*)f, g\right\rangle\Big|_{t=0} = \langle \delta_D(h \otimes k^*)f, g\rangle.$$

Hence,

$$\begin{aligned}
\langle \delta_D(h \otimes k^*)f, g\rangle &= \frac{d}{dt}\left(\left\langle f, e^{itD}k\right\rangle \left\langle e^{itD}h, g\right\rangle\right)\big|_{t=0} \\
&= \left\langle f, e^{itD}iDk\right\rangle \left\langle e^{itD}h, g\right\rangle \big|_{t=0} + \left\langle f, e^{itD}k\right\rangle \left\langle e^{itD}iDh, g\right\rangle \big|_{t=0} \\
&= \langle f, iDk\rangle \langle h, g\rangle + \langle f, k\rangle \langle iDh, g\rangle \\
&= \langle \langle f, iDk\rangle h, g\rangle + \langle \langle f, k\rangle iDh, g\rangle \\
&= \langle [h \otimes (iDk)^*]f, g\rangle + \langle [(iDh) \otimes k^*]f, g\rangle \\
&= \langle [(iDh) \otimes k^* + h \otimes (iDk)^*]f, g\rangle
\end{aligned}$$

As $f, g \in \mathcal{H}$ were arbitrary, $\delta_D(h \otimes k^*) = (iDh) \otimes k^* + h \otimes (iDk)^*$.

For general $n \in \mathbb{N}$, the rank-one operator $h \otimes k^*$ is n-times weakly D differentiable if and only if for every $f, g \in \mathcal{H}$ the map $t \mapsto \langle \alpha_t(h \otimes k^*)f, g\rangle$ is n-times continuously differentiable. As above, $\langle \alpha_t(h \otimes k^*)f, g\rangle = \left\langle f, e^{itD}k\right\rangle \left\langle e^{itD}h, g\right\rangle$ and, given $h, k \in \mathrm{Dom}(D^n)$, the functions $t \mapsto \left\langle f, e^{itD}k\right\rangle$ and $t \mapsto \left\langle e^{itD}h, g\right\rangle$ are n-times continuously differentiable, where

$$\frac{d^j}{dt^j}\left\langle f, e^{itD}k\right\rangle = \left\langle f, e^{itD}[(iD)^j k]\right\rangle \quad \text{and} \quad \frac{d^j}{dt^j}\left\langle e^{itD}h, g\right\rangle = \left\langle e^{itD}[(iD)^j h], g\right\rangle$$

for each $j = 1, ..., n$. Since the product of two n-times continuously differentiable functions is n-times continuously differentiable, $t \mapsto \langle \alpha_t(h \otimes k^*)f, g\rangle$ is n-times continuously differentiable. As $f, g \in \mathcal{H}$ were arbitrary, $h \otimes k^* \in \mathrm{Dom}(\delta_D^n)$, and a computation similar to the $n = 1$ case yields

$$\delta_D^n(h \otimes k^*) = \sum_{j=0}^{n} (iD)^{n-j} h \otimes [(iD)^j k]^*.$$

□

Notation 3.13. Given a subset $S \subseteq \mathcal{H}$, let $\mathcal{F}(S) := \mathrm{Span}\{f \otimes g^* : f, g \in S\}$.

Lemma 3.14. *If $S \subseteq \mathcal{H}$ is a dense subspace, then $\mathcal{F}(S)$ is norm-dense in $\mathcal{K}(\mathcal{H})$.*

The proof is an easy exercise which we leave to the reader.

Corollary 3.15. *For each* $n \in \mathbb{N}$, $\mathrm{Dom}(\delta_D^n) \cap \mathcal{F}(\mathcal{H})$ *is norm-dense in* $\mathcal{K}(\mathcal{H})$.

Proof. By Lemma 3.12, $\mathcal{F}(\mathrm{Dom}(D^n)) \subseteq \mathrm{Dom}(\delta_D^n) \cap \mathcal{F}(\mathcal{H})$. As $\mathrm{Dom}(D^n)$ is dense in $\mathcal{H}$ for each $n \in \mathbb{N}$ by Nelson's Analytic Vector Theorem, Lemma 3.14 implies $\mathcal{F}(\mathrm{Dom}(D^n))$ is norm-dense in $\mathcal{K}(\mathcal{H})$. Therefore, $\mathrm{Dom}(\delta_D^n) \cap \mathcal{F}(\mathcal{H})$ is norm-dense in $\mathcal{K}(\mathcal{H})$. □

Theorem 3.16. *For each* $n \in \mathbb{N}$, $\mathrm{Dom}(\delta_D^n)$ *is SOT-dense in* $\mathcal{B}(\mathcal{H})$.

Proof. As the norm topology is finer than the SOT on $\mathcal{B}(\mathcal{H})$,

$$\overline{\mathcal{F}(\mathcal{H}) \cap \mathrm{Dom}(\delta_D^n)}^{\mathrm{SOT}} \supseteq \overline{\mathcal{F}(\mathcal{H}) \cap \mathrm{Dom}(\delta_D^n)}^{\|\cdot\|} = \mathcal{K}(\mathcal{H})$$

by Corollary 3.15. Therefore, $\overline{\mathcal{F}(\mathcal{H}) \cap \mathrm{Dom}(\delta_D^n)}^{\mathrm{SOT}} = \overline{\mathcal{K}(\mathcal{H})}^{\mathrm{SOT}} = \mathcal{B}(\mathcal{H})$. □

3.4 C_o–Groups of Isometries and their Infinitesimal Generators

Theorem 3.16 strengthens Christensen's Theorem 3.3 and provides a way to construct elements in $\mathrm{Dom}(\delta_D^n)$ using elements from $\mathrm{Dom}(D^n)$. Given that the analytic vectors for D are dense in $\mathcal{H}$, we were led to wonder if the analytic vectors for δ_D (which are operators in $\mathcal{B}(\mathcal{H})$) were SOT-dense in $\mathcal{B}(\mathcal{H})$.

To relate the analytic vectors for D and δ_D as we related $\mathrm{Dom}(D^n)$ and $\mathrm{Dom}(\delta_D^n)$ in Lemma 3.12, we exploit an equivalent notion of analyticity for the one-parameter families for which D and δ_D are infinitesimal generators: $\{e^{itD}\}_{t\in\mathbb{R}}$ and $\{\alpha_t\}_{t\in\mathbb{R}}$, respectively. We first introduce the notion of analytic vectors for a general one-parameter family on a Banach space, and then we specialize to our setting.

Definition 3.17. Let X be a Banach space and let Y be a closed subspace of X^*. A one-parameter family $\{\tau_t\}_{t\in\mathbb{R}}$ of isometries on X into itself is called a $\sigma(X,Y)$-*continuous group of isometries of* X if

1. $\tau_0 = I$,
2. $\tau_{s+t} = \tau_s\tau_t$ for all $s, t \in \mathbb{R}$,
3. $t \mapsto \tau_t(x)$ is $\sigma(X,Y)$-continuous for all $x \in X$, i.e., $t \mapsto \psi(\tau_t(x))$ is continuous for all $x \in X$ and $\psi \in Y$, and
4. $x \mapsto \tau_t(x)$ is $\sigma(X,Y)$-continuous for all $t \in \mathbb{R}$.

Note that condition (4) in Definition 3.17 is needed as Y may not be invariant under the Banach space adjoint of τ_t acting on X^*. Given $\lambda > 0$, set $\Omega_\lambda := \{z \in \mathbb{C} : \mathrm{Im}(z) < \lambda\}$.

Definition 3.18. Given a $\sigma(X,Y)$-continuous group of isometries $\{\tau_t\}_{t\in\mathbb{R}}$, an element $x \in X$ is *analytic for* $\{\tau_t\}_{t\in\mathbb{R}}$ if there exists $\lambda > 0$ and a function $\varphi : \Omega_\lambda \to X$ such that

1. $\varphi(t) = \tau_t(x)$ for all $t \in \mathbb{R}$ and
2. $z \mapsto \psi(\varphi(z))$ is analytic on Ω_λ for all $\psi \in Y$.

Definition 3.19. Given a $\sigma(X,Y)$-continuous group of isometries $\{\tau_t\}_{t\in\mathbb{R}}$, the *infinitesimal generator* S for $\{\tau_t\}_{t\in\mathbb{R}}$ is the operator whose domain consists of all elements $x \in X$ such that there exists $x' \in X$ which satisfies

$$\lim_{t\to 0} \psi\left(\frac{\tau_t(x) - x}{t} - x'\right) = 0 \text{ for all } \psi \in Y. \qquad (*)$$

If $x \in \mathrm{Dom}(S)$, set $Sx := x'$, where x' satisfies condition $(*)$.

Proposition 3.20 below states that the two notions of analyticity in Definitions 2.14 and 3.18 are equivalent when S is the infinitesimal generator of $\{\tau_t\}_{t\in\mathbb{R}}$.

Proposition 3.20 (Bratteli-Robinson, [4]). *If $\{\tau_t\}_{t\in\mathbb{R}}$ is a $\sigma(X,Y)$-continuous group of isometries with infinitesimal generator S, then x is analytic for $\{\tau_t\}_{t\in\mathbb{R}}$ if and only if x is an analytic vector for S.*

Consider the Banach space $\mathcal{B}(\mathcal{H})$ along with the one-parameter group of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ given by $\alpha_t(x) = e^{itD}xe^{-itD}$ for all $x \in \mathcal{B}(\mathcal{H})$, $t \in \mathbb{R}$. The closed subspace of elementary vector functionals Y in $\mathcal{B}(\mathcal{H})^*$ recovers the WOT on $\mathcal{B}(\mathcal{H})$ as the $\sigma(X,Y)$-topology.

Proposition 3.21. *The family $\{\alpha_t\}_{t\in\mathbb{R}}$ is a WOT-continuous group of $*$-automorphisms with infinitesimal generator δ_D.*

It is straightforward to check WOT-continuity of the automorphism group $\{\alpha_t\}_{t\in\mathbb{R}}$ using the SOT-continuity of the unitary group $\{e^{itD}\}_{t\in\mathbb{R}}$. Furthermore, δ_D is the corresponding infinitesimal generator for $\{\alpha_t\}_{t\in\mathbb{R}}$ simply by definition of weak D-differentiability. As a corollary of Propositions 3.20 and 3.21, we have the following:

Corollary 3.22. *An element $x \in \mathcal{B}(\mathcal{H})$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$ if and only if $x \in \mathsf{A}(\delta_D)$, where $\mathsf{A}(\delta_D)$ denotes the set of analytic operators for δ_D.*

3.5 The Riesz Map and Density of Analytic Vectors

Initially, our strategy for proving SOT-density of the set of analytic vectors for δ_D in $\mathcal{B}(\mathcal{H})$ was to mimic the steps of Lemma 3.12—given $h,k \in \mathsf{A}(D)$, we wanted $h \otimes k^*$ to be analytic for δ_D. If $h,k \in \mathsf{A}(D)$, the equivalent notion of analyticity from Proposition 3.20 implies that for each $f,g \in \mathcal{H}$, the maps $t \mapsto \langle e^{itD}h, g\rangle$ and $t \mapsto \langle e^{itD}k, f\rangle$ extend to analytic functions on some strip in the complex plane. But then, the map $t \mapsto \langle f, e^{itD}k\rangle$ is co-analytic, and since $\langle \alpha_t(h\otimes k^*)f, g\rangle = \langle e^{itD}h, g\rangle \langle f, e^{itD}k\rangle$ is the product of an analytic function and a co-analytic function, we could not necessarily extend the map $t \mapsto \langle \alpha_t(h\otimes k^*)f, g\rangle$ to an analytic function on a strip in the complex plane.

To remedy the issue of co-analyticity for the function involving k, we utilize the Riesz map $\mathcal{R} : H \to H^*$ given by $h \mapsto \psi_h$, where

$$\psi_h(f) := \langle f, h \rangle \quad \text{for all } f \in \mathcal{H}.$$

Note that $\mathcal{R}$ is *anti*-unitary: $\langle \mathcal{R}f, \mathcal{R}g \rangle_{\mathcal{H}^*} = \langle g, f \rangle_{\mathcal{H}}$ for all $f, g \in \mathcal{H}$.

It is clear that conjugating a self-adjoint operator D by a unitary U results in another self-adjoint operator. Below we verify that conjugating D by $\mathcal{R}$ results in a self-adjoint operator.

Lemma 3.23. *Define $D^{\#} : \mathcal{R}(\mathrm{Dom}(D)) \to \mathcal{H}^*$ by $D^{\#}(\mathcal{R}h) := \mathcal{R}(Dh)$ for all $h \in \mathrm{Dom}(D)$. The map $D^{\#} := \mathcal{R}D\mathcal{R}^{-1}$ with $\mathrm{Dom}\big(D^{\#}\big) = \mathcal{R}(\mathrm{Dom}(D))$ is self-adjoint.*

Proof. We first show $D^{\#}$ is a linear symmetric operator. Given $h \in \mathrm{Dom}(D)$ and $\lambda \in \mathbb{C}$, observe

$$D^{\#}(\lambda \mathcal{R}h) = \left[\mathcal{R}D\mathcal{R}^{-1}\right](\lambda \mathcal{R}h) = [\mathcal{R}D](\overline{\lambda}h) = \mathcal{R}(\overline{\lambda}Dh) = \lambda[\mathcal{R}D\mathcal{R}^{-1}](\mathcal{R}h) = \lambda D^{\#}(\mathcal{R}h).$$

As $h \in \mathrm{Dom}(D)$ was arbitrary and $\mathrm{Dom}\big(D^{\#}\big) = \mathcal{R}(\mathrm{Dom}(D))$, we have $D^{\#}(\lambda\psi) = \lambda D^{\#}\psi$ for all $\psi \in \mathrm{Dom}\big(D^{\#}\big)$ and $\lambda \in \mathbb{C}$. It's easy to check additivity of $D^{\#}$, so $D^{\#}$ is linear. For $f, h \in \mathrm{Dom}(D)$,

$$\left\langle D^{\#}\mathcal{R}h, \mathcal{R}f \right\rangle = \langle \mathcal{R}Dh, \mathcal{R}f \rangle = \langle f, Dh \rangle = \langle Df, h \rangle = \langle \mathcal{R}h, \mathcal{R}Df \rangle = \left\langle \mathcal{R}h, D^{\#}\mathcal{R}f \right\rangle.$$

As $f, h \in \mathrm{Dom}(D)$ were arbitrary and $\mathrm{Dom}\big(D^{\#}\big) = \mathcal{R}(\mathrm{Dom}(D))$,

$$\left\langle D^{\#}\psi, \phi \right\rangle = \left\langle \psi, D^{\#}\phi \right\rangle \text{ for all } \psi, \phi \in \mathrm{Dom}\big(D^{\#}\big).$$

Therefore, $D^{\#}$ is symmetric. Note that $\mathcal{R}(\mathrm{Dom}(D))$ is dense in $\mathcal{H}^*$ since $\mathrm{Dom}(D)$ is dense in $\mathcal{H}$ and $\mathcal{R}$ is a continuous bijection. Thus, it suffices to show $\mathrm{Dom}\big((D^{\#})^*\big) \subseteq \mathrm{Dom}\big(D^{\#}\big)$. Recall that the domain of the adjoint of $D^{\#}$ is the set

$$\begin{aligned}
\mathrm{Dom}\big((D^{\#})^*\big) &= \{\phi \in \mathcal{H}^* : \text{ the map } \mathrm{Dom}\big(D^{\#}\big) \to \mathbb{C};\ \psi \mapsto \big\langle D^{\#}\psi, \phi\big\rangle \text{ is bounded}\} \\
&= \{\phi \in \mathcal{H}^* : \text{ the map } \mathcal{R}(\mathrm{Dom}(D)) \to \mathbb{C};\ \ \mathcal{R}h \mapsto \big\langle D^{\#}(\mathcal{R}h), \phi\big\rangle \text{ is bounded}\}. \\
&= \{\phi \in \mathcal{H}^* : \text{ the map } \mathcal{R}(\mathrm{Dom}(D)) \to \mathbb{C};\ \ \mathcal{R}h \mapsto \big\langle \mathcal{R}^{-1}\phi, \mathcal{R}^{-1}D^{\#}(\mathcal{R}h)\big\rangle \text{ is bounded}\}. \\
&= \{\phi \in \mathcal{H}^* : \text{ the map } \mathcal{R}(\mathrm{Dom}(D)) \to \mathbb{C};\ \ \mathcal{R}h \mapsto \big\langle \mathcal{R}^{-1}\phi, Dh\big\rangle \text{ is bounded}\}.
\end{aligned}$$

Hence, given $\phi \in \mathrm{Dom}\big((D^{\#})^*\big)$, the map $\mathcal{R}(\mathrm{Dom}(D)) \to \mathbb{C}$ defined by

$$\mathcal{R}h \mapsto \big\langle \mathcal{R}^{-1}\phi, Dh\big\rangle \text{ for all } h \in \mathrm{Dom}(D)$$

is a bounded linear functional. Then, since $\mathcal{R}$ is isometric, the composition

$$\begin{array}{ccccc}
\mathrm{Dom}(D) & \longrightarrow & \mathcal{R}(\mathrm{Dom}(D)) & \longrightarrow & \mathbb{C} \\
h & \mapsto & \mathcal{R}h & \mapsto & \langle \mathcal{R}^{-1}\phi, Dh\rangle
\end{array}$$

defines a bounded linear functional on the domain of D. By the definition of the domain of D^*, this implies $\mathcal{R}^{-1}\phi$ belongs to $\mathrm{Dom}(D^*)$. Further, self-adjointness of D implies $\mathrm{Dom}(D) = \mathrm{Dom}(D^*)$, so $\mathcal{R}^{-1}\phi \in \mathrm{Dom}(D)$. Since $\mathcal{R}$ is bijective, we conclude $\phi \in \mathcal{R}(\mathrm{Dom}(D)) = \mathrm{Dom}\big(D^{\#}\big)$. Therefore, $D^{\#}$ is self-adjoint. □

By Nelson's Analytic Vector Theorem, the set of analytic vectors $\mathsf{A}(D^{\#})$ is dense in $\mathcal{H}^*$. As $\mathcal{R}^{-1} : \mathcal{H}^* \to \mathcal{H}$ is a continuous bijection, it follows that $\mathcal{R}^{-1}[\mathsf{A}(D^{\#})]$ is dense in $\mathcal{H}$.

Notation 3.24. Given subsets $S_1, S_2 \subseteq \mathcal{H}$, let

$$\mathcal{F}(S_1, S_2) := \text{Span}\{f \otimes g^* : f \in S_1, g \in S_2\}.$$

Denote $\mathcal{F}(S_1, S_1)$ by $\mathcal{F}(S_1)$.

Lemma 3.25. *If $S_1, S_2 \subseteq \mathcal{H}$ are dense, then $\mathcal{F}(S_1, S_2)$ is norm-dense in $\mathcal{K}(\mathcal{H})$.*

The proof of Lemma 3.25 is a simple modification of the case when $S_1 = S_2$ in Lemma 3.14. By Lemma 3.25, $\mathcal{F}\left(\mathsf{A}(D), \mathcal{R}^{-1}[\mathsf{A}(D^\#)]\right)$ is norm-dense in $\mathcal{K}(\mathcal{H})$.

Proposition 3.26. *If $h \in \mathsf{A}(D)$ and $k \in \mathcal{R}^{-1}[\mathsf{A}(D^\#)]$, then $h \otimes k^* \in \mathsf{A}(\delta_D)$.*

Proof. Let $h \in \mathsf{A}(D)$ and $k \in \mathcal{R}^{-1}[\mathsf{A}(D^\#)]$. By Corollary 3.22, $h \otimes k^* \in \mathsf{A}(\delta_D)$ if and only if $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$. To prove $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$, we must find $\lambda > 0$ and a function $\varphi : \Omega_\lambda \to \mathcal{B}(\mathcal{H})$ such that

1. $\varphi(t) = \alpha_t(h \otimes k^*)$ for all $t \in \mathbb{R}$ and

2. $z \mapsto \langle \varphi(z)f, g\rangle$ is analytic on Ω_λ for all $f, g \in \mathcal{H}$.

We construct φ using the two functions obtained from analytic properties of h and k. As $h \in \mathsf{A}(D)$, Proposition 3.20 implies h is analytic for $\{e^{itD}\}_{t\in\mathbb{R}}$. Thus, there exists $\lambda_h > 0$ and a function $\varphi_h : \Omega_{\lambda_h} \to \mathcal{H}$ such that

1. $\varphi_h(t) = e^{itD}h$ for all $t \in \mathbb{R}$ and

2. $z \mapsto \langle \varphi_h(z), g\rangle$ is analytic on Ω_{λ_h} for all $g \in \mathcal{H}$.

As $k \in \mathcal{R}^{-1}[\mathsf{A}(D^\#)]$, there exists a unique $\zeta_k \in \mathsf{A}(D^\#)$ such that $k = \mathcal{R}^{-1}\zeta_k$. Since ζ_k is analytic for $D^\#$, it is analytic for $\{e^{itD^\#}\}_{t\in\mathbb{R}}$ by Proposition 3.20. Hence, there exists $\lambda_k > 0$ and a function $\varphi_k : \Omega_{\lambda_k} \to \mathcal{H}^*$ such that

1. $\varphi_k(t) = e^{itD^\#}\zeta_k$ for all $t \in \mathbb{R}$ and

2. $z \mapsto \langle \varphi_k(z), \mathcal{R}f \rangle$ is analytic on Ω_{λ_k} for all $f \in \mathcal{H}$.

Note that in (2), we simply identified $\mathcal{H}^*$ with $\mathcal{R}(\mathcal{H})$. Set $\lambda := \min\{\lambda_h, \lambda_k\}$, and fix $z \in \Omega_\lambda$. Define a map $[\cdot,\cdot] : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$[f,g] := \langle \varphi_h(z), g \rangle \langle \varphi_k(z), \mathcal{R}f \rangle \text{ for all } f, g \in \mathcal{H}.$$

Sesquilinearity of the inner products on $\mathcal{H}$ and $\mathcal{H}^*$ and antilinearity of $\mathcal{R}$ establishes that $[\cdot,\cdot]$ is a sesquilinear form. Moreover, for any $f, g \in \mathcal{H}$,

$$|[f,g]| = |\langle \varphi_h(z), g \rangle| \, |\langle \varphi_k(z), \mathcal{R}f \rangle| \leq \|\varphi_h(z)\| \, \|g\| \, \|\varphi_k(z)\| \, \|f\| \, .$$

As h, k, and z are fixed, $[\cdot,\cdot]$ defines a bounded sesquilinear form on $\mathcal{H}$. Thus, for each $z \in \Omega_\lambda$, the Riesz Representation Theorem for Bounded Sesquilinear Forms yields an operator $\varphi(z) \in \mathcal{B}(\mathcal{H})$ such that

$$\langle \varphi(z)f, g \rangle = [f,g] = \langle \varphi_h(z), g \rangle \langle \varphi_k(z), \mathcal{R}f \rangle \text{ for all } f, g \in \mathcal{H}.$$

As the two maps $z \mapsto \langle \varphi_h(z), g \rangle$ and $z \mapsto \langle \varphi_k(z), \mathcal{R}f \rangle$ are analytic on Ω_λ for all $f, g \in \mathcal{H}$, their product $z \mapsto \langle \varphi(z)f, g \rangle$ is analytic on Ω_λ for all $f, g \in \mathcal{H}$. Furthermore, for each $t \in \mathbb{R}$,

$$\langle \varphi(t)f, g \rangle = \left\langle e^{itD}h, g \right\rangle \left\langle e^{itD^\#}\zeta_k, \mathcal{R}f \right\rangle = \left\langle e^{itD}h, g \right\rangle \left\langle f, e^{itD}k \right\rangle = \langle \alpha_t(h \otimes k^*)f, g \rangle \, .$$

As $f, g \in \mathcal{H}$ were arbitrary, we have $\varphi(t) = \alpha_t(h \otimes k^*)$ for all $t \in \mathbb{R}$. Therefore, $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$ in the WOT. By equivalence of analyticity for $\{\alpha_t\}_{t\in\mathbb{R}}$ and δ_D, we conclude $h \otimes k^* \in \mathsf{A}(\delta_D)$. □

Theorem 1.1. *The set of analytic vectors for δ_D is SOT-dense in $\mathcal{B}(\mathcal{H})$.*

Proof. Proposition 3.26 implies $\mathcal{F}(\mathsf{A}(D), \mathcal{R}^{-1}[\mathsf{A}(D^{\#})])$ is contained in $\mathsf{A}(\delta_D)$, so

$$\mathcal{F}\left(\mathsf{A}(D), \mathcal{R}^{-1}[\mathsf{A}(D^{\#})]\right) \subseteq \mathsf{A}(\delta_D) \cap F(\mathcal{H}).$$

By Lemma 3.25 and Nelson's Analytic Vector Theorem, $\mathcal{F}\left(\mathsf{A}(D), \mathcal{R}^{-1}[\mathsf{A}(D^{\#})]\right)$ is norm-dense in $\mathcal{K}(\mathcal{H})$. Thus, $\mathsf{A}(\delta_D) \cap \mathcal{F}(\mathcal{H})$ is norm-dense in $\mathcal{K}(\mathcal{H})$. Therefore, $\mathsf{A}(\delta_D)$ is SOT-dense in $\mathcal{B}(\mathcal{H})$. $\square$

Chapter 4

Kernel Stabilization

The main theorem of this chapter, Theorem 1.2, states that for any self-adjoint operator D on a Hilbert space, $\ker \delta_D^n = \ker \delta_D$ for all $n \in \mathbb{N}$. We call this property *kernel stabilization.*

4.1 Motivating Example

Throughout section 4.1, we denote the standard orthonormal basis for $\ell^2(\mathbb{Z})$ by $\{\epsilon_j : j \in \mathbb{Z}\}$, and we denote the matrix representation of an operator $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$ with respect to the standard orthonormal basis by $[x_{rc}]$ where

$$x_{rc} := \langle x\epsilon_c, \epsilon_r \rangle \text{ for all } r, c \in \mathbb{Z}.$$

Example 4.1. Define $(Df)(j) := jf(j)$ for $f \in \mathrm{Dom}(D)$, where

$$\mathrm{Dom}(D) := \{f \in \ell^2(\mathbb{Z}) : \sum_{j\in\mathbb{Z}} j^2 \left|f(j)\right|^2 < \infty\}.$$

Then

(i) the operator D is self-adjoint.

(ii) an operator $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is n-times weakly D-differentiable if and only if for every

$k \leq n$, $x(\operatorname{Dom}(D^k)) \subseteq \operatorname{Dom}(D^k)$ and the matrix $[i^k(r-c)^k x_{rc}]$ with dense domain $\operatorname{Dom}(D^k)$ extends to a bounded operator on $\ell^2(\mathbb{Z})$. When either condition is satisfied,

$$[\delta_D^n(x)_{rc}]|_{\operatorname{Dom}(D^n)} = [i^n(r-c)^n x_{rc}].$$

(iii) for any $g \in \ell^\infty(\mathbb{Z})$, $\delta_D(M_g) = 0$.

(iv) for all $n \in \mathbb{N}$, $\ker \delta_D^n = \operatorname{diag}(\ell^\infty(\mathbb{Z}))$.

Proof. (i) See Example 7.1.5 of [22].

(ii) Matrix multiplication shows for any $r, c \in \mathbb{Z}$,

$$d^k(x)_{rc} = i^k(r-c)^k x_{rc}.$$

Given $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$ such that $x(\operatorname{Dom}(D^k)) \subseteq \operatorname{Dom}(D^k)$ for each $k \leq n$, the domain of $d^k(x)$ is $\operatorname{Dom}(D^k)$. Theorem 3.7 states x is n-times weakly D-differentiable if and only if for every $k \leq n$, $x(\operatorname{Dom}(D^k)) \subseteq \operatorname{Dom}(D^k)$ and $d^k(x)$ is bounded on $\operatorname{Dom}(D^k)$. It follows that x is n-times weakly D-differentiable if and only if $x(\operatorname{Dom}(D^k)) \subseteq \operatorname{Dom}(D^k)$ and $[d^k(x)_{rc}] = [i^k(r-c)^k x_{rc}]$ is bounded on $\operatorname{Dom}(D^k)$. As D is self-adjoint, $\operatorname{Dom}(D^k)$ is dense in $\ell^2(\mathbb{Z})$ for all $k \in \mathbb{N}$. Therefore, $[d^k(x)_{rc}]$ extends to a bounded matrix on all of $\ell^2(\mathbb{Z})$. By Theorem 3.7, the closure $\delta_D^n(x)$ is the extension of $[i^n(r-c)^n x_{rc}]$ to all of $\ell^2(\mathbb{Z})$.

(iii) Fix $g \in \ell^\infty(\mathbb{Z})$, and let $f \in \operatorname{Dom}(D)$. We show $M_g f \in \operatorname{Dom}(D)$. Observe

$$\sum_{j\in\mathbb{Z}} |j(M_g f)(j)|^2 = \sum_{j\in\mathbb{Z}} |jg(j)f(j)|^2 \leq \|g\|_\infty^2 \left(\sum_{j\in\mathbb{Z}} |jf(j)|^2 \right) < \infty.$$

As $f \in \mathrm{Dom}(D)$ was arbitrary, $M_g(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$, and hence, the commutator $[iD, M_g]$ is a well-defined linear operator on $\mathrm{Dom}(D)$. Furthermore, iD and M_g are diagonal matrices with complex entries (which commute), so the commutator $[iD, M_g]$ is simply a restriction of the 0 operator to $\mathrm{Dom}(D)$. Theorem 3.2 implies $M_g \in \mathrm{Dom}(\delta_D)$ and $\delta_D(M_g)$ is the extension of $[iD, M_g]$ to all of $\mathcal{H}$. In particular, $\delta_D(M_g) = 0$. Hence, $M_g \in \ker \delta_D$, and since $g \in \ell^\infty(\mathbb{Z})$ was arbitrary, $\mathrm{diag}(\ell^\infty(\mathbb{Z})) \subseteq \ker \delta_D$.

(iv) Part (c) quickly implies $\mathrm{diag}(\ell^\infty(\mathbb{Z})) \subseteq \ker \delta_D^n$ for all $n \in \mathbb{N}$. We now show if $\delta_D^n(x) = 0$, then $x \in \mathrm{diag}(\ell^\infty(\mathbb{Z}))$. If $x \in \mathrm{Dom}(\delta_D^n)$ and $\delta_D^n(x) = 0$, then $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$ and $\delta_D^n(x)_{rc} = 0$ for every $r, c \in \mathbb{Z}$. By part (b),

$$[\delta_D^n(x)_{rc}]|_{\mathrm{Dom}(D^n)} = [i^n(r-c)^n x_{rc}],$$

thus, $i^n(r-c)^n x_{rc} = 0$ for every $r, c \in \mathbb{Z}$. If $r \neq c$, it must be that $x_{rc} = 0$, i.e., x must be zero off the diagonal. As $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$, we conclude $x \in \mathrm{diag}(\ell^\infty(\mathbb{Z}))$. Therefore, $\ker \delta_D^n = \mathrm{diag}(\ell^\infty(\mathbb{Z}))$ for all $n \in \mathbb{N}$.

□

This kernel stabilization phenomenon initially appears unique to the setting of Example 4.1; the self-adjoint operator has a complete set of eigenvectors which forms our choice of orthonormal basis. However, Theorem 1.2 shows that this example is not unique; kernel stabilization holds for every self-adjoint operator on any Hilbert space.

4.2 General Kernel Stabilization of δ_D

Proposition 4.2. *Let $\mathcal{H}$ be a Hilbert space and D a self-adjoint operator. The algebra $\ker \delta_D$ is a von Neumann algebra.*

Proof. The identity I of $\mathcal{B}(\mathcal{H})$ is easily shown to be in $\ker\delta_D$. Let $x \in \ker\delta_D$. As $\mathrm{Dom}(\delta_D)$ is a $*$-algebra by Theorem 3.3, $x^* \in \mathrm{Dom}(\delta_D)$. Since δ_D is a $*$-derivation, $\delta_D(x^*) = \delta_D(x)^* = 0$. Therefore, $x^* \in \ker\delta_D$. Finally, if $x, y \in \ker\delta_D$, then $xy \in \mathrm{Dom}(\delta_D)$ and $\delta_D(xy) = \delta_D(x)y + x\delta_D(y) = 0$, so $xy \in \ker\delta_D$.

Let $(x_\lambda)_{\lambda\in\Lambda} \subset \ker\delta_D$ be a net converging in the WOT to some $x \in \mathcal{B}(\mathcal{H})$. We show $x \in \mathrm{Dom}(\delta_D)$ and $\delta_D(x) = 0$. Because $\delta_D(x_\lambda) = 0$ for all $\lambda \in \Lambda$, we trivially have $\delta_D(x_\lambda) \overset{\text{WOT}}{\to} 0$ as $\lambda \to \infty$. By Theorem 3.3, the graph of δ_D is WOT-closed. Therefore, $x \in \mathrm{Dom}(\delta_D)$ and $\delta_D(x) = 0$. We conclude $\ker\delta_D$ is a von Neumann algebra. □

Notation 4.3. Let $\mathscr{P}_D$ denote the collection of all spectral projections for D obtained through the Spectral Theorem for Unbounded Self-Adjoint Operators. Also, let

$$\mathcal{M}_D := \mathscr{P}_D''.$$

Lemma 4.4. *Suppose $x \in \mathcal{B}(\mathcal{H})$ satisfies $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$. If $P \in \mathscr{P}_D$, then*

$$[P, [D, x]]h = [D, [P, x]]h \textit{ for all } h \in \mathrm{Dom}(D).$$

Proof. Let $\mathcal{B}(\mathbb{R})$ be the bounded Borel functions on $\mathbb{R}$, and for $R \in \mathbb{R}$, define $\mathrm{id}_R : \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{id}_R(t) := \begin{cases} t; & \text{-}R \leq t \leq R \\ 0; & else \end{cases}.$$

The Spectral Theorem, stated as in Theorem 7.2.8 of [22], provides a bounded Borel functional calculus for D, that is, a $*$-homomorphism $\Phi_D : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ satisfying $\Phi_D(1) = I$,

$$\mathrm{Dom}(D) = \{h \in \mathcal{H} : \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)h\| < \infty\},$$

and

$$Dh = \lim_{R\to\infty} \Phi_D(\mathrm{id}_R)h \ \text{ for all } \ h \in \mathrm{Dom}(D).$$

We claim for each $P \in \mathscr{P}_D$, $P(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ and $PDh = DPh$ for all $h \in \mathrm{Dom}(D)$. Given $P \in \mathscr{P}_D$, there exists some Borel set $E \subseteq \mathbb{R}$ such that $P = \Phi_D(\chi_E)$. Note that

$$(\mathrm{id}_R \cdot \chi_E)(t) = \begin{cases} t; & \mathrm{t} \in E \cap [-R, R] \\ 0; & else \end{cases}.$$

Thus, for any $h \in \mathrm{Dom}(D)$,

$$\lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)Ph\| = \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)\Phi_D(\chi_E)h\| = \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R \cdot \chi_E)h\| \leq \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)h\| < \infty.$$

Therefore, $Ph \in \mathrm{Dom}(D)$, and as $h \in \mathrm{Dom}(D)$ was arbitrary, $P(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$. Furthermore,

$$\begin{aligned} \|DPh - PDh\| &= \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)\Phi_D(\chi_E)h - \Phi_D(\chi_E)\Phi_D(\mathrm{id}_R)h\| \\ &= \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R \cdot \chi_E)h - \Phi_D(\chi_E \cdot \mathrm{id}_R)h\| \\ &= \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R \cdot \chi_E)h - \Phi_D(\mathrm{id}_R \cdot \chi_E)h\| \\ &= 0. \end{aligned}$$

Given $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$, for any $h \in \mathrm{Dom}(D)$ we observe

$$\begin{aligned}
[P,[D,x]]h &= P(Dx - xD)h - (Dx - xD)Ph \\
&= PDxh - PxDh - DxPh + xDPh \\
&= DPxh - PxDh - DxPh + xPDh \\
&= DPxh - DxPh + xPDh - PxDh \\
&= D(Px - xP)h + (xP - Px)Dh \\
&= D(Px - xP)h - (Px - xP)Dh \\
&= [D,[P,x]]h
\end{aligned}$$

Hence, $[P,[D,x]]h = [D,[P,x]]h$ for all $h \in \mathrm{Dom}(D)$, and as $P \in \mathscr{P}_D$ was arbitrary, this equality holds for any spectral projection of D. □

Proposition 4.5. $\mathcal{M}_D \subseteq \ker \delta_D = \mathcal{M}'_D$.

Proof. Let $P \in \mathscr{P}_D$. By the previous lemma, $[D,P] = 0$ on $\mathrm{Dom}(D)$, so $P \in \mathrm{Dom}(\delta_D)$ by Theorem 3.2. Moreover, $\delta_D(P)$ is the bounded extension of $i(DP - PD)$ to all of $\mathcal{H}$, which is 0. Therefore, $P \in \ker \delta_D$. Because $\mathcal{M}_D$ is generated as a von Neumann algebra by the projections in $\mathscr{P}_D$, Proposition 4.2 implies $\mathcal{M}_D \subseteq \ker \delta_D$.

Let $x \in \ker \delta_D$. By Theorem 3.7, $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ and $\delta_D(x)|_{\mathrm{Dom}(D)} = [D,x]|_{\mathrm{Dom}(D)} = 0$. Then, by Theorem X.4.11 of [7], $xf(D) \subseteq f(D)x$ for any $f \in \mathcal{B}(\mathbb{R})$. In particular, when $f = \chi_E$ for some Borel subset $E \subseteq \mathbb{R}$ and P denotes the corresponding spectral projection for D, $xP = Px$. Hence, x commutes with all projections in $\mathscr{P}_D$, and as $\mathcal{M}_D$ is generated as a von Neumann algebra by these projections, it follows that $x \in \mathcal{M}_D'$.

Let $x \in \mathcal{M}'_D$. For each $t \in \mathbb{R}$, $e^{itD} \in \mathcal{M}_D$. Thus, $\alpha_t(x) = e^{itD}xe^{-itD} = x$ for all $t \in \mathbb{R}$. In particular, for any $h, k \in \mathcal{H}$, the function $t \mapsto \langle \alpha_t(x)h, k\rangle = \langle xh, k\rangle$ is constant, and thus is continuously differentiable with derivative 0. Therefore, $x \in \ker \delta_D$ by Proposition 3.5. □

We now present our kernel stabilization result.

Theorem 1.2. *If D is any self-adjoint operator on a Hilbert space $\mathcal{H}$, then for every $n \in \mathbb{N}$,*

$$\ker \delta_D^n = \ker \delta_D.$$

Proof. We first show $\ker \delta_D^2 = \ker \delta_D$. The inclusion $\ker \delta_D \subseteq \ker \delta_D^2$ is clear. Let $x \in \ker \delta_D^2$. Proposition 4.5 states $\ker \delta_D = \mathcal{M}'_D$. Thus, it suffices to prove $x \in \mathcal{M}'_D$, which holds if and only if $[P, x] = 0$ for every $P \in \mathscr{P}_D$. By Proposition 3.6, if $x \in \mathrm{Dom}(\delta_D^2)$, then

(i) $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$,

(ii) $\delta_D(x)(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$, and

(iii) $\delta_D^2(x)|_{\mathrm{Dom}(D)} = [iD, \delta_D(x)]$.

Since $\delta_D^2(x) = 0$, it must be that $[iD, \delta_D(x)] = 0$. Thus, Theorem X.4.11 of [7] implies $\delta_D(x)$ commutes with the bounded Borel functional calculus for D, so, in particular, $[P, \delta_D(x)] = 0$ for every $P \in \mathscr{P}_D$. Because $\delta_D(x)$ and P both preserve the domain of D, so does the commutator $[P, \delta_D(x)]$. Thus, Lemma 4.4 implies

$$0 = [P, \delta_D(x)]|_{\mathrm{Dom}(D)} = [P, [iD, x]]|_{\mathrm{Dom}(D)} = [iD, [P, x]]|_{\mathrm{Dom}(D)}.$$

As $[P, x] \in \mathcal{B}(\mathcal{H})$, $[P, x](\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$, and $[iD, [P, x]]$ is bounded on the domain of

D, Theorem 3.2 implies $[P,x] \in \ker \delta_D$. Hence, by Proposition 4.5, $[P,x] \in \mathcal{M}'_D$. Therefore,

$$\begin{aligned}
[P,x] &= (P+P^\perp)[P,x](P+P^\perp) \\
&= P[P,x]P + P[P,x]P^\perp + P^\perp[P,x]P + P^\perp[P,x]P^\perp \\
&= P[P,x]P + PP^\perp[P,x] + P^\perp P[P,x] + P^\perp[P,x]P^\perp \\
&= P(Px-xP)P + 0 + 0 + P^\perp(Px-xP)P^\perp \\
&= PxP - PxP + 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

As $P \in \mathscr{P}_D$ was arbitrary, $x \in \mathcal{M}'_D$. By Proposition 4.5, $x \in \ker \delta_D$.

We proceed by induction on n. The case when $n = 1$ is vacuous. Suppose $\ker \delta_D^k = \ker \delta_D$ for some $k \in \mathbb{N}$. Let $x \in \ker \delta_D^{k+1}$. Then $\delta_D(x) \in \ker \delta_D^k$, which equals $\ker \delta_D$ by the inductive hypothesis. Hence, $x \in \ker \delta_D^2$. Since we have already shown $\ker \delta_D^2 = \ker \delta_D$, we have $x \in \ker \delta_D$. Therefore, $\ker \delta_D^n = \ker \delta_D$ for all $n \in \mathbb{N}$. $\square$

Remark 4.6. Let $n \in \mathbb{N}$ be arbitrary, and let $x \in \mathcal{B}(\mathcal{H})$. By Christensen's Theorem 3.7, kernel stabilization of δ_D is equivalent to the following statement: If

(i) the domains of the iterated commutators $d^k(x)$ for $k = 1, ..., n$ contain a common core $\mathscr{C}$ for D,

(ii) $d^k(x)$ is bounded on $\mathscr{C}$ for all $k = 1, ..., n$, and

(iii) the continuous bounded extension of $d^n(x)$ to all of $\mathcal{H}$ belongs to $\mathcal{M}'_D$,

then $[iD,x]|_{\mathscr{C}} = 0$.

Less formally, if $\underbrace{[iD, ..., [iD,x]]}_{n \text{ times}}$ and all lower commutators are well-defined and bounded on a common core for D, then

$$\underbrace{[iD, ..., [iD, x]]}_{n \text{ times}} = 0 \text{ implies } [iD, x] = 0.$$

This rephrasing of Theorem 1.2 in the case when $n = 2$ is equivalent to Theorem 1.6.3 of [17] in the self-adjoint setting. Putnam's proof relies on techniques in the proof of Fuglede's Theorem, whereas our proof is direct. Establishing the equivalence of these statements requires use of Christensen's work in [5].

Equivalence of Kernel Stabilization to a Result of C.R. Putnam

Theorem 4.7 (Putnam, 1.6.3 [17])**.** Suppose D is normal and $x, y \in \mathcal{B}(\mathcal{H})$. If

1. $xD + y \subset Dx$ and

2. $yD \subset Dy$,

then $y = 0$.

We claim that when D is self-adjoint, Theorem 4.7 is equivalent to Theorem 1.2 in the case when $n = 2$. To show this, we show hypotheses (1) and (2) of Putnam's Theorem 4.7 are equivalent to the hypothesis in Theorem 1.2.

(1) Note that the domain of $xD + y$ is $\mathrm{Dom}(D)$ because y is bounded, and

$$\mathrm{Dom}(D)x = \{f \in \mathcal{H} : xf \in \mathrm{Dom}(D)\}.$$

To say $xD + y \subset Dx$ is to say that there is an inclusion of these operators' graphs.

Hence,

$$\begin{aligned}\Gamma(xD+y)\subset\Gamma(Dx) &\iff \{(h,xDh+yh):h\in\mathrm{Dom}(D)\}\subset\{(k,Dxk):k\in\mathrm{Dom}(Dx)\}\\ &\iff \mathrm{Dom}(D)\subset\mathrm{Dom}(Dx)\ \textbf{and}\ xDh+yh=Dxh\ \ \forall h\in\mathrm{Dom}(D)\\ &\iff \mathrm{Dom}(D)\subset\{f\in H:xf\in\mathrm{Dom}(D)\}\ \textbf{and}\ [D,x]h=yh\ \ \forall h\in\mathrm{Dom}(D)\\ &\iff x(\mathrm{Dom}(D))\subset\mathrm{Dom}(D)\ \textbf{and}\ [D,x]h=yh\ \ \forall h\in\mathrm{Dom}(D).\end{aligned}$$

(2) Similarly, $yD\subset Dy$ is an inclusion of these operators' graphs. Note that the domain of yD is the domain of D, while

$$\mathrm{Dom}(Dy)=\{f\in\mathcal{H}:yf\in\mathrm{Dom}(D)\}.$$

Thus,

$$\begin{aligned}\Gamma(yD)\subset\Gamma(Dy) &\iff \{(h,yDh):h\in\mathrm{Dom}(D)\}\subset\{(k,Dyk):k\in\mathrm{Dom}(Dy)\}\\ &\iff \mathrm{Dom}(D)\subset\mathrm{Dom}(Dy)\ \textbf{and}\ yDh=Dyh\ \ \forall h\in\mathrm{Dom}(D)\\ &\iff \mathrm{Dom}(D)\subset\{f\in H:yf\in\mathrm{Dom}(D)\}\ \textbf{and}\ [D,y]h=0\ \ \forall h\in\mathrm{Dom}(D)\\ &\iff y(\mathrm{Dom}(D))\subset\mathrm{Dom}(D)\ \textbf{and}\ [D,y]h=0\ \ \forall h\in\mathrm{Dom}(D).\end{aligned}$$

The content of Theorem 1.2 in the case when $n=2$ is $\ker\delta_D^2\subseteq\ker\delta_D$. We break the hypothesis that $x\in\ker_D^2$ into two simpler hypotheses:

(I) $x\in\mathrm{Dom}(\delta_D)$

(II) $y:=\delta_D(x)\in\mathrm{Dom}(\delta_D)$ **and** $\delta_D(y)=0$.

Below we rewrite (I) and (II) using Christensen's Theorem 3.2.

(I) By Theorem 3.2,

$$
\begin{aligned}
x \in \mathrm{Dom}(\delta_D) &\iff \exists y \in \mathcal{B}(\mathcal{H}) \text{ st. } [iD, x]|_{\mathrm{Dom}(D)} = y|_{\mathrm{Dom}(D)} \\
&\iff Dx - xD \text{ is well-defined on } \mathrm{Dom}(D) \\
&\qquad \textbf{and } \exists y \in \mathcal{B}(\mathcal{H}) \text{ s.t. } [iD, x]|_{\mathrm{Dom}(D)} = y|_{\mathrm{Dom}(D)} \\
&\iff x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D) \textbf{ and } \exists y \in \mathcal{B}(\mathcal{H}) \text{ s.t. } [iD, x]h = yh \ \ \forall h \in \mathrm{Dom}(D) \\
&\iff (1).
\end{aligned}
$$

(II) Again by Theorem 3.2,

$$
\begin{aligned}
y \in \mathrm{Dom}(\delta_D) \textbf{ and } \delta_D(y) = 0 &\iff [D, y] \text{ is well-defined on } \mathrm{Dom}(D) \textbf{ and } [D, y]|_{\mathrm{Dom}(D)} = 0 \\
&\iff y(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D) \textbf{ and } [D, y]h = 0 \ \ \forall h \in \mathrm{Dom}(D) \\
&\iff (2).
\end{aligned}
$$

We have established that the statement of Theorem 1.2 in the $n = 2$ case is equivalent to Theorem 4.7 in the self-adjoint setting.

The proofs of both Theorems 1.2 and 4.7 rely heavily on the Spectral Theorem for normal operators. However, the kernel stabilization result depends only on independently-proven facts about commutators of $x \in \mathcal{B}(\mathcal{H})$ with spectral projections for D, while Putnam's theorem is stated as a corollary to Fuglede's Theorem. Of course, Fuglede's Theorem makes great use of spectral projections for normal operators, but our proof is more direct. We then applied a simple inductive argument to get kernel stabilization for all higher powers of δ_D.

4.3 Applications

Abstract Derivations on C^*-algebras

Given a self-adjoint operator D, our proof of kernel stabilization of δ_D relied on the relationship between δ_D and commutation with D. Intuitively, then, kernel stabilization is likely to occur for a derivation δ on an abstract C^*-algebra that can be implemented, under an appropriate representation, as commutation with a self-adjoint operator. Theorem 1.3 provides sufficient conditions for when a derivation on a C^*-algebra has such a representation.

Under this representation, Bratteli and Robinson construct an essentially self-adjoint operator S which implements the derivation's action as commutation with S. Once this essentially self-adjoint operator is defined, we use its self-adjoint closure $D = \overline{S}$ to generate a corresponding weak-D derivation δ_D. We shall show δ_D extends $\delta \circ \pi$ and then apply Theorem 1.2 (kernel stabilization of δ_D) to obtain kernel stabilization of δ.

Theorem 1.3 (Bratteli-Robinson, 4 [3]). *Let δ be a derivation of a C^*-algebra $\mathcal{A}$, and assume there exists a state ω on $\mathcal{A}$ which generates a faithful cyclic representation (π, H, f) satisfying*

$$\omega(\delta(a)) = 0 \text{ for all } a \in \mathrm{Dom}(\delta).$$

Then δ is closable and there exists a symmetric operator S on $\mathcal{H}$ such that

$$\mathrm{Dom}(S) = \{h \in \mathcal{H} : h = \pi(a)f \ \text{ for some } a \in \mathcal{A}\}$$

and $\pi(\delta(a))h = [S, \pi(a)]h$, for all $a \in \mathrm{Dom}(\delta)$ and all $h \in \mathrm{Dom}(S)$. Moreover, if the set $\mathsf{A}(\delta)$ of analytic vectors for δ is dense in $\mathcal{A}$, then S is essentially self-adjoint on $\mathrm{Dom}(S)$. For $x \in \mathcal{B}(\mathcal{H})$ and $t \in \mathbb{R}$, define

$$\alpha_t(x) := e^{i\overline{S}t} x e^{-i\overline{S}t}$$

where $\overline{S}$ *denotes the self-adjoint closure of* S*. It follows that* $\alpha_t(\pi(\mathcal{A})) = \pi(\mathcal{A})$ *for all* $t \in \mathbb{R}$*, and* $\{\alpha_t\}_{t\in\mathbb{R}}$ *is a strongly continuous group of* $*$*-automorphisms with closed infinitesimal generator* $\widetilde{\delta}$ *equaling the closure of* $\pi \circ \delta|_{\mathsf{A}(\delta)}$*.*

The condition that there exist a state ω on $\mathcal{A}$ which satisfies $\omega(\delta(a)) = 0$ for all $a \in \mathrm{Dom}(\delta)$ physically represents the presence of an *equilibrium state* for the C^*-algebra $\mathcal{A}$ of observables for a physical system with time evolution described by δ. If δ were the infinitesimal generator for a one-parameter group of $*$-automorphisms $\{\beta_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$, then $\omega(\beta_t(a)) = \omega(a)$ for all $t \in \mathbb{R}$ would be an equivalent condition to require, and this condition more commonly describes an equilibrium state. However, δ is an abstract derivation on $\mathcal{A}$ with norm-dense domain, so there may not be a one-parameter group of $*$-automorphisms for which δ is the infinitesimal generator.

Under the representation π, however, δ is implemented by commutation with S, whose closure provides unitaries from which we can build a one-parameter group of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ on $\mathcal{B}(\mathcal{H})$. We relate the infinitesimal generator $\widetilde{\delta}$ for $\{\alpha_t\}_{t\in\mathbb{R}}$ in Theorem 1.3 to a derivation δ_u studied by Christensen.

Definition 4.8. Let D be a self-adjoint operator on a Hilbert space $\mathcal{H}$. An operator $x \in \mathcal{B}(\mathcal{H})$ is *uniformly D-differentiable* if there exists $y \in \mathcal{B}(\mathcal{H})$ such that

$$\lim_{t\to 0} \left\| \frac{e^{itD}xe^{-itD} - x}{t} - y \right\| = 0. \qquad (*)$$

We denote this by $x \in \mathrm{Dom}(\delta_u)$ and set $\delta_u(x) = y$, where y satisfies condition $(*)$.

Remark 4.9. Let S and $\widetilde{\delta}$ be as in Theorem 1.3, and let $D = \overline{S}$. Then $\widetilde{\delta}$ from Theorem 1.3 and δ_u from Definition 4.8 are the same derivations with the same domains.

Proposition 4.10. *If* D *is a self-adjoint operator, then* $\ker \delta_u = \ker \delta_D$.

Proof. Theorem 4.1 of [6] states $x \in \mathrm{Dom}(\delta_u)$ if and only if $x \in \mathrm{Dom}(\delta_D)$ and $t \mapsto \alpha_t(\delta_D(x))$ is norm-continuous. Moreover, δ_D extends δ_u. Thus, $\ker \delta_u \subseteq \ker \delta_D$.

Let $x \in \ker \delta_D$. Then $t \mapsto \alpha_t(\delta_D(x)) = 0$ is norm-continuous, and hence, $x \in \mathrm{Dom}(\delta_u)$. Moreover, $\delta_u(x) = [\delta_D|_{\mathrm{Dom}(\delta_u)}](x) = 0$. Therefore, $x \in \ker \delta_u$. We conclude $\ker \delta_D = \ker \delta_u$. □

Corollary 4.11. *For all $n \in \mathbb{N}$, $\ker \delta_u^n = \ker \delta_u$.*

Proof. Fix $n \in \mathbb{N}$ and let $x \in \ker \delta_u^n$. Then $x \in \mathrm{Dom}(\delta_u^n) \subseteq \mathrm{Dom}(\delta_D^n)$ and $\delta_D^n(x) = \delta_u^n(x) = 0$. Therefore, $x \in \ker \delta_D^n$, so by Theorem 1.2, $x \in \ker \delta_D$. By Proposition 4.10, $\ker \delta_D = \ker \delta_u$, so we conclude $x \in \ker \delta_u$. Thus, $\ker \delta_u^n = \ker \delta_u$ for all $n \in \mathbb{N}$, as claimed. □

Lemma 4.12. *If δ, $\mathcal{A}$, π, and $\widetilde{\delta}$ are as in Theorem 1.3, then*

$$\ker \widetilde{\delta}^n \cap \pi(\mathsf{A}(\delta)) = \pi(\ker \delta^n) \text{ for all } n \in \mathbb{N}.$$

Proof. Fix $n \in \mathbb{N}$. If $a \in \mathsf{A}(\delta)$, then $a \in \mathrm{Dom}(\delta^n)$ and $\delta^n(a) \in \mathsf{A}(\delta)$. Theorem 1.3 states $\widetilde{\delta}(\pi(b)) = \pi(\delta(b))$ for all $b \in \mathsf{A}(\delta)$. Thus, as $\delta^n(a) \in \mathsf{A}(\delta)$, we have $\widetilde{\delta}^n(\pi(a)) = \pi(\delta^n(a))$. Suppose $\widetilde{\delta}^n(\pi(a)) = 0$. Then $\pi(\delta^n(a)) = \widetilde{\delta}^n(\pi(a)) = 0$, and since π is faithful, $\delta^n(a) = 0$. Therefore, $\pi(a) \in \pi(\ker \delta^n)$.

Conversely, suppose $a \in \ker \delta^n$. Then $a \in \mathsf{A}(\delta)$ because $\delta^j(a) = 0$ for all $j \geq n$ and $\sum_{k=0}^{\infty} \frac{t^k}{k!} \left\|\delta^k(a)\right\| = \sum_{k=0}^{n-1} \frac{t^k}{k!} \left\|\delta^k(a)\right\| < \infty$ for any choice of $t > 0$. Similar to above, $\widetilde{\delta}^n(\pi(a)) = \pi(\delta^n(a)) = \pi(0) = 0$. Therefore, $\pi(a) \in \ker \widetilde{\delta}^n \cap \pi(\mathsf{A}(\delta))$. As $a \in \mathcal{A}$ was arbitrary, $\ker \widetilde{\delta}^n \cap \pi(\mathsf{A}(\delta)) = \pi(\ker \delta^n)$. Finally, because $n \in \mathbb{N}$ was arbitrary, this equality holds for all $n \in \mathbb{N}$. □

Theorem 4.13. *If δ, $\mathcal{A}$, π, $\widetilde{\delta}$, and S are as in Theorem 1.3, then* $\ker \delta^n = \ker \delta$.

Proof. Fix $n \in \mathbb{N}$, and let $a \in \ker \delta^n$. Then $a \in \mathsf{A}(\delta)$ and $\pi(a) \in \ker \widetilde{\delta}^n$ by Lemma 4.12. Note $\widetilde{\delta} = \delta_u$ where $D = \overline{S}$, so Proposition 4.11 implies $\ker \widetilde{\delta}^n = \ker \widetilde{\delta}$ for all $n \in \mathbb{N}$. Hence,

$\pi(a) \in \ker \widetilde{\delta} \cap \pi(\mathsf{A}(\delta))$. By another application of Lemma 4.12, we get $a \in \ker \delta$. Therefore, $\ker \delta^n = \ker \delta$ for all $n \in \mathbb{N}$. □

The Heisenberg Commutation Relation

Our second application of Theorem 1.2 gives a sufficient condition for when two self-adjoint operators which satisfy the Heisenberg Commutation Relation must both be unbounded.

Definition 1.5. Let A and B be two (possibly unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$, with domains $\mathrm{Dom}(A)$ and $\mathrm{Dom}(B)$, respectively. We say A and B *satisfy the Heisenberg Commutation Relation* (HCR) if there is a dense subspace K of $\mathcal{H}$ such that

(i) $K \subseteq \mathrm{Dom}([A, B])$ and

(ii) $[A, B]k = ik$ for all $k \in K$.

Definition 4.14. The classical example of a pair satisfying the HCR is the *Schrödinger pair*, the quantum mechanical position operator Q and momentum operator P on $L^2(\mathbb{R})$ from Examples 2.6 and 2.9.

Let $S(\mathbb{R})$ denote the *Schwartz space* on $\mathbb{R}$:

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall m, n \in \mathbb{N},\ \|Q^m P^n f\|_\infty < \infty\}.$$

Proposition X.6.5 of [7] shows $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, and it is clear from its definition that $S(\mathbb{R})$ is contained in $\mathrm{Dom}(Q) \cap \mathrm{Dom}(P)$ and is invariant under both Q and P. Hence, $S(\mathbb{R}) \subseteq \mathrm{Dom}([Q, P])$. Furthermore, $[Q, P]g = ig$ for all $g \in S(\mathbb{R})$. Therefore, Q and P satisfy the HCR.

If two operators are unitarily equivalent to a direct sum of copies of the Schrödinger pair, they are certainly both unbounded, and it is well-known that no two bounded operators may satisfy the HCR. Below is a well-known example of a pair of operators satisfying the HCR where one operator is bounded.

Example 4.15. For $f \in L^2([0,1])$, define $(Af)(x) = xf(x)$ for a.e. $x \in [0,1]$. In contrast to its unbounded analogue Q, the operator A is contractive. Let $AC([0,1])$ denote the set of functions which are absolutely continuous on $[0,1]$, and let

$$\mathrm{Dom}(B) = \{f \in AC[0,1] : f' \in L^2([0,1]),\ f(0) = f(1)\}.$$

For $g \in \mathrm{Dom}(B)$, define $Bg = -ig'$. Example X.1.12 of [7] shows the operator B with this particular domain is self-adjoint. Due to boundedness of A,

$$\mathrm{Dom}([A,B]) = \{f \in \mathrm{Dom}(B) : Af \in \mathrm{Dom}(B)\}.$$

Choose

$$K := \{f \in AC([0,1]) : f' \in L^2([0,1]),\ f(0) = f(1) = 0\}.$$

Example X.1.11 of [7] shows K is dense in $L^2([0,1])$ as it contains all polynomials p on $[0,1]$ satisfying $p(0) = p(1) = 0$. Furthermore, we claim K is invariant for A. Indeed, products of absolutely continuous functions are again absolutely continuous, so $(Ag)(x) = xg(x)$ for a.e. $x \in [0,1]$ defines an absolutely continuous function. The a.e.-defined derivative of Ag is equivalent to $Ag' + g$ by the product rule. Moreover, $Ag' + g$ belongs to $L^2([0,1])$ as $g' \in L^2([0,1])$ and $A \in \mathcal{B}(L^2([0,1]))$. Lastly,

$$(Ag)(0) = 0 \cdot g(0) = 0 = 1 \cdot 0 = 1 \cdot g(1) = (Ag)(1).$$

Thus, $AK \subseteq K$. As a result, $K \subseteq \mathrm{Dom}([A, B])$. For $k \in K$, observe

$$[A, B]k = A(-ik') - B(Ak) = -iAk' - (-i)[Ak' + k] = ik.$$

Therefore, A and B satisfy the HCR.

We claim the boundedness of the operators in Examples 4.14 and 4.15 is due to the relative size of $\mathrm{Dom}([Q, P])$ in $L^2(\mathbb{R})$ versus $\mathrm{Dom}([A, B])$ in $L^2([0, 1])$. In particular, $\mathrm{Dom}([A, B])$ does not contain a core for A or B, while $\mathrm{Dom}([Q, P])$ contains a core for both Q and P.

Theorem 1.6. *Let A and B be self-adjoint operators which satisfy the HCR on a dense subspace $K \subseteq \mathcal{H}$. If K is a core for A and B, then A and B are both unbounded.*

Proof. Suppose that K is a core for both A and B. It is well-known that A and B cannot both be bounded and satisfy the Heisenberg Relation. Thus, without loss of generality, the only possibilities are that A is bounded and B is unbounded, or both A and B are unbounded. Suppose that $A \in \mathcal{B}(\mathcal{H})$. Note that $[A, B]k = ik$ for all $k \in K$ if and only if $[iB, A]k = k$ for all $k \in K$.

As K is a core for B and $\|[iB, A]|_K\| = 1$, we have that $A \in \mathrm{Dom}(\delta_B)$. Furthermore, $\delta_B(A)$ is the continuous extension of the bounded and densely-defined operator $[iB, A]|_K$ to all of $\mathcal{H}$, and thus, $\delta_B(A) = I$. Trivially, $I \in \mathrm{Dom}(\delta_B)$ and $\delta_B(I) = 0$, so $A \in \mathrm{Dom}(\delta_B^2)$ and $\delta_B^2(A) = 0$. Since $A \in \ker \delta_B^2$, Theorem 1.2 implies $A \in \ker \delta_B$. But then

$$0 = \delta_B(A)|_K = [iB, A]|_K = I|_K,$$

which is absurd. Therefore, A cannot be bounded. We conclude that if A and B satisfy the HCR on a common core for A and B, then A and B must both be unbounded. $\square$

Chapter 5

A Covariant Stone-von Neumann Theorem

5.1 $(G, \mathcal{A}, \alpha)$-Heisenberg and Schrödinger Representations

Throughout, G is a locally compact abelian group with Haar measure μ and dual group $\widehat{G}$ with Haar measure $\hat{\mu}$. As defined in Definition 1.8, the Schrödinger representation (λ, V) for a locally compact abelian group G is an example of a Heisenberg representation for G. We seek to generalize the definition of this pair to a representation of a C^*-dynamical system $(G, \mathcal{A}, \alpha)$ on a Hilbert $\mathcal{A}$-module.

Definition 5.1. A $(G, \mathcal{A}, \alpha)$-*Heisenberg representation* is a quadruple (X, ρ, r, s) with the following properties:

(i) X is a full Hilbert $\mathcal{A}$-module.

(ii) $\rho : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ is a nondegenerate $*$-representation.

(iii) $r : G \to \mathcal{U}(\mathsf{X})$ is a (strictly continuous) unitary group representation.

(iv) $s : \widehat{G} \to \mathcal{U}(\mathsf{X})$ is a (strictly continuous) unitary group representation.

(v) $s_\gamma r_x = \gamma(x) r_x s_\gamma$ for all $x \in G$ and $\gamma \in \widehat{G}$.

(vi) (ρ, r) is a nondegenerate covariant homomorphism of $(G, \mathcal{A}, \alpha)$ into X.

(vii) $\rho(a)s_\gamma = s_\gamma\rho(a)$ for all $a \in \mathcal{A}$ and $\gamma \in \widehat{G}$.

When $\mathcal{A} = \mathbb{C}$, we recover the definition of a classical Heisenberg representation. To define the $(G, \mathcal{A}, \alpha)$-Schrödinger representation, consider the right Hilbert $\mathcal{A}$-module $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, defined in Example 2.30, which we recall here for convenience. For each $\phi \in C_c(G, \mathcal{A})$ and $a \in \mathcal{A}$, define

$$[\phi \bullet a](x) := f(x)\alpha_x(a) \text{ for all } x \in G.$$

Then $\bullet$ makes $C_c(G, \mathcal{A})$ into a right $\mathcal{A}$-module. For $\phi, \psi \in C_c(G, \mathcal{A})$, define

$$\langle \psi \,|\, \phi \rangle := \int_G \alpha_{x^{-1}} \left(\psi(x)^* \phi(x)\right) \, d\mu(x).$$

We denote the completion of $C_c(G, \mathcal{A})$ with respect to the induced norm $\|\cdot\|_\alpha := \|\langle \cdot \,|\, \cdot \rangle\|_{\mathcal{A}}^{1/2}$ by $\mathsf{L}^2(G, \mathcal{A}, \alpha)$. Next, consider the map $\mathsf{M} : \mathcal{A} \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ defined on $\phi \in C_c(G, \mathcal{A})$ by

$$[\mathsf{M}(a)\phi](x) := a\phi(x) \text{ for all } x \in G.$$

Proposition 5.2. $\mathsf{M} : \mathcal{A} \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ *is a well-defined nondegenerate $*$-representation.*

Proof. Fix $a \in \mathcal{A}$. First we show $\mathsf{M}(a)|_{C_c(G,\mathcal{A})}$ is bounded with respect to $\|\cdot\|_\alpha$, and by $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, A, \alpha)$, we may continuously extend $\mathsf{M}(a)$ to all of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$.

Recall that for any element d of a unital C^*-algebra $\mathcal{B}$ with unit e, $d^*d \leq_{\mathcal{B}} \|d\|^2 e$, where $\leq_{\mathcal{B}}$ is the ordering on the positive elements in $\mathcal{B}$. Let $\phi \in C_c(G, \mathcal{A})$. Using an approximate identity argument and Theorem 2.2.5(b) of [15], we have that

$$\phi(x)^*(a^*a)\phi(x) \leq_{\mathcal{A}} \phi(x)^* \|a^*a\| \, \phi(x) = \|a\|^2 \, \phi(x)^*\phi(x).$$

Observe

$$\begin{aligned}
\langle \mathsf{M}(a)\phi \,|\, \mathsf{M}(a)\phi\rangle &= \int_G \alpha_{x^{-1}}((a\phi(x))^* a\phi(x))\, d\mu(x) \\
&= \int_G \alpha_{x^{-1}}(\phi(x)^* a^* a\phi(x))\, d\mu(x) \\
&\leq_{\mathcal{A}} \int_G \alpha_{x^{-1}}(\|a\|_{\mathcal{A}}^2\, \phi(x)^*\phi(x))\, d\mu(x) \\
&= \|a\|_{\mathcal{A}}^2 \langle \phi \,|\, \phi\rangle
\end{aligned}$$

Theorem 2.2.5(c) of [15] implies $\|\langle \mathsf{M}(a)\phi \,|\, \mathsf{M}(a)\phi\rangle\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}}^2 \|\langle \phi \,|\, \phi\rangle\|_{\mathcal{A}}$. Therefore,

$$\|\mathsf{M}(a)\phi\|_{\alpha}^2 \leq \|a\|_{\mathcal{A}}^2 \|\phi\|_{\alpha}^2 ,$$

so $\mathsf{M}(a)|_{C_c(G,\mathcal{A})}$ is $\|\cdot\|_{\alpha}$-continuous. Similarly, so is $\mathsf{M}(a^*)$. For $\phi, \psi \in C_c(G, \mathcal{A})$,

$$\langle \psi \,|\, \mathsf{M}(a)\phi\rangle = \int_G \alpha_{x^{-1}}(\psi(x)^* a\phi(x))\, d\mu(x) = \int_G \alpha_{x^{-1}}([a^*\psi(x)]^*\phi(x))\, d\mu(x) = \langle \mathsf{M}(a^*)\psi \,|\, \phi\rangle .$$

As $\mathsf{M}(a)$ and $\mathsf{M}(a^*)$ are both $\|\cdot\|_{\alpha}$-continuous, this equality of inner products holds on arbitrary elements of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$. Therefore $\mathsf{M}(a^*) = \mathsf{M}(a)^*$, so $\mathsf{M}(a) \in \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$. Moreover, M is clearly linear, multiplicative, and $*$-preserving, so M is a well-defined $*$-representation of $\mathcal{A}$. We now show M is nondegenerate.

Fix $\phi \in C_c(G, \mathcal{A})$. As $\text{Range}(\phi) \subseteq \phi[\text{Supp}(\phi)] \cup \{0_{\mathcal{A}}\}$, and as $\overline{\text{Supp}(\phi)}$ is a compact subset of G, we see that $\text{Range}(\phi)$ is contained in a compact subset of $\mathcal{A}$. Compact subsets of metric spaces are separable, and subsets of separable subsets of metric spaces are separable, so in particular, $\text{Range}(\phi)$ is a separable subset of $\mathcal{A}$. Let D be a countable dense subset of $\text{Range}(\phi)$. If $\mathcal{B}$ denotes the C^*-subalgebra of $\mathcal{A}$ generated by $\text{Range}(\phi)$, then $\mathcal{B}$ is also the C^*-subalgebra of $\mathcal{A}$ generated by D. Hence, $\mathcal{B}$ is a separable C^*-algebra, which means that it possesses a sequential approximate identity $(e_n)_{n\in\mathbb{N}}$. Now, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|\phi - \mathsf{M}(e_n)\phi\|_\alpha &= \|\langle \phi - \mathsf{M}(e_n)\phi \,|\, \phi - \mathsf{M}(e_n)\phi\rangle\|_{\mathcal{A}}^{1/2} \\
&= \left\| \int_G \alpha_{x^{-1}}([\phi(x) - e_n\phi(x)]^* [\phi(x) - e_n\phi(x)]) \ d\mu(x) \right\|_{\mathcal{A}}^{\frac{1}{2}} \\
&\leq \left[\int_G \|\alpha_{x^{-1}}([\phi(x) - e_n\phi(x)]^* [\phi(x) - e_n\phi(x)])\|_{\mathcal{A}} \ d\mu(x) \right]^{\frac{1}{2}} \\
&= \left[\int_G \|[\phi(x) - e_n\phi(x)]^* [\phi(x) - e_n\phi(x)]\|_{\mathcal{A}} \ d\mu(x) \right]^{\frac{1}{2}} \\
&= \left[\int_G \|\phi(x) - e_n\phi(x)\|_{\mathcal{A}}^2 \ d\mu(x) \right]^{\frac{1}{2}}.
\end{aligned}$$

Next, notice for all $n \in \mathbb{N}$ and $x \in G$ that

$$\begin{aligned}
\|\phi(x) - e_n\phi(x)\|_{\mathcal{A}}^2 &\leq [\|\phi(x)\|_{\mathcal{A}} + \|e_n\phi(x)\|_{\mathcal{A}}]^2 \\
&\leq [\|\phi(x)\|_{\mathcal{A}} + \|e_n\|_{A}\|\phi(x)\|_{\mathcal{A}}]^2 \\
&\leq [\|\phi(x)\|_{\mathcal{A}} + \|\phi(x)\|_{A}]^2 \qquad (\text{As } \|e_n\|_{\mathcal{A}} \leq 1.) \\
&= 4\|\phi(x)\|_{\mathcal{A}}^2.
\end{aligned}$$

Hence, the $\mathbb{R}$-valued sequence of functions $\{\|\phi(\cdot) - e_n\phi(\cdot)\|_{\mathcal{A}}^2\}_{n\in\mathbb{N}}$ is dominated by the integrable function $x \mapsto 4\|\phi(x)\|_{\mathcal{A}}^2$. As this sequence converges pointwise to 0, the Lebesgue Dominated Convergence Theorem yields

$$\lim_{n\to\infty} \|\phi - \mathsf{M}(e_n)\phi\|_\alpha = 0.$$

Finally, an $\frac{\epsilon}{3}$-argument shows that for any $\Phi \in \mathsf{L}^2(G, \mathcal{A}, \alpha)$ and any $\epsilon > 0$, there exists an $a \in \mathcal{A}$ such that $\|\Phi - \mathsf{M}(a)\Phi\|_\alpha < \epsilon$. Therefore, M is nondegenerate. □

Next we define $u : G \to \mathcal{U}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$, where for each $\phi \in C_c(G, \mathcal{A})$,

$$[u_x \phi](y) := \alpha_x(\phi(x^{-1}y)) \text{ for all } y \in G.$$

A similar argument as in Proposition 5.2 shows that $u_x \in \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ with adjoint $u_x^* = u_{x^{-1}}$ for each $x \in G$. Note $u_x|_{C_c(G,\mathcal{A})} = \alpha_x \circ \mathsf{lt}_x$. Thus, as $\alpha_x \in \mathrm{Aut}(\mathcal{A})$ and $\mathsf{lt}_x \in \mathrm{Aut}(C_o(G, \mathcal{A}))$ are norm-continuous, strict continuity of the map $x \mapsto u_x|_{C_c(G,\mathcal{A})}$ follows immediately. Finally, $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ implies strict continuity holds for the mapping $x \mapsto u_x$. Therefore, $u : G \to \mathcal{U}(\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is a strictly continuous unitary group representation.

Last, consider $v : \widehat{G} \to \mathcal{U}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ given by $\gamma \mapsto v_\gamma$, which acts on $\phi \in C_c(G, \mathcal{A})$ by

$$[v_\gamma \phi](y) := \gamma(y)\phi(y) \text{ for all } y \in G.$$

Note $\|v_\gamma \phi - \phi\|_{C_c(G,\mathcal{A})} = \|\gamma \cdot \phi - \phi\|_{C_c(G,\mathcal{A})} = \|\gamma - \mathbf{1}\|_\infty \cdot \|\phi\|_{C_c(G,\mathcal{A})} \to 0$ as $\gamma \to 0$. By Corollary 2.33, we have $\|v_\gamma \phi - \phi\|_\alpha \to 0$ as $\gamma \to 0$. Therefore, $\gamma \mapsto v_\gamma|_{C_c(G,\mathcal{A})}$ is strongly, and thus strictly, continuous. By $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, strict continuity holds for the mapping $\gamma \mapsto v_\gamma$. We conclude $v : \widehat{G} \to \mathcal{U}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ is a strictly continuous unitary group representation.

Definition 5.3. The $(G, \mathcal{A}, \alpha)$-*Schrödinger representation* is the quadruple $(\mathsf{L}^2(G, \mathcal{A}, \alpha), \mathsf{M}, u, v)$.

When $\mathcal{A} = \mathbb{C}$, we recover the classical Schrödinger representation (λ, V) of G.

Proposition 5.4. *The* $(G, \mathcal{A}, \alpha)$*-Schrödinger representation is a* $(G, \mathcal{A}, \alpha)$*-Heisenberg representation.*

Proof. Fullness of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is established in Theorem 5.7, and nondegeneracy of M is given in Proposition 5.2. By above, u and v are (strictly continuous) unitary group representations of G and $\widehat{G}$, respectively. Fix $x \in G$ and $\gamma \in \widehat{G}$. Then for all $y \in G$ and $\phi \in C_c(G, \mathcal{A})$,

$$\begin{aligned} ([v_\gamma u_x]\phi)](y) &= \gamma(y) \cdot [u_x \phi](y) \\ &= \gamma(xx^{-1}y) \cdot \alpha_x(\phi(x^{-1}y)) \\ &= \gamma(x) \cdot \gamma(x^{-1}y) \cdot \alpha_x(\phi(x^{-1}y)) \\ &= \gamma(x) \cdot \alpha_x([\gamma \cdot \phi](x^{-1}y)) \\ &= \gamma(x)[u_x v_\gamma \phi](y). \end{aligned}$$

As $y \in G$ was arbitrary, $[v_\gamma u_x]\phi = \gamma(x) \cdot [u_x v_\gamma]\phi$ for all $\phi \in C_c(G, \mathcal{A})$, and as $\phi \in C_c(G, \mathcal{A})$ was arbitrary, this holds for any $\phi \in C_c(G, \mathcal{A})$. By $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ and $\|\cdot\|_\alpha$-continuity of both u_x and v_γ, we have $v_\gamma u_x = \gamma(x) \cdot u_x v_\gamma$. As $x \in G$ and $\gamma \in \widehat{G}$ were arbitrary, this equality holds for all $x \in G$ and $\gamma \in \widehat{G}$, so the pair (u, v) satisfies the Weyl Commutation Relation.

Next we show (M, u) is a covariant homomorphism for $(G, \mathcal{A}, \alpha)$. Fix $x \in G$ and $a \in \mathcal{A}$. For any $\phi \in C_c(G, \mathcal{A})$ and $y \in G$, observe

$$([u_x \mathsf{M}(a)]\phi)(y) = \alpha_x(a\phi(x^{-1}y)) = \alpha_x(a)\alpha_x(\phi(x^{-1}y)) = ([\mathsf{M}(\alpha_x(a))u_x]\phi)(y).$$

As $y \in G$ was arbitrary, $[u_x \mathsf{M}(a)]\phi = [\mathsf{M}(\alpha_x(a))u_x]\phi$. As $\phi \in C_c(G, \mathcal{A})$ was arbitrary, this holds for all $\phi \in C_c(G, \mathcal{A})$. By $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ and $\|\cdot\|_\alpha$-continuity of the adjointable operators u_x, $\mathsf{M}(a)$, and $\mathsf{M}(\alpha_x(a))$, we have $u_x \mathsf{M}(a) = \mathsf{M}(\alpha_x(a))u_x$. Since $x \in G$ and $a \in \mathcal{A}$ were arbitrary, this equality holds for all $x \in G$ and $a \in \mathcal{A}$. Therefore, (M, u) is a covariant homomorphism.

Last, for fixed $\gamma \in \widehat{G}$ and $a \in \mathcal{A}$, note that for each $\phi \in C_c(G, \mathcal{A})$,

$$([v_\gamma \mathsf{M}(a)]\phi)(y) = \gamma(y) \cdot a\phi(y) = a(\gamma(y) \cdot \phi(y)) = ([\mathsf{M}(a)v_\gamma]\phi)(y) \text{ for all } y \in G.$$

By similar reasoning as above, we have that $v_\gamma \mathsf{M}(a) = \mathsf{M}(a) v_\gamma$ for any $\gamma \in \widehat{G}$ and $a \in \mathcal{A}$. It follows that v and M are commuting representations. Therefore, $(\mathsf{L}^2(G, \mathcal{A}, \alpha), \mathsf{M}, u, v)$ is a $(G, \mathcal{A}, \alpha)$-Heisenberg representation. □

The ultimate goal of this chapter is to prove Theorem 1.11, which states that every $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation is unitarily equivalent to a direct sum of copies of the $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Schrödinger representation. We call this the "Covariant Stone-von Neumann Theorem."

5.2 Green's Imprimitivity Theorem

The Stone-von Neumann Theorem relies on the C^*-isomorphism $C_o(G) \rtimes_{\mathsf{lt}} G \cong \mathcal{K}(L^2(G))$. In [25] this isomorphism is given by the integrated form of the covariant pair (M, λ), where $M : C_o(G) \to \mathcal{B}(L^2(G))$ takes $f \in C_o(G)$ to the bounded multiplication operator M_f and $\lambda : G \to \mathcal{U}(L^2(G))$ is the left regular representation. Our required generalization of this isomorphism is achieved via Green's Imprimitivity Theorem and Proposition 3.8 of [18].

Definition 5.5 (Rieffel)**.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are C^*-algebras and X is a left Hilbert $\mathcal{C}$-module, a right Hilbert $\mathcal{D}$-module, and a $\mathcal{C}$-$\mathcal{D}$ bimodule. Then X is a $\mathcal{C}$-$\mathcal{D}$ *imprimitivity bimodule* if

(i) X is full as both a Hilbert $\mathcal{C}$-module and Hilbert $\mathcal{D}$-module and

(ii) ${}_{\mathcal{C}}\langle x \,|\, y \rangle \bullet z = x \bullet \langle y \,|\, z \rangle_{\mathcal{D}}$ for all $x, y, z \in \mathsf{X}$

where ${}_{\mathcal{C}}\langle \cdot \,|\, \cdot \rangle$ denotes the inner product on X as a left Hilbert $\mathcal{C}$-module and $\langle \cdot \,|\, \cdot \rangle_{\mathcal{D}}$ denotes the inner product on X as a right Hilbert $\mathcal{D}$-module.

Remark 5.6 (Brown-Mingo-Shen, 1.9 [21])**.** As a consequence of (ii), a $\mathcal{C}$-$\mathcal{D}$ imprimitivity bimodule X also satisfies

$$ {}_{\mathcal{C}}\langle x \bullet d \,|\, y \rangle = {}_{\mathcal{C}}\langle x \,|\, y \bullet d^* \rangle \text{ for all } x, y \in \mathsf{X},\ d \in \mathcal{D} $$

and

$$\langle c \bullet x \,|\, y \rangle_{\mathcal{D}} = \langle x \,|\, c^* \bullet y \rangle_{\mathcal{D}} \text{ for all } x, y \in \mathsf{X},\ c \in \mathcal{C}.$$

Moreover, the norms induced on X by $\mathcal{C}$ and $\mathcal{D}$ coincide: $\|x\|_{\mathcal{C}} = \|x\|_{\mathcal{D}}$ for all $x \in \mathsf{X}$.

Given a C^*-dynamical system $(G, \mathcal{A}, \alpha)$, let σ denote the "diagonal action" on $C_o(G, \mathcal{A})$ by G, i.e., for each $x \in G$, $\sigma_x = \alpha_x \circ \mathsf{lt}_x$. Below we state Green's Imprimitivity Theorem in our specific context.

Theorem 5.7 (Green's Imprimitivity Theorem)**.** *Let $\mathcal{B}_o := C_c(G \times G, \mathcal{A})$. If $(G, \mathcal{A}, \alpha)$ is a C^*-dynamical system, then $C_c(G, \mathcal{A})$ is a $\mathcal{B}_o$-$\mathcal{A}$ pre-imprimitivity bimodule with actions*

$$(b \bullet f)(y) = \int_G b(x, y)[\sigma_x(f)](y)\, d\mu(x) \textit{ for all } b \in \mathcal{B}_o,\ y \in G,$$

$$(f \bullet a)(x) = f(x)\alpha_x(a) \textit{ for all } a \in \mathcal{A},\ x \in G,$$

and inner products

$$[{}_{\mathcal{B}_o}\langle f \,|\, g \rangle](x, y) = [f \cdot \sigma_x(\overline{g})](y) = f(y)\alpha_x[g(x^{-1}y)^*] \textit{ for all } x, y \in G$$

$$\langle f \,|\, g \rangle_{\mathcal{A}} = \int_G \alpha_{x^{-1}}(f(x)^* g(x))\, d\mu(x).$$

Moreover, the completion Z of $C_c(G, \mathcal{A})$ with respect to the norms induced by $\mathcal{B}_o$ and $\mathcal{A}$ (which coincide) is a $\mathcal{B}$-$\mathcal{A}$ imprimitivity bimodule, where $\mathcal{B} := C_o(G, \mathcal{A}) \rtimes_\sigma G$ contains a dense copy of $\mathcal{B}_o$ and acts on Z by the extension of the action of $\mathcal{B}_o$ on $C_c(G, \mathcal{A})$.

Note that Z as a right Hilbert $\mathcal{A}$-module is precisely $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, so Green's Imprimitivity Theorem actually says $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is a $C_o(G, \mathcal{A}) \rtimes_\sigma G$-$\mathcal{A}$ imprimitivity bimodule.

Proposition 5.8 (Raeburn-Williams, 3.8 [18])**.** *If X is a $\mathcal{C}$-$\mathcal{D}$ imprimitivity bimodule, the map $\Phi : \mathcal{C} \to \mathcal{L}(\mathsf{X}_{\mathcal{D}})$ defined by $\Phi(c)x := c \bullet x$ for all $x \in \mathsf{X}$ is an isomorphism of $\mathcal{C}$ onto $\mathcal{K}(\mathsf{X}_{\mathcal{D}})$.*

Since $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is a $C_o(G, \mathcal{A}) \rtimes_\sigma \mathcal{A}$ - $\mathcal{A}$ imprimitivity bimodule, Proposition 5.8 implies $C_o(G, \mathcal{A}) \rtimes_\sigma G \cong \mathcal{K}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$, where $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is viewed as a right Hilbert $\mathcal{A}$-module. We now give an explicit definition of Φ in this setting. Consider the map $\Xi : C_o(G, \mathcal{A}) \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ defined on $\phi \in C_c(G, \mathcal{A})$ by

$$([\Xi(f)]\phi)(x) := f(x)\phi(x) \text{ for all } x \in G.$$

Note $\|[\Xi(f)]\phi\|_{C_c(G,\mathcal{A})} = \|f\phi\|_{C_c(G,\mathcal{A})} \leq \|f\|_{C_o(G,\mathcal{A})} \cdot \|\phi\|_{C_c(G,\mathcal{A})}$, so the operator $\Xi(f)|_{C_c(G,\mathcal{A})}$ is $\|\cdot\|_{C_c(G,\mathcal{A})}$-continuous. Following an argument similar to the proof of Proposition 5.2, $\Xi(f)|_{C_c(G,\mathcal{A})}$ is $\|\cdot\|_\alpha$-continuous, so we may continuously extend $\Xi(f)$ to act on all of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$. Checking $\Xi(f)^* = \Xi(f^*)$, where $f^*(x) = f(x^{-1})^*$ for each $x \in G$, confirms that $\Xi(f)$ is an adjointable operator on $\mathsf{L}^2(G, \mathcal{A}, \alpha)$. Therefore, Ξ is a well-defined $*$-representation of $C_o(G, \mathcal{A})$ on $\mathsf{L}^2(G, \mathcal{A}, \alpha)$.

To explicitly describe $\Phi : C_o(G, \mathcal{A}) \rtimes_\sigma G \xrightarrow{\cong} \mathcal{K}(\mathsf{L}^2(G, \mathcal{A}, \alpha)$, we also require the $\mathcal{A}$-valued *Fourier transform* $\mathcal{F}$ for $\widehat{G}$, where $\mathcal{F} : C_c(\widehat{G}, \mathcal{A}) \to C_o(G, \mathcal{A})$ is defined on $f \in C_c(\widehat{G}, \mathcal{A})$ by

$$[\mathcal{F}f](x) := \int_{\widehat{G}} f(\gamma)\gamma(x)\, d\hat{\mu}(\gamma) \text{ for all } x \in G.$$

Denote $\mathcal{F}f$ by $\hat{f}$. Consider the C^*-dynamical system $(\widehat{G}, \mathcal{A}, \iota)$ with trivial action ι. Note that $\mathcal{F}$ is just the restriction of the C^*-isomorphism $\varphi_2 : \mathcal{A} \rtimes_\iota \widehat{G} \xrightarrow{\cong} C_o(G, \mathcal{A})$ in Lemma 7.3 of [25] to the dense $*$-subalgebra $C_c(\widehat{G}, \mathcal{A})$ of $\mathcal{A} \rtimes_\iota \widehat{G}$.

Lemma 5.9. *The $*$-representation $\Xi : C_o(G, \mathcal{A}) \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ is equal to $(\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}$, where $\mathsf{M} \times v$ is the integrated form of the covariant homomorphism (M, v) for $(\widehat{G}, \mathcal{A}, \iota)$.*

Proof. Note that $\mathcal{F}(C_c(\widehat{G},\mathcal{A}))$ is dense in $C_o(G,\mathcal{A})$. Fix $f \in C_c(\widehat{G},\mathcal{A})$. For $\phi \in C_c(G,\mathcal{A})$,

$$[(\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}](\hat{f})\phi = [(\mathsf{M} \rtimes v)(f)]\phi = \underbrace{\left(\int_{\widehat{G}} \mathsf{M}(f(\gamma))v_\gamma \, d\hat{\mu}(\gamma)\right)}_{\in \mathcal{L}(\mathsf{L}^2(G,\mathcal{A},\alpha))} \phi = \underbrace{\int_{\widehat{G}} \mathsf{M}(f(\gamma))v_\gamma \phi \, d\hat{\mu}(\gamma)}_{\in C_c(G,\mathcal{A})},$$

where the last equality is a standard property of this vector-valued integral. The reader is referred to Section 1.5 of [25] for details. Since point evaluation is a linear functional on $C_o(G,\mathcal{A})$,

$$\begin{aligned}
\int_{\widehat{G}} \mathsf{M}(f(\gamma))[v_\gamma \phi](x) \, d\hat{\mu}(\gamma) &= \int_{\widehat{G}} f(\gamma)\gamma(x)\phi(x) \, d\hat{\mu}(\gamma) \\
&= \left(\int_{\widehat{G}} f(\gamma)\gamma(x) \, d\hat{\mu}(\gamma)\right) \phi(x) \\
&= \hat{f}(x)\phi(x) \\
&= [\Xi(\hat{f})\phi](x)
\end{aligned}$$

for every $x \in G$. As $x \in G$ was arbitrary, as was $\phi \in C_c(G,\mathcal{A})$, we have that

$$\Xi(\hat{f})|_{C_c(G,\mathcal{A})} = [(\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}](\hat{f})|_{C_c(G,\mathcal{A})}.$$

By density of $C_c(G,\mathcal{A})$ in $\mathsf{L}^2(G,\mathcal{A},\alpha)$, this equality holds on $\mathsf{L}^2(G,\mathcal{A},\alpha)$. Then, by density of $\mathcal{F}(C_c(\widehat{G},\mathcal{A}))$ in $C_o(G,\mathcal{A})$ and continuity of Ξ and $(\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}$, we have $\Xi(g) = [(M \rtimes v) \circ \mathcal{F}^{-1}](g)$ for all $g \in C_o(G,\mathcal{A})$. □

Having established $\Xi = (\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}$, we know that Ξ is nondegenerate. We now show (Ξ, u) is a covariant homomorphism of $(G, C_o(G,\mathcal{A}), \sigma)$ into $\mathcal{L}(\mathsf{L}^2(G,\mathcal{A},\alpha))$. Fix $x \in G$ and

$f \in C_c(\widehat{G}, \mathcal{A})$. Let $\phi \in C_c(G, \mathcal{A})$ be arbitrary. Then for all $y \in G$,

$$\begin{aligned}([u_x \Xi(\hat{f})]\phi)(y) &= \alpha_x(\hat{f}(x^{-1}y)\phi(x^{-1}y)) \\ &= \alpha_x(\hat{f}(x^{-1}y))\alpha_x(\phi(x^{-1}y)) \\ &= [\sigma_x(\hat{f})](y)\alpha_x(\phi(x^{-1}y)) \\ &= ([\Xi(\sigma_x(\hat{f}))u_x]\phi)(y).\end{aligned}$$

As $y \in G$ was arbitrary, $[u_x \Xi(\hat{f})]\phi = [\Xi(\sigma_x(\hat{f}))u_x]\phi$. Also, $\phi \in C_c(G, \mathcal{A})$ was arbitrary, and $C_c(G, \mathcal{A})$ is $\|\cdot\|_\alpha$-dense in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, so $u_x \Xi(\hat{f}) = \Xi[\sigma_x(\hat{f})]u_x$ as adjointable operators. By density of $\mathcal{F}(C_c(\widehat{G}, \mathcal{A}))$ in $C_o(G, \mathcal{A})$, $\|\cdot\|_{C_o(G,\mathcal{A})}$-continuity of Ξ and σ_x suffice to conclude $u_x \Xi(g) = [\Xi(\sigma_x(g))]u_x$ for all $g \in C_o(G, \mathcal{A})$. Thus, (Ξ, u) is a nondegenerate covariant homomorphism for $(G, C_o(G, \mathcal{A}), \sigma)$ whose integrated form yields a nondegenerate $*$-representation $\Xi \rtimes u : C_o(G, \mathcal{A}) \rtimes_\sigma G \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$.

Proposition 5.10. *The isomorphism* $\Phi : C_o(G, \mathcal{A}) \rtimes_\sigma G \to \mathcal{K}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ *in Proposition 5.8 is the integrated form* $\Xi \rtimes u$.

Proof. It suffices to check that $\Phi(F) = (\Xi \rtimes u)F$ for all $F \in C_c(G, C_o(G, \mathcal{A}))$ by density of $C_c(G, C_o(G, \mathcal{A}))$ in $C_o(G, \mathcal{A}) \rtimes_\sigma G$. Let $\phi, \psi \in C_c(G, \mathcal{A})$, and observe

$$
\begin{aligned}
\langle \phi \,|\, [\Xi \rtimes u](F)\psi \rangle &= \left\langle \phi \,\middle|\, \left(\int_G \Xi(F_y) u_y \, d\mu(y) \right) \psi \right\rangle \\
&= \int_G \langle \phi \,|\, [\Xi(F_y)u_y](\psi) \rangle \; d\mu(y) \qquad [\text{ by Lemma 2.51 }] \\
&= \int_G \left(\int_G \alpha_{x^{-1}} \left[\phi(x)^* \; F_y(x) \; \alpha_y(\psi(y^{-1}x)) \right] \; d\mu(x) \right) \; d\mu(y) \\
&= \int_G \left(\int_G \alpha_{x^{-1}} \left[\phi(x)^* \; F_y(x) \; \alpha_y(\psi(y^{-1}x)) \right] \; d\mu(y) \right) \; d\mu(x) \\
&= \int_G \alpha_{x^{-1}} \left[\phi(x)^* \left(\int_G F_y(x) \; \alpha_y(\psi(y^{-1}x)) \, d\mu(y) \right) \right] \; d\mu(x) \\
&= \int_G \alpha_{x^{-1}} \left[\phi(x)^* \left(\int_G F_y(x) \; [\sigma_y(\psi)](x) \, d\mu(y) \right) \right] \; d\mu(x) \\
&= \int_G \alpha_{x^{-1}} \left[\phi(x)^* (F \bullet \psi)(x) \right] \; d\mu(x) \qquad [\text{ by Green's Imprimitivity Theorem }] \\
&= \langle \phi \,|\, \Phi(F)\psi \rangle \, .
\end{aligned}
$$

By density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, we conclude $\Phi(F) = [\Xi \rtimes u](F)$. Moreover, $C_c(G, C_o(G, \mathcal{A}))$ is dense in $C_o(G, \mathcal{A}) \rtimes_\sigma G$, so we finally establish that $\Phi = \Xi \rtimes u$. □

The isomorphism $C_o(G) \rtimes_{\mathsf{lt}} G \cong \mathcal{K}(L^2(G))$ relates nondegenerate $*$-representations of $C_o(G) \rtimes_{\mathsf{lt}} G$ with the nicely classified nondegenerate $*$-representations of $\mathcal{K}(L^2(G))$. For our purposes, then, the utility of Proposition 5.8 follows only from having an analogous classification of representations of $\mathcal{K}(\mathsf{X})$ where X is a Hilbert $\mathcal{A}$-module for some C^*-algebra $\mathcal{A}$. Without more assumptions on $\mathcal{A}$, however, such a classification for representations of $\mathcal{K}(\mathsf{X})$ does not exist. Hence, we restrict our attention to Hilbert $\mathcal{K}(\mathcal{H})$-modules.

5.3 Representations of Hilbert $\mathcal{K}(\mathcal{H})$-modules

Henceforth, X denotes a Hilbert $\mathcal{K}(\mathcal{H})$-module. The main result of this section, Theorem 5.14, generalizes the following theorem to representations of $\mathcal{K}(\mathsf{X})$ as adjointable operators on

Hilbert $\mathcal{K}(\mathcal{H})$-modules. It will be useful to keep Lemma 2.56 in mind.

Theorem 5.11 (Arveson, 1.4.4 [2])**.** *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{K}(\mathcal{H})$, and let π be any nondegenerate representation of $\mathcal{A}$. Then there is an orthogonal family $\{\pi_i\}$ of irreducible subrepresentations of π such that $\pi = \sum_i \pi_i$, and each π_i is equivalent to a subrepresentation of the identity representation $id : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.*

Definition 5.12. Let $\mathcal{A}$ be a C^*-algebra. A projection $p \in \mathcal{A}$ is called *minimal* if and only if $p \neq 0_{\mathcal{A}}$ and the only sub-projections of p in $\mathcal{A}$ are $0_{\mathcal{A}}$ and p itself.

Note that the minimal projections in $\mathcal{K}(\mathcal{H})$ are simply the rank-one operators, and recall that **every** nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module is full by simplicity of $\mathcal{K}(\mathcal{H})$.

Lemma 5.13. *The C^*-algebra $\mathcal{K}(\mathsf{X})$ acts irreducibly on X, that is, X has no nontrivial $\mathcal{K}(\mathsf{X})$-invariant closed $\mathcal{K}(\mathcal{H})$-submodules.*

Proof. Suppose Y were a nontrivial $\mathcal{K}(\mathsf{X})$-invariant closed $\mathcal{K}(\mathcal{H})$-submodule of X. Let p be a rank-one projection in $\mathcal{K}(\mathcal{H})$. By Lemma 2.56, $\mathsf{Y} \bullet p$ and $\mathsf{X} \bullet p$ are Hilbert spaces, and furthermore, $\mathsf{Y} \bullet p$ is a closed subspace of $\mathsf{X} \bullet p$. We claim that $\mathsf{Y} \bullet p$ is $\mathcal{K}(\mathsf{X} \bullet p)$-invariant. Let $b \in \mathcal{K}(\mathsf{X} \bullet p)$. By Theorem 2.57, b has the form $a|_{\mathsf{X}\bullet p}$ for some $a \in \mathcal{K}(\mathsf{X})$. Thus,

$$b[\mathsf{Y} \bullet p] = a|_{\mathsf{X}\bullet p}[\mathsf{Y} \bullet p] = a[\mathsf{Y} \bullet p] = (a\mathsf{Y}) \bullet p \subseteq \mathsf{Y} \bullet p$$

by $\mathcal{K}(\mathcal{H})$-linearity of a. As $b \in \mathcal{K}(\mathsf{X}\bullet p)$ was arbitrary, $\mathsf{Y}\bullet p$ is $\mathcal{K}(\mathsf{X}\bullet p)$-invariant. Furthermore,

$$\overline{(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})} = \mathsf{Y}$$

by Proposition 2.59, so since Y is nontrivial, $\mathsf{Y} \bullet p$ must be nontrivial. Last, $\mathsf{Y} \bullet p$ is a proper subspace of $\mathsf{X} \bullet p$. Indeed, if $\mathsf{Y} \bullet p = \mathsf{X} \bullet p$, then applying Proposition 2.59 twice implies

$$\mathsf{Y} = \overline{(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})} = \overline{(\mathsf{X} \bullet p) \bullet \mathcal{K}(\mathcal{H})} = \mathsf{X},$$

which contradicts the assumption that Y is a proper $\mathcal{K}(\mathcal{H})$-submodule of X. Therefore, $\mathsf{Y} \bullet p$ is a $\mathcal{K}(\mathsf{X} \bullet p)$-invariant proper nontrivial closed subspace of $\mathsf{X} \bullet p$. This is a contradiction to the fact that given any Hilbert space $\mathcal{H}$, there are no $\mathcal{K}(\mathcal{H})$-invariant proper nontrivial closed subspaces of $\mathcal{H}$. Since $\mathsf{X} \bullet p$ is a Hilbert space, we have reached a contradiction. Thus, there can exist no nontrivial $\mathcal{K}(\mathsf{X})$-invariant closed $\mathcal{K}(\mathcal{H})$-submodules of X, so $\mathcal{K}(\mathsf{X})$ acts irreducibly on X. □

Theorem 5.14. *Let X and Y be Hilbert $\mathcal{K}(\mathcal{H})$-modules. If $\widetilde{\pi} : \mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{Y})$ is a nondegenerate $*$-representation, then $\widetilde{\pi}$ is unitarily equivalent to a direct sum of copies of the identity representation $id : \mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{X})$.*

Proof. Our proof is an adaptation of Arveson's proof of Theorem 5.11. Fix a rank-one projection $p \in \mathcal{K}(\mathcal{H})$, and consider the composition π given by

$$\pi : \mathcal{K}(\mathsf{X} \bullet p) \stackrel{\cong}{\to} \mathcal{K}(\mathsf{X}) \stackrel{\widetilde{\pi}}{\to} \mathcal{L}(\mathsf{Y}) \stackrel{\cong}{\to} \mathcal{B}(\mathsf{Y} \bullet p),$$

where $[(\Psi_{\mathsf{X}})|_{\mathcal{K}(\mathsf{X}\bullet p)}]^{-1} : \mathcal{K}(\mathsf{X} \bullet p) \stackrel{\cong}{\to} \mathcal{K}(\mathsf{X})$ and $\Psi_{\mathsf{Y}} : \mathcal{L}(\mathsf{Y}) \stackrel{\cong}{\to} \mathcal{B}(\mathsf{Y} \bullet p)$ are provided by Theorem 2.57. As $\widetilde{\pi}$ is nondegenerate and π is the composition of $\widetilde{\pi}$ with C^*-isomorphisms, π is also nondegenerate. Note that $\mathsf{X} \bullet p$ and $\mathsf{Y} \bullet p$ are both Hilbert spaces by Lemma 2.56, so in fact, π is a nondegenerate $*$-representation of the compact operators on the Hilbert space $\mathsf{X} \bullet p$ as bounded operators on the Hilbert space $\mathsf{Y} \bullet p$. Thus, by Theorem 5.11, there exists an index set J and a unitary $W : \oplus_{j \in J} \mathsf{X} \bullet p \to \mathsf{Y} \bullet p$ such that $\pi(a) = \mathrm{ad}_W \circ \oplus_j a$ for all $a \in \mathcal{K}(\mathsf{X} \bullet p)$. However, Theorem 2.57 does not necessarily lift W to a unitary $w : \oplus_j \mathsf{X} \to \mathsf{Y}$, so we proceed to construct the desired unitary $w : \oplus_j \mathsf{X} \to \mathsf{Y}$.

By Arveson's proof, there is a rank-one projection $q \in \mathcal{K}(\mathsf{X} \bullet p)$ such that $\pi(q) \neq 0$. Furthermore, Theorem 2.57 yields a minimal projection $E \in \mathcal{K}(\mathsf{X})$ such that $q = E|_{\mathsf{X}\bullet p}$. Since $\pi(q) \neq 0$, it must be that $\widetilde{\pi}(E) \neq 0$. By Corollary 2.54, there is a linear functional

$$f_q : \mathcal{K}(\mathsf{X} \bullet p) \to \mathbb{C} \text{ which satisfies } f_q(S)q = qSq \text{ for all } S \in \mathcal{K}(\mathsf{X} \bullet p).$$

Define a linear functional $g : \mathcal{K}(\mathsf{X}) \to \mathbb{C}$ by $g(T) := f_q(T|_{\mathsf{X}\bullet p})$. For each $T \in \mathcal{K}(\mathsf{X})$, notice

$$(ETE)|_{\mathsf{X}\bullet p} = E|_{\mathsf{X}\bullet p}\, T|_{\mathsf{X}\bullet p}\, E|_{\mathsf{X}\bullet p} = q(T|_{\mathsf{X}\bullet p})q = f_q(T|_{\mathsf{X}\bullet p})q = f_q(T|_{\mathsf{X}\bullet p})E|_{\mathsf{X}\bullet p} = [g(T)E]|_{\mathsf{X}\bullet p}.$$

By Theorem 2.57, we conclude $ETE = g(T)E$ for all $T \in \mathcal{K}(\mathsf{X})$.

Consider the $\mathcal{K}(\mathcal{H})$-submodule $E[\mathsf{X}]$ of X. Note that $E[\mathsf{X}]$ is nonzero since $E \neq 0$, and $E[\mathsf{X}]$ is closed because E is a projection. Thus, $E[\mathsf{X}]$ is a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module. Similarly, $\widetilde{\pi}(E)[\mathsf{Y}]$ is a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module. Hence, by Corollary 2.55, there exist $\xi \in E[\mathsf{X}]$ and $\eta \in \widetilde{\pi}(E)[\mathsf{Y}]$ such that $\langle \xi \,|\, \xi \rangle_{\mathsf{X}} = p$ and $\langle \eta \,|\, \eta \rangle_{\mathsf{Y}} = p$.

Define a map $w' : [\mathcal{K}(\mathsf{X})\xi]\bullet\mathcal{K}(\mathcal{H}) \to [\widetilde{\pi}(\mathcal{K}(\mathsf{X}))\eta]\bullet\mathcal{K}(\mathcal{H})$ by $\sum_{i=1}^{n} T_i(\xi\bullet a_i) \mapsto \sum_{i=1}^{n} \widetilde{\pi}(T_i)(\eta\bullet$

a_i). By virtue of being an isometry, w' is well-defined: for $T_1, ..., T_n \in \mathcal{K}(\mathsf{X})$, $a_1, ..., a_n \in \mathcal{K}(\mathcal{H})$,

$$\begin{aligned}
\left\| \left\langle \sum_{i=1}^n \widetilde{\pi}(T_i)(\eta \bullet a_i) \,\middle|\, \sum_{j=1}^n \widetilde{\pi}(T_j)(\eta \bullet a_j) \right\rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} &= \left\| \sum_{i,j=1}^n \langle \widetilde{\pi}(T_i)(\eta \bullet a_i) \,|\, \widetilde{\pi}(T_j)(\eta \bullet a_j) \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^n \langle \widetilde{\pi}(T_i)([\widetilde{\pi}(E)\eta] \bullet a_i) \,|\, \widetilde{\pi}(T_j)([\widetilde{\pi}(E)\eta] \bullet a_j) \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^n \langle \widetilde{\pi}(T_i E)(\eta \bullet a_i) \,|\, \widetilde{\pi}(T_j E)(\eta \bullet a_j) \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^n \left\langle \widetilde{\pi}(E T_j^* T_i E)(\eta \bullet a_i) \,\middle|\, \eta \bullet a_j \right\rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^n \left\langle \widetilde{\pi}(g(T_j^* T_i) E)(\eta \bullet a_i) \,\middle|\, \eta \bullet a_j \right\rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^n \overline{g(T_j^* T_i)} \, \langle \widetilde{\pi}(E)(\eta \bullet a_i) \,|\, \eta \bullet a_j \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^n \overline{g(T_j^* T_i)} \, \langle \eta \bullet a_i \,|\, \eta \bullet a_j \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^n \overline{g(T_j^* T_i)} \, a_i^* \, \langle \eta \,|\, \eta \rangle_{\mathsf{Y}} \, a_j \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^n \overline{g(T_j^* T_i)} \, a_i^* \, p \, a_j \right\|_{\mathcal{K}(\mathcal{H})}
\end{aligned}$$

Following a nearly identical computation yields

$$\left\| \left\langle \sum_{i=1}^n T_i(\xi \bullet a_i) \,\middle|\, \sum_{j=1}^n T_j(\xi \bullet a_j) \right\rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} = \left\| \sum_{i,j=1}^n \overline{g(T_j^* T_i)} \; a_i^* \; p \; a_j \right\|_{\mathcal{K}(\mathcal{H})}.$$

Therefore, w' is a surjective isometry which extends by continuity to $w' : \mathsf{X}' \to \mathsf{Y}'$, where

$$\mathsf{X}' := \overline{[\mathcal{K}(\mathsf{X})\xi] \bullet \mathcal{K}(\mathcal{H})} \;\text{ and }\; \mathsf{Y}' := \overline{[\widetilde{\pi}(\mathcal{K}(\mathsf{X}))\eta] \bullet \mathcal{K}(\mathcal{H})}.$$

Note that X' is a nonzero closed $\mathcal{K}(\mathsf{X})$-invariant $\mathcal{K}(\mathcal{H})$-submodule of X. Thus, by Lemma 5.13, $\mathsf{X} = \mathsf{X}'$. Hence, $w' : \mathsf{X} \to \mathsf{Y}'$ is a surjective isometry, which, moreover, is $\mathcal{K}(\mathcal{H})$-linear. Thus, $w' : \mathsf{X} \to \mathsf{Y}'$ is unitary.

We claim $w'T = [\widetilde{\pi}(T)|_{\mathsf{Y}'}]w'$ for all $T \in \mathcal{K}(\mathsf{X})$. Fix $T \in \mathcal{K}(\mathsf{X})$ and let $T_1, ..., T_n \in \mathcal{K}(\mathsf{X})$ and $a_1, ..., a_n \in \mathcal{K}(\mathcal{H})$ be arbitrary. Then

$$\begin{aligned} w'T\left(\sum_{i=1}^{n} T_i(\xi \bullet a_i)\right) &= w'\left(\sum_{i=1}^{n} TT_i(\xi \bullet a_i)\right) \\ &= \sum_{i=1}^{n} \widetilde{\pi}(TT_i)(\eta \bullet a_i) \\ &= \sum_{i=1}^{n} \widetilde{\pi}(T)\widetilde{\pi}(T_i)(\eta \bullet a_i) \\ &= \widetilde{\pi}(T)\left(\sum_{i=1}^{n} \widetilde{\pi}(T_i)(\eta \bullet a_i)\right) \\ &= \widetilde{\pi}(T)w'\left(\sum_{i=1}^{n} T_i(\xi \bullet a_i)\right) \end{aligned}$$

By density of $[\mathcal{K}(\mathsf{X})\xi]\bullet\mathcal{K}(\mathcal{H})$ in X and continuity of both $w'T$ and $(\widetilde{\pi}(T)|_{\mathsf{Y}'})w'$, we have $w'T = (\widetilde{\pi}(T)|_{\mathsf{Y}'})w'$. Thus, the map $\mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{Y}')$ given by $T \mapsto \widetilde{\pi}(T)|_{\mathsf{Y}'}$ is a nondegenerate $*$-representation of $\mathcal{K}(\mathsf{X})$ on Y' which is unitarily equivalent via w' to the identity representation $\mathrm{id} : \mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{X})$.

Complementability of Hilbert $\mathcal{K}(\mathcal{H})$-modules allows us to apply this argument to the subrepresentation $T \mapsto \widetilde{\pi}(T)|_{(\mathsf{Y}')^\perp}$ of $\widetilde{\pi} : \mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{Y})$. An exhaustive argument and application of Zorn's Lemma yields a family $\{\mathsf{Y}_j\}_{j\in J}$ of closed $\mathcal{K}(\mathcal{H})$-submodules of Y and unitaries $\{w_j : \mathsf{X} \to \mathsf{Y}_j\}_{j\in J}$ such that $\mathsf{Y} = \oplus_j \mathsf{Y}_j$. Then $w := \oplus_j w_j$ is a unitary from $\oplus_j \mathsf{X}$ onto Y such that $w[\oplus_j T] = \widetilde{\pi}(T)w$ for all $T \in \mathcal{K}(\mathsf{X})$. This completes the proof. $\square$

5.4 Correspondence of $(G, C_o(G,\mathcal{A}), \mathsf{lt}\otimes\alpha)$-Covariant Homomorphisms and $(G,\mathcal{A},\alpha)$-Heisenberg Representations

Let $(G,\mathcal{A},\alpha)$ be a dynamical system. Suppose $s : \widehat{G} \to \mathcal{U}(\mathsf{X})$ is a unitary group representation on a Hilbert $\mathcal{A}$-module X and $\rho : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ is a nondegenerate $*$-representation such that $\rho(a)s_\gamma = s_\gamma\rho(a)$ for all $a \in \mathcal{A}$, $\gamma \in \widehat{G}$. Then the integrated form $\rho \rtimes s : \mathcal{A} \rtimes_\iota \widehat{G} \to \mathcal{L}(\mathsf{X})$ is a nondegenerate $*$-representation by Proposition 2.50. Define $\Pi_{\rho,s}$ to be the composition $\Pi_{\rho,s} : C_o(G,\mathcal{A}) \xrightarrow{\mathcal{F}^{-1}} \mathcal{A} \rtimes_\iota \widehat{G} \xrightarrow{\rho\rtimes s} \mathcal{L}(\mathsf{X})$. As $\mathcal{F}^{-1}$ is a C^*-isomorphism and $\rho \rtimes s$ is a nondegenerate $*$-representation of $\mathcal{A} \rtimes_\iota \widehat{G}$, the map $\Pi_{\rho,s}$ is a nondegenerate $*$-representation of $C_o(G,\mathcal{A})$.

Theorem 5.15. *If (X,ρ,r,s) is a $(G,\mathcal{A},\alpha)$-Heisenberg representation, then $(\Pi_{\rho,s}, r)$ is a nondegenerate covariant homomorphism for $(G, C_o(G,\mathcal{A}),\sigma)$ into $\mathcal{L}(\mathsf{X})$.*

Proof. Fix $x \in G$ and $f \in C_c(\widehat{G},\mathcal{A})$, so $\hat{f} \in C_o(G,\mathcal{A})$. Let $\hat{x} : \widehat{G} \to \mathbb{C}$ denote the copy of $x \in G$ acting as an element of the dual of $\widehat{G}$ by $\hat{x}(\gamma) = \overline{\gamma(x)}$ for each $\gamma \in \widehat{G}$. For all $y \in G$, note

$$[\mathcal{F}(f\cdot\hat{x})](y) = \int_{\widehat{G}} f(\gamma)\overline{\gamma(x)}\gamma(y)\,d\hat{\mu}(\gamma) = \int_{\widehat{G}} f(\gamma)\gamma(x^{-1}y)\,d\hat{\mu}(\gamma) = \hat{f}(x^{-1}y) = [\mathsf{lt}_x(\hat{f})](y).$$

It follows that $\alpha_x \circ \mathcal{F}(f \cdot \hat{x}) \overset{(\star)}{=} \sigma_x(\hat{f})$ since $\sigma_x = \alpha_x \circ \mathsf{lt}_x$. Thus,

$$\begin{aligned}
r_x \Pi_{\rho,s}(\hat{f}) &= r_x \left(\int_{\widehat{G}} \rho(f(\gamma)) s_\gamma \, d\hat{\mu}(\gamma) \right) \\
&= \int_{\widehat{G}} r_x \rho(f(\gamma)) s_\gamma \, d\hat{\mu}(\gamma) \\
&= \int_{\widehat{G}} \rho[\alpha_x(f(\gamma))] \; r_x s_\gamma \, d\hat{\mu}(\gamma) \qquad [\text{ by covariance of } (\rho, r) \text{ }] \\
&= \int_{\widehat{G}} \rho[\alpha_x(f(\gamma))] \; \overline{\gamma(x)} s_\gamma r_x \, d\hat{\mu}(\gamma) \qquad [\text{ as } r \text{ and } s \text{ satisfy the WCR }] \\
&= \left(\int_{\widehat{G}} \rho[\alpha_x(f(\gamma) \, \overline{\gamma(x)})] s_\gamma \, d\hat{\mu}(\gamma) \right) r_x \\
&= \left(\int_{\widehat{G}} \rho[\alpha_x([f \cdot \hat{x}](\gamma))] s_\gamma \, d\hat{\mu}(\gamma) \right) \circ r_x \\
&= [(\rho \rtimes s)(\alpha_x \circ (f \cdot \hat{x}))] r_x \\
&= [(\rho \rtimes s) \circ \mathcal{F}^{-1}][\mathcal{F}(\alpha_x \circ (f \cdot \hat{x}))] r_x \\
&= \Pi_{\rho,s}[\alpha_x \circ \mathcal{F}(f \cdot \hat{x})] r_x \\
&= \Pi_{\rho,s}(\sigma_x(\hat{f})) r_x \qquad [\text{ by } (\star) \text{ }].
\end{aligned}$$

As $f \in C_c(\widehat{G}, \mathcal{A})$ was arbitrary and $\mathcal{F}(C_c(\widehat{G}, \mathcal{A}))$ is dense in $C_o(G, \mathcal{A})$, $\|\cdot\|_{C_o(G,\mathcal{A})}$-continuity of both $\Pi_{\rho,s}$ and σ_x imply $r_x \Pi_{\rho,s}(g) = \Pi_{\rho,s}(\sigma_x(g)) r_x$ for all $g \in C_o(G, \mathcal{A})$. Therefore, since $x \in G$ was arbitrary, $(\Pi_{\rho,s}, r)$ is a nondegenerate covariant homomorphism for $(G, C_o(G, \mathcal{A}), \sigma)$. $\square$

5.5 Proof of the Covariant Stone-von Neumann Theorem

Definition 5.16. Two $(G, \mathcal{A}, \alpha)$-Heisenberg representations (X, ρ, r, s) and (Y, τ, u, v) are *unitarily equivalent* if there exists a unitary $w : \mathsf{X} \to \mathsf{Y}$ such that

(i) $\tau = \mathrm{ad}_w \circ \rho$, that is, $\tau(a) = w \rho(a) w^{-1}$ for all $a \in \mathcal{A}$,

(ii) $u_x = w r_x w^{-1}$ for all $x \in G$, and

(iii) $v_\gamma = w s_\gamma w^{-1}$ for all $\gamma \in \widehat{G}$.

Theorem 1.11. *Every $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation is unitarily equivalent to a direct sum of copies of the $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Schrödinger representation.*

Proof. Given a $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation (X, ρ, r, s), Theorem 5.15 states $(\Pi_{\rho,s}, r)$ is a covariant homomorphism for $(G, C_o(G, \mathcal{K}(\mathcal{H})), \sigma)$. Since $\Pi_{\rho,s}$ is nondegenerate, the integrated form $\Pi_{\rho,s} \rtimes r$ is a nondegenerate $*$-representation of $C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G$ into $\mathcal{L}(\mathsf{X})$. Let $\mathsf{Z} := \mathsf{L}^2(G, \mathcal{K}(\mathcal{H}), \alpha)$, and recall Propositions 5.8 and 5.10 yield the isomorphism $\Xi \rtimes u : C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G \overset{\cong}{\to} \mathcal{K}(\mathsf{Z})$. Thus, the composition

$$\Theta : \mathcal{K}(\mathsf{Z}) \overset{(\Xi \rtimes u)^{-1}}{\longrightarrow} C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G \overset{\Pi_{\rho,s} \rtimes r}{\longrightarrow} \mathcal{L}(\mathsf{X})$$

is a nondegenerate $*$-representation of $\mathcal{K}(\mathsf{Z})$ as adjointable operators on the Hilbert $\mathcal{K}(\mathcal{H})$-module X. As Z and X are Hilbert $\mathcal{K}(\mathcal{H})$-modules, Theorem 5.14 implies Θ is unitarily equivalent to a direct sum of copies of the identity representation $\mathrm{id} : \mathcal{K}(\mathsf{Z}) \to \mathcal{L}(\mathsf{Z})$. Specifically, there exists a unitary $w : \mathsf{X} \to \oplus_j \mathsf{Z}$ such that $\mathrm{ad}_w \circ \Theta = \oplus_j \mathrm{id}$.

We claim $\mathrm{ad}_w \circ \rho = \oplus_j \mathsf{M}$, $\mathrm{ad}_w \circ r = \oplus_j u$, and $\mathrm{ad}_w \circ s = \oplus_j v$. Note that for any covariant homomorphism (π, q) for $(G, C_o(G, \mathcal{K}(\mathcal{H})), \sigma)$ into $\mathcal{L}(\mathsf{X})$, we have

$$(\mathrm{ad}_w \circ \pi) \rtimes (\mathrm{ad}_w \circ q) = \mathrm{ad}_w \circ (\pi \rtimes q) : C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G \to \mathcal{L}(\oplus_j \mathsf{Z}).$$

Thus, Proposition 2.52 implies

$$\Pi_{(\mathrm{ad}_w \circ \rho),\, (\mathrm{ad}_w \circ s)} \rtimes (\mathrm{ad}_w \circ r) = \mathrm{ad}_w \circ (\Pi_{\rho,s} \rtimes r) = \oplus_j (\Xi \rtimes u) = [\oplus_j \Xi] \rtimes [\oplus_j u].$$

By Proposition 2.50, the covariant homomorphisms $(\Pi_{(\mathrm{ad}_w \circ \rho),\, (\mathrm{ad}_w \circ s)}, \mathrm{ad}_w \circ r)$ and $(\oplus_j \Xi, \oplus_j u)$ for $(G, C_o(G, \mathcal{K}(\mathcal{H})), \sigma)$ must coincide since their integrated forms are the same nondegenerate $*$-representation of $C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G$ into $\mathcal{L}(\oplus_j \mathsf{Z})$. Therefore,

$$\Pi_{(\mathrm{ad}_w \circ \rho),\, (\mathrm{ad}_w \circ s)} = \oplus_j \Xi \quad \text{and} \quad \mathrm{ad}_w \circ r = \oplus_j u.$$

Recall $\Xi = (\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}$ by Lemma 5.9. Hence,

$$[(\mathrm{ad}_w \circ \rho) \rtimes (\mathrm{ad}_w \circ s)] \circ \mathcal{F}^{-1} = \Pi_{(\mathrm{ad}_w \circ \rho),\, (\mathrm{ad}_w \circ s)} = \oplus_j \Xi = [\oplus_j (\mathsf{M} \rtimes v)] \circ \mathcal{F}^{-1} = ([\oplus_j \mathsf{M}] \rtimes [\oplus_j v]) \circ \mathcal{F}^{-1}.$$

By another application of Proposition 2.50, we have that $\mathrm{ad}_w \circ \rho = \oplus_j \mathsf{M}$ and $\mathrm{ad}_w \circ s = \oplus_j v$, as desired. We conclude (X, ρ, r, s) is unitarily equivalent to a direct sum of copies of $(\mathsf{Z}, \mathsf{M}, u, v)$. $\square$

Chapter 6

Conclusions and Future Directions

6.1 Weak D-Antidifferentiability and Extended Derivations

Given an unbounded self-adjoint operator D on a Hilbert space $\mathcal{H}$, Christensen's work in [6] and [5] gives multiple equivalent conditions for when an operator $x \in \mathcal{B}(\mathcal{H})$ makes the commutator $[iD, x]$ defined and bounded on $\mathrm{Dom}(D)$. Recall that this family of operators is precisely $\mathrm{Dom}(\delta_D)$. A lingering question is when an operator $y \in \mathcal{B}(\mathcal{H})$ arises as the continuous extension of $[iD, x]|_{\mathrm{Dom}(D)}$ for some $x \in \mathcal{B}(\mathcal{H})$, which, by Christensen's work, is simply when $y \in \mathrm{Range}(\delta_D)$.

If $y \in \ker \delta_D$ is nonzero, then $y \notin \mathrm{Range}(\delta_D)$. Indeed, if $y = \delta_D(x)$ for some $x \in \mathrm{Dom}(\delta_D)$, then $\delta_D^2(x) = \delta_D(y) = 0$. Thus, $x \in \ker \delta_D^2$, which, by Theorem 1.2, implies $x \in \ker \delta_D$. This contradicts the assumption that $\delta_D(x) = y \neq 0$, so $\ker \delta_D \cap \mathrm{Range}(\delta_D) = \{0\}$. We are led to ask:

(1) If we extended δ_D to act on unbounded operators that are *affiliated* with $\mathcal{B}(\mathcal{H})$, would kernel stabilization for the extension Δ_D of δ_D still hold?

(2) Would operators in $\ker \Delta_D$ be weakly D-antidifferentiable if we allow for antiderivatives to be unbounded operators which are affiliated to $\mathcal{B}(\mathcal{H})$?

Our resounding answer to (1) is "no," and consequently our answer to (2) is "yes." Let

P be the momentum operator on $L^2(\mathbb{R})$ defined in Example 2.9, and let Q be the position operator on $L^2(\mathbb{R})$ defined in Example 2.6. Recall that the domains of P and Q contain the class of Schwartz functions $S(\mathbb{R})$, which is a core for both P and Q. Let $\mathscr{C}$ be any common core for P and Q. Ideally, we would define Δ_P so that $Q \in \mathrm{Dom}(\Delta_P)$, and

$$\Delta_P(Q)|_{\mathscr{C}} = [iP, Q]|_{\mathscr{C}} = I|_{\mathscr{C}}.$$

As $\mathscr{C}$ is dense in $L^2(\mathbb{R})$, we have $\Delta_P(Q) = I$, but $\Delta_P^2(Q) = \Delta_P(I) = 0$, so $\ker \Delta_P^2 \neq \ker \Delta_P$. Furthermore, we could say that a weak P-antiderivative of I is Q, or more generally, $Q + y$ where y is any element of $\ker \Delta_P$.

The notion of defining or extending a derivation on an algebra $\mathcal{A}$ of bounded operators to unbounded operators which are affiliated with $\mathcal{A}$ is studied in [11] of R. Kadison and Z. Liu. Specifically, Kadison and Liu consider the extensions of an arbitrary derivation δ on a von Neumann algebras $\mathcal{M}$ to a derivation Δ on the affiliated *Murray-von Neumann algebra* $\mathscr{A}_{\mathrm{f}}(\mathcal{M})$. The definition of their extended derivation in the case when $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\delta = \delta_D$ may be a fruitful place to begin in the quest for Δ_D.

6.2 Further Generalizations of the Stone-von Neumann Theorem

Thanks to D. Pitts, the Covariant Stone-von Neumann Theorem has an interesting interpretation we've not yet explored. Given a C^*-dynamical system $(G, \mathcal{K}(\mathcal{H}), \alpha)$, note that for each $x \in G$, $\alpha_x \in \mathrm{Aut}(\mathcal{K}(\mathcal{H}))$ must be implemented by unitary conjugation, i.e., there exists a unitary $U_x \in \mathcal{B}(\mathcal{H})$ such that $\alpha_x(a) = U_x a U_x^*$ for all $a \in \mathcal{K}(\mathcal{H})$. While $\{\alpha_x\}_{x\in G}$ is a norm-continuous group, the family $\{U_x\}_{x\in G}$ need not form a group. It does, however, satisfy a *2-cocycle* condition: $U_x U_y = \sigma(x,y) U_{xy}$ for all $x, y \in G$, where $\sigma : G \times G \to \mathbb{T}$ is a 2-cocycle. Then, the representation $G \to \mathcal{U}(\mathcal{H})$ given by $x \mapsto U_x$ defines a *projective unitary group representation*. So, we could consider our classification of representations of dynamical systems of the form $(G, \mathcal{K}(\mathcal{H}), \alpha)$ as a classification of projective unitary group representations.

Delving more deeply into this interpretation may offer some insight on how we can extend our Covariant Stone-von Neumann Theorem without attempting to replace $\mathcal{K}(\mathcal{H})$ with a more general C^*-algebra. On the other hand, if $\mathcal{A}$ were a C^*-algebra such that any nondegenerate $*$-representation of $\mathcal{K}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ decomposed as in Theorem 5.14, our statement of Theorem 1.11 would hold if we replaced $\mathcal{K}(\mathcal{H})$ with $\mathcal{A}$. Identifying C^*-algebras with this desirable representation property may require tools such as Morita equivalence and KK-theory.

As an application of Theorem 1.11 in its current form, we are able to classify all pairs of self-adjoint operators (A, B) on a Hilbert $\mathcal{K}(\mathcal{H})$-module X which satisfy the HCR on some dense $\{A, B\}$-analytic $\mathcal{K}(\mathcal{H})$-submodule of X. This extends Huang's main result in [9], and will appear in an article on the `arXiv` this summer.

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