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Sequential Differences in Nabla Fractional Calculus

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SEQUENTIAL DIFFERENCES IN NABLA FRACTIONAL CALCULUS

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SEQUENTIAL DIFFERENCES IN NABLA FRACTIONAL CALCULUS

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We study the composition of nabla fractional differences of unequal orders, known as "sequential" nabla fractional differences, of the form $\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}f(t)$ in the cases

1. $k < \mu < k+1$, $k+1 < \nu < k+2$, and $2k+1 < \mu+\nu < 2k+2$ for $k \in \mathbb{N}_0$,

2. $k < \mu < k+1$, $k-1 < \nu < k$, and $2k < \mu+\nu < 2k+1$ for $k \in \mathbb{N}_1$.

In this work, we develop rules for sequential nabla fractional differences and present connections between the sign of these sequential differences and the monotonicity of the function $f(t)$. We establish uniqueness of solutions to various initial value problems and boundary value problems involving sequential nabla fractional differences and give an explicit expression for the Green's functions. We conclude with an investigation of the properties of the Green's functions and some useful generalizations involving sequential nabla fractional differences.

To my grandmother, Rosemary, who nurtured a joy of critical thinking.

To my father, who modeled calculation in problem solving.

To my mother, who instilled in me the confidence to do anything.

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Chapter 1

Preliminaries

In this dissertation we consider discrete fractional boundary value problems of the form

$$\begin{cases} -\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}y(t) &= h(t), \quad t \in \mathbb{N}^{b}_{a+2k+3} \\ y(a+2k+2) &= 0, \\ y(b) &= 0 \end{cases} \tag{1.1}$$

where $k+1 < \nu < k+2$, $k < \mu < k+1$, and $2k+1 < \mu+\nu < 2k+2$ and $b-(a+2k+2) \in \mathbb{N}_1$. The composed difference in (1.1) is known as a *sequential nabla fractional difference.*

The field of continuous fractional calculus has been widely developed, seeing much growth just in the past decade. Comparatively, the field of discrete fractional calculus has progressed rather slowly. In recent years, however, a number of works have begun to contribute to the theoretical base of discrete fractional calculus. For example, Atici and Eloe [12] and Goodrich [24] analyzed discrete fractional initial value problems. Other recent advances in the field of discrete fractional calculus may be found in [4], [19], and [49]. Of particular note, a recent paper by Bachher, Abbas, & Sarkar [14] interestingly addressed modeling of porous material using fractional difference equations, and another work by Atici & Sengul [13] presented the modeling of cancer

tumor growth with the use of fractional difference equations. These imply, then, that there is some promise in using fractional difference equations to model physical problems, where changing the order of the difference to a fractional order may allow for a more accurate model.

In the theory of both differential and difference equations, fractional boundary value problems have been extensively researched. In the classical fractional discrete case (i.e., when we have just the single operator ∇_a^μ with $0 < \mu < 1$), problems have been considered in several recent works - see, for instance, [4], [18], [37] and the references therein. However, we are not aware of work thus far which attempts to consider sequential differences in discrete fractional boundary value problems. Due to the unique case sequential differences present in requiring different domains, it seems mathematically interesting to investigate these sorts of problems, and in this work we make an initial attempt to fill this gap in the existing literature.

This dissertation proceeds as follows. We begin by offering a light introduction concerning the field of discrete nabla fractional calculus, including some new theorems for the nabla Laplace transform to help us elegantly solve initial value problems. We then present what contributions have been made in the way of composition rules in the theory of discret fractional calculus, and proceed to examine and prove some advanced composition rules in order to better understand the relationships between two different types of nabla differences. Further, we study the connections between the sign of sequential nabla fractional differences and the monotonicity of the function. Such rules, identities, and relationships are useful when determining solution behavior of initial value problems and boundary value problems. The last chapter studies boundary value problems that involve sequential nabla difference equations.

1.1 Introduction to Discrete Nabla Fractional Calculus

The derivative is at the heart of mathematics and specifically calculus, so why not extend this idea? *What does it mean to take a half derivative? A π^{th} derivative?* These are the questions that sparked the area of fractional calculus. Indeed, L'Hôpital asked this first question in a 1695 letter to Leibniz. Throughout the centuries, the notion of a fractional derivative has been defined for all complex numbers. The theory of fractional calculus builds around this initial definition of the fractional derivative, often times following analogously the theory of continuous whole-order calculus as we know it. Fractional calculus has a wide variety of applications such as signal processing, diffusion problems [14], and modeling porous materials [17]. In particular, when we study fractional calculus over a discrete domain, we can produce equations which best model cancer growth [13].

1.2 Background

1.2.1 Nabla Whole Order Differences and Integrals

Within the study of discrete fractional calculus, we study real-valued functions defined on a shift of the natural numbers, either

$$\mathbb{N}_a = \mathbb{N}_0 + \{a\} = \{a, a+1, a+2, \dots\} \text{ or } \mathbb{N}_a^b := \{a, a+1, \dots, b\}$$

for $a, b \in \mathbb{R}$ fixed such that $b - a$ is a positive integer. Analogous to a whole-order derivative for real functions, there are two main types of difference functions. The first is the delta, or forward, difference, which is widely used in ordinary difference equations. However, the delta difference presents an often problematic shifting of domain - when we take the delta difference of a function the resulting domain has

empty intersection with the domain of the original function. In the case of the second fractional difference, domain shifts do not present as much of an issue [[37], p. 149]. In this work, we use this second fractional difference, called the nabla difference, or backwards difference, defined as follows.

Definition 1.1. *The nabla difference of* $f : \mathbb{N}_a \to \mathbb{R}$ *is defined by*

$$\nabla f(t) := f(t) - f(t-1), \quad t \in \mathbb{N}_{a+1}.$$

Differences of higher order $N \in \mathbb{N}$ are defined iteratively:

$$\nabla^N f(t) := \nabla \left(\nabla^{N-1} f \right)(t), \quad t \in \mathbb{N}_{a+N}.$$

For convenience, we often use the backward jump operator, defined as follows.

Definition 1.2. *The backward jump operator* $\rho : \mathbb{N}_{a+1} \to \mathbb{N}_a$ *is given by* $\rho(t) = t - 1$.

In nabla fractional calculus, the discrete version of the definite integral in single variable calculus is the nabla definite integral of a function defined as follows.

Definition 1.3. *The nabla definite integral of* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *is defined for* $t \in \mathbb{N}_a$ *by*

$$\int_a^t f(s) \nabla s := \sum_{s=a+1}^{t} f(s),$$

with the convention $\int_a^a f(t)\nabla t = 0$.

Many results of single variable calculus have analogous versions in the whole-order discrete case. In particular, we have a Fundamental Theorem of Nabla Calculus which gives a relationship between the nabla difference and the nabla definite integral. First, we define the notion of an antidifference.

Definition 1.4 (Definition 3.33, [37]). *Assume* $f : \mathbb{N}_{a+1}^b \to \mathbb{R}$. *We say* $F : \mathbb{N}_a^b \to \mathbb{R}$ *is a nabla antidifference of* $f(t)$ *on* $\mathbb{N}_a^b$ *provided*

$$\nabla F(t) = f(t), \quad t \in \mathbb{N}_{a+1}^b.$$

If $f : \mathbb{N}_{a+1}^b \to \mathbb{R}$, *then if we define* F *by*

$$F(t) := \int_a^t f(s)\nabla s, \quad t \in \mathbb{N}_a^b,$$

we have that $\nabla F(t) = f(t)$ *for* $t \in \mathbb{N}_{a+1}^b$, *that is,* $F(t)$ *is a nabla antidifference of* $f(t)$ *on* $\mathbb{N}_a^b$.

Theorem 1.5 (Theorem 3.37, [37]). ***(Fundamental Theorem of Nabla Fractional Calculus)*** *If* $f : \mathbb{N}_{a+1}^b \to \mathbb{R}$ *and* F *is any nabla antidifference of* f *on* $\mathbb{N}_a^b$ *(i.e.,* $\nabla F(t) = f(t)$, *for* $t \in \mathbb{N}_{a+1}^b$*), then*

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

1.2.2 Nabla Fractional Sums and Differences

In order to motivate nabla fractional sums and differences, we need to first define the whole-order nabla sum of a function $f : \mathbb{N}_{a+1} \to \mathbb{R}$. We define this sum by repeated integration.

Definition 1.6. *For* $f : \mathbb{N}_{a+1} \to \mathbb{R}$, *the whole-order nabla sum is defined by*

$$\nabla_a^{-n} f(t) := \int_a^t \int_a^{\tau_{n-1}} \cdots \int_a^{\tau_2} f(\tau_1)\nabla\tau_1\nabla\tau_2 \cdots \nabla\tau_{n-1},$$

where we use $-n$ *in the notation of the operator to signify the reverse operation of*

the nabla difference.

We will next state a more convenient formula for the whole-order sum, which condenses this repeated integration into a single integral expression. This will allow us to generalize to a sum of any positive order, and we will then define a fractional difference in terms of this fractional sum.

Theorem 1.7. *For $f : \mathbb{N}_{a+1} \to \mathbb{R}$, the whole-order nabla sum is given by*

$$\nabla_a^{-n} f(t) := \int_a^t \frac{(t-s+1)^{\overline{n-1}}}{(n+1)!} f(s) \nabla s$$

for $t \in \mathbb{N}_a$, $n \in \mathbb{N}_1$.

The rising function in the above theorem is given by the following definition.

Definition 1.8. *For $t \in \mathbb{N}_a$ and $n \in \mathbb{N}_1$, the fractional rising function is defined by*

$$t^{\overline{n}} = t(t+1)(t+2)\cdots(t+n-1),$$

read as "t to the n rising."

We can rewrite this definition using factorial functions when our domain is based at $a = 1$. This reformulation will be useful when we generalize the rising function in the next section:

$$t^{\overline{n}} = \frac{(t+n-1)!}{(t-1)!}.$$

In order to begin thinking about the fractional case of some of the previous results, such as considering a nabla difference of order 2.3 or order π, we need to define the Gamma function.

1.2.3 Extending to the Fractional Case

Definition 1.9. *The Gamma function is defined by*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

for $z \in \mathbb{C}$ such that the real part is positive.

We can integrate $\Gamma(z+1)$ by parts to find the following property.

Property 1.10. *Considering the Gamma function defined in Definition 1.9,*

1. $\Gamma(z+1) = z\Gamma(z)$ *for z such that the real part is positive.*

2. *For $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.*

Part (1) is used to extend the domain of the gamma function $\Gamma(z)$ to all complex numbers $z \notin \mathbb{C} \setminus \{\ldots, -2, -1, 0\}$.

Considering how the Gamma function extends the factorial function to values in $\mathbb{C} \setminus \{\ldots, -2, -1, 0\}$, it is natural to extend the rising function in terms of the Gamma function.

Definition 1.11. *The fractional rising function is defined by*

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)},$$

for $t \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$, $\alpha \in \mathbb{R}$, with the conventions

(i) $t^{\overline{\alpha}} = 0$ *provided $t \in \mathbb{Z}^-$ and $t + r \notin \mathbb{Z}^-$,*

(ii) $t^{\overline{0}} = 1$.

Following from the whole-order sum, we have the nabla fractional sum of order ν.

Definition 1.12. *For $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$,*

$$\nabla_a^{-\nu} f(t) = \int_a^t \frac{(t-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)} f(s)\nabla s, \quad t \in \mathbb{N}_{a+1}.$$

From this fractional sum we can define the nabla fractional difference (specifically, the Riemann-Liouville nabla fractional difference).

Definition 1.13 (Riemann-Liouville Nabla Difference)**.** *For $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$,*

$$\nabla_a^{\nu} f(t) := \nabla^N \nabla_a^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+N},$$

where $N = \lceil \nu \rceil$.

We can rewrite the above (Riemann-Liouville) ν^{th} order nabla fractional difference in a form more similar to the ν^{th} order nabla fractional sum. This form is widely more useful, so we state this as the next theorem.

Theorem 1.14. *Assume that $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, and $\nu \notin \mathbb{N}$. Let $N \in \mathbb{N}$ such that $N-1 < \nu < N$. Then the ν^{th} order nabla fractional difference is given by*

$$\nabla_a^{\nu} f(t) := \int_a^t (t-\rho(s))^{\overline{-\nu-1}} f(s)\nabla s = \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{-\nu-1}} f(s),$$

for each $t \in \mathbb{N}_{a+n}$.

There is another fractional difference, namely, the nabla Caputo fractional difference, which is also defined in terms of the fractional sum.

Definition 1.15. *For $\nu > 0$ and $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$,*

$$\nabla_{a*}^{\nu} f(t) := \nabla_a^{-(N-\nu)} \nabla^N f(t), \quad t \in \mathbb{N}_{a+1},$$

where $N = \lceil \nu \rceil$.

The research concerning the Riemann-Liouville nabla difference is more developed; however, the Caputo nabla difference has some nice properties the Riemann-Liouville nabla difference does not possess. One such property is that the Caputo ν^{th}-order difference of any constant function is zero for $\nu \geq 1$. This is only true in the Riemann-Liouville case when $\nu = k$ is a positive integer, or when the constant is zero.

1.3 Properties and Formulas for Nabla Differences

In this section we present many useful properties of nabla differences, nabla Taylor monomials, and Caputo nabla differences.

In the next theorem we collect some of the properties of the fractional nabla difference.

Theorem 1.16. *Assume* $f, g : \mathbb{N}_a \to \mathbb{R}$ *and* $\alpha, \beta \in \mathbb{R}$. *Then for* $t \in \mathbb{N}_{a+1}$,

(i) $\nabla(f(t) + g(t)) = \nabla f(t) + \nabla g(t)$,

(ii) $\nabla(f(t)g(t)) = f(\rho(t))\nabla g(t) + \nabla f(t)g(t)$,

(iii) $\nabla\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)\nabla f(t) - f(t)\nabla g(t)}{g(t)g(\rho(t))}$, *if* $g(t) \neq 0$, $t \in \mathbb{N}_{a+1}$.

We will make use of the following elementary result.

Theorem 1.17 (Integration by Parts)**.** *Given two functions* $u, v : \mathbb{N}_a \to R$ *and* $b, c \in \mathbb{N}_a$ *such that* $b < c$, *we have the integration by parts formula*

$$\int_b^c u(\rho(s))\nabla v(s)\nabla s = u(s)v(s)\Big|_b^c - \int_b^c v(s)\nabla u(s)\nabla s.$$

We also have a variety of power rules within nabla fractional calculus as follows.

Property 1.18. *For $p \in \mathbb{N}$ and values of $t, r,$ and α so that the values in the following equations make sense, we have*

(i) $\nabla(t-a)^{\overline{\alpha-1}} = (\alpha-1)(t-a)^{\overline{\alpha-2}}$,

(ii) $\nabla(\alpha-t)^{\overline{r}} = -r(\alpha-\rho(t))^{\overline{r-1}}$,

(iii) $\nabla^p(t-a)^{\overline{\alpha-1}} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p)}(t-a)^{\overline{\alpha-p-1}}$.

Proof. The proofs of (1) and (2) can be found in [37]. We prove part (3). Notice by repeated use of part (1) we have

$$\begin{aligned}
\nabla^p(t-a)^{\overline{\alpha-1}} &= \nabla^{p-1}\nabla(t-a)^{\overline{\alpha-1}} \\
&= \nabla^{p-2}\nabla(\alpha-1)(t-a)^{\overline{\alpha-2}} \\
&= \nabla^{p-2}(\alpha-1)(\alpha-2)(t-a)^{\overline{\alpha-3}} \\
&= \nabla^{p-3}(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^{\overline{\alpha-4}} \\
&\dots \\
&= \nabla^{p-p}(\alpha-1)(\alpha-2)\cdots(\alpha-p)(t-a)^{\overline{\alpha-p-1}} \\
&= (\alpha-1)(\alpha-2)\cdots(\alpha-p)(t-a)^{\overline{\alpha-p-1}} \\
&= \frac{\Gamma(\alpha)}{\Gamma(\alpha-p)}(t-a)^{\overline{\alpha-p-1}}.
\end{aligned}$$

□

1.3.1 Nabla Taylor Monomials

In this section we study the nabla Taylor monomials and give some of their important properties. These nabla Taylor monomials appear in the nabla version of Taylor's Theorem, and further prove useful in reformulating the nabla sum and nabla difference.

Definition 1.19. *For $\alpha \neq -1, -2, -3, \ldots,$ the α^{th}-order ∇-fractional Taylor monomial is given by*

$$H_\alpha(t,a) = \frac{(t-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)}.$$

In the next theorem we present some basic properties of fractional nabla Taylor monomials.

Theorem 1.20 (Theorem 3.57, [37])**.** *The following hold:*

(i) $H_\mu(a,a) = 0$,

(ii) $\nabla H_\mu(t,a) = H_{\mu-1}(t,a)$,

(iii) for $k \in \mathbb{N}_1$, $H_{-k}(t,a) = 0$, $t \in \mathbb{N}_a$,

provided the expressions are well-defined.

Now that we have defined the nabla Taylor monomial, we can rewrite some of our previous definitions and properties in terms of Taylor monomials, namely part (3) of Property 1.18 becomes

$$\nabla^p (t-a)^{\overline{\alpha-1}} = \Gamma(\alpha) H_{\alpha-p-1}(t,a),$$

and the definition of the nabla sum (Definition 1.12) becomes the following.

Definition 1.21. *The fractional ∇-sum of order $\alpha > 0$ for $f : \mathbb{N}_{a+1} \to \mathbb{R}$ is defined by*

$$\nabla_a^{-\alpha} f(t) = \int_a^t H_{\alpha-1}(t, \rho(s)) f(s) \nabla s, \quad t \in \mathbb{N}_a,$$

where by convention $\nabla_a^{-\alpha} f(a) = 0$.

We note that $\nabla_a^{-n} f(t)$ satisfies

$$\begin{cases} \nabla^n y(t) & = f(t) \\ \nabla^i y(a) & = 0, \qquad \text{for } i = 0, 1, \ldots, n-1. \end{cases}$$

The following result concerning Taylor monomials is useful.

Property 1.22. *For $\alpha > 0$,*

$$\nabla_a^{-\alpha} H_\nu(t,a) = H_{\nu+\alpha}(t,a).$$

Specifically, note that the above property holds for arbitrary a, and further, that the base of the operator need not match the value inside the Taylor monomial, meaning, for example, that the following holds.

Remark 1.23. *For $\alpha > 0$,*

$$\nabla_a^{-\alpha} H_\nu(t,a+1) = H_{\nu+\alpha}(t,a+1).$$

We also have a rule for taking a fractional nabla difference of a Taylor monomial.

Theorem 1.24 (Theorem 3.93, [37])**.** *Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that μ and $\mu - \nu$ are nonnegative integers. Then we have that*

$$\nabla_a^{\nu} H_\mu(t,a) = H_{\mu-\nu}(t,a).$$

Thus far, we have considered power rules concerning whole-order differences. We now move on to considering such a rule involving a fractional nabla sum.

Lemma 1.25. *Assume f is a real-valued function defined on $\mathbb{N}_{a+1}$, $\alpha > 0$, and $0 \leq N-1 < \nu \leq N$. Then*

$$\nabla_a^{-\alpha}(t-a)^{\overline{\nu}} = \Gamma(\nu+1)H_{\nu+\alpha}(t,a) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}(t-a)^{\overline{\nu+\alpha}}$$

for $\nu \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$ such that $\nu+\alpha \in \mathbb{Z}^+$, $t \in \mathbb{N}_a$.

Proof. The proof is given in [37] (Theorem 3.93 (iii)). It proceeds as follows:

$$\begin{aligned}
\nabla_a^{-\alpha}(t-a)^{\overline{\nu}} &= \nabla_a^{-\alpha}\frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}\Gamma(\nu+1) \\
&= \Gamma(\nu+1)\nabla_a^{-\alpha}H_\nu(t,a) \\
&= \Gamma(\nu+1)H_{\nu+\alpha}(t,a) \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}(t-a)^{\overline{\nu+\alpha}}.
\end{aligned}$$

□

1.4 Nabla Laplace Transforms

Just as in traditional calculus, the Laplace transform provides us with a nice way to solve initial value problems involving fractional nabla difference equations. In this section, we provide basic definitions and properties of the nabla Laplace transform operator $\mathcal{L}_a$ which are useful in solving various nabla fractional initial value problems.

We will use the convenient formula presented by Goodrich & Peterson [37].

Theorem 1.26 (Theorem 3.65, [37])**.** *Assume $f : \mathbb{N}_{a+1} \to \mathbb{R}$. Then*

$$\mathcal{L}_a\{f\}(s) = \sum_{k=1}^{\infty}(1-s)^{k-1}f(a+k).$$

for those values of s such that this infinite series converges.

We note that the above formula is well-defined for $s = 1$.

Theorem 1.27 (Corollary 3.75, [37]). *For $\nu \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$, we have that*

$$\mathcal{L}_a\{H_\nu(\cdot, a)\}(s) = \frac{1}{s^{\nu+1}}, \quad for |s-1| < 1.$$

Theorem 1.28 (Theorem 3.82, [37]). *Assume $\nu > 0$ and the nabla Laplace transform of $f : \mathbb{N}_{a+1} \to \mathbb{R}$ converges for $|s-1| < r$ for some $r > 0$. Then*

$$\mathcal{L}_a\{\nabla_a^{-\nu} f\}(s) = \frac{1}{s^\nu}\mathcal{L}_a\{f\}(s)$$

for $|s-1| < \min\{1, r\}$.

In order to present further nabla Laplace transform results, we need to define functions of exponential order.

Definition 1.29 (Definition 3.68, [37]). *A function $f : \mathbb{N}_{a+1} \to \mathbb{R}$ is said to be of exponential order $r > 0$ if there exists a constant $M > 0$ and a number $T \in \mathbb{N}_{a+1}$ such that*

$$|f(t)| \leq Mr^t, \quad \forall t \in \mathbb{N}_T.$$

Theorem 1.30 (Theorem 3.85, [37]). *Assume $f : \mathbb{N}_{a-n+1} \to \mathbb{R}$ is of exponential order $r > 0$. Then*

$$\mathcal{L}_a\{\nabla^n f\}(s) = s^n \mathcal{L}_a\{f\}(s) - \sum_{k=1}^{n} s^{n-k} \nabla^{k-1} f(a),$$

for $|s-1| < r$, for each $n \in \mathbb{N}_1$.

Lemma 1.31 (Shifting Base Lemma 3.88, [37]). *Given $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $n \in \mathbb{N}_1$, we have that*

$$\mathcal{L}_{a+n}\{f\}(s) = \left(\frac{1}{1-s}\right)^n \mathcal{L}_a\{f\}(s) - \sum_{k=1}^{n} \frac{f(a+k)}{(1-s)^{n-k+1}}.$$

Theorem 1.32. *Assume $\nu > 0$, $N-1 < \nu < N$ and $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$ is of exponential order $r > 0$. Then*

$$\mathcal{L}_a\{\nabla_a^\nu f\}(s) = s^\nu \mathcal{L}_a\{f\}(s),$$

for $|s-1| < r$ for each $N \in \mathbb{N}_1$.

Proof. By taking the Laplace transform of both sides and using the definition of the nabla difference, we have

$$\begin{aligned}
\mathcal{L}_a\{\nabla^\nu f\}(s) &= \mathcal{L}_a\{\nabla^N \nabla_a^{-(N-\nu)} f\}(s) \\
&= s^N \mathcal{L}_a\{\nabla_a^{-(N-\nu)} f\}(s) - \sum_{k=1}^{N} s^{N-k} \nabla^{k-1} \nabla_a^{-(N-\nu)} f(a) \\
&= \frac{s^N}{s^{N-\nu}} \mathcal{L}_a\{f\}(s) - \sum_{k=1}^{N} s^{N-k} \nabla^{k-1} \nabla_a^{-(N-\nu)} f(a) \\
&= s^\nu \mathcal{L}_a\{f\}(s) - \sum_{k=1}^{N} s^{N-k} \nabla^{k-1} 0 \\
&= s^\nu \mathcal{L}_a\{f\}(s),
\end{aligned}$$

where we used Theorems 1.30 and 1.28 and the convention

$$\nabla_a^{-(N-\nu)} f(a) = \sum_{a+1}^{a} H_{N-\nu-1}(a, \rho(s)) f(s) = 0.$$

□

Remark 1.33. *The above theorem holds for any a, meaning, as long as the base of the Laplace transform and the base of the nabla difference match, the result holds.*

The following theorem provides a formula for a Laplace transform of a nabla difference when the difference of the base of the transform and base of the difference is not necessarily 1.

Theorem 1.34 (Theorem 3.89, [37])**.** *Given $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $0 < \nu < 1$. Then we have*

$$\mathcal{L}_{a+1}\{\nabla_a^\nu f\}(s) = s^\nu \mathcal{L}_{a+1}\{f\}(s) - \frac{1-s^\nu}{1-s} f(a+1).$$

Given the formula above concerning a Laplace transform with a larger base than the nabla difference, we consider next what happens in a more general case, when the Laplace transform has a base which differs from the nabla difference by at least the ceiling of the power ν.

Theorem 1.35 (The $a+N$ Nabla Laplace Transform of a Nabla Fractional Difference)**.** *Assume the nabla Laplace transform of $f : \mathbb{N}_{a+1} \to \mathbb{R}$ exists for $|s-1| < r$, where $r > 0$ and $\nu > 0$. Further, pick $M \in \mathbb{N}_1$ such that $M - 1 < \nu \leq M$. Finally, assume for $k \in \mathbb{N}_0$ and $N \in \mathbb{N}_1$ that $N - k \geq M$. Then*

$$\begin{aligned}\mathcal{L}_{a+N}\left\{\nabla_{a+k}^\nu f\right\}(s) &= s^\nu \mathcal{L}_{a+N}\{f\}(s) \\ &\quad + \sum_{j=0}^{N-k-1} \Bigg[\frac{s^\nu f(a+k+j+1) - s^{N-k} \nabla_{a+k}^{-(N-k-\nu)} f(a+k+j+1)}{(1-s)^{N-k-j}} \\ &\qquad - s^j \nabla^{N-k-j-1} \nabla_{a+k}^{-(N-k-\nu)} f(a+N) \Bigg].\end{aligned}$$

Proof. Consider by definition of the nabla difference and the use of Theorems 3.82 and 3.85 and Lemma 3.88 [37],

$$\begin{aligned}
\mathcal{L}_{a+N}\{\nabla^{\nu}_{a+k}f\}(s) &= \mathcal{L}_{a+N}\left\{\nabla^{N-k}\nabla^{-(N-k-\nu)}_{a+k}f\right\}(s) \\
&= s^{N-k}\mathcal{L}_{a+N}\left\{\nabla^{-(N-k-\nu)}_{a+k}f\right\}(s) \\
&\qquad - \sum_{j=1}^{N-k} s^{N-k-j}\nabla^{j-1}\nabla^{-(N-k-\nu)}_{a+k}f(a+N) \\
&= s^{N-k}\left[\left(\frac{1}{1-s}\right)^{N-k}\mathcal{L}_{a+k}\left\{\nabla^{-(N-k-\nu)}_{a+k}f\right\}(s)\right. \\
&\qquad\qquad \left. - \sum_{j=1}^{N-k}\frac{\nabla^{-(N-k-\nu)}_{a+k}f(a+k+j)}{(1-s)^{N-k-j+1}}\right] \\
&\qquad - \sum_{j=1}^{N-k} s^{N-k-j}\nabla^{j-1}\nabla^{-(N-k-\nu)}_{a+k}f(a+N) \\
&= \left(\frac{s}{1-s}\right)^{N-k}\frac{1}{s^{N-k-\nu}}\mathcal{L}_{a+k}\{f\}(s) \\
&\qquad - s^{N-k}\sum_{j=1}^{N-k}\frac{\nabla^{-(N-k-\nu)}_{a+k}f(a+k+j)}{(1-s)^{N-k-j+1}} \\
&\qquad - \sum_{j=1}^{N-k} s^{N-k-j}\nabla^{j-1}\nabla^{-(N-k-\nu)}_{a+k}f(a+N) \\
&= \frac{s^{\nu}}{(1-s)^{N-k}}\mathcal{L}_{a+k}\{f\}(s) \\
&\qquad - \sum_{j=1}^{N-k}\left[\frac{s^{N-k}\nabla^{-(N-k-\nu)}_{a+k}f(a+k+j)}{(1-s)^{N-k-j+1}}\right. \\
&\qquad\qquad \left. + s^{N-k-j}\nabla^{j-1}\nabla^{-(N-k-\nu)}_{a+k}f(a+N)\right] \qquad (1.2)
\end{aligned}$$

Replacing a with $a+k$ and n with $N-k$ in the Shifting Lemma 1.31, we have

$$\left(\frac{1}{1-s}\right)^{N-k}\mathcal{L}_{a+k}\{f\}(s) = \mathcal{L}_{a+N}\{f\}(s) + \sum_{j=1}^{N-k}\frac{f(a+k+j)}{(1-s)^{N-k-j+1}}.$$

Therefore, when we use this equality to substitute into the equation (1.2) above, we have

$$\begin{aligned}
\mathcal{L}_{a+N}\left\{\nabla^{\nu}_{a+k}f\right\}(s) &= s^{\nu}\left[\mathcal{L}_{a+N}\{f\}(s) + \sum_{j=1}^{N-k}\frac{f(a+k+j)}{(1-s)^{N-k-j+1}}\right] \\
&\quad - \sum_{j=1}^{N-k}\left[\frac{s^{N-k}\nabla^{-(N-k-\nu)}_{a+k}f(a+k+j)}{(1-s)^{N-k-j+1}}\right. \\
&\qquad \left. + s^{N-k-j}\nabla^{j-1}\nabla^{-(N-k-\nu)}_{a+k}f(a+N)\right] \\
&= s^{\nu}\mathcal{L}_{a+N}\{f\}(s) \\
&\quad + \sum_{j=1}^{N-k}\left[\frac{s^{\nu}f(a+k+j) - s^{N-k}\nabla^{-(N-k-\nu)}_{a+k}f(a+k+j)}{(1-s)^{N-k-j+1}}\right. \\
&\qquad \left. - s^{N-k-j}\nabla^{j-1}\nabla^{-(N-k-\nu)}_{a+k}f(a+N)\right] \\
&= s^{\nu}\mathcal{L}_{a+N}\{f\}(s) \\
&\quad + \sum_{j=0}^{N-k-1}\left[\frac{s^{\nu}f(a+k+j+1) - s^{N-k}\nabla^{-(N-k-\nu)}_{a+k}f(a+k+j+1)}{(1-s)^{N-k-j}}\right. \\
&\qquad \left. - s^{j}\nabla^{N-k-j-1}\nabla^{-(N-k-\nu)}_{a+k}f(a+N)\right].
\end{aligned}$$

This completes the proof.

□

When $k = 0$ in Theorem 1.35, we have the following corollary, presented in [37]. This corollary presents the traditionally considered case of the nabla fractional difference operator with base a. While Goodrich & Peterson provide their own rigorous proof in [37], we show in this work how the corollary follows given our generalized Theorem 1.35.

Corollary 1.36 (Theorem 3.110, [37]). *Assume the nabla Laplace transform of $f : \mathbb{N}_{a+1} \to \mathbb{R}$ exists for $|s-1| < r$, $r > 0$, $\nu > 0$, and pick $N \in \mathbb{N}_1$ such that $N - 1 < \nu \leq N$. Then*

$$\begin{aligned}\mathcal{L}_{a+N}\left\{\nabla_a^\nu f\right\}(s) = s^\nu \mathcal{L}_{a+N} f(s) \\ + \sum_{j=0}^{N-1} \Bigg[\frac{s^\nu}{(1-s)^{N-j}} f(a+j+1) \\ - \frac{s^N}{(1-s)^{N-j}} \nabla_a^{-(N-\nu)} f(a+j+1) \\ - s^j \nabla^{N-j-1} \nabla_a^{-(N-\nu)} f(a+N) \Bigg].\end{aligned}$$

Proof. We show that the corollary follows from Theorem 1.35. Let $k = 0$, so that

$$\begin{aligned}\mathcal{L}_{a+N}\{\nabla_{a+0}^\nu f\}(s) &= s^\nu \mathcal{L}_{a+N}\{f\}(s) \\ &\quad + \sum_{j=0}^{N-0-1} \Bigg[\frac{s^\nu f(a+0+j+1) - s^{N-0} \nabla_{a+0}^{-(N-0-\nu)} f(a+0+j+1)}{(1-s)^{N-0-j}} \\ &\quad - s^j \nabla^{N-0-j-1} \nabla_{a+0}^{-(N-0-\nu)} f(a+N) \Bigg] \\ &= s^\nu \mathcal{L}_{a+N}\{f\}(s) \\ &\quad + \sum_{j=0}^{N-1} \Bigg[\frac{s^\nu f(a+j+1) - s^N \nabla_a^{-(N-\nu)} f(a+j+1)}{(1-s)^{N-j}} \\ &\quad - s^j \nabla^{N-j-1} \nabla_a^{-(N-\nu)} f(a+N) \Bigg].\end{aligned}$$

This concludes our proof.

$\square$

Chapter 2

Composing Nabla Fractional Differences

2.1 Elementary Composition Rules

Composition rules have been extensively studied in delta fractional calculus (see [37], [39]). However, less has been studied in the case of nabla fractional calculus. In this chapter, we investigate the relationships between various Riemann-Liouville nabla fractional differences and also relationships between Riemann-Liouville and Caputo nabla fractional differences. Nabla composition rules are not as straightforward in comparison to other analogous counterparts of traditional calculus that we have seen thus far. However, these rules and other related formulas are efficient, useful tools in solving initial value problems and boundary value problems as we will see in later chapters.

Proposition 2.1. *Let* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *and* $\alpha, \beta > 0$*. Then*

$$\nabla_a^{-\alpha}\nabla_a^{-\beta} f(t) = \nabla_a^{-(\alpha+\beta)} f(t) = \nabla_a^{-\beta}\nabla_a^{-\alpha} f(t)$$

for $t \in \mathbb{N}_a$*.*

Proof. Using the definition of the nabla sum, we have

$$\begin{aligned}
\nabla_a^{-\alpha}\nabla_a^{-\beta}f(t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\nabla_a^{-\beta}f(s) \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\cdot\frac{1}{\Gamma(\beta)}\sum_{\tau=a+1}^{s}(s-\rho(\tau))^{\overline{\beta-1}}f(\tau) \\
&= \sum_{\tau=a+1}^{t}\sum_{s=\tau}^{t}\frac{(t-\rho(s))^{\overline{\alpha-1}}(s-\rho(\tau))^{\overline{\beta-1}}}{\Gamma(\alpha)\Gamma(\beta)}f(\tau) \\
&= \frac{1}{\Gamma(\beta)}\sum_{\tau=a+1}^{t}\nabla_{\tau-1}^{-\alpha}(t-\rho(\tau))^{\overline{\beta-1}}f(\tau) \\
&= \frac{1}{\Gamma(\beta)}\sum_{\tau=a+1}^{t}\frac{\Gamma(\beta-1+1)}{\Gamma(\beta-1+\alpha+1)}(t-\rho(\tau))^{\overline{\beta-1+\alpha}}f(\tau) \\
&= \frac{1}{\Gamma(\alpha+\beta)}\sum_{\tau=a+1}^{t}(t-\rho(\tau))^{\overline{\alpha+\beta-1}}f(\tau) \\
&= \nabla_a^{-(\alpha+\beta)}f(t),
\end{aligned}$$

where we used Lemma 1.25 in the third to last step.

Note that starting with $\nabla_a^{-\beta}\nabla_a^{-\alpha}f(t)$ would reach the same conclusion.

□

The following lemma provides some basic nabla composition formulas.

Lemma 2.2. *Assume $f:\mathbb{N}_{a+1}\to\mathbb{R}$, $\alpha>0$, and $0\le N-1<\alpha\le N$. Then*

(i) $\nabla^k\nabla_a^{-\alpha}f(t)=\nabla_a^{k-\alpha}f(t)$, $k\in\mathbb{N}_0$, $t\in\mathbb{N}_{a+k}$,

(ii) $\nabla_a^{\alpha}\nabla_a^{-\alpha}f(t)=f(t)$, $t\in\mathbb{N}_{a+N}$.

Proof. The proof of (i) is given in [37] (Theorem 3.108). It proceeds as follows.

Consider $k = 1$ and $\alpha = 1$. Then

$$\begin{aligned}
\nabla\nabla_a^{-1}f(t) &= \nabla \int_a^t H_0(t,\rho(\tau))f(\tau)\nabla\tau \\
&= \nabla \int_a^t f(\tau)\nabla\tau \\
&= f(t), \quad t \in \mathbb{N}_{a+1}.
\end{aligned}$$

Now choose $N \in \mathbb{N}$ such that $N - 1 < \alpha \leq N$. For the case $\alpha = N$, we have

$$\begin{aligned}
\nabla^k\nabla_a^{-N} &= \nabla^{k-1}\left[\nabla\nabla_a^{-1}\nabla_a^{-(N-1)}f(t)\right] \\
&= \nabla^{k-1}\nabla_a^{-(N-1)}f(t) \\
&= \nabla^{k-2}\left[\nabla\nabla_a^{-1}\nabla_a^{-(N-2)}f(t)\right] \\
&= \nabla^{k-2}\nabla_a^{-(N-2)}f(t) \\
&\cdots \\
&= \nabla^{k-k}\nabla_a^{-(N-k)}f(t) \\
&= \nabla_a^{k-N}f(t), \quad t \in \mathbb{N}_{a+k}.
\end{aligned}$$

The rest of the proof comes from using $N - 1 < \alpha < N$ and replacing α with $-\alpha$ in the result for $\nabla^k\nabla_a^{\alpha}f(t) = \nabla_a^{k+\alpha}f(t)$.

Considering (ii), we have by the definition of the nabla difference that

$$\begin{aligned}
\nabla_a^{\alpha}\nabla_a^{-\alpha}f(t) &= \nabla^N\nabla_a^{-(N-\alpha)}\nabla_a^{-\alpha}f(t) \\
&= \nabla^N\nabla_a^{-(N-\alpha)-\alpha}f(t) \\
&= \nabla^N\nabla_a^{-N}f(t) \\
&= \nabla_a^{N-N}f(t) = f(t),
\end{aligned}$$

where we used Proposition 2.1 in the second step and part (i) in the fourth step.

□

Note that the above lemma provides a rule for composing a nabla difference with a nabla sum, both of which have the same order. We now present a rule involving such a composition with different orders.

Lemma 2.3. *Assume that* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *and* μ*,* $\nu > 0$ *with* $N-1 < \nu < N$*. Then*

$$\nabla_a^{\nu} \nabla_a^{-\mu} f(t) = \nabla_a^{\nu-\mu} f(t), \quad t \in \mathbb{N}_{a+N}.$$

Proof. Applying the definition of the nabla difference, we have

$$\begin{aligned}
\nabla_a^{\nu} \nabla_a^{-\mu} f(t) &= \nabla^N \nabla_a^{-(N-\nu)} \nabla_a^{-\mu} f(t) \\
&= \nabla^N \nabla_a^{-(N-\nu)-\mu} f(t) \\
&= \nabla_a^{N-(N-\nu)-\mu} f(t) \\
&= \nabla_a^{\nu-\mu} f(t),
\end{aligned}$$

where we used Proposition 2.1 and Lemma 2.2 (i).

□

2.2 Advanced Composition Rules

2.2.1 Composition Rules for Riemann-Liouville Nabla Differences

We now move to more advanced rules which cover previously unstudied compositions. First, we note that Proposition 2.1 showed us that composing two nabla sums is equivalent to adding the orders. However, this proposition was presented with both nabla sums being based at a. Similarly, Lemma 2.3 presented a rule for composing a

nabla sum with a nabla difference of same base a. We now examine the case when the bases differ by a natural number.

Theorem 2.4. *Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\alpha, \beta > 0$. Further, let $k \in \mathbb{N}_0$. Then*

$$\nabla_a^{-\alpha} \nabla_{a+k}^{-\beta} f(t) = \nabla_{a+k}^{-(\alpha+\beta)} f(t), \quad t \in \mathbb{N}_{a+k}.$$

Proof. Considering the differing bases a and $a+k$, we use the definition of the nabla sum to expand as follows:

$$\begin{aligned}
\nabla_a^{-\alpha} \nabla_{a+k}^{-\beta} f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \nabla_{a+k}^{-\beta} f(s) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \cdot \frac{1}{\Gamma(\beta)} \sum_{\tau=a+k+1}^{s} (s-\rho(\tau))^{\overline{\beta-1}} f(\tau) \\
&= \sum_{s=a+1}^{t} \sum_{\tau=a+k+1}^{s} \frac{(t-\rho(s))^{\overline{\alpha-1}} (s-\rho(\tau))^{\overline{\beta-1}}}{\Gamma(\alpha)\Gamma(\beta)} f(\tau) \\
&= \sum_{s=a+k+1}^{t} \sum_{\tau=a+k+1}^{s} \frac{(t-\rho(s))^{\overline{\alpha-1}} (s-\rho(\tau))^{\overline{\beta-1}}}{\Gamma(\alpha)\Gamma(\beta)} f(\tau) \\
&= \sum_{\tau=a+k+1}^{t} \sum_{s=\tau}^{t} \frac{(t-\rho(s))^{\overline{\alpha-1}} (s-\rho(\tau))^{\overline{\beta-1}}}{\Gamma(\alpha)\Gamma(\beta)} f(\tau) \\
&= \frac{1}{\Gamma(\beta)} \sum_{\tau=a+k+1}^{t} \nabla_{\rho(\tau)}^{-\alpha} (t-\rho(\tau))^{\overline{\beta-1}} f(\tau) \\
&= \frac{1}{\Gamma(\beta)} \sum_{\tau=a+k+1}^{t} \frac{\Gamma(\beta-1+1)}{\Gamma(\beta-1+\alpha+1)} (t-\rho(\tau))^{\overline{\beta-1+\alpha}} f(\tau) \\
&= \sum_{\tau=a+k+1}^{t} \frac{(t-\rho(\tau))^{\overline{\beta-1+\alpha}}}{\Gamma(\alpha+\beta)} f(\tau) \\
&= \sum_{\tau=a+k+1}^{t} H_{\alpha+\beta}(t,\rho(\tau)) f(\tau) \\
&= \nabla_{a+k}^{-(\alpha+\beta)} f(t),
\end{aligned}$$

where we used Lemma 1.25 in the fourth to last step.

□

Theorem 2.5. *Let* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *and* α, $\beta > 0$ *with* $N = \lceil \alpha \rceil$. *Then*

$$\nabla_a^{\alpha} \nabla_{a+k}^{-\beta} f(t) = \nabla_{a+k}^{\alpha-\beta} f(t), \quad t \in \mathbb{N}_{\max\{a+k,\, a+N\}}.$$

Proof. Applying the definition of the nabla difference, we have

$$\begin{aligned}
\nabla_a^{\alpha} \nabla_{a+k}^{-\beta} f(t) &= \nabla^N \nabla_a^{-(N-\alpha)} \nabla_{a+k}^{-\beta} f(t) \\
&= \nabla^N \nabla_{a+k}^{-(N-\alpha)-\beta} f(t) \\
&= \nabla_{a+k}^{\alpha-\beta} f(t).
\end{aligned}$$

where we used Theorem 2.4. This concludes our proof.

□

Notice that in Lemma 2.2, we had a composition involving a whole-order nabla difference, in which the powers of the differences combined nicely. However, the whole-order difference was the outermost operator. We now explore what happens when the whole-order difference is the inside operator. The following theorem presents the case when we have a whole-order difference of order 1 as the inside operator and a nabla sum as the outside operator.

Theorem 2.6. *For* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *and* $\alpha > 0$,

$$\nabla_a^{-\alpha} \nabla f(t) = \nabla \nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a), \quad t \in \mathbb{N}_{a+1}.$$

Proof. First note that using Property 1.18 (ii) we have

$$\begin{aligned}\nabla_s\left[(t-s)^{\overline{\alpha-1}}f(s)\right] &= \nabla_s\left((t-s)^{\overline{\alpha-1}}\right)f(s)+(t-\rho(s))^{\overline{\alpha-1}}\nabla_s f(s)\\ &= -(\alpha-1)(t-\rho(s))^{\overline{\alpha-2}}f(s)+(t-\rho(s))^{\overline{\alpha-1}}\nabla_s f(s).\end{aligned}$$

Now

$$\begin{aligned}\nabla_a^{-\alpha}\nabla f(t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\nabla_s f(s)\\ &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\rho(s))^{\overline{\alpha-1}}\nabla_s f(s).\end{aligned}$$

We proceed by integration by parts. Let

$$u(\rho(s)) = (t-\rho(s))^{\overline{\alpha-1}}$$

and

$$\nabla_s v(s) = \nabla_s f(s),$$

so that

$$\begin{aligned}u(s) &= (t-s)^{\overline{\alpha-1}},\\ \nabla_s u(s) &= -(\alpha-1)(t-\rho(s))^{\overline{\alpha-2}},\\ v(s) &= f(s).\end{aligned}$$

Then using Theorem 1.17, we have

$$\begin{aligned}
\nabla_s \left[(t-s)^{\overline{\alpha-1}} f(s)\right] &= -(\alpha-1)(t-\rho(s))^{\overline{\alpha-2}} f(s) + (t-\rho(s))^{\overline{\alpha-1}} \nabla_s f(s) \\
&= \frac{1}{\Gamma(\alpha)} \left[(t-s)^{\overline{\alpha-1}} f(s)\Big|_a^t + \int_a^t (\alpha-1)(t-\rho(s))^{\overline{\alpha-2}} f(s) \nabla s \right] \\
&= \frac{1}{\Gamma(\alpha)} \left[(t-s)^{\overline{\alpha-1}} f(s)\Big|_a^t + (\alpha-1) \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-2}} f(s) \right] \\
&= -\frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) + \frac{(\alpha-1)}{(\alpha-1)\Gamma(\alpha-1)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-2}} f(s) \\
&= -\frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) + \nabla_a^{-(\alpha-1)} f(t) \\
&= -\frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) + \nabla\nabla_a^{-\alpha} f(t),
\end{aligned}$$

where we used Lemma 2.2 (i) in the last step. This concludes our proof.

□

Remark 2.7. *Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\alpha > 0$, and $N = \lceil \alpha \rceil$. Then we can rewrite*

$$\nabla\nabla_a^{\alpha} f(t) = \nabla\nabla^N \nabla_a^{-(N-\alpha)} f(t) = \nabla^N \nabla\nabla_a^{-(N-\alpha)} f(t), \tag{2.1}$$

or by Theorem 2.6,

$$\nabla\nabla_a^{\alpha} f(t) = \nabla^N \left[\nabla_a^{-(N-\alpha)} \nabla f(t) + \frac{(t-a)^{\overline{N-\alpha-1}}}{\Gamma(N-\alpha)} f(a) \right] \tag{2.2}$$

for $t \in \mathbb{N}_{a+N}$.

This implies that the results of Theorem 2.6 are valid for any real α.

We now present the case when the inside operator is an arbitrary whole-order difference. We might hope that we could just interchange the order of the sum and

difference (allowing us to add the orders by Lemma 2.2 (ii)), but in reality we find some extra terms.

Theorem 2.8. *For* $f : \mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}$ *and* $p \in \mathbb{Z}^+$,

$$\nabla_a^{-\alpha}\nabla^p f(t) = \nabla^p \nabla_a^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{\alpha-p+k}}}{\Gamma(\alpha-p+k+1)} \nabla^k f(a),$$

for $t \in \mathbb{N}_{a+p}$.

Proof. We will proceed by induction. Consider $p = 1$. Then by Theorem 2.6,

$$\begin{aligned}\nabla_a^{-\alpha}\nabla^1 f(t) &= \nabla\nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) \\ &= \nabla\nabla_a^{-\alpha} f(t) - \sum_{k=0}^{1-1} \frac{(t-a)^{\overline{\alpha-1+k}}}{\Gamma(\alpha-1+k+1)} \nabla^k f(a).\end{aligned}$$

Thus the theorem holds for $p = 1$. Now assume our induction hypothesis holds for all $p \geq 1$. Then

$$\begin{aligned}\nabla_a^{-\alpha}\nabla^{p+1} f(t) &= \nabla_a^{-\alpha}\nabla^p(\nabla f(t)) \\ &= \nabla^p\nabla_a^{-\alpha}\nabla f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{\alpha-p+k}}}{\Gamma(\alpha-p+k+1)} \nabla^k(\nabla f(a)) \\ &= \nabla^p \left[\nabla\nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a)\right] - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{\alpha-p+k}}}{\Gamma(\alpha-p+k+1)} \nabla^{k+1} f(a) \\ &= \nabla^{p+1}\nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1-p}}}{\Gamma(\alpha-p)} f(a) - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{\alpha-p+k}}}{\Gamma(\alpha-p+k+1)} \nabla^{k+1} f(a) \\ &= \nabla^{p+1}\nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-p-1}}}{\Gamma(\alpha-p)} \nabla^0 f(a) - \sum_{k=1}^{p} \frac{(t-a)^{\overline{\alpha-p-1+k}}}{\Gamma(\alpha-p+k)} \nabla^k f(a) \\ &= \nabla^{p+1}\nabla_a^{-\alpha} f(t) - \sum_{k=0}^{(p+1)-1} \frac{(t-a)^{\overline{\alpha-(p+1)+k}}}{\Gamma(\alpha-(p+1)+k+1)} \nabla^k f(a),\end{aligned}$$

where we used Theorem 2.6 in the third step and Property 1.18 (iii) in the fourth step and shifted the sum in the fifth step. Thus by induction we have proven our result.

□

Note that the above theorem holds for all $\alpha \in \mathbb{R}$. However, $\alpha = 0$ is a special case, and we can prove it separately as follows.

Proposition 2.9. *For $\alpha = 0$ and $p \in \mathbb{Z}^+$,*

$$\nabla_a^{-\alpha} \nabla^p f(t) = 0,$$

where $f : \mathbb{N}_a \to \mathbb{R}$.

Proof. Let $\alpha = 0$. Considering the left hand side of the equality in Theorem 2.8, we have by definition of the nabla sum

$$\begin{aligned} \nabla_a^{-0} \nabla^p f(t) &= \int_a^t H_{0-1}(t, \rho(s)) \nabla^p f(s) \nabla s \\ &= \int_a^t 0 \cdot \nabla^p f(s) \nabla s \\ &= 0, \end{aligned}$$

where we used the fact that $H_{-k}(t, a) = 0$ for all $k \in \mathbb{N}_1$. Considering the right hand side of the equality in Theorem 2.8, we have

$$\begin{aligned} \nabla^p \nabla_a^{-0} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{0-p+k}}}{\Gamma(0-p+k+1)} \nabla^k f(a) &= \nabla^p \left[\int_a^t H_{-1}(t, \rho(s)) f(s) \nabla s \right] \\ &\quad - \sum_{k=0}^{p-1} H_{k-p}(t, a) \nabla^k f(a) \\ &= 0 - 0 = 0, \end{aligned}$$

where we again used the fact that $H_{-k}(t,a)=0$ for all $k\in\mathbb{N}_1$, since $k\leq p-1<p$ implies $k-p<0$ and $k-p\in\{-1,-2,\dots\}$. Thus we have achieved our result.

$\square$

Theorem 2.8 proves integral to developing the relationship between nabla differences. Indeed, the theorem is a generalization of a type of composition that has been lightly considered in the literature. Goodrich & Peterson studied two specific cases presented as the following corollaries. While Goodrich & Peterson provided rigorous proofs for these particular composition rules, we show in this work how these corollaries follow immediately from Theorem 2.8.

Corollary 2.10 (Lemma 3.147, [37]). *For $f:\mathbb{N}_a\to\mathbb{R}$ and $0<\nu<1$,*

$$\nabla_a^{-(1-\nu)}\nabla f(t)=\nabla\nabla_a^{-(1-\nu)}f(t)-H_{-\nu}(t,a)f(a),\quad t\in\mathbb{N}_{a+1}.$$

Corollary 2.11 (Corollary 3.148, [37]). *For $f:\mathbb{N}_a\to\mathbb{R}$ and $0<\nu<1$,*

$$\nabla_a^{-\nu}\nabla f(t)=\nabla\nabla_a^{-\nu}f(t)-H_{\nu-1}(t,a)f(a),\quad t\in\mathbb{N}_{a+1}.$$

The proofs use integration by parts and Lemma 2.2 (ii).

Similar to Theorem 2.8, we also present a rule for composing a nabla difference with a whole-order nabla difference.

Theorem 2.12. *Assume $f:\mathbb{N}_a\to\mathbb{R}$, $\nu>0$, and $k\in\mathbb{N}_0$. Then*

$$\nabla_{a+k}^{\nu}\nabla^k f(t)=\nabla_{a+k}^{\nu+k}f(t)-\sum_{j=0}^{k-1}\nabla^j f(a+k)H_{-\nu-k+j}(t,a+k),$$

for each $t\in\mathbb{N}_a$.

Now that we have established rules for composing a nabla sum with a whole-order nabla difference, we investigate nabla sums composed with fractional nabla differences. In order to do so, we need the following power rule.

Theorem 2.13 (n^{th} Power Rule)**.** *For $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$,*

$$\nabla^n \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} = \frac{(t-a)^{\overline{\alpha-1-n}}}{\Gamma(\alpha-n)}, \quad t \in \mathbb{N}_{a+n+1}.$$

Proof. We will proceed by induction. Consider the case $n = 1$:

$$\nabla^1 \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} = \frac{(\alpha-1)(t-a)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} = \frac{(t-a)^{\overline{\alpha-1-1}}}{\Gamma(\alpha-1)}.$$

Now assume the hypothesis holds for all $n \geq 1$. Then we can rewrite the $n+1$ order difference as

$$\nabla^{n+1} \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} = \nabla \nabla^n \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}.$$

Hence

$$\begin{aligned}
\nabla^1 \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} &= \nabla \frac{(t-a)^{\overline{\alpha-1-n}}}{\Gamma(\alpha-n)} \\
&= \frac{(\alpha-1-n)(t-a)^{\overline{\alpha-1-n-1}}}{(\alpha-n-1)\Gamma(\alpha-n-1)} \\
&= \frac{(t-a)^{\overline{\alpha-1-(n+1)}}}{\Gamma(\alpha-(n+1))}.
\end{aligned}$$

Thus we have achieved our result.

$\square$

Note that we can think of this power rule in terms of Taylor monomials:

$$\nabla^n H_{\alpha-1}(t,a) = H_{\alpha-1-n}(t,a).$$

We now move to our desired goal of a rule for composing a fractional nabla sum with a fractional nabla difference.

Theorem 2.14. *For $\alpha > 0$ where $N-1 < \alpha < N$ and $f : \mathbb{N}_{a+1} \to \mathbb{R}$,*

$$\nabla_a^{-\alpha}\nabla_a^{\alpha} f(t) = \begin{cases} f(t), & \alpha \notin \mathbb{N} \\ f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!}\nabla^k f(a), & \alpha = n \in \mathbb{N}, \end{cases}$$

for $t \in \mathbb{N}_{a+N}$.

Note that we could rewrite the sum as

$$\sum_{k=0}^{n-1} H_k(t,a)\nabla^k f(a).$$

Proof. Suppose $\alpha \notin \mathbb{N}$. Then by the definition of the nabla difference we have

$$\begin{aligned}
\nabla_a^{-\alpha}\nabla_a^{\alpha} f(t) &= \nabla_a^{-\alpha}\nabla^N \left[\nabla_a^{-(N-\alpha)} f(t)\right] \\
&= \nabla^N \nabla_a^{-\alpha}\left[\nabla_a^{-(N-\alpha)} f(t)\right] - \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{\alpha-N+k}}}{\Gamma(\alpha-N+k+1)}\nabla^k \left[\nabla_a^{-(N-\alpha)} f(a)\right] \\
&= \nabla^N \nabla_a^{-N} f(t) = \nabla_a^{N-N} f(t) = f(t),
\end{aligned}$$

where we used Theorem 2.8 in the third step and the fact that

$$\nabla_a^{-(N-\alpha)} f(a) = \sum_{s=a+1}^{a} H_{N-\alpha-1}(a, \rho(s)) f(s) = 0.$$

Now suppose $\alpha = n \in \mathbb{N}$. Then

$$\begin{aligned} \nabla_a^{-n} \nabla_a^{n} f(t) &= \nabla^n \nabla_a^{-n} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{n+k-n}}}{\Gamma(n+k-n+1)} \nabla^k f(a) \\ &= f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{\Gamma(k+1)} \nabla^k f(a), \end{aligned}$$

where we used Lemma 2.2 (i) in the last step. Thus we have reached our conclusion.

□

Notice that, similar to Lemma 2.2 (ii), we achieve the nice result of switching the order of the composition, and thus we are able to add the orders in the case $\alpha \notin \mathbb{N}$. However, we must still be careful in the case $\alpha \in \mathbb{N}$, where we switch the order of the composition, but must subtract a summation.

2.2.2 Compositions Involving Caputo Nabla Differences

Recall from Chapter 1 that the Caputo definition of the nabla difference places the whole-order difference as the inside operator rather than the outside operator. This often makes the Caputo difference easier to work with, but we must be careful to attend to the different domain that this definition requires.

We can now present a power rule involving the Caputo nabla difference due to some properties shown in the last section.

Theorem 2.15 (Caputo Power Rule)**.** *For* $1 \neq \beta > 0$ *and* $\alpha \geq 0$*,*

$$\nabla_{a*}^{\alpha}(t-a)^{\overline{\beta-1}} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\overline{\beta-\alpha-1}}. \tag{2.3}$$

Proof. First recall that

$$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}(t-a)^{\overline{\alpha+\mu}}$$

and

$$\nabla^N \frac{(t-a)^{\overline{N-\alpha-1}}}{\Gamma(N-\alpha)} = \frac{(t-a)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}.$$

Now suppose $1 \neq \beta > 0$ and $\alpha \geq 0$. Then we have by the definition of the Caputo fractional difference

$$\begin{aligned}
\nabla_{a*}^{\alpha}(t-a)^{\overline{\beta-1}} &= \nabla_a^{-(N-\alpha)}\nabla^N(t-a)^{\overline{\beta-1}} \\
&= \nabla^N\nabla_a^{-(N-\alpha)}(t-a)^{\overline{\beta-1}} - \sum_{k=0}^{N-1}\frac{(t-a)^{\overline{N-\alpha+k-N}}}{\Gamma(N-\alpha+k-N+1)}\nabla^k(a-a)^{\overline{\beta-1}} \\
&= \nabla^N\left[\frac{\Gamma(\beta-1+1)}{\Gamma(N-\alpha+\beta-1+1)}(t-a)^{\overline{N-\alpha+\beta-1}}\right] \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\overline{\beta-\alpha-1}},
\end{aligned} \tag{2.4}$$

where we used Theorem 2.8 in the second step, the fact that $\nabla^k(0)^{\overline{\beta-1}} = 0$ in the third step, and the n^{th} Power Rule in Theorem 1.18 in the last step.

By convention, dividing by $0, -1, -2, \ldots$ results in zero. So, when $\beta - 1 = \alpha - j$ for $j = 1, 2, \ldots, n$, equation (2.3) is zero.

□

We note that this theorem works out the same as in the Riemann-Liouville case. However, the Caputo nabla difference $\nabla_{a*}^{\alpha}(1) = 0$ while the Riemann-Liouville nabla

difference of 1 is

$$\begin{aligned}
\nabla_a^{\alpha}(1) &= \nabla^N \nabla^{-(N-\alpha)}(1) \\
&= \nabla^N \left[\int_a^t H_{N-\alpha-1}(t, \rho(s)) \cdot 1 \nabla s \right] \\
&= \nabla^N H_{N-\alpha}(t, a) \\
&= H_{-\alpha}(t, a).
\end{aligned}$$

The following identity is useful in transforming nabla Caputo fractional differences into equations in terms of Riemann-Liouville differences:

Theorem 2.16. *For $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$ and any $\alpha > 0$,*

$$\nabla_{a*}^{\alpha} f(t) = \nabla_a^{\alpha} f(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla^k f(a) = \nabla_a^{\alpha} f(t) - \sum_{k=0}^{N-1} H_{k-\alpha}(t, a) \nabla^k f(a).$$

In particular, when $0 < \alpha < 1$,

$$\nabla_{a*}^{\alpha} f(t) = \nabla_a^{\alpha} f(t) - \frac{(t-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(a) = \nabla_a^{\alpha} f(t) - H_{-\alpha}(t, a) f(a).$$

Proof. We notice by the definition of the Caputo fractional difference that

$$\begin{aligned}
\nabla_{a*}^{\alpha} f(t) &= \nabla_a^{-(N-\alpha)} \nabla^N f(t) \\
&= \nabla^N \nabla_a^{-(N-\alpha)} f(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{(N-\alpha)-N+k}}}{\Gamma(N-\alpha-N+k+1)} \nabla^k f(a) \\
&= \nabla_a^{\alpha} f(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla^k f(a),
\end{aligned}$$

where we used Theorem 2.8 in the second step and Lemma 2.2 (i) in the third step.

$\square$

This theorem is significant as it implies that the nabla Riemann-Liouville and Caputo fractional differences coincide in the particular case when f vanishes at a.

We are now equipped to consider a Caputo nabla difference composed with a Riemann-Liouville nabla difference.

Lemma 2.17. *For $\alpha > 0$ and $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$,*

$$\nabla_{a*}^{\alpha} \nabla_a^{-\alpha} f(t) = f(t), \quad t \in \mathbb{N}_{a+N}.$$

Proof. For $N = \lceil \alpha \rceil$,

$$\begin{aligned}
\nabla_{a*}^{\alpha} \nabla_a^{-\alpha} f(t) &= \nabla_a^{-(N-\alpha)} \nabla^N \nabla_a^{-\alpha} f(t) \\
&= \nabla_a^{-(N-\alpha)} \nabla_a^{N-\alpha} f(t) \\
&= f(t),
\end{aligned}$$

where we used Lemma 2.2 (i) in the second step, and Theorem 2.14.

□

Now, when we compose a nabla sum with a nabla difference, we do not always get the original function back. However, one of the conveniences of studying both Riemann-Liouville and Caputo nabla differences is that composing a Riemann-Liouville sum with either type of nabla difference yields a similar formula as the following theorem shows.

Theorem 2.18. *Suppose $\alpha > 0$ and $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$. Then*

$$\nabla_a^{-\alpha} \nabla_{a*}^{\alpha} f(t) = f(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a) = f(t) - \sum_{k=0}^{N-1} H_k(t,a) \nabla^k f(a),$$

for $t \in \mathbb{N}_{a+1}$. *In particular, if* $0 < \alpha \leq 1$, *then*

$$\nabla_a^{-\alpha}\nabla_{a*}^{\alpha}f(t) = f(t) - f(a).$$

Proof. By the definition of the Caputo difference, for $N = \lceil \alpha \rceil$ we have

$$\begin{aligned}
\nabla_a^{-\alpha}\nabla_{a*}^{\alpha}f(t) &= \nabla_a^{-\alpha}\nabla_a^{-(N-\alpha)}\nabla^N f(t) \\
&= \nabla_a^{-(\alpha+N-\alpha)}\nabla^N f(t) \\
&= \nabla_a^{-N}\nabla^N f(t),
\end{aligned}$$

and from here the proof proceeds as the proof of the case $\alpha \in \mathbb{N}$ in Theorem 2.14. Thus we achieve our result.

Now suppose $0 < \alpha < 1$. Then by the definition of the Caputo nabla difference we have

$$\begin{aligned}
\nabla_a^{-\alpha}\nabla_{a*}^{\alpha}f(t) &= \nabla_a^{-\alpha}\left[\nabla_a^{-(1-\alpha)}\nabla^1 f(t)\right] \\
&= \nabla_a^{-\alpha}\left[\nabla\nabla_a^{-(1-\alpha)}f(t) - H_{-\alpha}(t,a)f(a)\right] \\
&= \nabla_a^{-\alpha}\nabla\left[\nabla_a^{-(1-\alpha)}f(t)\right] - \nabla_a^{-\alpha}H_{-\alpha}(t,a)f(a).
\end{aligned}$$

Hence

$$\begin{aligned}
\nabla_a^{-\alpha}\nabla_{a*}^{\alpha}f(t) &= \nabla\nabla_a^{-\alpha}\nabla_a^{-(1-\alpha)}f(t) - H_{\alpha-1}(t,a)\nabla_a^{-(1-\alpha)}f(a) - H_{-\alpha+\alpha}(t,a)f(a) \\
&= f(t) - f(a),
\end{aligned}$$

where we used Corollaries 2.10 and 2.11 in the second step and fourth step, respectively. Further, we used the facts that $\nabla_a^{-(1-\alpha)}f(a) = 0$ and $H_0(t,a) = 1$.

Finally, suppose $\alpha = 1$. Then

$$\begin{aligned}
\nabla_a^{-1}\nabla_{a*}^1 f(t) &= \nabla_a^{-1}\left[\nabla_a^{-(1-1)}\nabla^1 f(t)\right] \\
&= \nabla_a^{-1}\nabla f(t) \\
&= \nabla\nabla_a^{-1} f(t) - \sum_{k=0}^{1-1} H_{1-1+k}(t,a)\nabla^k f(a) \\
&= f(t) - H_0(t,a)f(a) \\
&= f(t) - f(a).
\end{aligned}$$

Thus for the case $0 < \alpha \leq 1$,

$$\nabla_a^{-\alpha}\nabla_{a*}^{\alpha} f(t) = f(t) - f(a).$$

□

Since we see that we can achieve a similar formula with either type of nabla difference, we are allowed some flexibility in solving problems.

Chapter 3

Sequential Nabla Fractional Differences

3.1 Introduction

In this chapter we study composed nabla fractional differences of a special type known as *sequential nabla fractional differences.* By "sequential," we mean nabla differences with orders whose bounds are sequential, such as

$$\nabla_{a+1}^{\nu}\nabla_{a}^{\mu}x(t)$$

for $0 < \nu < 1$ and $1 < \mu < 2$. Further, we note that when we consider a sequential nabla difference in this work, we are assuming the bases of the differences differ by at least one integer, as in the example above, we have the bases a and $a+1$. Fractional sequential differences are relatively understudied in the field, especially when we consider nabla sequential differences. When we do consider compositions of different orders in nabla fractional calculus, we predominantly have results which involve the same base $a \in \mathbb{R}$ as seen in the previous chapter. Sequential fractional differences were first studied, to the author's knowledge, in 2012 by Girejko [22] and Goodrich [30]. These works involved sequential differences in delta fractional calculus. In subsequent years, delta sequential fractional differences were further studied by

Dahal & Goodrich [20] and Goodrich [33], [34], and [35], and also in [51], [23] , [32], [49], and [19]. Nabla sequential fractional differences were first fully considered in [21] in 2018. There has also been work with sequential differences in the continuous case, see [3], [15], [6], [7] ,[9], [50], and [54].

There is certainly a reason that the field has focused predominantly on problems using the same base a. When studying sequential differences with different bases, we must more carefully attend to domain shifts - often to a more strict domain. Indeed, all the theorems that follow hold on a domain at least $\mathbb{N}_{a+1}$, but often hold on a domain with a base larger than $a+1$.

In the work that follows, we consider sequential nabla fractional differences of the generalized form $\nabla_{a+k+1}^{\nu}\nabla_a^{\mu}f(t)$ for the case where $k<\mu<k+1$, $k+1<\nu<k+2$, $2k+1<\mu+\nu<2k+2$, $k\in\mathbb{N}_0$, as well as the case $k<\mu<k+1$, $k-1<\nu<k$, and $2k-1<\mu+\nu<2k$, $k\in\mathbb{N}_1$, the difference between these two cases being whether μ or ν is the larger order.

3.2 Sequential Nabla Differences of the Form $\nabla_{a+k+1}^{\nu+k}\nabla_a^{\mu-(k+1)}f(t)$

In this section we consider sequential nabla differences of the form $\nabla_{a+k+1}^{\nu+k}\nabla_a^{\mu-(k+1)}f(t)$, where each of the powers and the base of the outer difference vary with our choice of $k\in\mathbb{N}_0$. The following two theorems present valuable equalities for these general sequential nabla fractional differences.

Theorem 3.1. *We have two cases, each implying the same equality.*

(i) Suppose for $k\in\mathbb{N}_0$ that $k<\mu<k+1$, $k+1<\nu<k+2$, and $2k+1<\mu+\nu<2k+2$. Then for each $t\in\mathbb{N}_{a+2k+3}$,

$$\nabla_{a+k+1}^{\nu+k}\nabla_a^{\mu-(k+1)}f(t) = \nabla_a^{\mu+\nu-1}f(t) - \sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s).$$

(ii) Suppose for $k \in \mathbb{N}_1$ that $k < \mu < k+1$, $k-1 < \nu < k$, and $2k-1 < \mu+\nu < 2k$. Then for each $t \in \mathbb{N}_{a+2k+1}$,

$$\nabla_{a+k+1}^{\nu+k}\nabla_a^{\mu-(k+1)}f(t) = \nabla_a^{\mu+\nu-1}f(t) - \sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s).$$

Proof. We first carefully attend to domains. As we are considering the sequential difference

$$\nabla_{a+k+1}^{\nu+k}\nabla_a^{\mu-(k+1)}f(t) = \nabla^k\nabla_{a+k+1}^{\nu}\nabla_a^{\mu-(k+1)}f(t),$$

we can consider the domain of each of these differences to identify the domain of the entire composed, or sequential, difference. First, by definition, $\nabla^k g(t)$ is defined for $t \in \mathbb{N}_{a+k}$. Further notice that in each case, $k < \mu < k+1$, implying that $\mu-(k+1) < 0$. This means that our third difference is in fact a nabla sum of order $k+1-\mu$, which, by definition, is defined for $t \in \mathbb{N}_a$. This only leaves the middle nabla difference, which has a different domain for each case given above. For the first case, as $k+1 < \nu < k+2$, this difference is defined for $t \in \mathbb{N}_{(a+k+1)+(k+2)} = \mathbb{N}_{a+2k+3}$. For the second case, as $k-1 < \nu < k$, this difference is defined for $t \in \mathbb{N}_{(a+k+1)+(k)} = \mathbb{N}_{a+2k+1}$. As the ν^{th} difference is the most restrictive of the three in the composition, this provides our domain in each case.

We now proceed to the proof. Since $\nu + k > 0$ and ν is between two consecutive integers, we can apply the definition of the nabla difference to expand the following:

$$\begin{aligned}\nabla_{a+k+1}^{\nu+k}\nabla_a^{\mu-(k+1)}f(t) &= \int_{a+k+1}^{t} H_{-(\nu+k)-1}(t,\rho(\tau))\nabla_a^{-(k+1-\mu)}f(\tau)\nabla\tau \\ &= \int_{a+k+1}^{t} H_{-(\nu+k)-1}(t,\rho(\tau))\int_a^{\tau} H_{k-\mu}(\tau,\rho(s))f(s)\nabla s\nabla\tau.\end{aligned}$$

Thus we have

$$\begin{aligned}\nabla_{a+k+1}^{\nu+k}\nabla_a^{\mu-(k+1)}f(t) &= \sum_{\tau=a+k+2}^{t} H_{-(\nu+k)-1}(t,\rho(\tau))\sum_{a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s) \\ &= \sum_{\tau=a+1}^{t} H_{-(\nu+k)-1}(t,\rho(\tau))\sum_{a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s) \\ &\quad - \sum_{\tau=a+1}^{a+k+1} H_{-(\nu+k)-1}(t,\rho(\tau))\sum_{a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s) \\ &= \nabla_a^{\nu+k}\nabla_a^{\mu-(k+1)}f(t) \\ &\quad - \sum_{\tau=a+1}^{a+k+1} H_{-(\nu+k)-1}(t,\rho(\tau))\sum_{a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s) \\ &= \nabla_a^{\mu+\nu-1}f(t) - \sum_{\tau=a+1}^{a+k+1} H_{-(\nu+k)-1}(t,\rho(\tau))\sum_{a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s),\end{aligned}$$

where we noticed that $k < \mu < k+1$ implies $k+1-\mu > 0$, so that we could apply the definition of the nabla sum as well, along with Lemma 2.3. This concludes our proof.

$\square$

Two immediate corollaries follow from the above theorem, which were first presented by Dahal & Goodrich. We view Theorem 3.1 as a nice generalization of any case of order conditions for this specific type of composition. To that end, while Dahal & Goodrich provided their own rigorous proofs for the specific cases, we show in this work how the corollaries follow immediately from Theorem 3.1.

Corollary 3.2 (Lemma 2.9, [21]). *Suppose that* $0 < \mu < 1$*,* $1 < \nu < 2$*, and* $1 < \mu + \nu < 2$*. For each* $t \in \mathbb{N}_{a+3}$ *it holds that*

$$\nabla_{a+1}^{\nu} \nabla_a^{\mu-1} f(t) = \nabla_a^{\mu+\nu-1} f(t) - H_{-\nu-1}(t,a) f(a+1).$$

Proof. We now show that the corollary follows from Theorem 3.1. Suppose that $k = 0$ and consider the first set of parameter conditions in Theorem 3.1. Then $0 < \mu < 1$, $1 < \nu < 2$, and $1 < \mu + \nu < 2$. Then by Theorem 3.1, for each $t \in \mathbb{N}_{a+3}$,

$$\begin{aligned}
\nabla_{a+1}^{\nu} \nabla_a^{\mu-1} f(t) &= \nabla_a^{\mu+\nu-1} f(t) - \sum_{\tau=a+1}^{a+0+1} H_{-\nu-0-1}(t, \rho(\tau)) \sum_{s=a+1}^{\tau} H_{0-\mu}(\tau, \rho(s)) f(s) \\
&= \nabla_a^{\mu+\nu-1} f(t) - \sum_{\tau=a+1}^{a+1} H_{-\nu-1}(t, \rho(\tau)) \sum_{s=a+1}^{\tau} H_{-\mu}(\tau, \rho(s)) f(s) \\
&= \nabla_a^{\mu+\nu-1} f(t) - H_{-\nu-1}(t,a) H_{-\mu}(a+1,a) f(a+1) \\
&= \nabla_a^{\mu+\nu-1} f(t) - H_{-\nu-1}(t,a) f(a+1).
\end{aligned}$$

This concludes our proof.

□

Corollary 3.3 (Lemma 2.11, [21]). *Suppose that* $1 < \mu < 2$*,* $0 < \nu < 1$*, and* $1 < \mu + \nu < 2$*. For each* $t \in \mathbb{N}_{a+3}$ *it holds that*

$$\begin{aligned}
&\nabla_{a+2}^{\nu+1} \nabla_a^{\mu-2} f(t) \\
&\quad = \nabla_a^{\mu+\nu-1} f(t) \\
&\quad - \left[H_{-\nu-2}(t,a) + (2-\mu) H_{-\nu-2}(t,a+1) \right] f(a+1) - H_{-\nu-2}(t,a+1) f(a+2).
\end{aligned}$$

Proof. We now show that the corollary follows from Theorem 3.1.

Suppose $k = 1$ and consider the second set of parameter conditions in Theorem

3.1. Then $1 < \mu < 2$, $0 < \nu < 1$, and $1 < \mu + \nu < 2$. Then by Theorem 3.1, for each $t \in \mathbb{N}_{a+2(1)+1} = \mathbb{N}_{a+3}$,

$$\begin{aligned}\nabla_{a+2}^{\nu+1}\nabla_a^{\mu-2}f(t) &= \nabla_a^{\mu+\nu-1}f(t) - \sum_{\tau=a+1}^{a+1+1} H_{-\nu-1-1}(t,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{1-\mu}(\tau,\rho(s))f(s)\\ &= \nabla_a^{\mu+\nu-1}f(t) - \sum_{\tau=a+1}^{a+2} H_{-\nu-2}(t,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{1-\mu}(\tau,\rho(s))f(s).\end{aligned}$$

Hence

$$\begin{aligned}\nabla_{a+2}^{\nu+1}\nabla_a^{\mu-2}f(t) &= \nabla_a^{\mu+\nu-1}f(t) - H_{-\nu-2}(t,a)H_{1-\mu}(a+1,a)f(a+1)\\ &\quad - H_{-\nu-2}(t,a+1)\big[H_{1-\mu}(a+2,a)f(a+1)\\ &\qquad\qquad + H_{1-\mu}(a+2,a+1)f(a+2)\big]\\ &= \nabla_a^{\mu+\nu-1}f(t) - H_{-\nu-2}(t,a)f(a+1)\\ &\quad - H_{-\nu-2}(t,a+1)\left((2-\mu)f(a+1) + f(a+2)\right)\\ &= \nabla_a^{\mu+\nu-1}f(t)\\ &\quad - \left[H_{-\nu-2}(t,a) + (2-\mu)H_{-\nu-2}(t,a+1)\right]f(a+1)\\ &\quad - H_{-\nu-2}(t,a+1)f(a+2).\end{aligned}$$

Thus the corollary follows from Theorem 3.1, and this concludes our proof.

□

3.3 Sequential Nabla Differences of the Form $\nabla_{a+k+1}^{\nu}\nabla_a^{\mu}f(t)$

In this section we consider sequential nabla differences of the form $\nabla_{a+k+1}^{\nu}\nabla_a^{\mu}f(t)$. In this case the base of the outer difference differs from the base of the inner difference by an integer factor of at least 1.

The following two theorems present valuable equalities for these general sequential nabla fractional differences.

Theorem 3.4. *We have two cases, each implying the same equality.*

(i) Suppose for $k \in \mathbb{N}_0$ that $k < \mu < k+1$, $k+1 < \nu < k+2$, and $2k+1 < \mu+\nu < 2k+2$. Then for each $t \in \mathbb{N}_{a+2k+3}$,

$$\begin{aligned}\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}f(t) = \nabla^{\mu+\nu}_{a}f(t) \\ &- \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s)\right] \\ &- \left[\sum_{j=0}^{k} \nabla^{\mu+j-k-1}_{a} f(a+k+1) H_{-\nu-k+j-1}(t,a+k+1)\right].\end{aligned}$$

(ii) Suppose for $k \in \mathbb{N}_1$ that $k < \mu < k+1$, $k-1 < \nu < k$, and $2k-1 < \mu+\nu < 2k$. Then for each $t \in \mathbb{N}_{a+2k+1}$,

$$\begin{aligned}\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}f(t) = \nabla^{\mu+\nu}_{a}f(t) \\ &- \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s)\right] \\ &- \left[\sum_{j=0}^{k} \nabla^{\mu+j-k-1}_{a} f(a+k+1) H_{-\nu-k+j-1}(t,a+k+1)\right].\end{aligned}$$

Proof. We first carefully attend to domains. As we are considering the sequential difference

$$\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}f(t),$$

we can consider the domain of each of these differences to identify the domain of the entire composed, or sequential, difference. For the first case, as $k+1 < \nu < k+2$, the first difference is defined for $t \in \mathbb{N}_{(a+k+1)+(k+2)} = \mathbb{N}_{a+2k+3}$. As $k < \mu < k+1$, the

second difference is defined for $t \in \mathbb{N}_{a+k+1}$. Together, we see that the first difference has the more restrictive domain. For the second case, as $k-1 < \nu < k$, the first difference is defined for $t \in \mathbb{N}_{(a+k+1)+(k)} = \mathbb{N}_{a+2k+1}$. As $k < \mu < k+1$, the second difference is defined for $t \in \mathbb{N}_{a+k+1}$. Together, we see that the first difference has the more restrictive domain.

We now proceed to the proof. Assume the first case hypothesis, and let $g(t) := \nabla_a^{\mu-(k+1)} f(t)$. Then

$$
\begin{aligned}
\nabla_{a+k+1}^{\nu} \nabla_a^{\mu} f(t) &= \nabla^{k+1} \nabla_{a+k+1}^{\nu-(k+1)} \nabla^{k+1} \nabla_a^{\mu-(k+1)} f(t) \\
&= \nabla^{k+1} \nabla_{a+k+1}^{\nu-(k+1)} \nabla^{k+1} g(t) \\
&= \nabla^{k+1} \Bigg[\nabla_{a+k+1}^{\nu} g(t) \\
&\qquad - \sum_{j=0}^{k+1-1} \nabla^j g(a+k+1) H_{-(\nu-k-1)-(k+1)+j}(t, a+k+1) \Bigg] \\
&= \nabla^{k+1} \Bigg[\nabla_{a+k+1}^{\nu} \nabla_a^{\mu-(k+1)} f(t) \\
&\qquad - \sum_{j=0}^{k} \nabla^j \nabla_a^{\mu-(k+1)} f(a+k+1) H_{-\nu+j}(t, a+k+1) \Bigg] \\
&= \nabla \left[\nabla_{a+k+1}^{\nu+k} \nabla_a^{\mu-(k+1)} f(t) \right] \\
&\quad - \nabla^{k+1} \left[\sum_{j=0}^{k} \nabla^j \nabla_a^{\mu-(k+1)} f(a+k+1) H_{-\nu+j}(t, a+k+1) \right] \\
&= \nabla \left[\nabla_a^{\mu+\nu-1} f(t) - \sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t, \rho(\tau)) \sum_{s=a+1}^{\tau} H_{k-\mu}(\tau, \rho(s)) f(s) \right] \\
&\quad - \nabla^{k+1} \left[\sum_{j=0}^{k} \nabla_a^{\mu+j-k-1} f(a+k+1) H_{-\nu+j}(t, a+k+1) \right],
\end{aligned}
$$

so that

$$\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}f(t) = \nabla^{\mu+\nu}_{a}f(t) - \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s)\right]$$
$$- \sum_{j=0}^{k} \nabla^{\mu+j-k-1}_{a} f(a+k+1)H_{-\nu-k+j-1}(t,a+k+1),$$

where we used Theorem 2.12 with $k = k+1$ since $\nu-(k+1) > 0$, and Theorem 3.1.

Now consider the second case hypothesis, and let $g(t) := \nabla^{\mu-(k+1)}_{a}f(t)$. Then, noting that $\nu - k < 0$,

$$\begin{aligned}
\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}f(t) &= \nabla^{k}\nabla^{\nu-k}_{a+k+1}\nabla^{k+1}\nabla^{\mu-(k+1)}_{a}f(t) \\
&= \nabla^{k}\nabla^{\nu-k}_{a+k+1}\nabla^{k+1}g(t) \\
&= \nabla^{k}\left[\nabla^{\nu+1}_{a+k+1}g(t) - \sum_{j=0}^{k+1-1}\nabla^{j}g(a+k+1)H_{-\nu+k-(k+1)+j}(t,a+k+1)\right] \\
&= \nabla^{k}\Bigg[\nabla^{\nu+1}_{a+k+1}\nabla^{\mu-(k+1)}_{a}f(t) \\
&\qquad - \sum_{j=0}^{k}\nabla^{j}\nabla^{\mu-(k+1)}_{a}f(a+k+1)H_{-\nu+j-1}(t,a+k+1)\Bigg] \\
&= \nabla\nabla^{\nu+k}_{a+k+1}\nabla^{\mu-(k+1)}_{a}f(t) \\
&\quad - \sum_{j=0}^{k}\nabla^{\mu+j-k-1}_{a}f(a+k+1)H_{-\nu-k+j-1}(t,a+k+1) \\
&= \nabla\left[\nabla^{\mu+\nu-1}_{a}f(t) - \sum_{\tau=a+1}^{a+k+1}H_{-\nu-k-1}(t,\rho(\tau))\sum_{s=a+1}^{\tau}H_{k-\mu}(\tau,\rho(s))f(s)\right] \\
&\quad - \sum_{j=0}^{k}\nabla^{\mu+j-k-1}_{a}f(a+k+1)H_{-\nu-k+j-1}(t,a+k+1).
\end{aligned}$$

Hence

$$\nabla_{a+k+1}^{\nu}\nabla_a^{\mu}f(t) = \nabla_a^{\mu+\nu}f(t) - \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))f(s)\right]$$
$$-\sum_{j=0}^{k}\nabla_a^{\mu+j-k-1}f(a+k+1)H_{-\nu-k+j-1}(t,a+k+1).$$

Thus we have shown that with either set of order conditions, we arrive at the formula. This concludes our proof.

□

Again, two immediate corollaries follow from this theorem, which were first presented by Dahal & Goodrich. We view Theorem 3.4 as a nice generalization of any case of order conditions for this specific type of composition. To that end, while Dahal & Goodrich provided their own rigorous proofs for the specific cases, we show in this work how the corollaries follow immediately from Theorem 3.4.

Corollary 3.5 (Lemma 2.10, [21])**.** *Suppose that* $0 < \mu < 1$*,* $1 < \nu < 2$*, and* $1 < \mu + \nu < 2$*. For each* $t \in \mathbb{N}_{a+3}$ *it holds that*

$$\nabla_{a+1}^{\nu}\nabla_a^{\mu}f(t) = \nabla_a^{\mu+\nu}f(t) - H_{-\nu-2}(t,a)f(a+1) - H_{-\nu-1}(t,a+1)f(a+1).$$

Proof. We show that the corollary follows from Theorem 3.4.

Suppose that $k = 0$ and consider the first set of parameter conditions in Theorem 3.4. Then $0 < \mu < 1$, $1 < \nu < 2$, and $1 < \mu + \nu < 2$. Then by Theorem 3.4, for each $t \in \mathbb{N}_{a+3}$,

$$\begin{aligned}
\nabla_{a+1}^{\nu}\nabla_a^{\mu}f(t) &= \nabla_a^{\mu+\nu}f(t) - \nabla\left[\sum_{\tau=a+1}^{a+0+1} H_{-\nu-0-1}(t,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{0-\mu}(\tau,\rho(s))f(s)\right] \\
&- \left[\sum_{j=0}^{0} \nabla_a^{\mu+j-0-1}f(a+0+1)H_{-\nu-0+j-1}(t,a+0+1)\right] \\
&= \nabla_a^{\mu+\nu}f(t) - \nabla\left[\sum_{\tau=a+1}^{a+1} H_{-\nu-1}(t,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{-\mu}(\tau,\rho(s))f(s)\right] \\
&- \left[\sum_{j=0}^{0} \nabla_a^{\mu+j-1}f(a+1)H_{-\nu+j-1}(t,a+1)\right] \\
&= \nabla_a^{\mu+\nu}f(t) - \nabla H_{-\nu-1}(t,a)H_{-\mu}(a+1,a)f(a+1) \\
&\quad - \nabla_a^{\mu-1}f(a+1)H_{-\nu-1}(t,a+1) \\
&= \nabla_a^{\mu+\nu}f(t) - H_{-\nu-2}(t,a)f(a+1) - f(a+1)H_{-\nu-1}(t,a+1),
\end{aligned}$$

where we used the fact that

$$\nabla_a^{\mu-1}f(a+1) = f(a+1).$$

Thus the corollary follows from Theorem 3.4, and this concludes our proof.

□

Next we present the second of the corollaries, originally presented by Dahal & Goodrich with a case-specific proof.

Corollary 3.6 (Lemma 2.12, [21]). *Suppose that* $1 < \mu < 2$, $0 < \nu < 1$, *and* $1 < \mu + \nu < 2$. *For each* $t \in \mathbb{N}_{a+3}$ *it holds that*

$$
\begin{aligned}
&\nabla_{a+2}^{\nu}\nabla_a^{\mu}f(t)\\
&\quad= \nabla_a^{\mu+\nu}f(t)\\
&\quad+ \big[-H_{-\nu-3}(t,a) - (2-\mu)H_{-\nu-3}(t,a+1)\\
&\qquad - (2-\mu)H_{-\nu-2}(t,a+2) - (1-\mu)H_{-\nu-1}(t,a+2)\big]f(a+1)\\
&\quad+ \left[-H_{-\nu-3}(t,a+1) - H_{-\nu-2}(t,a+2) - H_{-\nu-1}(t,a+2)\right]f(a+2).
\end{aligned}
$$

Proof. We show that the corollary follows from Theorem 3.4. Recall the facts

$$\nabla_a^{\mu-2}f(a+2) = (2-\mu)f(a+1) + f(a+2)$$

and

$$\nabla_a^{\mu-1}f(a+2) = (1-\mu)f(a+1) + f(a+2).$$

Suppose that $k=1$ and consider the second set of parameter conditions in Theorem 3.4. Then $1<\mu<2$, $0<\nu<1$, and $1<\mu+\nu<2$. Then by Theorem 3.4, for each $t \in \mathbb{N}_{a+3}$,

$$
\begin{aligned}
\nabla_{a+2}^{\nu}\nabla_a^{\mu}f(t) = \nabla_a^{\mu+\nu}f(t) &- \nabla\left[\sum_{\tau=a+1}^{a+1+1} H_{-\nu-1-1}(t,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{1-\mu}(\tau,\rho(s))f(s)\right]\\
&- \left[\sum_{j=0}^{1} \nabla_a^{\mu+j-1-1}f(a+1+1)H_{-\nu+j-1-1}(t,a+1+1)\right].
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\nabla_{a+2}^{\nu}\nabla_a^{\mu}f(t) &= \nabla_a^{\mu+\nu}f(t) - \nabla\left[\sum_{\tau=a+1}^{a+2} H_{-\nu-2}(t,\rho(\tau))\sum_{s=a+1}^{\tau} H_{1-\mu}(\tau,\rho(s))f(s)\right] \\
&\quad - \left[\sum_{j=0}^{1}\nabla_a^{\mu+j-2}f(a+2)H_{-\nu+j-2}(t,a+2)\right] \\
&= \nabla_a^{\mu+\nu}f(t) \\
&\quad - \nabla\left[H_{-\nu-2}(t,a)H_{1-\mu}(a+1,a)f(a+1)\right. \\
&\qquad +H_{-\nu-2}(t,a+1)\left(H_{1-\mu}(a+2,a)f(a+1)\right. \\
&\qquad \left.\left. +H_{1-\mu}(a+2,a+1)f(a+2)\right)\right] \\
&\quad - \left[\nabla_a^{\mu-2}f(a+2)H_{-\nu-2}(t,a+2) + \nabla_a^{\mu-1}f(a+2)H_{-\nu-1}(t,a+2)\right] \\
&= \nabla_a^{\mu+\nu}f(t) - \nabla\left[H_{-\nu-2}(t,a)f(a+1)\right. \\
&\quad \left. +H_{-\nu-2}(t,a+1)\left((2-\mu)f(a+1)+f(a+2)\right)\right] \\
&\quad - \left[((2-\mu)f(a+1)+f(a+2))H_{-\nu-2}(t,a+2)\right. \\
&\quad \left. +((1-\mu)f(a+1)+f(a+2))H_{-\nu-1}(t,a+2)\right] \\
&= \nabla_a^{\mu+\nu}f(t) - \left[H_{-\nu-3}(t,a)f(a+1)\right. \\
&\quad \left. +H_{-\nu-3}(t,a+1)\left((2-\mu)f(a+1)+f(a+2)\right)\right] \\
&\quad - \left[((2-\mu)f(a+1)+f(a+2))H_{-\nu-2}(t,a+2)\right. \\
&\quad \left. +((1-\mu)f(a+1)+f(a+2))H_{-\nu-1}(t,a+2)\right] \\
&= \nabla_a^{\mu+\nu}f(t) \\
&\quad + \left[-H_{-\nu-3}(t,a) - (2-\mu)H_{-\nu-3}(t,a+1)\right. \\
&\qquad \left. -(2-\mu)H_{-\nu-2}(t,a+2) - (1-\mu)H_{-\nu-1}(t,a+2)\right]f(a+1) \\
&\quad + \left[-H_{-\nu-3}(t,a+1) - H_{-\nu-2}(t,a+2) - H_{-\nu-1}(t,a+2)\right]f(a+2).
\end{aligned}
$$

This concludes our proof. $\square$

Chapter 4

Monotonicity Results

4.1 Introduction

The work in in this chapter aims to demonstrate connections between the sign of sequential nabla fractional differences and the monotonicity of the function. This is particularly interesting due to the fact that, while $\nabla f(t) \geq 0$ would immediately tell us that the function f is increasing, the sign of a fractional derivative $\nabla_a^\nu f(t)$ does not necessarily tell us anything about the monotonicity of the function f. Thus investigating conditions under which we can achieve monotonicity is useful.

While uniform monotonicity is ideal, we do not always achieve this. In these instances, it is interesting to study the admissible parameter spaces as compared to the space on which the monotonicity of the function holds.

4.2 Uniform Monotonicity

In this first case, we do achieve uniform monotonicity with fairly few assumptions, as was presented by Dahal & Goodrich.

Theorem 4.1. *[21] Suppose that $0 < \mu < 1$, $1 < \nu < 2$, and $1 < \mu + \nu < 2$. In addition, assume that each of the following is true for the function $f : \mathbb{N}_{a+1} \to \mathbb{R}$.*

1. $f(a+1) \geq 0$

2. $\nabla f(a+2) \geq 0$

3. $\nabla_{a+1}^{\nu}\nabla_{a}^{\mu}f(t) \geq 0$*, for each* $t \in \mathbb{N}_{a+3}$

Then $\nabla f(t) \geq 0$ *for each* $t \in \mathbb{N}_{a+3}$*.*

4.3 Non-Uniform Monotonicity

The theorem above shows that uniform monotonicity holds under the given conditions for the case $0 < \mu < 1$, $1 < \nu < 2$, and $1 < \mu+\nu < 2$. However, for a slightly different set of order conditions, we do not get uniform monotonicity. To find monotonicity on at least a subspace, we must impose two more assumptions, namely a condition relating the value of the function at $a+2$ to its value at $a+1$.

Theorem 4.2. *[21] Suppose that* $1 < \mu < 2$*,* $0 < \nu < 1$*, and* $1 < \mu+\nu < 2$*. In addition, assume that each of the following is true for the function* $f : \mathbb{N}_{a+1} \to \mathbb{R}$*.*

1. $f(a+1) \geq 0$

2. $f(a+2) \geq 0$

3. $\nabla f(a+2) \geq 0$

4. $\nabla_{a+2}^{\nu}\nabla_{a}^{\mu}f(a+3) \geq 0$

5. $f(a+2) \leq Cf(a+1)$*, for some* $C \geq 1$

Then provided that $\nu \leq \frac{-\mu^2+3\mu-2}{2(C-1)}$*, it follows* $\nabla f(a+3) \geq 0$*.*

Notice that Dahal & Goodrich [21] established mixed order monotonicity results for the logical first two cases: first, the nice uniform monotonicity result for the set of

order conditions $1 < \nu < 2$, $0 < \mu < 1$, and $1 < \mu + \nu < 2$, and second, monotonicity on a subspace for the set of order conditions $0 < \nu < 1$, $1 < \mu < 2$, and $1 < \mu+\nu < 2$. We now present a monotonicity result that requires the exact same assumptions as we have in Theorem 4.2, but for a new set of order conditions, where $1 < \mu < 2$, $2 < \nu < 3$, and $3 < \mu + \nu < 4$.

Theorem 4.3. *Suppose that $1 < \mu < 2$, $2 < \nu < 3$, and $3 < \mu + \nu < 4$. In addition, assume that each of the following is true for the function $f : \mathbb{N}_{a+1} \to \mathbb{R}$.*

1. $f(a+1) \geq 0$
2. $f(a+2) \geq 0$
3. $\nabla f(a+2) \geq 0$
4. $\nabla^{\nu}_{a+2}\nabla^{\mu}_{a} f(a+3) \geq 0$
5. $f(a+2) \leq Cf(a+1)$, *for some* $C \geq 1$

Then provided that $\nu \leq \frac{-\mu^2+3\mu-2}{2(C-1)}$, it follows $\nabla f(a+3) \geq 0$.

Proof. We first use Theorem 3.4 to notice that

$$\begin{aligned}
\nabla^{\nu}_{a+2}\nabla^{\mu}_{a} f(t) = \nabla^{\mu+\nu}_{a} f(t) \\
&+ [-H_{-\nu-3}(t,a) - (2-\mu)H_{-\nu-3}(t,a+1) \\
&\quad -(2-\mu)H_{-\nu-2}(t,a+2) - (1-\mu)H_{-\nu-1}(t,a+2)]\, f(a+1) \\
&+ [-H_{-\nu-3}(t,a+1) - H_{-\nu-2}(t,a+2) - H_{-\nu-1}(t,a+2)]\, f(a+2).
\end{aligned}$$

Now we will rewrite the $\mu+\nu^{th}$ nabla fractional difference using Theorem 1.14 since $\mu+\nu>0$:

$$\begin{aligned}\nabla_a^{\mu+\nu} f(t) &= \int_a^t H_{-\mu-\nu-1}(t,\rho(s))f(s)\nabla s \\ &= \left[H_{-\mu-\nu-1}(t,a) + H_{-\mu-\nu}(t,a+1)\right] f(a+1) \\ &\qquad + \nabla f(t) + \sum_{\tau=a+2}^{t-1} H_{-\mu-\nu}(t,\rho(\tau))\nabla f(\tau).\end{aligned}$$

Now whenever $\nabla f(\tau) \geq 0$ it follows that $\sum_{\tau=a+2}^{t-1} H_{-\mu-\nu}(t,\rho(\tau))\nabla f(\tau) < 0$, and so by assumption (3) we have that

$$-\left[\sum_{\tau=a+2}^{t-1} H_{-\mu-\nu}(t,\rho(\tau))\nabla f(\tau)\right] \geq 0.$$

Thus using these two pieces together, it follows that

$$\begin{aligned}&\nabla f(a+3) \\ &\quad \geq -\sum_{\tau=a+2}^{a+2} H_{-\mu-\nu}(a+3,\rho(\tau))\nabla f(\tau) \\ &\qquad - \left[H_{-\mu-\nu-1}(a+3,a) + H_{-\mu-\nu}(a+3,a+1)\right] f(a+1) \\ &\qquad - \left[-H_{-\nu-3}(a+3,a) - (2-\mu)H_{-\nu-3}(a+3,a+1)\right. \\ &\qquad\qquad \left. -(2-\mu)H_{-\nu-2}(a+3,a+2) - (1-\mu)H_{-\nu-1}(a+3,a+2)\right] f(a+1) \\ &\qquad - \left[-H_{-\nu-3}(a+3,a+1) - H_{-\nu-2}(a+3,a+2) - H_{-\nu-1}(a+3,a+2)\right] f(a+2) \\ &\quad \geq - \left[H_{-\mu-\nu-1}(a+3,a) + H_{-\mu-\nu}(a+3,a+1)\right] f(a+1) \\ &\qquad - \left[-H_{-\nu-3}(a+3,a) - (2-\mu)H_{-\nu-3}(a+3,a+1)\right. \\ &\qquad\qquad \left. -(2-\mu)H_{-\nu-2}(a+3,a+2) - (1-\mu)H_{-\nu-1}(a+3,a+2)\right] f(a+1) \\ &\qquad - \left[-H_{-\nu-3}(a+3,a+1) - H_{-\nu-2}(a+3,a+2) - H_{-\nu-1}(a+3,a+2)\right] f(a+2).\end{aligned}$$

Hence

$$
\begin{aligned}
&\nabla f(a+3) \\
&\quad \geq -\left[H_{-\mu-\nu-1}(a+3,a) + H_{-\mu-\nu}(a+3,a+1)\right] f(a+1) \\
&\qquad \left[H_{-\nu-3}(a+3,a) + (2-\mu)H_{-\nu-3}(a+3,a+1)\right. \\
&\qquad\qquad \left. +(2-\mu)H_{-\nu-2}(a+3,a+2) + (1-\mu)H_{-\nu-1}(a+3,a+2)\right] f(a+1) \\
&\qquad \left[H_{-\nu-3}(a+3,a+1) + H_{-\nu-2}(a+3,a+2) + H_{-\nu-1}(a+3,a+2)\right] f(a+2).
\end{aligned} \tag{4.1}
$$

Considering the coefficient of $f(a+2)$ we have

$$
\begin{aligned}
&H_{-\nu-3}(a+3,a+1) + H_{-\nu-2}(a+3,a+2) + H_{-\nu-1}(a+3,a+2) \\
&\quad = \frac{2^{\overline{-\nu-3}}}{\Gamma(-\nu-2)} + \frac{1^{\overline{-\nu-2}}}{\Gamma(-\nu-1)} + \frac{1^{\overline{-\nu-1}}}{\Gamma(-\nu)} \\
&\quad = \frac{\Gamma(-\nu-1)}{\Gamma(2)\Gamma(-\nu-2)} + \frac{\Gamma(-\nu-1)}{\Gamma(1)\Gamma(-\nu-1)} + \frac{\Gamma(-\nu)}{\Gamma(1)\Gamma(-\nu)} \\
&\quad = \frac{(-\nu-2)\Gamma(-\nu-2)}{\Gamma(-\nu-2)} + 1 + 1 \\
&\quad = (-\nu-2) + 1 + 1 = -\nu < 0.
\end{aligned}
$$

Therefore the coefficient of $f(a+2)$ is negative, so when we multiply both sides of an inequality by this coefficient, the sign will flip. Recall assumption (5). Then we have that $f(a+2) \leq Cf(a+1)$ implies

$$
\begin{aligned}
&\Big[H_{-\nu-3}(a+3,a+1) + H_{-\nu-2}(a+3,a+2) + H_{-\nu-1}(a+3,a+2)\Big] f(a+2) \\
&\quad \geq \left[H_{-\nu-3}(a+3,a+1) + H_{-\nu-2}(a+3,a+2) + H_{-\nu-1}(a+3,a+2)\right] Cf(a+1).
\end{aligned}
$$

Using the facts

$$H_{-\nu-1}(a+3,a+2) = H_{-\nu-2}(a+3,a+2) = 1,$$

$$H_{-\nu-3}(a+3,a+1) = \frac{2^{\overline{-\nu-3}}}{\Gamma(-\nu-2)} = \frac{\Gamma(-\nu-1)}{\Gamma(2)\Gamma(-\nu-2)} = -\nu-2,$$

and

$$H_{-\nu-3}(a+3,a) = \frac{3^{\overline{-\nu-3}}}{\Gamma(-\nu-2)} = \frac{\Gamma(-\nu)}{\Gamma(3)\Gamma(-\nu-2)} = \frac{1}{2}(-\nu-1)(-\nu-2) = \frac{1}{2}(\nu+1)(\nu+2),$$

we in turn have

$$\begin{aligned}
&\Big[H_{-\nu-3}(a+3,a) + (2-\mu)H_{-\nu-3}(a+3,a+1)\\
&\quad + (2-\mu)H_{-\nu-2}(a+3,a+2) + (1-\mu)H_{-\nu-1}(a+3,a+2)\Big] f(a+1)\\
&\quad + \left[H_{-\nu-3}(a+3,a+1) + H_{-\nu-2}(a+3,a+2) + H_{-\nu-1}(a+3,a+2)\right] f(a+2)\\
&\geq \Big[H_{-\nu-3}(a+3,a) + (2-\mu)H_{-\nu-3}(a+3,a+1)\\
&\quad + (2-\mu)H_{-\nu-2}(a+3,a+2) + (1-\mu)H_{-\nu-1}(a+3,a+2)\Big] f(a+1)\\
&\quad + \left[H_{-\nu-3}(a+3,a+1) + H_{-\nu-2}(a+3,a+2) + H_{-\nu-1}(a+3,a+2)\right] Cf(a+1)\\
&= \Big[H_{-\nu-3}(a+3,a) + (C+2-\mu)H_{-\nu-3}(a+3,a+1)\\
&\quad + (C+2-\mu)H_{-\nu-2}(a+3,a+2) + (C+1-\mu)H_{-\nu-1}(a+3,a+2)]\Big] f(a+1)\\
&= \left[\frac{1}{2}(\nu+1)(\nu+2) + (C+2-\mu)\left[(-\nu-2)+1\right] + (C+1-\mu)\right] f(a+1)\\
&= \left[\frac{1}{2}(\nu+1)(\nu+2) + (C+2-\mu)(-1-\nu) + (C+1-\mu)\right] f(a+1)\\
&= \left[\frac{1}{2}\nu^2 + \left(\mu - C - \frac{1}{2}\right)\nu\right] f(a+1).
\end{aligned}$$

Now putting this into (4.1), we have

$$
\begin{aligned}
&\nabla f(a+3)\\
&\quad\geq -\left[H_{-\mu-\nu-1}(a+3,a)+H_{-\mu-\nu}(a+3,a+1)\right]f(a+1)\\
&\qquad\quad+\left[H_{-\nu-3}(a+3,a)+(2-\mu)H_{-\nu-3}(a+3,a+1)\right.\\
&\qquad\qquad\left.+(2-\mu)H_{-\nu-2}(a+3,a+2)+(1-\mu)H_{-\nu-1}(a+3,a+2)\right]f(a+1)\\
&\qquad+\left[H_{-\nu-3}(a+3,a+1)+H_{-\nu-2}(a+3,a+2)+H_{-\nu-1}(a+3,a+2)\right]f(a+2)\\
&\quad\geq\left[-\frac{3^{\overline{-\mu-\nu-1}}}{\Gamma(-\mu-\nu)}-\frac{2^{\overline{-\mu-\nu}}}{\Gamma(-\mu-\nu+1)}\right]f(a+1)+\left[\frac{1}{2}\nu^2+\left(\mu-C-\frac{1}{2}\right)\nu\right]f(a+1)\\
&\quad=-\frac{1}{2}\left[(1-\mu-\nu)(-\mu-\nu)-(1-\mu-\nu)+\frac{1}{2}\nu^2+\left(\mu-C-\frac{1}{2}\right)\right]f(a+1)\\
&\quad=\left[\frac{1}{2}(-2C\nu-\mu^2+3\mu+2\nu-2)\right]f(a+1).
\end{aligned}
$$

So, given assumption (1), $\nabla f(a+3)\geq 0$ if

$$
\begin{aligned}
(-2C+2)\nu &\geq \mu^2-3\mu+2\\
\nu &\leq \frac{-\mu^2+3\mu-2}{2(C-1)},
\end{aligned}
$$

where we used the fact $C>1$ so that $-2C+2<0$. Thus since this inequality is true by assumption, we have achieved our result.

$\square$

4.4 Further Work

For future directions, one may continue to explore establishing mixed order monotonicity results for different sets of order conditions for the sequential nabla differences. For the sequential nabla difference $\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}f(t)$, this chapter has only covered

the cases

1. $\nabla_{a+1}^{\nu}\nabla_{a}^{\mu}f(t)$ where $0 < \mu < 1$, $1 < \nu < 2$, and $1 < \mu + \nu < 2$,

2. $\nabla_{a+1}^{\nu}\nabla_{a}^{\mu}f(t)$ where $1 < \mu < 2$, $0 < \nu < 1$, and $1 < \mu + \nu < 2$, and

3. $\nabla_{a+2}^{\nu}\nabla_{a}^{\mu}f(t)$ where $1 < \mu < 2$, $2 < \nu < 3$, and $3 < \mu + \nu < 4$.

Note that (1) and (3) above match the form of Theorem 3.4 (i), and (2) matches the form of Theorem 3.4 (ii).

While it would be convenient to be able to establish a general monotonicity result for the nabla difference $\nabla_{a+k+1}^{\nu}\nabla_{a}^{\mu}f(t)$ for either case in Theorem 3.4, we conjecture that the amount of conditions needed to establish any sort of monotonicity, uniform or not, will increase as we increase the value k. Similarly, we notice that even though we considered the same sequential difference in (1) and (2) above, the set of order conditions presented in (2) where $\mu > \nu$ presented more of a challenge in establishing a monotonicity result - requiring more conditions and also achieving monotonicity on just a subspace of the admissible parameter space. This will likely be the case for any set of order conditions matching the form in Theorem 3.4 (ii).

Chapter 5

Initial Value Problems and Boundary Value Problems Involving Sequential Nabla Differences

5.1 Introduction

In this chapter we introduce the sequential nabla fractional difference equation

$$\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}y(t) = h(t). \tag{5.1}$$

Notice that this is the same form we considered in Theorem 3.4 in Chapter 3. In Section 5.2 we will consider the simple homogeneous difference equation

$$\nabla^{\nu}_{a+k}x(t) = 0,$$

and in Section 5.3 we consider the case $k = 0$ in equation (5.1) and prove that the solutions, with certain initial conditions, both exist and are unique. We further establish a variation of constants formula for a nonhomogeneous nabla fractional difference initial value problem, followed by considering a nonhomogeneous sequential nabla difference boundary value problem, deriving a Green's function for the corresponding homogeneous boundary value problem. We also show that the Green's function is

nonnegative and find its maximum.

In Section 5.4 we consider the case $k = 1$ of equation (5.1) and prove that the solutions, with certain initial conditions, both exist and are unique. We further establish a variation of constants formula for a nonhomogeneous nabla fractional difference initial value problem, followed by considering a nonhomogeneous sequential nabla difference boundary value problem, deriving a Green's function for the corresponding homogeneous boundary value problem. We also show that the Green's function is nonnegative and find its maximum. In Section 5.5 we briefly consider the general form in (5.1), proving existence and uniqueness of solutions.

Before moving on to the sequential nabla difference equation (5.1), we first consider a nabla difference equation of the form

$$\nabla^{\nu}_{a+k} x(t) = f(t) \tag{5.2}$$

for $k \in \mathbb{N}_0$ and $\nu > 0$. We note that there are many results involving similar initial value problems and boundary value problems, but for the specific case $k = 0$, and most often with the condition $0 < \nu < 1$ (e.g. [18], [37]).

5.2 The Homogeneous Difference Equation $\nabla^{\nu}_{a+k} x(t) = 0$

The following theorem provides us with a general solution to the homogeneous version of the nabla difference equation (5.2).

Theorem 5.1. *Assume $\nu > 0$, $k \in \mathbb{N}_0$, and $N - 1 < \nu \leq N$. Then a general solution of the homogeneous equation $\nabla^{\nu}_{a+k} x(t) = 0$ is given by*

$$x(t) = c_1 H_{\nu-1}(t, a+k) + c_2 H_{\nu-2}(t, a+k) + \cdots + c_N H_{\nu-N}(t, a+k), \quad t \in \mathbb{N}_{a+k}.$$

Proof. For $1 \leq j \leq N$,

$$\begin{aligned}\nabla_{a+k}^{\nu} H_{\nu-j}(t, a+k) &= \nabla^{j} \nabla_{a+k}^{\nu-j} H_{\nu-j}(t, a+k) \\ &= \nabla^{j} H_{(\nu-j)-(\nu-j)}(t, a+k) \\ &= \nabla^{j} H_{0}(t, a+k) \\ &= \nabla^{j} 1 = 0\end{aligned}$$

for $t \in \mathbb{N}_{a+k}$. Since these N solutions are linearly independent on $\mathbb{N}_{a+k}$, we have

$$x(t) = c_1 H_{\nu-1}(t, a+k) + c_2 H_{\nu-2}(t, a+k) + \cdots + c_N H_{\nu-N}(t, a+k)$$

is a general solution of $\nabla_{a+k}^{\nu} x(t) = 0$ on $\mathbb{N}_{a+k}$.

□

Note that the case $k = 0$ provides us with the following theorem as a corollary:

Theorem 5.2 (Theorem 3.95, [37]). *Assume $\nu > 0$ and $N - 1 < \nu \leq N$. Then a general solution of $\nabla_a^{\nu} x(t) = 0$ is given by*

$$x(t) = c_1 H_{\nu-1}(t, a) + c_2 H_{\nu-2}(t, a) + \cdots + c_N H_{\nu-N}(t, a)$$

for $t \in \mathbb{N}_a$.

We will now study a simple homogeneous nabla difference equation before moving on to sequential nabla fractional differences. For $1 < \nu \leq 2$, consider

$$\nabla_{a+1}^{\nu} x(t) = 0. \tag{5.3}$$

We next present some initial value problems involving this homogeneous nabla difference equation (5.3).

Theorem 5.3. *Assume* $1 < \nu \leq 2$. *Then the unique solution of the initial value problem*

$$\begin{cases} \nabla^{\nu}_{a+1}x(t) & = 0, \quad t \in \mathbb{N}_{a+3} \\ x(a+3) & = A_0 \\ \nabla x(a+3) & = A_1 \end{cases} \tag{5.4}$$

where $A_0, A_1 \in \mathbb{R}$, *is given by*

$$x(t) = [(2-\nu)A_0 + (\nu-1)A_1]\, H_{\nu-1}(t, a+1) + [(\nu-1)A_0 - \nu A_1]\, H_{\nu-2}(t, a+1)$$

for $t \in \mathbb{N}_{a+1}$.

Proof. Let $x(t)$ be the solution of the initial value problem (5.4). Then by Theorem 5.1,

$$x(t) = c_1 H_{\nu-1}(t, a+1) + c_2 H_{\nu-2}(t, a+1).$$

Using the initial conditions, we get that

$$x(a+2) = x(a+3) - \nabla x(a+3) = A_0 - A_1.$$

It follows that

$$x(a+2) = c_1 H_{\nu-1}(a+2, a+1) + c_2 H_{\nu-2}(a+2, a+1) = A_0 - A_1,$$

which implies $c_1 + c_2 = A_0 - A_1$. Since

$$\nabla x(t) = c_1 H_{\nu-2}(t, a+1) + c_2 H_{\nu-3}(t, a+1),$$

we use our second initial condition to see that

$$\begin{aligned}\nabla x(a+3) &= c_1 H_{\nu-2}(a+3, a+1) + c_2 H_{\nu-3}(a+3, a+1) \\ &= c_1 \frac{2^{\overline{\nu-2}}}{\Gamma(\nu-1)} + c_2 \frac{2^{\overline{\nu-3}}}{\Gamma(\nu-2)} \\ &= c_1 \frac{\Gamma(\nu)}{\Gamma(2)\Gamma(\nu-1)} + c_2 \frac{\Gamma(\nu-1)}{\Gamma(2)\Gamma(\nu-2)} \\ &= c_1(\nu-1) + c_2(\nu-2) \\ &= A_1.\end{aligned}$$

Solving the system

$$\begin{aligned} c_1 + c_2 &= A_0 - A_1 \\ (\nu-1)c_1 + (\nu-2)c_2 &= A_1 \end{aligned}$$

gives us that $c_1 = (2-\nu)A_0 + (\nu-1)A_1$ and $c_2 = (\nu-1)A_0 - \nu A_1$. Thus we have that the solution of the initial value problem (5.4) is given by

$$x(t) = \left[(2-\nu)A_0 + (\nu-1)A_1\right] H_{\nu-1}(t, a+1) + \left[(\nu-1)A_0 - \nu A_1\right] H_{\nu-2}(t, a+1)$$

for $t \in \mathbb{N}_{a+1}$. This concludes our proof.

$\square$

In fact, we can generalize this theorem in terms of the base as follows, to find the unique solution of any such initial value problem.

Theorem 5.4. *Assume $1 < \nu \leq 2$ and $k \in \mathbb{N}_0$. Then the unique solution of the initial value problem*

$$\begin{cases} \nabla^{\nu}_{a+k}x(t) &= 0, \quad t \in \mathbb{N}_{a+k+2} \\ x(a+k+2) &= A_0 \\ \nabla x(a+k+2) &= A_1 \end{cases} \tag{5.5}$$

where $A_0, A_1 \in \mathbb{R}$, is given by

$$x(t) = \left[(2-\nu)A_0 + (\nu-1)A_1\right] H_{\nu-1}(t, a+k) + \left[(\nu-1)A_0 - \nu A_1\right] H_{\nu-2}(t, a+k)$$

for $t \in \mathbb{N}_{a+k}$.

Proof. Let $x(t)$ be the solution of the initial value problem (5.5). Then by Theorem 5.1,

$$x(t) = c_1 H_{\nu-1}(t, a+k) + c_2 H_{\nu-2}(t, a+k).$$

Using the initial conditions, we get that

$$x(a+k+1) = x(a+k+2) - \nabla x(a+k+2) = A_0 - A_1.$$

It follows that

$$x(a+k+1) = c_1 H_{\nu-1}(a+k+1, a+k) + c_2 H_{\nu-2}(a+k+1, a+k) = A_0 - A_1,$$

which implies $c_1 + c_2 = A_0 - A_1$. Since

$$\nabla x(t) = c_1 H_{\nu-2}(t, a+k+1) + c_2 H_{\nu-3}(t, a+k+1),$$

it follows that

$$\begin{aligned}
\nabla x(a+k+2) &= c_1 H_{\nu-2}(a+k+2,a+k) + c_2 H_{\nu-3}(a+k+2,a+k)\\
&= c_1 \frac{2^{\overline{\nu-2}}}{\Gamma(\nu-1)} + c_2 \frac{2^{\overline{\nu-3}}}{\Gamma(\nu-2)}\\
&= c_1 \frac{\Gamma(\nu)}{\Gamma(2)\Gamma(\nu-1)} + c_2 \frac{\Gamma(\nu-1)}{\Gamma(2)\Gamma(\nu-2)}\\
&= c_1(\nu-1) + c_2(\nu-2)\\
&= A_1.
\end{aligned}$$

Solving the system

$$\begin{aligned}
c_1 + c_2 &= A_0 - A_1\\
(\nu-1)c_1 + (\nu-2)c_2 &= A_1
\end{aligned}$$

gives us

$c_1 = (2-\nu)A_0 + (\nu-1)A_1$ and $c_2 = (\nu-1)A_0 - \nu A_1$. Thus we have the form of our solution to the initial value problem (5.5) as

$$x(t) = \left[(2-\nu)A_0 + (\nu-1)A_1\right] H_{\nu-1}(t,a+k) + \left[(\nu-1)A_0 - \nu A_1\right] H_{\nu-2}(t,a+k)$$

for $t \in \mathbb{N}_{a+k}$.

$\square$

Note that $k=0$ gives us the well-studied case of an initial value problem involving a nabla difference with the traditional base a, and $k=1$ gives us the case examined in Theorem 5.3.

5.3 The Sequential Nabla Difference $\nabla_{a+1}^{\nu}\nabla_a^{\mu}x(t)$

In this section we introduce the sequential nabla fractional difference equation

$$-\nabla_{a+1}^{\nu}\nabla_a^{\mu}y(t) = h(t), \tag{5.6}$$

where $t \in \mathbb{N}_{a+3}^{b}$ for some real numbers a, b such that $b-(a+2) \in \mathbb{N}_1$, and $f : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$. We consider the set of order conditions where $1 < \nu < 2$, $0 < \mu < 1$, and $1 < \mu+\nu < 2$.

5.3.1 Existence and Uniqueness

In this section we prove an existence and uniqueness theorem for the nabla sequential difference initial value problem.

Theorem 5.5 (Existence & Uniqueness Theorem). *Let $x : \mathbb{N}_{a+1} \to \mathbb{R}$, $1 < \nu < 2$, $0 < \mu < 1$, and $1 < \mu+\nu < 2$. Then the fractional initial value problem*

$$\begin{cases} -\nabla_{a+1}^{\nu}\nabla_a^{\mu}x(t) &= f(t), \quad t \in \mathbb{N}_{a+3} \\ x(a+1) = A,\ x(a+2) &= B. \end{cases} \tag{5.7}$$

has a unique solution $x(t)$.

Proof. Consider the initial value problem (5.7). By Theorem 3.4 (i) with $k = 0$, we rewrite the nabla difference equation as

$$\begin{aligned} -\nabla_{a+1}^{\nu}\nabla_a^{\mu}x(t) &= -\left[\nabla_a^{\mu+\nu}x(t) - H_{-\nu-2}(t,a)x(a+1) - H_{-\nu-1}(t,a+1)x(a+1)\right] \\ &= -\nabla_a^{\mu+\nu}x(t) + H_{-\nu-2}(t,a)x(a+1) + H_{-\nu-1}(t,a+1)x(a+1) \\ &= f(t). \end{aligned}$$

Letting $t = a + 3$, we have

$$
\begin{aligned}
f(a+3) &= -\left(\nabla^{\nu}_{a+1}\nabla^{\mu}_{a}x\right)(a+3)\\
&= -\left(\nabla^{\mu+\nu}_{a}x\right)(a+3) + H_{-\nu-2}(a+3,a)x(a+1)\\
&\quad + H_{-\nu-1}(a+3,a+1)x(a+1)\\
&= -\sum_{s=a+1}^{a+3} H_{-\mu-\nu-1}(a+3,\rho(s))x(s) + H_{-\nu-2}(a+3,a)x(a+1)\\
&\quad + H_{-\nu-1}(a+3,a+1)x(a+1)\\
&= -H_{-\mu-\nu-1}(a+3,a)x(a+1) - H_{-\mu-\nu-1}(a+3,a+1)x(a+2)\\
&\quad - H_{-\mu-\nu-1}(a+3,a+2)x(a+3) + H_{-\nu-2}(a+3,a)x(a+1)\\
&\quad + H_{-\nu-1}(a+3,a+1)x(a+1)\\
&= -H_{-\mu-\nu-1}(a+3,a)x(a+1) - H_{-\mu-\nu-1}(a+3,a+1)x(a+2) - x(a+3)\\
&\quad + H_{-\nu-2}(a+3,a)x(a+1) + H_{-\nu-1}(a+3,a+1)x(a+1).
\end{aligned}
$$

Rearranging provides

$$
\begin{aligned}
x(a+3) &= -f(a+3) - H_{-\mu-\nu-1}(a+3,a)x(a+1) - H_{-\mu-\nu-1}(a+3,a+1)x(a+2)\\
&\quad + H_{-\nu-2}(a+3,a)x(a+1) + H_{-\nu-1}(a+3,a+1)x(a+1)\\
&= -f(a+3) - AH_{-\mu-\nu-1}(a+3,a) - BH_{-\mu-\nu-1}(a+3,a+1)\\
&\quad + AH_{-\nu-2}(a+3,a) + AH_{-\nu-1}(a+3,a+1).
\end{aligned}
$$

Thus the value of $x(a+3)$ is uniquely determined by the initial conditions $x(a+1) = A$ and $x(a+2) = B$ and the value of the given function $f(t)$.

To show $x(t)$ is uniquely determined on $\mathbb{N}_{a+1}$ we will proceed by induction. Suppose for $t \in \mathbb{N}^{t_0}_{a+1}$ there exists a unique solution, $x(t)$, to the initial value problem

(5.7), where $t_0 \in \mathbb{N}_{a+2}$. We will show that the value of $x(t_0+1)$ is uniquely determined by the values of $x(t)$ on $\mathbb{N}_{a+1}^{t_0}$.

$$\begin{aligned}
f(t_0+1) &= -\left(\nabla_{a+1}^{\nu}\nabla_a^{\mu}x\right)(t_0+1) \\
&= -\left(\nabla_a^{\mu+\nu}x\right)(t_0+1) + H_{-\nu-2}(t_0+1,a)x(a+1) \\
&\quad + H_{-\nu-1}(t_0+1,a+1)x(a+1) \\
&= -\sum_{s=a+1}^{t_0+1} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) + H_{-\nu-2}(t_0+1,a)x(a+1) \\
&\quad + H_{-\nu-1}(t_0+1,a+1)x(a+1) \\
&= -H_{-\mu-\nu-1}(t_0+1,t_0)x(t_0+1) - \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) \\
&\quad + H_{-\nu-2}(t_0+1,a)x(a+1) + H_{-\nu-1}(t_0+1,a+1)x(a+1).
\end{aligned}$$

Rearranging provides

$$\begin{aligned}
x(t_0+1) = -f(t_0+1) &- \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) + H_{-\nu-2}(t_0+1,a)x(a+1) \\
&+ H_{-\nu-1}(t_0+1,a+1)x(a+1).
\end{aligned}$$

Now by the induction hypothesis, all the values of $x(t)$ for $t \in \mathbb{N}_{a+1}^{t_0}$ are known. Therefore $x(t_0+1)$ is uniquely determined and hence $x(t)$ is the unique solution of the fractional initial value problem (5.7) on $\mathbb{N}_{a+1}^{t_0+1}$. Thus a unique solution of the initial value problem (5.7) exists on $\mathbb{N}_{a+1}$.

$\square$

5.3.2 Initial Value Problem

We begin this section by presenting an initial value problem involving a nonhomogeneous nabla difference equation with zero initial conditions. This section continues by presenting our Cauchy function and variation of constants formula for a nabla difference initial value problem. All these results will be drawn upon in the next section to establish the form of the Green's function.

Theorem 5.6. *Let $g : \mathbb{N}_{a+1} \to \mathbb{R}$ and $1 < \nu \leq 2$. Then for $t \in \mathbb{N}_{a+3}$ the initial value problem*

$$\begin{cases} -\nabla^{\nu}_{a+1}x(t) &= g(t), \quad t \in \mathbb{N}_{a+3} \\ x(a+3) &= 0 \\ \nabla x(a+3) &= 0 \end{cases} \tag{5.8}$$

has the unique solution

$$x(t) = -\nabla^{-\nu}_{a+1}g(t) + [g(a+2) + g(a+3)]\, H_{\nu-1}(t, a+1) - g(a+3)H_{\nu-2}(t, a+1).$$

Proof. Considering the initial conditions in (5.8), we begin by noting that

$$\begin{aligned} 0 = \nabla x(a+3) &= x(a+3) - x(a+2) \\ &= 0 - x(a+2) \end{aligned}$$

implies $x(a+2) = 0$.

Now by taking the nabla Laplace transform based at $a+3$ of both sides and implementing Theorem 1.35 with $k = 1$ and $N = 3$, we have

$$-\mathcal{L}_{a+3}\{g\}(s) = s^{\nu}\mathcal{L}_{a+3}\{x\}(s) + \sum_{j=0}^{3-1-1}\left[\frac{s^{\nu}x(a+1+j+1) - s^{3-1}\nabla_{a+1}^{-(3-1-\nu)}x(a+1+j+1)}{(1-s)^{3-1-j}} - s^{j}\nabla^{3-1-j}\nabla_{a+1}^{-(3-1-\nu)}x(a+3)\right].$$

Hence

$$\begin{aligned}-\mathcal{L}_{a+3}\{g\}(s) &= s^{\nu}\mathcal{L}_{a+3}\{x\}(s) + \sum_{j=0}^{1}\left[\frac{s^{\nu}x(a+2+j) - s^{2}\nabla_{a+1}^{-(2-\nu)}x(a+2+j)}{(1-s)^{2-j}} - s^{j}\nabla^{2-j}\nabla_{a+1}^{-(2-\nu)}x(a+3)\right]\\ &= s^{\nu}\mathcal{L}_{a+3}\{x\}(s) + \frac{s^{\nu}x(a+2) - s^{2}\nabla_{a+1}^{-(2-\nu)}x(a+2)}{(1-s)^{2}} - \nabla^{2}\nabla_{a+1}^{-(2-\nu)}x(a+3)\\ &\quad + \frac{s^{\nu}x(a+3) - s^{2}\nabla_{a+1}^{-(2-\nu)}x(a+3)}{(1-s)} - s\nabla\nabla_{a+1}^{-(2-\nu)}x(a+3)\\ &= s^{\nu}\mathcal{L}_{a+3}\{x\}(s),\end{aligned}$$

since $x(a+2) = x(a+3) = 0$. Now we are interested in an equation involving the Laplace transform based at $a+1$, as that is the base of our nabla difference. To transform our current equation, we use the Shifting Lemma 1.31, replacing a with $a+1$, where $n = 2$ as follows:

$$-\left(\frac{1}{1-s}\right)^{2}\mathcal{L}_{a+1}\{g\}(s) + \sum_{k=1}^{2}\frac{g(a+1+k)}{(1-s)^{2-k+1}} = s^{\nu}\left[\left(\frac{1}{1-s}\right)^{2}\mathcal{L}_{a+1}\{x\}(s) - \sum_{k=1}^{2}\frac{x(a+1+k)}{(1-s)^{2-k+1}}\right].$$

Hence

$$-\frac{1}{(1-s)^2}\mathcal{L}_a\{g\}(s)+\sum_{k=1}^{2}\frac{g(a+1+k)}{(1-s)^{2-k+1}}=\frac{s^\nu}{(1-s)^2}\mathcal{L}_{a+1}\{x\}(s)-\frac{s^\nu}{(1-s)^2}x(a+2)$$
$$-\frac{s^\nu}{(1-s)}x(a+3)$$
$$-\frac{1}{(1-s)^2}\mathcal{L}_a\{g\}(s)+\sum_{k=1}^{2}\frac{g(a+1+k)}{(1-s)^{2-k+1}}=\frac{s^\nu}{(1-s)^2}\mathcal{L}_{a+1}\{x\}(s),$$

again, since $x(a+2)=x(a+3)=0$. Expanding gives

$$\frac{s^\nu}{(1-s)^2}\mathcal{L}_{a+1}\{x\}(s)=-\frac{1}{(1-s)^2}\mathcal{L}_{a+1}\{g\}(s)+\frac{g(a+2)}{(1-s)^2}+\frac{g(a+3)}{(1-s)},$$

and then multiplying through by $(1-s)^2$ gives

$$\mathcal{L}_{a+1}\{x\}(s)=-\frac{1}{s^\nu}\mathcal{L}_{a+1}\{g\}(s)+\frac{1}{s^\nu}g(a+2)+\frac{1-s}{s^\nu}g(a+3)$$
$$\mathcal{L}_{a+1}\{x\}(s)=-\frac{1}{s^\nu}\mathcal{L}_{a+1}\{g\}(s)+\frac{1}{s^\nu}g(a+2)+\frac{1}{s^\nu}g(a+3)-\frac{1}{s^{\nu-1}}g(a+3).$$

This implies that

$$\begin{aligned}x(t)=&-\left(H_{\nu-1}(\cdot,a+1)*g(\cdot)\right)(t)+g(a+2)H_{\nu-1}(t,a+1)+g(a+3)H_{\nu-1}(t,a+1)\\&-g(a+3)H_{\nu-2}(t,a+1)\\=&-\nabla_{a+1}^{-\nu}g(t)+\left[g(a+2)+g(a+3)\right]H_{\nu-1}(t,a+1)-g(a+3)H_{\nu-2}(t,a+1).\end{aligned}$$

This completes the proof.

□

Our Green's function, along with the following theorem, will involve the Cauchy function which is defined below.

Definition 5.7. *Assume $1 < \nu < 2$, $0 < \mu < 1$, and $1 < \mu + \nu < 2$. We define the Cauchy function $x(t, \rho(s))$ for the homogeneous fractional equation*

$$\nabla_{a+1}^{\nu} \nabla_a^{\mu} y(t) = 0$$

to be the function $x : \mathbb{N}_{a+1} \times \mathbb{N}_{a+1} \to \mathbb{R}$ such that for each fixed $s \in \mathbb{N}_{a+1}$, $x(\cdot, \rho(s))$ is the unique solution of the fractional initial value problem

$$\begin{cases} \nabla_s^{\nu} \nabla_{\rho(s)}^{\mu} x(t) &= 0, \quad t \in \mathbb{N}_{s+2} \\ x(\rho(s)) = 0, \quad \nabla x(s) &= 1 \end{cases}$$

and is given by the formula

$$x(t, \rho(s)) = \sum_{\tau=s}^{t} \frac{(t-\rho(\tau))^{\overline{\mu-1}} (\tau - \rho(s))^{\overline{\nu-1}}}{\Gamma(\mu)\Gamma(\nu)} = H_{\mu+\nu-1}(t, \rho(s)), \quad t \in \mathbb{N}_{a+1}. \tag{5.9}$$

Note that by convention $x(t, \rho(s)) = 0$ for $t \leq \rho(s)$.

We now explore a sequential nabla difference initial value problem with a nonhomogeneous difference equation with zero initial conditions. The solution is given in terms of the Cauchy function.

Theorem 5.8. *Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $1 < \nu < 2$, $0 < \mu < 1$, and $1 < \mu + \nu < 2$. The solution to the fractional initial value problem*

$$\begin{cases} -\nabla_{a+1}^{\nu} \nabla_a^{\mu} y(t) &= f(t), \quad t \in \mathbb{N}_{a+3} \\ \nabla y(a+2) &= 0 \\ y(a+3) = \nabla y(a+3) &= 0 \end{cases} \tag{5.10}$$

is given by

$$y(t) = -\sum_{s=a+4}^{t} f(s)x(t, \rho(s))$$

where $x(t, \rho(s))$ is the Cauchy function (5.9).

Proof. Let $y(t)$ be the solution of the fractional initial value problem (5.10) and let $h(t) = \nabla_a^{\mu} y(t)$ where $0 < \mu < 1$. Then $h(t)$ is a solution of the initial value problem

$$\begin{cases} -\nabla_{a+1}^{\nu} h(t) &= f(t), \quad t \in \mathbb{N}_{a+3} \\ h(a+3) = \nabla_a^{\mu} y(a+3) &= 0 \\ \nabla h(a+3) = \nabla \nabla_a^{\mu} y(a+3) &= 0 \end{cases} \tag{5.11}$$

The initial conditions $y(a+3) = \nabla y(a+3) = 0$ imply

$$0 = \nabla y(a+3) = y(a+3) - y(a+2) = 0 - y(a+2),$$

which implies $y(a+2) = 0$. Since $\nabla y(a+2) = 0$,

$$0 = \nabla y(a+2) = y(a+2) - y(a+1) = 0 - y(a+1),$$

we have that $y(a+1) = 0$. In turn this implies

$$\begin{aligned} h(a+3) &= \nabla_a^{\mu} y(a+3) \\ &= \sum_{s=a+1}^{a+3} H_{-\mu-1}(a+3, \rho(s)) y(s) \\ &= H_{-\mu-1}(a+3, a) y(a+1) + H_{-\mu-1}(a+3, a+1) y(a+2) + y(a+3) \\ &= H_{-\mu-1}(a+3, a)(0) + H_{-\mu-1}(a+3, a+1)(0) + 0 = 0. \end{aligned}$$

Lastly, this implies

$$\begin{aligned}
\nabla h(a+3) = \nabla\nabla_a^{\mu} y(a+3) &= \nabla_a^{\mu+1} y(a+3) \\
&= \sum_{s=a+1}^{a+3} H_{-\mu-2}(a+3,\rho(s))y(s) \\
&= H_{-\mu-2}(a+3,a)y(a+1) + H_{-\mu-2}(a+3,a+1)y(a+2) \\
&\qquad + H_{-\mu-2}(a+3,a+2)y(a+3) \\
&= 0+0+0 = 0.
\end{aligned}$$

We also note the following equalities:

$$\begin{aligned}
H_{\nu-1}(t,a) &= \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} \\
&= \frac{(t-\rho(a+1))^{\overline{\nu-1}}}{\Gamma(\nu)},
\end{aligned}$$

$$\begin{aligned}
H_{\nu-1}(t,a) - H_{\nu-2}(t,a) &= \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} - \frac{(t-a)^{\overline{\nu-2}}}{\Gamma(\nu-1)} \\
&= \frac{\Gamma(t-a+\nu-1)}{\Gamma(t-a)\Gamma(\nu)} - \frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a)\Gamma(\nu-1)} \\
&= \frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a)\Gamma(\nu-1)}\left[\frac{(t-a+\nu-2)}{(\nu-1)} - 1\right] \\
&= \frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a)\Gamma(\nu-1)}\left[\frac{(t-a+\nu-2)-(\nu-1)}{(\nu-1)}\right] \\
&= \frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a)\Gamma(\nu-1)}(t-a-1) \\
&= \frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a-1)\Gamma(\nu)} \\
&= \frac{(t-a-1)^{\overline{\nu-1}}}{\Gamma(\nu)} \\
&= \frac{(t-\rho(a+2))^{\overline{\nu-1}}}{\Gamma(\nu)},
\end{aligned}$$

and

$$
\begin{aligned}
&H_{\nu-1}(t,a) - 2H_{\nu-2}(t,a) - H_{\nu-3}(t,a)\\
&\quad= \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)} - 2\frac{(t-a)^{\overline{\nu-2}}}{\Gamma(\nu-1)} - \frac{(t-a)^{\overline{\nu-3}}}{\Gamma(\nu-2)}\\
&\quad= \frac{\Gamma(t-a+\nu-1)}{\Gamma(t-a)\Gamma(\nu)} - 2\frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a)\Gamma(\nu-1)} + \frac{\Gamma(t-a+\nu-3)}{\Gamma(t-a)\Gamma(\nu-2)}\\
&\quad= \frac{\Gamma(t-a+\nu-3)}{\Gamma(t-a)\Gamma(\nu-2)}\left[\frac{(t-a+\nu-3)(t-a+\nu-2)}{(\nu-2)(\nu-1)} - 2\frac{(t-a+\nu-3)}{(\nu-2)} + 1\right]\\
&\quad= \frac{\Gamma(t-a+\nu-3)}{\Gamma(t-a)\Gamma(\nu-2)}\left[\frac{(t-a+\nu-3)}{(\nu-2)}\left[\frac{(t-a+\nu-2)}{(\nu-1)} - 2\frac{(\nu-1)}{(\nu-1)}\right] + 1\right]\\
&\quad= \frac{\Gamma(t-a+\nu-3)}{\Gamma(t-a)\Gamma(\nu-2)}\left[\frac{(t-a+\nu-3)}{(\nu-2)}\cdot\frac{(t-a-\nu)}{(\nu-1)} + 1\right]\\
&\quad= \frac{\Gamma(t-a+\nu-3)}{\Gamma(t-a)\Gamma(\nu-2)}\left[\frac{(t-a+\nu-3)(t-a-\nu)+(\nu-1)(\nu-2)}{(\nu-1)(\nu-2)}\right]\\
&\quad= \frac{\Gamma(t-a+\nu-3)}{\Gamma(t-a)\Gamma(\nu-2)}\left[\frac{(t-a-1)(t-a-2)}{(\nu-2)(\nu-1)}\right]\\
&\quad= \frac{\Gamma(t-a+\nu-3)}{\Gamma(t-a-2)\Gamma(\nu)}\\
&\quad= \frac{(t-a-2)^{\overline{\nu-1}}}{\Gamma(\nu)}\\
&\quad= \frac{(t-\rho(a+3))^{\overline{\nu-1}}}{\Gamma(\nu)}.
\end{aligned}
$$

By Theorem 5.6, the solution of the initial value problem (5.11) is given by

$$
\begin{aligned}
h(t) &= -\nabla_{a+1}^{-\nu}f(t) + [f(a+2)+f(a+3)]\,H_{\nu-1}(t,a+1) - f(a+3)H_{\nu-2}(t,a+1)\\
&= -\sum_{s=a+2}^{t} H_{\nu-1}(t,\rho(s))f(s) + [f(a+2)+f(a+3)]\,H_{\nu-1}(t,a+1)\\
&\qquad - f(a+3)H_{\nu-2}(t,a+1),
\end{aligned}
$$

so that

$$\begin{aligned}
h(t) &= -\sum_{s=a+2}^{t} H_{\nu-1}(t,\rho(s))f(s) + f(a+2)H_{\nu-1}(t,a+1) \\
&\quad + f(a+3)\left[H_{\nu-1}(t,a+1) - H_{\nu-2}(t,a+1)\right] \\
&= -\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) + f(a+3)\left[H_{\nu-1}(t,a+1) - H_{\nu-2}(t,a+1)\right] \\
&= -\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) + f(a+3)\left[\frac{\Gamma(\nu+t-a-2)}{\Gamma(\nu)\Gamma(t-a-1)} - \frac{\Gamma(\nu+t-a-3)}{\Gamma(\nu-1)\Gamma(t-a-1)}\right] \\
&= -\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) \\
&\quad + f(a+3)\left(\frac{\Gamma(\nu+t-a-3)}{\Gamma(\nu-1)\Gamma(t-a-1)}\right)\left[\frac{\Gamma(\nu+t-a-2)}{\Gamma(\nu)\Gamma(t-a-1)} - 1\right] \\
&= -\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) \\
&\quad + f(a+3)\left(\frac{\Gamma(\nu+t-a-3)}{\Gamma(\nu-1)\Gamma(t-a-1)}\right)\left[\frac{\nu+t-a-3}{\nu-1} - 1\right] \\
&= -\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) \\
&\quad + f(a+3)\left(\frac{\Gamma(\nu+t-a-3)}{\Gamma(\nu-1)\Gamma(t-a-1)}\right)\left[\frac{\nu+t-a-3-(\nu-1)}{\nu-1}\right] \\
&= -\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) + f(a+3)\left(\frac{\Gamma(\nu+t-a-3)}{\Gamma(\nu-1)\Gamma(t-a-1)}\right)\left[\frac{t-a-2}{\nu-1}\right] \\
&= -\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) + f(a+3)\left(\frac{\Gamma(\nu+t-a-3)(t-a-2)}{(\nu-1)\Gamma(\nu-1)(t-a-2)\Gamma(t-a-2)}\right) \\
&= -\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) + f(a+3)\left(\frac{\Gamma(\nu+t-a-3)}{\Gamma(\nu)\Gamma(t-a-2)}\right) \\
&= -\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) + f(a+3)H_{\nu-1}(t,a+2) \\
&= -\sum_{s=a+4}^{t} H_{\nu-1}(t,\rho(s))f(s) \\
&= -\nabla_{a+3}^{-\nu} f(t).
\end{aligned}$$

Composing each side of

$$\nabla_a^{\mu} y(t) = h(t) = -\sum_{s=a+4}^{t} H_{\nu-1}(t,\rho(s))f(s) = -\nabla_{a+3}^{-\nu} f(t)$$

with $\nabla_a^{-\mu}$ gives, by Theorem 2.4,

$$\begin{aligned} y(t) &= -\nabla_a^{-\mu}\nabla_{a+3}^{-\nu} f(t) \\ &= -\nabla_{a+3}^{-(\mu+\nu)} f(t) \\ &= -\sum_{s=a+4}^{t} H_{\mu+\nu-1}(t,\rho(s))f(s) \\ &= -\sum_{s=a+4}^{t} f(s)x(t,\rho(s)). \end{aligned}$$

Thus our solution to the initial value problem (5.10) is given in terms of the Cauchy function, and this completes the proof.

$\square$

Based on the solution we found to the initial value problem (5.10), we have a simple comparison theorem that follows.

Corollary 5.9. *Assume* $1 < \nu < 2$, $0 < \mu < 1$, *and* $1 < \mu + \nu < 2$ *and* $u(t), v(t)$ *satisfy*

$$\begin{aligned} \nabla_{a+1}^{\nu}\nabla_a^{\mu} u(t) &\geq \nabla_{a+1}^{\nu}\nabla_a^{\mu} v(t), \quad t \in \mathbb{N}_{a+3}, \\ u(a+1) &= v(a+1), \\ u(a+2) &= v(a+2). \end{aligned}$$

Then $u(t) \geq v(t)$ *on* $\mathbb{N}_{a+1}$.

Proof. Set $w(t) = u(t) - v(t)$ and for $t \in \mathbb{N}_{a+3}$ let

$$h(t) = -\nabla^{\nu}_{a+1}\nabla^{\mu}_{a}w(t) = -\left(\nabla^{\nu}_{a+1}\nabla^{\mu}_{a}u(t) - \nabla^{\nu}_{a+1}\nabla^{\mu}_{a}v(t)\right)$$
$$= \nabla^{\nu}_{a+1}\nabla^{\mu}_{a}v(t) - \nabla^{\nu}_{a+1}\nabla^{\mu}_{a}u(t) \leq 0.$$

Hence w solves the initial value problem

$$\begin{cases} -\nabla^{\nu}_{a+1}\nabla^{\mu}_{a}w(t) & = h(t) \\ \nabla w(a+2) & = 0 \\ w(a+3) = \nabla w(a+3) & = 0 \end{cases}$$

and Theorem 5.8 gives for $t \in \mathbb{N}_{a+1}$

$$w(t) = -\sum_{s=a+4}^{t} h(s)x(t,\rho(s)) = -\sum_{s=a+4}^{t} h(s)\sum_{\tau=s}^{t} H_{\mu-1}(t,\rho(\tau))H_{\nu-1}(\tau,\rho(s)) \geq 0,$$

since $h(t) \leq 0$ by assumption, $t \geq \tau$, and $\tau \geq s$. Therefore $u(t) \geq v(t)$ for $t \in \mathbb{N}_{a+1}$. This concludes our proof.

□

We now present our variation of constants formula for the initial value problem with the nonhomogeneous nabla difference equation based at $a+1$ and nonzero initial conditions.

Theorem 5.10. *Let $1 < \nu < 2$ and $y, h : \mathbb{N}_{a+1} \to \mathbb{R}$. Then for $t \in \mathbb{N}_{a+1}$ the initial value problem*

$$\begin{cases} -\nabla^{\nu}_{a+1}y(t) & = h(t), \quad t \in \mathbb{N}_{a+3} \\ y(a+2) & = A \\ y(a+3) & = B \end{cases}$$

has the unique solution

$$y(t) = -\nabla_{a+1}^{-\nu} h(t) + [A(1-\nu) + B + h(a+2) + h(a+3)]\, H_{\nu-1}(t, a+1) \\ + [A\nu - B - h(a+3)]\, H_{\nu-2}(t, a+1).$$

Proof. First we attend to the initial conditions:

$$\begin{aligned} y(a+2) &= -\nabla_{a+1}^{-\nu} h(a+2) + [A(1-\nu) + B + h(a+2) + h(a+3)]\, H_{\nu-1}(a+2, a+1) \\ &\quad + [A\nu - B - h(a+3)]\, H_{\nu-2}(a+2, a+1) \\ &= -\sum_{s=a+2}^{a+2} H_{\nu-1}(a+2, \rho(s))h(s) + A(1-\nu) + B + h(a+2) + h(a+3) \\ &\quad + A\nu - B - h(a+3) \\ &= -h(a+2) + A - A\nu + B + h(a+2) + h(a+3) + A\nu - B - h(a+3) \\ &= A, \end{aligned}$$

and

$$\begin{aligned} y(a+3) &= -\nabla_{a+1}^{-\nu} h(a+3) + [A(1-\nu) + B + h(a+2) + h(a+3)]\, H_{\nu-1}(a+3, a+1) \\ &\quad + [A\nu - B - h(a+3)]\, H_{\nu-2}(a+3, a+1) \\ &= -\sum_{s=a+2}^{a+3} H_{\nu-1}(a+3, \rho(s))h(s) + [A(1-\nu) + B + h(a+2) + h(a+3)]\,(\nu) \\ &\quad + [A\nu - B - h(a+3)]\,(\nu-1) \\ &= -\left[H_{\nu-1}(a+3, a+1)h(a+2) + H_{\nu-1}(a+3, a+2)h(a+3)\right] \\ &\quad + [A(1-\nu) + B + h(a+2) + h(a+3)]\,(\nu) \\ &\quad + [A\nu - B - h(a+3)]\,(\nu-1), \end{aligned}$$

so that

$$\begin{aligned}
y(a+3) &= -\left[\nu h(a+2) + h(a+3)\right] + A(\nu - \nu^2) + B\nu + \nu h(a+2) + \nu h(a+3) \\
&\qquad + A(\nu^2 - \nu) - B(\nu - 1) - h(a+3)(\nu - 1) \\
&= (-\nu + \nu)\, h(a+2) + (1 + \nu - \nu + 1)\, h(a+3) + \left(\nu - \nu^2 + \nu^2 - \nu\right) A \\
&\qquad + (\nu - \nu + 1)\, B \\
&= B,
\end{aligned}$$

where we used the facts

$$H_{\nu-1}(a+3, a+1) = \frac{((a+3)-(a+1))^{\overline{\nu-1}}}{\Gamma(\nu)} = \frac{2^{\overline{\nu-1}}}{\Gamma(\nu)} = \frac{\Gamma(\nu+1)}{\Gamma(\nu)} = \nu$$

and

$$H_{\nu-2}(a+3, a+1) = \frac{((a+3)-(a+1))^{\overline{\nu-2}}}{\Gamma(\nu-1)} = \frac{2^{\overline{\nu-2}}}{\Gamma(\nu-1)} = \frac{\Gamma(\nu)}{\Gamma(\nu-1)} = \nu - 1.$$

Now we find the form of the solution $y(t)$ by taking the Laplace transform based at $a+3$ of both sides:

$$\begin{aligned}
-\mathcal{L}_{a+3}\{\nabla^{\nu}_{a+1} y\}(s) &= \mathcal{L}_{a+3}\{h\}(s) \\
\mathcal{L}_{a+3}\{\nabla^{\nu}_{a+1} y\}(s) &= -\mathcal{L}_{a+3}\{h\}(s). \qquad (5.12)
\end{aligned}$$

First considering the left hand side of (5.12), we use the Shifting Lemma 1.31, replacing a with $a+1$ and letting $n = 2$ to obtain a form that is in terms of the Laplace transform based at $a+1$ in order to match the base of the operator:

$$
\begin{aligned}
\mathcal{L}_{a+3}\{\nabla^{\nu}_{a+1}y\}(s) &= \left(\frac{1}{1-s}\right)^2 \mathcal{L}_{a+1}\{\nabla^{\nu}_{a+1}y\}(s) - \sum_{k=1}^{2} \frac{\nabla^{\nu}_{a+1}y(a+1+k)}{(1-s)^{2-k+1}} \\
&= \left(\frac{1}{1-s}\right)^2 \mathcal{L}_{a+1}\{\nabla^{\nu}_{a+1}y\}(s) - \frac{\nabla^{\nu}_{a+1}y(a+2)}{(1-s)^2} - \frac{\nabla^{\nu}_{a+1}y(a+3)}{(1-s)} \\
&= \left(\frac{1}{1-s}\right)^2 \mathcal{L}_{a+1}\{\nabla^{\nu}_{a+1}y\}(s) - \frac{y(a+2)}{(1-s)^2} - \frac{\nabla^{\nu}_{a+1}y(a+3)}{(1-s)} \\
&= \frac{s^{\nu}}{(1-s)^2}\mathcal{L}_{a+1}\{y\}(s) - \frac{y(a+2)}{(1-s)^2} - \frac{\nabla^{\nu}_{a+1}y(a+3)}{(1-s)},
\end{aligned}
$$

where we used Theorem 1.32 and the fact that

$$
\nabla^{\nu}_{a+1}y(a+2) = \sum_{s=a+2}^{a+2} H_{-\nu-1}(a+2,\rho(s))y(s) = y(a+2).
$$

Similarly, with the right hand side of (5.12), we will also make a shift so that our equation is in terms of the Laplace transform based at $a+1$:

$$
\begin{aligned}
-\mathcal{L}_{a+3}\{h\}(s) &= -\left(\frac{1}{1-s}\right)^2 \mathcal{L}_{a+1}\{h\}(s) + \sum_{k=1}^{2} \frac{h(a+1+k)}{(1-s)^{2-k+1}} \\
&= -\left(\frac{1}{1-s}\right)^2 \mathcal{L}_{a+1}\{h\}(s) + \frac{h(a+2)}{(1-s)^2} + \frac{h(a+3)}{(1-s)}.
\end{aligned}
$$

Thus when we multiply through by $(1-s)^2$, equation (5.12) becomes

$$
\begin{aligned}
s^{\nu}\mathcal{L}_{a+1}\{y\}(s) &= -\mathcal{L}_{a+1}\{h\}(s) + h(a+2) + (1-s)h(a+3) + y(a+2) \\
&\quad + \nabla^{\nu}_{a+1}y(a+3)(1-s) \\
&= -\mathcal{L}_{a+1}\{h\}(s) + h(a+2) + (1-s)h(a+3) + y(a+2) \\
&\quad + \left[y(a+3) - \nu y(a+2)\right](1-s),
\end{aligned}
$$

where we used the fact that

$$\begin{aligned}
\nabla_{a+1}^{\nu} y(a+3) &= \sum_{s=a+2}^{a+3} H_{-\nu-1}(a+3,\rho(s))y(s) \\
&= H_{-\nu-1}(a+3,a+1)y(a+2) + H_{-\nu-1}(a+3,a+2)y(a+3) \\
&= \frac{2^{\overline{-\nu-1}}}{\Gamma(-\nu)} y(a+2) + y(a+3) \\
&= \frac{\Gamma(-\nu+1)}{\Gamma(2)\Gamma(-\nu)} y(a+2) + y(a+3) \\
&= y(a+3) - \nu y(a+2).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{L}_{a+1}\{y\}(s) = &-\frac{1}{s^{\nu}}\mathcal{L}_{a+1}\{h\}(s) + \frac{1}{s^{\nu}}h(a+2) + \frac{1}{s^{\nu}}h(a+3) - \frac{1}{s^{\nu-1}}h(a+3) \\
&+ \frac{1}{s^{\nu}}y(a+2) + \frac{1}{s^{\nu}}y(a+3) - \frac{1}{s^{\nu-1}}y(a+3) - \frac{\nu}{s^{\nu}}y(a+2) \\
&+ \frac{\nu}{s^{\nu-1}}y(a+2) \\
= &-\frac{1}{s^{\nu}}\mathcal{L}_{a+1}\{h\}(s) \\
&+ \left[h(a+2) + h(a+3) + y(a+2) + y(a+3) - \nu y(a+2)\right]\frac{1}{s^{\nu}} \\
&+ \left[-h(a+3) - y(a+3) + \nu y(a+2)\right]\frac{1}{s^{\nu-1}} \\
= &-\frac{1}{s^{\nu}}\mathcal{L}_{a+1}\{h\}(s) + \left[h(a+2) + h(a+3) + A + B - A\nu\right]\frac{1}{s^{\nu}} \\
&+ \left[-h(a+3) - B + A\nu\right]\frac{1}{s^{\nu-1}} \\
= &-\frac{1}{s^{\nu}}\mathcal{L}_{a+1}\{h\}(s) + \left[A(1-\nu) + B + h(a+2) + h(a+3)\right]\frac{1}{s^{\nu}} \\
&+ \left[A\nu - B - h(a+3)\right]\frac{1}{s^{\nu-1}}
\end{aligned}$$

which implies

$$y(t) = -\nabla_{a+1}^{-\nu} h(t) + [A(1-\nu) + B + h(a+2) + h(a+3)]\, H_{\nu-1}(t, a+1)$$
$$+ [A\nu - B - h(a+3)]\, H_{\nu-2}(t, a+1).$$

We end by noting that this solution satisfies the fractional difference equation due to the convention that $H_{-k}(t,a) = 0$ for $k \in \mathbb{N}$ as follows.

$$\begin{aligned}
-\nabla_{a+1}^{\nu} y(t) = -\nabla_{a+1}^{\nu} \Big[& - \nabla_{a+1}^{-\nu} h(t) \\
& + [A(1-\nu) + B + h(a+2) + h(a+3)]\, H_{\nu-1}(t, a+1) \\
& + [A\nu - B - h(a+3)]\, H_{\nu-2}(t, a+1) \Big] \\
= \nabla_{a+1}^{\nu} & \nabla_{a+1}^{-\nu} h(t) \\
& - [A(1-\nu) + B + h(a+2) + h(a+3)]\, \nabla_{a+1}^{\nu} H_{\nu-1}(t, a+1) \\
& - [A\nu - B - h(a+3)]\, \nabla_{a+1}^{\nu} H_{\nu-2}(t, a+1) \\
= h(t) & - [A(1-\nu) + B + h(a+2) + h(a+3)]\, H_{-1}(t, a+1) \\
& - [A\nu - B - h(a+3)]\, H_{-2}(t, a+1) \\
= h(t)&,
\end{aligned}$$

where we also used Lemma 2.2 (ii) and Theorem 1.24. We have therefore concluded our proof.

□

5.3.3 Green's Function for $\nabla_{a+1}^{\nu}\nabla_a^{\mu}x(t)$ Boundary Value Problem

In this section we will consider the nonhomogeneous nabla sequential difference boundary value problem

$$\begin{cases} -\nabla_{a+1}^{\nu}\nabla_a^{\mu}y(t) &= h(t), \quad t \in \mathbb{N}_{a+3}^{b} \\ y(a+2) &= 0, \\ y(b) &= 0, \end{cases} \tag{5.13}$$

and the corresponding homogeneous boundary value problem

$$\begin{cases} -\nabla_{a+1}^{\nu}\nabla_a^{\mu}y(t) &= 0, \quad t \in \mathbb{N}_{a+3}^{b} \\ y(a+2) &= 0, \\ y(b) &= 0, \end{cases} \tag{5.14}$$

for the particular set of order conditions $1 < \nu < 2$, $0 < \mu < 1$, and $1 < \mu + \nu < 2$, and $a, b \in \mathbb{R}$ with $b - (a+2) \in \mathbb{N}_1$.

Theorem 5.11. *Let $1 < \nu < 2$, $0 < \mu < 1$, and $1 < \mu + \nu < 2$, and further let $y, h : \mathbb{N}_{a+1} \to \mathbb{R}$. The fractional boundary value problem (5.13) where $a, b \in \mathbb{R}$ with $b - (a+2) \in \mathbb{N}_1$, has the unique solution*

$$y(t) = \sum_{s=a+3}^{t} h(s)G(t,s) = \int_{a+2}^{t} h(s)G(t,s)\nabla s$$

where

$$G(t,s) = \begin{cases} \frac{x(b,\rho(s))}{x(\rho(b),a+1)} x(\rho(t), a+1), & t \le s-1 \\ \frac{x(b,\rho(s))}{x(\rho(b),a+1)} x(\rho(t), a+1) - x(t, \rho(s)), & t \ge s \end{cases}$$

and $x(t, \rho(s))$ is the Cauchy function (5.9).

Proof. Let $x(t) = \nabla_a^{\mu} y(t)$, along with the conditions $A = x(a+2) = \nabla_a^{\mu} y(a+2)$ and $B = x(a+3) = \nabla_a^{\mu} y(a+3)$. Then $x(t)$ solves the initial value problem

$$\begin{cases} -\nabla_{a+1}^{\nu} x(t) &= h(t) \\ x(a+2) &= A \\ x(a+3) &= B. \end{cases}$$

Thus by the variation of constants formula in Theorem 5.10,

$$\begin{aligned} x(t) = -\nabla_{a+1}^{-\nu} h(t) &+ \left[A(1-\nu) + B + h(a+2) + h(a+3)\right] H_{\nu-1}(t, a+1) \\ &+ \left[A\nu - B - h(a+3)\right] H_{\nu-2}(t, a+1). \end{aligned}$$

We next compose both sides with the operator $\nabla_a^{-\mu}$ as follows. Since $\mu \notin \mathbb{N}$,

$$\begin{aligned} \nabla_a^{-\mu} x(t) &= -\nabla_a^{-\mu} \nabla_{a+1}^{-\nu} h(t) \\ &\quad + \left[A(1-\nu) + B + h(a+2) + h(a+3)\right] \nabla_a^{-\mu} H_{\nu-1}(t, a+1) \\ &\quad + \left[A\nu - B - h(a+3)\right] \nabla_a^{-\mu} H_{\nu-2}(t, a+1) \\ \nabla_a^{-\mu} \nabla_a^{\mu} y(t) &= -\nabla_{a+1}^{-(\mu+\nu)} h(t) + \left[A(1-\nu) + B + h(a+2) + h(a+3)\right] H_{\mu+\nu-1}(t, a+1) \\ &\quad + \left[A\nu - B - h(a+3)\right] H_{\mu+\nu-2}(t, a+1) \\ y(t) &= -\sum_{s=a+2}^{t} H_{\mu+\nu-1}(t, \rho(s)) h(s) \\ &\quad + \left[A(1-\nu) + B + h(a+2) + h(a+3)\right] H_{\mu+\nu-1}(t, a+1) \\ &\quad + \left[A\nu - B - h(a+3)\right] H_{\mu+\nu-2}(t, a+1), \end{aligned}$$

and hence

$$
\begin{aligned}
y(t) = &-\sum_{s=a+2}^{t} x(t,\rho(s))h(s) \\
&+ \left[A(1-\nu) + B + h(a+2) + h(a+3)\right] x(t,a+1) \\
&+ \left[A\nu - B - h(a+3)\right] \nabla x(t,a+1) \\
= &-\sum_{s=a+2}^{t} x(t,\rho(s))h(s) + h(a+2)x(t,a+1) \\
&+ \left[A - A\nu + B + h(a+3)\right] x(t,a+1) \\
&+ \left[A\nu - B - h(a+3)\right] x(t,a+1) \\
&- \left[A\nu - B - h(a+3)\right] x(t-1,a+1).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y(t) = &-\sum_{s=a+3}^{t} x(t,\rho(s))h(s) \\
&+ \left[A - A\nu + B + h(a+3) + A\nu - B - h(a+3)\right] x(t,a+1) \\
&- \left[A\nu - B - h(a+3)\right] x(t-1,a+1) \\
y(t) = &-\sum_{s=a+3}^{t} x(t,\rho(s))h(s) + Ax(t,a+1) \\
&- \left[A\nu - B - h(a+3)\right] x(\rho(t),a+1),
\end{aligned}
$$

where we used Theorem 2.4 and Remark 1.23 in the first step, and in the third to last step noticed that the second term is equivalent to the $s = a + 2$ term of the sum.

Letting $t = a + 2$ gives

$$\begin{aligned} 0 = y(a+2) &= -\sum_{s=a+3}^{a+2} x(a+2, \rho(s))h(s) + Ax(a+2, a+1) \\ &\quad - [A\nu - B - h(a+3)]\, x(a+1, a+1) \\ &= 0 + A - 0 = A. \end{aligned}$$

Thus we see that $A = 0$ and our solution becomes

$$y(t) = -\sum_{s=a+3}^{t} x(t, \rho(s))h(s) - [-B - h(a+3)]\, x(\rho(t), a+1).$$

Now let $C := -B - h(a+3)$. Letting $t = b$ gives

$$0 = y(b) = -\sum_{s=a+3}^{b} x(b, \rho(s))h(s) - Cx(\rho(b), a+1),$$

and solving for C gives

$$C = \frac{-\sum_{s=a+3}^{b} x(b, \rho(s))h(s)}{x(\rho(b), a+1)}.$$

Substituting this value for C into the formula for $y(t)$ gives us that

$$\begin{aligned} y(t) &= -\sum_{s=a+3}^{t} x(t, \rho(s))h(s) + \frac{x(\rho(t), a+1)}{x(\rho(b), a+1)} \sum_{s=a+3}^{b} x(b, \rho(s))h(s) \\ &= -\sum_{s=a+3}^{t} x(t, \rho(s))h(s) + \frac{x(\rho(t), a+1)}{x(\rho(b), a+1)} \sum_{s=a+3}^{t} x(b, \rho(s))h(s) \\ &\quad + \frac{x(\rho(t), a+1)}{x(\rho(b), a+1)} \sum_{s=t+1}^{b} x(b, \rho(s))h(s). \end{aligned}$$

Hence

$$\begin{aligned} y(t) = \sum_{s=a+3}^{t} h(s)\left[\frac{x(b,\rho(s))}{x(\rho(b),a+1)}x(\rho(t),a+1) - x(t,\rho(s))\right] \\ + \sum_{t=1}^{b} h(s)\left[\frac{x(b,\rho(s))}{x(\rho(b),a+1)}x(\rho(t),a+1)\right] \\ = \sum_{s=a+3}^{b} h(s)G(t,s), \end{aligned}$$

where

$$G(t,s) = \begin{cases} \frac{x(b,\rho(s))}{x(\rho(b),a+1)}x(\rho(t),a+1), & t \leq s-1 \\ \frac{x(b,\rho(s))}{x(\rho(b),a+1)}x(\rho(t),a+1) - x(t,\rho(s)), & t \geq s. \end{cases}$$

Therefore any solution to the boundary value problem (5.13) is given by the formula derived. Uniqueness of the solution $y(t)$ follows from Theorem 5.5.

□

5.3.4 Properties of the Green's Function

In this section we present some nice properties of the Green's function derived in the last section, namely the positivity of the Green's function and its maximum value.

Theorem 5.12. *The Green's function $G(t,s)$ for the boundary value problem (5.14) satisfies $G(t,s) \geq 0$ for $(t,s) \in \mathbb{N}_{a+2}^{b} \times \mathbb{N}_{a+3}^{b}$, and, specifically, $G(t,s) > 0$ for $(t,s) \in \mathbb{N}_{a+3}^{b-1} \times \mathbb{N}_{a+4}^{b-1}$.*

Proof. We will show for any fixed s that $G(t,s)$ increases from $G(a+2,s) = 0$ to a positive value at $t = s-1$ and then decreases to $G(b,s) = 0$. Let $s \in \mathbb{N}_{a+4}^{b}$ be fixed but arbitrary. First, we show that $G(a+2,s) = G(b,s) = 0$.

$$
\begin{aligned}
G(a+2,s) &= \frac{x(b,\rho(s))}{x(\rho(b),a+1)}x(\rho(a+2),a+1)\\
&= \frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(\rho(b),a+1)}H_{\mu+\nu-1}(a+1,a+1) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
G(b,s) &= \frac{x(b,\rho(s))}{x(\rho(b),a+1)}x(\rho(b),a+1) - x(b,\rho(s))\\
&= x(b,\rho(s)) - x(b,\rho(s))\\
&= 0.
\end{aligned}
$$

Now we will show that for each fixed s, the Green's function $G(t,s)$ increases with respect to t for values of t between $a+3$ and $s-1$. To do so, we consider the nabla difference with respect to t. For $t \geq a+3$,

$$
\begin{aligned}
\nabla_t G(t,s) &= \nabla_t \left[\frac{x(b,\rho(s))}{x(\rho(b),a+1)}x(\rho(t),a+1)\right]\\
&= \frac{x(b,\rho(s))}{x(\rho(b),a+1)}\nabla_t x(\rho(t),a+1)\\
&= \frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(\rho(b),a+1)}\nabla_t H_{\mu+\nu-1}(\rho(t),a+1)\\
&= \frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(\rho(b),a+1)}H_{\mu+\nu-2}(\rho(t),a+1)\\
&= \frac{\Gamma(\mu+\nu+b-s)\Gamma(\mu+\nu)\Gamma(b-a-2)\Gamma(\mu+\nu+t-a-4)}{\Gamma(\mu+\nu)\Gamma(b-s+1)\Gamma(\mu+\nu+b-a-3)\Gamma(\mu+\nu-1)\Gamma(t-a-2)}\\
&= \frac{\Gamma(\mu+\nu+b-s)\Gamma(b-a-2)\Gamma(\mu+\nu+t-a-4)}{\Gamma(b-s+1)\Gamma(\mu+\nu+b-a-3)\Gamma(\mu+\nu-1)\Gamma(t-a-2)} > 0,
\end{aligned}
$$

due to the following: recall by assumption that $1<\nu<2$, $0<\mu<1$, $1<\mu+\nu<2$, and $b-(a+2)\in\mathbb{N}_1$.

1. Since $s \leq b$ and $\mu, \nu > 0$, $\mu + \nu + b - s > 0$.

2. Since $b - (a+2) \in \mathbb{N}_1$, $b - a - 2 > 0$.

3. Since $t \geq a+3$, $t - a - 3 \geq 0$. Further, since $\nu > 1$, $\nu - 1 > 0$. Combined with $\mu > 0$, this gives us $(t-a-3) + (\nu - 1) + \mu = \mu + \nu + t - a - 4 > 0$.

4. Since $s \leq b$, $b - s + 1 > 0$.

5. Since $b - (a+2) \in \mathbb{N}_1$, $b - a - 3 \geq 0$. Combined with $\mu, \nu > 0$, we have $\mu + \nu + b - a - 3 > 0$.

6. Since $\nu > 1$, $\nu - 1 > 0$. Combined with $\mu > 0$, we have $\mu + \nu - 1 > 0$.

7. Since $t \geq a+3$, $t - a - 2 > 0$.

Therefore $G(t,s)$ is increasing for all values of t between $a+3$ and $s-1$. Since the Green's function is zero at $t = a+2$ and increases for t between $a+3$ and $s-1$, we have that $G(t,s) > 0$ for $t \in \mathbb{N}_{a+3}^{s-1}$ and $G(t,s) \geq 0$ for $t \in \mathbb{N}_{a+2}^{s-1}$. We now show that $G(t,s)$ is decreasing for values of t between s and b. Similarly, we consider the nabla difference with respect to t. For $t \geq s$,

$$\begin{aligned}
\nabla_t G(t,s) &= \nabla_t \left[\frac{x(b,\rho(s))}{x(\rho(b),a+1)} x(\rho(t),a+1) - x(t,\rho(s)) \right] \\
&= \nabla_t \left[\frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(\rho(b),a+1)} H_{\mu+\nu-1}(\rho(t),a+1) - H_{\mu+\nu-1}(t,\rho(s)) \right] \\
&= \nabla_t \left[\frac{H_{\mu+\nu-1}(b,s-1)}{H_{\mu+\nu-1}(b-1,a+1)} H_{\mu+\nu-1}(t-1,a+1) - H_{\mu+\nu-1}(t,s-1) \right] \\
&= \frac{H_{\mu+\nu-1}(b,s-1)}{H_{\mu+\nu-1}(b-1,a+1)} H_{\mu+\nu-1}(t-1,a+1) - H_{\mu+\nu-1}(t,s-1) \\
&\quad - \frac{H_{\mu+\nu-1}(b,s-1)}{H_{\mu+\nu-1}(b-1,a+1)} H_{\mu+\nu-1}(t-2,a+1) + H_{\mu+\nu-1}(t-1,s-1)
\end{aligned}$$

Hence

$$\nabla_t G(t,s) = C\Big[(b-s+1)^{\overline{\mu+\nu-1}}(t-a-2)^{\overline{\mu+\nu-1}} - (b-a-2)^{\overline{\mu+\nu-1}}(t-s+1)^{\overline{\mu+\nu-1}} \\ -(b-s+1)^{\overline{\mu+\nu-1}}(t-a-3)^{\overline{\mu+\nu-1}} + (b-a-2)^{\overline{\mu+\nu-1}}(t-s)^{\overline{\mu+\nu-1}}\Big], \tag{5.15}$$

where $C = \frac{1}{\Gamma(\mu+\nu)(b-a-2)^{\overline{\mu+\nu-1}}} > 0$ since $\mu+\nu > 1$ and $b-(a+2) \in \mathbb{N}_1$. We will show that the bracketed expression in (5.15) is less than or equal to zero. Rearranging and factoring, the claim that the bracketed expression is less than or equal to zero is equivalent to

$$(b-s+1)^{\overline{\mu+\nu-1}}\Big[(t-a-2)^{\overline{\mu+\nu-1}} - (t-a-3)^{\overline{\mu+\nu-1}}\Big] \\ \le (b-a-2)^{\overline{\mu+\nu-1}}\Big[(t-s+1)^{\overline{\mu+\nu-1}} - (t-s)^{\overline{\mu+\nu-1}}\Big].$$

Now since $s-1 \ge a+2$, we have $0 < b-(s-1) \le b-(a+2)$, it remains to show

$$(t-a-2)^{\overline{\mu+\nu-1}} - (t-a-3)^{\overline{\mu+\nu-1}} \le (t-s+1)^{\overline{\mu+\nu-1}} - (t-s)^{\overline{\mu+\nu-1}}. \tag{5.16}$$

Noting that

$$\begin{aligned}(t-s+1)^{\overline{\mu+\nu-1}} &= \frac{\Gamma(t-s+1+\mu+\nu-1)}{\Gamma(t-s+1)} \\ &= \frac{(t-s+\mu+\nu-1)\Gamma(t-s+\mu+\nu-1)}{(t-s)\Gamma(t-s)} \\ &= \frac{t-s+\mu+\nu-1}{t-s}(t-s)^{\overline{\mu+\nu-1}}\end{aligned}$$

and

$$
\begin{aligned}
(t-a-2)^{\overline{\mu+\nu-1}} &= \frac{\Gamma(t-a-2+\mu+\nu-1)}{\Gamma(t-a-2)} \\
&= \frac{(t-a-3+\mu+\nu-1)\Gamma(t-a-3+\mu+\nu-1)}{(t-a-3)\Gamma(t-a-3)} \\
&= \frac{t-a-3+\mu+\nu-1}{t-a-3}(t-a-3)^{\overline{\mu+\nu-1}},
\end{aligned}
$$

we have that (5.16) becomes

$$
\begin{aligned}
(t-a-3)^{\overline{\mu+\nu-1}}\left[\frac{t-a-3+\mu+\nu-1}{t-a-3}-1\right] &\leq (t-s)^{\overline{\mu+\nu-1}}\left[\frac{t-s+\mu+\nu-1}{t-s}-1\right] \\
(t-a-3)^{\overline{\mu+\nu-1}}\left[\frac{\mu+\nu-1}{t-a-3}\right] &\leq (t-s)^{\overline{\mu+\nu-1}}\left[\frac{\mu+\nu-1}{t-s}\right].
\end{aligned}
$$

Since $1<\mu+\nu<2$ by assumption, $\mu+\nu-1>0$. Thus it follows that

$$
\frac{(t-a-3)^{\overline{\mu+\nu-1}}}{t-a-3} \leq \frac{(t-s)^{\overline{\mu+\nu-1}}}{t-s}, \tag{5.17}
$$

since $s-1\geq a+2$, $s\geq a+3$. Thus we can show the inequality above holds by showing that the expression $\frac{(t-\tau)^{\overline{\mu+\nu-1}}}{t-\tau}$ is increasing in τ. To do so, we will consider the nabla difference with respect to τ. Using the quotient and power rules, we have

$$
\begin{aligned}
\nabla_\tau \frac{(t-\tau)^{\overline{\mu+\nu-1}}}{t-\tau} &= \frac{-(\mu+\nu-1)(t-\rho(\tau))^{\overline{\mu+\nu-2}}(t-\tau)-(t-\tau)^{\overline{\mu+\nu-1}}(-1)}{(t-\tau)(t-\rho(\tau))} \\
&= \frac{-(\mu+\nu-1)(t-\tau)^{\overline{\mu+\nu-1}}+(t-\tau)^{\overline{\mu+\nu-1}}}{(t-\tau)(t-\tau+1)} \\
&= \frac{(t-\tau)^{\overline{\mu+\nu-1}}(2-\mu-\nu)}{(t-\tau)(t-\tau+1)}, \qquad (5.18)
\end{aligned}
$$

where we used the fact that

$$\begin{aligned}(t-\rho(\tau))^{\overline{\mu+\nu-2}} &= (t-\tau+1)^{\overline{\mu+\nu-2}} \\ &= \frac{\Gamma(\mu+\nu+t-\tau-1)}{\Gamma(t-\tau+1)} \\ &= \frac{\Gamma(\mu+\nu+t-\tau-1)}{(t-\tau)\Gamma(t-\tau)} \\ &= \frac{(t-\tau)^{\overline{\mu+\nu-1}}}{t-\tau}.\end{aligned}$$

Now since $t-\tau \geq 0$ and $2-\mu-\nu > 0$, we have that (5.30) is greater than or equal to zero, and thus the expression $\frac{(t-\tau)^{\overline{\mu+\nu-1}}}{t-\tau}$ is increasing with respect to τ. This in turn implies that inequality (5.17) holds, and therefore the bracketed expression in (5.15) is less than or equal to zero. This tells us that the Green's function $G(t,s)$ is decreasing for values of t between s and b. Since the Green's function is zero at $t=b$ and is decreasing for values of t between s and b, this implies $G(t,s) > 0$ for $t \in \mathbb{N}_s^{b-1}$ and $G(t,s) \geq 0$ for $t \in \mathbb{N}_s^b$.

□

Now using both Theorem 5.11 and Theorem 5.12, we get the following useful result as a corollary.

Corollary 5.13. *Assume that $u(t)$ and $v(t)$ satisfy*

$$\begin{aligned}\nabla_{a+1}^{\nu}\nabla_a^{\mu}u(t) &\geq \nabla_{a+1}^{\nu}\nabla_a^{\mu}v(t), \quad t \in \mathbb{N}_{a+3}^b \\ u(a+2) &= v(a+2) \\ u(b) &= v(b).\end{aligned}$$

Then $u(t) \leq v(t)$ on $\mathbb{N}_{a+2}^b$.

Proof. Set $w(t) = u(t) - v(t)$ and let

$$h(t) := -\nabla^{\nu}_{a+1}\nabla^{\mu}_{a}w(t) = -\nabla^{\nu}_{a+1}\nabla^{\mu}_{a}u(t) + \nabla^{\nu}_{a+1}\nabla^{\mu}_{a}v(t) \leq 0, \quad t \in \mathbb{N}^{b}_{a+3}.$$

Thus it follows that $w(t)$ solves the boundary value problem

$$\begin{cases} -\nabla^{\nu}_{a+1}\nabla^{\mu}_{a}w(t) & = h(t), \quad t \in \mathbb{N}^{b}_{a+3} \\ w(a+2) & = 0 \\ w(b) & = 0. \end{cases}$$

By Theorem 5.11, the solution of this boundary value problem is given by

$$w(t) = \int_{a+2}^{b} G(t,s)h(s)\nabla s, \quad t \in \mathbb{N}^{b}_{a+2},$$

where $G(t,s) \geq 0$ by Theorem 5.12 and $h(s) \leq 0$.

Therefore $w(t) \leq 0$, implying $u(t) \leq v(t)$ for all $t \in \mathbb{N}^{b}_{a+2}$. This completes our proof.

□

We have shown in Theorem 5.12 that for any fixed value of s, $G(t,s) > 0$ for all values of $t \in \mathbb{N}^{b-1}_{a+3}$ and $G(t,s) \geq 0$ for all values of $t \in \mathbb{N}^{b}_{a+2}$. Note that though we have $G(t,s)$ is increasing up to $t = s-1$ and decreasing from $t = s$, it is yet uncertain whether the maximum of the Green's function $G(t,s)$ occurs at $t = s-1$ or $t = s$. We will consider this question in the proof of Theorem 5.14, in which we also find the exact value of the maximum dependent upon whether the quantity $b + a$ is even or odd.

Theorem 5.14. *The maximum of the Green's function $G(t,s)$ defined in Theorem 5.11 is given by*

$$G\left(\left\lfloor\frac{b+a+5}{2}\right\rfloor_a - 1, \left\lfloor\frac{b+a+5}{2}\right\rfloor_a\right)$$
$$= \begin{cases} \frac{1}{\Gamma(\mu+\nu)}(b-a-2)^{\overline{\mu+\nu-1}}\left(1+\frac{2(\mu+\nu-1)}{b-a-3}\right)\left[\left(\frac{b-a-3}{2}\right)^{\overline{\mu+\nu-1}}\right]^2, & \left\lfloor\frac{b+a+5}{2}\right\rfloor_a = \frac{b+a+5}{2} \\ \frac{1}{\Gamma(\mu+\nu)}(b-a-2)^{\overline{\mu+\nu-1}}\left[\left(\frac{b-a-2}{2}\right)^{\overline{\mu+\nu-1}}\right]_a^2, & \left\lfloor\frac{b+a+5}{2}\right\rfloor_a = \frac{b+a}{2}+2, \end{cases}$$

where the floor function $\lfloor\cdot\rfloor_a$ is the largest value in $\mathbb{N}_{a+3}^b$ that is less than or equal to its input.

Proof. We begin by determining whether the maximum of the Green's function, for a fixed t, occurs at $(s-1,s)$ or (s,s). First, via basic calculation, we have

$$\begin{aligned} G(s-1,s) &= \frac{x(b,\rho(s))}{x(\rho(b),a+1)}x(\rho(s-1),a+1) \\ &= \frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(\rho(b),a+1)}H_{\mu+\nu-1}(\rho(s-1),a+1) \\ &= \frac{1}{\Gamma(\mu+\nu)}\left[\frac{(b-s+1)^{\overline{\mu+\nu-1}}}{(b-a-2)^{\overline{\mu+\nu-1}}}(s-a-3)^{\overline{\mu+\nu-1}}\right]. \end{aligned}$$

Notice that the initial value $G(a+3,a+3)$ is equivalent to zero, and from the work in Theorem 5.12, we have seen that for certain values of s and t that the Green's function $G(t,s)$ is strictly increasing from 0, so we can conclude that 0 is not the maximum of the function. Thus we can consider the restriction $s > a+3$ in the following:

$$\begin{aligned} G(s,s) &= \frac{x(b,\rho(s))}{x(\rho(b),a+1)}x(\rho(s),a+1) - x(s,\rho(s)) \\ &= \frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(\rho(b),a+1)}H_{\mu+\nu-1}(\rho(s),a+1) - H_{\mu+\nu-1}(s,\rho(s)). \end{aligned}$$

Hence

$$
\begin{aligned}
G(s,s) &= \frac{1}{\Gamma(\mu+\nu)}\left[\frac{(b-s+1)^{\overline{\mu+\nu-1}}}{(b-a-2)^{\overline{\mu+\nu-1}}}(s-a-2)^{\overline{\mu+\nu-1}} - 1^{\overline{\mu+\nu-1}}\right] \\
&= \frac{1}{\Gamma(\mu+\nu)}\left[\frac{(b-s+1)^{\overline{\mu+\nu-1}}}{(b-a-2)^{\overline{\mu+\nu-1}}}(s-a-3)^{\overline{\mu+\nu-1}}\left(1+\frac{\mu+\nu-1}{s-a-3}\right) - \Gamma(\mu+\nu)\right] \\
&= \frac{1}{\Gamma(\mu+\nu)}\left[\frac{(b-s+1)^{\overline{\mu+\nu-1}}}{(b-a-2)^{\overline{\mu+\nu-1}}}(s-a-3)^{\overline{\mu+\nu-1}}\right] \\
&\quad + \frac{(\mu+\nu-1)(b-s+1)^{\overline{\mu+\nu-1}}(s-a-3)^{\overline{\mu+\nu-1}}}{(\mu+\nu-1)\Gamma(\mu+\nu-1)(b-a-2)^{\overline{\mu+\nu-1}}(s-a-3)} - \frac{\Gamma(\mu+\nu)}{\Gamma(\mu+\nu)} \\
&= \frac{1}{\Gamma(\mu+\nu)}\left[\frac{(b-s+1)^{\overline{\mu+\nu-1}}}{(b-a-2)^{\overline{\mu+\nu-1}}}(s-a-3)^{\overline{\mu+\nu-1}}\right] \\
&\quad + \frac{(b-s+1)^{\overline{\mu+\nu-1}}(s-a-3)^{\overline{\mu+\nu-1}}}{\Gamma(\mu+\nu-1)(b-a-2)^{\overline{\mu+\nu-1}}(s-a-3)} - 1.
\end{aligned}
$$

Therefore

$$
G(s,s) = G(s-1,s) + \frac{(b-s+1)^{\overline{\mu+\nu-1}}(s-a-3)^{\overline{\mu+\nu-1}}}{\Gamma(\mu+\nu-1)(b-a-2)^{\overline{\mu+\nu-1}}(s-a-3)} - 1,
$$

where we used the fact that

$$
\begin{aligned}
(s-a-2)^{\overline{\mu+\nu-1}} &= \frac{\Gamma(s-a-2+\mu+\nu-1)}{\Gamma(s-a-2)} \\
&= \frac{(s-a-3+\mu+\nu-1)\Gamma(s-a-3+\mu+\nu-1)}{(s-a-3)\Gamma(s-a-3)} \\
&= \frac{(s-a-3+\mu+\nu-1)}{(s-a-3)}(s-a-3)^{\overline{\mu+\nu-1}} \\
&= \left(1+\frac{\mu+\nu-1}{s-a-3}\right)(s-a-3)^{\overline{\mu+\nu-1}}.
\end{aligned}
$$

We will now show that the expression

$$\frac{(b-s+1)^{\overline{\mu+\nu-1}}(s-a-3)^{\overline{\mu+\nu-1}}}{\Gamma(\mu+\nu-1)(b-a-2)^{\overline{\mu+\nu-1}}(s-a-3)}-1$$

is negative so that $G(s,s) \leq G(s-1,s)$. Recall that $1 < \mu+\nu < 2$, so that $0 < \mu+\nu-1 < 1$, implying $\frac{1}{\Gamma(\mu+\nu-1)} < 1$. Further, since we assumed $s > a+3$, $s-1 > a+2$, so that $b-(s-1) = b-s+1 < b-a-2 = b-(a+2)$, which implies

$$\frac{(b-s+1)^{\overline{\mu+\nu-1}}}{(b-a-2)^{\overline{\mu+\nu-1}}} < 1.$$

We also have

$$\frac{(s-a-3)^{\overline{\mu+\nu-1}}}{s-a-3} = \frac{\Gamma(\mu+\nu+s-a-4)}{(s-a-3)\Gamma(s-a-3)} = \frac{\Gamma(\mu+\nu+s-a-4)}{\Gamma(s-a-2)} < 1$$

since $\mu+\nu-1 < 1$ implies $(\mu+\nu-1)+(s-a-3) < 1+(s-a-3)$, or more simply $\mu+\nu+s-a-4 < s-a-2$. Thus it follows that

$$\begin{aligned}
\frac{1}{\Gamma(\mu+\nu-1)} \cdot \frac{(b-s+1)^{\overline{\mu+\nu-1}}}{(b-a-2)^{\overline{\mu+\nu-1}}} \cdot \frac{(s-a-3)^{\overline{\mu+\nu-1}}}{s-a-3} &< 1 \\
\frac{(b-s+1)^{\overline{\mu+\nu-1}}(s-a-3)^{\overline{\mu+\nu-1}}}{\Gamma(\mu+\nu-1)(b-a-2)^{\overline{\mu+\nu-1}}(s-a-3)} &< 1 \\
\frac{(b-s+1)^{\overline{\mu+\nu-1}}(s-a-3)^{\overline{\mu+\nu-1}}}{\Gamma(\mu+\nu-1)(b-a-2)^{\overline{\mu+\nu-1}}(s-a-3)} - 1 &< 0.
\end{aligned}$$

Therefore for $s > a+3$, we have $G(s,s) < G(s-1,s)$.

Now we will maximize $G(s-1,s)$ for s values between $a+4$ and b. To do so we consider the nabla difference with respect to s.

This equates to

$$\begin{aligned}
&\nabla_s\left[(b-s+1)^{\overline{\mu+\nu-1}}(s-a-3)^{\overline{\mu+\nu-1}}\right]\\
&= (b-\rho(s)+1)^{\overline{\mu+\nu-1}}(\mu+\nu-1)(s-a-3)^{\overline{\mu+\nu-2}}\\
&\qquad -(\mu+\nu-1)(b-\rho(s)+1)^{\overline{\mu+\nu-2}}(s-a-3)^{\overline{\mu+\nu-1}}\\
&= (\mu+\nu-1)(b-s+\mu+\nu)(b-s+2)^{\overline{\mu+\nu-2}}(s-a-3)^{\overline{\mu+\nu-2}}\\
&\qquad -(\mu+\nu-1)(b-s+2)^{\overline{\mu+\nu-2}}(s-a-3)^{\overline{\mu+\nu-1}}\\
&= (\mu+\nu-1)(b-s+\mu+\nu)(b-s+2)^{\overline{\mu+\nu-2}}(s-a-3)^{\overline{\mu+\nu-2}}\\
&\qquad -(\mu+\nu-1)(b-s+2)^{\overline{\mu+\nu-2}}(\mu+\nu+s-a-5)(s-a-3)^{\overline{\mu+\nu-2}}\\
&= (\mu+\nu-1)(b-s+2)^{\overline{\mu+\nu-2}}(s-a-3)^{\overline{\mu+\nu-2}}\left[b-s+\mu+\nu-(\mu+\nu+s-a-5)\right]\\
&= (\mu+\nu-1)(b-s+2)^{\overline{\mu+\nu-2}}(s-a-3)^{\overline{\mu+\nu-2}}\left[b+a+5-2s\right], \qquad (5.19)
\end{aligned}$$

where we implemented the product rule in the first step, and in the second step we used the fact that

$$\begin{aligned}
(b-\rho(s)+1)^{\overline{\mu+\nu-1}} &= \frac{\Gamma(b-s+2+\mu+\nu-1)}{\Gamma(b-s+2)}\\
&= \frac{(b-s+2+\mu+\nu-2)\Gamma(b-s+2+\mu+\nu-2)}{\Gamma(b-s+2)},
\end{aligned}$$

which gives

$$\begin{aligned}
(b-\rho(s)+1)^{\overline{\mu+\nu-1}} &= \frac{(b-s+\mu+\nu)\Gamma(b-\rho(s)+1+\mu+\nu-2)}{\Gamma(b-s+2)}\\
&= (b-s+\mu+\nu)(b-s+2)^{\overline{\mu+\nu-2}},
\end{aligned}$$

and the fact

$$\begin{aligned}(s-a-3)^{\overline{\mu+\nu-1}} &= \frac{\Gamma(s-a-3+\mu+\nu-1)}{\Gamma(s-a-3)}\\ &= \frac{\Gamma(s-a-2+\mu+\nu-2)}{\Gamma(s-a-3)}\\ &= (s-a-2)^{\overline{\mu+\nu-2}}.\end{aligned}$$

In the third step we used the fact that

$$\begin{aligned}(s-a-3)^{\overline{\mu+\nu-1}} &= \frac{\Gamma(s-a-3+\mu+\nu-1)}{\Gamma(s-a-3)}\\ &= \frac{(s-a-3+\mu+\nu-2)\Gamma(s-a-3+\mu+\nu-2)}{\Gamma(s-a-3)}\\ &= (\mu+\nu+s-a-5)(s-a-3)^{\overline{\mu+\nu-2}}.\end{aligned}$$

Considering the expression in (5.19), we see that the first three terms of the product are positive due to the following. First, since $1 < \mu+\nu < 2$ we have $\mu+\nu-1 > 0$. Second, $s-1 \leq b$ implies $b-s+2 > 0$ and $\mu, \nu > 0$ implies $b-s+\mu+\nu > 0$, which together imply $\frac{\Gamma(\mu+\nu+b-s)}{\Gamma(b-s+2)} > 0$. Lastly, $s \geq a+4$ implies $s-a-4 \geq 0$ and $1 < \mu+\nu < 2$ implies $\mu+\nu-1 > 0$, which together imply $\mu+\nu+s-a-5 = (s-a-3)+(\mu+\nu-2) > 0$.

Now, the solution to the equation $b+a+5-2s = 0$ is given by $s = \frac{b+a+5}{2}$. To ensure that we consider appropriate values of s in the prescribed domain $\mathbb{N}_{a+3}^{b}$, we will consider

$$s = \left\lfloor \frac{b+a+5}{2} \right\rfloor_a.$$

If $s \leq \left\lfloor \frac{b+a+5}{2} \right\rfloor_a$, the difference $(b+a+5)-2s$ is positive, and thus the expression $\left[(b-s+2)^{\overline{\mu+\nu-1}}(s-a-3)^{\overline{\mu+\nu-1}}\right]$ is increasing. If $s \geq \left\lfloor \frac{b+a+5}{2} \right\rfloor_a$, the difference $(b+$

$a+5)-2s$ is negative, and thus the expression $\left[(b-s+2)^{\overline{\mu+\nu-1}}(s-a-3)^{\overline{\mu+\nu-1}}\right]$ is decreasing. This tells us that the maximum of the Green's function $G(t,s)$ for the boundary value problem (5.14) occurs at $s=\left\lfloor\frac{b+a+5}{2}\right\rfloor_a$.

Lastly, we note that there are two potential cases depending on whether the sum $b+a$ is even or odd. When $b+a$ is odd, $s=\left\lfloor\frac{b+a+5}{2}\right\rfloor_a=\frac{b+a+5}{2}$. Thus we have

$$\begin{aligned}
&G\left(\frac{b+a+5}{2}-1,\frac{b+a+5}{2}\right)\\
&\quad=\frac{x\left(b,\frac{b+a+5}{2}-1\right)}{x(\rho(b),a+1)}x\left(\frac{b+a+5}{2}-2,a+1\right)\\
&\quad=\frac{H_{\mu+\nu-1}\left(b,\frac{b+a+5}{2}-1\right)}{H_{\mu+\nu-1}(\rho(b),a+1)}H_{\mu+\nu-1}\left(\frac{b+a+5}{2}-2,a+1\right)\\
&\quad=\frac{1}{\Gamma(\mu+\nu)}\left[\frac{\left(\frac{2b-b-a-5+2}{2}\right)^{\overline{\mu+\nu-1}}}{(b-a-2)^{\overline{\mu+\nu-1}}}\left(\frac{b+a+5-2a-6}{2}\right)^{\overline{\mu+\nu-1}}\right]\\
&\quad=\frac{1}{\Gamma(\mu+\nu)(b-a-2)^{\overline{\mu+\nu-1}}}\left[\left(\frac{b-a-3}{2}\right)^{\overline{\mu+\nu-1}}\left(\frac{b-a-1}{2}\right)^{\overline{\mu+\nu-1}}\right]\\
&\quad=\frac{1}{\Gamma(\mu+\nu)(b-a-2)^{\overline{\mu+\nu-1}}}\left[\left(\frac{b-a-3}{2}\right)^{\overline{\mu+\nu-1}}\right]^2\left(1+\frac{2(\mu+\nu-1)}{b-a-3}\right), \qquad (5.20)
\end{aligned}$$

where we used in the last step the fact that

$$\begin{aligned}
\left(\frac{b-a-1}{2}\right)^{\overline{\mu+\nu-1}}&=\frac{\Gamma\left(\frac{b}{2}-\frac{a}{2}-\frac{1}{2}+\mu+\nu-1\right)}{\Gamma\left(\frac{b}{2}-\frac{a}{2}-\frac{1}{2}\right)}\\
&=\frac{\left(\frac{b}{2}-\frac{a}{2}-\frac{3}{2}+\mu+\nu-1\right)\Gamma\left(\frac{b}{2}-\frac{a}{2}-\frac{3}{2}+\mu+\nu-1\right)}{\left(\frac{b}{2}-\frac{a}{2}-\frac{3}{2}\right)\Gamma\left(\frac{b}{2}-\frac{a}{2}-\frac{3}{2}\right)}\\
&=\frac{\left(\frac{b-a-3+2(\mu+\nu-1)}{2}\right)\Gamma\left(\frac{b}{2}-\frac{a}{2}-\frac{3}{2}+\mu+\nu-1\right)}{\left(\frac{b-a-3}{2}\right)\Gamma\left(\frac{b}{2}-\frac{a}{2}-\frac{3}{2}\right)}\\
&=\frac{[b-a-3+2(\mu+\nu-1)]\,\Gamma\left(\frac{b}{2}-\frac{a}{2}-\frac{3}{2}+\mu+\nu-1\right)}{(b-a-3)\Gamma\left(\frac{b}{2}-\frac{a}{2}-\frac{3}{2}\right)}\\
&=\left(1+\frac{2(\mu+\nu-1)}{b-a-3}\right)\left(\frac{b-a-3}{2}\right)^{\overline{\mu+\nu-1}}.
\end{aligned}$$

When $b+a$ is even, $s = \left\lfloor \frac{b+a+5}{2} \right\rfloor_a = \frac{b+a}{2} + 2$. Thus we have

$$\begin{aligned}
&G\left(\frac{b+a}{2}+1, \frac{b+a}{2}+2\right) \\
&= \frac{x\left(b, \frac{b+a}{2}+2-1\right)}{x(\rho(b), a+1)} x\left(\frac{b+a}{2}+1-1, a+1\right) \\
&= \frac{H_{\mu+\nu-1}\left(b, \frac{b+a}{2}+1\right)}{H_{\mu+\nu-1}(\rho(b), a+1)} H_{\mu+\nu-1}\left(\frac{b+a}{2}, a+1\right) \\
&= \frac{1}{\Gamma(\mu+\nu)} \left[\frac{\left(\frac{2b-b-a-2}{2}\right)^{\overline{\mu+\nu-1}}}{(b-a-2)^{\overline{\mu+\nu-1}}} \left(\frac{b+a-2a-2}{2}\right)^{\overline{\mu+\nu-1}} \right] \\
&= \frac{1}{\Gamma(\mu+\nu)(b-a-2)^{\overline{\mu+\nu-1}}} \left[\left(\frac{b-a-2}{2}\right)^{\overline{\mu+\nu-1}} \left(\frac{b-a-2}{2}\right)^{\overline{\mu+\nu-1}} \right] \\
&= \frac{1}{\Gamma(\mu+\nu)(b-a-2)^{\overline{\mu+\nu-1}}} \left[\left(\frac{b-a-2}{2}\right)^{\overline{\mu+\nu-1}} \right]^2. \qquad (5.21)
\end{aligned}$$

Note that both (5.20) and (5.21) equate to a constant in terms of $b-a$ and $\mu+\nu$. Thus the maximum of the Green's function $G(t,s)$ is given by

$$\begin{aligned}
&G\left(\left\lfloor \frac{b+a+5}{2} \right\rfloor_a - 1, \left\lfloor \frac{b+a+5}{2} \right\rfloor_a\right) \\
&= \begin{cases} \frac{1}{\Gamma(\mu+\nu)}(b-a-2)^{\overline{\mu+\nu-1}} \left(1 + \frac{2(\mu+\nu-1)}{b-a-3}\right) \left[\left(\frac{b-a-3}{2}\right)^{\overline{\mu+\nu-1}}\right]^2, & \left\lfloor \frac{b+a+5}{2} \right\rfloor_a = \frac{b+a+5}{2} \\ \frac{1}{\Gamma(\mu+\nu)}(b-a-2)^{\overline{\mu+\nu-1}} \left[\left(\frac{b-a-2}{2}\right)^{\overline{\mu+\nu-1}}\right]^2, & \left\lfloor \frac{b+a+5}{2} \right\rfloor_a = \frac{b+a}{2} + 2. \end{cases}
\end{aligned}$$

This concludes our proof.

□

5.4 The Sequential Nabla Difference $\nabla_{a+2}^{\nu} \nabla_a^{\mu} x(t)$

In this section we introduce the sequential nabla fractional difference equation

$$\nabla_{a+2}^{\nu} \nabla_a^{\mu} y(t) = h(t). \qquad (5.22)$$

We predominantly consider the case where $2 < \nu < 3$, $1 < \mu < 2$, and $3 < \mu + \nu < 4$, but also consider the set of order conditions $1 < \mu < 2$, $0 < \nu < 1$, and $1 < \mu + \nu < 2$. Aside from a more strict domain and an increased number of boundary conditions, these results are similar to those found in the previous section. Specifically, the only significant difference in the arguments is the influence the order conditions have on whether we are considering a nabla sum or nabla difference at various steps of a proof, thus altering the formulas we rely on.

5.4.1 Existence and Uniqueness

In this section we prove two existence and uniqueness theorems for the nabla sequential difference initial value problem, each with different sets of order conditions.

Theorem 5.15 (Existence & Uniqueness Theorem)**.** *Assume $x : \mathbb{N}_{a+1} \to \mathbb{R}$, $1 < \mu < 2$, $2 < \nu < 3$, and $3 < \mu + \nu < 4$. Then the fractional initial value problem*

$$\begin{cases} -\nabla^{\nu}_{a+2}\nabla^{\mu}_{a}x(t) &= f(t), \quad t \in \mathbb{N}_{a+5} \\ x(a+1) &= A, \\ x(a+2) &= B, \\ x(a+3) &= C, \end{cases}$$

has a unique solution $x(t)$.

Proof. By Theorem 3.4 (i) with $k = 1$,

$$-\nabla^{\nu}_{a+2}\nabla^{\mu}_{a}x(t) = -\Bigg[\nabla^{\mu+\nu}_{a}x(t) - \nabla\left[\sum_{\tau=a+1}^{a+2} H_{-\nu-2}(t,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{1-\mu}(\tau,\rho(s))x(s)\right] \\ - \left[\sum_{j=0}^{1} \nabla^{\mu+j-2}_{a}x(a+2)H_{-\nu+j-2}(t,a+2)\right]\Bigg].$$

Hence

$$
\begin{aligned}
-\nabla_{a+2}^{\nu}\nabla_a^{\mu}x(t) &= -\nabla_a^{\mu+\nu}x(t) \\
&\quad + \nabla\left[H_{-\nu-2}(t,a)H_{1-\mu}(a+1,a)x(a+1)\right. \\
&\qquad\quad \left. +H_{-\nu-2}(t,a+1)\sum_{s=a+1}^{a+2}H_{1-\mu}(a+2,\rho(s))x(s)\right] \\
&\quad + \nabla_a^{\mu-2}x(a+2)H_{-\nu-2}(t,a+2) + \nabla_a^{\mu-1}x(a+2)H_{-\nu-1}(t,a+2) \\
&= -\nabla_a^{\mu+\nu}x(t) \\
&\quad + \nabla\left[H_{-\nu-2}(t,a)x(a+1)\right. \\
&\qquad\quad \left. +H_{-\nu-2}(t,a+1)\left(H_{1-\mu}(a+2,a)x(a+1)+x(a+2)\right)\right] \\
&\quad + \nabla_a^{\mu-2}x(a+2)H_{-\nu-2}(t,a+2) + \nabla_a^{\mu-1}x(a+2)H_{-\nu-1}(t,a+2) \\
&= -\nabla_a^{\mu+\nu}x(t) + H_{-\nu-3}(t,a)x(a+1) \\
&\quad + H_{-\nu-3}(t,a+1)H_{1-\mu}(a+2,a)x(a+1) + H_{-\nu-3}(t,a+1)x(a+2) \\
&\quad + \left[-\mu(1-\mu)(2-\mu)x(a+1)+x(a+2)\right]H_{-\nu-2}(t,a+2) \\
&\quad + \left[-\mu(1-\mu)x(a+1)+x(a+2)\right]H_{-\nu-1}(t,a+2) \\
&= -\nabla_a^{\mu+\nu}x(t) \\
&\quad + \left[H_{-\nu-3}(t,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(t,a+1)\right. \\
&\qquad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(t,a+2) \\
&\qquad \left. -\mu(1-\mu)H_{-\nu-1}(t,a+2)\right]x(a+1) \\
&\quad + \left[H_{-\nu-3}(t,a+1) + H_{-\nu-2}(t,a+2) - H_{-\nu-1}(t,a+2)\right]x(a+2),
\end{aligned}
$$

where we used the following equalities.

First,

$$\begin{aligned}
\nabla_a^{\mu-2}x(a+2) &= \nabla_a^{-(2-\mu)}x(a+2)\\
&= \sum_{s=a+1}^{a+2} H_{2-\mu-1}(a+2,\rho(s))x(s)\\
&= H_{1-\mu}(a+2,a)x(a+1) + H_{1-\mu}(a+2,a+1)x(a+2)\\
&= \frac{\Gamma(3-\mu)}{\Gamma(-\mu)}x(a+1) + x(a+2)\\
&= -\mu(1-\mu)(2-\mu)x(a+1) + x(a+2),
\end{aligned}$$

and second,

$$\begin{aligned}
\nabla_a^{\mu-1}x(a+2) &= \sum_{s=a+1}^{a+2} H_{-(\mu-1)-1}(a+2,\rho(s))x(s)\\
&= H_{-\mu}(a+2,a)x(a+1) + H_{-\mu}(a+2,a+1)x(a+2)\\
&= \frac{\Gamma(2-\mu)}{\Gamma(-\mu)}x(a+1) + x(a+2)\\
&= -\mu(1-\mu)x(a+1) + x(a+2).
\end{aligned}$$

Letting $t = a+4$, we have

$$\begin{aligned}
f(a+4) = &-\nabla_a^{\mu+\nu}x(a+4)\\
&+ [H_{-\nu-3}(a+4,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(a+4,a+1)\\
&\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(a+4,a+2)\\
&\quad -\mu(1-\mu)H_{-\nu-1}(a+4,a+2)]\, x(a+1)\\
&+ [H_{-\nu-3}(a+4,a+1) + H_{-\nu-2}(a+4,a+2)\\
&\quad -H_{-\nu-1}(a+4,a+2)]\, x(a+2),
\end{aligned}$$

so that

$$
\begin{aligned}
f(a+4) = &-\sum_{a+1}^{a+4} H_{-\mu-\nu-1}(a+4,\rho(s))x(s) \\
&+ [H_{-\nu-3}(a+4,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(a+4,a+1) \\
&\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(a+4,a+2) \\
&\quad -\mu(1-\mu)H_{-\nu-1}(a+4,a+2)]\, x(a+1) \\
&+ [H_{-\nu-3}(a+4,a+1) + H_{-\nu-2}(a+4,a+2) \\
&\quad -H_{-\nu-1}(a+4,a+2)]\, x(a+2) \\
= &-H_{-\mu-\nu-1}(a+4,a)x(a+1) - H_{-\mu-\nu-1}(a+4,a+1)x(a+2) \\
&- H_{-\mu-\nu-1}(a+4,a+2)x(a+3) - H_{-\mu-\nu-1}(a+4,a+3)x(a+4) \\
&+ [H_{-\nu-3}(a+4,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(a+4,a+1) \\
&\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(a+4,a+2) \\
&\quad -\mu(1-\mu)H_{-\nu-1}(a+4,a+2)]\, x(a+1) \\
&+ [H_{-\nu-3}(a+4,a+1) + H_{-\nu-2}(a+4,a+2) \\
&\quad -H_{-\nu-1}(a+4,a+2)]\, x(a+2) \\
= &-H_{-\mu-\nu-1}(a+4,a)x(a+1) - H_{-\mu-\nu-1}(a+4,a+1)x(a+2) \\
&- H_{-\mu-\nu-1}(a+4,a+2)x(a+3) - x(a+4) \\
&+ [H_{-\nu-3}(a+4,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(a+4,a+1) \\
&\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(a+4,a+2) \\
&\quad -\mu(1-\mu)H_{-\nu-1}(a+4,a+2)]\, x(a+1) \\
&+ [H_{-\nu-3}(a+4,a+1) + H_{-\nu-2}(a+4,a+2) \\
&\quad -H_{-\nu-1}(a+4,a+2)]\, x(a+2).
\end{aligned}
$$

Rearranging gives us that

$$\begin{aligned}x(a+4) = -f(a+4) &- H_{-\mu-\nu-1}(a+4,a)x(a+1) - H_{-\mu-\nu-1}(a+4,a+1)x(a+2)\\ &- H_{-\mu-\nu-1}(a+4,a+2)x(a+3)\\ &+ [H_{-\nu-3}(a+4,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(a+4,a+1)\\ &\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(a+4,a+2)\\ &\quad -\mu(1-\mu)H_{-\nu-1}(a+4,a+2)]\, x(a+1)\\ &+ [H_{-\nu-3}(a+4,a+1) + H_{-\nu-2}(a+4,a+2)\\ &\quad -H_{-\nu-1}(a+4,a+2)]\, x(a+2).\end{aligned}$$

Thus the value of $x(a+4)$ is uniquely determined by the initial conditions $x(a+1) = A$, $x(a+2) = B$, and $x(a+3) = C$ and the value of the given function $f(t)$.

To show $x(t)$ is uniquely determined on $\mathbb{N}_{a+1}$ we will proceed by induction. Suppose there exists a unique solution to the initial value problem, $x(t)$, for $t \in \mathbb{N}_{a+1}^{t_0}$, where $t_0 \in \mathbb{N}_{a+2}$. We will show that the value of $x(t_0+1)$ is uniquely determined by the values of $x(t)$ on $\mathbb{N}_{a+1}^{t_0}$. Again using Theorem 3.4 (i) with $k = 1$ to rewrite the sequential difference, we have

$$\begin{aligned}f(t_0+1) = -\nabla_a^{\mu+\nu}x(t_0+1)&\\ &+ [H_{-\nu-3}(t_0+1,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(t_0+1,a+1)\\ &\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(t_0+1,a+2)\\ &\quad -\mu(1-\mu)H_{-\nu-1}(t_0+1,a+2)]\, x(a+1)\\ &+ [H_{-\nu-3}(t_0+1,a+1) + H_{-\nu-2}(t_0+1,a+2)\\ &\quad -H_{-\nu-1}(t_0+1,a+2)]\, x(a+2).\end{aligned}$$

Hence

$$
\begin{aligned}
f(t_0+1) = &- \sum_{s=a+1}^{t_0+1} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) \\
&+ [H_{-\nu-3}(t_0+1,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(t_0+1,a+1) \\
&\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(t_0+1,a+2) \\
&\quad -\mu(1-\mu)H_{-\nu-1}(t_0+1,a+2)]\, x(a+1) \\
&+ [H_{-\nu-3}(t_0+1,a+1) + H_{-\nu-2}(t_0+1,a+2) \\
&\quad -H_{-\nu-1}(t_0+1,a+2)]\, x(a+2). \\
= &-H_{-\mu-\nu-1}(t_0+1,t_0)x(t_0+1) - \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) \\
&+ [H_{-\nu-3}(t_0+1,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(t_0+1,a+1) \\
&\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(t_0+1,a+2) \\
&\quad -\mu(1-\mu)H_{-\nu-1}(t_0+1,a+2)]\, x(a+1) \\
&+ [H_{-\nu-3}(t_0+1,a+1) + H_{-\nu-2}(t_0+1,a+2) \\
&\quad -H_{-\nu-1}(t_0+1,a+2)]\, x(a+2) \\
= &-x(t_0+1) - \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) \\
&+ [H_{-\nu-3}(t_0+1,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(t_0+1,a+1) \\
&\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(t_0+1,a+2) \\
&\quad -\mu(1-\mu)H_{-\nu-1}(t_0+1,a+2)]\, x(a+1) \\
&+ [H_{-\nu-3}(t_0+1,a+1) + H_{-\nu-2}(t_0+1,a+2) \\
&\quad -H_{-\nu-1}(t_0+1,a+2)]\, x(a+2).
\end{aligned}
$$

Rearranging provides

$$\begin{aligned}
x(t_0+1) = -f(t_0+1) - \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) \\
&+ \left[H_{-\nu-3}(t_0+1,a) - \mu(1-\mu)(2-\mu)H_{-\nu-3}(t_0+1,a+1)\right. \\
&\quad -\mu(1-\mu)(2-\mu)H_{-\nu-2}(t_0+1,a+2) \\
&\quad \left.-\mu(1-\mu)H_{-\nu-1}(t_0+1,a+2)\right]x(a+1) \\
&+ \left[H_{-\nu-3}(t_0+1,a+1) + H_{-\nu-2}(t_0+1,a+2)\right. \\
&\quad \left.-H_{-\nu-1}(t_0+1,a+2)\right]x(a+2).
\end{aligned}$$

Now by the induction hypothesis, all the values of $x(t)$ for $t \in \mathbb{N}_{a+1}^{t_0}$ are known. Therefore $x(t_0+1)$ is uniquely determined and hence $x(t)$ is the unique solution of the fractional initial value problem on $\mathbb{N}_{a+1}^{t_0+1}$. Thus a unique solution exists on $\mathbb{N}_{a+1}$.

$\square$

We now present an existence and uniqueness theorem for the different set of order conditions on μ and ν.

Theorem 5.16 (Existence & Uniqueness Theorem)**.** *Assume $x : \mathbb{N}_{a+1} \to \mathbb{R}$, along with the set of order conditions $1 < \mu < 2$, $0 < \nu < 1$, and $1 < \mu + \nu < 2$. The fractional initial value problem*

$$\begin{cases}
-\nabla_{a+2}^{\nu}\nabla_{a}^{\mu}x(t) &= f(t), \quad t \in \mathbb{N}_{a+3} \\
x(a+1) &= A, \\
x(a+2) &= B.
\end{cases}$$

has a unique solution $x(t)$.

Proof. By Theorem 3.4 (ii) with $k = 1$,

$$
\begin{aligned}
-\nabla_{a+2}^{\nu}\nabla_a^{\mu}x(t) = -\Big[&\nabla_a^{\mu+\nu}x(t) \\
&+ \left[H_{-\nu-3}(t,a) - (2-\mu)H_{-\nu-3}(t,a+1) - (2-\mu)H_{-\nu-2}(t,a+2)\right. \\
&\quad \left. -(1-\mu)H_{-\nu-1}(t,a+2)\right]x(a+1) \\
&+ \left[-H_{-\nu-3}(t,a+1) - H_{-\nu-2}(t,a+2) - H_{-\nu-1}(t,a+2)\right]x(a+2)\Big] \\
= -\nabla_a^{\mu+\nu}x(t)& \\
&- \left[H_{-\nu-3}(t,a) - (2-\mu)H_{-\nu-3}(t,a+1) - (2-\mu)H_{-\nu-2}(t,a+2)\right. \\
&\quad \left. -(1-\mu)H_{-\nu-1}(t,a+2)\right]x(a+1) \\
&- \left[-H_{-\nu-3}(t,a+1) - H_{-\nu-2}(t,a+2) - H_{-\nu-1}(t,a+2)\right]x(a+2).
\end{aligned}
$$

Letting $t = a + 3$, we have

$$
\begin{aligned}
f(a+3) = -\nabla_a^{\mu+\nu}&x(a+3) \\
&- \left[H_{-\nu-3}(a+3,a) - (2-\mu)H_{-\nu-3}(a+3,a+1)\right. \\
&\quad \left. -(2-\mu)H_{-\nu-2}(a+3,a+2) - (1-\mu)H_{-\nu-1}(a+3,a+2)\right]x(a+1) \\
&- \left[-H_{-\nu-3}(a+3,a+1) - H_{-\nu-2}(a+3,a+2)\right. \\
&\quad \left. -H_{-\nu-1}(a+3,a+2)\right]x(a+2) \\
= -&\sum_{s=a+1}^{a+3} H_{-\mu-\nu-1}(a+3,\rho(s))x(s) \\
&- \left[H_{-\nu-3}(a+3,a) - (2-\mu)H_{-\nu-3}(a+3,a+1) - (2-\mu)\right. \\
&\quad \left. -(1-\mu)\right]x(a+1) \\
&- \left[-H_{-\nu-3}(a+3,a+1) - 1 - 1\right]x(a+2).
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(a+3) = &-H_{-\mu-\nu-1}(a+3,a)x(a+1) - H_{-\mu-\nu-1}(a+3,a+1)x(a+2) \\
&- H_{-\mu-\nu-1}(a+3,a+2)x(a+3) \\
&- [H_{-\nu-3}(a+3,a) - (2-\mu)H_{-\nu-3}(a+3,a+1) - (2-\mu) \\
&\quad -(1-\mu)]\, x(a+1) \\
&- [-H_{-\nu-3}(a+3,a+1) - 2]\, x(a+2) \\
= &-H_{-\mu-\nu-1}(a+3,a)x(a+1) - H_{-\mu-\nu-1}(a+3,a+1)x(a+2) - x(a+3) \\
&- [H_{-\nu-3}(a+3,a) - (2-\mu)H_{-\nu-3}(a+3,a+1) - (2-\mu) \\
&\quad -(1-\mu)]\, x(a+1) \\
&- [-H_{-\nu-3}(a+3,a+1) - 2]\, x(a+2).
\end{aligned}
$$

Rearranging terms, we get

$$
\begin{aligned}
x(a+3) = &-f(a+3) - H_{-\mu-\nu-1}(a+3,a)x(a+1) - H_{-\mu-\nu-1}(a+3,a+1)x(a+2) \\
&- [H_{-\nu-3}(a+3,a) - (2-\mu)H_{-\nu-3}(a+3,a+1) - (2-\mu) \\
&\quad -(1-\mu)]\, x(a+1) \\
&- [-H_{-\nu-3}(a+3,a+1) - 2]\, x(a+2).
\end{aligned}
$$

Thus the value of $x(a+3)$ is uniquely determined by the initial conditions $x(a+1) = A$ and $x(a+2) = B$ and the value of the given function $f(t)$.

To show $x(t)$ is uniquely determined on $\mathbb{N}_{a+1}$ we will proceed by induction. Suppose there exists a unique solution to the initial value problem, $x(t)$, for $t \in \mathbb{N}_{a+1}^{t_0}$, where $t_0 \in \mathbb{N}_{a+2}$. We will show that the value of $x(t_0+1)$ is uniquely determined by the values of $x(t)$ on $\mathbb{N}_{a+1}^{t_0}$. Again using Theorem 3.4 (ii) with $k = 1$ to rewrite the

sequential difference, we have

$$
\begin{aligned}
f(t_0+1) = -\nabla_a^{\mu+\nu}x(t_0+1)& \\
&- [H_{-\nu-3}(t_0+1,a) - (2-\mu)H_{-\nu-3}(t_0+1,a+1) \\
&\quad -(2-\mu)H_{-\nu-2}(t_0+1,a+2) - (1-\mu)H_{-\nu-1}(t_0+1,a+2)]\,x(a+1) \\
&- [-H_{-\nu-3}(t_0+1,a+1) - H_{-\nu-2}(t_0+1,a+2) \\
&\quad -H_{-\nu-1}(t_0+1,a+2)]\,x(a+2) \\
= -\sum_{s=a+1}^{t_0+1} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s)& \\
&- [H_{-\nu-3}(t_0+1,a) - (2-\mu)H_{-\nu-3}(t_0+1,a+1) - (2-\mu) \\
&\quad -(1-\mu)]\,x(a+1) \\
&- [-H_{-\nu-3}(t_0+1,a+1) - 2]\,x(a+2).
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(t_0+1) &= -H_{-\mu-\nu-1}(t_0+1,t_0)x(t_0+1) - \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) \\
&\quad - [H_{-\nu-3}(t_0+1,a) - (2-\mu)H_{-\nu-3}(t_0+1,a+1) - (2-\mu) \\
&\qquad -(1-\mu)]\,x(a+1) \\
&\quad - [-H_{-\nu-3}(t_0+1,a+1) - 2]\,x(a+2) \\
&= -x(t_0+1) - \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) \\
&\quad - [H_{-\nu-3}(t_0+1,a) - (2-\mu)H_{-\nu-3}(t_0+1,a+1) - (2-\mu) \\
&\qquad -(1-\mu)]\,x(a+1) \\
&\quad - [-H_{-\nu-3}(t_0+1,a+1) - 2]\,x(a+2).
\end{aligned}
$$

Rearranging provides

$$\begin{aligned}
x(t_0+1) = -f(t_0+1) - &\sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))x(s) \\
&- [H_{-\nu-3}(t_0+1,a) - (2-\mu)H_{-\nu-3}(t_0+1,a+1) - (2-\mu) \\
&\quad -(1-\mu)]\, x(a+1) \\
&- [-H_{-\nu-3}(t_0+1,a+1) - 2]\, x(a+2).
\end{aligned}$$

Now by the induction hypothesis, all the values of $x(t)$ for $t \in \mathbb{N}_{a+1}^{t_0}$ are known. Therefore $x(t_0+1)$ is uniquely determined and hence $x(t)$ is the unique solution of the fractional initial value problem on $\mathbb{N}_{a+1}^{t_0+1}$. Thus a unique solution exists on $\mathbb{N}_{a+1}$.

□

5.4.2 Initial Value Problem

We begin this section by presenting an initial value problem involving a nonhomogeneous nabla difference equation with zero initial conditions. This section continues by presenting our Cauchy function and variation of constants formula for a nabla difference initial value problem. All these results will be drawn upon in the next section to establish the Green's function.

Theorem 5.17. *Let $g : \mathbb{N}_{a+1} \to \mathbb{R}$ and $2 < \nu \leq 3$. Then for $t \in \mathbb{N}_{a+5}$ the initial value problem*

$$\begin{cases}
\nabla_{a+2}^{\nu} x(t) &= g(t), \quad t \in \mathbb{N}_{a+5} \\
\nabla x(a+4) &= 0 \\
\nabla x(a+5) &= x(a+5) = 0
\end{cases}$$

has the unique solution

$$x(t) = \nabla_{a+2}^{-\nu} g(t) - [g(a+3) + g(a+4) + g(a+5)]\, H_{\nu-1}(t, a+2)$$
$$+ [g(a+4) + 2g(a+5)]\, H_{\nu-2}(t, a+2) - g(a+5) H_{\nu-3}(t, a+2).$$

Proof. We begin by noting that

$$\begin{aligned} 0 = \nabla x(a+5) &= x(a+5) - x(a+4) \\ &= 0 - x(a+4), \end{aligned}$$

which implies $x(a+4) = 0$, and

$$\begin{aligned} 0 = \nabla x(a+4) &= x(a+4) - x(a+3) \\ &= 0 - x(a+3) \end{aligned}$$

implies $x(a+3) = 0$. Now by taking the Laplace transform based at $a+5$ of both sides and implementing Theorem 1.35 with $k = 1$ and $N = 3$, we have

$$\begin{aligned} \mathcal{L}_{a+5}\{g\}(s) &= s^{\nu} \mathcal{L}_{a+5}\{x\}(s) \\ &\quad + \sum_{j=0}^{5-2-1} \left[\frac{s^{\nu} x(a+2+j+1) - s^{5-2} \nabla_{a+2}^{-(5-2-\nu)} x(a+2+j+1)}{(1-s)^{5-2-j}} \right. \\ &\quad \left. - s^{j} \nabla^{5-2-j-1} \nabla_{a+2}^{-(5-2-\nu)} x(a+5) \right] \\ &= s^{\nu} \mathcal{L}_{a+5}\{x\}(s) + \sum_{j=0}^{2} \left[\frac{s^{\nu} x(a+2+j) - s^{2} \nabla_{a+1}^{-(2-\nu)} x(a+2+j)}{(1-s)^{2-j}} \right. \\ &\quad \left. - s^{j} \nabla^{2-j} \nabla_{a+1}^{-(2-\nu)} x(a+5) \right], \end{aligned}$$

so that

$$\begin{aligned}\mathcal{L}_{a+5}\{g\}(s) = s^\nu \mathcal{L}_{a+5}\{x\}(s) &+ \frac{s^\nu x(a+3) - s^3 \nabla_{a+2}^{-(3-\nu)} x(a+3)}{(1-s)^3} - \nabla^2 \nabla_{a+2}^{-(3-\nu)} x(a+5) \\ &+ \frac{s^\nu x(a+4) - s^3 \nabla_{a+2}^{-(3-\nu)} x(a+4)}{(1-s)^2} - s \nabla \nabla_{a+2}^{-(3-\nu)} x(a+5) \\ &+ \frac{s^\nu x(a+5) - s^3 \nabla_{a+2}^{-(3-\nu)} x(a+5)}{1-s} - s^2 \nabla_{a+2}^{-(3-\nu)} x(a+5) \\ = s^\nu \mathcal{L}_{a+5}\{x\}(s),\end{aligned}$$

since $x(a+3) = x(a+4) = x(a+5) = 0$.

Now we are interested in an equation involving the Laplace transform based at $a+2$, as that is the base of our nabla difference. To transform our current equation, we use the Shifting Lemma 1.31, replacing a with $a+2$ and letting $n = 3$ as follows. First, the left hand side becomes

$$\mathcal{L}_{a+5}\{g\}(s) = \frac{1}{(1-s)^3} \mathcal{L}_{a+2}\{g\}(s) - \sum_{k=1}^{3} \frac{g(a+2+k)}{(1-s)^{3-k+1}},$$

and the right hand side becomes

$$\begin{aligned}s^\nu \mathcal{L}_{a+5}\{x\}(s) &= s^\nu \left[\left(\frac{1}{1-s} \right)^3 \mathcal{L}_{a+2}\{x\}(s) - \sum_{k=1}^{3} \frac{x(a+2+k)}{(1-s)^{3-k+1}} \right] \\ &= \frac{s^\nu}{(1-s)^3} \mathcal{L}_{a+2}\{x\}(s) - \frac{s^\nu}{(1-s)^3} x(a+3) \\ &\quad - \frac{s^\nu}{(1-s)^2} x(a+4) - \frac{s^\nu}{1-s} x(a+5) \\ &= \frac{s^\nu}{(1-s)^3} \mathcal{L}_{a+2}\{x\}(s),\end{aligned}$$

since $x(a+3) = x(a+4) = x(a+5) = 0$.

Expanding and then multiplying through by $(1-s)^3$ gives

$$\begin{aligned}
\frac{s^\nu}{(1-s)^3}\mathcal{L}_{a+2}\{x\}(s) &= \frac{1}{(1-s)^3}\mathcal{L}_{a+2}\{g\}(s) - \frac{g(a+3)}{(1-s)^3} - \frac{g(a+4)}{(1-s)^2 - \frac{g(a+5)}{1-s}} \\
\mathcal{L}_{a+2}\{x\}(s) &= \frac{1}{s^\nu}\mathcal{L}_{a+2}\{g\}(s) - \frac{1}{s^\nu}g(a+3) - \frac{1-s}{s^\nu}g(a+4) \\
&\quad - \frac{(1-s)^2}{s^\nu}g(a+5) \\
\mathcal{L}_{a+2}\{x\}(s) &= \frac{1}{s^\nu}\mathcal{L}_{a+2}\{g\}(s) - \frac{1}{s^\nu}g(a+3) - \frac{1}{s^\nu}g(a+4) + \frac{1}{s^{\nu-1}}g(a+4) \\
&\quad - \frac{1}{s^\nu}g(a+5) + \frac{2}{s^{\nu-1}}g(a+5) - \frac{1}{s^{\nu-2}}g(a+5).
\end{aligned}$$

This implies that

$$\begin{aligned}
x(t) = &\left(H_{\nu-1}(\cdot, a+2) * g(\cdot)\right)(t) - \left[g(a+3) + g(a+4) + g(a+5)\right] H_{\nu-1}(t, a+2) \\
&+ \left[g(a+4) + 2g(a+5)\right] H_{\nu-2}(t, a+2) - g(a+5)H_{\nu-3}(t, a+2) \\
&+ g(a+3)H_{\nu-2}(t, a+1),
\end{aligned}$$

so that

$$\begin{aligned}
x(t) = \nabla_{a+2}^{-\nu}g(t) &- \left[g(a+3) + g(a+4) + g(a+5)\right] H_{\nu-1}(t, a+2) \\
&+ \left[g(a+4) + 2g(a+5)\right] H_{\nu-2}(t, a+2) - g(a+5)H_{\nu-3}(t, a+2) \\
&+ g(a+3)H_{\nu-2}(t, a+1).
\end{aligned}$$

This completes the proof.

□

Our Green's function, along with the following theorem, will involve the Cauchy function which is defined below.

Definition 5.18. *Assume $2 < \nu < 3$, $1 < \mu < 2$, and $3 < \mu + \nu < 4$. We define the Cauchy function $x(t, \rho(s))$ for the homogeneous fractional equation*

$$\nabla^{\nu}_{a+2}\nabla^{\mu}_{a}y(t) = 0$$

to be the function $x : \mathbb{N}_{a+1} \times \mathbb{N}_{a+1} \to \mathbb{R}$ such that for each fixed $s \in \mathbb{N}_{a+1}$, $x(\cdot, \rho(s))$ is the unique solution of the fractional initial value problem

$$\begin{cases} \nabla^{\nu}_{s+1}\nabla^{\mu}_{\rho(s)}x(t) &= 0, \quad t \in \mathbb{N}_{s+4} \\ x(\rho(s)) = 0, \quad \nabla x(s) &= 1 \end{cases}$$

and is given by the formula

$$x(t, \rho(s)) = H_{\mu+\nu-1}(t, \rho(s)), \quad t \in \mathbb{N}_{a+1}. \tag{5.23}$$

Note that by convention $x(t, \rho(s)) = 0$ for $t \leq \rho(s)$.

We now explore a sequential nabla difference initial value problem with a nonhomogeneous difference equation with zero initial conditions. The solution is given in terms of the Cauchy function.

Theorem 5.19. *Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $2 < \nu < 3$, $1 < \mu < 2$, and $3 < \mu + \nu < 4$. The solution to the fractional initial value problem*

$$\begin{cases} \nabla^{\nu}_{a+2}\nabla^{\mu}_{a}y(t) &= f(t), \quad t \in \mathbb{N}_{a+5} \\ \nabla y(a+j) &= 0, \quad \text{for } j = 2, 3, 4, 5 \\ y(a+5) &= 0, \end{cases}$$

is given by

$$y(t) = \sum_{s=a+6}^{t} f(s)x(t,\rho(s))$$

where $x(t,\rho(s))$ is the Cauchy function (5.23).

Proof. Let $y(t)$ be the solution of the fractional initial value problem and let $h(t) = \nabla_a^{\mu} y(t)$ where $1 < \mu < 2$. Then $h(t)$ is a solution of the initial value problem

$$\begin{cases} \nabla_{a+2}^{\nu} h(t) & = f(t), \quad t \in \mathbb{N}_{a+5} \\ h(a+5) & = \nabla_a^{\mu} y(a+5) = 0 \\ \nabla h(a+4) & = \nabla \nabla_a^{\mu} y(a+4) = 0 \\ \nabla h(a+5) & = \nabla \nabla_a^{\mu} y(a+5) = 0. \end{cases}$$

The initial conditions $y(a+5) = \nabla y(a+5) = 0$ imply

$$\begin{aligned} 0 = \nabla y(a+5) &= y(a+5) - y(a+4) \\ &= 0 - y(a+4), \end{aligned}$$

which implies $y(a+4) = 0$. Since $\nabla y(a+4) = 0$,

$$\begin{aligned} 0 = \nabla y(a+4) &= y(a+4) - y(a+3) \\ &= 0 - y(a+3), \end{aligned}$$

we have that $y(a+3) = 0$. Similarly, we have that our initial conditions imply $y(a+2) = y(a+1) = 0$.

In turn this implies

$$\begin{aligned}
h(a+5) &= \nabla_a^{\mu} y(a+5) \\
&= \sum_{s=a+1}^{a+5} H_{-\mu-1}(a+5,\rho(s))y(s) \\
&= H_{-\mu-1}(a+5,a)y(a+1) + H_{-\mu-1}(a+5,a+1)y(a+2) \\
&\quad + H_{-\mu-1}(a+5,a+2)y(a+3) + H_{-\mu-1}(a+5,a+3)y(a+4) + y(a+5) \\
&= H_{-\mu-1}(a+5,a)(0) + H_{-\mu-1}(a+5,a+1)(0) \\
&\quad + H_{-\mu-1}(a+5,a+2)(0) + H_{-\mu-1}(a+5,a+3)(0) + (0) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\nabla h(a+4) &= \nabla\nabla_a^{\mu} y(a+4) \\
&= \nabla_a^{\mu+1} y(a+4) \\
&= \sum_{s=a+1}^{a+4} H_{-\mu-2}(a+4,\rho(s))y(s),
\end{aligned}$$

which implies that

$$\begin{aligned}
\nabla h(a+4) &= H_{-\mu-2}(a+4,a)y(a+1) + H_{-\mu-2}(a+4,a+1)y(a+2) \\
&\quad + H_{-\mu-2}(a+4,a+2)y(a+3) + y(a+4) \\
&= H_{-\mu-2}(a+4,a)(0) + H_{-\mu-2}(a+4,a+1)(0) \\
&\quad + H_{-\mu-2}(a+4,a+2)(0) + (0) \\
&= 0.
\end{aligned}$$

Lastly, this implies that

$$\begin{aligned}
\nabla h(a+5) &= \nabla\nabla_a^{\mu} y(a+5)\\
&= \nabla_a^{\mu+1} y(a+5)\\
&= \sum_{s=a+1}^{a+5} H_{-\mu-2}(a+5,\rho(s))y(s)\\
&= H_{-\mu-2}(a+5,a)y(a+1) + H_{-\mu-2}(a+5,a+1)y(a+2)\\
&\quad + H_{-\mu-2}(a+5,a+2)y(a+3) + H_{-\mu-2}(a+5,a+3)y(a+4)\\
&\quad + y(a+5)\\
&= H_{-\mu-2}(a+5,a)(0) + H_{-\mu-2}(a+5,a+1)(0) + H_{-\mu-2}(a+5,a+2)(0)\\
&\quad + H_{-\mu-2}(a+5,a+3)(0) + (0) = 0.
\end{aligned}$$

We also note the following equalities:

$$\begin{aligned}
H_{\nu-1}(t,a+2) - H_{\nu-2}(t,a+2) &= \frac{(t-a-2)^{\overline{\nu-1}}}{\Gamma(\nu)} - \frac{(t-a-2)^{\overline{\nu-2}}}{\Gamma(\nu-1)}\\
&= \frac{\Gamma(t-a-2+\nu-1)}{\Gamma(t-a-2)\Gamma(\nu)} - \frac{\Gamma(t-a-2+\nu-2)}{\Gamma(t-a-2)\Gamma(\nu-1)}\\
&= \frac{\Gamma(t-a-2+\nu-2)}{\Gamma(t-a-2)\Gamma(\nu-1)}\left[\frac{(t-a-2+\nu-2)}{(\nu-1)} - 1\right],
\end{aligned}$$

so that

$$\begin{aligned}
H_{\nu-1}(t,a+2) - H_{\nu-2}(t,a+2) &= \frac{\Gamma(t-a-2+\nu-2)}{\Gamma(t-a-2)\Gamma(\nu-1)}\left[\frac{t-a-2+\nu-2-\nu+1}{\nu-1}\right]\\
&= \frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a)\Gamma(\nu-1)}(t-a-1)\\
&= \frac{\Gamma(t-a-3+\nu-1)(t-a-3)}{(t-a-3)\Gamma(t-a-3)\Gamma(\nu)}\\
&= \frac{(t-a-3)^{\overline{\nu-1}}}{\Gamma(\nu)} = H_{\nu-1}(t,\rho(a+4)),
\end{aligned}$$

and

$$\begin{aligned}
&H_{\nu-1}(t,a+2) - 2H_{\nu-2}(t,a+2) + H_{\nu-3}(t,a+2)\\
&\quad= \frac{(t-a-2)^{\overline{\nu-1}}}{\Gamma(\nu)} - 2\frac{(t-a-2)^{\overline{\nu-2}}}{\Gamma(\nu-1)} + \frac{(t-a-2)^{\overline{\nu-3}}}{\Gamma(\nu-2)}\\
&\quad= \frac{\Gamma(t-a-2+\nu-1)}{\Gamma(t-a-2)\Gamma(\nu)} - 2\frac{\Gamma(t-a-2+\nu-2)}{\Gamma(t-a-2)\Gamma(\nu-1)} + \frac{\Gamma(t-a-2+\nu-3)}{\Gamma(t-a-2)\Gamma(\nu-2)}\\
&\quad= \frac{\Gamma(t-a-2+\nu-3)}{\Gamma(t-a-2)\Gamma(\nu-2)}\left[\frac{(t-a-2+\nu-3)(t-a-2+\nu-2)}{(\nu-2)(\nu-1)}\right.\\
&\qquad\qquad\left. -2\frac{(t-a-2+\nu-3)}{(\nu-2)} + 1\right]\\
&\quad= \frac{\Gamma(t-a-2+\nu-3)}{\Gamma(t-a-2)\Gamma(\nu-2)}\left[\frac{(t-a-2+\nu-3)}{(\nu-2)}\left(\frac{(t-a-2+\nu-2)}{(\nu-1)}\right.\right.\\
&\qquad\qquad\left.\left. -2\frac{(\nu-1)}{(\nu-1)}\right) + 1\right]\\
&\quad= \frac{\Gamma(t-a-2+\nu-3)}{\Gamma(t-a-2)\Gamma(\nu-2)}\left[\frac{(t-a-2+\nu-3)}{(\nu-2)}\cdot\frac{(t-a-2-\nu)}{(\nu-1)} + 1\right]\\
&\quad= \frac{\Gamma(t-a-2+\nu-3)}{\Gamma(t-a-2)\Gamma(\nu-2)}\left[\frac{(t-a-2+\nu-3)(t-a-2-\nu) + (\nu-1)(\nu-2)}{(\nu-1)(\nu-2)}\right]\\
&\quad= \frac{\Gamma(t-a-2+\nu-3)}{\Gamma(t-a-2)\Gamma(\nu-2)}\left[\frac{(t-a-3)(t-a-4)}{(\nu-2)(\nu-1)}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
H_{\nu-1}(t,a+2) - 2H_{\nu-2}(t,a+2) + H_{\nu-3}(t,a+2) &= \frac{\Gamma(t-a-2+\nu-3)}{\Gamma(t-a-4)\Gamma(\nu)}\\
&= \frac{(t-a-4)^{\overline{\nu-1}}}{\Gamma(\nu)}\\
&= H_{\nu-1}(t,\rho(a+5)).
\end{aligned}$$

By Theorem 5.17, the solution of this initial value problem is given by

$$\begin{aligned}
h(t) = \nabla_{a+2}^{-\nu} f(t) &- \left[f(a+3) + f(a+4) + f(a+5)\right] H_{\nu-1}(t,a+2)\\
&+ \left[f(a+4) + 2f(a+5)\right] H_{\nu-2}(t,a+2) - f(a+5) H_{\nu-3}(t,a+2).
\end{aligned}$$

Hence

$$\begin{aligned}
h(t) = &\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) - [f(a+3)+f(a+4)+f(a+5)]\, H_{\nu-1}(t,a+2) \\
&+ [f(a+4)+2f(a+5)]\, H_{\nu-2}(t,a+2) - f(a+5)H_{\nu-3}(t,a+2) \\
= &\sum_{s=a+2}^{t} H_{\nu-1}(t,\rho(s))f(s) - f(a+2)H_{\nu-1}(t,a+1) \\
&- f(a+3)\,[H_{\nu-1}(t,a+1) + H_{\nu-2}(t,a+1)] \\
= &\sum_{s=a+3}^{t} H_{\nu-1}(t,\rho(s))f(s) - f(a+3)H_{\nu-1}(t,a+2) \\
&- f(a+4)\,[H_{\nu-1}(t,a+2) - H_{\nu-2}(t,a+2)] \\
&- f(a+5)\,[H_{\nu-1}(t,a+2) - 2H_{\nu-2}(t,a+2) + H_{\nu-3}(t,a+2)] \\
= &\sum_{s=a+4}^{t} H_{\nu-1}(t,\rho(s))f(s) - f(a+4)\,[H_{\nu-1}(t,a+2) - H_{\nu-2}(t,a+2)] \\
&- f(a+5)\,[H_{\nu-1}(t,a+2) - 2H_{\nu-2}(t,a+2) + H_{\nu-3}(t,a+2)] \\
= &\sum_{s=a+6}^{t} H_{\nu-1}(t,\rho(s))f(s) \\
= &\nabla_{a+5}^{-\nu} f(t).
\end{aligned}$$

Composing each side with $\nabla_a^{-\mu}$ gives, by Theorem 2.4,

$$\begin{aligned}
y(t) = \nabla_a^{-\mu}\nabla_{a+5}^{-\nu} f(t) &= \nabla_{a+5}^{-(\mu+\nu)} f(t) \\
&= \sum_{s=a+6}^{t} H_{\mu+\nu-1}(t,\rho(s))f(s) \\
&= \sum_{s=a+6}^{t} f(s)x(t,\rho(s)).
\end{aligned}$$

This completes the proof.

□

Following from this theorem, we get a simple but useful comparison theorem involving sequential differences as a corollary.

Corollary 5.20. *Assume* $2 < \nu < 3$, $1 < \mu < 2$, *and* $3 < \mu + \nu < 4$ *and* $u(t)$, $v(t)$ *satisfy*

$$\begin{aligned}
\nabla^{\nu}_{a+2}\nabla^{\mu}_{a}u(t) &\geq \nabla^{\nu}_{a+2}\nabla^{\mu}_{a}v(t), \quad t \in \mathbb{N}_{a+5},\\
u(a+1) &= v(a+1),\\
u(a+2) &= v(a+2),\\
u(a+3) &= v(a+3),\\
u(a+4) &= v(a+4),\\
u(a+5) &= v(a+5).
\end{aligned}$$

Then $u(t) \geq v(t)$ *on* $\mathbb{N}_{a+1}$.

Proof. Set $w(t) = u(t) - v(t)$ and for $t \in \mathbb{N}_{a+5}$ let

$$h(t) = \nabla^{\nu}_{a+2}\nabla^{\mu}_{a}w(t) = \nabla^{\nu}_{a+2}\nabla^{\mu}_{a}u(t) - \nabla^{\nu}_{a+2}\nabla^{\mu}_{a}v(t) \geq 0.$$

Hence $w(t)$ solves the initial value problem

$$\begin{cases}
\nabla^{\nu}_{a+2}\nabla^{\mu}_{a}w(t) &= h(t), \quad t \in \mathbb{N}_{a+5}\\
\nabla w(a+4) = \nabla w(a+3) = \nabla w(a+2) &= 0\\
w(a+5) = \nabla w(a+5) &= 0,
\end{cases}$$

and Theorem 5.19 gives for $t \in \mathbb{N}_{a+1}$ that

$$w(t) = \sum_{s=a+6}^{t} h(s)x(t, \rho(s)) = \sum_{s=a+6}^{t} h(s)H_{\mu+\nu-1}(t, \rho(s)) \geq 0,$$

since $h(t) \geq 0$ by assumption and $t \geq s$. Therefore $u(t) \geq v(t)$ for $t \in \mathbb{N}_{a+1}$.

□

We now present our variation of constants formula for the initial value problem with the nonhomogeneous nabla difference equation based at $a+2$ and nonzero initial conditions.

Theorem 5.21. *Let $2 < \nu < 3$ and $y, h : \mathbb{N}_{a+1} \to \mathbb{R}$. Then for $t \in \mathbb{N}_{a+1}$ the initial value problem*

$$\begin{cases} -\nabla_{a+2}^{\nu} y(t) &= h(t), \quad t \in \mathbb{N}_{a+5} \\ y(a+3) &= A \\ y(a+4) &= B \\ y(a+5) &= C \end{cases}$$

has the unique solution

$$\begin{aligned} y(t) = -\nabla_{a+2}^{-\nu} h(t) + \Bigg[& h(a+3) + h(a+4) + h(a+5) \\ & + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C \Bigg] H_{\nu-1}(t, a+2) \\ - \big[& h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C \big] H_{\nu-2}(t, a+2) \\ + \Bigg[& h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C \Bigg] H_{\nu-3}(t, a+2). \end{aligned}$$

Proof. First we attend to the initial conditions to ensure that the solution satisfies each condition. We proceed by plugging in the values $a+3$, $a+4$, and $a+5$ respectively to check that we arrive at the appropriate values.

$$
\begin{aligned}
y(a+3) &= -\nabla_{a+2}^{-\nu} h(a+3) \\
&\quad + \Big[h(a+3) + h(a+4) + h(a+5) \\
&\qquad + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\Big] H_{\nu-1}(a+3, a+2) \\
&\quad - \big[h(a+4) + 2h(a+5) \\
&\qquad + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\big] H_{\nu-2}(a+3, a+2) \\
&\quad + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\nu-3}(a+3, a+2) \\
&= -\sum_{s=a+3}^{a+3} H_{\nu-1}(a+3, \rho(s)) h(s) \\
&\quad + \left[h(a+3) + h(a+4) + h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\right] \\
&\quad - \big[h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\big] \\
&\quad + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] \\
&= -h(a+3) + h(a+3) + h(a+4) + h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) \\
&\quad + B(1-\nu) + C - h(a+4) - 2h(a+5) - A(\nu^2 - 2\nu) - B(1-2\nu) \\
&\quad - 2C + h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C \\
&= A, \\
y(a+4) &= -\nabla_{a+2}^{-\nu} h(a+4) \\
&\quad + \Big[h(a+3) + h(a+4) + h(a+5) \\
&\qquad + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\Big] H_{\nu-1}(a+4, a+2) \\
&\quad - \big[h(a+4) + 2h(a+5) \\
&\qquad + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\big] H_{\nu-2}(a+4, a+2) \\
&\quad + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\nu-3}(a+4, a+2),
\end{aligned}
$$

so that

$$
\begin{aligned}
y(a+4) &= -\sum_{s=a+3}^{a+4} H_{\nu-1}(a+4,\rho(s))h(s) \\
&\quad + \Bigg[h(a+3)+h(a+4)+h(a+5) \\
&\qquad\quad +A\left(1-\frac{3}{2}\nu+\frac{v^2}{2}\right)+B(1-\nu)+C\Bigg]\frac{\Gamma(\nu+1)}{\Gamma(\nu)} \\
&\quad - \left[h(a+4)+2h(a+5)+A(\nu^2-2\nu)+B(1-2\nu)+2C\right]\frac{\Gamma(\nu)}{\Gamma(\nu-1)} \\
&\quad + \left[h(a+5)+A\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)-B\nu+C\right]\frac{\Gamma(\nu-1)}{\Gamma(\nu-2)} \\
&= -H_{\nu-1}(a+4,a+2)h(a+3)-h(a+4) \\
&\quad + \Bigg[h(a+3)+h(a+4)+h(a+5) \\
&\qquad\quad +A\left(1-\frac{3}{2}\nu+\frac{v^2}{2}\right)+B(1-\nu)+C\Bigg]\frac{\Gamma(\nu+1)}{\Gamma(\nu)} \\
&\quad - \left[h(a+4)+2h(a+5)+A(\nu^2-2\nu)+B(1-2\nu)+2C\right]\frac{\Gamma(\nu)}{\Gamma(\nu-1)} \\
&\quad + \left[h(a+5)+A\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)-B\nu+C\right]\frac{\Gamma(\nu-1)}{\Gamma(\nu-2)} \\
&= -\nu h(a+3)-h(a+4) \\
&\quad + \Bigg[h(a+3)+h(a+4)+h(a+5) \\
&\qquad\quad +A\left(1-\frac{3}{2}\nu+\frac{v^2}{2}\right)+B(1-\nu)+C\Bigg]\frac{\Gamma(\nu+1)}{\Gamma(\nu)}(\nu) \\
&\quad - \left[h(a+4)+2h(a+5)+A(\nu^2-2\nu)+B(1-2\nu)+2C\right](\nu-1) \\
&\quad + \left[h(a+5)+A\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)-B\nu+C\right](\nu-2) \\
&= h(a+3)[-\nu+\nu]+h(a+4)[-1+\nu-\nu+1] \\
&\qquad + h(a+5)[\nu-2\nu+2+\nu-2] \\
&\qquad + A\left[\left(1-\frac{3}{2\nu}+\frac{\nu^2}{2}\right)\nu-(\nu-1)(\nu^2-2\nu)+(\nu-2)\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)\right] \\
&\qquad + B[\nu-\nu^2-(\nu-1-2\nu^2+2\nu)-\nu^2+2\nu]+C[\nu-2(\nu-1)+\nu-2] \\
&= B,
\end{aligned}
$$

and

$$
\begin{aligned}
y(a+5) = &-\nabla_{a+2}^{-\nu}h(a+5) \\
&+\Bigg[h(a+3)+h(a+4)+h(a+5) \\
&\qquad +A\left(1-\frac{3}{2}\nu+\frac{v^2}{2}\right)+B(1-\nu)+C\Bigg]H_{\nu-1}(a+5,a+2) \\
&-\big[h(a+4)+2h(a+5) \\
&\qquad +A(\nu^2-2\nu)+B(1-2\nu)+2C\big]H_{\nu-2}(a+5,a+2) \\
&+\left[h(a+5)+A\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)-B\nu+C\right]H_{\nu-3}(a+5,a+2) \\
= &-\sum_{s=a+3}^{a+5}H_{\nu-1}(a+5,\rho(s))h(s) \\
&+\Bigg[h(a+3)+h(a+4)+h(a+5) \\
&\qquad +A\left(1-\frac{3}{2}\nu+\frac{v^2}{2}\right)+B(1-\nu)+C\Bigg]\frac{\nu^2+\nu}{2} \\
&-\big[h(a+4)+2h(a+5)+A(\nu^2-2\nu)+B(1-2\nu)+2C\big]\frac{\nu^2-\nu}{2} \\
&+\left[h(a+5)+A\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)-B\nu+C\right]\frac{(\nu-1)(\nu-2)}{2} \\
= &\,h(a+3)\left[\frac{1}{2}\big(-(\nu^2+\nu)+(\nu^2+\nu)\big)\right]+h(a+4)\left[-\nu+\frac{\nu^2+\nu}{2}-\frac{\nu^2-\nu}{2}\right] \\
&+h(a+5)\left[-1+\frac{\nu^2+\nu}{2}-(\nu^2-\nu)+\frac{(\nu-1)(\nu-2)}{2}\right] \\
&+A\Bigg[\frac{1}{2}\Bigg(\left(1-\frac{3}{2}\nu+\frac{\nu^2}{2}\right)(\nu^2+\nu)-(\nu^2-2\nu)(\nu^2-\nu) \\
&\qquad +\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)(\nu-1)(\nu-2)\Bigg)\Bigg] \\
&+B\left[\frac{1}{2}\big((1-\nu)(\nu^2+\nu)-(1-2\nu)(\nu^2-\nu)+(\nu-1)(\nu-2)\big)\right] \\
&+C\left[\frac{1}{2}\big(\nu^2+\nu-2(\nu^2-\nu)+(\nu-1)(nu-2)\big)\right] \\
= &\,C,
\end{aligned}
$$

where we used the facts

$$H_{\nu-1}(a+5,a+2) = \frac{\Gamma(3+\nu-1)}{\Gamma(3)\Gamma(\nu)} = \frac{\Gamma(\nu+2)}{2\Gamma(\nu)} = \frac{(\nu+1)\nu}{2} = \frac{\nu^2+\nu}{2}$$

and

$$H_{\nu-1}(a+5,a+3) = \frac{\Gamma(2+\nu-1)}{\Gamma(2)\Gamma(\nu)} = \frac{\Gamma(\nu+1)}{\Gamma(\nu)} = \nu.$$

Now we find the form of the solution $y(t)$ by taking the Laplace transform based at $a+5$ of both sides:

$$\begin{aligned} -\mathcal{L}_{a+5}\{\nabla^{\nu}_{a+2}y\}(s) &= \mathcal{L}_{a+5}\{h\}(s) \\ \mathcal{L}_{a+5}\{\nabla^{\nu}_{a+2}y\}(s) &= -\mathcal{L}_{a+5}\{h\}(s). \end{aligned} \tag{5.24}$$

First considering the left hand side of (5.24), we use the Shifting Lemma 1.31, replacing a with $a+2$ and letting $n=3$ to obtain a form that is in terms of the Laplace transform based at $a+2$ in order to match the base of the operator:

$$\begin{aligned} \mathcal{L}_{a+5}\{\nabla^{\nu}_{a+2}y\}(s) &= \left(\frac{1}{1-s}\right)^3 \mathcal{L}_{a+2}\{\nabla^{\nu}_{a+2}y\}(s) - \sum_{k=1}^{3} \frac{\nabla^{\nu}_{a+2}y(a+2+k)}{(1-s)^{3-k+1}} \\ &= \left(\frac{1}{1-s}\right)^3 \mathcal{L}_{a+2}\{\nabla^{\nu}_{a+2}y\}(s) - \frac{\nabla^{\nu}_{a+2}y(a+3)}{(1-s)^3} - \frac{\nabla^{\nu}_{a+2}y(a+4)}{(1-s)^2} \\ &\quad - \frac{\nabla^{\nu}_{a+2}y(a+5)}{(1-s)} \\ &= \left(\frac{1}{1-s}\right)^3 \mathcal{L}_{a+2}\{\nabla^{\nu}_{a+2}y\}(s) - \frac{y(a+3)}{(1-s)^3} - \frac{(-\nu y(a+3)+y(a+4))}{(1-s)^2} \\ &\quad - \frac{\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)y(a+3) - \nu y(a+4) + y(a+5)}{(1-s)}, \end{aligned}$$

so that

$$\mathcal{L}_{a+5}\{\nabla^{\nu}_{a+2}y\}(s) = \frac{s^{\nu}}{(1-s)^3}\mathcal{L}_{a+2}\{y\}(s) - \frac{y(a+3)}{(1-s)^3} - \frac{(-\nu y(a+3)+y(a+4))}{(1-s)^2}$$
$$-\frac{\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)y(a+3)-\nu y(a+4)+y(a+5)}{(1-s)},$$

where we used Theorem 1.32 and the facts

$$\nabla^{\nu}_{a+2}y(a+3) = \sum_{s=a+3}^{a+3} H_{-\nu-1}(a+3,\rho(s))y(s) = y(a+3)$$

and

$$\nabla^{\nu}_{a+2}y(a+4) = \sum_{s=a+3}^{a+4} H_{-\nu-1}(a+4,\rho(s))y(s) = -\nu y(a+3)+y(a+4)$$

and

$$\nabla^{\nu}_{a+2}y(a+5) = \sum_{s=a+3}^{a+5} H_{-\nu-1}(a+5,\rho(s))y(s) = \left(\frac{\nu^2}{2}-\frac{\nu}{2}\right)y(a+3)-\nu y(a+4)+y(a+5).$$

Similarly, with the right hand side of (5.24), we will also shift so that our equation is in terms of the Laplace transform based at $a+2$, using the Shifting Lemma (1.31), replacing a with $a+2$ and letting $n=3$:

$$\begin{aligned}-\mathcal{L}_{a+5}\{h\}(s) &= -\left(\frac{1}{1-s}\right)^3\mathcal{L}_{a+2}\{h\}(s) + \sum_{k=1}^{3}\frac{h(a+2+k)}{(1-s)^{3-k+1}}\\ &= -\left(\frac{1}{1-s}\right)^3\mathcal{L}_{a+2}\{h\}(s) + \frac{h(a+3)}{(1-s)^3} + \frac{h(a+4)}{(1-s)^2} + \frac{h(a+5)}{(1-s)}.\end{aligned}$$

Thus when we multiply through by $(1-s)^3$, equation (5.24) becomes

$$\begin{aligned}
s^{\nu}\mathcal{L}_{a+2}\{y\}(s) &= -\mathcal{L}_{a+2}\{h\}(s) + h(a+3) + (1-s)h(a+4) + (1-s)^2h(a+5) \\
&\quad + y(a+3) + [-\nu y(a+3) + y(a+4)](1-s) \\
&\quad + \left[\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) y(a+3) - \nu y(a+4) + y(a+5)\right](1-s)^2 \\
&= -\mathcal{L}_{a+2}\{h\}(s) + h(a+3) + h(a+4) + h(a+5) \\
&\quad + \left(1 - \nu + \frac{\nu^2}{2} - \frac{\nu}{2}\right) y(a+3) + (1-\nu)y(a+4) + y(a+5) \\
&\quad + s\left[-h(a+4) - 2h(a+5) + \left(\nu - 2\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right)\right) y(a+3)\right. \\
&\qquad \left. + (-1+2\nu)y(a+4) - 2y(a+5)\right] \\
&\quad + s^2\left[h(a+5) + \left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) y(a+3) - \nu y(a+4) + y(a+5)\right] \\
&= -\mathcal{L}_{a+2}\{h\}(s) + h(a+3) + h(a+4) + h(a+5) \\
&\quad + A\left(1 - \frac{3}{2}\nu + \frac{\nu^2}{2}\right) + B(1-\nu) + C \\
&\quad - s\left[h(a+4) + 2h(a+5) + A\left(\nu^2 - \nu\right) + B(1-2\nu) + 2C\right] \\
&\quad + s^2\left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right].
\end{aligned}$$

Finally, dividing both sides by s^{ν} gives us that

$$\begin{aligned}
\mathcal{L}_{a+2}\{y\}(s) &= -\frac{1}{s^{\nu}}\mathcal{L}_{a+2}\{h\}(s) \\
&\quad + \frac{1}{s^{\nu}}\left[h(a+3) + h(a+4) + h(a+5)\right. \\
&\qquad \left. + A\left(1 - \frac{3}{2}\nu + \frac{\nu^2}{2}\right) + B(1-\nu) + C\right] \\
&\quad - \frac{1}{s^{\nu-1}}\left[h(a+4) + 2h(a+5) + A\left(\nu^2 - \nu\right) + B(1-2\nu) + 2C\right] \\
&\quad + \frac{1}{s^{\nu-2}}\left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right],
\end{aligned}$$

which implies that

$$
\begin{aligned}
y(t) = -\nabla_{a+2}^{-\nu}h(t) + \Bigg[& h(a+3) + h(a+4) + h(a+5) \\
& +A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\Bigg] H_{\nu-1}(t, a+2) \\
& - \left[h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\right] H_{\nu-2}(t, a+2) \\
& + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\nu-3}(t, a+2).
\end{aligned}
$$

We end by noting that this solution satisfies the fractional difference equation due to the convention that $H_{-k}(t,a) = 0$ for $k \in \mathbb{N}$ as follows.

$$
\begin{aligned}
-\nabla_{a+2}^{\nu} y(t) = -\nabla_{a+2}^{\nu}\Bigg[& -\nabla_{a+2}^{-\nu}h(t) + \Bigg[h(a+3) + h(a+4) + h(a+5) \\
& +A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) \\
& +B(1-\nu) + C] H_{\nu-1}(t, a+2) \\
& - [h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu) \\
& +B(1-2\nu) + 2C] H_{\nu-2}(t, a+2) \\
& + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\nu-3}(t, a+2)\Bigg] \\
= {} & \nabla_{a+2}^{\nu}\nabla_{a+2}^{-\nu}h(t) \\
& + \Bigg[h(a+3) + h(a+4) + h(a+5) \\
& +A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\Bigg] \nabla_{a+2}^{\nu} H_{\nu-1}(t, a+2) \\
& - [h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu) \\
& +B(1-2\nu) + 2C] \nabla_{a+2}^{\nu} H_{\nu-2}(t, a+2) \\
& + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] \nabla_{a+2}^{\nu} H_{\nu-3}(t, a+2),
\end{aligned}
$$

so that

$$
\begin{aligned}
-\nabla_{a+2}^{\nu} y(t) = h(t) + \Bigg[& h(a+3) + h(a+4) + h(a+5) \\
& +A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\Bigg] H_{-1}(t, a+2) \\
& - \left[h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu)\right. \\
& \quad \left. +B(1-2\nu) + 2C\right] H_{-2}(t, a+2) \\
& + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{-3}(t, a+2) = h(t),
\end{aligned}
$$

where we also used Lemma 2.2 (ii) and Theorem 1.24. This concludes our proof.

□

5.4.3 Green's Function for $\nabla_{a+2}^{\nu}\nabla_a^{\mu}x(t)$ Boundary Value Problem

In this section we will consider the nonhomogeneous nabla sequential difference boundary value problem

$$
\begin{cases}
-\nabla_{a+2}^{\nu}\nabla_a^{\mu}y(t) & = h(t), \quad t \in \mathbb{N}_{a+5}^{b} \\
y(a+3) & = 0 \\
y(a+4) & = 0 \\
y(b) & = 0
\end{cases}
\tag{5.25}
$$

and the corresponding homogeneous boundary value problem

$$
\begin{cases}
-\nabla_{a+2}^{\nu}\nabla_a^{\mu}y(t) & = 0, \quad t \in \mathbb{N}_{a+5}^{b} \\
y(a+3) & = 0 \\
y(a+4) & = 0 \\
y(b) & = 0
\end{cases}
\tag{5.26}
$$

for the particular set of order conditions $2 < \nu < 3$, $1 < \mu < 2$, and $3 < \mu + \nu < 4$, and $a, b \in \mathbb{R}$ with $b - (a+4) \in \mathbb{N}_1$.

Theorem 5.22. *Let $2 < \nu < 3$, $1 < \mu < 2$, and $3 < \mu + \nu < 4$, and further let $y, h : \mathbb{N}_{a+1} \to \mathbb{R}$. The fractional boundary value problem (5.25) where $a, b \in \mathbb{R}$ with $b - (a+4) \in \mathbb{N}_1$, has the unique solution*

$$y(t) = \sum_{s=a+5}^{t} h(s)G(t,s) = \int_{a+4}^{t} h(s)G(t,s)\nabla s$$

where

$$G(t,s) = \begin{cases} \frac{x(b,\rho(s))}{x(b-2,a+2)} x(t-2,a+2), & t \leq s-1 \\ \frac{x(b,\rho(s))}{x(b-2,a+2)} x(t-2,a+2) - x(t,\rho(s)), & t \geq s \end{cases}$$

and $x(t,\rho(s))$ is the Cauchy function (5.23).

Proof. Let $x(t) = \nabla_a^{\mu} y(t)$ where $1 < \mu < 2$, and let

$$A = x(a+3) = \nabla_a^{\mu} y(a+3),$$

$$B = x(a+4) = \nabla_a^{\mu} y(a+4),$$

$$C = x(a+5) = \nabla_a^{\mu} y(a+5).$$

Then $x(t)$ solves the initial value problem

$$\begin{cases} -\nabla_{a+2}^{\nu} x(t) &= h(t), \quad t \in \mathbb{N}_{a+5} \\ x(a+3) &= A \\ x(a+4) &= B \\ x(a+5) &= C. \end{cases}$$

Thus by the Variation of Constants Theorem 5.21,

$$
\begin{aligned}
x(t) = -\nabla_{a+2}^{-\nu} h(t) + \Big[& h(a+3) + h(a+4) + h(a+5) \\
& + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\Big] H_{\nu-1}(t, a+2) \\
& - \left[h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\right] H_{\nu-2}(t, a+2) \\
& + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\nu-3}(t, a+2).
\end{aligned}
$$

Composing both sides with the operator $\nabla_a^{-\mu}$ gives, since $\mu \notin \mathbb{N}$,

$$
\begin{aligned}
\nabla_a^{-\mu} x(t) = -\nabla_a^{-\mu}\nabla_{a+2}^{-\nu} h(t) + \Big[& h(a+3) + h(a+4) + h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) \\
& + B(1-\nu) + C\Big] \nabla_a^{-\mu} H_{\nu-1}(t, a+2) \\
& - \big[h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu) \\
& \quad + B(1-2\nu) + 2C\big] \nabla_a^{-\mu} H_{\nu-2}(t, a+2) \\
& + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] \nabla_a^{-\mu} H_{\nu-3}(t, a+2) \\
\nabla_a^{-\mu}\nabla_a^{\mu} y(t) = -\nabla_{a+2}^{-(\mu+\nu)} h(t) + \Big[& h(a+3) + h(a+4) + h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) \\
& + B(1-\nu) + C\Big] H_{\mu+\nu-1}(t, a+2) \\
& - \big[h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu) \\
& \quad + B(1-2\nu) + 2C\big] H_{\mu+\nu-2}(t, a+2) \\
& + \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\mu+\nu-3}(t, a+2).
\end{aligned}
$$

Hence

$$
\begin{aligned}
y(t) = &- \sum_{s=a+3}^{t} H_{\mu+\nu-1}(t,\rho(s))h(s) + H_{\mu+\nu-1}(t,a+2)h(a+3) \\
&+ \left[h(a+4) + h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right)\right. \\
&\qquad \left. + B(1-\nu) + C\right] H_{\mu+\nu-1}(t,a+2) \\
&- \left[h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu)\right. \\
&\qquad \left. +B(1-2\nu) + 2C\right] H_{\mu+\nu-2}(t,a+2) \\
&+ \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\mu+\nu-3}(t,a+2),
\end{aligned}
$$

where we used Theorem 2.4 and Remark 1.23 in the second step. Noticing that the second term above is equivalent to the $s = a+3$ term of the sum, our solution becomes

$$
\begin{aligned}
y(t) = &- \sum_{s=a+4}^{t} H_{\mu+\nu-1}(t,\rho(s))h(s) \\
&+ \left[h(a+4) + h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\right] H_{\mu+\nu-1}(t,a+2) \\
&- \left[h(a+4) + 2h(a+5) + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\right] H_{\mu+\nu-2}(t,a+2) \\
&+ \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\mu+\nu-3}(t,a+2) \\
= &- \sum_{s=a+4}^{t} H_{\mu+\nu-1}(t,\rho(s))h(s) + h(a+4)\left[H_{\mu+\nu-1}(t,a+2) - H_{\mu+\nu-2}(t,a+2)\right] \\
&+ \left[h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\right] H_{\mu+\nu-1}(t,a+2) \\
&- \left[2h(a+5) + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\right] H_{\mu+\nu-2}(t,a+2) \\
&+ \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\mu+\nu-3}(t,a+2).
\end{aligned}
$$

Hence

$$\begin{aligned}
y(t) = &- \sum_{s=a+5}^{t} H_{\mu+\nu-1}(t,\rho(s))h(s) \\
&+ \left[h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\right] H_{\mu+\nu-1}(t,a+2) \\
&- \left[2h(a+5) + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\right] H_{\mu+\nu-2}(t,a+2) \\
&+ \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] H_{\mu+\nu-3}(t,a+2) \\
= &- \sum_{s=a+5}^{t} x(t,\rho(s))h(s) \\
&+ \left[h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\right] x(t,a+2) \\
&- \left[2h(a+5) + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\right] \nabla x(t,a+2) \\
&+ \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] \nabla^2 x(t,a+2),
\end{aligned}$$

where we noticed in the first step that the second term is equivalent to the $s = a+4$ term of the sum. Continuing to simplify our solution, we have

$$\begin{aligned}
y(t) = &- \sum_{s=a+5}^{t} x(t,\rho(s))h(s) \\
&+ \left[h(a+5) + A\left(1 - \frac{3}{2}\nu + \frac{v^2}{2}\right) + B(1-\nu) + C\right] x(t,a+2) \\
&- \left[2h(a+5) + A(\nu^2 - 2\nu) + B(1-2\nu) + 2C\right] (x(t,a+2) - x(\rho(t),a+2)) \\
&+ \left[h(a+5) + A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right)\right. \\
&\qquad \left. - B\nu + C\right] (x(t,a+2) - 2x(t-1,a+2) + x(t-2,a+2)),
\end{aligned}$$

so that

$$
\begin{aligned}
y(t) = &-\sum_{s=a+5}^{t} x(t,\rho(s))h(s) \\
&+ \Bigg[h(a+5) + A\left(1-\frac{3}{2}\nu+\frac{v^2}{2}\right) + B(1-\nu) + C - 2h(a+5) - A(\nu^2-2\nu) \\
&\qquad -B(1-2\nu) - 2C + h(a+5) + A\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right) - B\nu + C\Bigg] x(t,a+2) \\
&+ \Bigg[2h(a+5) + A(\nu^2-2\nu) + B(1-2\nu) + 2C \\
&\qquad -2\left(h(a+5) + A\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right) - B\nu + C\right)\Bigg] x(\rho(t),a+2) \\
&+ \left[h(a+5) + A\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right) - B\nu + C\right] x(t-2,a+2) \\
y(t) = &-\sum_{s=a+5}^{t} x(t,\rho(s))h(s) + \left[-h(a+5) + A\left(1-\nu\right)\right] x(t,a+2) \\
&+ \left[2h(a+5) + B\right] x(\rho(t),a+2) + \left[A\left(\frac{\nu^2}{2}-\frac{\nu}{2}\right) - B\nu + C\right] x(t-2,a+2).
\end{aligned}
$$

In order to see that we had a term equivalent to the $s = a+4$ term of the sum earlier in this proof, we relied on the fact that

$$
\begin{aligned}
&H_{\mu+\nu-1}(t,a+2) - H_{\mu+\nu-2}(t,a+2) \\
&\qquad = \frac{\Gamma(\mu+\nu+t-a-3)}{\Gamma(\mu+\nu)\Gamma(t-a-2)} - \frac{\Gamma(\mu+\nu+t-a-4)}{\Gamma(\mu+\nu-1)\Gamma(t-a-2)} \\
&\qquad = \frac{\Gamma(\mu+\nu+t-a-4)}{\Gamma(\mu+\nu-1)\Gamma(t-a-2)} \left[\frac{\mu+\nu+t-a-4}{\mu+\nu-1} - 1\right] \\
&\qquad = \frac{\Gamma(\mu+\nu+t-a-4)}{\Gamma(\mu+\nu-1)\Gamma(t-a-2)} \left[\frac{t-a-3}{\mu+\nu-1}\right] \\
&\qquad = \frac{\Gamma(\mu+\nu+t-a-4)(t-a-3)}{(\mu+\nu-1)\Gamma(\mu+\nu-1)(t-a-3)\Gamma(t-a-3)} \\
&\qquad = \frac{\Gamma(\mu+\nu+t-a-4)}{\Gamma(\mu+\nu)\Gamma(t-a-3)}.
\end{aligned}
$$

Returning to the main proof, letting $t = a + 3$ gives

$$\begin{aligned}
0 = y(a+3) = &-\sum_{s=a+5}^{a+3} x(a+3, \rho(s))h(s) \\
&+ \left[-h(a+5) + A\left(1-\nu\right)\right] x(a+3, a+2) \\
&+ \left[2h(a+5) + B\right] x(\rho(a+3), a+2) \\
&+ \left[A\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] x((a+3)-2, a+2) \\
= &\; 0 + A(1-\nu) - h(a+5) \\
= &\; A(1-\nu) - h(a+5).
\end{aligned}$$

Thus we have found the value $A = \frac{h(a+5)}{1-\nu}$.

Our solution now becomes

$$\begin{aligned}
y(t) = &-\sum_{s=a+5}^{t} x(t, \rho(s))h(s) \\
&+ \left[-h(a+5) + \left(\frac{h(a+5)}{1-\nu}\right)(1-\nu)\right] x(t, a+2) \\
&+ \left[2h(a+5) + B\right] x(\rho(t), a+2) \\
&+ \left[\left(\frac{h(a+5)}{1-\nu}\right)\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] x(t-2, a+2) \\
= &-\sum_{s=a+5}^{t} x(t, \rho(s))h(s) \\
&\left[-h(a+5) + \left(\frac{h(a+5)}{1-\nu}\right)(1-\nu)\right] x(t, a+2) \\
&+ \left[2h(a+5) + B\right] x(\rho(t), a+2) \\
&+ \left[\left(\frac{h(a+5)}{1-\nu}\right)\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) - B\nu + C\right] x(t-2, a+2),
\end{aligned}$$

so that

$$
\begin{aligned}
y(t) = &- \sum_{s=a+5}^{t} x(t, \rho(s))h(s) \\
&+ [2h(a+5) + B]\, x(\rho(t), a+2) \\
&- \left[\left(\frac{\nu h(a+5)}{2}\right) + B\nu - C\right] x(t-2, a+2).
\end{aligned}
$$

Letting $t = a + 4$ gives

$$
\begin{aligned}
0 = y(a+4) = &- \sum_{s=a+5}^{a+4} x(t, \rho(s))h(s) \\
&+ [2h(a+5) + B]\, x(\rho(a+4), a+2) \\
&- \left[\left(\frac{\nu h(a+5)}{2}\right) + B\nu - C\right] x((a+4)-2, a+2) \\
= &\ 0 + 2h(a+5) + B \\
= &\ 2h(a+5) + B.
\end{aligned}
$$

Thus we see that $B = -2h(a+5)$ and so our solution becomes

$$
\begin{aligned}
y(t) &= - \sum_{s=a+5}^{t} x(t, \rho(s))h(s) + [2h(a+5) + (-2h(a+5))]\, x(\rho(t), a+2) \\
&\quad - \left[\left(\frac{\nu h(a+5)}{2}\right) + (-2h(a+5))\nu - C\right] x(t-2, a+2) \\
&= - \sum_{s=a+5}^{t} x(t, \rho(s))h(s) - \left[-\frac{3}{2}\nu h(a+5) - C\right] x(t-2, a+2).
\end{aligned}
$$

Now let $\alpha := -\frac{3}{2}\nu h(a+5) - C$. Letting $t = b$ gives

$$
0 = y(b) = - \sum_{s=a+5}^{b} x(b, \rho(s))h(s) - \left[-\frac{3}{2}\nu h(a+5) - C\right] x(b-2, a+2),
$$

and solving for α gives

$$\alpha = \frac{-\sum_{s=a+5}^{b} x(b,\rho(s))h(s)}{x(b-2,a+1)}.$$

Substituting this value for α into the formula for $y(t)$ gives us that

$$\begin{aligned}
y(t) &= -\sum_{s=a+5}^{t} x(t,\rho(s))h(s) + \frac{x(t-2,a+2)}{x(b-2,a+2)} \sum_{s=a+5}^{b} x(b,\rho(s))h(s) \\
&= -\sum_{s=a+5}^{t} x(t,\rho(s))h(s) + \frac{x(t-2,a+2)}{x(b-2,a+2)} \sum_{s=a+5}^{t} x(b,\rho(s))h(s) \\
&\quad + \frac{x(t-2,a+2)}{x(b-2,a+2)} \sum_{s=t+1}^{b} x(b,\rho(s))h(s) \\
&= \sum_{s=a+5}^{t} h(s) \left[\frac{x(b,\rho(s))}{x(b-2,a+2)} x(t-2,a+2) - x(t,\rho(s)) \right] \\
&\quad + \sum_{s=t+1}^{b} h(s) \frac{x(b,\rho(s))}{x(b-2,a+2)} x(t-2,a+2) \\
&= \sum_{s=a+5}^{b} h(s)G(t,s),
\end{aligned}$$

where

$$G(t,s) = \begin{cases} \frac{x(b,\rho(s))}{x(b-2,a+2)} x(t-2,a+2), & t \leq s-1 \\ \frac{x(b,\rho(s))}{x(b-2,a+2)} x(t-2,a+2) - x(t,\rho(s)), & t \geq s. \end{cases}$$

Therefore any solution to the boundary value problem (5.25) is given by the formula derived. Uniqueness of the solution $y(t)$ follows from Theorem 5.15.

□

5.4.4 Properties of the Green's Function

In this section we present some nice properties of the Green's function derived in the last section, namely the positivity of the Green's function and its maximum value.

Theorem 5.23. *The Green's function $G(t,s)$ for the boundary value problem (5.26) satisfies $G(t,s) \geq 0$ for $(t,s) \in \mathbb{N}_{a+4}^{b} \times \mathbb{N}_{a+5}^{b}$, and, specifically, $G(t,s) > 0$ for $(t,s) \in \mathbb{N}_{a+5}^{b-1} \times \mathbb{N}_{a+6}^{b-1}$.*

Proof. We will show for any fixed s that $G(t,s)$ increases from $G(a+2,s) = 0$ to a positive value at $t = s-1$ and then decreases to $G(b,s) = 0$. Let $s \in \mathbb{N}_{a+4}^{b}$ be fixed but arbitrary.

First, we show that $G(a+3,s) = G(a+4,s) = G(b,s) = 0$.

$$\begin{aligned} G(a+3,s) &= \frac{x(b,\rho(s))}{x(b-2,a+2)} x(a+3-2,a+2) \\ &= \frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(b-2,a+2)} H_{\mu+\nu-1}(a+1,a+2) \\ &= 0, \end{aligned}$$

$$\begin{aligned} G(a+4,s) &= \frac{x(b,\rho(s))}{x(b-2,a+2)} x(a+4-2,a+2) \\ &= \frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(b-2,a+2)} H_{\mu+\nu-1}(a+2,a+2) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} G(b,s) &= \frac{x(b,\rho(s))}{x(b-2,a+1)} x(b-2,a+2) - x(b,\rho(s)) \\ &= x(b,\rho(s)) - x(b,\rho(s)) \\ &= 0. \end{aligned}$$

Now we will show that for each fixed s, the Green's function $G(t,s)$ increases with respect to t for values of t between $a+5$ and $s-1$. To do so, we consider the nabla

difference with respect to t. For $t \geq a+5$,

$$\begin{aligned}
\nabla_t G(t,s) &= \nabla_t \left[\frac{x(b,\rho(s))}{x(b-2,a+2)} x(t-2,a+2) \right] \\
&= \frac{x(b,\rho(s))}{x(b-2,a+2)} \nabla_t x(t-2,a+2) \\
&= \frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(b-2,a+2)} \nabla_t H_{\mu+\nu-1}(t-2,a+2) \\
&= \frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(b-2,a+2)} H_{\mu+\nu-2}(t-2,a+2) \\
&= \frac{\Gamma(\mu+\nu+b-s)\Gamma(\mu+\nu)\Gamma(b-a-4)\Gamma(\mu+\nu+t-a-6)}{\Gamma(\mu+\nu)\Gamma(b-s+1)\Gamma(\mu+\nu+b-a-5)\Gamma(\mu+\nu-1)\Gamma(t-a-4)} \\
&= \frac{\Gamma(\mu+\nu+b-s)\Gamma(b-a-4)\Gamma(\mu+\nu+t-a-6)}{\Gamma(b-s+1)\Gamma(\mu+\nu+b-a-5)\Gamma(\mu+\nu-1)\Gamma(t-a-4)} > 0,
\end{aligned}$$

due to the following: recall by assumption that $1 < \nu < 2$, $0 < \mu < 1$, $1 < \mu+\nu < 2$, and $b-(a+2) \in \mathbb{N}_1$.

1. Since $s \leq b$ and $\mu, \nu > 0$, $\mu+\nu+b-s > 0$.

2. Since $b-(a+4) \in \mathbb{N}_1$, $b-a-4 > 0$.

3. Since $t \geq a+5$, $t-a-5 \geq 0$. Further, since $\nu > 2$, $\nu-1 > 0$. Combined with $\mu > 1 > 0$, this gives us $(t-a-5)+(\nu-1)+\mu = \mu+\nu+t-a-6 > 0$.

4. Since $s \leq b$, $b-s+1 > 0$.

5. Since $b-(a+4) \in \mathbb{N}_1$, $b-a-5 \geq 0$. Combined with $\mu, \nu > 0$, we have $\mu+\nu+b-a-5 > 0$.

6. Since $\nu > 2$, $\nu-1 > 0$. Combined with $\mu > 1 > 0$, we have $\mu+\nu-1 > 0$.

7. Since $t \geq a+5$, $t-a-4 > 0$.

Therefore $G(t,s)$ is increasing for all values of t between $a+5$ and $s-1$. Since the Green's function is zero at $t = a+3$ and $t = a+4$ and increases for t between $a+5$

and $s-1$, we have that $G(t,s)>0$ for $t\in\mathbb{N}_{a+5}^{s-1}$ and $G(t,s)\geq 0$ for $t\in\mathbb{N}_{a+3}^{s-1}$. We now show that $G(t,s)$ is decreasing for values of t between s and b. Similarly, we consider the nabla difference with respect to t. For $t\geq s$,

$$\begin{aligned}\nabla_t G(t,s) &= \nabla_t\left[\frac{x(b,\rho(s))}{x(b-2,a+2)}x(t-2,a+2)-x(t,\rho(s))\right]\\ &= \nabla_t\left[\frac{H_{\mu+\nu-1}(b,\rho(s))}{H_{\mu+\nu-1}(b-2,a+2)}H_{\mu+\nu-1}(t-2,a+2)-H_{\mu+\nu-1}(t,\rho(s))\right]\\ &= \nabla_t\left[\frac{H_{\mu+\nu-1}(b,s-1)}{H_{\mu+\nu-1}(b-2,a+2)}H_{\mu+\nu-1}(t-2,a+2)-H_{\mu+\nu-1}(t,s-1)\right]\\ &= \frac{H_{\mu+\nu-1}(b,s-1)}{H_{\mu+\nu-1}(b-2,a+2)}H_{\mu+\nu-1}(t-2,a+2)-H_{\mu+\nu-1}(t,s-1)\\ &\quad -\frac{H_{\mu+\nu-1}(b,s-1)}{H_{\mu+\nu-1}(b-2,a+2)}H_{\mu+\nu-1}(t-3,a+2)+H_{\mu+\nu-1}(t-1,s-1).\end{aligned}$$

Hence

$$\begin{aligned}\nabla_t G(t,s) = \beta\Big[&(b-s+1)^{\overline{\mu+\nu-1}}(t-a-3)^{\overline{\mu+\nu-1}}-(b-a-4)^{\overline{\mu+\nu-1}}(t-s+1)^{\overline{\mu+\nu-1}}\\ &-(b-s+1)^{\overline{\mu+\nu-1}}(t-a-4)^{\overline{\mu+\nu-1}}+(b-a-4)^{\overline{\mu+\nu-1}}(t-s)^{\overline{\mu+\nu-1}}\Big],\end{aligned} \tag{5.27}$$

where $\beta=\frac{1}{\Gamma(\mu+\nu)(b-a-4)^{\overline{\mu+\nu-1}}}>0$ since $\mu+\nu>1$ and $b-(a+4)\in\mathbb{N}_1$. We will show that the bracketed expression in (5.27) is less than or equal to zero. Rearranging and factoring, the claim that the bracketed expression is less than or equal to zero is equivalent to

$$\begin{aligned}&(b-s+1)^{\overline{\mu+\nu-1}}\left[(t-a-3)^{\overline{\mu+\nu-1}}-(t-a-4)^{\overline{\mu+\nu-1}}\right]\\ &\qquad\leq (b-a-2)^{\overline{\mu+\nu-1}}\left[(t-s+1)^{\overline{\mu+\nu-1}}-(t-s)^{\overline{\mu+\nu-1}}\right].\end{aligned}$$

Now since $s-1 \geq a+4$, we have $0 < b-(s-1) \leq b-(a+4)$, it remains to show

$$(t-a-3)^{\overline{\mu+\nu-1}} - (t-a-4)^{\overline{\mu+\nu-1}} \leq (t-s+1)^{\overline{\mu+\nu-1}} - (t-s)^{\overline{\mu+\nu-1}}. \tag{5.28}$$

Noting that

$$\begin{aligned}(t-s+1)^{\overline{\mu+\nu-1}} &= \frac{\Gamma(t-s+1+\mu+\nu-1)}{\Gamma(t-s+1)}\\ &= \frac{(t-s+\mu+\nu-1)\Gamma(t-s+\mu+\nu-1)}{(t-s)\Gamma(t-s)}\\ &= \frac{t-s+\mu+\nu-1}{t-s}(t-s)^{\overline{\mu+\nu-1}}\end{aligned}$$

and

$$\begin{aligned}(t-a-3)^{\overline{\mu+\nu-1}} &= \frac{\Gamma(t-a-3+\mu+\nu-1)}{\Gamma(t-a-3)}\\ &= \frac{(t-a-4+\mu+\nu-1)\Gamma(t-a-4+\mu+\nu-1)}{(t-a-4)\Gamma(t-a-4)}\\ &= \frac{t-a-4+\mu+\nu-1}{t-a-4}(t-a-4)^{\overline{\mu+\nu-1}},\end{aligned}$$

we have that (5.28) becomes

$$\begin{aligned}(t-a-4)^{\overline{\mu+\nu-1}}\left[\frac{t-a-4+\mu+\nu-1}{t-a-4}-1\right] &\leq (t-s)^{\overline{\mu+\nu-1}}\left[\frac{t-s+\mu+\nu-1}{t-s}-1\right]\\ (t-a-4)^{\overline{\mu+\nu-1}}\left[\frac{\mu+\nu-1}{t-a-4}\right] &\leq (t-s)^{\overline{\mu+\nu-1}}\left[\frac{\mu+\nu-1}{t-s}\right].\end{aligned}$$

Since $3 < \mu+\nu < 4$ by assumption, $\mu+\nu-1 > 0$. Thus it follows that

$$\frac{(t-a-4)^{\overline{\mu+\nu-1}}}{t-a-4} \leq \frac{(t-s)^{\overline{\mu+\nu-1}}}{t-s}, \tag{5.29}$$

since $s-1 \geq a+4$, $s > a+4$. Thus we can show the inequality above holds by

showing that the expression $\frac{(t-\tau)^{\overline{\mu+\nu-1}}}{t-\tau}$ is increasing in τ. To do so, we will consider the nabla difference with respect to τ. Using the quotient and power rules, we have

$$\begin{aligned}
\nabla_\tau \frac{(t-\tau)^{\overline{\mu+\nu-1}}}{t-\tau} &= \frac{-(\mu+\nu-1)(t-\rho(\tau))^{\overline{\mu+\nu-2}}(t-\tau)-(t-\tau)^{\overline{\mu+\nu-1}}(-1)}{(t-\tau)(t-\rho(\tau))} \\
&= \frac{-(\mu+\nu-1)(t-\tau)^{\overline{\mu+\nu-1}}+(t-\tau)^{\overline{\mu+\nu-1}}}{(t-\tau)(t-\tau+1)} \\
&= \frac{(t-\tau)^{\overline{\mu+\nu-1}}(2-\mu-\nu)}{(t-\tau)(t-\tau+1)}, \qquad (5.30)
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
(t-\rho(\tau))^{\overline{\mu+\nu-2}} &= (t-\tau+1)^{\overline{\mu+\nu-2}} \\
&= \frac{\Gamma(\mu+\nu+t-\tau-1)}{\Gamma(t-\tau+1)} \\
&= \frac{\Gamma(\mu+\nu+t-\tau-1)}{(t-\tau)\Gamma(t-\tau)} \\
&= \frac{(t-\tau)^{\overline{\mu+\nu-1}}}{t-\tau}.
\end{aligned}$$

Now since $t-\tau \geq 0$ and $2-\mu-\nu > 0$, we have that (5.28) is greater than or equal to zero, and thus the expression $\frac{(t-\tau)^{\overline{\mu+\nu-1}}}{t-\tau}$ is increasing with respect to τ. This in turn implies that inequality (5.29) holds, and therefore the bracketed expression in (5.15) is less than or equal to zero. This tells us that the Green's function $G(t,s)$ is decreasing for values of t between s and b. Since the Green's function is zero at $t=b$ and is decreasing for values of t between s and b, this implies $G(t,s) > 0$ for $t \in \mathbb{N}_s^{b-1}$ and $G(t,s) \geq 0$ for $t \in \mathbb{N}_s^b$.

Thus we have shown that $G(t,s)$ is strictly positive for $t \in \mathbb{N}_{a+5}^{b-1}$ and $G(t,s) \geq 0$ for $t \in \mathbb{N}_{a+3}^b$. This concludes our proof.

$\square$

Now using both Theorem 5.22 and Theorem 5.23, we get the following useful result as a corollary.

Corollary 5.24. *Assume for* $2 < \nu < 3$, $1 < \mu < 2$, *and* $3 < \mu + \nu < 4$ *that* $u(t)$ *and* $v(t)$ *satisfy*

$$\begin{aligned} \nabla_{a+2}^{\nu}\nabla_a^{\mu}u(t) &\geq \nabla_{a+2}^{\nu}\nabla_a^{\mu}v(t), \quad t \in \mathbb{N}_{a+5}^{b} \\ u(a+3) &= v(a+3) \\ u(a+4) &= v(a+4) \\ u(b) &= v(b). \end{aligned}$$

Then $u(t) \leq v(t)$ *on* $\mathbb{N}_{a+3}^{b}$.

Proof. Set $w(t) = u(t) - v(t)$ and let

$$h(t) := -\nabla_{a+2}^{\nu}\nabla_a^{\mu}w(t) = -\nabla_{a+2}^{\nu}\nabla_a^{\mu}u(t) + \nabla_{a+2}^{\nu}\nabla_a^{\mu}v(t) \leq 0, \quad t \in \mathbb{N}_{a+5}^{b}.$$

Thus it follows that $w(t)$ solves the boundary value problem

$$\begin{cases} -\nabla_{a+2}^{\nu}\nabla_a^{\mu}w(t) &= h(t), \quad t \in \mathbb{N}_{a+5}^{b} \\ w(a+3) &= 0 \\ w(a+4) &= 0 \\ w(b) &= 0. \end{cases}$$

By Theorem 5.22, the solution of this boundary value problem is given by

$$w(t) = \int_{a+4}^{b} G(t,s)h(s)\nabla s, \quad t \in \mathbb{N}_{a+3}^{b},$$

where $G(t,s) \geq 0$ by Theorem 5.23 and $h(s) \leq 0$. Therefore $w(t) \leq 0$, implying $u(t) \leq v(t)$ for all $t \in \mathbb{N}_{a+3}^{b}$. This completes our proof.

□

We have shown in Theorem 5.23 that for any fixed value of s, $G(t,s) > 0$ for all values of $t \in \mathbb{N}_{a+5}^{b-1}$ and $G(t,s) \geq 0$ for all values of $t \in \mathbb{N}_{a+3}^{b}$. Note that though we have that $G(t,s)$ is increasing up to $t = s-1$ and decreasing from $t = s$, it is yet uncertain whether the maximum of the Green's function $G(t,s)$ occurs at $t = s-1$ or $t = s$.

5.5 The Generalized Sequential Nabla Difference $\nabla_{a+k+1}^{\nu}\nabla_a^{\mu}x(t)$

In this section we consider the sequential nabla difference equation

$$\nabla_{a+k+1}^{\nu}\nabla_a^{\mu}x(t) = f(t).$$

We consider the case where for $k \in \mathbb{N}_0$, the orders satisfy $k < \mu < k+1$, $k+1 < \nu < k+2$, and $2k+1 < \mu+\nu < 2k+2$, and $x : \mathbb{N}_{a+1} \to \mathbb{R}$. Recall that this is the same form we considered in Theorem 3.4 (i) in Chapter 3.

We will prove an existence and uniqueness theorem for the above difference equation. We note that as we increase k, the more initial conditions we require.

Theorem 5.25 (Generalized Existence & Uniqueness Theorem). *Suppose $k \in \mathbb{N}_0$, $k < \mu < k+1$, $k+1 < \nu < k+2$, $2k+1 < \mu+\nu < 2k+2$, and $x : \mathbb{N}_{a+1} \to \mathbb{R}$. The fractional initial value problem*

$$\begin{cases} -\nabla_{a+k+1}^{\nu}\nabla_a^{\mu}x(t) = f(t), & t \in \mathbb{N}_{a+2k+3} \\ x(a+j) = c_j, & \textit{for } j = 1,2,\ldots,k+2 \end{cases} \tag{5.31}$$

has a unique solution $x(t)$.

Proof. We begin by expanding the sequential difference equation in (5.31). By Theorem 3.4 (i),

$$
\begin{aligned}
-\nabla_{a+k+1}^{\nu}\nabla_a^{\mu}x(t) &= -\Bigg(\nabla_a^{\mu+\nu}x(t) - \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right] \\
&\qquad - \left[\sum_{j=0}^{k}\nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(t,a+k+1)\right]\Bigg) \\
&= -\nabla_a^{\mu+\nu}x(t) + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right] \\
&\qquad + \left[\sum_{j=0}^{k}\nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(t,a+k+1)\right].
\end{aligned}
$$

Letting $t = a+2k+3$, we have

$$
\begin{aligned}
f(a+2k+3) &= -\nabla_a^{\mu+\nu}x(t) \\
&\qquad + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(a+2k+3,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right] \\
&\qquad + \left[\sum_{j=0}^{k}\nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(a+2k+3,a+k+1)\right] \\
&= -\sum_{s=a+1}^{a+2k+3} H_{-\mu-\nu-1}(a+2k+3,\rho(s))y(s) \\
&\qquad + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(a+2k+3,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right] \\
&\qquad + \left[\sum_{j=0}^{k}\nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(a+2k+3,a+k+1)\right].
\end{aligned}
$$

Hence

$$\begin{aligned}
-\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}x(t) &= -H_{-\mu-\nu-1}(a+2k+3,a+2k+2)x(a+2k+3)\\
&\quad - \sum_{s=a+1}^{a+2k+2} H_{-\mu-\nu-1}(a+2k+3,\rho(s))y(s)\\
&\quad + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(a+2k+3,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right]\\
&\quad + \left[\sum_{j=0}^{k} \nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(a+2k+3,a+k+1)\right]\\
&= -x(a+2k+3) - \sum_{s=a+1}^{a+2k+2} H_{-\mu-\nu-1}(a+2k+3,\rho(s))y(s)\\
&\quad + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(a+2k+3,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right]\\
&\quad + \left[\sum_{j=0}^{k} \nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(a+2k+3,a+k+1)\right].
\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}
x(a+2k+3) &= -f(a+2k+3) - \sum_{s=a+1}^{a+2k+2} H_{-\mu-\nu-1}(a+2k+3,\rho(s))y(s)\\
&\quad + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(a+2k+3,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right]\\
&\quad + \left[\sum_{j=0}^{k} \nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(a+2k+3,a+k+1)\right]
\end{aligned}$$

Thus the value of $x(a+2k+3)$ is uniquely determined by the initial conditions $x(a+j)=c_j$ for $j=1,2,\ldots,k+2$ and the value of the given function $f(t)$.

To show $x(t)$ is uniquely determined on $\mathbb{N}_{a+1}$ we will proceed by induction. Suppose for $t\in\mathbb{N}_{a+1}^{t_0}$ there exists a unique solution $x(t)$ to the initial value problem (5.31), where $t_0\in\mathbb{N}_{a+2k+2}$.

We will show that the value of $x(t_0+1)$ is uniquely determined by the values of $x(t)$ on $\mathbb{N}_{a+1}^{t_0}$.

$$\begin{aligned}
f(t_0+1) &= -\nabla_a^{\mu+\nu}x(t) + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t_0+1,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right] \\
&\quad + \left[\sum_{j=0}^{k} \nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(t_0+1,a+k+1)\right] \\
&= -\sum_{s=a+1}^{t_0+1} H_{-\mu-\nu-1}(t_0+1,\rho(s))y(s) \\
&\quad + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t_0+1,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right] \\
&\quad + \left[\sum_{j=0}^{k} \nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(t_0+1,a+k+1)\right] \\
&= -H_{-\mu-\nu-1}(t_0+1,t_0)x(t_0+1) \\
&\quad - \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))y(s) \\
&\quad + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t_0+1,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right] \\
&\quad + \left[\sum_{j=0}^{k} \nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(t_0+1,a+k+1)\right] \\
&= -x(t_0+1) - \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))y(s) \\
&\quad + \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t_0+1,\rho(\tau))\sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right] \\
&\quad + \left[\sum_{j=0}^{k} \nabla_a^{\mu+j-k-1}x(a+k+1)H_{-\nu+j-k-1}(t_0+1,a+k+1)\right].
\end{aligned}$$

Rearranging the terms provides

$$\begin{aligned}x(t_0+1) = -f(t_0+1) &- \sum_{s=a+1}^{t_0} H_{-\mu-\nu-1}(t_0+1,\rho(s))y(s) \\ &+ \nabla\left[\sum_{\tau=a+1}^{a+k+1} H_{-\nu-k-1}(t_0+1,\rho(\tau)) \sum_{s=a+1}^{\tau} H_{k-\mu}(\tau,\rho(s))x(s)\right] \\ &+ \left[\sum_{j=0}^{k} \nabla_a^{\mu+j-k-1} x(a+k+1) H_{-\nu+j-k-1}(t_0+1,a+k+1)\right].\end{aligned}$$

Now by the induction hypothesis, all the values of $x(t)$ for $t \in \mathbb{N}_{a+1}^{t_0}$ are known. Therefore $x(t_0+1)$ is uniquely determined and hence $x(t)$ is the unique solution of the fractional initial value problem on $\mathbb{N}_{a+1}^{t_0+1}$. Thus a unique solution exists on $\mathbb{N}_{a+1}$.

□

5.6 Further Work

For future directions, one may continue to explore various sequential nabla difference initial value problems and, perhaps of most interest, develop more properties of the Green's functions of both the $\nabla_{a+1}^{\nu}\nabla_a^{\mu}x(t)$ sequential difference boundary value problem and the $\nabla_{a+2}^{\nu}\nabla_a^{\mu}x(t)$ sequential difference boundary value problem. This is largely due to the fact that Green's functions, once found, can aid in solving numerous nonhomogeneous versions of difference equations, and we can use their properties to establish behavior of solutions. We note that this thesis focused predominantly on the case where the orders of the sequential difference satisfied case (i) of Theorem 3.4, namely for $k \in \mathbb{N}_0$, order conditions of the form $k < \mu < k+1$, $k+1 < \nu < k+2$, and $2k+1 < \mu+\nu < 2k+2$. To that effect, one could develop similar results to those of this chapter in case (ii) of Theorem 3.4, where the orders satisfy the conditions $k < \mu < k+1$, $k-1 < \nu < k$, and $2k-1 < \mu+\nu < 2k$ for $k \in \mathbb{N}_1$.

Paralleling the work in Sections 5.3 and 5.4, we could expand Section 5.5 by obtaining results for the generalized sequential nabla difference $\nabla^{\nu}_{a+k+1}\nabla^{\mu}_{a}x(t)$ with the set of order conditions $k<\mu<k+1$, $k+1<\nu<k+2$, and $2k+1<\mu+\nu<2k+2$, meaning the establishment of a Green's function and corresponding properties that are valid for any k on the appropriate domain. The domain is the most notable element when generalizing, since each time we increase our value k, we require more initial conditions and boundary conditions to achieve results. However, in noticing the similarities amongst theorems in both Sections 5.3 and 5.4, we conjecture that nice generalizations can be produced. Indeed, we conjecture that with the appropriate boundary conditions, we can generalize the Green's function in this case to be

$$G_k(t,s)=\begin{cases}\frac{x(b,\rho(s))}{x(b-(k+1),a+(k+1))}x(t-(k+1),a+(k+1)), & t\leq\rho(s)\\ \frac{x(b,\rho(s))}{x(b-(k+1),a+(k+1))}x(t-(k+1),a+(k+1)-x(t,\rho(s)), & t\geq s\end{cases}$$

As was mentioned above, we could also examine the general form for the set of order conditions in case (ii) of Theorem 3.4 and establish existence and uniqueness of solutions, a variation of constants formula, and a Green's function and corresponding properties.

5.7 Conclusion

This work has contributed to filling a gap in the current field of discrete fractional calculus - presenting much-needed composition rules for the foundation of the theory, before moving on to focus on the concept of sequential nabla fractional differences. We first saw, with the proper conditions, that we can use the sign of the sequential difference of a function to confirm something about the monotonicity of the function. The bulk of our contribution to the current theory was with the study of sequen-

tial nabla difference boundary value problems - establishing Green's functions and properties which are useful for discovering characteristics of solutions. Finally, we conjectured a generalized form of a Green's function for the k case of the boundary value problem given appropriate conditions. Due to the unique composition of sequential nabla fractional differences, and thus the requirements on domain that they present, we see promise in their use of modeling physical problems.

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