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Reflections on Inquiry-Based Learning in a First-Year Algebra Classroom: Implications for Practitioners

Gregory P. Sand
University of Nebraska-Lincoln, gregory.sand@ops.org

REFLECTIONS ON INQUIRY-BASED LEARNING IN A FIRST-YEAR ALGEBRA CLASSROOM: IMPLICATIONS FOR PRACTITIONERS

by

Gregory P. Sand

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REFLECTIONS ON INQUIRY-BASED LEARNING IN A FIRST-YEAR ALGEBRA CLASSROOM: IMPLICATIONS FOR PRACTITIONERS

Gregory P. Sand, Ed.D.

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Advisor: Stephen A. Swidler

This self-study examines the learning by the teacher and students in a first-year Algebra class in a medium sized Midwestern city. The author, a secondary mathematics teacher, overviews two problems of practice and outlines methodology used to study these problems within the standard curriculum of first-year Algebra taught using inquiry-based methodology. Three units of study, Inequalities, Exponents, and Parabolas are analyzed to identify the learning of students and the mathematical and pedagogical learning of the teacher. Emergent mathematical and pedagogical themes from the analysis chapters are discussed with implications for practice.

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CHAPTER 1

INTRODUCTION

Math literacy is a civil right.

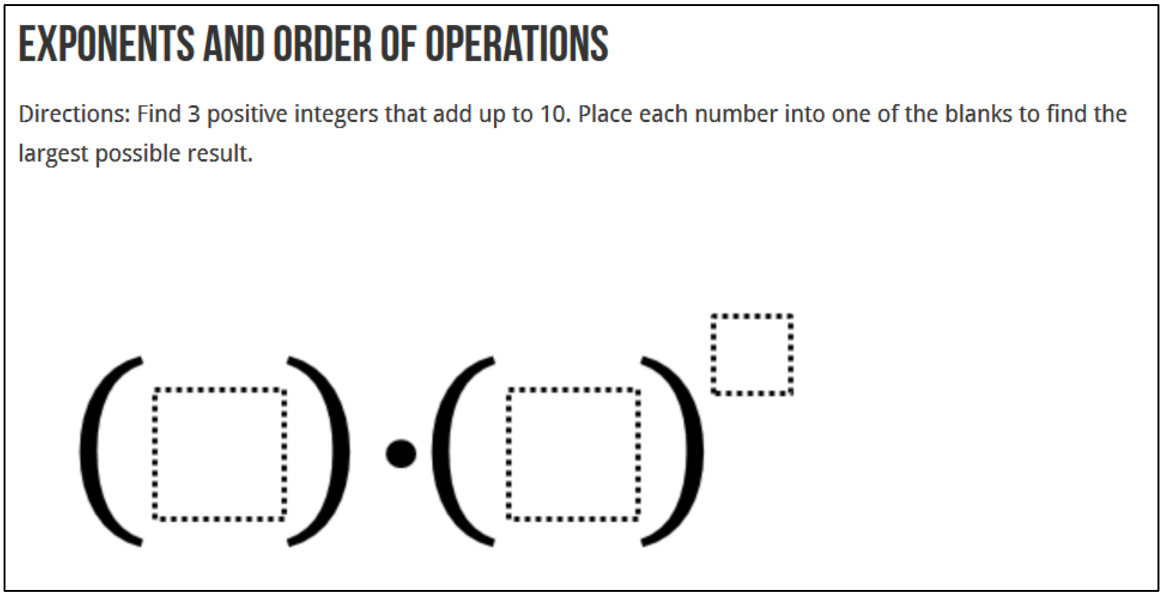
-Bob Moses

"Mr. Sand, I don't get it," rang Isabel's voice across the room. It was the third day of class and my return to middle school was not starting the way that I hoped it would.

I started the 2016 – 2017 school year with many goals. Keeping my sanity while teaching in two different school buildings was near the top of my list. More importantly was my desire to teach first-year Algebra employing techniques that I had developed and refined over the past fifteen years teaching advanced mathematics courses to high school honors math students.

Principle to my teaching is viewing students as mathematicians. Viewing students as mathematicians means believing that they are capable of making generalizations based on patterns, finding connections between concepts, writing careful and precise definitions, and applying concepts to solve problems given proper classroom support. I utilize small groups and student-centered learning tasks to accomplish this in my high school courses and planned to do the same this year with middle schoolers.

It would have been easy to start this new year off in typical middle school fashion with reviews of prior learning and homework assignments to refresh old skills. Instead, from the first day, I wanted to establish a classroom environment that encouraged risk taking and student involvement. To begin this process, I randomly placed the students into groups of three or four and presented them with a task involving exponents and operations (Figure 1.1).



EXPONENTS AND ORDER OF OPERATIONS

Directions: Find 3 positive integers that add up to 10. Place each number into one of the blanks to find the largest possible result.

$$(\square)\cdot(\square)^{\square}$$

Figure 1.1 – Opening prompt

I chose this problem because it involved multiple concepts that students presumably engaged with last year, during seventh grade. The first step in the problem requires understanding the definition of positive integers and selecting values with a given constraint. Next, each of the three numbers is placed in one of the squares in the expression. The resulting expression has to be evaluated while attending to the correct order of operations. By observing the students' attempts to solve this problem, I hoped to gain an initial sense of which students had secured knowledge of the order of operations, something we would revisit but needed to be mastered quickly if we were to move ahead in the curriculum.

Moreover, I wanted to give students the chance to engage in a problem that I thought was pretty easy to understand but provided no direct means of solution, i.e., they could see the situation and possible solutions but could not quickly determine the actual solution, kind of like a brainteaser. I gave them some time to read, think, and ask questions about the problem. I asked the groups to come up with solutions. I felt confident after the students asked a few questions that they would be able to handle the demands of this problem and offer up initial possible solutions that would serve as targets for other students to better.

Then Isabel raised her hand and said she did not "get it."

I asked her to explain to me the part of the problem she did not understand.

"All of it. I just don't get it," Isabel responded.

This was not how I wanted my first lesson to go and it immediately gave me a sinking feeling about my plans for the year, to engage my students in mathematical tasks as a means of learning mathematics. My practice had not prepared me for this type of reaction. I had grown used to honors, upper-level mathematics students who had developed habits necessary to playfully dabble in this type of ambiguous problem. Why not eighth graders? Isabel's statement was an abrupt reintroduction to working with "regular" eighth grade Algebra students who had not yet developed the habits to solve this type of problem. These students were much different than I expected; I had hoped that because they were eighth graders in first-year Algebra, they would be active thinkers interested in original ideas.

Upon further reflection, Isabel's reaction seemed obviously rational to me. I had a classroom full of students who were beginning their first experience with formal mathematics. Developing the thinking and reasoning skills necessary to solve this type of problem was critical for my class to function the way I had envisioned, with students doing mathematics to learn mathematics. This is not a strategy, or an attitude for that matter, that I would have considered in my first experience teaching middle school. Twenty-two years earlier when I started my career, I had a very "fixed" mindset about learning mathematics and saw students as either having inherent math abilities or not. My job was to get the "good" math students ready for the next class and the less good ones to cause as few disruptions as possible. However, in my experience teaching upper-level math with high achieving students, I had come to see students as mathematicians in their own right and felt the same could be true about eighth graders.

Not wanting to give up my vision on the third day of class, I did the age-old practice of math teachers and worked with Isabel to break the problem down into smaller parts until she understood how to approach it. Sitting down in a chair with her group, I asked, “Isabel, what part don’t you get?”

Isabel sounded exasperated. “All of it, I just don’t get it.”

“Do you know what positive integers are?”

“Not really.”

“Okay,” I replied, “positive integers are counting numbers. 1, 2, 3, 4, and so on.”

“That makes sense.”

“Now, the problem asks you to pick three that add up to 10.”

“So,” Isabel queried, “I just pick them?”

“Yes,” I assured, “Can you come up with three?”

“Really, I just pick them?” Isabel doubtfully asked. “Ugh, okay. How about 1, 3, 7?”

“Do they add up to 10?”

“No, 11. So 1, 3 and 6.”

“Now, can you substitute them into the expression and find a value?”

“Yeah,” Isabel confidently responded, “I can do that.”

I walked away from Isabel’s group and thought about the conversation. When Isabel told me that she did not get it, I needed to get some sense of her confusion. *Not getting what? All of it?* So, I started at the beginning of the problem and thought about the foundational qualities of the problem. When she asked if she should just pick them, I took this as a sign that she needed some sort of permission to engage in a mathematical task without direct guidance.

Isabel was perfectly able to take on the task and told me that she can “do that.” The class spent two days playing with this problem, sharing answers, finding methods to find the larger solutions, and proposing justifications about why a particular solution was optimal. This interaction with Isabel and the ensuing classroom discourse helped me gauge the development of my students’ ability to solve problems, take risks, discuss ideas with peers, perform calculations, communicate mathematically, and propose theories.

That sinking feeling dissipated over the course of the lesson, but it was an early indicator to me that all students may be like Isabel, students who needed to develop thinking skills and required me to consider ways to scaffold lessons if this year was going to go as planned.

• • •

Three Ideas for Research

I arrived in this eighth grade classroom as a result of a dissertation committee meeting on December 3, 2015. We discussed three different options for my research the following year. The first idea that I presented was a self-study on the process of teacher learning and knowledge as it related to modern reform efforts in mathematics education. One of my major goals was to make visible the idea that content knowledge is necessary but not sufficient for good teaching.

The second topic that I was curious about involved an area of instruction that affects my daily practice, first-year Algebra teaching in middle school. I mainly work with students who normally complete Advanced Placement Calculus BC (the BC course is equivalent to Calculus 1 and Calculus 2 at the university level) by the end of their junior year. They are bright, hardworking, and talented. However, every year they struggle with the same fundamental concepts and skills – equivalent forms of expressions and algebraic manipulation of equations and inequalities – that have their roots traced back to what they learned in seventh and eighth grade. I wanted to embrace the role of detached researcher by either designing an instrument to measure both pedagogical and content knowledge or by conducting interviews with those teachers to gain an understanding of the problems within their practice.

Finally, I am fortunate to have the opportunity to teach the Higher-Level Mathematics course as a part of my high school's International Baccalaureate (IB) program. I have noticed through teaching the course that the nature of the curriculum, designed and moderated externally by IB, forces tremendous growth in my students as mathematicians. This growth occurs because the course requires mathematically authentic inquiry methods where students engage in problems that make use of mathematical habits of mind instead of memorization and regurgitation. My interest was drawn to exploring the question: "How much are they growing and in what ways?" This could be an opportunity to examine what happens when a high school mathematics teacher tries to act upon a growth mindset.

Finding a Focus

These three ideas began to merge in my mind and led me to pose questions that could be explored. Could a high school mathematics teacher teach a "normal" eighth grade, first-year Algebra class using practices refined in teaching advanced classes with high-achieving students? What could I learn about the challenges middle school practitioners face in teaching Algebra that shape students' later learning and lead to continuing problems when I encounter them two, three, even four years later? Could I treat "regular" eighth graders in a "regular" algebra class as I had come to treat my high-achieving students, holding them to similar expectations and perhaps thwarting those problems? Could eighth graders learn and demonstrate algebraic knowledge conceptually, numerically, graphically and analytically similar to the requirements of the mathematics curriculum of Advance Placement or International Baccalaureate? Could I plan and develop lessons for first-year Algebra that emphasized learning in these areas?

The synthesis of these ideas and potential answers to these questions led me to teach first-year Algebra to regular eighth grade students during the 2016 – 2017 school year. I would be spending the first part of my day at a middle school near the high school where I taught my advanced courses. The goal was to understand the mathematical and pedagogical learning that occurs while teaching eighth grade Algebra employing the modern pedagogical techniques used in my current practice.

This idea represented elements of my three proposed ideas. It was a self-study, an area of research that I was first exposed to when I read Maggie Lampert's (2001) remarkable work, *Teaching Problems and the Problems of Teaching*. Her teaching and methodology mirrored my own reflective practices and my attitudes toward mathematics as an adventurous landscape. This allowed me to conceive an investigation into first-year Algebra instruction as a teacher instead of as a researcher. It granted me the opportunity to observe the growth in my students over the course of a school year. These ideas helped to form the foundation for my research and practice with these students.

As a pre-service teacher, I learned about reform efforts by the National Council of Teachers of Mathematics (NCTM). At that time, the reforms emphasized conceptual understanding, problem solving, and classroom discourse in place of teacher centered classrooms. In addition to these reforms, the first-year Algebra classroom saw the introduction of alternative tools for teaching like Algeblocks and Hands-on Equations. These reforms strongly influenced my early learning about math teaching by presenting me with a vision of teaching that was in contrast to how I learned mathematics as a student.

I spent the early years of my professional life teaching middle school math. This work was greatly influenced by the work I did as an undergraduate with NCTM's *Professional Standards for Teaching Mathematics* (1991) as well as methods courses and student teaching. I moved forward with a model of teaching that emphasize the NCTM reform ideas but was still teacher centered. As a teacher, I planned lessons, chose the activities, focused on the end product, and assessed and graded my students' performance.

My early attempts at creating a student-centered classroom resulted in the creation of a teacher-centered one. While I made use of activities, my teaching was focused on students finding the right answer. I am sure that this work introduced problems similar to those that I observed in my current students. My current practice is affected on a daily basis by misunderstandings and misconceptions that trace their origins back to first-year Algebra.

Years of teaching and reflection led me to the conclusion that the model I was using needed further transformation. While I had shifted the focus of instruction from passive receipt by students to active engagement, further inversion of this model would allow for more meaningful learning and better opportunities for reasoning by students by embracing a constructivist stance towards teaching. Using the work of Vygotsky (1962, 1978), Piaget (1952), Dewey (1929), Vico (1710), Rorty (1982), and Bruner (1960, 1966, 1973), the constructivist teacher works from the assumption that students are active learners who are making sense of current experiences based on prior knowledge, experiences, and beliefs (Jenkins, 2006). Learning is viewed as a deeply personal experience in which meaning is constructed by each individual at the conceptual level (Davis, Edmunds, & Kelly-Bateman 2008).

I begin planning lessons with my students' prior knowledge and experiences in mind. Building off of that knowledge, I design activities that allow students the opportunity to experience the ideas on their own. Students share what they discover while I facilitate the discussion. Once all ideas have been shared, the focus of the lesson shifts to me as I help the students understand which ideas are the most important and bring closure to the lesson. The end goal of any lesson is student learning and growth in understanding of a topic. Fundamental to these lessons is the use of a conceptual basis to build procedural skills.

Current reform efforts continue to highlight the need for this type of instruction. *Principles to Actions*, published in 2014 by NCTM built on the earlier work of *Professional Standards for Teaching Mathematics* (1991) and *Principles and Standards for School Mathematics* (2000). It continued to press forward the idea that student learning, not teacher instruction, should be the focus of daily classroom instruction.

From my first year of teaching in 1996 to my current practice, I have experimented, failed, changed, failed, succeeded, failed, redesigned, failed, reflected, failed, succeeded, and failed more with this type of teaching. More than a few times I have found myself questioning if this is too demanding for students.

Six years ago, I had the opportunity to teach in my school's International Baccalaureate Program. This is a curriculum that is aligned with much of what NCTM has set forth as best practices in mathematics education. Emphasis is placed on reasoning, sense making, conjecture, and proof. Each year I have two classes of HL (High Level) Math students, a year one cohort and a year two cohort. Because I have two years to work with each group of students, I have had the opportunity to refine my practice and have found some real success after many failures. These were the practices that I brought with me in the fall of 2016.

While I was developing my own practice, first-year Algebra in the middle schools had gone from a class reserved for the chosen few to a class for the masses. It became easy to use the middle school teachers as scapegoats for why our freshman, starting in either Geometry or second-year Algebra, struggled with concepts that we as high school teachers assumed they had mastered. My high school was not alone in noticing these struggles. Nationally, students who take Algebra in eighth grade are failing to achieve proficiency at the rates documented prior to this expansion on NAEP and other assessments (Loveless, 2008). My current practice is affected on a daily basis by misunderstandings and misconceptions that trace their origins back to first-year Algebra. Reflecting on my early career, I realized that I had caused problems similar to those that I observed in my current students.

It is one thing to notice a problem and place the blame on the teachers who are tasked with completing the challenge to teach more students first-year Algebra than ever before. It is another to take on the challenge of teaching a first-year Algebra course myself at one of my current school's feeder middle schools. Over the course of the 2016 – 2017 school year, I planned, taught, documented via daily fieldnotes, video recorded and collected student artifacts in an eighth grade Algebra course made up of a set of randomly selected students from the school.

The importance of studying Algebra instruction in terms of the reform of mathematics education is critical due to the need for mathematics in the modern economy. The technical society we live in demands a greater amount of mathematics knowledge in order for students to have access to the careers of the twenty-first century (World Economic Forum, 2016). Because of this, the tools for solving computational and algebraic problems must be more widely available to all students.

While my planning of the course would follow the district mandated curriculum and pacing guide, the model of instruction that I employed would not. The district instructional model is teacher-centered instruction where students are passive receives of information. This is conventional instruction in my school district and represents standard teaching practice across all disciplines and grade levels. This model provides teachers a structure for instruction that allows for a certain amount of autonomy in reaching district goals. However, this model does not align with *Principals to Actions* in other than the most abstract sense. Teachers who use this model are paradoxically faithfully meeting expectations.

In this research, I pursued a kind of natural experiment where I operated within the district curriculum in a first-year Algebra class. As often as possible, my students would be asked to engage in the doing of mathematics as a way to learn mathematics rather than act as passive receivers of knowledge. On occasion I would present a teacher-directed lesson, but this was the exception not the norm. My standard of teaching became the focus of my research as I reflected on the work in a classroom focused on learning mathematics by doing mathematics.

Upon completion of the year, I reflected on the work that I had done and chose three focus units: Inequalities, Exponents, and Parabolas. Initially, these three units represented the best opportunity to examine my teaching because they represented units of study that were not covered in previous courses and occurred late enough in the year that our classroom norms had been established.

Reflecting on the mathematics and pedagogy that I re-learned as I taught these units revealed a complexity that I had taken for granted after years of teaching in my current practice. Inequalities proved to be a unit that contained complex mathematical foundations and included elements of all concepts studied in the chapters prior to it. While teaching about exponents, I realized they represent a distinct algebra, unique from the algebra studied in prior units. Parabolas involved interacting with distinct layers of knowledge, connecting representation with vocabulary and computation.

Overview of this Dissertation

This written examination of my work focuses on the problems that I discovered when I examined the data I collected. In Chapter 2, I discuss two research-identified problems of practice in teaching Algebra. These two problems, equivalence and the equals sign and functions and variables, were identified through interviews with middle school Algebra teachers. They were then verified by a review of research on the topics.

Chapter 3 outlines the research questions and methodology used to conduct this research. During the course of the 2016 – 2017 school year, I video recorded daily lessons, collected student artifacts, and kept daily fieldnotes. Chapter 4 provides an orientation to my daily work as a teacher at my research site.

The analysis of my teaching of Inequalities, Exponents, and Parabolas are presented in Chapter 5 – 7 respectively. During these chapters, I focus on elements of my teaching that revealed themselves through the data as areas of struggle in the preparation, teaching, and learning of mathematics. In each chapter, I discuss three specific issues that emerged from the data relevant to the teaching and learning of first-year Algebra, and I include mathematical proofs that I completed to deepen my knowledge of concepts that I was teaching. This story is told through specific classroom interactions in the course of my daily practice.

I conclude this work in Chapter 8 looking across Chapters 5 – 7. The result of this analysis was the identification of two broadly defined dimensions of learning: my re-learning of the mathematics of first-year Algebra and my re-learning of how to teach it. The dimensions span topics in the three analysis chapters and result in identifying implications for policy and practice in the teaching of first-year Algebra.

CHAPTER 2

PROBLEM OF PRACTICE

In this chapter I describe two problems of practice in teaching first-year Algebra: student understanding of equivalent forms and the equals sign, and functions and variables. These problems emerged from educator interviews within my school district and are reflected in modern literature. I conclude by connecting them to my research.

Educator Interviews

In August 2015, the Director of Secondary Mathematics for my school district, J. Harrington, observed three classes at the high school where I teach. These classes were Pre-Algebra for English Language Learners, Precalculus, and Calculus 3. During each lesson he noted that the students could not move past a similar idea (J. Harrington, personal communication, August 25, 2015).

Students in the Pre-Algebra course were reviewing adding and subtracting of fractions. When the problems transitioned from ones with common denominators to ones with unequal denominators, the students quickly became frustrated. These types of problems require students to find common denominators by multiplying each fraction by another fraction that is numerically equivalent to 1. For example, in the problem below

$$\frac{1}{3} + \frac{1}{4} = \left(\frac{1}{3}\right)\left(\frac{4}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{3}\right) = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

students multiply each of the original fractions, $\frac{1}{3}$ and $\frac{1}{4}$, by two different fractions, $\frac{4}{4}$ and $\frac{3}{3}$, that are each equivalent to 1.

The Precalculus students were simplifying fractional radical expressions by utilizing properties of exponents. When working with problems where the radical was a square root, the class could handle the problems easily (i.e. $\sqrt{\frac{7x^3}{xy^2}}$). The lesson was derailed when they encountered problems of the same form with higher indices (i.e. $\sqrt[3]{\frac{7x^3}{xy^2}}$). Simplifying these types of problems requires a number of processes, but most importantly, students must determine what to multiply the expression by so that the denominator will not contain a radical when expressed in its final equivalent form. In the case of the problem,

$$\sqrt[3]{\frac{7x^3}{xy^2}} = \sqrt[3]{\frac{7x^3}{xy^2}} \cdot \sqrt[3]{\frac{x^2y}{x^2y}} = \sqrt[3]{\frac{7x^5y}{x^3y^3}} = \frac{\sqrt[3]{7x^5y}}{xy} = \frac{x\sqrt[3]{7x^2y}}{xy} = \frac{\sqrt[3]{7x^2y}}{y},$$

the choice in the second step of multiplying the expression by

$$\sqrt[3]{\frac{x^2y}{x^2y}}$$

requires different skills but fully rests upon the idea of multiplying by 1.

During the Calculus 3 course, students were working through the derivation of the decomposition of acceleration,

$$a(t),$$

into its normal and tangential components, or

$$a_T\vec{T} + a_N\vec{N}\,.$$

This process relies on symbolic manipulation of a set of equations. Similar to the previous two mentioned classes, the students in Calculus 3 struggled with the multiplication by the fraction

$$\frac{\|N\|}{\|N\|},$$

or a well disguised value equivalent to 1. All three courses observed by the Director unveiled the same struggle that dramatically different students faced during their math classes.

Struggles with these key ideas were also evident in first-year Algebra courses taught within the district. In February 2016, interviews with two colleagues, middle school teachers in my district and NOYCE Master Teacher Fellows, Jill Luschen and Phil Lafluer, revealed student difficulties with these ideas. When asked, "What do your students find most difficult to learn in first-year Algebra?" both teachers responded with similar statements (J. Luschen and P. Lafluer, personal communication, February 9, 2016).

Jill, a recognized "master teacher" and seventh grade first-year Algebra teacher, told me, "Students struggle with symbolic manipulation." She cited the example of writing the formula for finding the surface area of a cylinder into a form that solves for the height in terms of the radius and surface area. Students begin with the formula

$$SA = 2\pi r^2 + 2h\pi r\ ,$$

and are asked to transform it into

$$h = \frac{SA - 2\pi r^2}{2h\pi}.$$

To accomplish this, students must employ the standard solving algorithms that they developed earlier in the year. She identified the area of struggle for most students as a difficulty in working almost entirely with variables.

This can be easily illustrated in the first step that most students use to solve this problem. Beginning with

$$SA = 2\pi r^2 + 2h\pi r \text{ ,}$$

most students will subtract $2\pi r^2$ from both sides, which visually looks like

$$SA - 2\pi r^2 = 2\pi r^2 + 2h\pi r - 2\pi r^2 \text{ ,}$$

and simplifies to

$$SA - 2\pi r^2 = 2h\pi r \text{ .}$$

In this example, the students are writing the equation in an equivalent form by subtracting a quantity from both sides that results in a value of 0 on the right side. She stated, “While students understand the mechanics of the process, they get lost in all the letters because they don’t understand what they are doing; they just do the steps” (J. Luschen, personal communication, February 9, 2016). Jill’s statement made me wonder about her students’ thinking and understanding of the algebra that underlies the processes they were utilizing.

Phil Lafluer, an eighth grade Algebra teacher and recognized “master teacher,” made a similar statement about student struggles with abstraction. When observing his class working with simplifying exponential expressions (i.e. $\frac{x^3y^{-2}z^4}{x^{-2}y^6z^2}$), it became clear the challenges of working with purely mathematical structures caused tension in the room. While the students were able to "give the steps," they could not express why they were doing what they were doing (P. Lafluer, personal communication, February 9, 2016).

The Director of Secondary Mathematics and two middle school Algebra teachers in my district had all credibly observed that students do not conceptually understand equivalent forms and how they are used in symbolical manipulation, or how variables are used to construct functions. These two problems are also reflected in current scholarship in the field.

Equivalence and the Equals Sign

First-year Algebra has been the focus of reform efforts and research in mathematics education for over four decades. Specifically, one concept fundamental to student understanding and success receiving significant consideration is that of equivalence and the equals sign. Early work by Behr, Erlwanger, and Nicholas (1980) and Kieran (1981) laid the foundation for later research establishing ways students interpret the equals sign and the effects those interpretations have on success in first-year Algebra.

Behr et al. (1980) studied students in first through sixth grades to investigate misconceptions of equivalence and the equals sign. Utilizing a series of unstructured interviews, the researchers discovered that students tend not to view the equals sign as a sign of equivalence, but instead as a command to carry out computations from left to right. As a result, students struggle making sense of equations that are not of the form *a*

$+ b = c$. The authors conclude that this internalized understanding may affect students' ability to learn other mathematical concepts.

Kieran (1981) preformed a cross-sectional analysis to examine how the equals sign is understood by students ranging from preschool through college. She discovered that the idea of the equals sign as a command to perform an operation starts before formal education begins and continues throughout high school. Students were observed to have established this conceptualization prior to entering the primary grades and it undermined their understanding and success in algebra.

Baroody and Ginsbury (1983) studied first through third grade students participating in an individualized curriculum that consisted of a series of games focusing on one or two concepts at a time. Their results suggest that students' difficulties with equivalence are partly due to early mathematical experiences that produce an understanding of addition as a process that functions in only one direction. The results suggest the way students interpret symbolic representations of mathematical concepts is dependent on earlier learning experiences and influences success in algebra.

Later research has supported these findings. An operational view of the equals sign by students is supported by the research of Alibali (1999) with third and fourth grade students; Faulkner, Levi and Carpenter (1999) with first through sixth grade students; and McNeil and Alibabli (2005) with third through fifth grade students. Evidence from work by Carpenter, Franke and Levi (2003) and Seo and Ginsburg (2003) continues to suggest that the origins of these misunderstandings are in part due to students' elementary school learning experiences.

Knuth, Stephens, McNeil, and Alibali (2006) examined middle school students' understanding of the equals sign as it relates to their performance in solving equations. This study was accomplished by having students from sixth, seventh, and eighth grades complete assessments that measured both their understanding of the equals sign as well as their success in solving linear equations. The results of their research show that a strong relationship exists between students' conceptual understanding of the equals sign and success in solving linear equations. Additionally, students who had a strong relational understanding of the equals sign and no formal algebra instruction solved equations more successfully than students who had algebra instruction and a computational view of the equals sign.

Alibali, Knuth, Hattikudur, McNeil, and Stephens (2007) conducted a longitudinal study over a three-year period collecting data from a group of eighty-one students. This study measured students' understanding of the equals sign, their performance in solving linear equations, and changes in students' understanding of the equals sign and success in solving equations over time. Data from this research indicates that students' development of a more advanced conceptual understanding of the equals sign is associated with an improved performance in solving linear equations.

The results of these two studies were supported by the work of Booth and Koedinger (2008) in a study of forty-nine high school students taking a first-year Algebra course. Students were given an assessment that measured both the ability to solve linear equations and conceptual knowledge of ideas determined to be critical for success in algebra. The research suggests that when students have incorrect or incomplete understanding of the equals sign, they have difficulty in successfully solving linear equations. They also found that increasing conceptual knowledge of the equals sign increases overall learning.

Recent work by Matthews, Rittle-Johnson, McEldoon, and Taylor (2012); Byrd, McNeil, Chesney, and Matthews (2015); Knuth, Stephens, Blanton, and Gardiner (2016) supports the earlier research involving algebra and the equals sign. These studies focused on work done in the primary classroom that sets the foundation for success in first-year Algebra. Results continue to reinforce the relationship between success in solving linear equations and a conceptual understanding of the equals sign.

The results of nearly forty years of research show the importance of shifting student understanding of the equals sign from a computational one to a relational one for success in first-year Algebra. This is a problem that readily manifests itself in the first-year Algebra classroom in the Omaha Public Schools. Interviews conducted with Jill Luschen and Phil Lafluer in February 2016 provide a practitioner's point of view on these issues.

When Jill stated, "While students understand the mechanics of the process, they get lost in all the letters because they don't understand what they are doing; they just do the steps, (J. Luschen, personal communication, February 9, 2016)" she is addressing a concept that is fundamental to algebra, i.e. symbolic manipulation. Phil's observation that students were "able to give the steps but not understand what they were doing (P. Lafluer, personal communication, February 9, 2016)" when simplifying exponential expressions, reflects the same issue with a group of eighth grade Algebra students.

As the research demonstrates, these struggles with algebra come from a computational view of the equals sign as a command to execute a computational task. Because both of these teachers are working with middle school students in algebra, this is a point of view that would have been established in the primary grades or earlier (Behr, Erlwanger, & Nicholas (1980), Kieran (1981), Baroody & Ginsbury (1983), Alibali (1999), Faulkner, Levi & Carpenter (1999), McNeil & Alibabli (2005), Carpenter, Franke & Levi (2003), and Seo & Ginsburg (2003)). The encouraging results of recent research suggests that offering students a relational view of the equals sign improves learning in the algebra classroom (Knuth, Stephens, McNeil, & Alibali (2006), Alibali, Knuth, Hattikudur, McNeil, & Stephens (2007), Booth & Koedinger (2008), Matthews, Rittle-Johnson, McEldoon, & Taylor (2012), Byrd, McNeil, Chesney, & Matthews (2015), Knuth, Stephens, Blanton, & Gardiner (2016)).

While this research addresses the broad topic of equivalent relations and the equals sign, the earlier problem that was identified (utilizing the properties of 0 and 1) is a subset of this larger issue. Mathematically, symbolic manipulation of equations is accomplished by using different forms of the numbers 0 and 1. The reviewed research illustrates that for students to reason and make sense of this process, they must possess a relational view of the equals sign and not a computational one.

Functions and Variables

Fundamental to working with algebraically equivalent expressions is an understanding of the variables present within the expressions and how those expressions are used to construct mathematical functions. Understanding of variables and functions allows for the generalization of properties necessary for algebra to be conceptualized by learners as a qualitative exercise.

Early work in understanding how children interpret variables in mathematics was done by Collis (1975) and Küchemann (1978). Collis's work in understanding how young students interpret letters in mathematics class was refined by Küchemann in his design of an assessment instrument to reveal how students manage the demands of different mathematical tasks. He developed a set of six stages for describing how letters can be used mathematically (p. 23). These are the stages:

- Letter Evaluated
- Letter Not Used
- Object
- Specific Unknown
- Generalized Numbers
- Used as a variable

Küchemann groups these into four distinct levels. Level one consists of the first three stages and is considered to be the lowest level of understanding. The second level of understanding, treating a variable as a specific unknown, allows students to solve more complex problems. However, students at level two struggle with generalized concepts involving variables. The third level comes when students allow for variables to take on multiple values. Students with level four understanding are able to understand and interpret variables in differing contexts.

Through a series of interviews and algebra tests with twelve- to fifteen-year old students, Warren (1999) was able to support Küchemann's stages of understanding. Furthermore, Warren outlined several misconceptions that students possess about variables. One such misconception is the need for "closure" in which students feel the need for their answer to be a singular element (i.e. instead of a+b, students wrote ab). Another misconception discovered by Warren was that students assign values to letters based on their position in the alphabet.

Additional research has shown a number of other false beliefs about variables. Booth (1988) found that first-year Algebra students use variables to represent a unit or label instead of a quantity (i.e. *m* is minutes instead of *m* is the number of minutes). Stacey and MacGregor (1997) suggest that one of the reasons behind this problem may be the teacher's word choice when they choose variables as the first letter of what they represent. Stephens (2005) demonstrates that many students believe that different variables cannot hold the same value when entering first-year Algebra.

In 2003, Trigueros and Ursini built on Küchemann's work outlining three major interpretations of variables by students. The researchers focused on:

- Variables as specific unknowns
- Variables as general numbers
- Variables in functional relationships

In contrast to Küchemann's work, Trigueros and Ursini did not connect these conceptualizations to differentiated levels of understanding by students. Instead, their study focused on student interpretation of variables and whether or not each of these is valid.

The researchers found that when students view variables as specific unknowns, they are comfortable using “variables to factor, simplify, transpose or balance equations” (Trigueros and Ursini, 2003, p. 3). Students who view variables as general numbers can also “factor, simplify, expand and rearrange expression (ibid),” while those who view them as functional relationships can also understand global relationships between two quantities. Trigueros and Ursini present strong evidence that students who are unable to differentiate between variables as specific unknowns and variables as general numbers possess difficulties in understanding variables in functional relationships.

These struggles and misunderstandings about variables lead to difficulties with functions and the ability of students to work with and understand general forms of equations. This problem was noted by both middle school algebra teachers through classroom observations. In Jill’s class, the students were asked to transform the equation

$$SA = 2\pi r^2 + 2h\pi r$$

into the form

$$h = \frac{SA - 2\pi r^2}{2h\pi} .$$

One issue that students had with this equation was a lack of understanding about equivalent form, but another could be a fundamental misunderstanding of the use of symbols. Students who hold a view of variables as specific unknowns would not see the need for rearrangement. Others who see the variables as labels would find the exercise meaningless because they would not understand what the equation itself represents.

This misunderstanding of variables was also present in Phil's classroom. When students were confronted with the problem of simplifying exponential expressions (e.g. $\frac{x^3y^{-2}z^4}{x^{-2}y^6z^2}$), they were able to give steps without comprehending what they were doing. This lack of understanding could easily come from each individual student's level of interpretation of the variable.

Research has shown that teachers can help students improve their understanding of variables and develop deeper levels of conceptual awareness. Classroom discussions are one such method for increasing student conceptualization. Lodholz (1999) found that verbalizing thinking helps to "externalize the students' thoughts, makes them public, and provides the teacher with an invaluable tool for assessing students' understanding of concepts" (p. 55). To be successful in algebra, students need to learn to "use symbols as a language in which they can express their own ideas" (Lodholz, p. 55). Wagner and Parker (1993) indicted, "Students can work with variables without fully understanding the power and flexibility of literal symbols" (p. 330). Therefore, this knowledge does not need to be complete for a student to successfully learn algebra.

A Note on Inquiry-Based Learning

Throughout this dissertation, I employ the term "inquiry-based learning." This term and its usage are commonplace in STEM education and seemingly self-explanatory to reform-minded teachers. However, I recognize that it may not be hegemonic in public middle and high school practices. Here I use it in the most general sense to indicate that it is an idea- and student-centered orientation where students and I negotiate questions and cooperatively investigate mathematical concepts during daily instruction. Cultivating student questions is as important as posing teacher questions as they reveal the limits of extant knowledge and become starting points for instruction. Possessing this knowledge helps students take risks and undermines the fear of being publicly humiliated by their errors. Additionally, this type of teaching reflects how mathematicians authentically work in their practice.

Defining Inquiry-Based Learning

Inquiry-based learning, known in many different forms, is an approach to instruction that is a subcategory of inductive approaches to teaching and learning (Prince and Felder, 2006). While inquiry-based instruction can be traced to the teachings of Socrates and Confucius, work by Dewey (1933), Bruner (1960), Piaget (1972), Vygotsky (1962), and Schwab (1960) influences current pedagogical practice within constructivist learning philosophy in two domains: cognitive and social. The key tenant of constructivism is that an individual learner actively constructs knowledge and skills through experiences and interactions within the environment (Bruner 1960).

Dewey (1933) promoted an experiential learning pedagogy in which children are active, inquisitive learners rather than passive receivers of knowledge. Cognitive constructivism draws from the work of Piaget (1972) who proposed that individuals must construct their own knowledge built through experience. Social constructivism builds from Vygotsky's (1978) work focusing on learning through cultural history, social context, and language. This work includes the concept of the Zone of Proximal Development, which argues that individuals can, with the help of a more experienced peer, master concepts and ideas that they cannot understand on their own. Schwab (1960) called for inquiry to be divided into four distinct levels: Confirmation Inquiry, Structured Inquiry, Guided Inquiry, and Open Inquiry.

Bruner (1960) proposed a five-step cycle of inquiry where teachers and students begin by posing questions about a situation. These questions are then investigated in a variety of situations that lead to the generalization of observations. Generalizations are discussed and reflected upon, leading to more questions. This cycle is a guide to inquiry-based instruction, not a rigid process to be followed. Inquiry-based learning is better characterized as situated learning where learning happens as a function of the activity, context, or culture of the classroom. The learner moves from the periphery to the center within a community of practice (Lave & Wagner, 1991).

Reflecting on the work of these researchers along with my own personal and professional efforts has led me to a working definition of inquiry-based instruction as an approach to teaching and learning that beings with students engaging in problem-solving, making observations, or answering questions in order to develop individual understanding of concepts. Individual conclusions are shared first within small groups of students and then with the whole class. While students are working in small groups, the teacher provides clarification regarding the demands of the task. Once small groups are ready to share their conclusions, the teacher facilitates the discussion, helping students discern critical information from secondary and tertiary ideas.

This definition reflects four ideas principle to inquiry-based instruction. First, learners are the center of the process while the teacher, resources, and technology are organized to support them. Second, learning activities involve student questioning, reasoning, and sense making of information. Third, teachers facilitate the learning process by seeking to understand their students' interactions with the concepts being studied. Finally, learning goals emphasize the development of student reasoning and sense making relative to conceptual understanding.

Historically, mathematicians have viewed studying mathematics as an inquiry process often referred to as "digging deeply" or "conceptually understanding" a topic. This is a key principle in my instruction, and this dissertation is a reflection on my efforts to enact these ideas in the daily instruction of a normal first-year Algebra course within the constraints of district mandated curriculum.

Connection to My Research

My own practice is affected by misunderstandings that trace their roots back to first-year Algebra. Conversations with other teachers and the Director of Secondary Mathematics in my district helped me to understand that these problems were not unique to my classroom, content, building, or grade level. A review of modern literature identifies these issues as student misunderstanding of equivalent forms and the equals sign, and student misunderstanding of variables and the assembling of variables into functions.

These two issues point to areas of inquiry in a qualitative self-study. I spent the 2016 – 2017 school year teaching one section of first-year Algebra seeking to understand how these two issues emerge and present themselves to a teacher and students. Moreover, I wanted to see how this occurs in the context of enacting student-centered teaching that is highly content focused. Thus, two broad questions guided this self-study:

- How do students respond to inquiry-based instruction in a standard first-year Algebra course?
- What are the intellectual and practical demands on a teacher trying to enact inquiry-based instruction in first-year Algebra?

First, I wanted to know how eighth graders would respond to inquiry-based instruction. When I last worked with this population seventeen years ago, my classroom was teacher-centered and focused on covering content. While I wanted my early practice to be student-centered, it was not; and I knew it could have been. This research has allowed me to see for myself how regular, eighth grade students would respond inquiry-based instruction in a student-centered classroom. I also wanted to know if these methods would actually advance student learning of first-year Algebra and if this would be revealed in conventional assessments.

Second, I had come to a point in my own professional learning that I recognized student learning cannot be separated from a teachers' own math and pedagogical knowledge, or the knowledge math teachers are required to have in order to teach in an inquiry fashion. In a study of practice as the practitioner, it is necessary to ask about what is asked of the teacher, both intellectually and practically. Having not taught first-year Algebra at any level in the last decade or worked with eighth graders in the last seventeen years, the research required me to document my preparation, teaching, and reflection of how students and I responded to these efforts throughout the district standard curriculum.

Taken together, these two foci of the research point to the inseparability of teacher knowledge and student knowledge. Because of this interdependence, methodology was designed to collect data simultaneously from both sources. I analyzed data with this relationship in mind moving between evidence of teacher knowledge and student knowledge when searching for themes.

CHAPTER 3

METHODOLGY

In this chapter, I outline the methodology used throughout the study. Pertinent to this study, I discuss the purpose, setting, data collection methods, and data analysis techniques. The purpose outlines the areas of analysis and guiding questions. Jefferson Middle School, a diverse school located in a medium-sized Midwestern city, served as the setting for this experience. I collected data from participant observations, field notes and journaling, artifacts of student work, video recordings of classes, and professional communications. My teaching and reflections on my teaching of inequalities, exponents and parabolas served as the primary instruments for analysis. Data was verified by triangulation.

Purpose

The purpose of this research is to examine how inquiry-based instructional techniques will support student learning and improve achievement in first-year Algebra through the specific topics of inequalities, exponents, and parabolas *and* what is required of a teacher to carry this out. It is now generally accepted that improved learning and achievement in algebra, specifically by the eighth grade level, can lead to significant long-term benefits for student achievement and support their readiness for secondary level mathematics. This research is guided by two questions:

- How do students respond to inquiry-based instruction in a standard first-year Algebra course?

- What are the intellectual and practical demands on a teacher trying to enact inquiry-based instruction in first-year Algebra?

In an effort to explore these questions, I searched for a setting where I could collect sufficient data. In April of 2016, I negotiated entry into a middle school in the same district in which I am currently employed and obtained an assignment of teaching one section of first-year Algebra.

The Setting

I was assigned a class at Jefferson Middle School, located in a medium-sized Midwestern city. Jefferson is a school of approximately eight hundred students in sixth, seventh, and eighth grades. During the 2016-2017 school year, there were approximately:

- 800 students
- 35% White
- 15% Hispanic
- 34% African American
- 15% Asian
- 1% Native (Demographics, 2016)

Additional demographics of the school include:

- 72% Free/Reduced Lunch
- 20% Special Education
- 12% Current English Language Learners

- 17% Former English Language Learners (Demographics, 2016)

Part of my negotiation in gaining entry into Jefferson was ensuring that a class set of students was randomly assigned from the set of all eighth grade students eligible to enroll in first-year Algebra. In this case, a student is defined as eligible to enroll in the course if either Pre-Algebra was successfully completed during the seventh grade year, or first-year Algebra was unsuccessfully completed during the seventh grade year.

At the beginning of the 2016 – 2017 school year, twenty students were assigned to the course. Eighteen of the twenty assigned students enrolled at Jefferson Middle School on the first day of class, August 17, 2016. One of the eighteen students left the class during October and one student was added into the class during November.

Data Collection

Prior to the first day of class, I approached a school counselor who agreed to present the student consent form on the first day of class so that students did not feel pressured to participate in the research. Students were informed of their right to consent to be a part of this study as well as their right to withdraw from the study at any time. Parents were informed of the study via a letter mailed home prior to the beginning of the school year. I collected all parent and student forms during the first three days of class.

Required data was collected through a variety of qualitative techniques. During the course of normal instruction, participant observations yielded daily retrospective fieldnotes, the analysis of documentary artifacts of student written work, and video recordings of whole class and small group instruction yielded additional supplemental data. Professional communication in conversations and emails between Jill Luschen, Phil Lafluer, and Kenzi Mederos served as another source of data.

Participant Observation

During the course of normal teaching, I observed actions and reactions by students and myself before, during, and after a lesson. These observations were both verbal and nonverbal in nature in my attempt to capture student responses to inquiry-based learning and the demands that I encountered during the act of teaching. I took notes *in situ*, quickly jotting down observations to minimize the impact on normal classroom interactions. After class, these notes were transferred into my journal for reflection.

Cataloging participant observations proved to be a learning process that was more challenging than I anticipated. All teachers make mental notes during the course of teaching. For purposes of data collection, I took these notes more deliberately than in my normal practice. The challenge was delineating my observations between informative and secondary information.

In the early weeks, I had very little idea where to put my focus while making observations. Transferring this daily data into my fieldnotes allowed me to reflect on the different facets of the classroom and identify the components that were important to learning. I was then better prepared to focus on those ideas the next day in class. This process of improvement continued throughout the school year.

Fieldnotes and Journaling

Beginning on August 10, 2016, during the opening of school meetings and preparation time, I kept fieldnotes. This reflective note taking continued daily throughout the school year. These were an elaboration of the notes I took during teaching combined with reflections on the planning and execution of lessons.

I wrote in my journal daily immediately after the class period ended and then later, during time spent planning the next day's lesson. Each entry began with the current date which allowed me to match each journal entry with the correct lesson, video, student work and professional conversations. At the beginning of each quarter, I began my journal in a new notebook. Previous journals were stored in a locked cabinet.

Writing after a lesson allowed me to reflect on the successes of the lesson, to identify areas of concern in terms of student learning and behavior, to prepare for planning the next lesson, and to return ready for more purposeful observations. Writing during planning time allowed me to capture my thought processes, identify student learning goals, notice connections between the current lesson and previous and future lessons, and anticipate student reaction to different parts of a lesson.

Artifacts of Student Work

Student work was collected throughout the school year. These artifacts included practice assignments, formative assessments, and summative assessments. Practice assignments were ungraded tasks that included but were not limited to individual daily homework, small group problem sets, and other classroom activities. Formative assessments included all graded work that occurred prior to the unit exam. Summative assessments were district common assessments that were provided by the school district and required to be completed by all students enrolled in first-year Algebra.

Practice and formative work was photocopied and cataloged by chapter and date in file folders and would then be returned to the students. District common assessments were not returned to students due to district policy. Therefore, the original copies were cataloged by chapter and date in file folders along with the practice and formative work. This cataloging aligned the data to the journal entries, video recordings, and professional conversations. These copies were stored in a locked cabinet.

Video Recording

Upon receipt of all parent consent and student assent forms, I began recording the class on a daily basis. The first day of recording was August 24, 2016. Recording continued until the final day of school, May 26, 2017. The camera was placed in the back of the room allowing for the entire class to be seen in the recording. An external microphone was used to make sure student conversations were clearly recorded. Data was stored on SD drives and transferred weekly to an encrypted removable hard drive. The hard drive was stored in a locked cabinet.

Daily recordings required two or three separate files due to the quality of the video. The video files were labeled by day, month, and part to allow the files to be identified chronologically. Classroom videos were reviewed and annotated in an Excel spreadsheet allowing for the identification of topics and activities ensuring data correlation between videos, journals, student work, and professional conversations.

Professional Communications

Throughout the school year, I had a number of conversations with Jill Luschen, a NOYCE Master Teacher, Kenzi Mederos, a teacher at the research site and Math in the Middle graduate, and Phil Lafluer, a NOYCE Master Teacher and the teacher whose room I shared during the 2016 – 2017 school year.

Phil and I planned lessons on a daily basis throughout the school year. We discussed ideas in person and over email. Personal conversations were recorded in a journal and emails were downloaded and stored on an encrypted hard drive. Kenzi, Jill, and I discussed issues of teaching first-year Algebra in the middle school prior to my teaching during the 2016 – 2017 school year as well as over email during the school year. Our conversations were recorded in a journal and emails were downloaded and stored on an encrypted hard drive. This data was stored in a locked cabinet.

Data analysis

After gathering the data, three units were chosen for examination and reflection:

1. Inequalities
2. Exponents
3. Parabolas

These units were chosen because they represented topics that were not deeply covered in the Pre-Algebra course that most of the students had taken the prior year, and they represented concepts and skills necessary for success in upper level mathematics courses. Claims presented in the analysis chapters were chosen because they were reflected in at least three of the sources of data.

In this study, my teaching and reflections on my teaching are the primary instruments for analysis. To ensure that this study is valid, the method of verification used involved the triangulation of data. Triangulation of data refers to, "Comparing and cross checking the data" (Merriam, 2009).

When the research was complete, I reviewed and annotated each video lesson. From the annotations, I observed themes in student learning and teacher planning. Themes from lessons were then cross referenced with fieldnotes and student work to determine which themes would be analyzed. Data analysis was ongoing throughout the reflection allowing for continued triangulation.

I first identified issues to be explored in one type of data. Once identified, I found myself confronted with more questions about the veracity of the claim which forced me to look back at other forms of data. Every step forward with a particular claim would send me back into the data for evidence to support that claim. Once I had data from at least three different sources, I moved ahead by ensuring the evidence that I had collected triangulated across the different types of data I possessed.

CHAPTER 4

OVERVIEW OF THE 2016-2017 SCHOOL YEAR AND THREE TOPICS FOR ANALYSIS

This chapter outlines the 2016 – 2017 school year. I spent one period each day teaching at Jefferson Middle School and the reminder of the day teaching at my high school. I taught a standard first-year Algebra course while following the district mandated curriculum and pacing guide. Jefferson follows a blended scheduling format with each class meeting for forty-five minutes three days a week and two block days with each class meeting one of the two days for ninety minutes. Daily and weekly classroom norms emerged and are detailed. I describe the process and justification for the selection of the analysis topics with emphasis placed on reasoning and problem solving, and the language and notation of mathematics.

Back to Middle School

In the fall 2016, I found myself in a place I had left in 2001, the middle school classroom. Five of my first six years of teaching were spent working with middle school students and the experience was invaluable. I had the opportunity to teach students of all abilities and motivations. During those years I worked with students who were preparing to compete in national mathematics competitions and others who struggled with basic computation.

This return trip to teaching eighth grade was much different. I had taught at my current high school since 2002, and I had certain responsibilities that I needed to maintain. My course load included both International Baccalaureate and post AP

Calculus mathematics courses. These courses required both training and education beyond the typical classroom teacher, making it critical that I be available to teach those courses at the high school.

There were also concerns that my presence at particular middle schools would be viewed as a recruiting effort. Because of these two issues, I pursued one of the middle schools that serves the same region of the city as my high school. There was the added benefit that this middle school also serves as a partner school to my high school with the International Baccalaureate program.

I negotiated with my building principal, district officials, and the principal of Jefferson Middle School to make this unique placement and schedule possible. Instead of moving to a different school for this research experience, I was granted permission to spend time at both schools. My day started at the middle school during the first period of their school day. I would then return to my high school to teach the remainder of the day. When I received the official assignment, I was excited to find out that I would be sharing a classroom with another NOYCE Master Teacher Fellow, Phil Lafluer. Phil and I had worked together for the past seven years in both the Master Teacher Fellowship and as leaders of professional development for fellow math teachers in our district.

Jefferson Middle School

As described in Chapter 3, I was assigned to teach one class of first-year Algebra at Jefferson Middle School. Jefferson is a school of approximately eight hundred students with an ethnically diverse population and more than seven in ten qualifying for free or reduced lunch. About one in three students are non-native English speakers.

The school week is a blend of traditional and block scheduling. On Monday, Tuesday, and Friday, students attend eight classes each lasting about forty-five minutes. Students attend half of their classes on Wednesday in a block period format, each lasting about ninety minutes. They attend the other half of the classes on Thursday. Because my Algebra class met during the first period of the day, we met Monday, Tuesday, Wednesday, and Friday during a usual week. Some changes in the schedule happened throughout the year; for example, during the first and last weeks of each semester, classes met for forty-five minutes every day instead of four out of five.

District Mandated Curriculum

The curriculum of the course was dictated by my school district along with a general pacing guide. Eleven units of study were required for the course as shown in Figure 4.1. These units aligned by title and content with the chapters of the textbook that I would be using with the students throughout the year, *Algebra 1* by Glencoe (Carter et al., 2014). Additionally, district standard assessments were mandated to be given at the conclusion of each chapter. Students were allowed to review their scored work on these chapter tests, but they were not allowed to retain the scored tests.

Algebra 1-2: Unit Titles/Concepts

Semester 1	Pacing Days	Pacing Block
Unit 0: Preparing for Algebra	4	2
Unit 1: Expressions, Equations, and Functions	18	9
Unit 2: Linear Equations	14	7
Unit 3: Linear Functions	12	6
Unit 4: Equations of Linear Functions	10	5
Unit 5: Linear Inequalities	12	6
Unit 6: Systems of Linear Equations	14	7
Semester 2 (*NOTE: Due to State Testing, and Acuity extra days are built in to pacing guide)	Pacing Days	Pacing Block
Unit 0: Preparing for Algebra	6	3
Unit 7: Exponents	12	6
Unit 8: Quadratic Expressions and Equations	16	8
Unit 9: Quadratic Functions and Equations	16	8
Unit 10: Statistics and Probability	12	6

Figure 4.1 – District pacing guide as it appears in curriculum material

Daily and Weekly Routines

Prior to the start of the school year, I planned a daily and weekly routine with Phil. Having been away from the middle school classroom for fifteen years, Phil's advice on topics such as classroom norms and lesson pacing proved invaluable. The classroom routine emerged and more or less remained the same after a few adjustments during the early part of the year. Our day went as follows:

1. Students arrived to class at the opening of the school day.
2. Morning announcements and the Pledge of Allegiance were read over the intercom.
3. Students spent the first five minutes of class working in small groups helping each other with problems from the previous night's homework. Any problems no one could solve would be written on the white board at the front of the room.
4. While groups were working, I took attendance, checked in with any students who had been absent recently, and monitored the groups.
5. After going over homework, we moved on to the lesson for the day.
6. On Wednesdays we would use part of the additional time as preparation for the state math exam that would be given in the spring as well as a brain break at the half way point of the period.

Once established, this routine was maintained throughout the year and provided a daily reminder to the students that our classroom was collaborative and not individual. Daily lessons included multiple points where the students worked together to analyze situations, solve problems, explain reasoning, and inquire into the mathematics that they were studying. By starting class off with the students working in small groups on homework, conversations about mathematics were the first event of the period making further discussions easier and more natural for the small groups.

Three Topics for Analysis

The district mandated curriculum lists eleven units of study. Within these units are a myriad of potential topics for research. Any unit of first-year Algebra contains topics and sub-topics worthy of academic scrutiny. For instance, Unit 2 is the study of solving linear equations where the final topic is solving equations symbolically. I could have examined how students connected their mathematical understanding of solving one-variable equations to solving symbolic ones.

From the planned eleven units of study in the curriculum, I chose three for the analysis portion in the subsequent chapters. The three units are Unit 5: Linear Inequalities, Unit 7: Exponents, and Unit 9: Quadratic Functions and Equations. To choose these, I examined the data I had collected throughout the school year. After reviewing my notes, student work, and a sample of the video recorded lessons, these three units emerged because they connected to both my problem of practice and research questions. They were also convenient to study because the volume of data I had collected in these units allowed for a robust analysis of my learning and student learning.

While there were many topics worthy of study, these three units presented a great opportunity given the development of the students during the course as mathematical thinkers and learners. These were chosen because they contained material that was not covered in prior course, allowing me to study student learning and my teaching without students entirely relying on prior learning.

Teaching these three units was interesting to me both mathematically and pedagogically. Mathematically, studying inequalities requires the use of nearly every concept and procedure developed prior to the unit, e.g. solving linear inequalities uses the same procedures as solving linear equations. Exponents form a distinct algebraic structure that interacts with the algebra of linear equations, e.g. simplifying expressions with both variables and coefficients. Parabolas presented the opportunity to build a complex set of mathematical relationships that serve as a bridge between first-year Algebra and upper-level mathematics, e.g. the vertex of a parabola is a maximum or minimum value.

There were characteristics unique to each unit that made them compelling to teach. Inequalities offered me the opportunity to connect common ideas to mathematical representations when designing lessons. The result was the opportunity for students to make sense of mathematical concepts through authentic situations. Teaching properties of exponents was an opportunity for students to discover algebraic properties in a mathematically authentic way. The complexity and volume of information learned about parabolas was challenging to teach in a way that was still approachable for students.

The eighteen students in the class had all successfully completed a Pre-Algebra course in the prior year. Three of the eighteen students began in a first-year Algebra course at the beginning of seventh grade; two were moved to Pre-Algebra within the first six weeks of the first semester, and the third was required to repeat the course. The Pre-Algebra curriculum in my district covered similar material contained within units 0 – 4 of the first-year Algebra course. Phil had warned me that some of the students would rely on their knowledge from Pre-Algebra during those chapters and not pay attention in class.

This meant that the first topic that would be new to all but one of the students was Unit 5 and made it and any subsequent chapters ideal for study of my teaching because it would not be directly influenced by what was learned in the previous year.

Reasoning and Problem Solving

Each of these units contained distinct concepts and processes of abstract reasoning and problem solving. In Unit 5: Linear Inequalities, the students worked with equations that presented families of solutions instead of particular values solved for, e.g. $x > 5$ instead of $x = 5$. Unit 7: Exponents exposed students to an algebraic structure that was different from the previous ones studied, e.g. product of powers compared to multiplicative property of equality. Unit 9: Quadratic Functions and Equations engaged students in the algebra of parabolas, which utilizes the algebra of linear equations as a part of its analysis, e.g. completing the square.

Language and Notation of Mathematics

These three units also presented the opportunity to study how students interacted with the language and notation of mathematics critical to future success in further mathematical study. These interactions were foreign to any of their previous experiences in school or their lives outside of the classroom. Additionally, the structure of the algebra studied in these units is distinct. The algebra of linear inequalities and quadratic functions both make use of the algebra of linear equations in their development but have different goals. The algebra of exponents is distinct from the algebra of linear equations. Also, it does not lend itself easily to the modeling of ideas that are relevant to eighth grade students.

The three chapters that follow are an analysis of my teaching of Inequalities, Exponents, and Parabolas. My focus within these chapters are the problems and struggles that I discovered while examining the data I collected over the course of the school year. Some of these issues reflect the problems that current literature has identified in learning algebra, while others are specific to my practice. They represent the story of my 2016-2017 school year and act as a guide to my year-long learning experience returning to the middle school classroom after over a decade away from it.

CHAPTER 5

INEQUALITIES: OUR FIRST TRIP INTO THE UNKNOWN

All proportions, every arrangement of quantity, is alike to the understanding, because the same truths result to it from all; from greater from lesser, from equality and inequality.

-Edmund Burke

In this chapter I explore my teaching of inequalities, the fifth unit I worked through with my class from November 28, 2016 to December 20, 2016. This unit became an ideal place to begin examining my teaching in earnest, because it marked the first point in the school year that the material did not overlap with the prior course. The two problems of practice I had previously identified appeared throughout the unit. First, that equivalent forms of equations are used in solving one-variable inequalities, absolute value inequalities, and in rewriting two-variable inequalities into slope-intercept form. Second, that variables and functions like linear and absolute value inequalities are woven throughout the unit. Using these problems of inequalities as a basis for examination of students' work, I identified three areas of concern:

1. appropriately changing the direction of the inequality symbol during algebraic commonplaces,
2. graphing solutions to one variable inequalities, and
3. representing problems algebraically with an inequality.

Introducing the Unit

The students and I had spent the last four months in this classroom establishing norms, and they were slowly transitioning from the early feeling out period any teacher experiences into a more comfortable, community environment. On the whole, membership in the class was stable. Two students originally in the class had moved away, and one student joined the class late due to a scheduling issue.

The Inequalities Unit marked the point in the curriculum when I knew things were about to become difficult for a large portion of my class. During many conversations with Phil, I learned that the first four chapters of my first-year Algebra course mirrored the curriculum of the Pre-Algebra course that all but one of my students completed during the prior school year. Sabrina was the only student that did not take Pre-Algebra. She took first-year Algebra as a seventh grader but had been required to retake the course due to attendance issues.

Having completed units on solving equations and graphing lines, I was still unsure which students were learning new concepts and which were relying on prior knowledge from the previous course. This new unit on inequalities provided me the opportunity to gain some insight into the issue. Over the course of the next three weeks, the students engaged in study that made use of the major concepts developed during the first four chapters. As prescribed per the district curriculum, we explored the following topics:

1. Solving one-variable inequalities with addition and subtraction
2. Solving one-variable inequalities with multiplication and division

3. Solving one-variable multistep inequalities

4. Solving compound inequalities

5. Solving absolute value inequalities

6. Solving two-variable inequalities

This list of curricular topics presented me with my first challenge in planning the unit. I needed to bridge the author's word choice, intended to present teachers with an easily understood list of topics, to the mathematics that each represented. Prior to first-year Algebra, my students' primary experience with inequalities was using them to signify the relationship between two known quantities. A limited introduction to variables and inequalities occurs in Pre-Algebra. During our unit, however, they were not going to be determining the relationship between quantities. Instead, the relationship between two different expressions would be dictated and the students would need to learn that this relationship represented a set of solutions relative to the variable. Additionally, they would be using properties similar to those used to solve equations, but because the problems involved inequalities, there we subtle and important differences.

Understanding the Content

Looking over the list of topics, I concluded that the critical concept I needed to establish was the fundamental difference between working with and solving inequalities as compared to the equalities we had worked with in prior units. For example, when solving inequalities, students were confronted by solution sets that were either infinitely large or empty. This stands in contrast to our work in Unit 2 on solving linear equations. During that unit, students worked with problems that were initially of the form

$$ax + b = c,$$

and primarily had one solution. The equations became more complex during the unit, first of the form

$$ax + b = cx + d,$$

and then incorporating distributive property as either

$$a(x + b) = d$$

or

$$\frac{x + b}{c} = d.$$

Although the above two equations seemed to be treated differently by my students, they are mathematically equivalent if

$$a = \frac{1}{c}.$$

Once a variable is present on both sides of an equation, the possibility of infinite solutions or zero solutions becomes valid. The students first worked with these two ideas on October 5, 2016 (Planning notes). The idea of two valid solutions was developed shortly after the idea of infinite solutions or zero solutions on October 10 (Planning notes) when students were introduced to solving absolute value equations such as

$$|ax + b| = c,$$

where the equations were transformed into two equivalent equations that needed to be solved:

$$ax + b = c$$

and

$$ax + b = -c.$$

Absolute value equations also reinforce the idea of one solution when

$$c = 0,$$

and no solutions when

$$c < 0.$$

Units 3 and 4 reinforced the concept of infinite solutions when we graphed linear equations in slope-intercept form

$$y = mx + b,$$

standard form

$$Ax + By = C,$$

and point-slope form

$$y - y_1 = m(x - x_1).$$

While this type of understanding of solutions was one goal of the unit according to the district unit planner, I knew that a majority of the students held incomplete understandings about solving equations and graphing lines which could create difficulties when we applied them to inequalities (Journal, November 20, 2016). First, many students worked from a mindset that equations have a single solution and became confused when confronted with any case that resulted in anything else. Second, solving absolute value equations quickly became a rote skill exercise; students split the equations into their two equivalent forms and solved them. The same was true with graphing equations. The students graphed lines using the slope and y-intercept when working with equations in slope-intercept or point-slope form or identified the x- and y- intercepts when graphing equations in standard form.

Opening Activities

Keeping this in mind, I sought an opening activity for the unit. I chose to employ a technique I had been using throughout the semester and start the unit with an inquiry-based activity introducing a problem that made use of many of the skills students would be developing during the unit. We started the unit on November 28 with a three-act lesson, Buying Snacks, about a father and son who go to the grocery store to buy snacks. During the first act, students watched them go shopping, and I asked them what they noticed and what questions came to mind. In the second act, students learned how much the snacks cost and how much money the father and son had to spend. During the final act, students viewed possible combinations of snack purchases and total costs. After we finished the three videos, I challenged the students to model the situation with an equation.

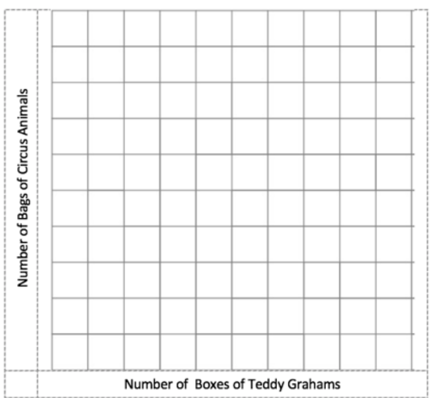
To help my students stay organized, I gave them a note sheet to record combinations that worked and did not work. A blank copy is shown in Figure 5.1. Students displayed their solutions on a graph, and we worked through finding all of the possible solutions to the problem.

Algebra 1-2
Teddy Grahams and Circus Animals.

From the video, we know the following combinations and their totals.

Combinations that work	Combinations that don't work

On the graph below, mark the combinations that work with a dot. Find all other combinations that are possible to purchase with $20. Can you find them by just using the graph?



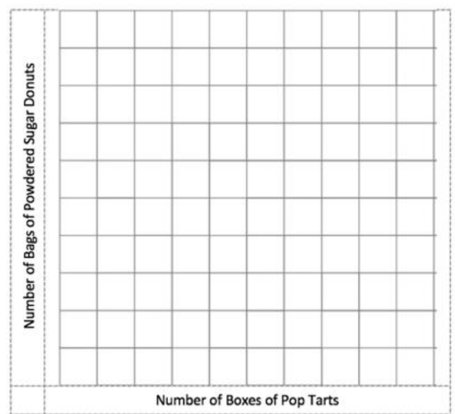
<u>Other combinations that work</u>

<u>Extension</u>

I'm going shopping for breakfast food. Pop Tarts cost $1.88 per box and Powdered Sugar Donuts $2.39 per bag. I only have $15 to spend on them. Find at least three combinations that work and at least three combinations that don't work.

Combinations that work	Combinations that don't work

On the graph below, mark the combinations that work with a dot. Find all other combinations that are possible to purchase with $15.



<u>Other combinations that work</u>

Figure 5.1 – Student note sheet for Buying Snacks activity

During this discussion, Juan, who appeared to have a knack for posing just the right question to move the discussion along, asked a question that created the conceptual framework for the entire unit. He asked, “Does he need to spend all of his money?”

1	Juan:	Mr. Sand, does he need to spend all of his money?
2	Greg Sand:	That’s a good question. What does the rest of the class think?
3	Seamus:	Well, yeah, don’t you want to buy as much food as possible?
4	GS:	Does anyone agree? Disagree?
5	Madison:	Well you can buy a lot of food, but that doesn’t mean you will spend all your money. That’s hard to do with those prices.
6	Philip:	Yeah, I don’t know if you can ever spend exactly $20. But he doesn’t have to spend all of it
7	GS:	So, what’s the answer to Juan’s question?
8	Philip:	No, he does not.
9	GS:	Why?
10	Philip:	Well, he could buy just a few, heck even just one, or some of each.
11	GS:	Okay, anyone unsure about that answer? Well then, let’s move forward.

This dialogue illustrates how the students used their personal experiences to make sense of the Buying Snacks activity. Seamus (line 3) connected the idea of spending all of the money as a way to maximize the number of snacks purchased. Madison (line 5) connected the idea of buying as many as possible with not being able to spend exactly twenty dollars. Philip (line 10) continued the discussion by sharing other options that could be purchased while not spending all of the money.

This discussion was an opportunity to informally introduce a majority of the new concepts and notation that we would use throughout the unit. The ideas shared and discussed by the students created an example of why students are asked to shade solutions on the xy-axis. The opening activity also prompted the students' conception of multiple solutions.

I then moved forward with another activity the next day, November 29, to reinforce the need for displaying the solutions to inequalities. During the Buying Snacks activity, I approached teaching the idea of inequalities through student questions. For the second day's activity, Riding the Kingda Ka, I wanted to confront the idea of inequalities directly, but still maintain an authentic approach to the concept. To accomplish this goal, I searched for a YouTube video of the highest roller coaster on earth at the time, the Kingda Ka ("Kingda Ka," 2016).

At some point during the discussion about the ride, Savannah asked, "How tall do you have to be to ride it?"

Anticipating this question, I shared that a rider needs to be at least 56 inches tall. I then asked the students several questions: "If you need to be at least 56 inches tall to ride this roller coaster, then what are the different heights that can ride? Is there a maximum height that can ride this coaster? How can we write out all of these answers?" Using these questions as prompts, I presented the idea of displaying solutions on a number line as well as the use of open and closed points to indicate the inclusion of the endpoint on the white board at the front of the room. My goal was to informally introduce how solutions are represented so that when the students were required to do it later it would not be a new concept.

An Early Issue

It was at this moment during the lesson that I made the choice to transition to purely algebraic equations instead of the conceptually-based ones that could have been used as transitions to formal equations. We only had three weeks before winter break began, so time was precious. Additionally, in the middle of December students were required to take the mathematics portion of the MAP test during class as a mid-year data point. Upon reflection, the choice to not spend more time solving authentic problems that could be modeled as inequalities like the Buying Snacks problem instead of solving symbolic ones was a choice I should have made differently. With more time, I could have constructed an example like, “In thirteen years I will be more than 50 years old. How old could I be right now? What is the youngest I could be?” rather than the exercise: Solve for x: $x + 13 \geq 50$.

I should have spent more time on opening activities similar to the Snack Problem and Riding the Kingda Ka. Instead of having the opportunity to help students develop deeper conceptual foundations for one- and two-variable inequalities, I moved forward to the more conventional topics of solving and graphing their solutions quicker than I wanted.

Major Issues Emerge

Rather than call this an opportunity lost, I instead used this decision to reinforce algebraic manipulation in the solving of one-step equations. What I used to see as frustrating "re-teaching," I came to realize was actually the time when skills become refined and automated. Refining these types of mathematical habits offered an opportunity to reinforce the conceptual basis of what we do in algebra. Solving inequalities makes use of a similar set of properties to solving equalities. For example, when solving equalities, we can use the additive property of equality, or

$$\text{If } a = b\text{, then } a + c = b + c.$$

When solving inequalities, we can use the additive property of inequality, or

$$\text{If } a < b \text{ and } c < d\text{, then } a + c < b + d.$$

Most teachers and textbooks use a more conventional version of this property:

$$\text{If } a < b\text{, then } a + c < b + c.$$

This transition caused my students to struggle with the other main idea I introduced that day, solving one-step inequalities with multiplication and division. At the heart of this concept was an idea that my students would struggle with until the final day of the unit, switching the direction of the inequality symbol when multiplying by a negative value.

My approach to this particular skill was conventional. I began with a true statement, for instance:

$$1 < 5.$$

I asked students working in small groups to notice what happened to the inequality relationship between the numbers when they added, subtracted, multiplied and divided by different values (November 29 lesson plan). After each operation, I asked them if the relationship was still true. One small group discussed how the relationship changed when they multiplied by a negative.

1 Clifton: Hey Mr. Sand, I think we did something wrong.
2 Greg Sand: What do you mean?
3 Jordan: Well, we multiplied by -3 and its wrong.
4 GS: Explain to me what's wrong.
5 Isabel: Well, it was $1 < 5$ but now its $-3 < -15$ and that's not right.
6 GS: Can I share this with the class and see what they think?
7 Isabel: Sure, but I don't know why, we messed something up.
8 GS: I don't think so; this is really a neat idea here. Let me help you understand what's going on.

This interaction with a small group was typical of the work I did with students during instruction. I would give teams a task and then move around the room monitoring the work and answering questions. To encourage discussion within groups, students asked me questions about things no one in the group could explain to the others. Clifton, a leader and risk-taker, gladly took on the responsibility of getting my attention for the group. His statement to me summarized what his group was feeling; they did not do anything incorrectly, but their result did not make sense to them. Isabel, a hard-working, focused student who lacked self-confidence, articulated what the group did, a role she often assumed during the year. After talking with the small group, I brought the class together to share what they discovered.

1 Greg Sand: One of the groups ran into a problem. Can someone from your group explain what happened?
2 Jordan: We started with $1 < 5$ and multiplied by -3 and now it's just wrong. -3 is not smaller than -15.
3 GS: Can anyone make sense of this for me?
4 Deng: No, it's true. -3 is less than -15.
5 Philip: You're wrong, negatives work backwards.
6 GS: Let's use the number line to figure this out. Where are the smaller numbers located?
7 Deng: On the left side.
8 GS: So, which is further left, -3 or -15?

9	Deng:	Oh, -15 so its smaller. That's hard to think about.
10	GS:	I agree. However, [to the whole class] what could we do to this relationship to keep it true?
11	Haley:	Well, -15 is smaller, so couldn't we flip the symbol?

When Jordan (line 2) stated, "It's just wrong," I knew that he understood the ordinal relationship between -3 and -15 as well as how to interpret the less than symbol, $<$. Deng (line 4) had a misunderstanding about the order of negative numbers which Philip (line 5) attempted to correct. To help Deng determine which value was larger, I (line 6) referred to the number line and asked, "Where are the smaller numbers located?" Even after Deng (line 9) determined that -15 was smaller, his statement, "That's hard to think about" told me that he was developing his own understanding of the order of negative numbers. Haley offered a solution (line 11) when she stated, "Couldn't we flip the symbol?" which resolved the issue. I brought the idea to the entire class for a large group conversation.

In addition to sharing what one group had discovered, the whole class conversation allowed the other groups to gain insight into a concept they may not have otherwise noticed. Jordan, who struggled throughout the year with shutting down when the material got too complex, did an excellent job of explaining the issue to his group as shown in line 2.

Deng worked through the misunderstanding with me. His habit of always looking for the fastest way to complete any problem created many misunderstandings. His willingness to engage in the problem, however, gave me the opportunity to help him slow down and explore a topic deeply, reinforcing his understanding of the ordering of negative integers. I understood from his comment (line 9) that he was trying to make sense out of the ordering of negative numbers, and it challenged him. The suggestion that Haley made (line 11) was typical of her work in class; she frequently made observations that were not clear to her peers.

During this large group interaction, my students encountered the idea of switching the direction of the inequality symbol, attempted to make sense of that change in the relationship, and then generalized the result into a rule. But where in mathematics does this idea come from and how can we say it is always true?

Mathematics Behind Switching the Inequality

This idea comes out of the foundations of algebra in advanced mathematics: groups and fields. These studies concern the algebraic structures of one or more operations interacting with a given set. A group is a set with an operation that satisfies four conditions: closure, associativity, identity, and inverse. A field is a set with two operations that are commutative and associative. Each operation has identity and inverse elements, and at least one operation must distribute across the other. The real number algebra my students studied is proven true from this branch of mathematics.

Early student experiences during the primary grades explore the field of addition and multiplication over the set of natural numbers. Problems of the form

$$a + b = c \text{ and } ab = c, \quad a, b, c \in \mathbb{R}$$

are introduced as different versions of "fill in the blank" or "find the value of the symbol." These experiences are extended when students encounter the field of rational numbers with multiplication and addition introduced in my state during fourth grade. First-year Algebra students encounter the field of real numbers. In the algebra book my eighth graders used, the algebraic properties of real numbers as shown on pages 285 – 287 (Carter, 2014) are a restatement of the field axioms.

A field is called an ordered field with $\leq$ if and only if for any arbitrary element $\alpha, \beta \in S$ where S is an arbitrary set such that:

$$a \leq b \text{ implies } a + c \leq b + c \qquad \textbf{(1)}$$

$$0 \leq a \text{ and } 0 \leq b \text{ implies } 0 \leq ab \qquad \textbf{(2)}$$

These two statements are the axioms of ordered sets, and properties of inequalities can be proven from them. Two of particular interest are the inequality inverse properties of addition and multiplication. The additive inverse inequality property states that,

> If $a \leq b$, then a's additive inverse, denoted $-a$ and b's additive inverse, denoted $-b$ have the order $(-a) \geq (-b)$.

The proof that follows serves two purposes. First, it is the mathematical justification that proves this property true. Second, it is an example of deepening my understanding of the mathematics that I am teaching. Because this proof is developmentally inappropriate for students, I chose a numeric activity to help students discover this property.

<u>Proof</u>

Let $a, b \in \mathbb{R}$ where $\mathbb{R}$ is the ordered field of real numbers with addition, multiplication and $\leq$. Because $\mathbb{R}$ is an ordered field, by **(1)** If

$$a \leq b,$$

then

$$a + (-a) \leq b + (-a)$$

where $-a$ is the additive inverse of a.

Because

$$a + (-a) = 0\,,$$

$$0 \leq b + (-a).$$

Similarly, by adding $-b$ to both sides

$$0 + (-b) \leq b + (-a) + (-b)$$

and

$$(-b) \leq (-a)$$

or by symmetry

$$(-a) \geq (-b).$$

■

This proof is informative to a teacher because it is the property understood as, "Multiply both sides by a negative and switch the sign." However, it is a property that is verified by addition and not multiplication. Perhaps this is a root cause of student misconception. This type of mathematics is beyond the scope of a conventional first-year Algebra course. The results, switching the direction of the inequality when multiplying or dividing by a negative number, are a part of the mechanics needed to solve inequalities.

Areas of Concern in Teaching and Learning Inequalities

My reflection on this unit of study uncovered three major areas of concern that emerged in students' struggles and that *appeared to be* mechanical in nature: changing the direction of the inequality appropriately, graphing solutions, and writing analytic forms of solutions. The next three sections illustrate how I came to these conclusions by examining not only the recordings of daily classroom teaching, but also my own fieldnotes and the students' work.

Changing the Direction of the Inequality

The notes that I made from November 29 through December 5 show the difficulty some of my students had with this concept. On November 29, the first day that the students encountered this idea, I noted that, "The students dictated to me the notation and symbols that we used to write models representing situations. When we tried to make sense of the changes in the notation numerically, I encountered a number of confused looks from the students." At the time I was convinced this was a minor misunderstanding, but it became a mistake my students repeated throughout the unit.

Later, on November 30, I wrote, "Philip tried to explain to the class about why you flip the symbol when you multiply by a negative. It would make a great textbook answer, but it seemed to make sense only to the kids who already understood it. I'm worried that about two–thirds of the kids don't understand this." I was struck by how detailed and through a statement Philip was able to make. But one student's understanding was not enough to help everyone; instead, I should have had other students explain it in a different way as well.

During class on December 5, I engaged the students in an error analysis activity in order to discuss valid and invalid steps in solving linear inequalities. One of the problems contained only one error, forgetting to switch the sign. My notes after that lesson were, "Three of the five groups couldn't find the error in problem #3. They told me, 'There isn't an error.' I'm not sure how to correct this without just telling them to change its direction." This dilemma is as much of a challenge to me now as it was then. I think that a better lesson on this concept at the beginning of the unit and more purposeful review would have helped reinforce the idea of switching the direction of the inequality symbol.

Student responses to quiz question #2. The students' struggles with correctly changing the direction of the inequality symbol when solving inequalities were further demonstrated in their work. The following are examples of two problems that the students attempted to solve with varying degrees of success. The first problem was from a quiz taken on November 30:

Solve the following inequality

$$-4r < 22.$$

To solve this problem, students needed to

1. Divide both sides by -4
2. Change the direction of the inequality symbol

The solution to this problem is $r > -5.5$.

Figure 5.2 shows Madison's work solving the problem while illustrating the ideal set of algebraic maneuvers. Madison first indicates that she is dividing both sides by -4 and then completes the problem by writing $-\frac{22}{4}$ as -5.5 and switching the direction of the inequality symbol.

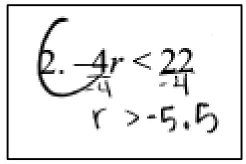


Figure 5.2 – Madison's response

Figures 5.3 and 5.4 show examples of student work that contain correct numerical values on the right side of the inequality, but do not have the inequality symbol facing the correct direction. When interviewing students on December 2, 2016, I asked each of them to explain how they solved the problem. Clifton, who completed the problem in Figure 5.3 said, "I just divided 22 by -4. Oops, I guess I forgot to flip the sign."

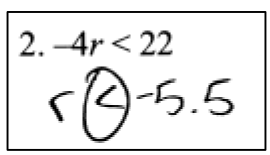


Figure 5.3 - Clifton's response

I followed up by asking why he needed to flip the sign.

He responded, "Because that's what you're supposed to do when you divide by a negative" (Class video, December 2). Clifton's response told me that he knew what to do but failed in this one instance.

Later when I asked Deng, who completed the problem in Figure 5.4, the same question, he said, "I dunno. I just thought about what times -4 is 22. I tried a few numbers and figured out it would be -5.5." I then asked if that was the only solution. He said, "Yeah, there are more, but I'm not sure what they are" (Class video, December 2). Deng's comment worries me now more than before. At the time, I thought that he had taken a novel approach to solve the problem.

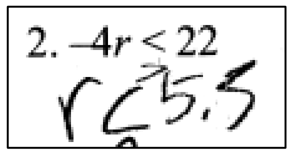


Figure 5.4 – Deng's response

Reflecting on this now, he seemed to be viewing the problem as an equality. When he said, "I tried a few numbers and figured out it would be -5.5," this told me he was not solving the problem algebraically but numerically. If he solved it algebraically, he would have divided 22 by -4. Instead, he substituted values for r until he found an answer to the equality, not the inequality.

Student responses to exam question #10. On the unit exam, students were asked to solve the following absolute value inequality:

Solve and graph the solution set

$$|3 - 2x| \geq 1.$$

Solving this problem requires that the students

1. understand it is to be rewritten into two equivalent problems, each with a different inequality symbol.

2. solve each equation. While there are multiple correct ways to solve it, most students will first subtract 3 from both sides and then divide by -2.
3. remember to invert the direction of the inequality symbol when they divide by -2.
4. graph the solution set after completing the problem.

Figure 5.5 shows Evan's work solving the problem where he uses the ideal set of algebraic maneuvers to solve it. The problem requires several algebraic manipulations, including splitting it into two equations and inverting the inequality sign to find the correct solution.

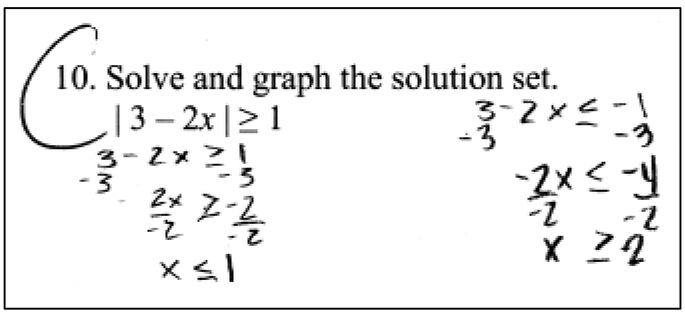


Figure 5.5 – Evan's response

Figure 5.6 shows an inconsistent application of inverting the inequality symbol. Joshua correctly writes the two equations needed to solve the original inequality, and each equation is solved correctly except for the final step. The equation on the left shows an incorrect final step by failing to change the inequality symbol. The equation on the right is solved correctly.

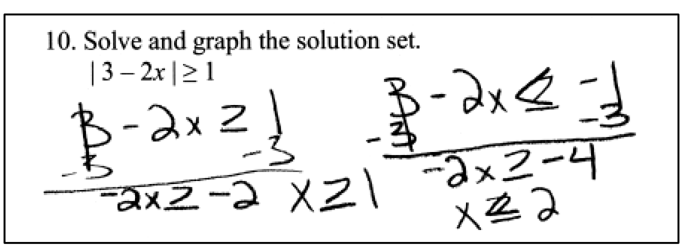


Figure 5.6 – Joshua's response

Figure 5.7 shows Kiera successfully solving the problem except for the final step. She failed to invert the inequality symbol in both equations.

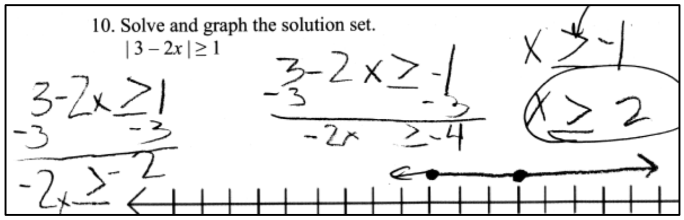


Figure 5.7 – Kiera's response

On the unit exam, there were three problems that required students to invert the inequality symbol. Eighteen students completed the exam. Seven of the students correctly solved all three problems. Six of the remaining eleven attempted all three problems, but none solved them correctly. The remaining five students incorrectly solved the first of the three problems and did not attempt the others.

Reflections on Changing the Direction of the Inequality

While teaching the first part of this unit, I wanted to stay true to norms I had established earlier in the year, notably, presenting a situation that my students could relate to as a starting point for new content. I began the unit with two situations that lent themselves to modeling results with inequalities. First, the class encountered a father and son buying snacks for a party. Second, students answered questions about who could ride a roller coaster. During the second lesson, I wanted the students to make sense of when to change the direction of the inequality symbol. Clearly my attempt to teach this was not as successful as I had imagined it would be. But why does this matter?

Teaching mathematics will always require teachers to justify new material that lies beyond the students' existing knowledge. Prior to exploring the mathematics behind switching the direction of the inequality symbol, I had never encountered the proof that I presented earlier. I found myself struggling with the dilemma of helping my students make sense of the mathematical relationships without formal justification.

In this particular case, I had students examine the interactions of given quantities that have an unequal relationship when performing valid algebraic operations. My hope was realized by the group of seven students who demonstrated this knowledge on our unit exam. I am left to question if they retained the understanding of why they made this move or if they had just transformed it into a rote skill without any thought about why or how they did it.

But this is only a portion of my class. The other eleven students either did not attempt the problem or executed it incorrectly. Why? This is a much more difficult question to unravel. For some students, it was clearly the end of a very long semester. This exhaustion affected their daily performance and understanding of a topic that was based on several other skills from earlier in the year. Consider the steps, shown on page 43, needed to solve the problem shown in Figure 5.5. Each step is necessary to successfully solve the problem, and if a student is insecure about any of these moves, the entire problem will be incorrect. This level of sophistication is expected in the early part of the curriculum and is present throughout the majority of a first-year Algebra course.

Options to address this concern. How could this have been overcome? I am left to consider three alternatives. One alternative is to find a way to create more time within the class to work on this fundamental idea. The modern classroom is beset by any number of outside demands for time: time to take state and district mandated assessments, time to prepare for those assessments, time to develop deep conceptual understanding of the concepts being studied, and time to productively struggle with those concepts so that a deep understanding can be achieved.

The second alternative is to return to the traditional model of teaching: I tell you what to do, you watch me do it, and then practice it on your own. In this model, time is less of an issue because student sense-making and understanding is secondary to the creation of a product that may or may not be understood or cared about by the student. However, if they can memorize a set of steps that allows them to correctly solve a problem, then standardized tests scores look good, and the public can notice what a good job the teachers are doing because of a score report in the newspaper.

Perhaps the best choice comes to mind after much reflection. As I was reviewing student work, I realized that I was lacking artifacts beyond formal assessments and classroom recordings from many of the students who failed to complete the problems correctly on the chapter test. What does this mean? In simple terms, students did not consistently complete daily practice to reinforce concepts and skills developed in class. It is easy to blame the students' lack of development on their failure to complete homework. Could I have hedged against this?

I found myself reexamining my lessons from this point forward in the unit and was forced to see my role in their underdevelopment due to my choice of warm-ups and examples. When teaching new concepts, it is easy to fall into the trap of choosing problems that stay away from numerical values that students struggle with. When teaching students how to solve equations, some teachers avoid examples that include negatives and rationales so that students can master the procedures. This avoidance keeps students from developing computational fluency with different types of numbers.

After years of these otherwise well-intended actions, the inevitable result is students with underdeveloped skills and limited understandings.

As teachers, we sometimes create self-fulfilling prophecies. I cannot count the number of times I have thought to myself, "Well, they can't work with fractions, so I'll just have them solve equations with integers because they need to get the process down first. I don't want to overwhelm them." Good intentions lead to unintended consequences. With every choice like this, I am contributing to the diminishing likelihood of success for my students in later math courses.

Examining my choices in planning lessons. During daily instruction within this unit, I presented to students thirty different problems to solve either individually or in small groups. Of those thirty, only six of them contained a negative coefficient. With that small of a ratio of problems, why should I be surprised with the results? I did not give my students adequate opportunity to practice this skill during my time with them. Once I came to terms with their lack of understanding of this skill, I should have increased the number of these types of problems instead of avoiding them.

I realized at the time that my students were struggling with this idea. In my journal from December 17, 2016 as I was planning the chapter review I wrote, "I have to plan problems that will allow me to do some reteaching. Especially problems that have the variable on both sides as well as multiplying or dividing by negatives. Some but not all of the class understands this." My reaction to this issue was avoidance.

On December 19, 2016, I wrote the following example on the board for the students to solve:

Solve and graph the solutions to

$$2(x-4) \geq 5x - 13.$$

1	Greg Sand:	At this point in the problem, we have variables on both sides [$2x - 8 \geq 5x - 14$ is written on the board]. What's a good next choice in solving this equation?
2	Aisha:	Subtract $5x$ from both sides.
3	Jordan:	Ugh, that means we get a negative. I hate that!
4	GS:	I know some of you don't always remember to flip the inequality when we divide by a negative, so we can make a choice here to avoid getting a negative.

Rather than confront this issue directly, I offered students a way to avoid it. Jordan's comment on line 3 was typical of the class. The students generally disliked negatives and fractions. Many of the strategies I shared with them throughout the year were chosen to help them solve problems while avoiding negative or rational numbers. By avoiding negative coefficients, the students missed the opportunity to reinforce a skill that many had not mastered. This is one example of a choice I made that left my students underdeveloped.

Graphing Solutions to One Variable Inequalities

In addition to the issue of inverting the inequality symbol, the students also struggled with graphing solutions. On the second day of the unit, the students and I watched a video showing the world's tallest roller coaster from the front seat of the first car. After the anticipated comments and questions (Fieldnotes, November 29, 2017) came the one that I chose to focus on, "How tall do you need to be to ride?" I shared with them the following information as shown in Figure 5.8:

Kingda Ka

Website | Directions

4.7 ★★★★★ 268 Google reviews

Roller coaster in Jackson Township, New Jersey

Kingda Ka is a steel accelerator roller coaster located at Six Flags Great Adventure in Jackson, New Jersey, United States. Wikipedia

Address: 1 Six Flags Blvd, Jackson, NJ 08527

Height: 456′

Max speed: 128 mph

Height restriction: 4′ 7″

G-force: 5 g

Opened: May 21, 2005

Track length: 3,118′

Figure 5.8 – Rollercoaster information

After translating the height restriction into a symbolic form

$$H \geq 56"$$

I showed the students the conventional way to display the solutions. The problems that followed over the next two days required students to display their solutions in a similar manner. My notes from those days included these statements:

> Some students can't read the inequalities correctly and are struggling to graph them. One group of students told me that they noticed that they imagined the inequality sign like an arrow and drew their graph in that direction. It seems like the class is split in half. One group understands how this works and the other is frustrated that they can't make sense out of connecting the symbols to the graph. (Fieldnotes, November 29)

For an introductory lesson, this seemed like the usual level of understanding of a topic. It would have been helpful to find out how many of the students who were struggling with graphing misunderstood the inequality symbol. I am sure that would have been a place to help students make sense out of how to graph the solution.

On December 2, I presented the class a different type of situation, one that required them to display solutions that had both an initial and a terminal value as shown in Figure 5.9. In this case, I was challenging the students to consider a compound inequality without using any of the terminology. Once I felt that some students in each of the groups understood how to answer this question, I showed them a set of graphs and had each group write an inequality that matched it.

*Figure **Error! No text of specified style in document.**5.9 – Lesson prompt*

It was when we moved to solving and graphing compound inequalities that a lack of understanding about the connection between the solution to an inequality and the graph of its solution became apparent to me. My notes from that day included a complaint that "Some students don't have a reflexive set of moves to solve equations. They still are asking, what do I do here?" It became clear to me that my work that week on graphing solutions was disappearing under the additional demand required to solve linear inequalities. Many of my students were unable to graph the solutions that earlier in the week (or day) they had been able to verbally and graphically demonstrate to me.

My fieldnotes from class reflect the difficultly that students had with the subtly complex language used in the examples in class. Translating words into symbols was part of their struggle in accurately graphing solutions to inequalities.

> "While the students had strong background knowledge about inequalities, they struggled to translate their words into symbols. Students couldn't tell that the phrases 'x is greater than n' and 'n is smaller than x' were the same regions when we graphed the solutions. One group of students noticed that the inequality symbol looked like an arrow. Deng told me that he figured out that the direction of the solutions was the same direction that the arrow pointed. Will this cause problems later? Is it okay for students to notice tricks?" (Fieldnotes, November 29)

Moments akin to this are part of every teacher's practice. Some students are struggling with the fundamental concepts being taught while others are noticing patterns to become more efficient. It occurs to me now that I should have noted which students had excelled with these types of tasks earlier in the year. I could have made sure to include at least one of these students in each group to help their peers.

On November 30, I wrote, "The kids seem to be okay with solving simple inequality equations, but they are struggling with graphing them." On December 5, I wrote, "During our quiz review many of the kids are still making mistakes with both open and closed dots as well as marking the direction of the solution." On December 12, I wrote, "The same graphing issues are plaguing class and slowing things down." Looking back at my fieldnotes, it is clear that I knew there was a problem. This is why it is important for a teacher to both keep notes and reflect on them over the course of a unit. Themes emerge in daily reflections and those themes should guide adjustments to instruction.

Classroom discourse. Two different moments from the recorded highlights of class illustrate this issue as well. The first interaction was during class on November 30, while the students were solving and graphing multistep inequalities.

1	Greg Sand:	So, you've gotten to the solution?
2	Jordan:	Yeah, it's x is greater than 5.
3	Philip:	No, it's greater than or equal to.
4	Jordan:	Oh, yeah sorry.
5	GS:	What's the problem?
6	Jordan:	I just don't get how to graph them.
7	GS:	Well, let's walk through the big ideas first. You said that the solution was x is greater than 5. Is 5 a solution?
8	Jordan:	Ummm.
9	Philip:	Yeah it is, duh.
10	GS:	Remember that we need to use a closed dot to show it's included, so go ahead and mark that. Now, where on your number line are the values greater than 5?
11	Jordan:	To the right.
12	GS:	So, we shade in that direction to show that is where the solutions are.

This conversation took place early in our work on graphing solutions to inequalities. I dialogued with Jordan while Philip, who was sitting near Jordan's group, shared his thoughts on the problem. Although Philip wanted to answer the questions I asked Jordan, it made it more difficult for me to identify the source of Jordan's misunderstanding. Because it was early in the unit, I walked Jordan through the process by asking him questions to help develop an internal dialogue that could guide him through other problems.

The second interaction was from our quiz review on December 5. Students were solving and graphing inequalities on individual white boards. After they had solved the problem and graphed the solution, they held up their individual white boards so I could check and give feedback on their work.

1	Greg Sand:	So, I'm looking around the room and noticing that there are many different solutions to this problem. Can someone explain why they are correct?
2	Jessi:	Well, I added 11 to both sides and then divided by 3. That gave me x was smaller than 9.
3	GS:	Does anyone disagree? Okay, so now I want each of you to look at your solutions and see if your graph shows x is less than 9. When you're ready, show me your solutions. I'm seeing a number of different solutions. Some of you have closed dots, some open. Some are shading to the left and others to the right.

Student responses to exam question #9. These inconsistences persisted until the end of the unit. A review of student work on the chapter test showed that no student was able to graph both of the one-variable inequalities correctly. The students made a variety of mistakes when graphing their solutions. The examples shown are from students who correctly solved the problem algebraically but made an error in graphing the solution. This is the original problem:

Solve the compound inequality and graph the solution set:

$$9.\quad -4 \leq n \text{ and } 3n + 1 < -2.$$

This problem requires students to utilize a number of skills from the unit. First, they must solve each part of the equation separately. The first half of the equation

$$-4 \leq n$$

does not require any algebraic manipulation. Students either need to be comfortable with reading it as "-4 is less than n" or they need to rewrite the inequality as,

$$n \geq -4$$

and interpret the solution in a form that is more comfortable for most, "n is greater than or equal to -4."

The second half of the equation

$$3n + 1 < -2$$

can be solved most efficiently by subtracting 1 from both sides and then dividing by 3. The resulting solution is

$$n < -1.$$

Because this statement includes the word "and," both statements must be true simultaneously. Thus, n can be any value greater than or equal to -4 and less than -1. Figure 5.10 shows the correct graphical solution to the problem.

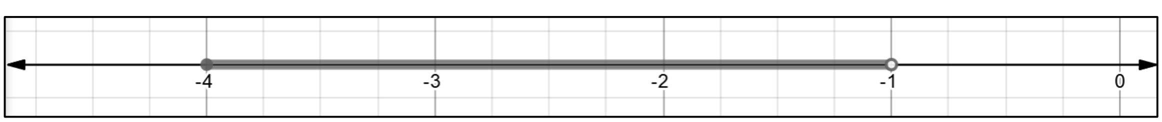


Figure 5.10 – Solution set to exam problem 9

The following three samples of student work were chosen because each of the students found the correct analytic solution to the problem but failed to correctly graph the solutions. Figure 5.11 shows Kiera's work correctly labeling the points as open (at -1) and closed (at -4) but failing to shade the correct region. In this case, it is most likely that she did not correctly interpret the meaning of the inequality symbols as they relate to the values of the solution.

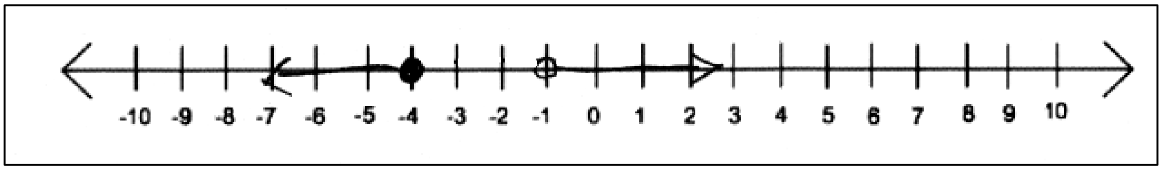


Figure 5.11 – Kiera's graph

Chloe's response is shown in Figure 5.12. She shaded the correct region, but incorrectly labeled the point at 1 instead of at -1 and as open instead of closed. While the algebraic solving of the equation showed a solution of $n < -1$, she both mismarked the point and did not use the correct notation to indicate the solution was not included.

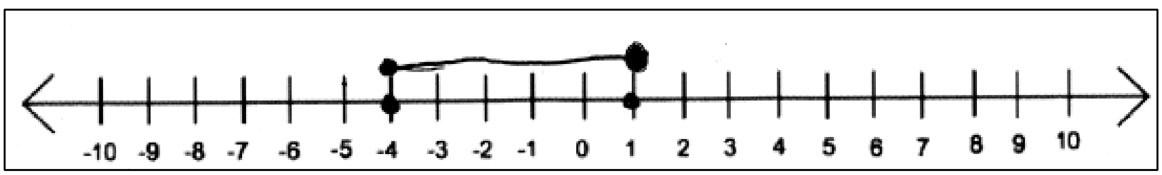


Figure 5.12 – Chloe's graph

Jordan's response is shown Figure 5.13. He correctly marked the point at -1 with an open dot; however, he incorrectly shaded the solution region. This error could either be a failure to acknowledge the second part of the solution ($-4 \leq n$) or a misinterpretation of the inequality because it was originally written in a form that started with the value instead of the variable.

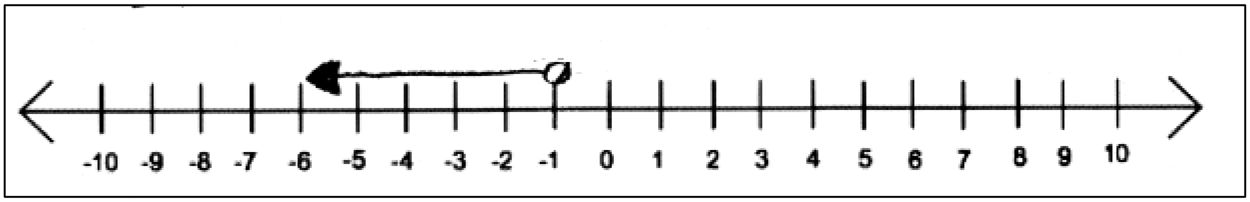


Figure 5.13 – Jordan's graph

Reflecting on this issue, I am shocked at how much my students struggled with this idea. When I approached this unit, I looked at it as an opportunity to refine my students' skills developed earlier in the year when solving equations. Instead, I discovered the students did not have the level of mastery I assumed they had from studying similar topics earlier in the year.

The first five topics studied in Unit 5 required almost identical procedures to those studied in Unit 2: Linear Equations, Unit 3: Linear Functions, and Unit 4: Equations of Linear Functions. Solving inequalities added two new dimensions to the procedures needed to successfully solve the problems from the previous units. First, students must be aware of when the inequality relationship is changed by multiplying by a negative. Second, students must display their solutions on a number line.

My awareness of these added dimensions was reflected in my notes at the beginning of the unit. On November 29, I noted there was "significant growth by the students from Chapter 2 in terms of solving equations. I hope it will hold up tomorrow." On November 30, I wrote, "Solving equations issues are still present. It is not an automatic process for the students. I hope that the new ideas that are a part of solving inequalities will force them into becoming more automatic with the processes and procedures needed to solve them." Because graphing inequalities involves solving inequalities, students who struggle solving inequalities using similar procedures to solving equations will struggle to produce correct graphs. I embraced this opportunity to reteach old skills, but it was another potential source of student misunderstanding.

Learning math is a nonlinear process. My notes here point to an idea that I have become more aware of in my teaching throughout this experience: learning math is a nonlinear process. The beauty and structure of mathematics allows it to be presented and justified in a formal, logical method. In algebra, variables lead to expressions. Equivalent expressions lead to solving equations. Solving equations leads to solving inequalities. The list could go on forever. However, student learning of algebra is nonlinear. While my students learned to solve equations, that knowledge was not secured during those previous chapters. Instead, as the year progressed, my students revisited old skills in the context of new ones. This helped them transition the process from slow and intentional to automatic.

I am also left to question my students' preparedness for this topic. I noticed early on that they had grown and improved in their ability to solve equations, but it did not seem like this growth was sufficient. When the mathematics required them to do more with a skill that they had not quite mastered, two critical and additional ideas were either not understood or were executed incorrectly.

This issue brings a larger concern to mind. It is extremely difficult for a teacher to diagnose the problems that students have in solving inequalities when there are so many different potential sources. I will use problem number 8 from the unit exam to illustrate this issue.

Solve and graph the following inequality:

$$8.\ 5(p+2)-2(p-1) \geq 7p+4$$

To solve this problem, students must use the distributive property, correctly multiply integer values, combine like terms with different signs, collect the variables to one side and the constants to the other, and then divide by the coefficient of the variable which may or may not require the inequality symbol to change direction. If any one of these steps is incorrectly executed, the student will fail to solve the problem. I should also mention that the type setting of this problem caused an additional problem for some students. Rather than distribute 5 to the quantity $p+2$, some of the students read it as 8.5 and distributed that.

Reflections on Graphing Solutions to One Variable Inequalities.

Teaching is a complex process and helping students grow and improve is critical to their success in both the short and long term. Each student presents the teacher with a different set of strengths and weaknesses. Finding out what works for each of them to be successful requires a different mindset in daily practice. In my classroom, I utilized small group work combined with large group discussion. Discussion allowed me to understand what my students were thinking and then respond individually to the needs of each student.

This type of teaching is challenging. It is easy to understand why some teachers default to a teacher-centered classroom. If the teacher is the one telling the students what to do and how to do it, then the teacher has control. The teacher is imposing the way that they make sense out of a particular topic onto the students. The teacher controls what questions are asked and answered. The process of learning mathematics becomes a set of steps that are to be memorized, practiced, and regurgitated. When a student makes an error, correcting that error is a simple process of referring back to the procedure being practiced.

Reflecting on this issue of my students' failure to correctly graph solutions to one-variable inequalities, I cannot identify one particular source of the problem. When a teacher seeks to create a learner-centered classroom, student understanding becomes the most important idea. Because of this, the teacher must take the time to listen to how an individual student understands an idea. From there, correcting mistakes made by a student is a process of uncovering how they understand the problem they are solving and correcting any misconceptions.

Representing Problems Algebraically with an Inequality

Throughout the unit I utilized a number of situations that lent themselves to being modeled by an inequality. The first activity was a father and son going to the store to buy snacks. My students were required to list the different combinations of snacks that they could buy with twenty dollars. One of the goals of this activity was to help students link numerical and graphical representations. Haley's work on the first half of the problem is shown in Figure 5.14.

This is a typical example of student work on this activity. The students were able to identify both examples and non-examples that solved this problem and use the graph to generate other examples of combinations that worked. They then listed the combinations as ordered pairs. Hayley's example in Figure 5.14 illustrates this, including how she labeled the ordered pair to define what each value represented.

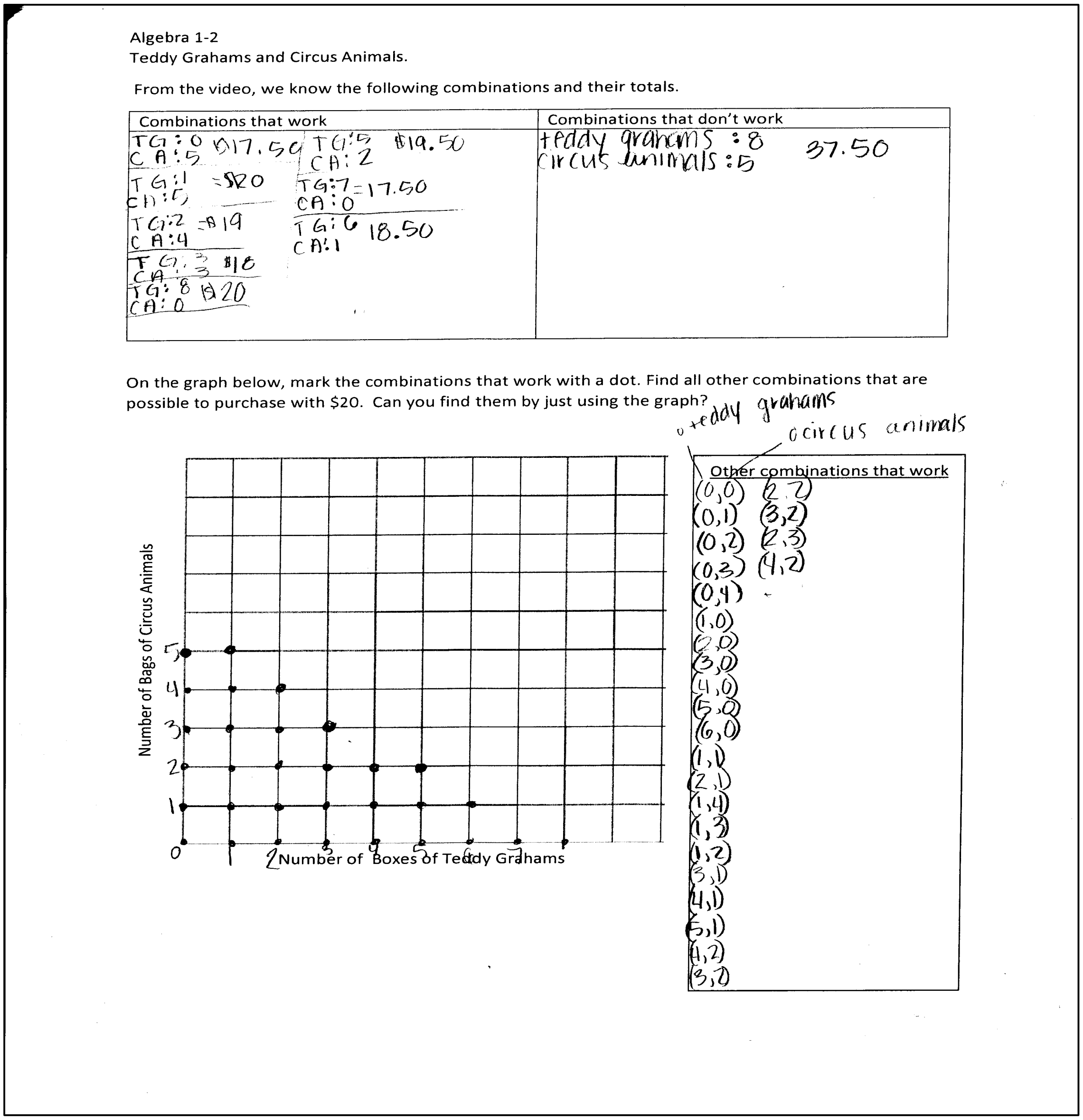


Figure 5.14 – Haley's work

The challenge occurred throughout the first semester. When I attempted to transition the students from graphical and numerical representations into an analytic one, the class quickly became frustrated. The following dialogue took place at the end of class on November 28.

1	Greg Sand:	What we have listed and graphed here are the solutions to this situation. My question to you is, how can we write this as an equation?
2	Jordan:	What do you mean?
3	GS:	Let's start at the beginning. What are the two items we're buying?
4	Evan:	Teddy Grahams and Animal Crackers.
5	GS:	Okay. Is there only one answer?
6	Evan:	No, there are lots of answers.
7	GS:	Right. If there can be many values that work, then this is where we can use a variable to represent each of these quantities. What variable should we use?
8	Isabel:	x and y
9	GS:	Really? x and y? Man, there are so many other letters we could choose from. Alright. What should each represent?
10	Madison:	No, let's use a for Animal Crackers and t for Teddy Grahams.
11	GS:	Let's be more precise, let a represent the number of bags of animal crackers and t represent the number of boxes of Teddy Grahams. Now, how can we write an expression that shows us how much we will pay in total?
12	Haley:	Um, $\$3.49a + \$2.49t$
13	GS:	Let me write that down. Now, here's the new idea. We can't spend more than $\$20$ so we can represent this with an inequality. We can spend $\$20$ or less, so in this situation, we will use the less than or equal to symbol.
14	Jordan:	I don't get it. How did you write that? Why do you need to do that?
15	Deng:	Yeah Mr. Sand, that doesn't make any sense to me.
16	Juan:	Why couldn't we just have left it as a list or a graph? That makes sense, this is confusing.

This conversation highlights a struggle about half of the students had throughout the school year with writing equations to represent mathematical problems of the form:

$$ax + by = c\ .$$

This knowledge led me to take a careful, precise approach to working through this problem. I began with specific questions to help the students identify the important information (lines 3, 5, and 7). Knowing that my students struggled with specifically defining variables, I made sure to help them specifically define them in line 11. Jordan (line 14), Deng (line 15), and Juan (line 16) were willing to share their frustration and misunderstanding with the idea. Jordan's frustration with the task manifested itself when he questioned the need for the equation. Deng was not able to complete the problem quickly, so he made one of his favorite statements, "That doesn't make any sense to me." Juan was comfortable working with the solutions graphically and numerically, but he struggled transitioning to an analytic form.

At that point in class, the bell was about to ring. I assigned them a similar task where they were to determine how many boxes of pop tarts and bags of doughnuts they could buy for fifteen dollars. Just like the first task, I asked them to display the solutions both numerically and graphically. I was unable to shake the feeling, however, that this issue of my students being comfortable modeling situations with equations had not improved.

After class on November 30, I noted in my journal that, "The only questions over homework today were about problems that were of the form

$$ax + b \leq c \text{ (or some variation).}$$

Students didn't ask about solving equations, just modeling." On December 2, I noted that, "Word problems that mirror equations in slope-intercept form are still a struggle for the students." After our mid-chapter quiz, I wrote, "Only four of the eighteen students were able to complete the two word problems correctly." Problems that I had noticed at the beginning of the unit did not improve by the first chapter quiz. Since performance did not improve, I should have had my students practice these types of problems more throughout the remainder of the chapter.

Student responses to exam problem #24. This problem persisted on the unit exam. Shown below is problem 24 from the unit exam:

> You have at most $200 to spend on shirts and jeans for school. Shirts cost $20 each and jeans cost $25 each.
>
> (a) Write an inequality to represent the number of shirts and jeans you can buy.
>
> (b) Graph the inequality and shade the region that represents reasonable solutions only.
>
> (c) Interpret the mean of the graph.

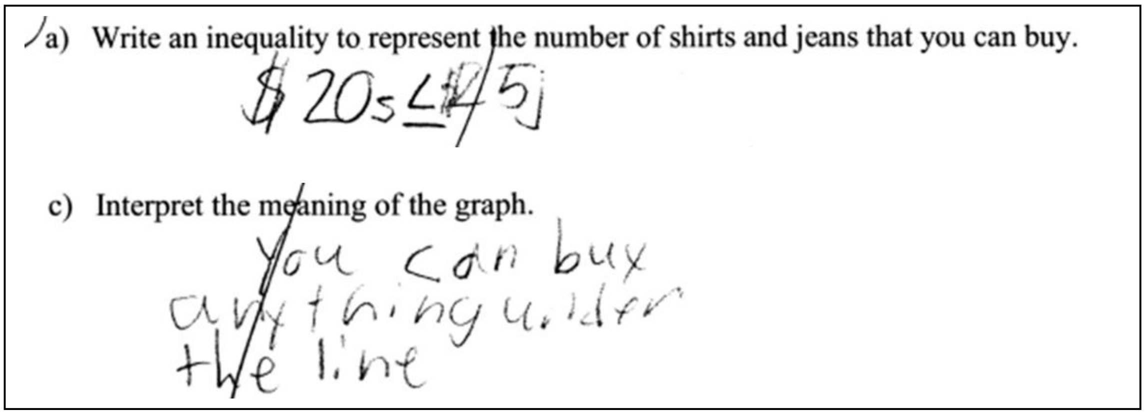
The correct solution to part (a) of this problem is

$$\$20x + \$25y \leq \$200$$

where x is the number of shirts purchased and y is the number of pairs of jeans purchased. Other variables were also acceptable.

The examples of student work that follow show the answers to parts (a) and (c). These answers point out how students attempted to model the situation and how they interpreted the meaning of the graph.

Figure 5.15 shows how Garrett relates the two different variables in the situation. He does not define the variable and does not include the total amount spent in the situation. The interpretation of the graphical solution is partially correct in this situation. Garrett does not acknowledge that only solutions at lattice points are valid, nor is it noted that solutions outside of the first quadrant are invalid.


a) Write an inequality to represent the number of shirts and jeans that you can buy.

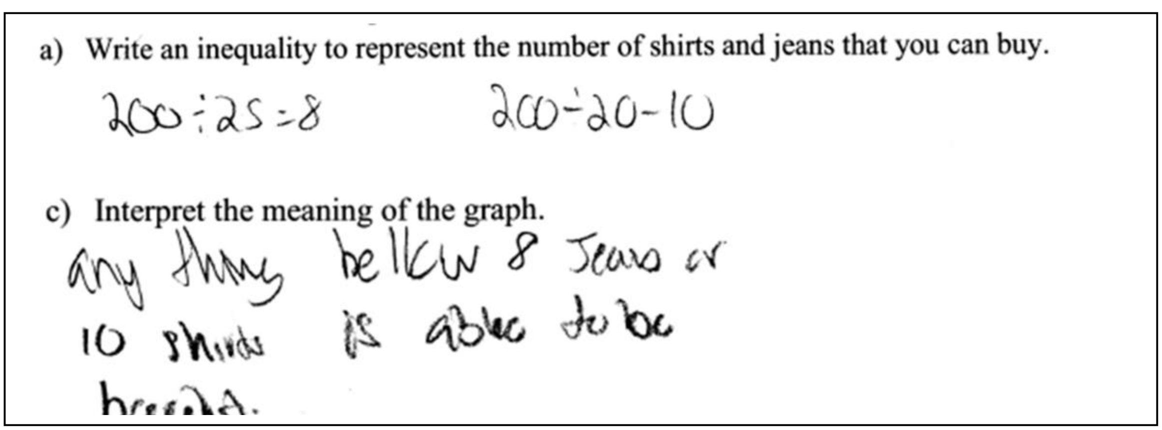
$20s ≤ 45j

c) Interpret the meaning of the graph.

You can buy anything under the line

Figure 5.15 – Garrett's response

Aisha's response to this problem is shown in Figure 5.16. She solves for the maximum number of each item that could be purchased separately but fails to use a variable to construct an inequality to model the situation. In part (c), she does not take into account combinations of both items together.


a) Write an inequality to represent the number of shirts and jeans that you can buy.

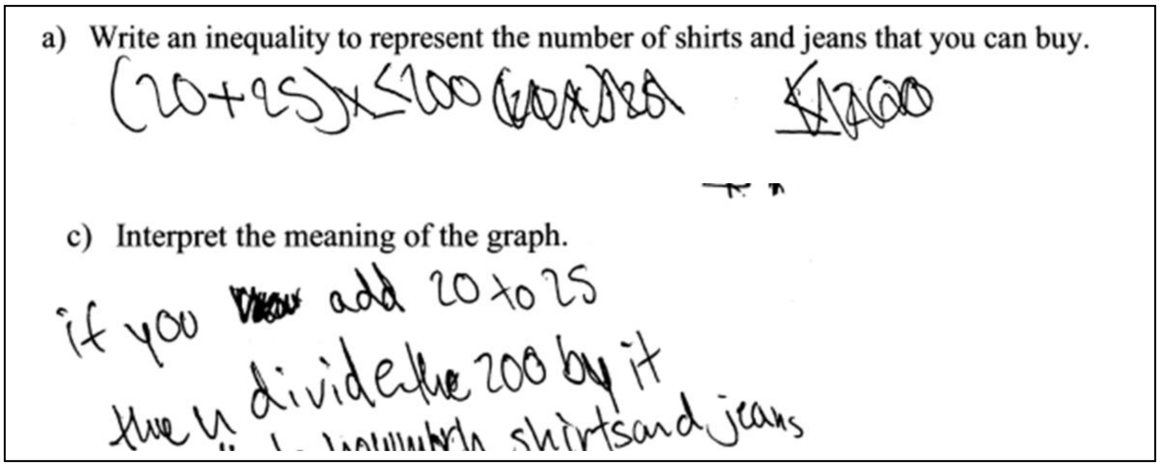
200 ÷ 25 = 8   200 ÷ 20 = 10

c) Interpret the meaning of the graph.

any thing bellow 8 Jeans or 10 shirts is able to be brought.

Figure 5.16 – Aisha's response

Deng attempts to model the situation using only one variable as shown in Figure 5.17. He takes into account the total amount spent and uses the correct inequality for the situation. The response to part (c) shows that he failed to consider that different quantities of each item could be purchased, which reflects the choice to use one variable instead of two.



a) Write an inequality to represent the number of shirts and jeans that you can buy.

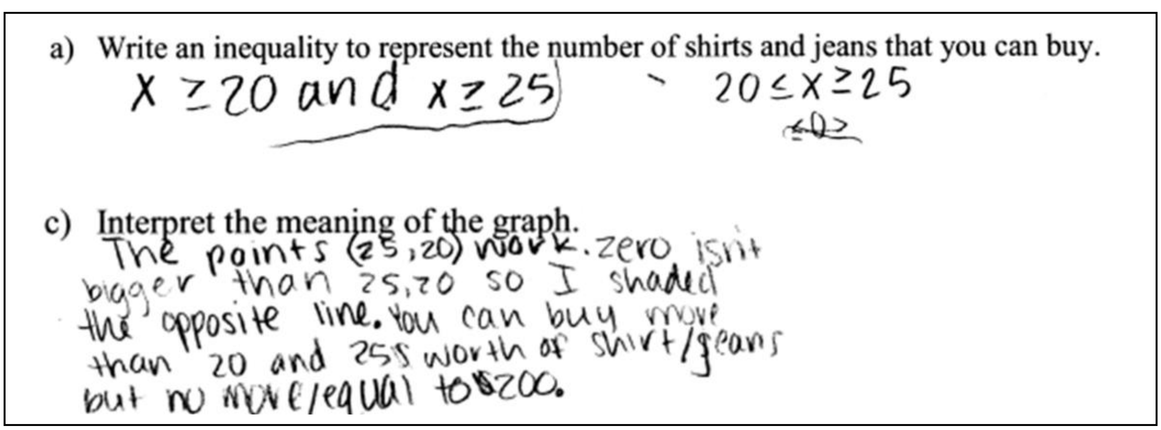
$(20+25)x \leq 200$

c) Interpret the meaning of the graph.

if you ~~then~~ add 20 to 25 then divide the 200 by it ... shirts and jeans

Figure 5.17 – Deng's response

Figure 5.18 shows Madison's work as she attempts to use one variable to model the problem. The response to part (c) shows a lack of understanding of the question. Madison does not appear to understand the relationship between the two variables that she is considering or how to model them as an inequality, and therefore is unable to build an accurate model for this situation. However, in part (c) she expresses part of her



a) Write an inequality to represent the number of shirts and jeans that you can buy.

$x \geq 20$ and $x \geq 25$ $\quad 20 \leq x \geq 25$

c) Interpret the meaning of the graph.

The points (25,20) work. zero isn't bigger than 25,20 so I shaded the opposite line. You can buy more than 20 and 25$ worth of shirt/jeans but no more/equal to $200.

Figure 5.18 – Madison's response

solution as an ordered pair. This may point to a lack of understanding of variables and ordered pairs.

Looking back at the way this issue has manifested itself throughout this unit, the first issue that needed more thorough treatment was defining variables when solving modeling problems. From the first day of the unit until the final one, my students were unsure about this fundamental starting point for problem solving. This is an issue that I can trace back to the first unit on variables and expressions. It is also a skill that I assumed the students would be comfortable with based on the Pre-Algebra curriculum from the prior year. Assumptions are one thing, practice is another.

During the course of the unit, I presented the students with three different modeling situations that have already been discussed in this chapter. My focus with these activities was to help students connect authentic situations with mathematical concepts. These activities formed the basis of many of the classroom discussions we had throughout the unit. As often as possible, when a student struggled to understand a mathematical idea, I would attempt to help them by recalling one of the situations we had already encountered.

It became clear to me that I needed to spend more time within each of these situations emphasizing the importance of properly defining variables. This is not a difficult task, but it is one that would have given my students a better chance of success on modeling problems. This is just another of the small but critical moves as a teacher that I should have made to improve my students’ depth of learning.

This issue also highlights the difficultly my students had connecting ideas from outside the mathematics classroom with the curricular goals of the unit. It was clear to me in each of the three experiences that the students could verbalize the idea, give numerical examples, and interpret graphical representations. The challenge for them was to write an algebraic inequality that represented the situation.

Reflections on Representing Problems Algebraically with an Inequality

One of my goals throughout the course was to allow students the opportunity to make sense of what we were studying. While they were able to make sense of concepts, the time that I invested in this left me without the time to properly address other major issues. Issues that I have outlined throughout this chapter. What is the solution?

It is tempting to say that I should have focused on skills and procedures allowing my students to successfully produce solutions. They could have done that without any understanding of what they had learned. This choice would rob students of the opportunity to understand what they are doing and why they are doing it. Instead, I should have been more aware of the choices I was making as a teacher in order to better prepare my students for success over the course of the unit and the entire year.

Conclusion

Throughout my examination of our work in this chapter, I identified three areas of concern, or issues having to do with students

1. appropriately changing the direction of the inequality symbol during algebraic commonplaces,
2. graphing solutions to one variable inequalities,

3. representing problems algebraically with an inequality.

These three issues have origins in fundamental understandings of equivalent forms, variables, and functions. My conclusions are based off of an examination of my planning journal, fieldnotes, classroom video recordings, and student work. While the origin of these problems is similar, the way that the problems manifest themselves in student work is as varied as the interventions required by the teacher to correct the misunderstandings.

I continue this work in the next chapter by examining my teaching and the resulting student issues in our unit on Exponents. Like inequalities, exponents are a necessary but insufficient part of studying first-year Algebra. While the concepts and processes involved in simplifying exponential expressions appear different from those used when working with linear inequalities, they rely on the same fundamental ideas of equivalence and the equal sign, and variables and functions.

CHAPTER 6

EXPONENTS: A NEW ALGEBRA

The greatest shortcoming of the human race is our inability to understand the exponential function.

-Albert Allen Bartlett

In Chapter 5, I identified three issues in my teaching of solving inequalities that emerged from reflecting on the evidence I gathered throughout the school year. These three issues relate directly back to the problems of equivalence and the equal sign, and variables and functions as identified in modern research in the teaching of first-year Algebra. Throughout this chapter, I examine my teaching of the algebra of exponents and focus specifically on three concerns, two relating to student learning and one relating to my planning and preparation. During this chapter I will explore three topics:

1. Students confronting the delineation of the algebra of exponents from the algebra of real numbers,
2. their difficulties with negative exponents, and
3. my struggle planning and teaching an algebra topic that is both authentic to the mathematics and appropriate for the students.

Teachers in my district have identified this unit as one in which students historically struggle to understand the properties that are utilized in the simplification of expressions involving constants, variables, and exponents.

The Importance of Exponents

Exponents hold a significant place in both theoretical and applied mathematics. For students, this first-year Algebra experience transforms their understanding of exponents from numeric to abstract. Working with monomials makes more readily apparent the underlying properties of exponents that are usually not formalized in prior math courses.

Typically, students are introduced to exponents as a form of repeated multiplication; the notation,

$$a^b,$$

implies that the base

$$a,$$

is multiplied by itself

$$b$$

times, where

$$a, b \in \mathbb{Z}^+.$$

During first-year Algebra, this definition is expanded to include non-positive integer values for the exponent and variables as bases, while rational bases and exponents are usually taught during second-year Algebra.

Variable Exponents

Another transition in the form of exponential equations occurs during second-year Algebra where the base is constant and the exponent is variable. These types of expressions are used to build growth and decay equations of the form

$$y = Ca^{kx},$$

that model physical phenomena like radioactive half-life, population growth, or compound interest. These models are utilized across multiple curricular areas.

Students who take first-year Algebra during eighth grade are on track to take a calculus course during their senior year. Exponents play a significant role throughout both Differential and Integral Calculus. Their uses include, but are not limited to, simplifying polynomial expressions, solving equations, finding inverses, finding derivatives and anti-derivatives, symbolically representing derivative rules, and determining areas and volumes of irregular regions.

Introducing the Unit

The end of winter break is bittersweet. It means two weeks away from work. It means time with family and friends to recharge and refresh both mind and attitude. After the strain of first semester, I welcomed this time.

I was amazed at how much effort it took to split time between two buildings. Starting the day at the middle school was not difficult, neither was driving over to my high school for the second half of the day. What I found most difficult was not really being a part of either building, but instead just moving back and forth between them.

Once January 6 arrived, I had to set those feelings aside and return my focus to teaching. I started the school year with twenty students on my roster and finished first semester with eighteen of them. Three students had moved away during the first semester, and one student had joined the class. Although experience had taught me otherwise, I was hopeful that the class would stay together throughout the second semester.

The challenge ahead was a short chapter on the algebraic properties of exponents. From January 9 through January 31, the students studied five different properties and applied those properties to scientific notation. The district curriculum required students to be able to work with the following properties:

1. Product of Powers: $a^m a^n = a^{n+m}$
2. Power of a Product: $(ab)^m = a^m b^m$
3. Quotient of Powers: $\frac{a^m}{a^n} = a^{m-n}$
4. Zero Exponent: If $a \neq 0$, then $a^0 = 1$.
5. Negative Exponent: If $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$

The challenge this unit presented was one that I had learned through experience: students in the past had struggled with these properties. First-year Algebra at the middle school level prior to this year was novel to me. I had always placed the blame on the high school students that I worked with when they were unable to solve problems that I believe should have been mastered during first-year Algebra. They were a convenient scapegoat. My thoughts usually centered around the idea that if they were smart enough, they would have learned it in middle school. This is a terribly fixed mindset, but it is where I was at the time.

Teachers' Perspectives

I was curious to learn how my middle school students would deal with this unit. I went to three teachers whom I had sought counsel from in the past when I was designing lessons, Phil Lafluer, eighth grade math teacher at Lewis and Clark Middle School and NOYCE Master Teacher Fellow; Kenzi Mederos, seventh and eighth grade math teacher at Lewis and Clark Middle School and Math in the Middle graduate; and Jill Luschen, seventh and eighth grade math teacher at Buffett Magnet Middle School. The common theme coming from all three was, "Properties of exponents are just too abstract for the students. It's hard for kids to work with an idea that doesn't analogize to the real world" (Field notes, January 6, 2017).

Armed with this knowledge, I was stuck trying to help students make sense of something that I am comfortable with but find it difficult to see how students are not able to understand. This is often the case when teaching mathematics. Early in my teaching career, I found myself lauded for helping students understand mathematics the way I understood it. But at some point, I came to the realization that it was more important for the students to make sense of mathematics themselves and not just in the way that I do. My role was to help them validate their own understanding by using mathematics as the judge of correctness.

Opening Activities

With this way of thinking as my guide, I moved forward planning the unit by looking for lessons to help my students establish what I viewed as a critical foundational idea: the behavior of exponential relationships is fundamentally different than that of the linear equations, functions, and inequalities we had worked with during first semester. I chose four activities for the first three days of class that I hoped would help students draw this conclusion.

The Rice Problem

On January 9, I introduced the students to the first activity. It was a classic from mathematics, the Rice Problem. My students read the prompt shown in Figure 6.1.

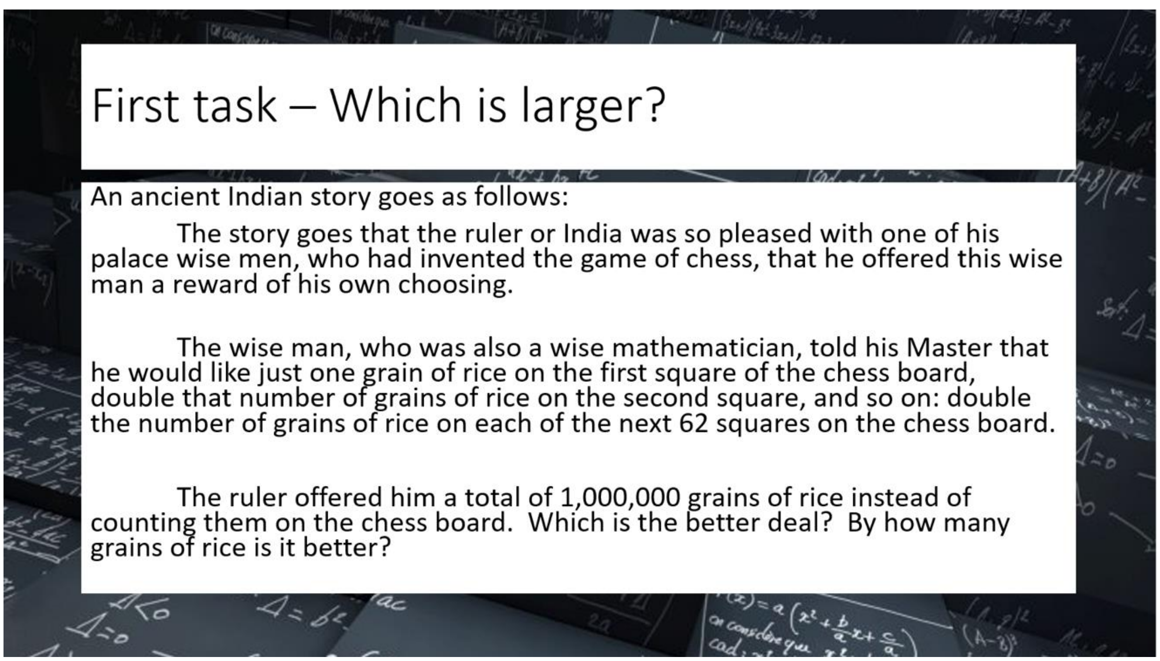


Figure 6.1 – Opening prompt

I chose this activity because I thought it would help students notice how quickly exponential growth occurs. Additionally, the students had been placed in new groups at the start of the new chapter. This prompt provided me the opportunity to observe how the new groups worked together.

The way many of the students determined the solution helped build an early connection to the Product of Powers Property (see page 82). The following dialogue occurred after the students had read the prompt. They were given five minutes to discuss it in their small groups. I began by asking if anyone had a guess as to the solution.

1	Garrett:	It's got to be the chess board.
2	Greg Sand:	Why?
3	Garrett:	I remember this problem from 6^{th} grade, and I know it's a trick question. I just don't know by how much.
4	GS:	[To the entire class] Okay teams, let's see what we can figure out. I'm guessing that you'll all want calculators, so come grab one if you need it. [Talking with a small group] How are you finding the solution?
5	Isabel:	Well, every square is double the previous, so I'm just multiplying by 2 each time.
6	GS:	Are you writing down the answers?
7	Jessi:	Oh right, he gets all the rice, not just the last square. Darn it!
8	GS:	[After some time has passed to the entire class] Does anyone have a solution?
9	Jordan:	I can't find one; it's impossible.
10	GS:	Why?
11	Jordan:	The number is too big for my calculator; it can't do it.
12	GS:	Did anyone else have that problem? [Lots of nodding heads] Why is this a problem?
13	Garrett:	Every time you double the number, it gets twice as big. When I do it 62, no, 63 times, its gets huge!
14	GS:	Can you give me an example?
15	Garrett:	Well, on square 20, there is over half a million grains of rice. So, on square 21 there will be over a million! I don't know how many more there will be, but it will be a lot.
16	GS:	Let me write out the answers without multiplying them out, maybe we can see a pattern. [I write out $1, 2, 2^2, 2^3 \ldots 2^{63}$]. I noticed that many of you were doubling, that's the same

thing as multiplying by 2 each time. Does anyone notice a pattern in the list?

17 Seamus: Each time you multiply by two, the power goes up by 1.

This reveals some students' use of recursive relationships to reason and understand the mathematics within this problem. When asked how she found the solution, Isabel noted (line 5) that she was "just multiplying by 2 each time." She made sense out of the problem by connecting each future term to the previous one.

This understanding was only partial because Jessi (line 7) reminded the group that, "He gets all the rice, not just the last square."

Garrett demonstrates recursive logic when he explains to the class (line 15) how immense the number is. He stated, "On square 20, there is over half a million grains of rice. So, on square 21 there will be over a million! I don't know how many more there will be, but it will be a lot."

These comments by the students explained to me their foundational understanding of the relationship between identical bases and exponents when they are multiplied. Because of the inherent limitations of the technology and the statements made by Isabel and Garrett, I displayed the remaining amounts as powers of 2. At that moment Seamus (line 17) noticed, "Each time you multiply by 2, the power goes up by 1," which demonstrated his awareness that multiplying a power of 2 by 2 would increase the value of the exponent by 1, noted as $(2^n)(2) = 2^{n+1}$.

However, I felt reservations that this numerical understanding would translate into a more general symbolic understanding, i.e. $(a^m)(a^n) = a^{m+n}$. While I wrote in my journal that this had been "a great activity," I was not sure "that this numerical activity will help them express it symbolically. I hope that I can use this activity in the future to prompt their thinking” (Fieldnotes, January 9, 2017). What I called “numerical” is important because it is the calculation that we are representing symbolically, but it is limited because it does not develop students’ abstract thinking and reasoning. The larger goal of this unit, and algebra as a whole, is to continuously support algebraic thinking and reasoning to strengthen students’ foundation for increasingly challenging work that will confront them in the remainder of this course and future mathematics courses.

Visualizing Exponential Growth

The second activity of the day was designed to aid students’ algebraic thinking by reinforcing their understanding of exponents, specifically, of the effect of increasing or decreasing the exponent by one. The class watched a video clip called “Powers of 10." Considered a classic from 1977, it begins with a couple resting on a blanket in a park in Chicago. An initial, one square meter view of the couple then zooms out so that every ten seconds the view is ten times larger than the previous one. It stops at a square with an area of 10^{24} square meters, an inconceivable size. The video then reverses direction, returns to the park, and zooms in on the hand of one of the people on the blanket. It shows powers of 10 from 10^0 to 10^{-16}, another challenging number to understand.

At the power of –16, I paused the video and asked the students if they had ever seen negative exponents. I posed this because I was both unsure about any prior experiences that the students had with negative exponents and hopeful that it could provide another point of entry to making sense of exponents and how they become intuitively rational with repeated examination. Their responses and our discussion followed.

1	Haley:	Yeah, we saw them in Physical Science
2	Greg Sand:	Wow, that's great. Where?
3	Kiera:	We use them in Scientific Notation.
4	GS:	Why?
5	Kiera:	When the numbers are really small, you know, like 0.0005. When we move the decimal place to the right, we're supposed to use negative exponents.
6	GS:	Let me write that down. [Writes 0.0005 on the board] Can you show me what you mean?
7	Kiera:	I move the decimal point four places to the right, so its 5×10^{-4}.

Haley (line 1) shared with her peers that they had seen negative exponents in a different course. Then Kiera, lines 3, 5, and 7, demonstrated to the class the procedure they were taught to express numbers in scientific notation utilizing negative exponents. Her ability to execute the procedure revealed to me that she understood negative exponents as a way to express numbers very close to zero and very small numbers.

While Kiera showed an emerging sense of negative exponents, I became concerned that her initial understanding may cause difficulties with the concepts I would be presenting to her and the other students later in the chapter. This is an example of a discipline, such as science, creating both an exposure to a mathematical idea and at the same time developing a discipline-specific conceptualization, like scientific notation, which could cause an incomplete understanding of negative exponents. This may then lead to difficulties when a more general conceptualization is offered in a mathematics course.

This activity, like the Rice problem, provided a way to introduce the Products of Powers Property. Each time the view became ten times larger, the exponent increased by one. Each time the view became ten times smaller, the exponent decreased by one which allowed me to introduce the Quotient of Powers Property.

Fry's Bank Account

Knowing that these first two activities appealed to me as a mathematician and learner, but not necessarily to all students, I looked for an activity that I thought would resonate with my students. I stumbled upon a second-year Algebra lesson based on a scene from the Fox cartoon, *Futurama* (Cohen & Groening, 1999). In this clip the main character, Fry, who was awoken from suspended animation after hundreds of years, goes to the bank to see how much money he has. The challenge to students is to determine how much money is in his account.

This type of problem—compound interest—is approachable for students but difficult to compute without using the standard formula properly. My goal was to engage the students in a conversation about how we would calculate it, not to actually calculate the solution. To my surprise a few students, Savannah, Juan, and Joshua, remembered the episode, and we were able to discuss the answer with the class from memory. While it would have been easy to become frustrated by my students' prior knowledge interfering with the lesson, I used it as an opportunity to allow students to take the lead in class discussion. When I showed the video, the following conversation took place.

1	Savannah:	Oh, I know the answer!
2	Greg Sand:	Really?
3	Savanah:	Yeah, this one is great. He has so much money.

4	Juan:	I saw this episode like just the other day. Fry is loaded.
5	Joshua:	He’s got like a billion dollars.
6	GS:	Can one of you explain it to me?
7	Savannah:	So, Fry didn’t have much money, he was a pizza delivery guy. But, his account made interest. And that interest made interest. He was in suspended animation for a long time, so it just kept adding up.

My initial goal of talking about the idea of the problem failed because a set of my students knew the solution from the episode. However, here it did allow Savannah, who was not confident as an algebra student, the opportunity to speak with authority about a problem. I believe her verbalization of how compound interest works (line 7) was more effective than I would have been saying the same thing or attempting to construct the formula for compound interest,

$$I = P\left(1 + \frac{r}{n}\right)^{nt},$$

from this situation.

During this part of class, my role changed from facilitating a conversation about a concept to interpreting and verifying a student idea about mathematics. This type of shift is a critical part of the daily work a math teacher must do. During any lesson, I need to be prepared to hear what a student says, interpret and make sense of the language that they use, connect it to a greater mathematical truth, and then bring that idea to them in a way that either validates or critiques their thinking as developmentally appropriate. This small interaction with Savannah highlighted for me the importance of content knowledge in teaching first-year Algebra beyond what is taught in the course.

Folding Paper to the Moon

The fourth and final introductory lesson the class engaged in was titled Folding Paper to the Moon. In this exploration, I asked students to determine two values, the thickness of a single piece of paper and how many times a piece of paper would need to be folded in order to reach the moon. To do this, students repeatedly folded a piece of notebook paper in half until it was thick enough to be measured in centimeters. Following the same theme as the other lessons, my goal was to help students make sense out of how exponents grow at different rates than the linear relationships we had encountered in the past.

These early lessons were different than the previous introductory lessons I had used in other chapters throughout the year. As a norm in my class, I tried to design and choose activities that emphasized student reasoning and sense-making throughout the course of the unit over the presentation of my own. As I shared in Chapter 4, the students worked on a problem that led them to graphing two variable inequalities. Unlike the activities in Chapter 4 that connected applications to graphing, numerical, and analytic solutions, this chapter did not lend itself to that approach because we were studying properties of exponents without any applications.

Transitioning to Formal Understanding

After three days of working on these four activities, I moved the class into the formal study of the properties on January 12. I then encountered the challenge my colleagues had expressed. Thinking back to their comments, I noted in my journal, “How do I help students make sense of an idea that doesn't lend itself to concrete examples?" My approach was to build the idea inductively with students.

Discovering the Product of Powers Property

I began with the following two exercises: First, as shown in Figure 6.2, I had students write exponents in expanded form. Next, I had the students convert from expanded form into exponential form (Figure 6.3). My goal was to remind students of what exponential notation means as well as to provide them with a strategy to use moving forward.

Review

Write out each of the following in expanded form.

x^5

3^8

$a^3b^2c^4$

$(2w^3)^4$

Figure 6.2 – Learning examples

Write out each of the following in exponential form.

$y \cdot y \cdot y \cdot y$

$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$

$g \cdot g \cdot g \cdot h \cdot h \cdot h \cdot h \cdot h \cdot j$

$3x^5 \cdot 3x^5 \cdot 3x^5 \cdot 3x^5 \cdot 3x^5 \cdot 3x^5$

Figure 6.3 – Learning examples

After this review I presented students with the following prompt, shown in Figure 6.4, with the goal of doing a small set of problems so they could recognize a pattern and come to a conclusion using inductive logic.

New Ideas for today

Simplify each expression by:

A. Writing the problem in expanded form
B. Rewrite the expanded problem in exponential form

Part 1 – Product of Powers

1. $(x^3)(x^4)$ 2. $(y^5)(y^2)$ 3. $(a^2b^3)(a^4b^2)$

Figure 6.4 – Learning prompt

The use of inductive logic in this case is mathematically appropriate because of the way in which these properties are proven for natural numbers. In Figure 6.4, problem 1, I wanted the students to work through the problem as shown in Figure 6.5.

$$(x^3)(x^4) = (x \cdot x \cdot x)(x \cdot x \cdot x \cdot x) = x^7$$

Figure 6.5 – Anticipated solution to problem 1

Similar work then followed with the second and third problems. My hope was that they would notice the pattern and complete a conjecture about the Products of Powers, namely that

$$(a^m)(a^n) = a^{m+n}.$$

Mathematical Justification of Product of Powers Property. As part of my review of the mathematics necessary to teach exponents, I wanted to explore the proofs of the properties of exponents. I hoped to gain insight into different ways to create lessons that would allow students to discover them. The inductive proof of the Product of Powers provided me with on such insight. This insight guided the decisions to use inductive logic throughout the design of the unit's activities.

Inductive logic is a habit of mathematicians and scientists in which patterns are generalized. These generalizations can be proved using a proof by induction and is used to prove the Product of Powers property for real number bases and natural number exponents. The following proof helped deepen my mathematical knowledge and guided my pedagogical decisions.

Proof

Let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$ with

$$a^1 = a$$

and define

$$a^{n+1} = a^n a.$$

Consider

$$a^m a^n$$

with the goal to prove that

$$a^m a^n = a^{m+n}.$$

Base case $(n = 1)$

$$a^m a = a^m a^1 = a^{m+1}$$

is true as defined above.

By the induction hypothesis, assume that the statement to be proven is true for $n = k$.

Thus,

$$a^m a^k = a^{m+k} \,.$$

So

$$a^m a^{k+1} = a^m(a^k a)$$

as defined above,

$$a^m(a^k a) = (a^m a^k)a$$

by the associative property of multiplication,

$$(a^m a^k)a = (a^{m+k})a$$

by the induction hypothesis,

$$(a^{m+k})a = a^{(m+k)+1}$$

as defined above, and

$$a^{(m+k)+1} = a^{m+(k+1)}$$

by the associative property of addition. Thus,

$$a^m a^{k+1} = a^{m+(k+1)} \,.$$

Thus it is true for $n = k + 1$

and therefore,

$$a^m a^k = a^{m+k} \,.$$

$\blacksquare$

A similar proof can be written for negative exponents. A more thorough proof for real number exponents comes out of real analysis and offered me no valuable mathematical insights that I could apply to teaching exponents. This exercise provided me insight into teaching methods that were authentic to the mathematics used to prove the ideas.

Students Confronting the Delineation of the Algebra of Exponents from the Algebra of Real Numbers

Students began to work through the examples shown in Figure 6.4. I moved from group to group to listen to discussions and offer advice. The class' reaction to the initial examples made me believe that the method was effective. While the students were working through the three examples:

$$(x^3)(x^4),$$

$$(y^5)(y^2),$$

and

$$(a^2b^3)(a^4b^2),$$

I had the following conversation with one group of four students. I asked the students to explain what patterns they noticed and how they observed patterns in the problems. By focusing on patterns and creating generalizations from them, I hoped to develop and highlight the mathematical habit of inductive logic. This was also an opportunity for me to help the group slow down and explain their thinking instead of making quick conclusions and not retaining what they had discovered.

1	Greg Sand:	What pattern do you see?
2	Garrett:	It seems too easy.
3	GS:	Huh?
4	Garrett:	I mean, you just have to add the powers, right?
5	GS:	Explain that to me
6	Isabel:	Look at the first one, x^3 times x^4 is x^7. That's just $3 + 4 = 7$. Am I right?
7	GS:	What do the rest of you think?
8	Joshua:	Seems right to me.
9	GS:	Why don't you look at the third example and then I'll check in with you to see if your solution is correct. [Walks away from group and returns after about 2 minutes] What do you think now?
10	Garrett:	Okay, we're right.
11	GS:	Talk me through the third example.
12	Jessi:	Well, there are two a's in the first part and four $a's$ in the second part. Also, there's three b's in the first part and two $b's$ in the second part. So there's a total of six $a's$ and five $b's$.
13	GS:	Let's try and be more exact. If the variables are the same when we multiply them, we add the exponents.
14	Garrett:	Like I said, we're right.

Here is a tension that happens when I create opportunities for students to discover mathematical properties. In line 4 when Garrett states, "You just have to add the powers," he has come to a valid yet incomplete conclusion. He has not developed the precise language to properly express those conclusions in a way that fully expresses the idea, but he is ready to move on from the idea. While I believe that he is fundamentally correct, I wanted to find out if he and his group really understood and could generalize this idea. To do this, I asked them to explain their conclusion (line 5), so Isabel (line 6) explained to me her work on the first example. I knew that they needed to be able to handle working with this idea when multiplying different bases, so I asked them to try the third example (line 9). The third problem was chosen for this reason. When Jessi (line

12) explained her group's solution to this problem, I offered a more general statement about what they were noticing.

Throughout this dialogue, I could not help but notice my students' focus on the idea of being right. This drive for correctness and validation could have undercut the work I was trying to accomplish with them during this portion of the lesson. My focus on thinking through the process, understanding why their conclusions were valid, and finally expressing it in a mathematically complete way was an attempt to deflect the immature concern about being correct in exchange for an emergent understanding of why they were correct. Without taking the time to develop a more precise understanding, fundamental misunderstandings can occur, an idea that I illustrate later in the chapter.

Discovering the Power of a Product Property

I continued this approach with the Power of a Product property and felt like my students were making sense of the ideas. The examples I used with the class are shown in Figure 6.6. After the students worked through the first example,

$$(c^3)^4,$$

Part 2 – Power of a Product		
1. $(c^3)^4$	2. $(d^2)^5$	3. $(v^4w^2)^3$

Figure 6.6 – Learning prompts

I shared the solution that I hoped they would discover, shown in Figure 6.7. My secondary goal for this example was to see if the students would be able to connect the Product of Powers Property with the Power of a Product Property. Like inductive thinking, learning to link one idea to the next is an integral part of learning and doing mathematics.

$$(c^3)^4 = (c^3)(c^3)(c^3)(c^3) = c^{12}$$

Figure 6.7 – Anticipated solution to problem 1

During this lesson on January 12, I quickly became aware that the students were confusing the properties of real numbers with the properties of exponents. The first time I noticed this was during the example problems (see Figure 6.8) that the students worked through after we had made our general statements about the properties they had discovered. As the students attempted to simplify these problems, two different groups made a set of errors with this common theme.

$(6n^3)(2n^7)$	$(3y^4)(7y^5)$	$(3pt^3)(p^3t^4)$
$(q^4)^{10}$	$(2p^3)^4$	$(-5f^4r^9)^3$

Figure 6.8 – Example problems

In the first example, the students were asked to simplify the following expression:

$$(6n^3)(2n^7).$$

To accomplish this, the students must first multiply the coefficients and then express the variable to a single exponent. The result is

$$12n^{10}.$$

After having time to work on the problem, I asked different groups in the class to share their solutions rather than share the answer myself. When I had solicited one solution, I made the choice to ask other groups for their solutions with multiple goals in mind. First, I always try to keep myself from being the arbiter of correctness; mathematics takes that role. It is a challenge for students to be wrong in front of their peers, especially in middle school. By not stating if any of the groups' answers were right or wrong, I allowed for students to take risks without fear of immediate judgement. Second, I could not have realized student misunderstandings if I did not allow others to share their solutions. Finally, when groups had shared their solutions, I worked the problem with them, allowing the students to self-assess and identify any errors that they had made without making those errors public and causing embarrassment among their peers. The students speaking in the following dialogue represent three different groups.

1	Greg Sand:	Let's check and see how we did with the first problem. Can someone share their solution?
2	Jessi:	I got $12n^{10}$.
3	GS:	Does anyone agree? Any other solutions?
4	Deng:	We got $8n^{10}$.
5	GS:	Any others?
6	Seamus:	We got $12n^{21}$.
7	GS:	Any others? [Pause] Okay, so how do we decide who's right? [Pause] Let's write this problem out in expanded form and see from there.

Deng's group added both the exponents and coefficients which told me that the group was treating the coefficients as exponents. Seamus's group multiplied both informing me that the exponents were being viewed like coefficients. This type of error occurred in the second, fifth, and sixth examples, respectively.

$$(3y^4)(7y^5),$$

$$(2p^3)^4$$

$$(-5f^4r^9)^3$$

Classroom recordings documented solutions to the second example of $10y^9$ and $21y^{20}$, solutions of $8p^{12}$ and $16p^{27}$ to the fifth, and $-15f^{12}r^{27}$ to the sixth example.

These concerns were reflected in my fieldnotes on January 12. Part of my reflections on the day included, "Several of the students are struggling to keep the rules separate. While this was expected, I was surprised by which students had difficulties today. Some kids have had a tough time all year while for others it was the first time." After reviewing the video, ten of the eighteen students persistently made one of these two errors throughout the unit. Seeing the errors reflected in different forms of data verified the problem and became a call to action to help students develop mastery of this skill.

Ongoing Development and Review

During the next three days of class (January 13, January 18 and January 20) I had the students spend part of each class doing individual practice through a small set of problems, four to six questions each, that would allow me to check for understanding. I referred to these exercises as "Quick Quizzes." Overall, the number correct increased each day. There were some problems, however, that continued to cause trouble for the same group of students as from the lesson on January 12.

2. $(5t^2)(3t^5)$

4. $(4pq^3)^2$

Figure 6.9 – Problems from quick quiz 1

On Quick Quiz number 1, problems 2 and 4 (shown in Figure 6.9) required the students to perform operations on both exponents and coefficients. A review of the recording of class that day showed a slight improvement in performance, with only eight students getting one or both of these problems incorrect. The errors made were of the same type that I had noticed on January 12, i.e. $15t^{15}$ or $8t^8$.

Quick Quizzes 2 and 3 saw similar errors. Figures 6.10 and 6.11 show the problems that solicited those errors. A review of the recording of class from those dates showed a continued improvement by the students, but by the end of class on January 20, three students were still making the same errors.

1. $(3r^2)(5r^4)$	2. $(-4f^3g)(2f^2g^4)$
3. $(2y^2z^3)^3$	4. $(-3pq^3)^2$

Figure 6.10 – Problems from quick quiz 2

3. $(-3uv^3)^3$

4. $(7rs^2t^3)(-3r^2s^2t)$

5. $(2g^4h^3)^3(5g^2h)^2$

Figure 6.11 – Problems from quick quiz 3

In the short term, the students generally improved. But when we reached the unit exam, many of these errors reemerged. The following samples of student work illustrate the different errors the students made. Set one was problem number 15 from the unit exam (see Figure 6.12). The original problem read, “Simplify completely. Your answer should not contain any negative exponents.”

Problem #15:

$$(6a^4)(5a^3)$$

To simplify the problem, students must simplify the coefficients using standard multiplication and the variables using the Properties of Exponents. This yields a simplified form of

$$30a^7.$$

In the first example, Jessi (Figure 6.12) correctly adds the exponents but adds the coefficients instead of multiplying them. In the second example, Chloe (Figure 6.13) correctly multiplies the coefficients but multiplies the exponents instead of adding them. The third example (Figure 6.14) shows that Aisha switches the rules, adding the coefficients and multiplying the exponents.

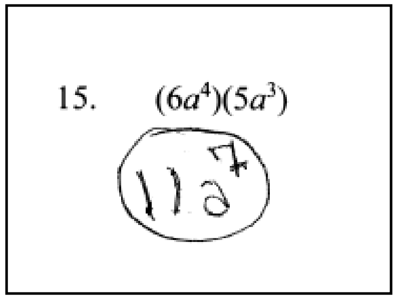


Figure 6.12 – Jessi's response

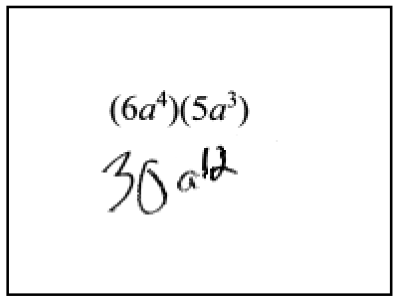


Figure 6.13 – Chloe's response

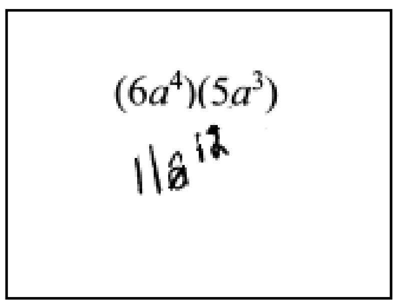


Figure 6.14 – Aisha's response

The second example of student work (Figure 6.13) is from problem 16 from the same quiz. The instructions for this problem were the same as problem 15.

Problem #16:

$$(5a^6y^3)^2.$$

To simplify the problem, a student must square each component of the expression,

$$(5^2)(a^6)^2(y^3)^2,$$

which is equivalent to

$$25a^{12}y^{6}.$$

Evan (Figure 6.15) correctly simplifies the exponents but multiplies the coefficient by 2 instead of multiplying it by itself. The second sample (Figure 6.16) is from Isabel's test. She fails to use either property correctly. Figure 6.17 shows Seamus's work; he only applies the exponent to the coefficient and does so incorrectly.

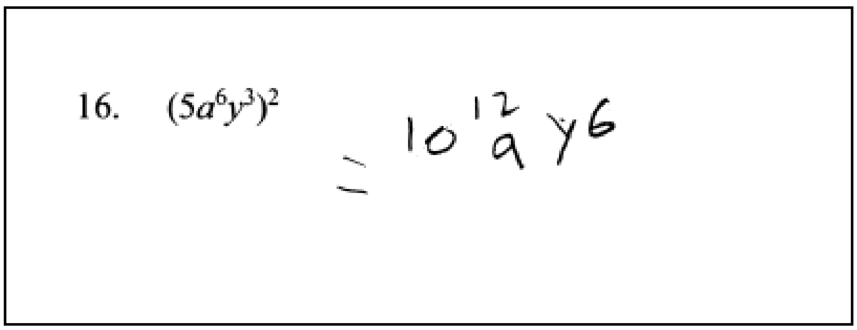


Figure 6.15 – Evan's response

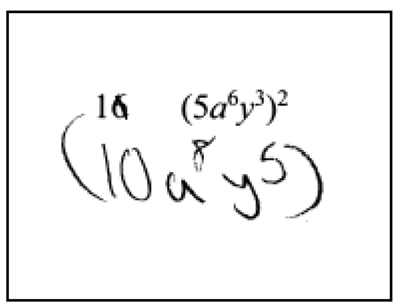


Figure 6.16 –Isabel's response

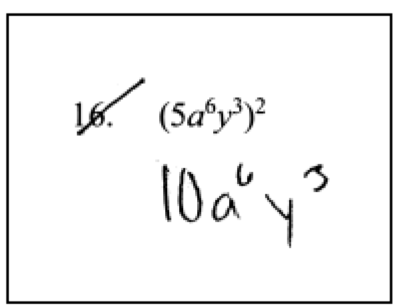


Figure 6.17 – Seamus's response

A third example is problem number 25. The students were asked to add two numbers in scientific notation. The prompt read, "Compute. Write the answer in scientific notation."

Problem #25:

$$(2 \times 10^3) + (8 \times 10^3).$$

Philip's response (Figure 6.18) and Clifton's response (Figure 6.19) illustrate two common examples of student errors.

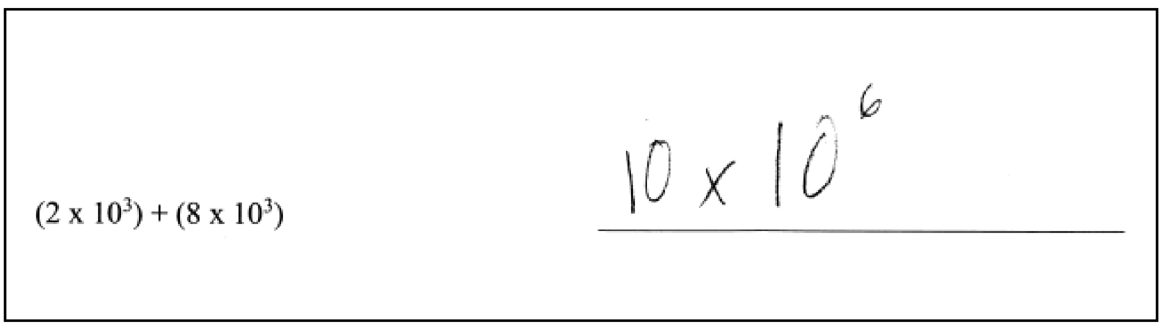


Figure 6.18 – Philip's response

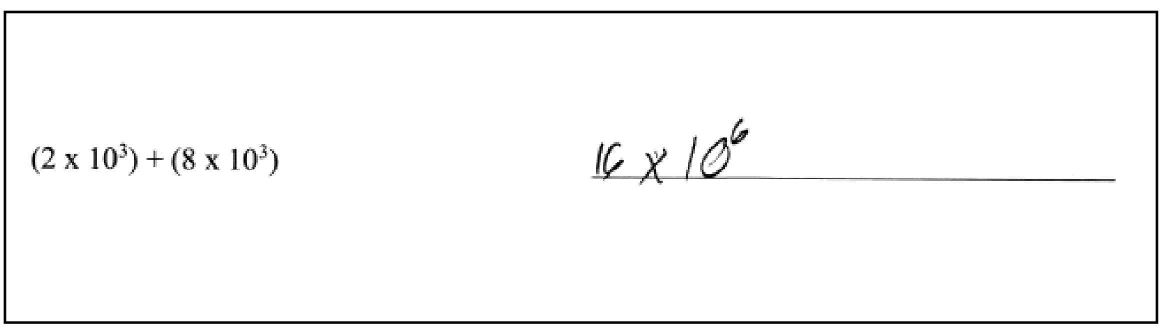


Figure 6.19 – Clifton's response

Each of the examples illustrates a misapplication of the properties of exponents. Philip correctly calculates the coefficient but applies the Product of Powers to an addition problem and arrives at 10^6 instead of 10^3. He simply misapplied a property. In the second example, Clifton incorrectly calculates both the coefficient and the exponent. He multiplies the coefficients instead of adding them and adds the exponents.

A sample of student work from the exam illustrates a problem in which the students were asked to analyze an error. Figures 6.20 and 6.21 display common examples of incorrect student responses.

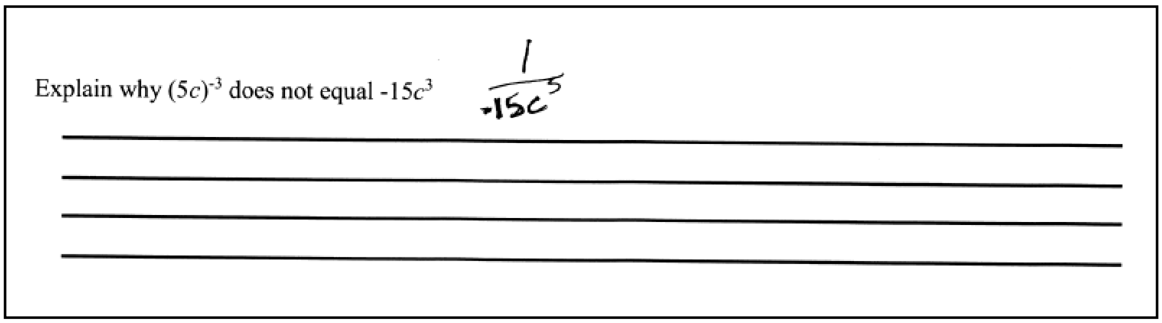


Figure 6.20 – Deng's response

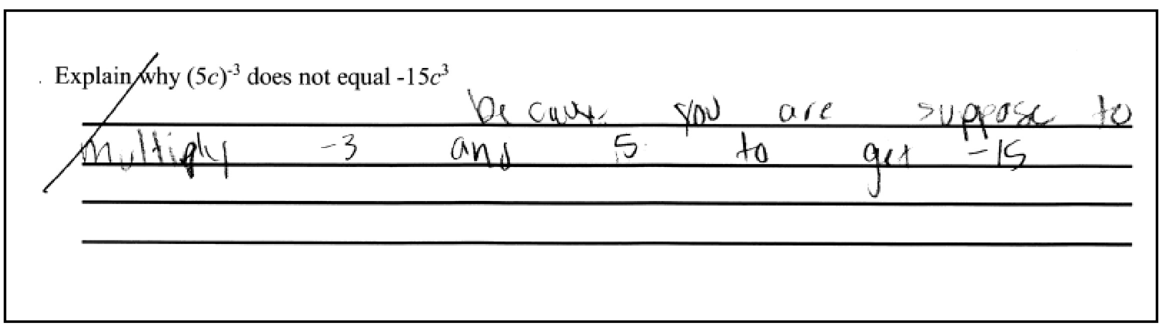


Figure 6.21– Kiera's response

The first example shows how Deng correctly applies the exponent to the variable but not to the coefficient. In the second, Kiera is unable to determine her error in the problem.

Reflections on Students Confronting the Delineation of the Algebra of Exponents from the Algebra of Real Numbers

As I explored this data, it became clear to me that a single issue – the students' confusion between the computation properties of real numbers and exponents – originated in at least fifteen places in students' individual and common experiences. Some students, for example Jessi and Chloe (Figure 6.12 and 6.13), fell into a habit of overgeneralization. They would learn a new rule and want to apply it to everything. They seemed hurried to complete the task, which hindered an opportunity to compare the two different algebraic systems with which they were interacting.

Jordan would try hard, but struggled when presented with new content. I expected him to develop more mathematically. Instead, he continued to be frustrated when dealing with multiple ideas at the same time. He saw a conflict between the two different algebras and could not reconcile them. I have a note about his comment in my journal on January 18. "Do you want me to add or to multiply? Just tell me which one!" It revealed to me the frustration that he must feel every day.

Savannah seemed to be the type of student that I hear every teacher has, who gradually withdrew from the course. She was comfortable early in the year when we worked with topics she was comfortable with. When her knowledge from Pre-Algebra expired, she struggled with classroom activities and stopped regularly participating in discussions. Regardless of emails, phone calls home, and arranged support with other team math teachers, I could not find a way to help her. She was resistant throughout.

Learning from student misconceptions. The first two features of this misconception illustrate the importance of knowing the relationship between mathematical content and the learning tasks and pedagogical responses I had. When I was working with students who were able to overcome misconceptions, I made it a habit to ask the question, "What does this mean?" I put the emphasis on how the students made sense and understood the mathematics at the conceptual level. Whenever I identified a misconception, it was always my goal to dig deeper into the mathematics. When a student made a computational mistake on a Power of a Product, I would write out the expression as a series of products and try to use the Product of Powers property. If that did not work, I would write out each power as a series of multiplications and return to the inductive approach. Most of my students did not employ this technique independently, and I should have encouraged students who struggled to employ it.

These issues also highlight the need for productive struggle in a classroom. No students demonstrated perfect understanding of this idea the first time or even the second. Instead, after covering the ideas of Product of Powers and Powers of Products, we worked back through them three additional times. Each encounter allowed me to give the students more time to make mistakes in a low stakes environment where they could get feedback from their peers and myself.

Earlier in the chapter, I shared an interaction with Garrett, Isabel, Joshua and Jessi. In this discussion the group observed the property and doubted its validity. I discussed a second example and then left them to work through the third. They made a mathematical conclusion, but still needed additional examples to convince themselves it was true.

Student Difficulties with Negative Exponents

After working through properties of exponents, another concept loomed on the horizon, negative exponents. It is a source of frustration when I teach high school and college math courses that students come to my classroom without understanding what is implied by the notation

$$f^{-1}.$$

The concept is often taught incompletely at its introduction in first-year Algebra where students understand it as changing the position of the base in a fraction. This happens again when students are exposed to it in second-year Algebra and Precalculus courses with different conceptual meanings.

For example, in a second-year Algebra course when students study matrices, the symbol

$$A^{-1}$$

implies the multiplicative inverse of matrix A. In Precalculus,

$$\sin^{-1} x$$

is equivalent to the

$$\arcsin x$$

or a function that will output an angle between 0 and π radians for a given value between -1 and 1. Neither of these are the same as students' first introduction to negative exponents, which occurs during first-year Algebra.

This is also a concept that I treated casually in my early years of teaching by simply telling students to switch the position of the variable in the fraction. This is an expiring rule in mathematics, or a rule that is true only in a limited sense and then becomes invalid in later math courses. I do not think I fully appreciated my own role in undermining students' understanding and long-term learning in exchange for short term gains. This authentic concept is really about inverses.

Inverses are significant throughout first-year Algebra. They are primarily used in solving and manipulating equations. When students learn to solve two step equations of the form

$$ax + b = c,$$

they are generally taught to first subtract b from both sides and then divide both sides by a.

Solving equations is so common place and straight forward that its complexity can easily be overlooked. Solving the previous equations requires the use of two different inverses, each of which can be denoted by the "-1" exponent. When b is subtracted from both sides, this is equivalent to adding the additive inverse of b, or b^{-1}, to both sides. Similarly, dividing both sides by a is equivalent to multiplying both sides by the multiplicative inverse of a, or a^{-1}.

Although this seems simple, it presents challenges for students during the introduction of inverse functions when a function (f) is given, and the inverse of the function (f^{-1}) is required to be found so that

$$f \circ f^{-1} = f^{-1} \circ f = x.$$

This idea is connected to the prior examples through the concepts of operation, identity, and inverse. In addition, the identity is zero and the inverse is $-b$. With multiplication the identity is 1 and the inverse is $\frac{1}{b}$.

Most operations that students are exposed to in mathematics courses are binary operations, or operations that act on two elements and result in a single element. Some operations have an identity element, or an element that when used with any other element along with the operation yields the non-identity element regardless of the order of the elements within the operation. For example, using addition and the set of real numbers, the identity is zero because for a real number a,

$$a + 0 = 0 + a = a.$$

This is not true for all real numbers in subtraction because

$$a - 0 \neq 0 - a.$$

An inverse is an element denoted a^{-1}, that is unique to any given element a, in context of a given operation such that a and a^{-1} yield the identity for the operation. Using addition as the operation,

$$a + a^{-1} = a^{-1} + a = 0.$$

With multiplication as the operation,

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

In first-year Algebra, we note a^{-1} as $-a$ with addition, and a^{-1} as $\frac{1}{a}$ with multiplication.

Students can remain unaware of these mathematical intricacies when they are forced to transform their conceptualization of an idea during upper level math courses from a specific instance to a more general concept. This creates the same set of problems as any expiring rule. Students can move quite logically from thinking they understand an idea to believing they do not understand. They question the validity of the content as well as teacher competency.

On the other hand, how did this affect my teaching? I found myself trapped down a proverbial rabbit hole. On its surface, this is a simple idea to express to students. *To make a negative exponent positive in an expression, simply change the variable's position from numerator to denominator or vice-versa.* I could tell students this, have them practice it a few dozen times, and move on knowing that my students could correctly answer the few questions on our unit exam that they are asked to simplify.

On the other hand, I was left without a way to introduce this idea conceptually. The lesson on negative exponents was scheduled for January 18 and 20. I wrote in my journal prior to the lesson on January 11, "Talked with the Physical Science teacher today about how and when the students learn scientific notation. He emphasizes to the students that negative exponents represent really small numbers. I need to make use of this piece of prior knowledge." I was searching for places that my students might have encountered the idea so I could access their prior knowledge.

When I had ideas about lessons, I used my journal as a place to record them. My note on January 12 is an example of this. "Don't forget about using the pattern of powers to help the kids understand negative exponents." I remembered a lesson I had given and wanted to try it in a new way. I made use of both of these strategies in presenting the idea to my students.

Introducing Negative Exponents

I planned this lesson using a Think – Notice – Wonder framework. Students were given a few minutes to write down two things they thought about the pattern, two things they noticed about the pattern, and two things they wondered about the pattern. On January 18, I began the lesson with the prompt shown in Figure 6.22:

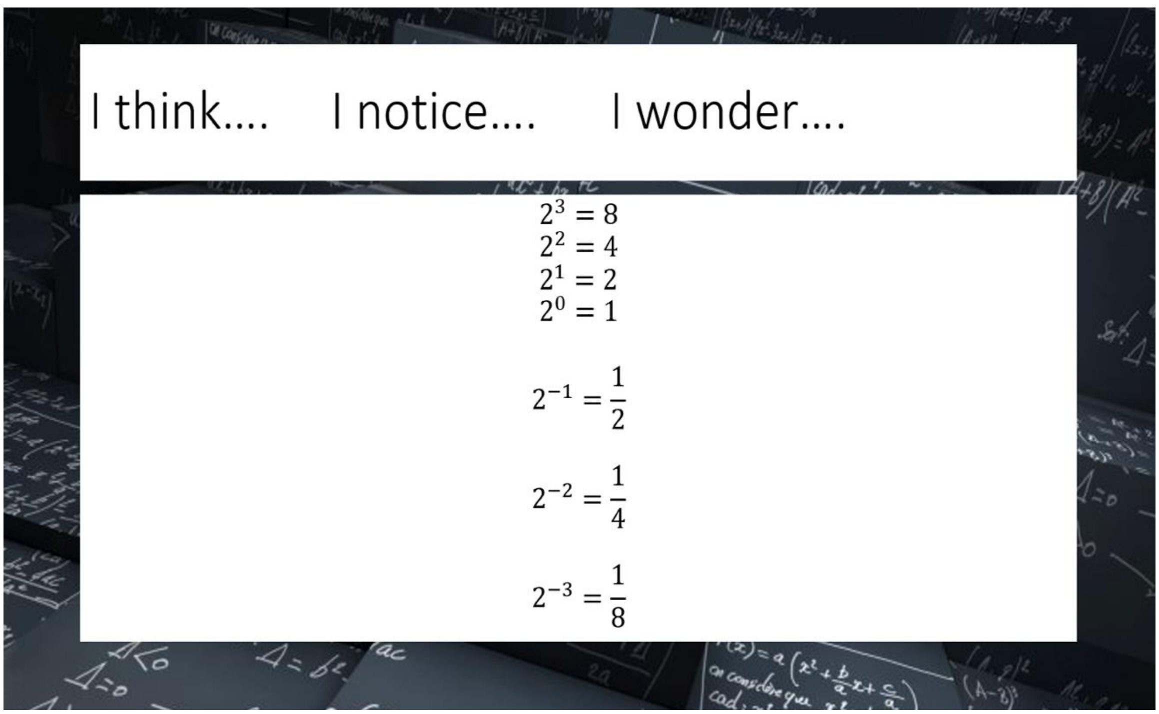


Figure 6.22 – Lesson opening prompt

I chose this activity to allow my students to observe and make inquiries about the mathematics in front of them, i.e. the power increase as you multiply by 2. This was a shift in my instruction from having the students do mathematics to having them make observations. Prior to this, I probably would have written this list on the board and told them what was important. I found this technique helpful when introducing new ideas when the students lacked any prior knowledge to which I could directly connect. This permitted me to listen to how students made sense of a mathematical idea during their first encounter with it.

Students were sitting in pairs, and I gave them a few minutes to make observations. While I did not give them any specific items to focus on, I hoped they would notice different qualities of the equivalent relationships that involved the negative exponents. This was also another opportunity to make use of the recursive relationships of exponents they had utilized earlier in the chapter. The following dialogue ensued and represents how students made sense of negative exponents.

1	Greg Sand:	What do you notice about the numbers as you move up or down the list?
2	Jordan:	They double when you go up, oh my goodness.
3	Philip:	Or they double negatively when you go down.
4	GS:	How do you double negatively?
5	Philip:	Fractions!
6	GS:	I'm still not sure what you mean by double negatively, hmmmmm. Let's come together as a large group and share what we've noticed. Jordan, what did you mean by doubling?
7	Jordan:	The numbers double as you go up the list.
8	GS:	Let me say this back to you, as you move up the list, the powers increase by 1 and you double the value. As you move down the list, what happens?
9	Philip:	You double it negatively.
10	Haley:	Mr. Sand, are you dividing by 2?
11	GS:	Philip, is that what you meant?
12	Philip:	Oh, yeah, that's right.
13	GS:	So, let me write this down. When you move up a power, you double the value and when you move down a power you half it. What else does someone notice?
14	Jessi:	All of the numbers are either even or 1.
15	GS:	Okay, anything else?
16	Madison:	Any of the powers that have a negative value are fractions.

This conversation revealed several challenges to me. Early on Philip (line 3) attempted to verbalize his understanding of negative exponents when he stated, "[Or] they double negatively when you go down." He stated that again in line 9. I wanted to honor Philip's idea, but much like Garrett's conclusion about Product of Powers, he lacked the precise language to express the idea. Haley (line 10) offered a more mathematically accurate observation that Philip (line 12) agreed with and better explained his thinking. These moments in classroom conversations are critical because the students are building upon each other's ideas, refining conclusions and understandings, and developing academic habits that extend beyond this class.

At the end of this portion of dialogue, Madison noticed that all negative exponents resulted in fractions. This is the idea that I wanted to bring to the students' attention. Negative exponents can be easily viewed as another rule to remember, but instead of casually presenting it as an impersonal idea, I attempted to create an opportunity for my students to make the conclusion on their own. Student-made conclusions can shift the way students view learning mathematics from something someone else has done into something that they are capable of doing.

Later examples and generalizations. At this point of the lesson, the students and their partners worked through the following two examples, shown in Figure 6.23, in an attempt to use what they had noticed during the opening example.

$3^3 = 27$	$a^3 = a^3$
$3^2 = 9$	$a^2 = a^2$
$3^1 =$	$a^1 =$
$3^0 =$	$a^0 = \quad , a \neq 0$
$3^{-1} = \frac{\square}{\square}$	$a^{-1} = \frac{\square}{\square}$
$3^{-2} = \frac{\square}{\square}$	$a^{-2} = \frac{\square}{\square}$
$3^{-3} = \frac{\square}{\square}$	$a^{-3} = \frac{\square}{\square}$

Figure 6.23 – Student examples

Next, they attempted the conjecture shown in Figure 6.24 to see how they generalized the new knowledge.

Conclusions

$a^0 =$ __________ as long $a \neq 0$.

$a^{-p} =$ __________ as long $a \neq 0$

Figure 6.24 – Conjecture to complete

When the groups had completed their conclusions, I brought the class together as a large group and asked for volunteers to share what they wrote. Haley and Jessi shared their thoughts.

1	Greg Sand:	It seems like all of the groups have finished. Let's come together as a large group and share what you came up with.
2	Haley:	a^0 is one.
3	GS:	Okay, let me write that down. Does anyone need help making more sense of this idea?
	Jordan:	Can a be any number?
4	GS:	Great question. Can anyone think of a number that wouldn't work with our pattern?

5	Isabel:	Zero.
6	GS:	Why?
7	Isabel:	It says it on the paper.
8	GS:	True. Let's try and make more sense. Jordan, when we lowered the power by one, what happened to the numbers on the right side?
9	Jordan:	You divide. Oh, you can't divide by zero.
10	GS:	Awesome. Anyone need more on that idea? Okay, what about the second conclusion?
11	Jessi:	It's one over a^p
12	GS:	Let me write that down. Any clarification here?
13	Jordan:	And you can't do that with zero because you can't have zero in the bottom.
14	GS:	Agreed.
15	Jordan:	I think I've got this.

The students had demonstrated that they were adept at making conclusions that reflected their lack of experience and formal mathematical language. However, the conclusions the students made here were better than I expected. Haley (line 2) and Jessi (line 11) both shared mathematically correct conclusions based on their experiences during class. Jordan was able to make sense out of these two ideas in a dialogue that was public to the entire class. Although I am sure other students had similar questions, having a student like Jordan, who was willing to take the risk of making his struggle public, allowed less willing students to have similar doubts addressed.

My hope is that this type of classroom synergy can be a positive and constructive force in improving student learning. Haley and Jessi were able to show their growth in expressing their conclusions from earlier in the chapter. Jordan's understanding of the topic was improved by his peers' ideas (rather than mine). Moments like this in class illustrate that students can do rigorous mathematical work and can improve their ability to make and communicate conclusions. It furthermore shows that my role can be that of facilitator of those ideas without having to simply tell them what to do; something that I hoped to enact with students.

I had high hopes moving forward that the students would be able to understand and apply this concept. But by the next day, it became clear that the students still harbored misunderstandings regarding negative exponents.

During the second day of the lesson, the students worked through three examples and then were assigned a set of eight problems. Their efforts on the three problems foreshadowed the challenges to come. Figure 6.25 shows the three problems they were given.

Examples

Simplify each of the following expressions by writing them without negative exponents.

$\frac{1}{r^{-3}}$ $\qquad\qquad$ $\frac{x^2}{y^{-3}}$ $\qquad\qquad$ $\frac{g^{-2}h^3}{j^5k^{-4}}$

Figure 6.25 – Student practice problems

Upon reviewing the video of that lesson, nearly all of the students were able to complete the first problem successfully. However, an interaction with Deng and the conclusion that he made forced me to pause and reconsider that there might be fundamental conceptual misunderstandings about the notation used to express exponential relationships and how it is manipulated.

1	Greg Sand:	Can you explain to me how you got your answer to the first problem?
2	Deng:	Since it's a negative exponent, I just moved it to the top.
3	GS:	What about the third problem?

4	Deng:	I moved the -2 from the top to the bottom and the -4 from the bottom to the top.
5	GS:	What about the variable? Why did you just move the exponent?
6	Deng:	Well, the rule is that if the power is negative then you switch the position in the fraction.
7	GS:	The position of what?
8	Deng:	Oh, I just moved the exponent, but not the variable. They go together, right?
9	GS:	Yes, you need to remember that the exponent and the variable are paired up, you can't separate them like that.
10	Deng:	I'll try and remember.

As Deng and I discussed how he simplified the different expressions, his response to me (line 4) uncovered a misunderstanding. When he stated that he "…moved the -2 from the top to the bottom and the -4 from the bottom to the top," I noticed that he only mentioned the exponents and not the variables. This indicated that he did not view each expression as a whole that expressed an idea and needed to be kept as a coherent unit. Instead, he viewed the exponent separate from the variable (line 8), but realized his error when I restated it to him (line 9). This interaction revealed another issue that would persist throughout the chapter. My students struggled to understand negative exponents when simplifying expressions. Their struggles manifested in a number of forms.

Examining student work. In addition to this conversation, I also made notes in my journal about these issues. On January 23, students completed a short, ungraded check for understanding where they attempted to complete six problems. Student results on problem number 6, Figure 6.26, caught my attention due to the errors they made.

$$6.\ m^3n^{-2}p^{-4}$$

Figure 6.26 – Student practice problem

Writing on January 23, "About half of the students moved the negative exponents to the denominator but left the exponents negative. This should be an easy point to reteach." This appeared to be a normal issue of teaching with an easy correction. However, it became an ongoing issue.

I observed this error again on our chapter quiz on two different questions. Question 7 on our chapter quiz instructed students to,

> Simplify the following expressions. Your final solution should not include negative exponents.
>
> $$x^{-2}$$
>
> The expected response was
>
> $$\frac{1}{x^2}.$$

Two samples of student work illustrate the common errors. In Figure 6.27, Clifton correctly positioned the variable in the denominator, but failed to write the exponent as a positive value. Six of the eighteen quizzes contained this error.

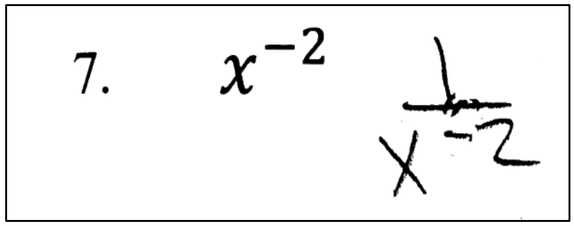


Figure 6.27 – Clifton's response

In Figure 6.28, Seamus wrote the expression without a negative exponent. Four of the eighteen students responded in this way.

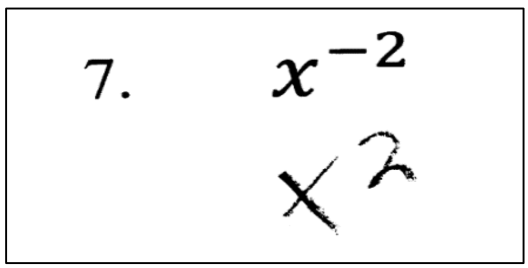


Figure 6.28- Seamus' response

The next problem on the quiz had the same instructions

$$\frac{a^{-9}}{b^{-7}}\ .$$

The expected response was

$$\frac{b^{7}}{a^{9}}\ .$$

Figure 6.29, Savannah's response, shows her rewriting the exponential expressions as binomials. Three different students made this same error which made me

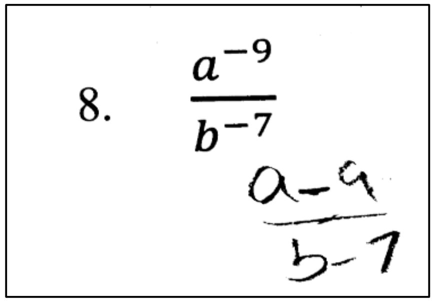


Figure 6.29 – Savannah's response

question what type of understanding of exponents they had developed.

Chloe wrote (Figure 6.30) the expression as two different expressions with no operation indicated between them. Two different students made this error.

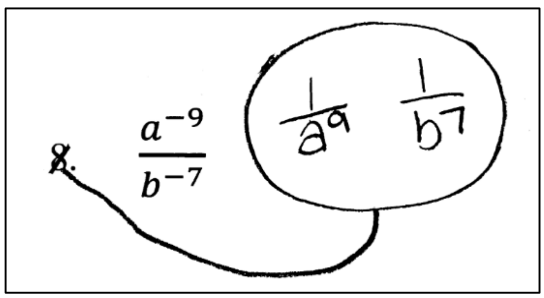


Figure 6.30 – Chloe's response

Figure 6.31 shows Deng's work. First, he did not correctly simplify the expression by combining the exponents into a single exponent. Second, he failed to write the expression without a negative exponent. Two different students made these same mistakes.

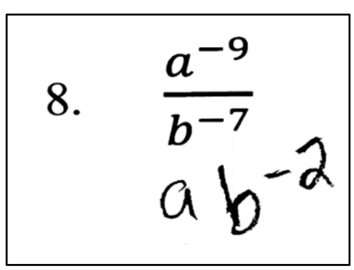


Figure 6.31 – Deng's response

The following two samples illustrate how the students properly applied the properties of exponents except for failing to simplify the negative exponents. Each problem was preceded by the instruction,

Simplify each expression. Final answers cannot include negative exponents.

Problem 14 read

$$4x^0y^{-2} \cdot 4y^{-2}$$

with an expected solution of

$$\frac{16}{y^4}.$$

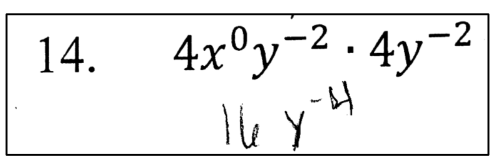
Problem 16 read

$$(2u^{-3}v^{-2})^4$$

with an expected solution of

$$\frac{16}{u^{12}v^8}.$$

In Figure 6.32, Evan correctly calculated the coefficient and applied both the zero exponent and the Product of Powers Property correctly. However, he did not correctly simplify the expression so that it did not have negative exponents.


14. $4x^0y^{-2} \cdot 4y^{-2}$

$16y^{-4}$

Figure 6.32 – Evan's response

Kiera's work, shown in Figure 6.33, contains a similar error to the work shown in Figure 6.31. In this sample, she applied the Power of a Product rule correctly to the coefficient and variables but failed to correctly simplify the expression.

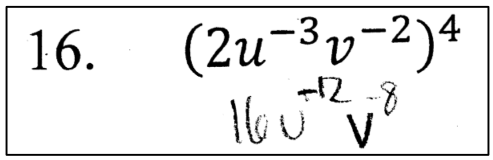


Figure 6.33 – Kiera's response

Figure 6.34 illustrates a misunderstanding that I first noticed during the second day of my lesson on negative exponents (see the transcript after Figure 6.24). Here,

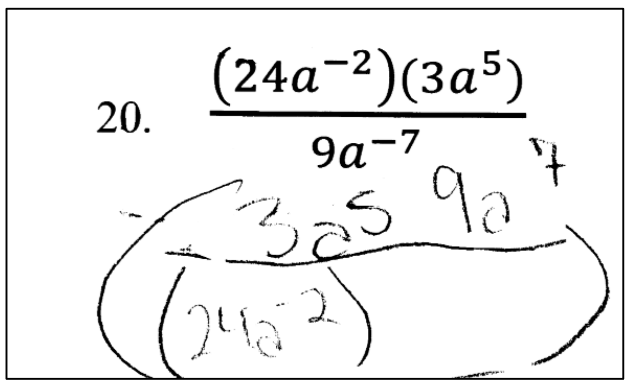
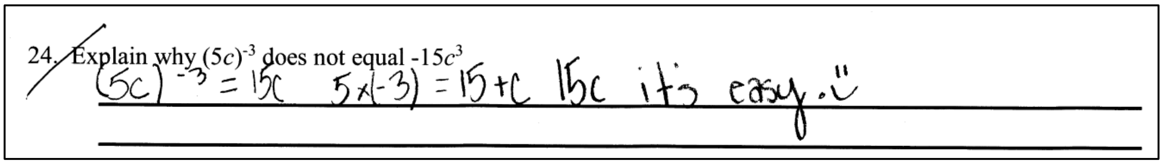


Figure 6.34 – Chloe's response

Chloe rearranged the expression to remove the negative exponents. She moved both the coefficient and variable into a different position in the fraction instead of only the variable.

The final set of student work was referenced earlier in this chapter (see Figure 6.15). These five samples are shown in Figures 6.35 to 6.37. The sample in Figure 6.35

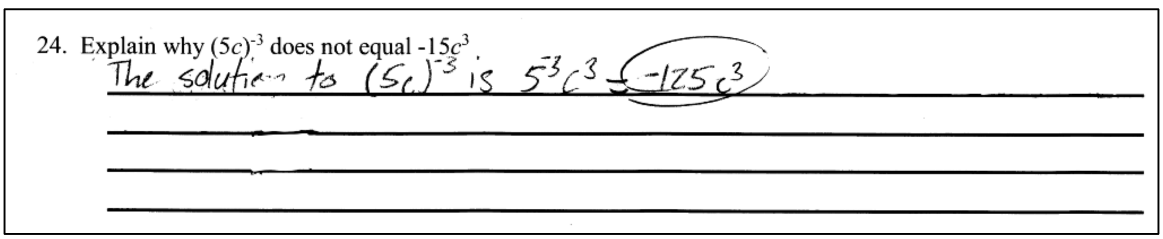
shows how Savannah treated the exponent similar to a coefficient. She also displayed a misunderstanding of variables and expressions.


24. Explain why $(5c)^{-3}$ does not equal $-15c^3$

$(5c)^{-3} = 15c$ $5\times(-3) = 15+c$ 15c it's easy.

Figure 6.35 – Savannah's response

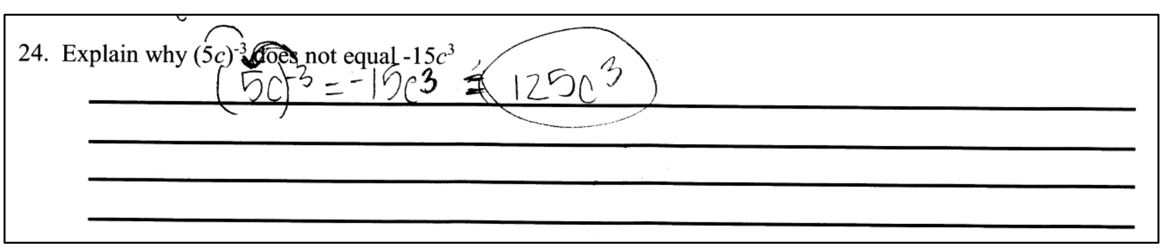
Joshua's work in Figure 6.36 shows a student who partially understands the Power of Products Property. He does not properly distribute the -3 to both terms (the variable is raised to the third power, not -3) and incorrectly simplifies 5^{-3} as -125 and not $\frac{1}{125}$.


24. Explain why $(5c)^{-3}$ does not equal $-15c^3$

The solution to $(5c)^{-3}$ is $5^{-3}c^3 = -125c^3$

Figure 6.36 – Joshua's response

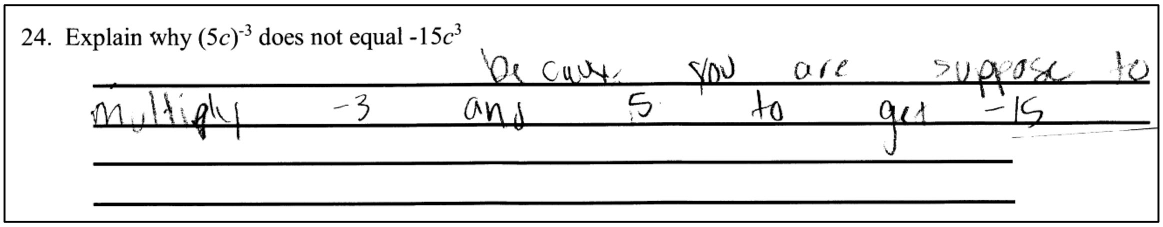
Aisha's work, shown in Figure 6.37, demonstrates two different attempted solutions to the problem. One solution is a recopy of the original, while the second shows a lack of acknowledgement of the negative exponent.


24. Explain why $(5c)^{-3}$ does not equal $-15c^3$

$(5c)^3 = -15c^3 \neq 125c^3$

Figure 6.37 – Aisha's response

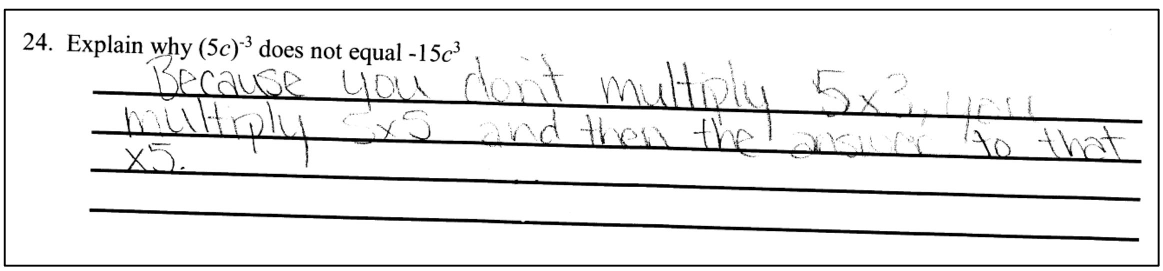
In Figure 6.38, Isabel expresses a misunderstanding of the definition of exponents. She multiplied the exponent and its coefficient.


24. Explain why $(5c)^{-3}$ does not equal $-15c^3$

because you are suppose to multiply -3 and 5 to get -15

Figure 6.38 – Isabel's response

Figure 6.39 illustrates how Madison correctly understands the meaning of exponents. However, she does not address the effect of the negative sign on the exponent.


24. Explain why $(5c)^{-3}$ does not equal $-15c^3$

Because you don't multiply 5x3, you multiply 5x5 and then the answer to that x5.

Figure 6.39 – Madison's response

Reflections on Student Difficulties with Negative Exponents

Each of these interactions, journal notes, and samples of student work support my observation that students struggled to understand negative exponents. The potential sources of these problems are as varied as the misconceptions, including foundational misunderstandings and over-generalization.

Types of Misunderstandings

The first type of misunderstanding that I noticed was illustrated in the dialogue immediately after Figure 6.19. In this conversation, the student moved the exponent instead of the variable and the exponent. Figures 6.23, 6.25, 6.28, 6.29, and 6.31 each present a different version of this error. Although the individual errors are unique, each comes from the same foundational misunderstanding about exponents.

These types of errors can be especially frustrating for an algebra teacher because while there is always an expected amount of reteaching of prior material during a course, there is also an expectation of a certain level of student knowledge and understanding. This occurs throughout math education and is often the source of frustration for both teachers and students. When I work with freshman and sophomore second-year Algebra students at the high school level, they are often shocked that they are expected to remember what they learned in middle school. This is similarly true in Precalculus and Calculus courses.

I often have conversations with my junior and senior International Baccalaureate students about the habits that they formed in seventh through tenth grades that led to this. A common admission from them is, "Yeah, I just memorized what I needed to know for the quiz or test and then promptly forgot it." How do I get them to overcome these habits? My primary method is to focus on the thinking and reasoning skills that allow students to make mathematically valid conclusions instead of just memorizing facts.

This mindset creates tension for me because my curriculum department requires that teachers post well defined learning goals. My learning goals with students focus more on how they learn than what they learn. This allows me a different way to answer the age-old question, "When am I ever going to use this?" Instead of making a focus on skills, I make an effort to focus on mathematical reasoning and critical thinking. These should benefit any student.

Over-generalization. The error made in Figure 6.24 shows an example of a student who is overgeneralizing her first understanding of negative exponents. While I mentioned this type of mistake in a prior portion of this chapter, it is something to be aware of as a mathematics teacher. While this particular mistake is easy to catch, there are many more that are unintentionally propagated by teachers. These are often referred to as expiring rules in mathematics.

For example, in probability theory, if the probability of an event occurring is zero, it is often stated that it means it cannot happen. For example, if a real number is randomly chosen on the number line, the probability of that number being chosen is

$$\frac{1}{\infty} \approx 0.$$

Any given number can be chosen. The probability that a particular value is selected is 0. However, any particular value can be chosen. Thus, when an event has a probability of 0, it means it either cannot happen or is extremely unlikely to happen.

As this example demonstrates, a mathematics teacher must have a breadth of knowledge about the subject. Without that knowledge, I can unintentionally damage the long-term learning of my students by making or validating statements that are not true.

Incomplete simplification. The student errors in figures 6.24, 6.26, and 6.33 all illustrate a failure to fully simplify the expression. These are little mistakes that happen every day in teaching but force me to ask two questions. First, why do we simplify for the sake of simplifying? One of the goals of this type of problem is to prepare students for solving problems in future courses. These types of problems do not occur, however, until second-year Algebra. The immediate goal is to simplify exponential expressions which represents the essence of algebra. This is writing equivalent forms of expressions and equations. It is also a different mathematical structure from the algebra used to solve linear equations. Helping students realize that there is not just one type of algebra opens their thinking to a much larger world of possibilities.

The second question that I have always struggled with is how to score this type of problem. If students show that they can utilize some of the properties of exponents, then they have demonstrated some learning. This does not mean that they completely understand everything that is required of them. This means that assessing them is more complex than a simple right or wrong grade. While I do not have a good answer to this, it is something that I question when assigning a grade to a student that reflects his or her learning and not just how many problems on a test or quiz he or she correctly answered.

My Struggle Planning and Teaching an Algebra Topic that is both Authentic to the Mathematics and Appropriate for the Students

Throughout this chapter, I have tried to show not only what my students were learning, but also how I taught it. This type of teaching has the goal of building mathematical thinking and reasoning in the students over developing only algebraic skills. It also allows me to embrace the uncertainty inherent in teaching mathematics.

During the unit, I created activities with the goal of being both mathematically authentic and student-centered. Figures 6.2 and 6.3 show the first pair of activities. Students were asked to convert between exponential and expanded forms. The goal of this activity was to activate prior knowledge about exponents before important conclusions were made.

The activity shown in Figure 6.4 built off of this first activity to help students discover the Product of Powers Property. While this Property was one of the curricular goals of the unit, it was my hope to allow students the opportunity to utilize inductive logic to make mathematical conclusions. These types of activities are shown in figures 6.6 and 6.23. This allowed me to help students build their thinking skills through activities that followed a consistent theme.

Reflections on My Struggle Planning and Teaching an Algebra Topic that is both Authentic to the Mathematics and Appropriate for the Students

My experiences from this type of teaching have made me aware of some issues that I need to be mindful of in the future. First, the conclusion of any lesson becomes critical. When students are working in teams and making conclusions, it is impossible to ensure that every student will make valid conclusions from the lesson. At the end of each lesson, it was critical that I collected thoughts from students and then synthesized them into a clear, valid conclusion that students could understand.

I was also confronted with the issue of how deep into the mathematics to take them. As I referenced earlier, it is easy to create misunderstandings and expiring rules for students. In this unit I struggled with the idea of negative exponents. A constant, variable, or expression raised to the negative one power has different meaning based on the context of the operation unified by a deeper mathematical truth. How deep into that understanding can we take first-year Algebra students? How do we address these issues without overwhelming them? I do not have any definitive answer to these questions, but they need to be considered by any teacher of mathematics regardless of the level of teaching to prevent long-term student misunderstanding and potential sabotage of student success in upper level mathematics courses.

Conclusion

Throughout my examination of student work and my preparation in this chapter, I identified three student issues upon reflection:

1. Student issues with the delineation between the algebra of exponents and the algebra of real numbers
2. Student difficulties with negative exponents
3. My struggle planning and teaching an algebra topic that is both authentic to the mathematics and appropriate for the students

The two student issues have origins in fundamental understandings of equivalent forms, variables, and functions. My conclusions are based off of an examination of my planning journal, field notes, classroom video recordings, and student work. The manifestation of these problems is as diverse as the interventions required to correct these misconceptions. I continue this work in the next chapter by examining my teaching and the resulting student issues in our unit on Parabolas. Parabolas are modeled by quadratic functions, the first function that students deeply explore that involves an exponent.

CHAPTER 7

PARABOLAS: ONE FINAL JOURNEY

Knowing is a process, not a product.

-Jerome Bruner

In Chapters 5 and 6, I explored issues surrounding student learning and teaching of inequalities and exponents, examining and reflecting on data that I had collected throughout the year. These issues include student understanding of equivalent forms and the equal sign, as well as variables and functions as reflected in modern literature in the teaching of first-year Algebra. In this chapter I continue this work by examining the final unit of the year, a numeric, graphic, and analytic exploration of parabolas. This unit presented students with the opportunity to connect the results of computational exercises to algebraic manipulation. In this chapter, I explore the following topics:

1. Student confusion with transformations of a parabola,
2. their difficulties interpreting notation, and
3. my struggle scaffolding activities to support students in overcoming the confusion and challenges of interpretation.

Introducing the Unit

With six weeks of school left, I found myself facing a dilemma every educator faces at the end of the school year, too much to teach and not enough time. According to our district pacing guide, I had one unit over parabolas and another on probability and statistics to cover with my students. I was confronted with a choice common in math education, exposure to content versus developing a depth of knowledge.

On the one hand, it is hard to argue against the importance of probability and statistics in today's society (Sowey & Petocz, 2017). It is critical that students learn how to understand and interpret the statistical data that fills our lives, and I felt leaving this topic untaught would be a disservice to my students.

On the other hand, parabolas are a topic that become important as students move forward in both mathematics and science. For most of my students, Geometry is the next course they take. Because of my prior experience teaching Geometry, I knew that they needed to be comfortable solving equations in the form of $ax^2 = b$ and $ax^2 + bc + c = 0$, in order to solve, for example, problems involving the Pythagorean Theorem and congruent and similar figures. Physical sciences make regular use of quadratic functions when studying phenomena like motion and electricity.

Prior to this unit, we had studied polynomials and learned to factor quadratics during Unit 8 from March 2 to April 5. My assessment informed me that students could use more exposure and practice with these techniques to prepare them for topics in later courses. Additionally, the unit contained rich algebraic topics that act as a preview of second-year Algebra, e.g., how transformations of parent graphs are demonstrated analytically.

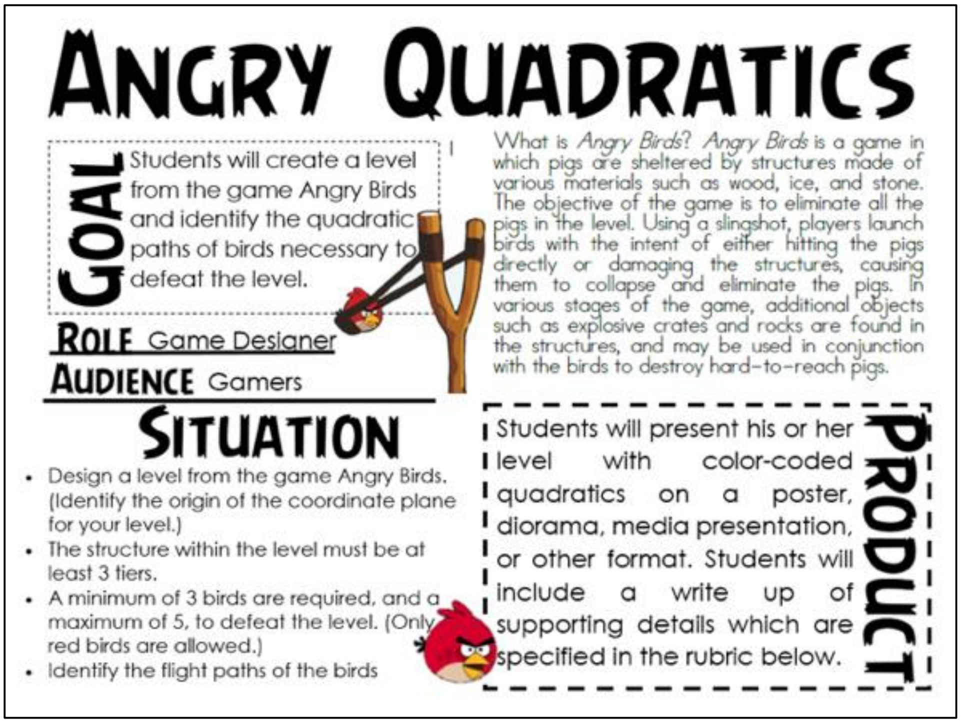
Phil and I spent a week debating the best possible choice. We knew that six weeks would allow us to develop a deep and rich understanding of one of the topics. The decision was made easier as Phil and I reflected on one portion of the year, our weekly work preparing the students for the state math exam. Every week during our block day, we spent part of the class period working through four to six exam preparation questions. The majority of the problem sets contained questions that are covered in the Probability and Statistics unit. Deciding that we had provided sufficient practice with these concepts, we moved forward with studying the parabolas unit. This choice was later supported when we realized that we were going to lose at least four days to district and state mandated assessments that were given during class time.

Understanding the Curriculum

With that issue resolved, I reflected back on prior unit planning to determine the best way to cover the necessary material. Utilizing a technique that I had made use of throughout the year, I focused on concepts occurring at the end of the unit to decide how to plan it. I knew that I wanted to do a project with the kids, and I was fortunate that another teacher in the building had a perfect project for the unit. The students were going to design a level of the popular mobile game *Angry Birds*. In this game, the player launches birds at structures in each level in an attempt to knock them down. The birds' flight roughly follows a parabolic path. In addition to designing the level, the students would present the equations of the parabolas that would allow the player to win the level they designed using three to five birds.

Unit Project

This project required the students to become comfortable with both the algebra and geometry of parabolas. I handed students the project outline shown in Figures 7.1 and 7.2 on April 17. First, the students needed to design a level anticipating how one bird



ANGRY QUADRATICS

GOAL
Students will create a level from the game Angry Birds and identify the quadratic paths of birds necessary to defeat the level.

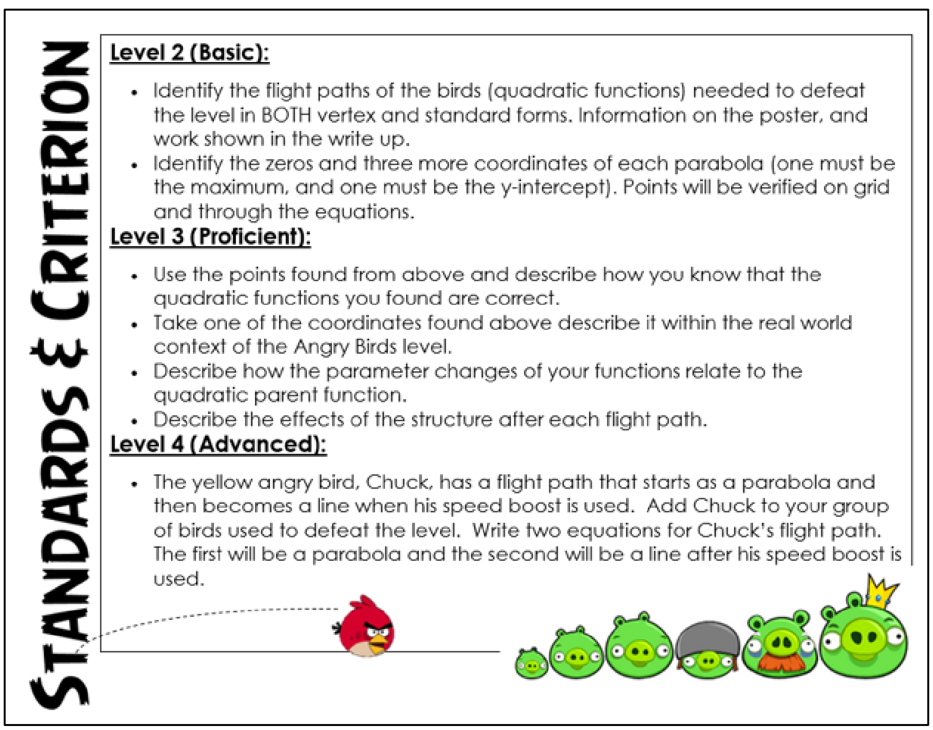
ROLE Game Designer
AUDIENCE Gamers

What is *Angry Birds*? *Angry Birds* is a game in which pigs are sheltered by structures made of various materials such as wood, ice, and stone. The objective of the game is to eliminate all the pigs in the level. Using a slingshot, players launch birds with the intent of either hitting the pigs directly or damaging the structures, causing them to collapse and eliminate the pigs. In various stages of the game, additional objects such as explosive crates and rocks are found in the structures, and may be used in conjunction with the birds to destroy hard-to-reach pigs.

SITUATION
- Design a level from the game Angry Birds. (Identify the origin of the coordinate plane for your level.)
- The structure within the level must be at least 3 tiers.
- A minimum of 3 birds are required, and a maximum of 5, to defeat the level. (Only red birds are allowed.)
- Identify the flight paths of the birds

PRODUCT
Students will present his or her level with color-coded quadratics on a poster, diorama, media presentation, or other format. Students will include a write up of supporting details which are specified in the rubric below.

Figure 7.1 – Front of project handout



STANDARDS & CRITERION

Level 2 (Basic):
- Identify the flight paths of the birds (quadratic functions) needed to defeat the level in BOTH vertex and standard forms. Information on the poster, and work shown in the write up.
- Identify the zeros and three more coordinates of each parabola (one must be the maximum, and one must be the y-intercept). Points will be verified on grid and through the equations.

Level 3 (Proficient):
- Use the points found from above and describe how you know that the quadratic functions you found are correct.
- Take one of the coordinates found above describe it within the real world context of the Angry Birds level.
- Describe how the parameter changes of your functions relate to the quadratic parent function.
- Describe the effects of the structure after each flight path.

Level 4 (Advanced):
- The yellow angry bird, Chuck, has a flight path that starts as a parabola and then becomes a line when his speed boost is used. Add Chuck to your group of birds used to defeat the level. Write two equations for Chuck's flight path. The first will be a parabola and the second will be a line after his speed boost is used.

Figure 7.2 – Back of project handout

striking a structure would affect the rest of the structures. Once the potential moves were planned out, the students would then place their design on an *xy*-axis and determine the equations of each of their parabolas. To accomplish these tasks, students required knowledge of the game *Angry Birds*, the geometric qualities of parabolas, and different ways to algebraically represent them.

My best preparation to teach a topic is to explore and understand the foundational mathematics for a concept. In reviewing the many particular concepts associated with parabolas, e.g. vertex, zeros, etc., I was struck by the complexity and volume of information. These are concepts that I use every day in my high school classroom and am casual with their usage. Because I was introducing my middle school students to this idea, I decided to explore the foundations of parabolas to enrich my conceptual understanding.

The Mathematics of Parabolas

The mathematics of parabolas has its origins in conic sections. Conic sections are the family of curves formed by the intersection of a plane with two right cones that share a vertex and open in opposite directions. As a function, these cones are commonly modeled with the equation $x^2 + y^2 = z^2$. The three conic sections formed in this manner are the parabola (image 1 in Figure 7.3), the ellipse (image 2) and the hyperbola (image 3). Also shown in picture 2 is a circle, which is a specialized form of an ellipse.

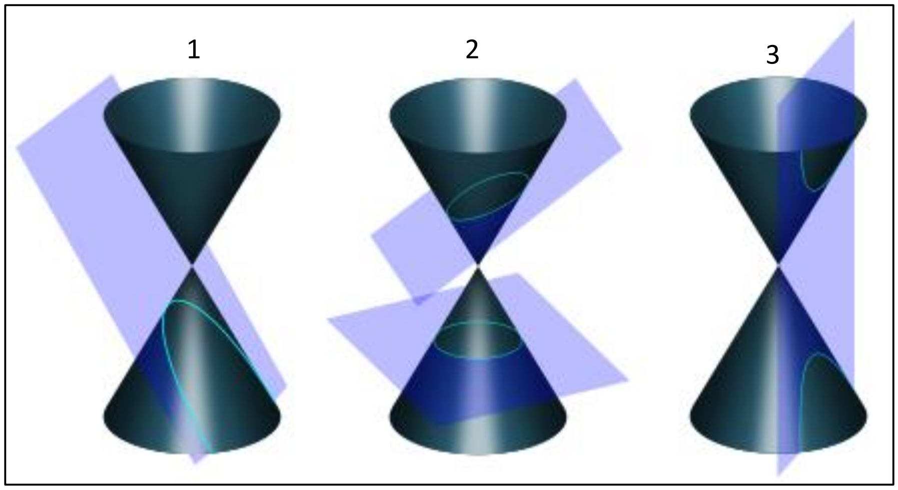


Figure 7.3 – The conic sections

Each of these curves are modeled in two dimensions and form the family of quadratics. Equivalently, a parabola is also defined as the locus of all points which are equidistant from a fixed point call the "focus" and a fixed line call the "directrix." It is from this definition that the algebraic form is derived as shown in the following proof.

Proof

Consider the following diagram (Figure 7.4) with point A being the focus and the line below A the directrix.

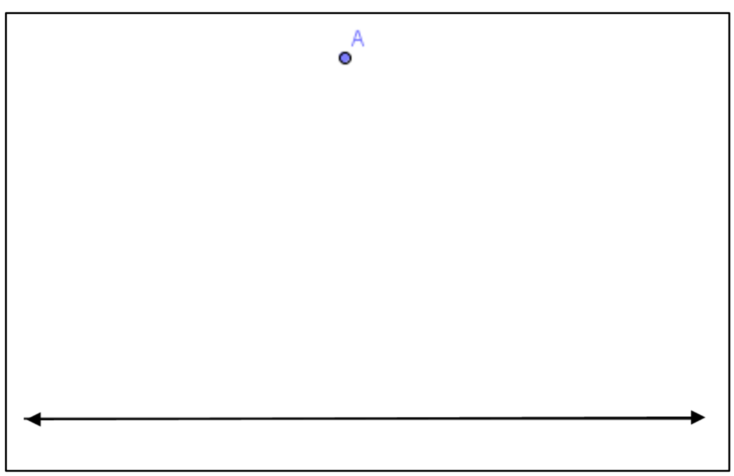


Figure 7.4 – Given information for the proof

Orienting these on the xy-axis as illustrated in Figure 7.5 allows for point A to be placed $(0, p)$ and the directrix defined as the line $y = -p$.

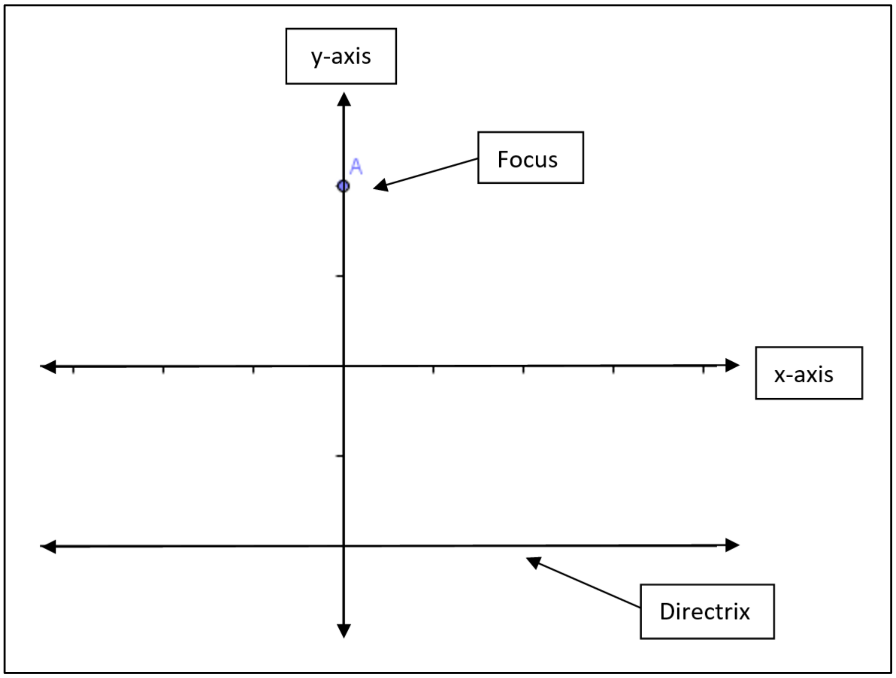


Figure 7.5 – Given information, oriented and labeled

The goal of the proof is to show that any point, (x, y), where $\{x, y \in \mathbb{R}\}$, that lies on the parabola is equidistant from $(0, p)$, and the line

$$y = -p$$

also satisfies the equation

$$y = \frac{px^2}{4}.$$

The choice of the placement of the axes allows for this specific case to be sufficient.

Conversely, it must be shown that a point satisfying the equation

$$y = \frac{px^2}{4}$$

is equidistant from $(0, p)$ and the line

$$y = -p.$$

For this proof, assume that $p > 0$ (The proof for $p < 0$ is similar and excluded).

Let the point (x, y) be equidistant from $(0, p)$ and

$$y = -p$$

as shown in Figure 7.6. Thus, the distance from

$$(x, y)$$

to the line

$$y = -p$$

is the perpendicular distance between the point and the line. This distance can be calculated in one dimension due to the orientation of the graph, and is

$$|y - (-p)| \text{ or } |y + p|.$$

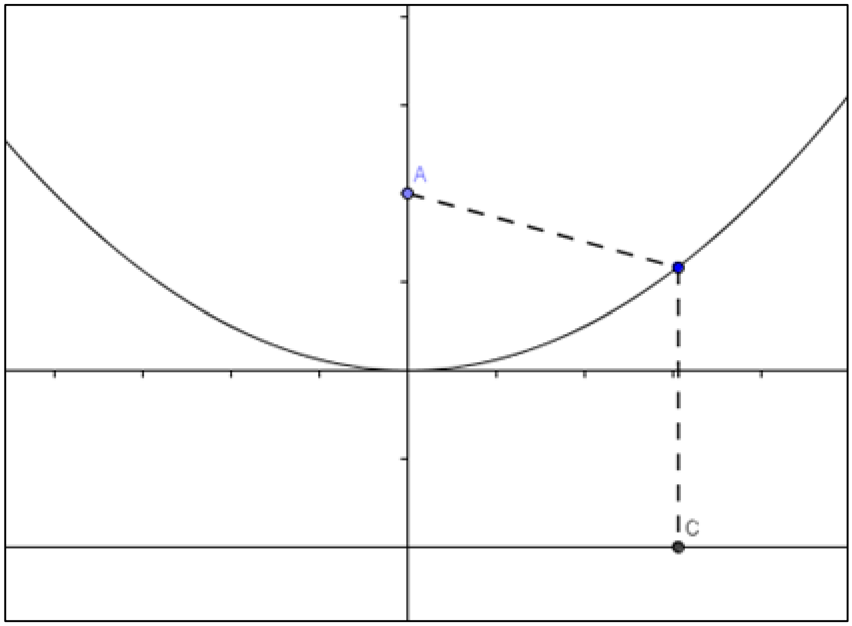


Figure 7.6 – Distances between focus, parabola and directrix

The distance from the points (x, y) to $(0, p)$ is the two-dimensional distance

$$\sqrt{(x-0)^2 + (y-p)^2}\ .$$

Due to the assumption that (x, y) is equidistant from $(0, p)$ and $y = -p$,

$$|y+p| = \sqrt{(x-0)^2 + (y-p)^2}\ .$$

Squaring both sides yields

$$(y+p)^2 = (x-0)^2 + (y-p)^2\,.$$

Expanding the squared quantities results in

$$y^2 + 2yp + p^2 = x^2 + y^2 - 2yp + p^2 \ .$$

Simplifying this equation yields

$$x^2 = 4yp$$

or

$$y = \frac{px^2}{4}.$$

By reversing the steps of this proof, the second half of the proof is completed. Thus, any point on a parabola that is equidistant from both the focus and directrix satisfies the equation

$$y = \frac{px^2}{4}.$$

Because we are only considering real values, for any point

$$(x_0, y_0)$$

that satisfies the equation

$$y = \frac{px^2}{4},$$

the point

$$(-x_0, y_0)$$

will also satisfy the equation. The only point that does not have this property is the point

$$(0, 0)$$

because zero is its own opposite value. This point is defined as the vertex of the parabola. Thus, there exists a line of symmetry for a parabola that passes through the vertex and focus and is perpendicular to the directrix. These are shown in the Figure 7.7.

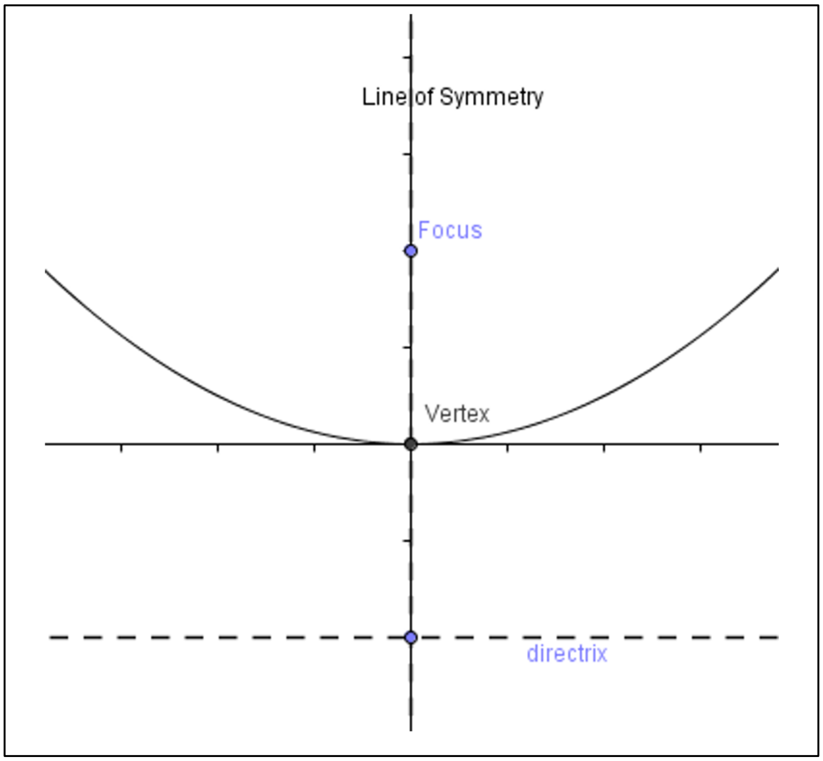


Figure 7.7 – Components of a parabola

The equation can be transformed from the form

$$y = \frac{px^2}{4}$$

into a more general form by applying function transformations. For any function

$$y = f(x),$$

the function

$$y = af(x)$$

is a vertical stretch of a, the function

$$y = f(x - b)$$

is a translation of the graph b spaces horizontally, and the function

$$y = f(x) + c$$

is a translation of the graph c spaces vertically. Thus, to move the vertex from the point

$$(0, 0)$$

to the point

$$(b, c)$$

and allowing for a vertical stretch of a, the equation

$$y = \frac{px^2}{4}$$

is transformed into

$$y = \frac{ap(x-b)^2}{4} + c.$$

∎

This transformed equation is commonly studied in three different but equivalent forms:

(1) Standard form: $y = ax^2 + bx + c, a \neq 0$

(2) x-Intercept form: $y = \alpha(x - x_1)(x - x_2), \alpha \neq 0$

(3) Vertex form: $4p(y-k)=(x-h)^2$

It is these three forms that my first-year Algebra students would be introduced to during this unit in preparation to complete this project.

The Importance of Parabolas

Parabolas are a topic in first-year Algebra with far-reaching implications. Typical topics explored during a study of parabolas include, but are not limited to,

1. Graphing parabolas in standard and vertex form.
2. Identifying critical elements of the graph (i.e. vertex, line of symmetry, etc.).
3. Using the discriminant to determine the number and type of zeros.
4. Factoring the parabola or using the Quadratic Formula to identify the zeros of the graph.
5. Transforming the parent graph based on changing parameters within the equation.

These concepts are utilized in math and science courses throughout high school and beyond. During Geometry, students solve quadratic equations in order to solve problems involving right triangles, similar figures, segment lengths of circles, and measurement problems involving plane and space figures. For my students, the procedures needed to solve these problems were first developed in the chapter immediately prior to this unit and then refined throughout this final chapter.

The use of these skills continues throughout high school mathematics. During second-year Algebra, students further explore conic sections, e.g. hyperbolas, ellipses, etc., model situations involving quadratic relationships, and solve polynomials requiring factoring which include quadratics as a subset. Precalculus continues this study with further emphasis on transformations of parent functions.

Parabolas are present in physics when they are used to model vertical, horizontal, and projectile motion problems. Vertical and horizontal motion problems use similar equations taking the form:

$$s(t) = -\frac{gt^2}{2} + v_0 t + s_0$$

where

$$g = \text{gravity},$$

$$v_0 = \text{initial velocity},$$

and

$$s_0 = \text{inital position}.$$

When working with projectile motion, a vector equation is used to model both vertical and horizontal motion simultaneously of the form:

$$r(t) = \langle |v_0| t \cos\theta, -\frac{gt^2}{2} + |v_0| t \sin\theta + s_0 \rangle$$

where

$$\theta = \text{angle of the projectile with the x} - \text{axis},$$

and all other variables are as defined earlier. Other physical phenomena, like electricity, are also modeled using quadratics.

While this unit on parabolas represented the end of my course, it also served as a transition point for my students from the type of thinking required in first-year Algebra to that of advanced mathematics present in the typical high school classroom.

Building Foundational Knowledge

To build this knowledge set with my students, I turned my attention to managing a number of dilemmas. I wrote four questions representing these dilemmas in my journal:

- How do I introduce my students to this mathematical topic in an authentic way that will motivate the large volume of new terminology that is necessary for them to demonstrate their knowledge?
- How do I help them keep a focus on the finely detailed procedure required by this topic?
- How do I ensure that all of my students understand the game, how it works, and how parabolas fit into it?
- How do I keep their attention focused on class and coursework during the final six weeks of school? (Journal, April 17)

The first choice I made was to start most days of class by giving the students an opportunity to play the game. I was fortunate to own a copy of the game we could play on the classroom computer as well as a smart board that would allow the students to stand at the front of the room and play along with their classmates. This stood in contrast to the normal way most of my students played games, isolated on their cell phones. There was the additional bonus that my algebra class was the first period of the day, so the game

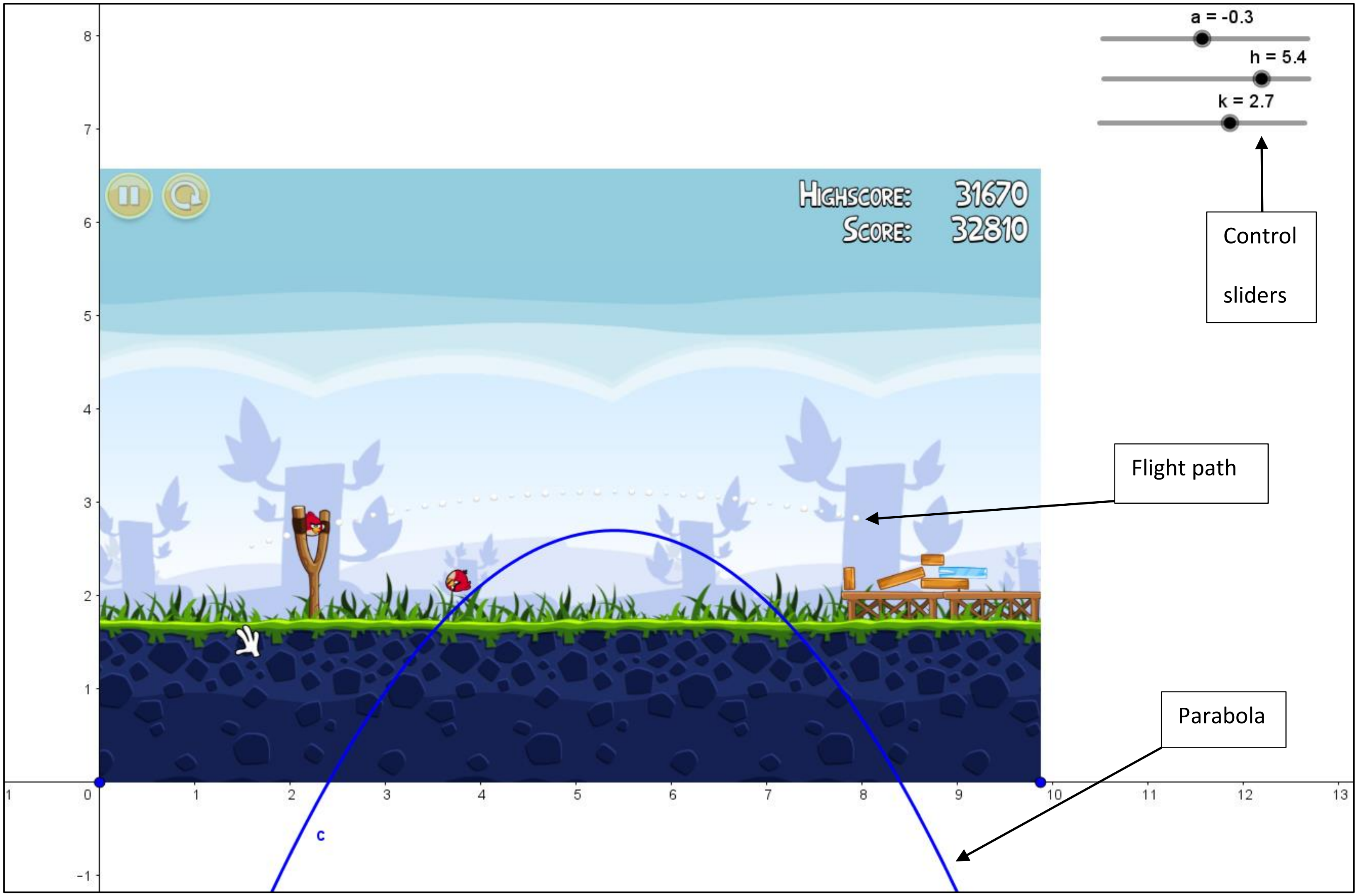


Figure 7.8 – A picture of the first level of Angry Birds inserted into Geogebra

was waiting for them as they entered the classroom. We usually had about 10 minutes of time to play before class started.

To prepare them for this, I designed the following lessons using a screen shot from *Angry Birds* inserted into a Geogebra file. Geogebra is a free, web-based math application widely used in math classrooms. Figure 7.8 is the first game level available to the player with the flight path of a bird shown as a white dotted line. The blue parabola was inputted in vertex form $y = a(x - h)^2 + k$ along with sliders that would allow the students to try and match the flight path. I did not mention to the students anything about the function, only the name of the curve.

On the first day of the unit, April 17, I used this activity to give the students an overview of the project they would be working on throughout the unit. Additionally, I knew that this unit would require developing a new vocabulary associated with the topic as well as the concepts connected to those terms. To that end, I wanted to develop the ideas prior to introducing the terminology, such as vertex and axis of symmetry.

Curve Matching

To get the activity started, I asked for a volunteer and had a student experiment with the sliders to try and match the curve. The conversation between the students focused on determining the effect changing a slider had on the curve.

1	Juan:	Here, I'll do it. Okay, let me try one. How about the top one?
2	Greg Sand:	What happened?
3	Savannah:	It made flat and then it went up.
4	GS:	When was it a line?
5	Juan:	Hang on, let me check. Zero!
6	Madison:	What do the other ones do?
7	Juan:	I'll try h.

8	Joshua:	That’s not right, it went sideways. You screwed it up.
9	GS:	But what did changing h do to the shape?
10	Juan	It moved it left and right. (moves h more) Is it in the right spot?
11	Savannah:	Too far; no, no, no; move it left.
12	Juan:	It’s fine. Let me try k.
13	Jordan:	It moves up and down. That’s weird.
14	GS:	What makes it weird?
15	Jordan:	I dunno, it’s one slider moves it side to side and the other up and down. It’s just weird.

This was the type of conversation I had hoped for in order to point out to the students this major concept in our final unit. “Dunno” and “Weird” here suggest that Jordan struggled making sense of the effect that altering two independent parameters was having on the graph of the parent function. As Juan varied the parameters (lines 1 and 7) the students made observations (lines 3, 8, 10, and 13) about changes to the graph. These observations contained mathematically valid conclusions. When Savannah (line 3) noted that the parabola was, “Flat and then it went up,” she was describing the effect that changing the lead coefficient has on the direction that the parabola opens.

Joshua noted (line 8), “It went sideways.” He observed that changing the parameter within the functions shifted its position horizontally. Juan made a similar observation (line 10). Jordan noticed (line 13) that, “It moves up and down,” or, when he changed the parameter added to the function, the positions of the graph shifted vertically. All this pointed to the “weirdness” of function transformations and the graphical representations.

Reflecting on Curve Matching

These student observations about transformations of a graph without formal terminology was a deliberate choice I made when designing this activity. By creating an activity that helped students notice how the graph was transformed by changing the parameters of the equation, students had a mathematical experience without formal mathematical language or notation. Eventually, a formal vocabulary would be introduced, and this experience would be used to help students understand and make sense of the definitions.

A parameter is an example of vocabulary. While the students had studied the effect of varying parameters while graphing lines, I did not introduce the term "parameter" to them. The idea was there, but the deliberate focus on terminology was not. As much as I wanted to bring their attention to this concept (and the more powerful idea of function transformation), it soon became clear to me that my students struggled to connect their own understanding of transformations of a parabola to the algebraic representation of them.

Student Confusion with Transformations of a Parabola

Prior to starting the unit, I was worried about this idea in general. I wrote,

> After many conversations with Phil, I find myself asking: What do the kids need to notice during a lesson? What will they notice? How do we connect noticing to formal mathematics? How does this time of year become an issue to work through? How do I find a balance between procedural and conceptual knowledge? (Journal, April 13, 2017)

These concerns guided much of my planning over the course of the unit. As I look back, it seems these concerns were well-founded, specifically my concern about students connecting their own observations to formal mathematics.

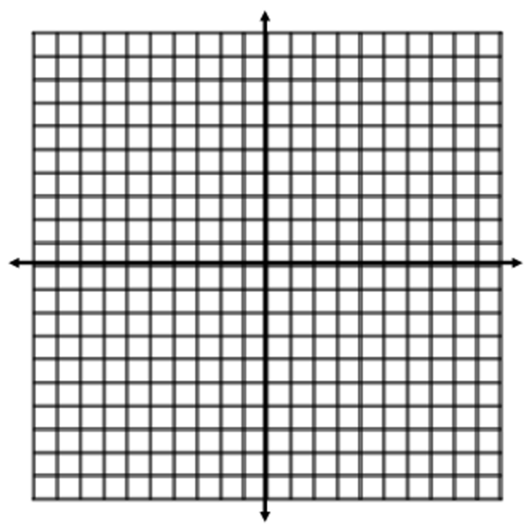
Graphing Calculator Exploration

The first formal lesson on this topic occurred on April 25. Each student was given a TI-84 graphing calculator, a common piece of mathematical technology in 2017, and three sets of parabolas to graph in which one of the three parameters was changed. Figure 7.9 shows one example that I used. After graphing the set, students were asked to reflect on the results. My goal for this lesson was to help students observe how changing each of the parameters in the equation of a parabola affects its graph when the equation is written in vertex form.

How does changing "h" effect the graph?

1. Using your TI-84, graph each of the following parabolas
2. Sketch all the graphs on the grid to the right.
3. Label each graph location of the vertex and the "h" value from the equation

$y = x^2$
$y = (x-2)^2$
$y = (x-4)^2$
$y = (x+1)^2$
$y = (x+3)^2$



Reflection: What effect does "h" have on the graph of a parabola?

Figure 7.9 – An example of a student task from transformations lesson

On the second day of the lesson, I brought the class together as a large group to process what they had learned. We summarized their thinking as shown in Figure 7.10.

This was my opportunity to allow students to present their ideas in their own words and then transition them to the formal mathematical notation and language.

Summary of Exploration of Vertex form of a Parabola

Vertex form of a Parabola

$y = a(x - h)^2 + k$

How does a effect the graph of $y = x^2$?

How does h effect the graph of $y = x^2$?

How does k effect the graph of $y = x^2$?

These effects on graph of $y = x^2$ are ________________________________. We describe these ______________________ in relation to the ______________________ ____________________.

Figure 7.10 - Summary from transformations lesson

After the groups had taken time to make conclusions in their own words, I brought the class together to share as a large group. I was interested in how they would verbalize the transformations of their graphs. This was their first genuine experience with transformations of parent graphs. While they had worked with equations of lines, that experience focused on identifying the slope and y-intercept, not on how the graph of

$$y = x$$

related to the graph of

$$y = mx + b.$$

I asked students from each group to share their conclusions to the three questions in Figure 7.10.

1	Deng:	We noticed that when a is positive it looked like a ‘U’ and when it was negative is looked like an ‘N’.
2	Greg Sand:	Okay, let me write that down. Does anyone have a different observation?
3	Haley:	Well, this is hard to say, but when the number was larger, the parabola got narrower, except when it was negative, then when it was smaller it narrowed. Also, when there’s a fraction, it gets wider.
4	GS:	Anyone have a different observation? Formally, the way we refer to what you noticed is that when $a > 0$ the parabola opens up, when $a < 0$ the parabola opens down. Also, when $\|a\| > 1,$ the parabola become narrower and when $0 < \|a\| < 1$, the parabola becomes wider. This is called a vertical stretch.

In this discussion, Deng (line 1) and Haley (line 3) made observations in their own words. When Deng stated, “When a is positive is looked like a ‘U’ and when it was negative is looked like an ‘N,’” he was making a connection between a parameter and how it is manifested on the graph. This allowed me to introduce the formal representation of his observation.

Haley deepened the connection when she stated, “When the number was larger, the parabola got narrower, except when it was negative, then when it was smaller it narrowed. Also, when there’s a fraction, it gets wider.” Here, she connected the magnitude of the parameter to its effect on the parent graph. My role in the conversation (line 4) was to connect their observations to formal mathematics notation and terminology. Rather than offer the vocabulary first as is traditional, I resisted the urge and allowed the students to find connections to their experiences and let the need for expression over what they had noted dictate when the terms were introduced. This was difficult and uncertain because I could only guess if my students would notice the mathematically important elements.

At this point of the lesson, I expected that students would be able to use the conclusions they had drawn (Figure 7.10) to describe transformations of parent graphs. Figure 7.11 shows the examples that students worked through. I expected them to read the first equation and note that the graph shifted three units to the right, the second equation translated the original graph four units left and one unit up, and the final example flipped the parabola upside down and shifted it right one and down four units. The examples were written to be increasingly difficult.

Describe how the graphs of each function are related to the graph of $y = x^2$?

$y = (x-3)^2$ $\qquad$ $y = (x+4)^2 + 1$ $\qquad$ $y = -(x-1)^2 - 4$

Figure 7.11 – Student practice problems

While the students were working with their partners, I noticed that many were struggling with this topic when I moved them from the summary activity to guided practice. The observations the students made from the graphing calculators were not translating into an immediate application of the ideas of simultaneous transformations. The following conversation illustrates this issue as the group tried to transfer what they had learned in the activity to a set of problems.

1	Isabel:	Mr. Sand, we need help. This just doesn't make any sense.
2	Greg Sand:	Let's see if we can figure out where you are stuck. Tell me what you know.
3	Jessi:	We don't get any of it.
4	GS:	Okay, let's look at our general form. What are the three different numbers that we are looking at?
5	Jessi:	In the second problem, we've got 4, 1 and 2.

6	GS:	Now be careful with the exponent. Remember we're basing our observations on the original graph, $y = x^2$. Try and focus on the numbers that are different.
7	Isabel:	But what are we supposed to do here?
8	GS:	Let's go back to our summary. What does the number inside of the parentheses do?
9	Seamus:	Moves it left and right, but it's kinda backwards because plus is left and minus is right.
10	GS:	Okay, so in this example it's a plus 4, so how does it change the graph?
11	Clifton:	Left four spaces.
12	GS:	So far so good, is everyone okay with the answer? [nodding heads] What about the plus 1 on the outside?

At this moment, I am scaffolding ideas with student to help them connect what they learned to new problems. By connecting ideas, students learn to see math as a coherent study instead of individual, loosely related topics. Just because a few students are comfortable making connections, other students need time to develop. Isabel and Jessi helped each other think through the next problem.

13	Isabel:	Up 1? Down 1? I'm not sure.
14	GS:	Let's compare to our general form. How does adding a number on the outside affect our graph?
15	Jessi:	It moves it up or down, depending on if it's positive or negative.
16	GS:	So, what's the effect of adding 1?
17	Isabel:	Up 1.
18	GS:	Looks great, try the next one on your own, I'll check back in a minute.

At this moment I realized that I should have included examples in the graphing calculator exploration that included multiple transformations. It never occurred to me that students would not automatically understand these examples. This dialogue enabled me to better understand that students had difficulty connecting all three transformations at once.

Members of the group expressed their frustration (lines 1, 3, 7, and 13), but did not quit on trying to solve the problems. When Jessi (line 5) answered my question about the numbers they needed to focus on by saying, “We’ve got 4, 1 and 2,” I realized that there was some confusion about what parts of the equation were important. To help clarify their understanding, I brought them back to the goal of the lesson by asking the effect of each of the parameters (lines 8, 10, and 14), redirecting them from their confusion to the tools that they needed to solve the problem. This type of thinking is critical in their preparation for an upper-level mathematics classroom where class sizes are larger and content is presented more quickly.

Once the students were able to focus on the critical parts of the equation, they were able to express the transformations. I asked (line 8), “What does the number inside of the parentheses do?”

Seamus responded (line 9), “Moves it left and right” with a mathematically valid response as well as an observation (lines 9), “but it’s kinda backwards because plus is left and minus is right.” His reflection is one that is commonly made by second-year Algebra students when they work with transformation of functions, nothing that I had expected a first-year Algebra student to notice.

Clifton (line 11) was able to utilize Seamus’ answer when he correctly noted that it moved, “Left four spaces.” This type of synergy is what I had hoped to accomplish by having the students work in small groups. My role in this exchange was to help the students come back to knowledge they had produced and apply it to these particular situations.

Similar conversations happened between me and two of the four other groups. Some initial confusion was expected, but student misunderstanding did not improve the way I had hoped between this lesson and the chapter quiz.

Student Responses to Chapter Quiz Questions

The chapter quiz included a few questions involving transformations of parabolas. Two problems best illustrated that my students were struggling to understand the algebraic form of transformed parabolas. The first was question 3 (Figure 7.12). Sixteen of eighteen students attempted this question and nine correctly answered with a response

Describe how the graph relates to the graph of $y = x^2$

3. $y = (x - 2)^2$

Figure 7.12 – Quiz problem 3

similar to, “Moves the graph two units to the right” which is a written explanation of the transformation.

The following samples represent different errors the students made in answering this question. Chloe’s work is shown in Figure 7.13. This is an example of a response by a student who has confused the vertical and horizontal transformations. This was a mistake made by three of the eighteen students who completed the quiz.

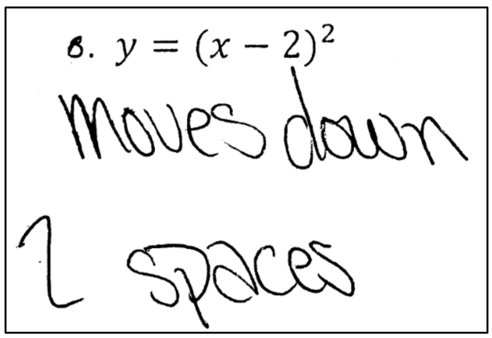


Figure 7.13 – Chloe’s response

Joshua’s response is shown in Figure 7.14 and is an example of a student who has confused how the sign accompanying the constant relates to the transformation. Six of the eighteen students who completed the quiz made this error. It suggests that students understand that it is a horizontal transformation but are confused about the direction.

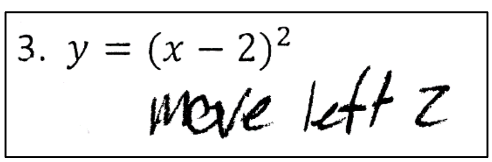
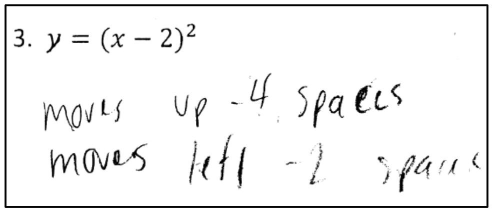


Figure 7.14 – Joshua’s response

Sabrina’s response is shown in Figure 7.15. She was the only student to make this type of mistake. This error seems to indicate that she is unsure what to do and has attempted to build an answer based on the values in the problem and the operation.


3. $y = (x - 2)^2$

moves up -4 spaces
moves left -2 spaces

Figure 7.15 – Sabrina's response

Each of these samples of student work are incorrect solutions. By examining the variety of errors made by students, I uncovered several misunderstandings. Each of these misunderstandings can be addressed through a tailored intervention based on student thinking.

The second problem that illustrates my students' struggles is problem 10. The original problem is shown in Figure 7.16. This problem was attempted by ten of the eighteen students who completed the quiz and correctly answered by two students with a solution similar to

$$y = (x + 3)^2 - 2.$$

10. The picture at the right shows the graph of $y = x^2$ and the graph of another equation after two transformations. Write the equation of the new graph in vertex form.

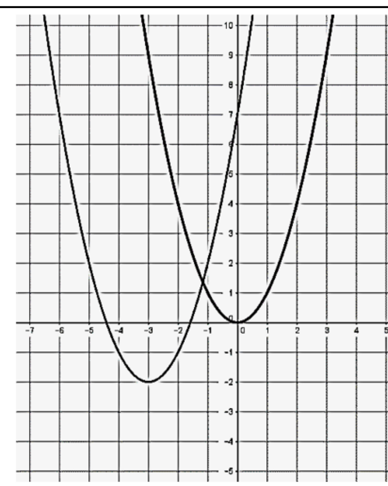
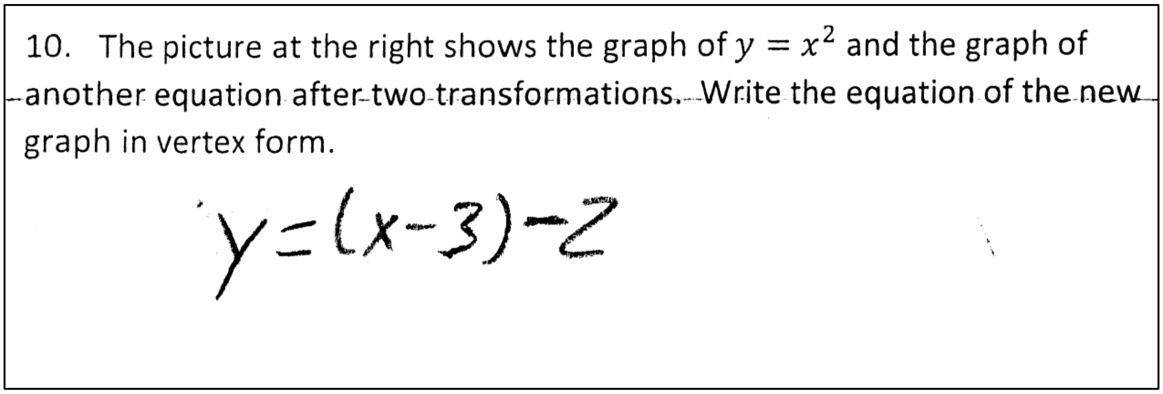


Figure 7.16 – Quiz problem 10

Garrett's response shown in Figure 7.17 is an example of a student who correctly found the vertex but incorrectly constructed the equation by writing $x - 3$

instead of $x + 3$. Three different students made this error in response to the question.


10. The picture at the right shows the graph of $y = x^2$ and the graph of another equation after two transformations. Write the equation of the new graph in vertex form.

$y = (x-3)-2$

Figure 7.17 – Garrett's response

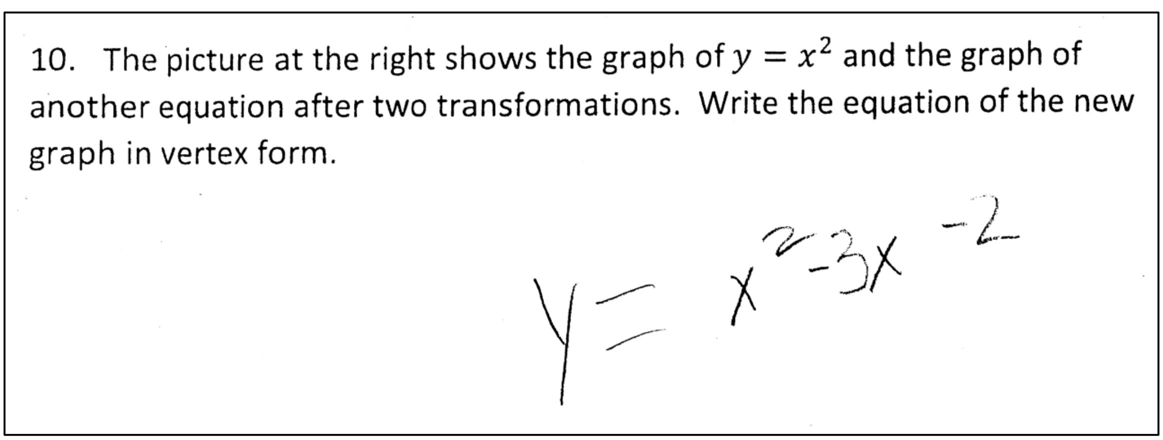
In Figure 7.18, Evan attempted to use the standard form of the parabola,

$$y = ax^2 + bx + c$$

instead of the vertex form

$$y = a(x - h)^2 + k$$

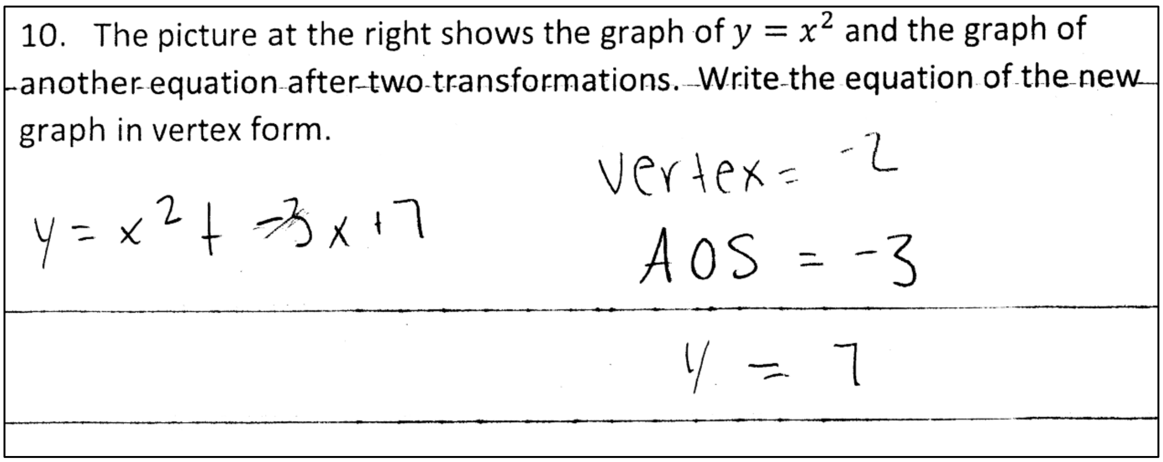
to write the solution. This error was made by two students.


10. The picture at the right shows the graph of $y = x^2$ and the graph of another equation after two transformations. Write the equation of the new graph in vertex form.

$y = x^2 - 3x - 2$

Figure 7.18 – Evan's response

Figures 7.19, 7.20 and 7.21 show errors made by three different students. In Figure 7.19, Haley identified the x value of the vertex, the axis of symmetry, and the y-intercept and attempted to used them to construct an equation in standard form. Isabel's response, shown in Figure 7.20, shows that she incorrectly used a linear model for the equation. Shown in Figure 7.21, Aisha wrote a quadratics and then attempted to use a linear model.



10. The picture at the right shows the graph of $y = x^2$ and the graph of another equation after two transformations. Write the equation of the new graph in vertex form.

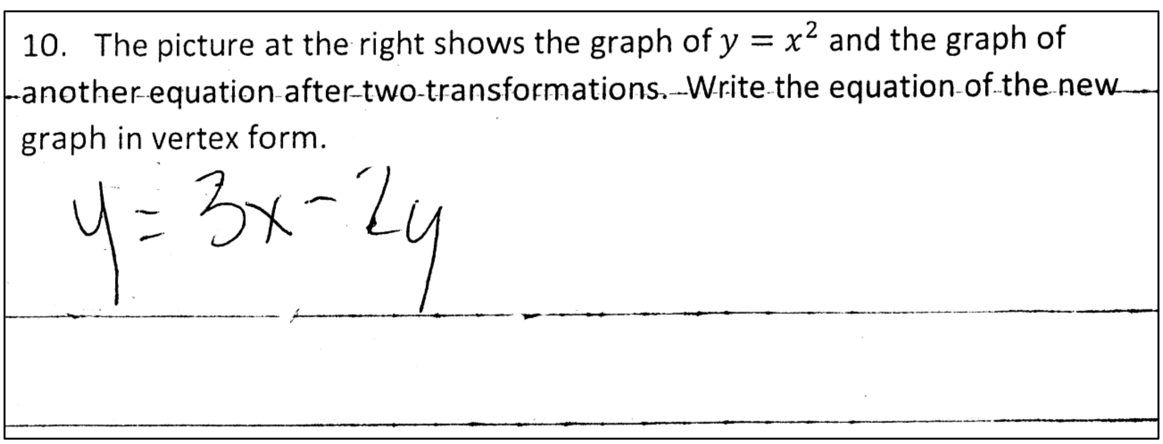
$y = x^2 + \cancel{-3}x + 7$

vertex = -2

AOS = -3

y = 7

Figure 7.19 – Haley's response



10. The picture at the right shows the graph of $y = x^2$ and the graph of another equation after two transformations. Write the equation of the new graph in vertex form.

$y = 3x - 2y$

Figure 7.20 – Isabel's response

10. The picture at the right shows the graph of $y = x^2$ and the graph of another equation after two transformations. Write the equation of the new graph in vertex form.

$y = x^2$

$y = 3x - 2$

Figure 7.21 – Aisha's response

These struggles continued on the chapter exam. The exam was a multiple-choice test that included two problems utilizing the vertex form of a parabola. Of the eighteen students that completed the test, five correctly answered one of the problems and seven students correctly answered the second question.

Reflections on Student Confusion with Transformations of a Parabola

This topic caused my students difficulty and I never felt able to sufficiently resolve their misunderstandings about transformations. At the core of this topic is the larger concept that becomes a center piece for working in upper level high school mathematics (Second-Year Algebra, Precalculus and Trigonometry) with transformations of functions.

Benefits of Transformations

Working with transformations provides students access to information, e.g. location of the vertex, that would otherwise be computationally challenging to determine and enables students to become more efficient problem solvers with functions. For example, if I asked a student to find the vertex of a parabola, in this unit they had three choices. One, they could build a table of values, graph the parabola, and as long as the values were integers, find the vertex from the graph of the parabola. However, this method is inefficient and inaccurate, forcing students to spend excessive time and effort doing calculations and hoping for accuracy.

As a second choice, they could use the standard form of a parabola $y = ax^2 + bx + c$, find the x-value of vertex using the standard algorithm

$$x = -\frac{b}{2a},$$

and then substitute that value into the original equation to find the y-value. This also reduces the problem to computation. The third choice is to use the vertex form of a parabola and interpret the equation. This method is the most efficient, even if the student is required to complete the square to transform it from standard form to vertex form.

Algebra as a qualitative exercise. Behind this argument is the larger issue of student misunderstanding of the field of mathematics. My experience as a teacher has taught me that for many students (and teachers), mathematics is largely viewed as a computational activity. While arithmetic is a subset of mathematics and a necessary part of algebra, it is not algebra. When solving linear equations, students focus on, “adding something to both sides,” or, “multiplying something on both sides.” Solving equations becomes a computational exercise. Instead, algebra is the manipulation of the equation, not the actual addition. It is the ability to manipulate the relationships in the equations that is critical, not the manipulation itself. The same thing is true in parabolas. The goal must focus on understanding the qualities of an equation, not on the reduction of the mathematics to a purely computational exercise.

Helping students make this transition is one critical way algebra teachers can prepare students for upper level mathematics. This transformation in thinking is part of the maturation successful students go through and those who struggle rarely make. Recently in a Calculus class I was teaching, students were given:

If $\int_0^5 f(x)dx = 4$, then what is $\int_0^5 [f(x) + 2]dx$?

Those who successfully solved the problem took one of two approaches: they either used properties of definite integrals and rewrote it as

$$\int_0^5 f(x)dx + \int_0^5 2dx$$

and solved it geometrically, or they used the properties of transformations of functions (a vertical translation of 2) and solved the problem. Neither approach is purely computational, and both required some type of conceptual understanding and symbolic manipulation.

Changes to Improve Student Learning

I am left to wonder how I could have improved my algebra students' performance on these kinds of tasks. More work on these in daily warm-ups would have been helpful. I could have reduced the number of topics we studied so that they could understand fewer concepts better, but that opens up another discussion that has been bothering me as I reflect on this experience, *What really delineates Pre-Algebra from Algebra? Is students'* *success in first-year Algebra related to their ability to keep up with the pace of the material?*

My students' struggle with this idea reflects a deeper and more important transition that I was trying to scaffold during this unit, the move from casual, informal language used to describe observations into more formal academic language and finally to mathematical notation. This examination of my students' struggle with the vertex form of a parabola illustrates this issue. Later lessons in this unit also exposed this problem. My students were successful in describing mathematical concepts in their own language, but most failed to express them using mathematical notation.

Student Difficulties Interpreting Notation

Two sets of lessons illustrate the difficulties that students had interpreting notation. The first lesson occurred on the day after the curve-fitting activity shown in Figure 7.8. The second lesson occurred two weeks later when I was introducing students to solving quadratics by graphing. During both of these lessons, the students made observations based on what they noticed, and then attempted to translate their observations into mathematical notation.

Noticing Geometric Characteristics of Parabolas

Once we finished with the curve-fitting activity as shown in Figure 7.8, I had the students partner up for a noticing activity. I left the Geogebra file open and formed a parabola that opened up, and I asked them to work with their partner to write down three different qualities of the graph. After one minute, I changed the parabola to one that opened down and asked them to repeat the activity, but this time without restating anything they had written down the first time. I then went from group to group and listed on the board what they had written down. Here is our list:

- Is the same on the left and right/symmetric
- Has a low or high point
- Goes up and down or down then up
- Is positive and negative
- Doesn't have a slope
- Looks like a U or an N
- Goes down at both ends or up at both ends
- Sometimes crosses the x-axis twice
- Isn't a line

(April 17 video)

We completed the list as class ended, so I wrote the list in my journal and transcribed a copy for later use.

The next day in class, I gave the students the opportunity to graph a parabola by constructing a table of values. Once they finished drawing a graph, I listed eight questions relating to some of the geometric characteristics they had noticed the day before. Figure 7.22 shows the first of the two parabolas that they graphed including the organizational table I provided to scaffold their thinking through the computation.

Graph each of the following parabolas by using a table. Use the graph to answer the questions that follow.

1. $y = x^2 - 9$

x	x^2	$y = x^2 - 9$
-2		
-1		
0		
1		
2		

a. What is the maximum y-value of the parabola?
b. What is the minimum y-value of the parabola?
c. Does the parabola open up or down?
d. Draw in the parabola's line of symmetry.
e. What is the domain of the parabola?
f. What is the range of the parabola?
g. For what x-values are the y-values negative?
h. For what x-values are the y-values zero?

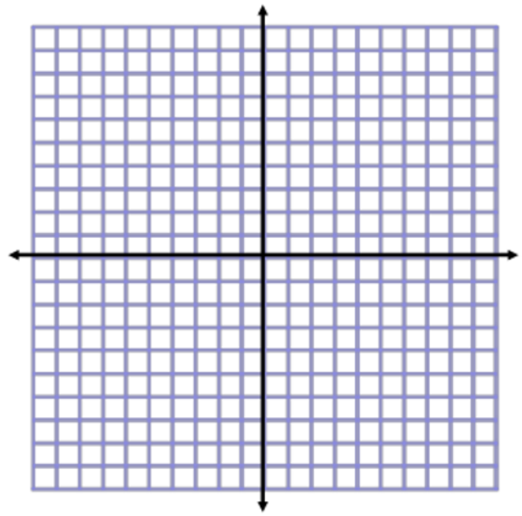


Figure 7.22 – Student classroom exercise

As we worked through the first example, I explained how to use the table to find values of the function. We then reviewed how to plot points and drew in our parabola. As a whole group, we talked through the questions. The conversation consisted of a few students offering answers to the questions, with some confusion about questions a, c, e, f, and h. For the students, the most difficult part of the lesson came when I wrote the answers to d, e, f, g and h using mathematical notation. Questions d, e, and f relied on prior knowledge that had been introduced in Unit 3. The answers to questions g and h required a comfort with compound inequalities, a topic that we briefly studied in Unit 5.

On April 18, Jordan ask me to help him understand how we wrote our final answers for question d. During our time working as a large group processing this problem, I had a student draw the line of symmetry on the graph in front of the class, and then I reviewed with students how to write the equation of a vertical line. Jordan used a phrase that was common throughout the year, "This just doesn't make any sense." I proceeded to ask him a series of questions to help him better express what he did and did not understand.

1	Jordan:	Mr. Sand, I need help. This just doesn't make any sense.
2	Greg Sand:	Which part doesn't make sense?
3	Jordan:	All of it.
4	GS:	Okay, so let's choose one part and work through it. Why don't you pick one?
5	Jordan:	Um, how about d?
6	GS:	Tell me what you understand.
7	Jordan:	So, I get the whole line of symmetry thing, that's where it's the same on both sides. But why did we write the answer as $x = 0$?
8	GS:	What direction does the line go?
9	Jordan:	Up and down.
10	GS:	True. I'm going to use the word vertical to describe it.
11	Jordan:	Oh yeah, but don't y's go up and down? How come it's $x = 0$?
12	GS:	Anytime we write the equation of a vertical line, it's always of the form x equals some number. Every x value on the line is the same.
13	Jordan:	But why can't we just draw the line and call it good. I know where it is.
14	GS:	We're trying to express the answer formally; this is the best way to write it.
15	Jordan:	I don't get it.

During this interaction, I asked Jordan a series of questions to see if I could gain insight into what he understood and what really confused him (lines 2 and 4). Once he chose the problems to focus on (line 5), I was able to find out what he understood (lines 9 – 10). When he said, "I get the whole line of symmetry thing, that's where it's the same on both sides," I became confident that he understood the foundational idea that we were using to graph the parabolas. However, when he asked, "Why did we write the answer as $x = 0$?"(line 7) I realized that he was not able to connect a vertical line where all values of x are equal to zero to the notation $x = 0$. We covered this concept earlier in the year, and even though we worked together to understand why vertical equations had the form $x = k$, Jordan was unable to apply it (line 15).

It would have been easy for me to give in to Jordan's desire to leave part of the problem incomplete (line 13), but if I had allowed him to leave the activity without fully completing it, then he would have been left with an incomplete view of the complexity of parabolas and the analysis of their graphs. Also, while it seems like a reasonable move to allow him to draw in the line without writing the equation, I would have missed an opportunity to reinforce concepts that will appear in subsequent math courses. By keeping the focus on completing the problem, Jordan was struggling through an idea that he seemed not to fully understand.

Diagnosing Misunderstandings

When students could not completely understand the answer to a question, I found myself attempting to uncover critical knowledge that was impeding their learning. As we discussed the solution to part (g) of the problem, Kiera highlighted the part of the graph that was below the x-axis. I then showed the students how to express the question, "For what x values are the y-values negative," mathematically as:

$$\text{Find all values of x such that } y < 0$$

and then the solution

$$-3 < x < 3.$$

Deng then asked me to help him understand why I wrote the solution this way.

1	Deng:	Mr. Sand can you explain something to me?
2	Greg Sand:	Happy to, which part?

3 Deng: This part of question g. You asked a question about the y-values but you wrote the answer using x's.

4 GS: That's true. Let's reread the question. It says, "For what x-values are the y-values negative." Is that confusing?

5 Deng: No, I get that and when you colored it, that made sense. The y-values are negative below this line [points to the x-axis] but how does $-3 < x < 3$ tell me that?

6 GS: Let's examine the graph closer. Do you see how it crosses the x-axis at -3 and goes below until it gets to 3 and then goes above? That's where the y-values are negative.

7 Deng: I don't know, but I'll be okay with it.

At the beginning, Deng (line 3) expressed confusion about the form of the answer when he said I had asked a question about the y-values but wrote the answer using x's. I was curious if he understood what the question was asking, but I felt satisfied when he was able to restate (line 5) what we did, "I get that and when you colored it … The y-values are negative below this line [points to the x-axis]." I tried to help him connect the graphical representation of the problem to the interval of the solution (line 6), but he expressed that he still was not sure (line 7).

I felt that Deng understood the problem and was okay with the graphical representation of the solution. Comfort with a graphical solution is necessary but not sufficient for a student to be proficient in first-year Algebra. While traditionally there is an emphasis on the analytic form of solutions, I asked students to developed analytic, numeric, and graphical understandings. For students to be prepared for the rigors of upper level mathematics, they need to be able to connect how they understand a topic in all three ways. I attempted to help Deng (lines 12 – 15) connect his graphical understanding to a numeric one when I asked him, "Do you see how it crosses the x-axis at -3 and goes below until it gets to 3 and then goes above?" Although I did not explicitly write out numerical values, I hoped that by noting values along the $x-$axis I could connect his graphical understanding to an analytic one through a numerical approach.

Reflecting on these two lessons, I noted that, “It seems like the kids understood what the question was asking and knew how to answer it, but were unable to understand the mathematical representation of the solution.” (Journal, April 18) I was impressed with my students’ ability to answer these question, but I was worried that transitioning into formal mathematical representation would inhibit student learning while they struggled to make sense out of the notation.

Analyzing the Flight of an Angry Bird

The second lesson that I want to bring attention to occurred when we were solving quadratic equations by graphing. Although this technique is at times inefficient, it presents the opportunity for students to connect the analytic work of solving quadratics with the graphic representation of the problem. The lesson for May 1 began with two images from *Angry Birds* placed on an xy-axis along with a parabola as shown in Figure 7.23. This allowed me to both keep the students focused on their chapter project and create the opportunity for discussion about solving quadratics graphically without using any analytic techniques. That transition would come later in the class period.

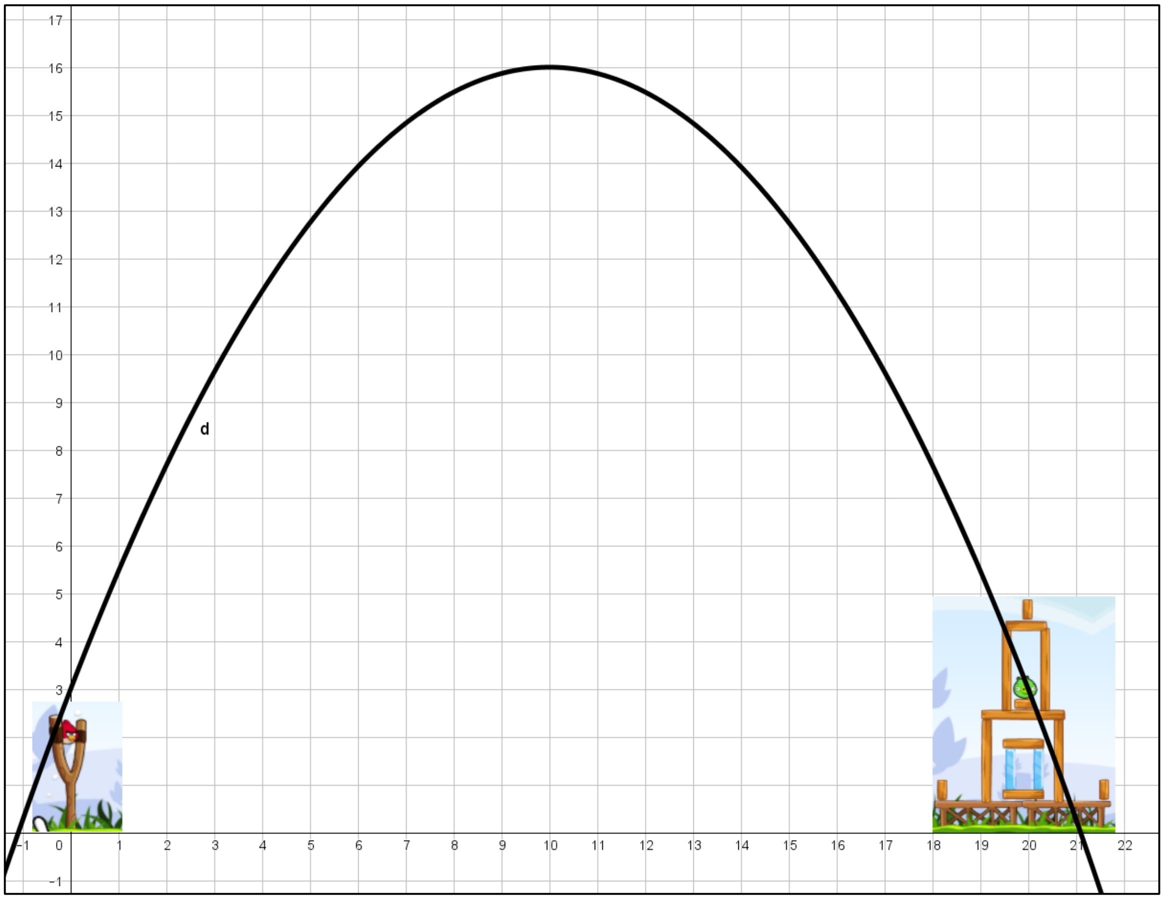


Figure 7.23 – Prompt shown to students

I gave students a set of questions to solve based on the picture. Haley's paper is shown below in Figure 7.24 with both the questions and her responses. After each student had the opportunity to individually write their own responses, the students shared their responses in small groups, next with the entire class, and then I translated their solutions into mathematical notation as part of class discussion.

For example, when Haley answered question 2, she wrote, "14 feet, he is that height again when he's 14 feet away from the sling shot." I expressed her solution with function notation as

$$f(6) = 14$$

and

$$f(14) = 14.$$

Use the graph to answer the following questions.

1. What is his height when he is 10 feet from the sling shot? Is he that height again? When?
   16 feet, he's never that height again.
2. What is his height when he is 6 feet from the sling shot? Is he that height again? When?
   14 feet, he is that height again when he's 14 ft away from the sling shot.
3. What is his height when he is 3 feet from the sling shot? Is he that height again? When?
   ≈9.8 feet, he is that height again when he's 17ft away from the sling shot.
4. How far away from the sling shot is he when he is 16 feet high? Does this happen again? When?
   10 feet away, he's never that height again.
5. How far away from the sling shot is he when he is 14 feet high? Does this happen again? When?
   6 feet away he's that height when he 14 feet away.
6. How far away from the sling shot is he when he is 5 feet high? Does this happen again? When?
   ≈1 ft away, he is that height when he's ≈20 feet away.
7. How far away from the sling shot is he when he is 20 feet high? Does this happen again? When?
   He never gets to 20 feet height

Figure 7.24 – Haley's responses to the prompts

Haley's responses to these prompts were similar to the rest of the class. She was able to understand questions in context that she may have struggled with if only expressed mathematically. Question 5 could be communicated as,

Find all values of x so that $f(x) = 14$.

My ultimate goal was to help students notice that the value of a parabolic function can be the same for multiple x values, or

$$f(a) = f(b),$$

where

$$a \neq b.$$

At the end of the activity, I asked students what questions 1 – 3 had in common and what questions 4 – 7 had in common. This was the opportunity to engage them in a meta-cognitive activity.

1	Greg Sand:	Nice job with the answers everyone. Next, I want to help you guys make connections between these questions. At the bottom of the note sheet I have broken the problems into two groups. Does anyone want to make a guess about what makes problems 1-3 a group?
2	Haley:	So, in 1, 2 and 3 you tell us how far away the bird is and ask us to tell you how high he is.
3	GS:	Okay, anyone agree with that? [Lots of heads nodding] Is there another way to say that?
4	Madison:	It's like you tell us the x-value and we need to give you the y-value.
5	GS:	Alright, does that make sense? Any questions about that statement?
6	Haley:	Is that the same as what I said?
7	Isabel:	Yeah, I like the first way. The second way confuses me.
8	GS:	Let me say that they are two ways of saying the same thing. We need to be okay with the idea that we are being asked to find a y-value given an x-value, but both statements mean the same thing.
9	Isabel:	I still like the first way better.
10	GS:	I understand, but we won't always be working with problems in context like this, we need to be okay using the math notation.

Processing the activity. I opened the discussion not being sure about what solutions the students would offer. Earlier in the year, the students gave answers that had little to do with the mathematics we studied and more to do with tertiary ideas like the color of the bird in the diagram or the location of the pig. At this point in the year, the maturation of our class discussions was apparent when the first shared idea (line 2) was, "In 1, 2 and 3 you tell us how far away the bird is and ask us to tell you how high he is."

Haley's statement made it clear to me that she understood the idea that I was trying to highlight in context of the problem.

Madison extended Haley's thinking by stating the idea (line 4) using formal mathematical terminology when she stated, "You tell us the x-value and we need to give you the y-value." These two connected ideas from Haley and Madison helped me realize that there had been significant growth in the quality of my students' mathematical conclusions.

Resistance to formal notation. Isabel (line 9) shared her resistance to expressing the idea analytically when she shared, "I still like the first way better." This moment was interesting for me to reflect on because she preferred discussing the problem in context instead of analytically. Much of my teaching experience shows that students prefer analytic problems where they are required to execute an algorithm to working through word problems in which they have to use mathematics in context. Isabel's comment is similar to Deng's (page 180, line 5) when he expressed his understanding of the problem in context but not analytically. Isabel, like Deng, shows necessary but insufficient understanding of a mathematical idea.

The discussion continued in a similar fashion about why problems 4 – 7 formed a group with similar results. In my Journal from May 1, I wrote, "This is becoming a frustrating habit. The students are okay with the language in context of a problem, but we aren't successfully moving to formal language."

Student Responses to Exam Questions

Although reading and understanding formal math notation was important to student success on the unit exam, two problems stood out as exemplars of this issue. The results of the exam made it difficult to determine how many understood the problems versus how many randomly guessed.

The first problem is shown in Figure 7.25. This problem required students to find the vertex of the parabola and use the x-coordinate to write the line of symmetry. Every student was able to demonstrate this skill during class given a graph of a parabola but struggled when asked to express it mathematically. After reviewing the results of the exam, only five of the eighteen students who completed the exam answered it correctly. Random guessing on the problem would yield 4.5 correct answers, so this result was no better than guessing.

2. What is the equation of the axis of symmetry of the graph of $y = x^2 + 6x - 7$?
   a. $x = 6$
   b. $x = -3$
   c. $x = 3$
   d. $x = -6$

Figure 7.25 – Unit exam problem 2

The second problem from the exam is shown in Figure 7.26. In this problem, students had to translate the word roots into the idea of finding the x-values of the points where the graph crosses the x-axis, a similar idea to determining where the graph is negative (p. 169). For this problem, seven of the eighteen students who completed the exam correctly answered the question. This result is only slightly better than would be expected from random guessing.

4. What are the roots of the quadratic equation whose related function is graphed below?
   a. -1, 3
   b. -1, 1
   c. -3, 1
   d. 1, 3

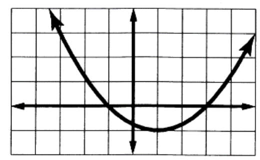


Figure 7.26 – Unit exam problem 4

Reflections on Student Difficulties Interpreting Notation

My students struggled during this final unit translating what they noticed about mathematical ideas into formal notation. While student struggles at this point in their mathematical development is not unexpected, it is crucial for future success that they go through these struggles. It is almost unbelievable how quickly students must advance their thinking from the time they enter a first-year Algebra course to just two short years later when they enter a second-year Algebra course. This struggle becomes critical to their mathematical development, and students should not be sheltered from it in the name of short-term success.

The Importance of Notational Fluency

Most of my students were moving on from first-year Algebra to Geometry for their first year in high school. Geometry is a course of logic and reasoning in which symbolic representation of ideas is a daily part of the class. It can be taught using a form of direct instruction where the teacher presents postulates and theorems in formal language and notation and then explains to the students what they mean. The course can also be taught experientially, allowing the students to draw their own conclusions and eventually express them formally. My experience has been that most teachers blend these two methods. In any case, students need to be exposed to these ideas for any chance of success in upper-level courses.

In Figure 7.24, I shared student responses to prompts about a figure. My goal was to help move them towards a notation about function points where functions are equal, or a prompt like,

For what values a, b does $f(a) = f(b)$?

While this notation is not critical for the unit, it is one that is used throughout courses at and beyond the second-year Algebra level. For example, this idea occurs fairly regularly in my Calculus courses. When we study mean value theorem, the students are eventually confronted with a statement similar to,

If $f(b) = f(a)$, then there exists a value c between a and b such that $f'(c) = 0$.

The beginning of this theorem is the same idea that was a central part of this lesson. It also proved to be the most misunderstood idea by the class. I assumed that my students would be comfortable making sense of the idea on their own, and I was wrong. More importantly, it is an idea that may or may not have been introduced five years prior.

It is impossible to state with certainty if first-year Algebra teachers are by themselves at fault for this misunderstanding. The critical question that keeps coming to the forefront for me is: *How many choices do teachers make that are designed to promote short-term gains but have long-term detrimental effects on our students' mathematical success?* If I had chosen to work on probability and statistics rather than parabolas, I would have sent them on to Geometry with little exposure to quadratics. Without that development, these students would have been ill-equipped to solve any problem involving Pythagorean Theorem much less more sophisticated problems. The Geometry teacher is then forced to choose either to push through the material without helping the students understand it or to slow the course down to teach these ideas and thus miss out on critical information developed at the end of the course.

Year after year these difficulties accumulate and the pedagogical issues compound. Students move from class to class thinking they have learned the necessary material for the next course. As students approach problems in future courses with knowledge developed in prior courses, they function as problem solvers and mathematical thinkers based in part on their learning. It is a common mistake to view students as unmotivated or disengaged when in reality they may be inadequately equipped for the expected tasks due to choices made by teachers over years of school.

This is not a new lesson, but a cautionary warning about the choices we make as teachers. I have been guilty of this in my early teaching career. It is a painful echo in my memory of solving equations with only integers because my students struggled working with rational numbers. I question now whether I caused harm in the long-term by avoiding challenging concepts under the misguided attempt to teach them the limited idea that was needed for the day.

Reflections on the Importance of Notational Fluency

Reflecting on this issue brings me back again to a central theme in my practice, content and curricular knowledge are critical. As a teacher, I have mathematical knowledge for teaching. This knowledge helps me understand how the concepts I am teaching one day fit into a larger mathematical landscape. However, I also have a curricular knowledge that helps me understand how the lessons from my course fit into the broader sequence of ideas that my students will be exposed to over their high school and college careers. These two ideas work in tandem to help me make decisions about what to teach and how to teach it. It is those decisions that I want to bring forward in the final portion of this chapter.

My Struggle Scaffolding Activities to Support Students in Overcoming the Confusion and Challenges of Interpretation

Throughout this school year, I found myself learning to adjust how I designed lessons to lead students toward the goals of a lesson. During this unit I quickly discovered that my students we not prepared for the complex computations necessary to explore algebraic relationships. Not having enough computers available for the students to use graphing software also created another barrier (in addition to the district assessment which required students to graph by hand). I made the choice to scaffold the complex tasks into smaller, more manageable parts. It has become clear to me as I reflect on this part of my teaching that scaffolding a lesson in a way that balances the supports that students need to accomplish tasks with the rigor that helps them develop was more challenging than I expected.

For example, Figure 7.22 has two differently scaffolded tasks. First, I created an organizational table designed to help students through the computational work, a kind of short hand. While this function was not particularly difficult, the subsequent problems were more rigorous. In Figure 7.27, my goal was to give my students a way to organize their thinking and allow them to more easily connect numerical and graphical representations. My past teaching experiences with complex computational tasks similar to these has been that previous students struggled to precisely execute the calculations.

x	x^2	$-x^2$	$-4x$	$y = -x^2 - 4x + 5$

Figure 7.27 – Scaffolded function evaluation

This difficulty with precision resulted in a lack of complete information necessary to notice the mathematical relationships between the middle set of columns and the final column.

When the students were working on this portion of the lesson, I found myself explaining how to use the chart more than I was actually helping them learn the important parts of the lesson. During this conversation from April 18, I stopped to work with a small group that was struggling to understand how I had designed the table. Neither Aisha nor Chloe were students who usually asked me questions during class, so their confusion surprised me.

1	Chloe:	Mr. Sand, we need help. How do I use this table?
2	Greg Sand:	Here’s the idea. You choose some x −values: -2, -1, 0, 1 and 2. So the first column is to square them, so go ahead and do that.
3	Chloe:	Okay, that’s done.
4	GS:	Now, the third column has us make them opposite, go ahead with that.
5	Aisha:	Do we multiply them by four next?
6	GS:	Notice that column is the original x, so multiply them by -4, not the $-x^2$ values.

7	Chloe:	Then how do we finish?
8	GS:	We'll take the numbers out of the $-x^2$ and the $-4x$ column and combine them with the $+5$ to find the y −value.
9	Aisha:	Wow, that's a lot of work just for one number.
10	GS:	Yeah it is, but don't give up. There's some really good stuff to learn coming up.

During this conversation, I took a very direct approach (line 2) to help Aisha and Chloe make use of the table. Telling her just seemed right in the moment and the short-term goal for her. The goal of the task was to help my students connect numerical and graphical understandings of parabolas. It was clear to me when Aisha asked (line 5), "Do we multiply them by four next?" that she was viewing the process as a lateral one, moving from one column to the next, not fully understanding how the column headings were connected to the computation.

Chloe shared her insecurity in completing the task (line 7) when after working through each of the sub-steps to complete one full evaluation of the function, she asked, "Then how do we finish?" Aisha seemed aware of the complexity of the task and able to execute it (line 9) when she stated, "That's a lot of work just for one number." This organizational tool was causing as many problems as it was solving.

Challenges in Scaffolding Function Evaluation

The difficulties in scaffolding this particular task were two-fold. First, I needed to establish the computational processes used to calculate a single value of the function. Second, I had to figure out the best way to organize the process. In this case, I chose a table modeled off of ones I have used in upper level math courses (i.e. when discovering the relationships between the polar functions and their graphs, I would create a table of each value needed to graph a function like $r = 2\cos 3\theta$). While it would have been easier to hand the students a graphing calculator, I felt that a few exercises would allow students to maintain a connection between the numerical relationships of the domain and range and how those relationships manifest themselves when the points are plotted on the xy-axis.

This conversation is typical of that work. My notes on that day reflect the difficulty my students had with this task.

> So that took much longer than I needed it too. In my attempt to make things easier, I have made them much more difficult. The kids struggled so much doing the operations that they missed the larger point of it. I'll have to revisit the big ideas tomorrow. (April 18)

To me, it seemed like the work I had done to make the activity easier actually made it more difficult. The computational challenges interfered with the mathematics that I had intended the students to discover.

Another example of how I worked to scaffold a lesson is shown in Figures 7.28 and 7.29. This lesson involved the students using a graphing calculator to generate the graphs quickly and then notice what had changed in each graph relative to the parent graph. I designed the prompt shown in Figure 7.28 so students could examine how changing the lead coefficient affected the graph. Figure 7.29 was designed to help students notice how changing the constant being added to the function transformed the graph.

$y = x^2$
$y = 2x^2$
$y = 3x^2$
$y = \frac{1}{2}x^2$
$y = -3x^2$
$y = -4x^2$
$y = -\frac{1}{3}x^2$

Figure 7.28 – Scaffolded graphing task

$y = x^2$
$y = x^2 + 1$
$y = x^2 + 3$
$y = x^2 - 2$
$y = x^2 - 4$

Figure 7.29 – Scaffolded graphing task

When I designed this lesson, my goal was to create learning tasks that would allow an opportunity for the students to notice three different transformations of the graph of a parabola in vertex form. Each task was completed successfully by the majority of the students. The problems with my structure did not become clear until the start of class the next day when I attempted to lead them through a discussion that assembled all of the properties into one general form.

None of the students had completed all of the tasks on the previous day, so after giving them time to finish, I presented them with the prompt shown in Figure 7.30.

Summary of Exploration of Vertex form of a Parabola

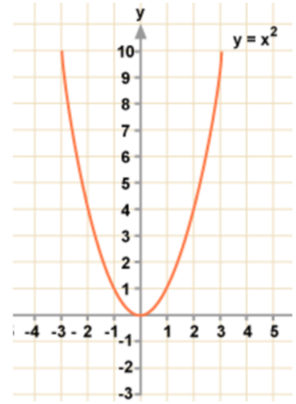
Vertex form of a Parabola

$y = a(x - h)^2 + k$

How does a effect the graph of $y = x^2$?

How does h effect the graph of $y = x^2$?

How does k effect the graph of $y = x^2$?



These effects on graph of $y = x^2$ are __________________________________. We describe these _______________________ in relation to the ________________________ ______________________.

Figure 7.30 – Summary prompts

The dialogue is from April 26 during the middle of class and begins after I have collected observations and conclusions from the students. During this conversation between Jordan, Philip, and myself, I realized I had missed opportunities to help my students make conclusions I had assumed they would make, in particular, the idea that multiple transformations could happen at the same time. Each of the activities focused on one of the three transformations (vertical shift, horizontal shift, and vertical stretch), but none of the activities presented students with the opportunity to observe simultaneous transformations.

1	Greg Sand:	Thanks for all of the input on your conclusions. What I want to do now is put all of the ideas together.
2	Jordan:	You mean they can happen all at the same time?
3	GS:	Well yes, why wouldn't they?
4	Jordan:	Well, we did three different things. How can they all get put together?
5	GS:	I guess we didn't look at one of those, but we will. The point is, they can all work together. For example, we said h

moves the graph left and right and k moves things up and down. Can't we do both at the same time?

6 Philip: That makes sense, but what would it look like? How do both happen at the same time?

When Jordan (line 2) asked me, "They can happen all at the same time?" I immediately realized my mistake in designing this lesson. Jordan's question (line 4) caught my attention. When he said, "How can they all get put together?" I realized the idea never occurred to him. In my work on deconstructing the main idea into pieces that the students could handle, I had forgotten to give them the chance to observe the phenomena together. In my journal notes for that day I wrote:

> Well, I think I self-sabotaged that lesson. Once again in spite of my hopes to create a lesson that was approachable to my students, I didn't anticipate the issues that would come up because of my choices. (April 26, 2017)

These two examples best illustrate problems that my attempts at scaffolding lessons created. In my attempts to create lessons that allowed students to make sense out of the mathematics we were studying, I failed to give them a complete picture. The pieces were there, but without examples that assembled the knowledge as a coherent unit, they fell short of their potential.

Reflections on My Struggle Scaffolding Activities to Support Students in Overcoming the Confusion and Challenges of Interpretation

Scaffolding is a great tool for teaching, especially when you have students that will engage in student-centered lessons but have not developed the skills of noticing the important mathematical ideas that are being presented. It also requires that a teacher refine the lesson each time it is taught. It is that refinement that I lacked during this year-long experience. I taught my middle school students, reflected in my journal on what had and would happen, and then prepared the next lesson. In hindsight it may have been better for me to teach two classes at the middle school so that between them I could reflect and redesign parts of the lesson. That seems to be a major component in good lesson design, reflection and refinement. I have used this type of teaching throughout my career, and my success with it has happened after multiple times presenting the lesson to different groups of students.

These types of lessons also need to be tailored to students based on their particular needs. Techniques that work one period may not work with another. Ideas that resonate with one cohort of students may fail the following year. Good teaching is not static or formulaic, but dynamic and ever changing. It is hard work, but if the goal is to determine where students are, meet them there, and move them to where they need to be, then the work is well worth it. These are lessons that do not come out of a particular text and will be the same even though the students change.

Teaching mathematics is a constant struggle between helping students gain a deep understanding of the ideas being studied and the demands of curriculum for results that show up nicely on standardized assessments. As I have come to realize through my reflection on my teaching of parabolas, short-term gains often undermine longer-term results.

One particular outcome of learning parabolas, similar to other parts of mathematics, is that both the teacher and the student must consider multiple ideas simultaneously. This involves multiple layers of knowledge interacting to make sense of concepts. I discovered these interactions as I made sense of the mathematics before, during, and after I presented lessons and activities to my students.

Reflecting on my work with parabolas, I discovered that I viewed parabolas as a set of three layers. The layers consist of the equation of a parabola, the terminology associated with the different features of a parabola, and the computational processes dictated by specific vocabulary based on the form of the equation. Understanding this perspective allowed me to design lessons about parabolas that helped my students learn in a genuine way that was authentic to the mathematics and vocabulary.

CHAPTER 8

REFLECTIONS ON RE-LEARNING TO TEACH ALGEBRA

The only way to learn mathematics is to do mathematics.

-Paul Halmos

In Chapters 5, 6 and 7, I analyzed my teaching of Inequalities, Exponents and Parabolas during the 2016 – 2017 school year. Three contemporaneous interactions emerged as fundamental to my work: my own interaction with the mathematics, my interaction with my students, and students' interaction with the mathematics. While each of these are distinct and can be discussed discretely, they were in constant motion and happening more or less simultaneously, influencing each other. My interaction with the mathematics came about from a need to deepen my own understanding of familiar content, which led me to re-think my students' interaction with that content as I was working interactively with them. This braiding of interactions reaches into my life-long respect for mathematics and its richness, complexity, and unknowability; for my respect in the awesome power of students' minds; and for my fears that I will fail to give both their due.

While teaching first-year Algebra to eighth grade students, I worked to design genuine lessons that respected their ideas, thoughts, and interactions with the mathematics they were learning. I accomplished this by attending to the complexity of inequalities, to the algebra of exponents, and to the interdependence of parabolas with the associated vocabulary. To allow learning to be approachable and authentic, I joined math teachers across the years by breaking concepts down into manageable portions and reassembling them with my students through learning activities.

Because of this emphasis on students' interaction with the mathematics they studied, my interactions with students primarily focused on content. Conversations between myself and students centered on either diagnosing a misunderstanding or validating thinking. My interaction with the different concepts we studied prepared me to thoughtfully consider my students' mathematical ideas, connect the ideas to particular mathematical concepts, and help them transition from an informal to a formal understanding.

While embracing ambitious teaching yet respecting the necessary content to prepare my students for the rigors of high school mathematics, I was able to identify within and through my practice two interwoven dimensions of in-service teacher learning relevant to algebra instruction and to math education policy and practice. These two broad categories emerged from my analyses in Chapters 5 – 7, re-learning the mathematics of first-year Algebra and how this induced, and was sometimes induced by re-learning how to teach first-year Algebra. The two categories inhere interrelated themes that recurred in my teaching units on Inequalities, Exponents and Parabolas.

Re-learning the Mathematics of First-Year Algebra

I have developed sufficiently strong mathematical knowledge to teach the highest levels that high schools offer (e.g. Calculus, Linear Algebra, Differential Equations). The last twenty-two years I have taught mathematics and embraced being a "math geek." I have always seen math as interesting, fun, fascinating, and something to be discovered in new and different ways.

Sixteen years of teaching upper-level high school math told me that I had a thorough understanding of the mathematics required at that level. Yet, in even the most basic parts of algebra, things that I had not thought much about because they seemed so simple forced me to pause and consider them because they were fundamental to later mathematical work. In returning to first-year Algebra, I had to reacquaint myself with foundational ideas; it was like rediscovering things long lost and being left in awe at their complexity. To be able to take these ideas to my students in ways that were authentic to the mathematics, I was forced to understand them myself.

The mathematics in first-year Algebra is deceptively complex, and my own mathematical experiences have taught me that the most fundamental mathematical concepts require the most complex logic to prove true. The complexity of these ideas can be lost due to their simple appearance and cause them to be distilled into a series of mindless steps simply to be learned and reproduced.

For example, when solving this simple equation

$$3x + 4 = 19,$$

the standard algorithm dictates subtracting 4 from both sides of the equation and then dividing the result by 3. The Additive Property of Equality allows any number to be added to both sides of an equation, and the fact that every real number has an additive inverse makes this computation possible. The same is true about dividing by 3 except the properties relate to multiplication instead of addition. While the computation appears simple, the mathematics behind these maneuvers are actually quite sophisticated.

As I approached teaching this class, I encountered the mathematics in a variety of ways in my attempts to cultivate a deep understanding. I found myself often getting reacquainted with ideas that I had become so familiar with that I took their substance for granted. Sometimes I explored topics with an intensity I never had in an attempt to solidify my knowledge. My exploration into the mathematics outlined within the curriculum led me to make choices both in the preparation and act of teaching.

Inequalities

Like my students, I had to make sense out of when and why the inequality symbol changes directions. This is such a fundamental feature of basic algebra that its complexity can easily get lost in the race to move forward with curriculum. It can be easily forgotten how, from an eighth grader's perspective, this fundamental element might be a challenge to grasp and use resourcefully in problem solving in later math courses.

The relation of students' prior learning to the content at hand, and the use of different types of situations in my efforts to provide authentic problem-solving activities became my orienting concerns. My re-learning of inequalities in order to teach them led me to pursue different ways to help students make sense out of changing the direction of the inequality symbol, to create connections between my students' prior and current learning, and to design lessons that utilized discrete and continuous situations.

Changing Direction

When I began preparing for the Inequalities Unit (Chapter 5), I first had to come to terms with an idea that had bothered me for as long as I could remember: the relation between two expressions changes when multiplied by a negative number. I never had to explain how or why until now, and I was forced to find ways to make sense of it.

A conventional means of making sense of this is to use a number line. If

$$a < b,$$

then b is further to the right on the number line than a. If the inequality is multiplied by -1, then the relationship needs to change because $-a$ will lie further to the right than $-b$. This is true because b is further to the right of 0 than a, so $-b$ will be further to the left of 0 than $-a$. This type of argument is useful in making sense of the idea, but it is inadequate in proving the idea true.

This argument, however, is mathematically insufficient. I dug deeper and found myself learning about the axioms of ordered fields. Out of this exploration came a proof (p. 60 – 62) that multiplying both sides of an inequality requires changing the direction of the symbol. I never intended sharing this with my students, but to feel prepared to teach this fundamental idea in more than a mechanical way, I needed this understanding. I was shocked to discover that this relationship was proven out of additive and not multiplicative properties.

Prior Knowledge

It is easy for me to take for granted that mathematics is formed by building ideas upon each other. In my Calculus class we begin with limits. The ideas of limits are used to develop and prove derivatives. Derivatives and limits are used to develop and prove integrals. Derivatives and integrals answer very different questions but share the common foundation of limits and have computational similarities.

As I prepared to teach the Inequalities Unit, it became clear to me that first-year Algebra took a similar path in its development, something that I had overlooked coming into this school year. When I first surveyed the topics, they appeared to form a traditional curriculum. We began with expressions, equations, and functions; moved on to solving one variable linear equations; and graphed two variable linear equations. As I reflected back on work with my students, I realized that this unit served as a place where all of the mathematics we had studied so far could be utilized.

Solving linear inequalities, either simple or compound, requires the same set of skills (p. 50) that solving linear equations requires. The knowledge developed in solving absolute value equations (p. 51) and graphing lines (p. 51) is utilized in solving absolute value inequalities and finding the graphical solution to two variable inequalities. While these skills may seem to form a simple list, they represent a significant amount of content knowledge in first-year Algebra.

Discrete and Continuous Modeling

While I have found it helpful to connect mathematical concepts to problems, I need to be cautious when designing any task that is both mathematically rigorous and authentic in the way the data is displayed. Knowing this forced me to reexamine and understand the difference between continuous and discrete variables.

A variable is continuous if between any two values another value could exist. Any variable that is not continuous is discrete. For example, on page 74, I showed my class a picture of a speed limit sign in South Dakota. Because between any two speeds another speed could exist, this is an example of a continuous variable. Buying Snacks (p. 53) is an example of a discrete variable because I am not guaranteed to find another valid solution between any two pairs of solutions.

Prior to teaching this class, I would have treated both problems the same. In the speed limit problem, I would have generated a graph similar to the one on page 64 which would correctly display all possible legal speeds. The snack problem would have also been treated as if it were continuous; I would have graphed the line, tested a point, and shaded the region that contained all the solutions.

The snack task becomes much richer, however, when it represents a discrete set of solutions. The solutions are only the ordered pairs which are non-negative integers because buying zero of either product is a valid solution. Equally germane to the situation is that it is impossible to buy a negative number of either snack. Therefore, in this case, only ordered pairs along either axis and the first quadrant are valid solutions, not all the points that shading the region indicates.

Exponents

It dawned on me, while planning and teaching, that exponents can be viewed as a distinct algebra. That is, exponents are their own algebraic realm, obeying the rules of algebra as a kind of subgenre that become a teaching resource. I was able to appreciate anew the properties of exponents, which led me to connect the ideas of negative exponents and inverses. My insight into the mathematics of exponents helped me create activities that empowered students to discover the algebra of exponents, validated my use of inductive logic to help students make conjectures, and supported my design of lessons about negative exponents in a way that honored the mathematics.

Algebra of Exponents

During most of my years of teaching, I thought about first-year Algebra as a single form of algebra, or the study of the manipulation of symbols to simplify expressions, manipulate equations, and solve for unknown quantities. In Chapter 5, as I described my work preparing to teach the exponents unit, I realized that I was preparing to teach a distinct form of algebra that is different than what I had taught previously, i.e., exponents are their own kind of algebra or a subset of algebra. This was a major insight for me.

The first six units of our class focused on working with the algebra of linear equations, which is built on the properties of equality. These properties are true when solving equations but are useless when simplifying expressions that involve exponents. The properties of exponents (p. 93) represent a distinct form of algebra, and are fundamentally different than the properties of equality. For example, if I multiply the equation

$$3x = 18$$

by $1/3$, I multiply 3 and 18 by $1/3$ which results in the equation

$$x = 6.$$

However, if I multiply each side by 3^{-1}, then to simplify $(3^{-1})(3)$, I add the exponents which results in $3^0 = 1$. Although this difference is subtle, it is a distinct type of algebra with different rules that can be discovered and utilized.

Proving Properties of Exponents

Once I realized that I was working with a different type of algebra, I realized that I had never proven the properties of exponents. Because first-year Algebra only worked with integer valued exponents, I put my focus on the proof for only that case. The result of this work (p. 107 – 109) helped me to understand how the idea is developed through an inductive process.

Induction, or generalizing patterns, is a powerful tool for mathematicians. It is often the beginning of the work necessary to prove an idea deductively. Deductive thinking is important, but often comes after an idea has been explored inductively, allowing for the different facets of an idea to be discovered. Playing with an idea and noticing patterns should be the first place the work with a concept occurs.

Negative Exponents and Inverses

Having made sense out of the algebra of exponents and how to prove the properties of exponents, I had one last mathematical concept to struggle with, the connection between negative exponents and inverse. As I wrote earlier (p. 124), it is easy to treat negative exponents and inverse function notation like homonyms, two symbols that appear the same but have two different meanings. My own mathematical experience has taught me that similar notations represent similar concepts.

Algebraically, the symbol

$$a^{-n}$$

is equivalent to

$$\frac{1}{a^n},$$

and the symbol

$$\frac{1}{a^{-n}}$$

is equivalent to

$$a^n.$$

However, I was unsure how to relate these to inverse notation, or

$$a^{-1}.$$

During my work preparing for Product of a Power,

$$(a^m)^n = a^{mn},$$

I realized that

$$a^{-n} = (a^n)^{-1} = (a^{-1})^n.$$

By changing how I looked at the notation, I transformed my interpretation of the notation from an integer raised to a negative exponent,

$$a^{-n},$$

to the inverse of an integer to a power,

$$(a^n)^{-1}$$

or the inverse of a number raised to a power,

$$(a^{-1})^n \, .$$

These two small shifts allowed me to unify the two ideas into one coherent concept. Discovering this connection within the concept revealed new ways to develop classroom tasks as illustrated in Chapter 5 and discussed below.

Parabolas

As I became reacquainted with the vocabulary necessary to study parabolas, I was struck by how the knowledge developed in the unit related to upper-level mathematics. This realization prompted me to rethink how students learn about parabolas so they could build intellectual resources crucial for studying upper-level mathematics.

From my content re-learning, I was able to create activities that allowed for a natural introduction and practice of vocabulary. These lessons developed ideas that I had taken for granted over years of teaching. I noticed and seized an opportunity to scaffold lessons as an assist for students who were balancing new vocabulary with rigorous content demands.

Words are Critical

The study of parabolas involves mathematical concepts and the vocabulary that represents those ideas. This language of parabolas is more than just a list of words that need to be learned, spelled correctly, and regurgitated. Instead, they are words tied to mathematical meanings and are necessary to communicate specific ideas in the context of larger conceptual understanding. These terms relate to different computational processes that reveal mathematical ideas.

Studying parabolas can include terms like vertex, line of symmetry, focus, directrix, x-intercepts, y-intercepts, maximum, minimum, and zeros. Each of these terms is carefully defined and connects to mathematical formulas and operations. For example, the x-intercepts are the x-coordinates of a graph when the y-coordinate is zero. A parabola can have either zero, one, or two x-intercepts. This is accomplished computationally by substituting zero for y and solving the resulting one-variable quadratic equation by either factoring, completing the square, or utilizing the quadratic formula.

As I prepared for this unit, I realized that I needed to simultaneously unlearn and relearn these terms. In my current practice, these ideas are automatic, requiring a vague mention of the idea before moving forward with a lesson. I had become so familiar with these ideas that I lacked the precision necessary to properly define the words and connect them to their mathematical content. I had to revisit the terms and their connections to computational processes through my own mathematical work on the topic before I could begin to design lessons for my students.

Connections to Upper-Level Mathematics

As I redeveloped my own understanding of the terminology of parabolas and how they connect to computational processes, I was astounded by the connections between parabolas and upper level mathematics. The terminology and concepts studied at the end of a first-year Algebra course provide a tool necessary to study Geometry and foundational understandings for upper-level mathematics.

Earlier I mentioned that solving one-variable quadratics can be accomplished by factoring, completing the square, or using the quadratic formula. In Geometry, quadratics are used to solve problems relating but not limited to Pythagorean Theorem, similar figures, area, and volume.

A parabola always has a maximum or minimum value, depending on the direction that it opens. Because a largest or smallest value of the function exists, optimal solutions exist to problems that can be modeled with quadratics. The topic of optimization appears in second-year Algebra and Calculus. The direction that a parabola opens, either up or down, is related to the concavity of a curve, an idea that appears in Precalculus and Calculus. These examples represent only a portion of the ideas that are part of studying parabolas in further coursework.

Re-Learning How to Learn

My exploration into critical vocabulary and connections between parabolas and upper-level mathematics was part of a larger issue that I was working through, re-learning how to study parabolas. Learning about parabolas meant understanding how to develop the necessary foundational ideas and how they link together into coherent concepts that result in meeting curricular goals.

I began by examining the broader mathematical idea (conics) and understanding how the particular ideas (parabolas) emerged from this general framework. The proof, included in Chapter 7, was the result of trying to understand why the equations of parabolas took the forms that they did. From this work I was able to make sense of the other formulas associated with parabolas (i.e. locating the x −coordinate of the vertex at $x = -\frac{b}{2a}$). By understanding the language of parabolas—and the appropriate use of the language in the modeling and solving of parabolas—it was easier to make connections to upper level mathematics. Seeing the complexity and importance of the language helped me design and implement appropriate learning tasks and set the stage for subsequent coursework.

Re-learning How to Teach First-Year Algebra

Reacquainting myself with the mathematics taught in first-year Algebra was the first step I took in preparation for teaching. The second and equally important step was re-learning how to teach first-year Algebra. This is material that was an integral part of my everyday practice. Solving inequalities, simplifying exponents, and working with parabolas are elements within the mathematics that I usually teach.

Doing this required me to make the familiar strange. Conversations with Phil throughout the year helped me gain a current practitioner's perspective on student knowledge and understanding. I gained further insight through classroom discussions where students shared their understanding and sense-making during lessons. Reflecting on these discussions in my journal writing helped me deepen my understanding of how students were learning.

Inequalities

Like my students, I had to make sense out of when and why the inequality symbol changes directions. This is such a fundamental feature of basic algebra that its complexity can easily be lost in the race to move on to bigger things. But rushing through this foundational concept is a disservice to students who are still struggling to grasp and use resourcefully ideas they will utilize again and again in the problem solving of second-year Algebra.

My re-learning of inequalities in order to teach them uncovered new and unique ways I could help my students make sense out of changing the direction of the inequality symbol, create connections between their prior and current learning, and design lessons utilizing discrete and continuous situations.

If when solving inequalities, the relationship changes when both expressions are multiplied by a negative number, which is a direct implication of the axioms of an ordered field, then students' understanding is enhanced by a teacher who understands how to connect the abstract proof to numerical and algebraic examples utilizing prior knowledge through discrete and continuous problem-solving situations. These ideas came to me through the mathematics I relearned in preparation for teaching and were implemented in the lessons throughout the unit. Linking my own learning with student learning allowed me to make choices that were mathematically authentic and created opportunities for students to make sense of critical concepts.

Changing Direction

The mathematical concepts I revisited to deepen my own understanding were not appropriate for my students. The proof offered me insight into how to construct an activity that created the opportunity for students to notice this property. In Chapter 5, I highlighted this activity (p. 57 – 58) and a dialogue between myself and a small group as well as large group discussion. We began with a true statement and manipulated it with algebraic properties. Instead of knowing the field axioms and working towards the goal of the proof, my students worked with the Algebraic Properties of Inequality until they reached a false statement. This false statement caused the small group I talked with to pause and reflect on their work. They then shared this conclusion with the rest of the class.

The choices I made in this lesson were a direct result of the mathematics that I learned. Instead of telling students the rule, I used my experiences to create a lesson that provided the opportunity for my students to come to this conclusion on their own. My role throughout the lesson was to offer guidance and enable my students to verify their thinking. By allowing students to draw their own conclusions and share their thinking with each other, the entire class was able to develop an understanding of a fundamental concept that was critical to the computational work throughout the unit.

Prior Knowledge

The lesson on changing the direction of the inequality symbol was the first of many lessons that accessed students' prior knowledge both from previous courses and concepts developed during first semester. To help my students learn new concepts throughout the unit, I formulated activities that connected prior knowledge with current learning as well as provided new experiences that could be referenced in subsequent lessons.

While it is easy to say that mathematics builds on itself, the challenge in teaching mathematics is in helping students connect a new concept with ideas that they have already acquired. In the Buying Snacks activity (p. 53), my students modeled a situation, found valid and invalid numerical solutions, and graphed the solutions. Later in the unit, this lesson was revisited, connecting the model to graphing a two-variable linear equation and shading the solution set. Shading the solution set was taught earlier in the unit (p. 60) in terms of one-variable inequalities. My awareness of particular conceptual connections helped me to guide my students through difficult portions of individual lessons by redirecting them to prior knowledge.

Discrete and Continuous Modeling

By becoming more aware of authentic mathematical modeling of situations, I was able to more carefully choose activities that required the use of an inequality and correctly represented the results. When asking for the graphical representation of who can ride the Kingda Ka (p. 73), or speed limits in South Dakota (p. 74), it was appropriate to shade the solution set because it represented a continuous variable. Because it was a discrete situation, representing the solutions to the Buying Snacks activity (p. 53) using only points was mathematically valid.

I utilized these experiences when I introduced my students to graphing two-variable inequalities. To help them understand why one particular region was shaded, I combined elements of both types of problems. By default, two-variable inequalities are continuous, so the region must be shaded. However, it is easier to determine the proper region by using a test point which is an acknowledgement that the set of discrete solutions to an inequality represents a subset of the continuous solutions.

Exponents

If exponents represent a distinct form of algebra separate from the algebra of linear equations, and if proving properties of integer exponents is accomplished using induction, then learning about exponents through an inductive approach offers students a way to connect a definition of exponents to the exponential properties that are studied in first-year Algebra. Knowing this allowed me to design activities and structure classroom discourse in such a way that the class would notice important properties and draw conclusions permitting individual students to make sense out of these ideas. This process was critical when negative exponents were introduced, while being mindful of the connection between negative exponents and inverses, avoiding the creation of expiring mathematical rules.

Algebra of Exponents

As I described in Chapter 5, exponents have their own kind of algebra, meaning the rules for manipulating exponents are distinct from the properties of linear equations. For integer exponents, these rules can be noticed and proven inductively. Using these two mathematical ideas as my guide, I was able to design lessons that reflected these ideas. The lesson (p. 93 – 96) that transitioned students from numerical noticing to formal symbolic work exemplifies this type of reasoning. My students took on the role of mathematicians, working with well-chosen examples that would help them discern these patterns. As the student discussion took place (p. 97), I was able to bring attention to the properties they noticed.

Noticing Properties of Exponents

Throughout this chapter, I designed lessons that took advantage of inductive logic. My students' role as mathematicians was limited to using inductive logic as a means to discern the properties. Because proofs by induction are inappropriate in first-year Algebra, I chose to leave that part of the lesson out. Noticing the properties created two different moves that I made use of later in the chapter. When my students were unsure about simplifying an expression (p. 106), I returned them to expanded form. Once they drew a conclusion, I was able to bring focus to writing their conclusions mathematically.

Negative Exponents and Inverses

By allowing my students to take on the role of mathematician throughout this unit, I was able to find a different way to introduce negative exponents without offering an incomplete definition. The student discussion on page 120 highlights how my students made sense out of what they saw. These ideas were not refined mathematical conclusions, but authentic displays of how students think, perceive, and wonder about mathematics. I listened to their ideas, translated their ideas into formal mathematical language, and allowed them to have ownership of their conclusions as they connected to the mathematics we were exploring.

Parabolas

If the language of parabolas is critical to learning about their mathematical foundations, then students' learning is enhanced when that vocabulary emerges authentically during explorations of parabolas. Students need the language to make the leap from learning about mathematics to doing mathematics. While doing the mathematics required to study parabolas, my students were able to build a foundation of knowledge that will benefit them as they move on to upper-level mathematics.

Developing Vocabulary

Mathematical language is more than just labeling parts of problems and equations; it is central to thinking mathematically. It is impossible to think through parabolas without the language of parabolas. One cannot, for instance, go from conic sections to parabolas without knowing when and how to employ the terminology.

In my class, students developed the necessary terminology by noticing features of a parabola. As students observed different elements of the graph (p. 199 – 200), I introduced the terms informally as part of the conversation. These informal introductions allowed me to formally introduce and define these words later in the lesson through activation of prior knowledge. By connecting my mathematical knowledge with the features that my students noticed, I allowed the language of parabolas to emerge in a natural way, fostering connections between formal mathematical language and the students' observations. The language of parabolas presented me the first opportunity to scaffold learning in way that supported my students' learning and understanding about parabolas.

Scaffolded Learning

The mathematics that we studied throughout this unit required specific language and computational processes dictated by the vocabulary. To help my students balance the mathematical demands studying parabolas require, I implemented a scaffolding process in individual lessons and the unit as a whole. The scaffolding process allowed me to focus on connecting individual or small group observations to general mathematical principles.

At times this process became the focus of the lesson (p. 199) instead of acting in support of the lesson. When this occurred, I found myself needing to assist students with processes that I thought would be automatic. These student struggles were not unexpected and required me to shift my focus from developing general concepts to addressing specific issues. It was critical that I diagnose the type of question a student or small group was asking so that I could address the need and help transition them from their struggles with an individual part towards a more general understanding.

Developing Knowledge

While my own knowledge of parabolas was well established, I had to find a way to help my students develop their own knowledge in an authentic way. The result of this effort was a series of lessons that connected student observations of the features of parabolas to formalization of the necessary terms while connecting those terms to computational processes. These lessons were developed to unify numerical, analytic, and graphical representations of parabolas.

To write these lessons, I found it necessary to maintain two stances toward parabolas. First, I had to develop a deep and rich understanding of the mathematics of parabolas so that I would be able to connect individual lessons in a framework that allowed my students to balance conceptual understanding with computation. Second, I had to find a way to make parabolas unfamiliar to me, otherwise my own knowledge would make it difficult to understand how students would be developing the ideas themselves.

What I did not Know

Over twenty-two years of teaching, I have learned a great deal about mathematics, students, and teaching. My return to the eighth grade classroom included acknowledging that my experiences had not fully prepared me for the complex landscape that I encountered. Confronting issues of mathematics, students, and teaching presented learning opportunities both during and while reflecting on my teaching of first-year eighth grade Algebra.

Foundational Proofs

My experience with mathematics has taught me that the most fundamental proofs are the most difficult. These difficulties can include determining how to start the proof, understanding how the pieces of the argument connect, or recognizing that it is complete. In my preparation to teach Inequalities, Exponents, and Parabolas, I realized that I either had not proven the fundamental ideas that I was teaching or did not fully understand the proofs. Each of the analysis chapters contain foundational proofs of the algebra we studied. I completed the proofs to deepen my own mathematical knowledge of these topics prior to teaching them. By completing these proofs, I hoped to gain insight into how to teach these concepts to my students.

Some of the mathematical work that I completed directly influenced my teaching. During the Exponents unit, I discovered that the proofs of the properties of integer exponents could be completed using a proof by induction. This proof guided my design of activities to help students notice and make sense of the properties. Although completing the proof assisted my design of the lesson, I did not present a full proof to my students because I did not believe it would deepen students' understanding.

I was not always successful in connecting my mathematical learning to my teaching. As I was attempting to mathematically justify why an inequality symbol changes direction when multiplied by a negative number, I wrote a proof based on the axioms of an ordered field. I dismissed this proof as too abstract to aid student learning and instead chose a numerical activity hoping students would make choices to notice this property. Upon deeper reflection, I realize that I could have designed a numerical activity based off of the axiom proof that would have allowed students to more readily notice the change in the relationship.

Connecting mathematical proof to teaching practice is something that I did not understand early in my career. I saw many of the mathematics courses I had taken during both undergraduate and graduate work as disconnected from my teaching unless I was presenting the proofs themselves. The mathematical experiences I encountered while teaching a first-year eighth grade Algebra course allowed me to bridge the divide between pure mathematics and mathematics for teaching.

Community of Learners

I have been teaching at my current high school for over fifteen years. During that time, I have come to understand the culture of learning that exists within the honors classes and honors students in the building. When a new group of students enters my classroom each year, I have a well-founded idea of the knowledge that they possess. This was not true at Jefferson Middle School. Before the first day of class, I realized that knowing mathematics and understanding high school honors students did not sufficiently prepare me to understand my eighth grade Algebra students as a distinct community of learners.

To learn about this new community, I turned to the people who best understood it, teachers within the building. I knew Phil from my experience in the NOYCE Master Teacher Fellowship and shared a room with him throughout the school year. He possessed knowledge that I lacked about building norms, teaching methodology, and prior course work the students had already encountered. Phil and I talked over email multiple times a day about lesson pacing, scaffolding of activities, and potential student reaction to lessons. My teaching became more effective because of his knowledge, and we planned together ways to connect building expectations to inquiry-based teaching.

During lesson planning, he helped me predict how students may react to particular activities that I may not have anticipated. When I proposed teaching the Fry’s Bank Account activity, I thought that only a few students would recognize the cartoon *Futurama.* He warned me that I would be surprised by how many students knew the show and that someone might know the answer. The discussion that I shared from that lesson in Chapter 6 confirmed his prediction.

This occurred again when we planned the *Angry Birds* project presented in Chapter 7. I thought that this game was old enough that not all of the students would be familiar with it. A lack of familiarity could have made the project less effective. Phil assured me that the students would be more than familiar enough with the game. The first few days of the unit confirmed his conclusions and allowed me to focus on the activity rather than having to teach the game first.

Managing Mathematics for Student Learning

As the topics became more complex and interdependent throughout the year, I found myself searching for ways to manage student interactions with mathematics in order to optimize student learning. I knew the mathematics, but I did not necessarily understand how these students made sense of the mathematical topics they were encountering for the first time. Phil provided insight during the planning phase, but I also needed to learn how to support students when they reacted in ways that neither of us anticipated.

In Chapter 7 I shared my use of scaffolding while evaluating complex functions in order to graph parabolas. I had utilized this technique in the past with high school students, and while the framework made sense to me, early on in the lesson it did not make sense to the students. I needed to spend time ascertaining from the students what did not make sense to them in order to support a more independent learning environment. It would have been easy to perform the calculations for them; however, this would have left the misunderstandings unaddressed.

I addressed similar issues in Chapter 6 when the students struggled to use multiple properties to simplify complex exponential expressions and again in Chapter 7 when the students struggled with simultaneous transformations of parabolas. As I learned to diagnose particular student misunderstandings, I was better able to support independent work by students and allow activities to progress. These interventions occurred during discussions between small groups of students and myself, as well as individual conversations with students. Supporting student learning in this manner is consistent with inquiry-based teaching and enabled me to grow in my understanding of which concepts my students securely understood and those that they needed to continue to improve.

Implications

I returned to the eighth grade classroom to teach first-year Algebra in an inquiry-based, student-centered, ambitious way within the constraints of district mandated curriculum. I developed and taught lessons while respecting course progression so that my students would be prepared to study advanced mathematics in high school and beyond. If I am in any way accurate in my observations and interpretations of the data, then this process has revealed two broadly categorized dimensions of learning relevant to the teaching and learning of first-year Algebra with implications for policy and practice. These implications offer guidance for professional development for first-year Algebra teachers who desire to transform or refine their practice.

Practice Mathematics

It seems fair to expect an art teacher to create art, an English teacher to read and write, and a Spanish teacher to speak and write in Spanish. The same should be true for algebra teachers. It should be expected that algebra teachers do algebra as a way to practice and refine content knowledge. I have displayed my own attempts at this by placing an emphasis on mathematics throughout this research.

In the span of my career as an educator, I have found the most effective professional development activities for math teachers involve doing mathematics. I completed my Master of Science in Mathematics in order to deepen my own knowledge. My work within the Robert Noyce Master Teacher Fellowship involved taking additional mathematics courses focused on teaching math. These opportunities improved my own conceptual understanding and enhanced my teaching practice involving particular topics of larger mathematical domains. Professional learning of mathematics exposes teachers to different ways of helping students learn mathematics.

Focus on Student Learning

When teaching in a student-centered classroom, it is critical that the teacher shift the focus from personal knowledge and understanding to the development of student reasoning and sense-making about particular topics. Because we are teachers, we have already demonstrated that we know the topic. Instead of passing on how we make sense of a topic, we need to provide each individual we work with the opportunity to do it for themselves. It is not about us; it is about the students.

Shifting the burden of understanding mathematics from a passing down of skills to a personal development, students gain the opportunity to cultivate and practice habits of mind that extend beyond the mathematics classroom. Students have the ability to make and test general statements based on observed patterns. They learn to pose questions that challenge the thinking of others. They become independent thinkers and critical consumers of knowledge. This shift also allows for an easy answer to the question, "When are we ever going to use this?" Every student needs to learn to think and reason, and teaching mathematics in a student-centered fashion provides the perfect opportunity to develop these skills.

Every Student is a Mathematician

As teachers engaged with content providing students occasion to reason and make sense of the mathematics that they are studying, it is incumbent upon them to view students as mathematicians. Viewing students as mathematicians means helping them engage in the habits of mathematicians: making generalizations based on patterns, finding connections between concepts, writing careful and precise definitions, and applying concepts to solve problems. Instruction in a student-centered classroom is enhanced when students' thoughts are treated as valid mathematical conclusions, even when vocabulary is lacking to formally express the emerging idea. The teacher can act as an expert by diagnosing student misunderstandings, finding the mathematical ideas being expressed, and offering guidance through complex and challenging tasks.

Understand What You Can Control

Although I had spoken with teachers about making sure that their lessons include examples that are rigorous enough help students grow mathematically, I had never taken the time to examine the problems I selected for the students to work on in class. I noticed that during my work with inequalities, my students struggled with the idea of inverting the inequality sign when multiplying or dividing by a negative. I did not realize until I had reflected upon it that I failed to give them adequate practice in our classwork to reinforce that skill.

It is too easy for any of us to place the blame on the students for their failure to practice a skill like this independently. The old cliché "Practice makes perfect" only applies if the practice is informed. A student who practices a skill incorrectly only reinforces their own misunderstanding. Our daily work in the classroom with informal assessment should guide our instruction in the short term. Any practice would benefit from the opportunity to examine all work done throughout a unit after a summative assessment is given so the teacher understands the impact of their daily instruction on student learning.

Awareness of Student Struggles with Prior Concepts During Current Learning

One structural quality of mathematics that attracted me to the subject is that most mathematical knowledge builds on itself. The work I did with students during our unit on exponents emphasized this idea. Once the students had established that $a^m \cdot a^n = a^{m+n}$, making sense of $(a^m)^n = a^{mn}$ could be accomplished by using the first property. While I had been aware of my students' struggles keeping the properties of real numbers separate from the properties of exponents, I did not give it the proper emphasis during instruction.

This school year I have applied this lesson to my daily instruction. After grading formative assessments, I take note of the errors the students are making and use it as an opportunity to both reteach and warn. This year I find myself making statements like, "Now we've struggled with this idea in the past, so let's be careful at this point to make sure we're okay with what's going on." This is tricky territory because there is a huge difference between pointing out faults and using them as teachable moments. Students can receive the wrong message if we do not emphasize that being aware of our mistakes helps us grow instead of seeing them as liabilities.

Mathematical Content and Pedagogy are Self-Reinforcing

Reflecting on my experience teaching first-year Algebra allowed me to observe my intellectual work between understanding the content I was teaching and how I would be teaching it. I realized that this work is not a simple implication (e.g. If I learn the content, then I will be ready to teach it), but instead that teaching algebra conceptually requires me to move back and forth between these two domains. The content and the pedagogy reinforce each other. Because of this, I believe that as teachers we should learn both in tandem during organized professional development experiences.

Learning Algebra is Non-Linear

While I have pondered the idea for years, this experience crystalized the reality that learning math is a non-linear activity despite the fact that most curriculum is organized in a purely linear fashion. Algebra (and all of mathematics) is in part an exercise in logic, one idea building on another. My experience has been that most curriculum resources over emphasize this idea and present mathematics in a logical, complete, and sanitized form, devoid of the joy and excitement of discovering a new relationship within a pattern of numbers or family of functions. If learning is viewed in terms of the standard curricular resources, then it is easy to believe that learning mathematics should follow that structure; it does not.

Students encounter similar linear concepts during a first-year Algebra course. When I taught the inequalities unit, I realized that it was an opportunity to reinforce nearly every topic that was a part of the prior units. Before this realization, I became frustrated that my students did not automatically remember how to solve equations or graph lines. I had to let go of the mindset that once I taught and assessed a particular topic, my students had mastered it. Instead, I shifted toward the mindset that every encounter with a prior topic was an opportunity to refine understanding and master procedures. This change allowed me to see my students' learning of mathematics as an ongoing process of growth towards mastery.

Grow in Your Understanding of Vertical Curriculum

The work that the class did with parabolas exposed a disconnect between what my students knew and what they were able to express mathematically. While examining classroom discussion analyzing the graph (p. 174), my students were able to give different points on the graph where the $x-$values were different but the $y-$values were the same when they were presented in words. They struggled to connect those words to the symbolic statement

$$f(a) = f(b).$$

It would be easy to dismiss the notation as unnecessary because my students were able to solve the problem.

This type of notation, however, plays a larger and larger role for students as they move through high school mathematics curriculum. If students are never exposed to an opportunity to develop the ability to read and decode mathematical notation in first-year Algebra, then in the next course for most students, Geometry, the nature of communication in the course (almost purely symbolic) will be foreign to them. They do not need to master this type of communication, but they do need the opportunity to develop it.

Knowing these connections requires a math teacher to understand how their course fits into the overall structure of a student's mathematics education. Students taking first-year Algebra in eighth grade puts them on course to take Calculus during their senior year. Strong preparation with an awareness of classes to come can help students better prepare for advanced math courses which in turn can give them access to high paying careers in STEM fields. A lack of awareness can lead a teacher to make choices to avoid difficult material without understanding the long-term effects it can have on students.

Find Ways to Keep a Conceptual and Computational Balance

One trend that I noticed during my year in the eighth grade classroom was the emphasis my students place on computation. They really wanted mathematics to be a computational exercise that began with a problem and ended with an answer. These were the same students that struggled later in the year because they wanted to memorize a large set of rules to follow rather than understand why they were executing the procedures they had chosen.

My experiences with eighth grade students in a first-year Algebra classroom have forced me to examine my role in helping students transition from their prior mathematical experiences to the real work of algebra. While I placed heavy emphasis on coming up with activities that created an opportunity for mathematical representation, I did not reemphasize why we were doing it. Students are more capable of being a part of the instructional conversation than we realize. This type of meta-conversation about learning is an opportunity to help students understand why we are doing what we are doing and how it connects to the mathematics we are studying.

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