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Approximation Via Degree Reduction of Nonlinearities with Applications to Turbulent Flows, Flame Fronts, and Magnetohydrodynamics

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APPROXIMATION VIA DEGREE REDUCTION OF NONLINEARITIES WITH APPLICATIONS TO TURBULENT FLOWS, FLAME FRONTS, AND MAGNETOHYDRODYNAMICS

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APPROXIMATION VIA DEGREE REDUCTION OF NONLINEARITIES WITH APPLICATIONS TO TURBULENT FLOWS, FLAME FRONTS, AND MAGNETOHYDRODYNAMICS

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We perform an analytical and computational investigation on the effectiveness of a locally bounded truncation function, which we call a calming function, when applied to the nonlinear terms of several dissipative partial differential equations. In particular, the 3D Navier-Stokes equations of incompressible fluid flow, the 2D Kuramoto-Sivashinsky equations of laminar flame fronts, and the 2D MHD-Boussinesq equations of magnetohydrodynamics. Each of these equations have open questions about the global existence and uniqueness of their solutions. These calming functions effectively reduce the algebraic degree of select nonlinear terms, thus one can verify global wellposedness for these "calmed systems". More specifically, in this work we show analytically in this work that the solutions to the calmed systems are globally well-posed, have higher-order regularity, and converge to solutions of the original models on short-time intervals as an introduced parameter in the calmed system tends to 0. We obtain additional results in the case of the 3D Calmed Navier-Stokes equations: when applying calming to the nonlinear term written in its rotational form, we find that the dynamical system generated by the calmed NSE in the rotational form possesses both an energy identity and a global attractor. Moreover, for calmed Navier-Stokes written either in its advective form or rotational form, we show that strong solutions to the calmed equations converge to strong solutions of the NSE

without assuming their existence, providing a new proof of the short-time existence of strong solutions to the 3D Navier-Stokes equations.

DEDICATION

Dedicated to my loving parents.

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Chapter 1

Introduction

1.1 Motivation

A major obstacle in the mathematical analysis and computational modeling of fluid dynamics is the rapid intensification of small length scales. This is due to the advective nonlinear term in the governing equations for turbulent fluids, the 3D Navier-Stokes equations. Many approaches have focused on mitigating this growth by introducing stronger diffusion, or by mollifying or filtering the nonlinear term. These strategies essentially involve some form of smoothing. But is smoothing the only method to control the Navier-Stokes equations? Derivatives can also grow via another mechanism: multiplication, which can lead to the generation of smaller length scales[1]. In this work, we introduce a novel modification to the incompressible Navier-Stokes equations (NSE), and other dissipative partial differential equations, that tempers the effect of the algebraic multiplication without introducing a smoothing operator. Specifically, we limit the advective velocity by smoothly truncating it, a process we call "algebraic calming" or simply "calming," since it effectively reduces the algebraic degree of the nonlinearity.

[1] For example, consider $g(x) = \sin(x)+\cos(x)$. It is straightforward to show that $\|\frac{d^n}{dx^n} g\|_{L^\infty} = \sqrt{2}$ for all $n \in \mathbb{N}$, but $\|\frac{d}{dx} g^n\|_{L^\infty} \geq n$, and hence $\frac{d}{dx} g^n$ grows without bound as $n \to \infty$.

Calming has several advantages over smoothing; namely:

- There is no need to modify the boundary conditions, the system is globally well-posed, in both 2D and 3D, with standard homogeneous Dirichlet (i.e., "no-slip") or periodic boundary conditions.
- The calmed system is of the same derivative order as the original system, as there are no modifications to the derivatives introduced.
- The "calming" modification is an entirely local operation, which may be more efficient than, e.g., mollification or filtering in computational settings, especially in the setting of parallel processing. (There is also no auxilliary equation to handle, such as in the case of the $k-\epsilon$ or $k-\omega$ models.)

In Chapter 3 we prove that the calmed NSE are globally well-posed in 3D with no-slip (i.e., physical) boundary conditions, and that their solutions converge, as the calming parameter $\epsilon \to 0^+$, to strong solutions of the Navier-Stokes equations on the time interval of existence and uniqueness of the latter. In addition to this, we show that there is no need to assume the existence of strong solutions to the Navier-Stokes equations *a priori*. In particular, via calming, we provide a new independent proof of the existence of strong solutions to the 3D Navier-Stokes equations. These same results also hold for the "calmed rotational Navier-Stokes equations" (calmed rNSE) in which the nonlinearity is first written as $\boldsymbol{\omega} \times \mathbf{u}$ and then calming is applied. For calmed rNSE we prove that under an additional assumption on the calming function, the resulting system satisfies exactly the same energy equality as for strong solutions to the NSE, in addition to enjoying the aforementioned properties of the calmed NSE. We then use this energy equality to prove that the calmed rNSE has a compact global attractor.

We obtain similar results in Chapter 4 for a velocity-vorticity formulation of the NSE with calming applied to the vortex stretching term, and furthermore we obtain an energy inequality for this calmed system. In Chapter 5 we study the MHD-Boussinesq system with a calmed Ohmic heating term, a multi-physics model for turbulent fluids with dynamic temperature and magnetic field. Despite the Ohmic heating term providing a destabilizing effect to the dynamics of the temperature, once the calming mechanism is applied one is able to prove global wellposedness for strong solutions, in addition to being able to demonstrate the short-time convergence of the calmed system to the original system. In Chapter 6 we examine the 2D Kuramoto-Sivashinsky equations (KSE), another equation whose global well-posedness is unknown and which acts as a model for flame fronts and other reaction-diffusion systems. We were able to show that calmed KSE is in fact globally well-posed for weak solutions and showed the convergence of solutions of this calmed system to solutions of KSE on short time intervals. Moreover, we provided computational results displaying the similarities in the dynamics between the two systems and supplied quantitative evidence that the convergence rates obtained section 3.4 were sharp. We then extended our results to work for a calmed version of scalar-valued KSE.

1.2 Main Results

1.2.1 The calmed Navier-Stokes Equations

We begin by defining what we mean by weak and strong solutions.

Definition 1.2.1 (Weak solution)**.** Let $T > 0$, $\mathbf{u}_0 \in H$ and let $\mathbf{f} \in L^2(0, T; V')$. We say that $\mathbf{u}$ is a *weak solution* to calmed NSE (3.1.4) or calmed rNSE (3.1.5) on the interval $[0,T]$ if $\mathbf{u}$ satisfies equation (3.2.1a) for all $\mathbf{v} \in V$ in the sense of $L^2((0,T))$ with $\mathbf{u} \in C([0,T];H)$ and $\partial_t \mathbf{u} \in L^2(0,T;V')$. Furthermore, we require (3.2.1b) to be satisfied in the sense of $C([0,T];H)$.

Definition 1.2.2 (Strong solution)**.** Let $T > 0$, $\mathbf{u}_0 \in V$, and let $\mathbf{f} \in L^2(0,T;H)$. We say that $\mathbf{u}$ is a *strong solution* to calmed NSE (3.1.4) or calmed rNSE (3.1.5) on the interval $[0,T]$ if $\mathbf{u}$ is a weak solution and also $\mathbf{u} \in C([0,T];V) \cap L^2(0,T;H^2 \cap V)$ with time derivative $\partial_t \mathbf{u} \in L^2(0,T;H)$ and initial data satisfied in the sense of $C([0,T];V)$.

We now state our results on the global well-posedness of solutions to calmed Navier-Stokes and calmed rotational Navier-Stokes.

Theorem 1.2.3 (Global existence of weak solutions to calmed systems)**.** *Let* $\mathbf{u}_0 \in H$*,* $T > 0$*, and let* $\mathbf{f} \in L^2(0,T;V')$ *be given. Suppose, for* $\epsilon > 0$*,* $\boldsymbol{\zeta}^\epsilon$ *is a calming function which satisfies conditions 1, 2, and 3 of Definition 2.1.1. Then weak solutions to calmed NSE or calmed rNSE* (3.2.1) *exist on* $[0,T]$*.*

Theorem 1.2.4 (First-order regularity of calmed systems)**.** *Let* $T > 0$*. Suppose that* $\mathbf{u}_0 \in V$ *and that* $\mathbf{f} \in L^2(0,T;H)$*. Consider a weak solution* $\mathbf{u}$ *to calmed NSE or calmed rNSE* (3.2.1) *on the interval* $[0,T]$*. Then* $\mathbf{u} \in C([0,T];V) \cap L^2(0,T;H^2 \cap V)$ *and* $\partial_t \mathbf{u} \in L^2(0,T;H)$*.*

Theorem 1.2.5 (Global well-posedness of strong solutions to calmed systems)**.** *Let* $T > 0$*,* $\mathbf{u}_0, \in V$*, and let* $\mathbf{f} \in L^2(0,T;H)$*. Suppose* $\boldsymbol{\zeta}^\epsilon$ *is a calming function which satisfies conditions 1, 2, and 3 of Definition 2.1.1. Then there exists a strong solution* $\mathbf{u} \in C([0,T];V) \cap L^2(0,T;H^2 \cap V)$ *to calmed NSE* (3.1.4) *and calmed rNSE* (3.1.5) *which depends continuously on its initial data and is unique in the class of weak solutions.*

For our calmed systems we also have the convergence of (3.1.4) (resp. (3.1.5)) to (3.1.1) (resp. (3.1.3)) on short time intervals.

Theorem 1.2.6 (Convergence)**.** *Let $T > 0$, and let $\boldsymbol{\zeta}^\epsilon$ be a calming function satisfying conditions 1, 2, and 3 of Definition 2.1.1, where $\beta \geq 1$ is the minimal value for which 3 holds. Suppose*

$$\mathbf{u} \in C([0,T];V) \cap L^{2\beta}(0,T;H^2 \cap V) \tag{1.2.1}$$

is a strong solution to the 3D Navier-Stokes equation written either in its velocity form (3.1.1) *or rotational form* (3.1.3) *with initial data $\mathbf{u}_0 \in V$ and forcing term $\mathbf{f} \in L^2(0,T;H)$. Suppose $\mathbf{u}^\epsilon \in C([0,T];V) \cap L^2(0,T;H^2 \cap V)$ is a solution to the 3D calmed NSE* (3.1.4) *(resp. 3D calmed rNSE* (3.1.5)*) with the same initial data $\mathbf{u}_0$ and forcing term $\mathbf{f}$. Then*

$$\|\mathbf{u} - \mathbf{u}^\epsilon\|_{L^\infty V} \leq K\epsilon^\alpha, \tag{1.2.2}$$

where $K > 0$ is a constant depending only on $\Omega, \nu, \beta, \|\mathbf{u}\|_{L^\infty V}, \|\triangle \mathbf{u}\|_{L^2}, T$, and α, β are determined by the choice of $\boldsymbol{\zeta}^\epsilon$ and are as given by condition 3 of Definition 2.1.1.

In section 3.5 we show that in fact, strong solutions to the calmed systems are Cauchy with respect to the calming parameter $\epsilon > 0$ and that the limit point obtained from this sequence is itself a strong solution to 3D Navier-Stokes.

Theorem 1.2.7 (Existence of Strong Solutions to Navier-Stokes)**.** *For each $\epsilon > 0$, let $\boldsymbol{\zeta}^\epsilon$ be a calming function satisfying conditions 1, 2, and 3 of Definition 2.1.1 and let $\mathbf{u}^\epsilon$ be a strong solution to calmed NSE* (3.1.4) *or calmed rNSE* (3.1.5) *with initial data $\mathbf{u}_0 \in V$ and forcing term $\mathbf{f} \in L^\infty(0,\infty;L^2)$. Suppose $T > 0$ is the maximal time* *for which*

$$\sup_{\epsilon>0}\sup_{0\le t\le T}\|\nabla\mathbf{u}^\epsilon(t)\|_{L^2}\le\sqrt{2}\,\|\nabla\mathbf{u}_0\|_{L^2}$$

is valid. Then,

1. *The sequence* $\{\mathbf{u}^\epsilon\}_{\epsilon>0}$ *is Cauchy in* $L^\infty H\cap L^2V$.

2. *The limit point of the sequence,* $\mathbf{u}$, *is a strong solution to the* 3*D Navier-Stokes equations* (3.1.1) *or* (3.1.3) *on the interval* $[0,T]$.

While calmed NSE and calmed rNSE share the same properties of global well-posedness, we can see a key distinction between the two in the next theorem when $\boldsymbol{\zeta}^\epsilon$ is assumed to be pointwise parallel.

Theorem 1.2.8 (Energy identity for weak solutions of calmed rNSE (3.1.5))**.** *Let* $\nu>0$, $\epsilon>0$, $\mathbf{u}_0\in H$, *and* $\mathbf{f}\in L^2(0,T;V')$ *be given. Suppose* $\boldsymbol{\zeta}^\epsilon$ *satisfies conditions 1, 2, 3, and 4 of Definition 2.1.1. Let* $\mathbf{u}^\epsilon$ *be a weak solution to calmed rNSE* (3.1.5). *Then* $\mathbf{u}^\epsilon$ *the satisfies the energy equalities*

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^\epsilon\|_{L^2}^2+\nu\|\nabla\mathbf{u}^\epsilon\|_{L^2}^2=\langle\mathbf{f},\mathbf{u}^\epsilon\rangle\,. \tag{1.2.3}$$

and

$$\|\mathbf{u}^\epsilon(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla\mathbf{u}^\epsilon(s)\|_{L^2}^2\,ds=\|\mathbf{u}_0\|_{L^2}^2+2\int_0^t\langle\mathbf{f}(s),\mathbf{u}^\epsilon(s)\rangle\,ds. \tag{1.2.4}$$

Remark 1.2.9. Combining Theorems 1.2.6 and 1.2.8, one can easily show that strong solutions to the Navier-Stokes equations enjoy an energy equality, by passing to a limit as $\epsilon\to 0$ in (1.2.4). Hence, our approach can be seen as an alternate proof of this well-known fact.

From these energy identities we deduce the existence of a global attractor, under the assumption that $\mathbf{f}$ is time-independent.

Theorem 1.2.10 (Existence of a global attractor)**.** *Let* $\boldsymbol{\zeta}^\epsilon$ *be a calming function which satisfies conditions 1, 2, 3, and 4 of Definition 2.1.1. If* $\mathbf{u}_0 \in H$ *and* $\mathbf{f} \in H$ *then the dynamical system on* H *generated by calmed rNSE* (3.1.5) *has a global attractor* $\mathcal{A}$ *on* H*.*

Remark 1.2.11. All of the above results hold, *mutatis mutandis*, in the case of periodic boundary conditions, after suitable modification imposing a mean-free condition.

1.2.2 The calmed Navier-Stokes Equations in Velocity-Vorticity Formulation

Definition 1.2.12. for $T > 0$, we say $(\mathbf{u}, \mathbf{w})$ is a weak solution to (4.1.3) if

$$\begin{aligned} \mathbf{u}, \mathbf{w} &\in C(0,T;H) \cap L^2(0,T;V), \\ \partial_t \mathbf{u} &\in L^2(0,T;V^{-1}), \\ \partial_t \mathbf{w} &\in L^{\frac{4}{3}}(0,T;V^{-1}), \end{aligned}$$

if $\mathbf{u}$ satisfies equations (4.1.4a) for all $\mathbf{v}_1 \in V$ in the sense of $L^2(0,T)$ and $\mathbf{w}$ satisfies equations (4.1.4b) for all $\mathbf{v}_2 \in V$ in the sense of $L^4(0,T)$.

Definition 1.2.13. for $T > 0$, we say $(\mathbf{u}, \mathbf{w})$ is a strong solution to (4.1.3) if $(\mathbf{u}, \mathbf{w})$ is a weak solution and, additionally,

$$\begin{aligned} \mathbf{u} &\in C(0,T;V) \cap L^2(0,T;V^2), \\ \mathbf{w} &\in C(0,T;H) \cap L^2(0,T;V), \\ \partial_t \mathbf{u} &\in L^2(0,T;H), \end{aligned}$$

$$\partial_t \mathbf{w} \in L^2(0,T;V).$$

if $\mathbf{u}$ satisfies equations (4.1.4a) for all $\mathbf{v}_1 \in H$ in the sense of $L^2(0,T)$ and $\mathbf{w}$ satisfies equations (4.1.4b) for all $\mathbf{v}_2 \in V$ in the sense of $L^2(0,T)$, and if (4.1.4c) is satisfied in the sense of $C([0,T];V) \times C([0,T];H)$.

Theorem 1.2.14. *Let $\boldsymbol{\zeta}^\epsilon$ be a calming function as defined in 2.1.1 and let $T > 0$. For initial data $(\mathbf{u}_0, \mathbf{w}_0) \in H \times H$, time $T > 0$, and forcing term $\mathbf{f} \in L^2(0,T;V^{-1})$, solutions to (4.1.3) exist on the interval $[0,T]$.*

Theorem 1.2.15. *For initial data $(\mathbf{u}_0, \mathbf{w}_0) \in V \times H$, time $T > 0$, and forcing term $\mathbf{f} \in L^2(0,T;H)$, there exists a strong solution $(\mathbf{u}, \mathbf{w})$ to (4.1.4) on the interval $[0,T]$. Furthermore, the solution $(\mathbf{u}, \mathbf{w})$ is unique.*

Theorem 1.2.16. *For $(\mathbf{u}_0, \mathbf{w}_0) \in V \times H$, $\mathbf{f} \in L^2(0,T;H)$, let $(\mathbf{u}, \boldsymbol{\omega})$ be a strong solution to VVNSE (4.1.1) on $[0,T]$ for $T > 0$ prior to some maximal time of existence, under the assumption that*

$$\int_0^T \|\boldsymbol{\omega}\|_{L^{2\beta}}^{2\beta} \|\triangle \mathbf{u}\|_{L^2}^2 < \infty. \tag{1.2.5}$$

Suppose $(\mathbf{u}^\epsilon, \mathbf{w}^\epsilon)$ is a strong solution to cVV (4.1.3). Then $\mathbf{u}^\epsilon \to \mathbf{u}$ in $C([0,T];V)$ and $\mathbf{w}^\epsilon \to \boldsymbol{\omega}$ in $C([0,T];H)$ as $\epsilon \to 0$.

Theorem 1.2.17. *For strong solutions $(\mathbf{u}, \mathbf{w})$ with initial data $(\mathbf{u}, \mathbf{w}_0) \in V \times H$, $t \geq 0$ and zero forcing term, $\mathbf{u}$ satisfies the energy equalities*

$$\frac{d}{dt} \|\mathbf{u}\|_{L^2}^2 + 2\nu \|\nabla \mathbf{u}\|_{L^2}^2 = \|\mathbf{u}\|_{L^2}^2, \tag{1.2.6}$$

and

$$\|\mathbf{u}(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla \mathbf{u}\|_{L^2}^2 \, ds = \|\mathbf{u}_0\|_{L^2}^2 \tag{1.2.7}$$

and $\mathbf{w}$ *satisfies the energy inequality*

$$\|\mathbf{w}(t)\|_{L^2} \leq \|\mathbf{w}_0\|_{L^2} + \left(\frac{t}{2\nu}\right)^{\frac{1}{2}} \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \left(\|\mathbf{u}_0\|_{L^2}^2 - \|\mathbf{u}(t)\|_{L^2}^2\right)^{\frac{1}{2}}. \tag{1.2.8}$$

1.2.3 The MHD-Boussinesq Equations with Calmed Ohmic Heating term

We state in this section our major theorems and the outline of the remaining of this paper.

Our first major result is the global existence of a unique regular solution to System 5.1.1.

Theorem 1.2.18 (Global well-posedness of the 2D MHD-Boussinesq System with calmed Ohmic heating)**.** *For $s \geq 2$, given arbitrary time $T > 0$, and initial conditions $u_0, b_0 \in H^s \cap V$, $\theta_0 \in H^s$, there exists a unique solution (u, b, θ) to System 5.1.1, where*

$$u, b \in C([0,T]; H^s \cap V) \cap L^2((0,T); H^{s+1} \cap V)$$

with

$$\partial_t u, \partial_t b \in L^2(0, T; V'),$$

and

$$\theta \in C([0,T]; H^s) \cap L^2((0,T); H^{s+1})$$

with

$$\partial_t \theta \in L^2(0, T; H^{-1})$$

In order to obtain the error estimates of the solution to System 5.1.1 compared to that of System 5.1.2, we need the following local-in-time well-posedness results of the original system.

Theorem 1.2.19 (Short-time existence of the regular solution to the original MHD-Boussinesq System with Ohmic heating.)**.** *For $s \geq 2$, U_0, $B_0 \in H^s \cap V$ and $\Theta_0 \in H^s$, there exists a unique solution (U, B, Θ) to System 5.1.2, with*

$$U, B \in L^\infty([0, T_1); H^s \cap V) \cap L^2((0, T_1); H^{s+1} \cap V),$$

and

$$\Theta \in L^\infty([0, T_1); H^s) \cap L^2((0, T_1); H^{s+1}),$$

where T_1 depends on $g, \nu, \mu, \kappa, \alpha$ and initial data.

The next theorem concerns the convergence of the solution to System 5.1.1 with calmed Ohmic heating to that of the original Boussinesq-MHD system without the calming mechanism (5.1.2), on the time-interval of existence of solutions of the latter.

Theorem 1.2.20 (Error analysis and convergence of the solution of (5.1.1) to that of (5.1.2))**.** *For $s \geq 2$, let (U, B, Θ) be the solution to System* (5.1.2) *satisfying the conditions of Theorem 1.2.19 for $T_1 > 0$ with initial data $U_0, B_0 \in H^s \cap V$, $\Theta_0 \in H^s$. Assume that $\boldsymbol{\zeta}^\epsilon$ is a calming function which is Lipschitz and satisfies* (2) *and* (2.1.1)*, and let (u, b, θ) be the solution to* (5.1.1) *satisfying the conditions of Theorem 1.2.19 for $T_2 > 0$ with initial data*

$$u_0 = U_0, b_0 = B_0, \theta_0 = \Theta_0.$$

Select $T \in (0, \min\{T_1, T_2\})$. Then we have for all $t \in [0,T]$ that

$$|U(t) - u(t)|^2 + |B(t) - b(t)|^2 + \|\Theta(t) - \theta(t)\|_{L^2}^2 \le c_1 \epsilon^{2\gamma},$$
$$\int_0^T \|U(t) - u(t)\|_{H^2}^2 + \|B(t) - b(t)\|_{H^2}^2 + \|\Theta(t) - \theta(t)\|_{H^1}^2 \, dt \le c_2 \epsilon^{2\gamma},$$

where the constants c_1 and c_2 depend on $g, \nu, \mu, \kappa, \alpha$, $\|\triangle U\|_{L^2}, \|\triangle B\|_{L^2}, \|\nabla \Theta\|_{L^2}$, and T. In particular,

$$|U(t) - u(t)| + |B(t) - b(t)| + \|\Theta(t) - \theta(t)\|_{L^2} \to 0 \ \ as \ \ \epsilon \to 0^+.$$

Remark 1.2.21. Note that all of the results in Section 1.2.3 also hold easily *mutatis mutandis* for the so-called “two-and-a -half dimensional” case, that is the case where $x = (x_1, x_2)$ is still two-dimensional, but the the outputs are three-dimensional, i.e.,

$$u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$$

and

$$b = b(x,t) = (b_1(x,t), b_2(x,t), b_3(x,t)).$$

This is because the only role dimenionality plays in our analysis is in Sobolev estimates, which depend only on the input dimension. For the sake of simplicty, we present only the 2D case.

1.2.4 The calmed Kuramoto-Sivashinsky Equations

Definition 1.2.22. Let $\mathbf{u}_0 \in L^2(\mathbb{T}^2)$ and let $T > 0$. We say that $\mathbf{u}$ is a *weak solution* to calmed KSE (6.1.3) on the interval $[0,T]$ if $\mathbf{u} \in L^2([0,T]; H^2(\mathbb{T}^2)) \cap C([0,T]; L^2(\mathbb{T}^2))$, $\partial_t \mathbf{u} \in L^2(0,T; H^{-2}(\mathbb{T}^2))$, and $\mathbf{u}$ satisfies (6.1.3a) in the sense of

$L^2(0,T;H^{-2}(\mathbb{T}^2))$ and satisfies (6.1.3b) in the sense of $C([0,T];L^2(\mathbb{T}^2))$.

Theorem 1.2.23 (Global Well-Posedness)**.** *Let* $\mathbf{u}_0 \in L^2(\mathbb{T}^2)$*, let* $T > 0$ *and fix* $\epsilon > 0$*. Suppose* $\boldsymbol{\zeta}^\epsilon$ *is a calming function which satisfies Conditions 1 and 2 of Definition 2.1.1. Then weak solutions to* (6.1.3) *on* $[0,T]$ *exist, are unique, and depend continuously on the initial data in* $L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$*.*

Theorem 1.2.24 (Regularity)**.** *Suppose that* $\boldsymbol{\zeta}^\epsilon$ *is is calming function which satisfies Conditions 1, and 2 of 2.1.1. Let* $m \in \{1,2\}$*, and suppose that* $\mathbf{u}$ *is a weak solution to* (6.1.3) *on* $[0,T]$ *for some* $T > 0$*. If* $\mathbf{u}_0 \in H^m(\mathbb{T})$*, then* $\mathbf{u} \in L^\infty(0,T;H^m(\mathbb{T}^2)) \cap L^2(0,T;H^{m+2}(\mathbb{T}))$*.*

Theorem 1.2.25 (Convergence)**.** *Given* $\mathbf{u}_0 \in L^2(\mathbb{T})$*, let*

$$\mathbf{u} \in C([0,T];L^2(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T})). \tag{1.2.9}$$

be the corresponding weak solution of (6.1.2) *with maximal time of existence and uniqueness* $T^* > 0$ *and with* $T \in (0,T^*)$*. Suppose* $\boldsymbol{\zeta}^\epsilon$ *satisfies Conditions 1 and 2 of Definition 2.1.1. Furthermore, suppose* $\boldsymbol{\zeta}^\epsilon$ *satisfies Condition 3, so that* (2.1.1) *holds for some fixed* $C, \alpha > 0$ *and any* $\beta \in (0,3]$*. Let* $\mathbf{u}^\epsilon$ *be the corresponding weak solution of* (6.1.3) *with calming function* $\boldsymbol{\zeta}^\epsilon$ *and initial data* $\mathbf{u}_0$*. Then for any* $\epsilon > 0$*, it holds that*

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^\infty(0,T;L^2)} \le K\epsilon^\alpha,$$
$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^2(0,T;H^2)} \le K'\epsilon^\alpha,$$

where $K, K' > 0$ *depend on* T*,* β*, and various norms of* $\mathbf{u}$*, but not on* ϵ *or* α*.*

Definition 1.2.26. Let $\phi_0 \in L^2(\mathbb{T}^2)$ and let $T > 0$. We say that ϕ is a *weak solution* to (6.1.4) on the interval $[0,T]$ if $\phi \in L^2([0,T]; H^2(\mathbb{T}^2)) \cap C([0,T]; L^2(\mathbb{T}^2))$, $\partial_t \phi \in L^2(0,T; H^{-2}(\mathbb{T}^2))$, and ϕ satisfies (6.1.4a) in the sense of $L^2(0,T; H^{-2}(\mathbb{T}^2))$ and satisfies (6.1.4b) in the sense of $C([0,T]; L^2(\mathbb{T}^2))$.

Theorem 1.2.27 (Global Well-posedness in scalar form)**.** *Let initial data $\phi_0 \in L^2(\mathbb{T}^2)$ be given, and let $T > 0$, $\epsilon > 0$ be fixed. Suppose $\boldsymbol{\zeta}^\epsilon$ is a calming function which satisfies Conditions 1 and 2 of Definition 2.1.1. Then weak solutions to* (6.1.4) *on* $[0,T]$ *exist, are unique, and depend continuously on the initial data in $L^\infty(0,T; L^2(\mathbb{T}^2)) \cap L^2(0,T; H^2(\mathbb{T}^2))$.*

Theorem 1.2.28 (Convergence in scalar form)**.** *Choose $\phi_0 \in L^2(\mathbb{T})$ and let ϕ be the corresponding weak solution of the scalar KSE* (6.1.1) *with maximal time of existence T^*. We assume that ϕ is in the natural energy space: for $T < T^*$,*

$$\phi \in C([0,T]; L^2(\mathbb{T})) \cap L^2(0,T; H^2(\mathbb{T})). \tag{1.2.10}$$

Suppose $\boldsymbol{\zeta}^\epsilon$ satisfies 1, 2, and 3 of Definition 2.1.1, so that there exists C, $\alpha > 0$, and $\beta \in \left(0, \frac{3}{2}\right]$ for which (2.1.1) *holds. and let ϕ^ϵ be the corresponding weak solution of the scalar calmed KSE* (6.1.4) *with calming function $\boldsymbol{\zeta}^\epsilon$ and with initial data ϕ_0. Consider the convergence of ϕ^ϵ to ϕ on the interval $[0,T]$. The difference $\phi^\epsilon - \phi$ satisfies*

$$\|\phi^\epsilon - \phi\|_{L^\infty(0,T;L^2)} \leq K\epsilon^\alpha,$$
$$\|\phi^\epsilon - \phi\|_{L^2(0,T;H^2)} \leq K'\epsilon^\alpha,$$

where $K, K' > 0$ depend on T, β, and various norms of ϕ, but not on ϵ or α.

Chapter 2

Preliminaries

In this chapter, we define the function spaces that we will be working in and the notation used throughout. We will also introduce the inequalities that will be used in each chapter along with any theorems or lemmas that will be used. Throughout this work, we use the notation $\partial_t = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$ to represent the partial time derivative and the partial derivative in the direction of x_i, respectively. Additionally, we use C to represent a constant which may change from line to line.

For chapters 6, 4, and 5, we work in the space of functions which are periodic on the boundary. For chapter 3, we work in the space of functions defined on bounded domains $\Omega \subseteq \mathbb{R}^3$ with no-slip conditions (zero boundary data).

2.1 Calming Functions

Here we explain the exact requirements for $\boldsymbol{\zeta}^\epsilon$ to be a calming function.

Definition 2.1.1. We say $\boldsymbol{\zeta}^\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ is a *calming function* if the following three conditions hold:

1. $\boldsymbol{\zeta}^\epsilon$ is Lipschitz continuous with Lipschitz constant 1.

2. For $\epsilon > 0$ fixed, $\boldsymbol{\zeta}^\epsilon$ is bounded.

3. There exists $C > 0$, $\alpha > 0$ and $\beta \geq 1$ such that for any $\mathbf{x} \in \mathbb{R}^3$,

$$|\boldsymbol{\zeta}^\epsilon(\mathbf{x}) - \mathbf{x}| \leq C\epsilon^\alpha |\mathbf{x}|^\beta \tag{2.1.1}$$

In some instances we may impose a fourth constraint on $\boldsymbol{\zeta}^\epsilon$:

4. For any $\epsilon > 0$ and for each $\mathbf{x} \in \mathbb{R}^3$ there exists $\lambda^\epsilon(\mathbf{x}) \in \mathbb{R}$ such that $\boldsymbol{\zeta}^\epsilon(\mathbf{x}) = \lambda^\epsilon(\mathbf{x})\mathbf{x}$. That is, $\boldsymbol{\zeta}^\epsilon(\mathbf{x})$ is parallel to $\mathbf{x}$.

Remark 2.1.2. The lower bound on β is necessary to satisfy condition 2 and inequality (2.1.1) of $\boldsymbol{\zeta}^\epsilon$. Using the triangle inequality and (2.1.1), we may write

$$|\mathbf{x}| \leq C\epsilon^\alpha |\mathbf{x}|^\beta + \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty},$$

which, when $|\mathbf{x}|$ is sufficiently large, fails to be valid for $\beta < 1$.

Any function which satisfies Conditions 1, 2, and 3 of Definition 2.1.1 is a calming function. To make things concrete, we consider several forms of calming functions; namely,

$$\boldsymbol{\zeta}^\epsilon(\mathbf{x}) = \begin{cases} \boldsymbol{\zeta}_1^\epsilon(\mathbf{x}) := \frac{\mathbf{x}}{1+\epsilon|\mathbf{x}|}, & \text{or} \\ \boldsymbol{\zeta}_2^\epsilon(\mathbf{x}) := \frac{\mathbf{x}}{1+\epsilon^2|\mathbf{x}|^2}, & \text{or} \\ \boldsymbol{\zeta}_3^\epsilon(\mathbf{x}) := \frac{1}{\epsilon}\arctan(\epsilon\mathbf{x}), & \text{or} \\ \boldsymbol{\zeta}_4^\epsilon(\mathbf{x}) := q^\epsilon(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}, & \end{cases} \tag{2.1.2}$$

where the arctangent in $\boldsymbol{\zeta}_3^\epsilon$ acts component-wise;

$$\arctan\left((z_1, z_2, z_3)^T\right) = (\arctan(z_1), \arctan(z_2), \arctan(z_3))^T,$$

and for $\boldsymbol{\zeta}_4^\epsilon$, we define $\boldsymbol{\zeta}_4^\epsilon(\mathbf{0}) = \mathbf{0}$, and

$$q^\epsilon(r) = \begin{cases} r, & 0 \le r < \frac{1}{\epsilon}, \\ -\frac{\epsilon}{2}\left(r - \frac{2}{\epsilon}\right)^2 + \frac{3}{2\epsilon}, & \frac{1}{\epsilon} \le r < \frac{2}{\epsilon}, \\ \frac{3}{2\epsilon}, & r \ge \frac{2}{\epsilon}. \end{cases} \tag{2.1.3}$$

Note that $\boldsymbol{\zeta}^\epsilon(\mathbf{x}) \to \mathbf{x}$ for all $\mathbf{x} \in \Omega$ (i.e., pointwise) as $\epsilon \to 0^+$, and $\boldsymbol{\zeta}_i^\epsilon \in C^1$ for $i = 1, \ldots, 4$.

We indicate in the next proposition the extent to which our examples of a calming function (stated in (2.1.2)) satisfy the conditions of Definition 2.1.1.

Proposition 2.1.3. *Consider $\boldsymbol{\zeta}_i^\epsilon$ as described in* (2.1.2).

For $i = 1, 2, 4$, $\boldsymbol{\zeta}_i^\epsilon$ satisfies Conditions 1-4 of Definition 2.1.1. For $i = 3$, $\boldsymbol{\zeta}_i^\epsilon$ satisfies Conditions 1, 2, and 3 of Definition 2.1.1. In particular, the following explicit bounds hold for $\epsilon > 0$.

1. *For $\boldsymbol{\zeta}_1^\epsilon$,*
$$\|\boldsymbol{\zeta}_1^\epsilon\|_{L^\infty} = \frac{1}{\epsilon} \text{ and } |\boldsymbol{\zeta}_1^\epsilon(\mathbf{x}) - \mathbf{x}| \le \epsilon\, |\mathbf{x}|^2 .$$

2. *For $\boldsymbol{\zeta}_2^\epsilon$,*
$$\|\boldsymbol{\zeta}_2^\epsilon\|_{L^\infty} = \frac{1}{2\epsilon} \text{ and } |\boldsymbol{\zeta}_2^\epsilon(\mathbf{x}) - \mathbf{x}| \le \epsilon^2\, |\mathbf{x}|^3 .$$

3. *For $\boldsymbol{\zeta}_3^\epsilon$,*
$$\|\boldsymbol{\zeta}_3^\epsilon\|_{L^\infty} = \frac{\sqrt{n}\pi}{2\epsilon} \text{ and } |\boldsymbol{\zeta}_3^\epsilon(\mathbf{x}) - \mathbf{x}| \le \epsilon^2\, |\mathbf{x}|^3 .$$

4. *For $\boldsymbol{\zeta}_4^\epsilon$,*
$$\|\boldsymbol{\zeta}_4^\epsilon\|_{L^\infty} = \frac{3}{2\epsilon} \text{ and } |\boldsymbol{\zeta}_4^\epsilon(\mathbf{x}) - \mathbf{x}| \le \epsilon\, |\mathbf{x}|^2 .$$

Furthermore, we can determine

Lemma 2.1.4. *Suppose that* $\Omega \subseteq \mathbb{R}^n$ *is a bounded domain or* $\Omega \equiv \mathbb{T}^n$*. Suppose that* $\boldsymbol{\zeta}^\epsilon$ *satisfies Conditions 1 and 2 of 2.1.1. Then the following statements hold.*

(i) Given $1 \leq p \leq \infty$*, if* $\mathbf{u} \in L^p(\Omega)$ *then* $\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \in L^p(\Omega)$ *and* $\boldsymbol{\zeta}^\epsilon$ *is Lipschitz in* $L^p(\Omega)$ *with Lipschitz constant* 1.

(ii) Fix $\mathbf{u}, \mathbf{w} \in L^2(0,T;L^2(\Omega))$ *and* $T > 0$*, let* $I_{\mathbf{u},\mathbf{w}} : L^2(0,T;H^1(\Omega)) \to \mathbb{R}$ *be the map*

$$I_{\mathbf{u},\mathbf{w}}(\boldsymbol{\phi}) = \int_0^T \left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla \right) \boldsymbol{\phi}, \mathbf{w} \right) dt. \tag{2.1.4}$$

Then $I_{\mathbf{u},\mathbf{w}}$ *is a bounded linear operator.*

Proof. (i). The result follows immediately from the definition of the L^p norm and from Condition 1 of Definition 2.1.1.

(ii). Let $\zeta_j^\epsilon(\mathbf{u})$ denote the j-th component of $\boldsymbol{\zeta}^\epsilon(\mathbf{u})$.

For $\boldsymbol{\phi} \in L^2(0,T;H^1(\mathbb{T}^2))$, we estimate

$$\begin{aligned}
|I_{\mathbf{u},\mathbf{w}}(\boldsymbol{\phi})| &\leq \sum_{j=1}^n \int_0^T \left| \left(\zeta_j^\epsilon(\mathbf{u}) \partial_j \boldsymbol{\phi}, \mathbf{w} \right) \right| dt \\
&\leq \sum_{j=1}^2 \int_0^T \left\| \zeta_j^\epsilon(\mathbf{u}) \right\|_{L^\infty} \left\| \partial_j \boldsymbol{\phi} \right\|_{L^2} \left\| \mathbf{w} \right\|_{L^2} dt \\
&\leq \left\| \boldsymbol{\zeta}^\epsilon \right\|_{L^\infty} \int_0^T \left\| \boldsymbol{\phi} \right\|_{H^1} \left\| \mathbf{w} \right\|_{L^2} dt \\
&\leq \left\| \boldsymbol{\zeta}^\epsilon \right\|_{L^\infty} \left\| \mathbf{w} \right\|_{L^2(0,T;L^2)} \left\| \boldsymbol{\phi} \right\|_{L^2(0,T;H^1)}
\end{aligned}$$

by the Cauchy-Schwarz inequality. This concludes the proof. ∎

2.2 No-Slip Boundary Conditions

Here we lay out the standard notation that will be used in Chapter 3. We assume that $\Omega \subset \mathbb{R}^3$ is a bounded, open, connected set with C^2 boundary. Furthermore, we assume Ω is convex, so that there exists $c_1, c_2 > 0$ for which

$$c_1 \left\| A\mathbf{u} \right\|_{L^2} \leq \left\| \mathbf{u} \right\|_{H^2} \leq c_2 \left\| A\mathbf{u} \right\|_{L^2}, \tag{2.2.1}$$

where A is defined in (2.2.2) (see, e.g., [24, 41]). Let $C_c^\infty(\Omega)$ denote the space of smooth, compactly supported test functions from Ω to $\mathbb{R}^3$, and let $H_0^1(\Omega) \equiv H_0^1$ denote the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. More specifically, we have

$$H_0^1 = \{\mathbf{u} \in H^1(\Omega) : \mathbf{u}|_{\partial\Omega} = \mathbf{0}\}$$

We set

$$\mathcal{V} = \{\boldsymbol{\phi} \in C_c^\infty(\Omega) : \quad \nabla \cdot \boldsymbol{\phi} = 0\},$$

and let H and V be the closure of $\mathcal{V}$ in $L^2(\Omega)$ and $H^1(\Omega)$, respectively.

We also denote the (real) L^2 inner-product and H^m Sobolev norm by

$$(\mathbf{u}, \mathbf{v}) := \sum_{i=1}^{3} \int_\Omega u_i(\mathbf{x}) v_i(\mathbf{x})\, d\mathbf{x}, \qquad \|\mathbf{u}\|_{H^m} := \left(\sum_{|\alpha| \leq m} \|D^\alpha \mathbf{u}\|_{L^2}^2 \right)^{\frac{1}{2}},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $D^\alpha \mathbf{u} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \mathbf{u}$. For brevity, we will use the notation $L^2(\Omega) \equiv L^2$ and $H^m(\Omega) \equiv H^m$ throughout.

We denote by $L^p(0, T; X)$ the space of Bochner integrable functions from $[0, T]$ to

X with norm given by

$$\|\mathbf{u}\|_{L^p(0,T;X)} \equiv \|\mathbf{u}\|_{L^pX} := \left(\int_0^T \|\mathbf{u}\|_X^p \right)^{1/p}.$$

Let $P_\sigma : L^2(\Omega) \to H$ be the Leray-Helmholtz orthogonal projection of $L^2(\Omega)$ onto H. Define the Stokes operator $A : \mathcal{D}(A) \subset H \to H$ as

$$A := -P_\sigma \triangle \tag{2.2.2}$$

with domain $\mathcal{D}(A) := H^2(\Omega) \cap V$. The operator A is known to be positive-definite, self-adjoint, and with compact inverse A^{-1} in H. From the Hilbert-Schmidt Theorem we obtain a sequence of eigenfunctions $\{\mathbf{w}_j\}_{j=1}^\infty$ of A^{-1}, which are also eigenfunctions of A, with corresponding eigenvalues $\{\lambda_j\}_{j=1}^\infty$ such that $\{\mathbf{w}_j\}_{j=1}^\infty$ is an orthonormal basis of H and the sequence $\{\lambda_j\}_{j=1}^\infty$ is positive, monotone increasing, and tend toward infinity, so that $A\mathbf{w}_j = \lambda_j \mathbf{w}_j$ with $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ and $\lim_{j\to\infty} \lambda_j = +\infty$. For further discussion see, e.g., [20, 90, 105]. For all $\mathbf{u} \in V$, we define the norm on V by

$$\langle A\mathbf{u}, \mathbf{u} \rangle = \|A^{1/2}\mathbf{u}\|_{L^2} = \|\nabla \mathbf{u}\|_{L^2}^2 .$$

Denote by P_m the projection onto the first m eigenfunctions of A,

$$P_m \mathbf{u} = \sum_{j=1}^m u_j \mathbf{w}_j. \tag{2.2.3}$$

This yields the following estimate: for $\mathbf{u} \in H^s(\Omega)$, $s > 0$,

$$\|(I - P_m)\,\mathbf{u}\|_{L^2}^2 \leq \lambda_m^{-s} \|\mathbf{u}\|_{H^s}^2 . \tag{2.2.4}$$

For $\mathbf{u} \in C_c^\infty(\Omega)$ and $\mathbf{v} \in \mathcal{V}$, we define the nonlinear term $B(\mathbf{u}, \mathbf{v})$ by

$$B(\mathbf{u}, \mathbf{v}) := P_\sigma \left((\mathbf{u} \cdot \nabla) \, \mathbf{v} \right). \tag{2.2.5}$$

The term $B(\cdot, \cdot)$ can be extended continuously to a bounded bilinear operator $B : H_0^1 \times V \to V'$. Similarly, we can define a trilinear operator $b : H_0^1 \times V \times V \to \mathbb{R}$ by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle \tag{2.2.6}$$

for all $\mathbf{u} \in H_0^1$ and $\mathbf{v}, \mathbf{w} \in V$.

Remark 2.2.1. In Chapter 3, for System (3.1.3) we also consider the nonlinear term defined by

$$B(\mathbf{u}, \mathbf{v}) := P_\sigma \left((\nabla \times \mathbf{v}) \times \mathbf{u} \right). \tag{2.2.7}$$

We use the same symbol for both expressions as there is no quantitative difference in the analysis between (2.2.5) and (2.2.7) with regards to the global wellposedness and convergence of Systems (3.1.1) and (3.1.3).

Remark 2.2.2. In Chapter 5, we use the notation $\mathcal{B}$ for the nonlinear term to avoid any confusion with the magnetic field B in System (5.1.2).

2.3 Periodic Boundary Conditions

We denote the 2-dimensional and 3-dimensional torus as $\mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2 = [0, 2\pi)^2$ and $\mathbb{T}^3 := \mathbb{R}^3/(2\pi\mathbb{Z})^3 = [0, 2\pi)^3$, respectively. For $n \in \{2, 3\}$, $\mathbf{u} : \mathbb{T}^n \to \mathbb{R}^n$ satisfies $\mathbf{u}(\mathbf{x} + 2\pi\mathbf{e}_j) = \mathbf{u}(\mathbf{x})$ for $j = 1, \ldots, n$, where $\mathbf{e}_j$ is the j-th unit basis vector of $\mathbb{R}^n$. We denote the set of real vector-valued L^2 functions on $\mathbb{T}^n$ by

$$L^2(\mathbb{T}^n) := \left\{ \mathbf{u} : \mathbb{T}^n \to \mathbb{R}^n \;\middle|\; \mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \hat{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},\ \overline{\hat{\mathbf{u}}_{\mathbf{k}}} = \hat{\mathbf{u}}_{-\mathbf{k}},\ \text{and} \sum_{\mathbf{k}\in\mathbb{Z}^n} |\hat{\mathbf{u}}_{\mathbf{k}}|^2 \infty \right\}$$

(with the usual convention of equivalence up to sets of measure zero). We also denote the (real) L^2 inner-product and H^s Sobolev norm, $s \in \mathbb{R}$, by

$$(\mathbf{u}, \mathbf{v}) := \sum_{i=1}^{n} \int_{\mathbb{T}^n} u_i(\mathbf{x}) v_i(\mathbf{x})\, d\mathbf{x}, \qquad \|\mathbf{u}\|_{H^s} := \left(\sum_{\mathbf{k}\in\mathbb{Z}^n} (1 + |\mathbf{k}|)^{2s} |\hat{\mathbf{u}}_{\mathbf{k}}|^2 \right)^{1/2},$$

and the corresponding space $H^s(\mathbb{T}^n) = \left\{ \mathbf{u} \in L^2(\mathbb{T}^n) \middle| \quad \|\mathbf{u}\|_{H^s} < \infty \right\}$. In this setting, we have

$$\mathcal{V} = \left\{ \boldsymbol{\phi} \in C_c^\infty(\mathbb{T}^n) : \quad \int_{\mathbb{T}^n} \boldsymbol{\phi}\, d\mathbf{x} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{\phi} = 0 \right\},$$

and we denote H and V as the closures of $\mathcal{V}$ in $L^2(\mathbb{T}^n)$ and $H^1(\mathbb{T}^n)$, respectively. In Chapter 4 we will use the notation $V^k \equiv V \cap H_0^k(\mathbb{T}^3)$ and $V^{-k} \equiv [V \cap H_0^k(\mathbb{T}^3)]'$, and we will set $V^0 = H$.

The definition for the mappings P_σ, A, $B(\cdot,\cdot)$, and $b(\cdot,\cdot,\cdot)$ are identical in this setting as in the no-slip boundary case, except with $\Omega \equiv \mathbb{T}^n$. However, we remark that on $\mathbb{T}^n$ we have the commutativity property $-P_\sigma \triangle = -\triangle P_\sigma$. Thus, for all $\mathbf{u} \in D(A)$

$$A\mathbf{u} = -P_\sigma \triangle \mathbf{u} = -\triangle P_\sigma \mathbf{u} = -\triangle \mathbf{u}$$

We now turn our attention to the operators that will be used in Chapter 6.

The space $L^2(\mathbb{T}^n)$ has an orthogonal basis of eigenfunctions of the Laplacian operator $-\triangle$ given by

$$\left\{e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_j : \mathbf{k} \in \mathbb{Z}^n, \quad j = 1, \ldots, n\right\}$$

with corresponding eigenvalues $\left\{|\mathbf{k}|^2 : \mathbf{k} \in \mathbb{Z}^n\right\}$. For $m \in \mathbb{N}$, we denote $P_m : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ to be the projection onto finitely many eigenfunctions of the operator $-\triangle$:

$$P_m\mathbf{u} = \sum_{\substack{\mathbf{k}\in\mathbb{Z}^n \\ |\mathbf{k}|\leq m}} \hat{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Denote $Q_m := I - P_m$. For these projections we have the following estimates: given any $\mathbf{u} \in H^s(\mathbb{T})$, $s > 0$,

$$\|(-\triangle)^s P_m\mathbf{u}\|_{L^2} \leq m^s \|P_m\mathbf{u}\|_{L^2} \tag{2.3.1}$$

$$\|Q_m\mathbf{u}\|_{L^2} \leq \frac{1}{m^s}\|\mathbf{u}\|_{H^s}. \tag{2.3.2}$$

2.4 Inequalities

Here, we list several of the inequalities that are used throughout this body of work.

We first state Poincaré's inequality for functions which are zero on the boundary of $\Omega \subseteq \mathbb{R}^n$ (or, equivalently, mean-free on $\mathbb{T}^n$):

$$\|\mathbf{u}\|_{L^2}^2 \leq \lambda_1^{-1} \|\nabla\mathbf{u}\|_{L^2}^2 \quad \text{for all } \mathbf{u} \in V, \tag{2.4.1}$$

$$\|\nabla\mathbf{u}\|_{L^2}^2 \leq \lambda_1^{-1} \|A\mathbf{u}\|_{L^2}^2 \quad \text{for all } \mathbf{u} \in \mathcal{D}(A). \tag{2.4.2}$$

We also frequently use Agmon's inequality on bounded domains: for $s_1 < 1 < s_2$, and for $\theta \in (0,1)$ such that $\theta s_1 + (1-\theta)s_2 = 1$,

$$\|\mathbf{u}\|_{L^\infty} \leq C\|\mathbf{u}\|_{H^{s_1}}^{\theta}\|\mathbf{u}\|_{H^{s_2}}^{1-\theta}, \tag{2.4.3}$$

and the Gagliardo-Nirenberg-Sobolev interpolation inequality (see, e.g., [103, p. 11]) in $\mathbb{R}^n$ for $1 \leq p, q < \infty$,

$$\|\mathbf{u}\|_{L^p} \leq C \|\mathbf{u}\|_{L^q}^{\theta} \|D^\alpha \mathbf{u}\|_{L^2}^{1-\theta}, \qquad \frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{|\alpha|}{n}\right). \tag{2.4.4}$$

Also, in Chapter 6, we repeatedly apply the following interpolation inequality: Using integration by parts, the Cauchy-Schwarz inequality, and Young's inequality, we obtain, for any $\delta > 0$, the estimate

$$\|\nabla \mathbf{u}\|_{L^2}^2 \leq \frac{1}{2\delta} \|\mathbf{u}\|_{L^2}^2 + \frac{\delta}{2} \|\triangle \mathbf{u}\|_{L^2}^2 . \tag{2.4.5}$$

The remaining inequalities are used in Chapter 5 and are valid on $\mathbb{T}^2$:

A special case of (2.4.4) in 2D is Ladyzhenskaya's inequality: for all $u \in V$,

$$\|u\|_{L^4}^2 \leq c\|u\|_{L^2}|u|, \tag{2.4.6}$$

Moreover, we have the following inequalities and identities that are valid for functions on $\mathbb{T}^n$:

Lemma 2.4.1. *For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, it holds (in two-or-three-dimensions) that*

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle_{V'} = -\langle B(\mathbf{u},\mathbf{w}), \mathbf{v}\rangle_{V'}, \tag{2.4.7a}$$

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{v}\rangle_{V'} = 0. \tag{2.4.7b}$$

Also (in two-dimensions only), for all $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *in the largest spaces* H, V, *or* $\mathcal{D}(A)$, *for which the right-hand sides of the inequalities below are finite,*

$$|\langle B(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{V'}| \leq C\,\|\mathbf{u}\|_{L^2}^{1/2}\,\|\mathbf{u}\|_{H^1}^{1/2}\,\|\mathbf{v}\|_{H^1}\,\|\mathbf{w}\|_{L^2}^{1/2}\,\|\mathbf{w}\|_{H^1}^{1/2} \tag{2.4.8a}$$

$$|\langle B(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{V'}| \leq C\,\|\mathbf{u}\|_{L^2}^{1/2}\,\|\mathbf{u}\|_{H^1}^{1/2}\,\|\mathbf{v}\|_{H^1}^{1/2}\,\|A\mathbf{v}\|_{L^2}^{1/2}\,\|\mathbf{w}\|_{L^2}, \tag{2.4.8b}$$

$$|\langle B(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{V'}| \leq C\,\|\mathbf{u}\|_{L^2}^{1/2}\,\|A\mathbf{u}\|_{L^2}^{1/2}\,\|\mathbf{v}\|_{H^1}\,\|\mathbf{w}\|_{L^2}. \tag{2.4.8c}$$

Moreover, due to the periodic boundary conditions, it holds (in two-dimensions) that

$$\langle B(w,w), Aw\rangle = 0,\ w \in \mathcal{D}(A), \tag{2.4.9}$$

and the following Jacobi identity holds

$$\langle B(u,w), Aw\rangle + \langle B(w,u), Aw\rangle + \langle B(w,w), Au\rangle = 0. \tag{2.4.10}$$

2.5 Theorems and Lemmas

Here we provide a list of the theorems and lemmas used in each chapter.

We first state the following uniform Grönwall's inequality, proved in [47] (see also [31] and the references therein)

Lemma 2.5.1. *Suppose that* $Y(t)$ *is a locally integrable and absolutely continuous function that satisfies the following:*

$$\frac{dY}{dt} + \alpha(t)Y \leq \beta(t), \qquad a.e.\ \text{on}\ (0,\infty),$$

such that

$$\liminf_{t\to\infty}\int_t^{t+\tau}\alpha(s)\,ds\geq\gamma,\qquad\limsup_{t\to\infty}\int_t^{t+\tau}\alpha^-(s)\,ds<\infty,$$

and

$$\lim_{t\to\infty}\int_t^{t+\tau}\beta^+(s)\,ds=0,$$

for fixed $\tau>0$*, and* $\gamma>0$*, where* $\alpha^-=\max\{-\alpha,0\}$ *and* $\beta^+=\max\{\beta,0\}$*. Then,* $Y(t)\to 0$ *at an exponential rate as* $t\to\infty$*.*

We also make repeated use of the Lions-Magenes lemma (see, e.g., [65] or [105, Ch. 3, Lemma 1.2]), which states:

Lemma 2.5.2. *Let* V, H, V' *be three Hilbert spaces such that*

$$V\subseteq H\equiv H'\subseteq V'$$

with each inclusion being a continuous embedding. If a function $\mathbf{u}$ *belongs to* $L^2(0,T;V)$ *and its derivative* $\partial_t\mathbf{u}$ *belongs to* $L^2(0,T;V')$*, then* $\mathbf{u}$ *is almost everywhere equal to a funciton continuous from* $[0,T]$ *into* H *and the equality*

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2=\langle\partial_t\mathbf{u},\mathbf{u}\rangle$$

holds in the scalar distribution on $(0,T)$*.*

Similarly, we use the Aubin-Lions (also called Aubin-Lions-Simon) Compactness Lemma throughout this work to show the convergence of our Galerkin approximations (see [95, Corollary 4, pg. 85]):

Lemma 2.5.3. *Suppose $X \subseteq B \subseteq Y$ with X compactly embedded in B. Let F be bounded in $L^\infty(0,T;X)$ and let $\partial_t F$ be bounded in $L^r(0,T;Y)$ for $r > 1$. Then F is relatively compact in $C([0,T];B)$.*

To recover the pressure term found in Systems (3.1.4) and (3.1.5) in Chapter 3, we will use a result of de Rham, which states for $\mathbf{f} \in C_c^\infty(\Omega)$,

$$\mathbf{f} = \nabla p \text{ for some } p \in C_c^\infty(\Omega) \text{ if and only if } \langle \mathbf{f}, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{v} \in \mathcal{V}. \tag{2.5.1}$$

See, e.g., [105, 108].

To obtain convergence of the calmed systems in each chapter we require the use of the following abstract bootstrapping principle (see, e.g., [103, p. 20]):

Lemma 2.5.4. *Let $T > 0$. Assume that two statements $C(t)$ and $H(t)$ with $t \in [0,T]$ satisfy the following conditions:*

(a) If $H(t)$ holds for some $t \in [0,T]$, then $C(t)$ holds for the same t;

(b) If $C(t)$ holds for some $t_0 \in [0,T]$, then $H(t)$ holds for t in a neighborhood of t_0;

(c) If $C(t)$ holds for $t_m \in [0,T]$ and $t_m \to t$, then $C(t)$ holds;

(d) $H(t)$ holds for at least one $t_1 \in [0,T]$.

Then $C(t)$ holds for all $t \in [0,T]$.

Chapter 3

The 3D Navier-Stokes Equations

3.1 Introduction

The three-dimensional (3D) incompressible constant-density Navier-Stokes equations (NSE) are given by

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\triangle\mathbf{u} + \mathbf{f} & \text{in } \Omega\times(0,T), & (3.1.1a) \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,T), & (3.1.1b) \\ \mathbf{u}\big|_{\partial\Omega} = \mathbf{0} & \text{on } \partial\Omega\times(0,T) & (3.1.1c) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, & (3.1.1d) \end{cases}$$

Here, $\mathbf{u} : \Omega\times[0,T]\to\mathbb{R}^3$ is the fluid velocity, $p : \Omega\times[0,T]\to\mathbb{R}$ is the (kinematic) pressure, and $\mathbf{f} : \Omega\times[0,T]\to\mathbb{R}^3$ is a body force. The domain $\Omega\subset\mathbb{R}^3$ is a bounded, open, connected set with C^2 boundary.

Note that, using the vector identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \tfrac{1}{2}\nabla|\mathbf{u}|^2, \tag{3.1.2}$$

one may formally rewrite (3.1.1) in the following equivalent rotational form (rNSE),

$$\begin{cases} \partial_t \mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} + \nabla \pi = \nu \triangle \mathbf{u} + \mathbf{f} & \text{in } \Omega \times (0,T), \quad (3.1.3a) \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \times (0,T), \quad (3.1.3b) \\ \mathbf{u}\big|_{\partial\Omega} = \mathbf{0} & \text{on } \partial\Omega \times (0,T), \quad (3.1.3c) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, \quad (3.1.3d) \end{cases}$$

where we have denoted the vorticity by $\boldsymbol{\omega} := \nabla \times \mathbf{u}$ and the Bernoulli pressure (or "dynamic pressure") as $\pi := p + \frac{1}{2}|\mathbf{u}|^2$. The term $\boldsymbol{\omega} \times \mathbf{u}$ is sometimes called the Lamb vector.

We use a bounded smooth truncation function — that we call a "calming function" when used in this context — that approximates the identity as the "calming parameter" $\epsilon \to 0^+$. We propose a calming-function approach to the 3D NSE. In particular, we propose two modifications of the Navier-Stokes system. The first is based on the form (3.1.1). Continuing the same approach we employed in [28], we introduce the following system that we call the calmed Navier-Stokes equations (calmed NSE).

$$\begin{cases} \partial_t \mathbf{u} + (\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla)\mathbf{u} + \nabla p = \nu \triangle \mathbf{u} + \mathbf{f} & \text{in } \Omega \times (0,T), \quad (3.1.4a) \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times (0,T), \quad (3.1.4b) \\ \mathbf{u}\big|_{\partial\Omega} = \mathbf{0} & \text{on } \partial\Omega \times (0,T) \quad (3.1.4c) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, \quad (3.1.4d) \end{cases}$$

One can see (3.1.4) as a modification of (3.1.1) in the spirit of Leray (see, e.g., [8, 10, 16, 17, 32, 43, 46, 64, 111] and many others), except that our modification does not mollify the nonlinearity but is instead a local truncation of the advective velocity.

While we show in the present work that the calming modification of (3.1.4) allows for a proof of global well-posedness and other desirable properties, it is clear that such a modification would have a different energy balance than that of Navier-Stokes, as the nonlinear term does not vanish in standard energy calculations. Therefore, we also consider a related modification of the rotational form (3.1.3) which locally limits the strength of the rotational term. Namely, we propose the following system, which we call the calmed rotational Navier-Stokes equations (calmed rNSE).

$$\begin{cases} \partial_t \mathbf{u} + (\nabla \times \mathbf{u}) \times \boldsymbol{\zeta}^\epsilon(\mathbf{u}) + \nabla \pi = \nu \triangle \mathbf{u} + \mathbf{f} & \text{in } \Omega \times (0,T), \quad (3.1.5\text{a}) \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times (0,T), \quad (3.1.5\text{b}) \\ \mathbf{u}\big|_{\partial\Omega} = \mathbf{0} & \text{on } \partial\Omega \times (0,T) \quad (3.1.5\text{c}) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega. \quad (3.1.5\text{d}) \end{cases}$$

Due to the presence of the calming function, one cannot rewrite calmed NSE as calmed rNSE using (3.1.2) as we do for NSE and rNSE. Thus, while they are both modifications of the Navier-Stokes equations which are similar, we treat them as different systems. However, system (3.1.5) is an interesting object to study in its own right. Thanks to the well-known geometric identity for the cross product,

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B} = 0, \tag{3.1.6}$$

one discovers exceptional features of System (3.1.5) when $\boldsymbol{\zeta}^\epsilon$ is suitably chosen. Namely, when $\boldsymbol{\zeta}^\epsilon(\mathbf{x})$ can be expressed as a scalar multiple of $\mathbf{x}$ pointwise we deduce that (3.1.5) possesses both an energy identity (Theorem 1.2.3) and its dynamical system has a global attractor (Theorem 1.2.10).

Remark 3.1.1. Applying a bounded truncation operator to the nonlinear term in 3D

Navier Stokes was also considered by Yoshida and Giga [113] and by the authors of [12] in the study of the globally modified Navier-Stokes Equations (GMNSE) (see also, [11, 13, 25, 50, 51, 72, 92, 114, 115]). In those works, the following system was studied.

$$\begin{cases} \partial_t \mathbf{u} + \min\left\{1, N\|\nabla \mathbf{u}\|_{L^2}^{-1}\right\}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\triangle\mathbf{u} + \mathbf{f} & \text{in } \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,T), \\ \mathbf{u}\big|_{\partial\Omega} = \mathbf{0} & \text{on } \partial\Omega\times(0,T) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, \end{cases}$$

For GMNSE, solutions converge to a solution of 3D Navier-Stokes as parameter N tends to infinity. This system is similar to calmed NSE (3.1.4) in that it bounds the nonlinear term as the velocity $\mathbf{u}$ gets large in a certain sense. However, our modification has several advantages over GMNSE. Namely, that the calming functions in the present work are defined pointwise and only bound the solution $\mathbf{u}$ in regions where $|\mathbf{u}(\mathbf{x},t)|$ is greater than approximately ϵ^{-1}, whereas the modification in GMNSE affects the solution globally. Also, whenever $\|\nabla\mathbf{u}\|_{L^2}\to\infty$, the nonlinearity in GMNSE vanishes entirely, but for calmed NSE this would only cause the large values of $|\mathbf{u}(\mathbf{x},t)|$ to be truncated locally. Moreover, our calming parameter depends on $\mathbf{u}$ while the GM function depends on $\nabla\mathbf{u}$, hence the manner in which we control the nonlinearity is different. In a future work, we will examine differences between these two systems computationally.

3.2 Existence of Weak Solutions for Calmed NSE

The proofs of existence, uniqueness, convergence, etc. are essentially identical for both equations (3.1.4) and (3.1.5). (The only phenomenological difference examined in this paper is the rotational form (3.1.5) has an energy identity, but for (3.1.4), this is unknown.) Therefore, we adopt a unified abstract notation which allows us to handle both equations simultaneously.

For either (3.1.4) or (3.1.5), the weak formulation can be written as follows: Given $\mathbf{u}_0 \in L^2(0,T;H)$ and $\mathbf{f} \in L^2(0,T;V')$, find $\mathbf{u} \in L^2(0,T;V)$ which satisfies

$$\langle \partial_t \mathbf{u}, \mathbf{v}\rangle + \langle \nu A\mathbf{u}, \mathbf{v}\rangle + \langle B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}), \mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in V, \tag{3.2.1a}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{3.2.1b}$$

where the Stokes operator A is defined by (2.2.2) and the nonlinear term $B(\cdot,\cdot)$ is defined in either advective (2.2.5) or rotational form (2.2.7). We note that the uniqueness of weak solutions is an open problem, similar to the situation regarding 3D Navier-Stokes. However, unlike the 3D Navier-Stokes case, we are able to prove the global existence of strong solutions.

We will prove the existence of solutions to (3.2.1) via Galerkin approximation. For $\mathbf{u}_0 \in H$, the system

$$\begin{cases} \partial_t \mathbf{u}_m = -\nu A\mathbf{u}_m - P_m B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m) + P_m \mathbf{f}, & (3.2.2a) \\ \mathbf{u}_m(0, \mathbf{x}) = P_m \mathbf{u}_0(\mathbf{x}) & (3.2.2b) \end{cases}$$

is locally Lipschitz in $P_m(H)$ provided that $\boldsymbol{\zeta}^\epsilon$ is Lipschitz (see [28], Lemma 3.2). So for each $m \in \mathbb{N}$, there is some $T_m > 0$ for which a unique solution to (3.2.2) exists.

3.2.1 Proof of Theorem 1.2.3

Let $\mathbf{u}_m$ be a solution to (3.2.2) on some maximum interval of existence $[0, T_m]$ with $T_m > 0$. Taking the inner product of (3.2.2) with $\mathbf{u}^m$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\mathbf{u}_m\right\|_{L^2}^2 + \nu\left\|\nabla\mathbf{u}_m\right\|_{L^2}^2 &= -\left(P_m B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m), \mathbf{u}_m\right) + \langle P_m\mathbf{f}, \mathbf{u}_m\rangle \\ &= -\left(B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m), \mathbf{u}_m\right) + (\mathbf{f}, \mathbf{u}_m).\end{aligned}$$

Now, using Hölder's Inequality and Young's Inequality,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\mathbf{u}_m\right\|_{L^2}^2 + \nu\left\|\nabla\mathbf{u}_m\right\|_{L^2}^2 \\ &\leq \left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}\left\|\nabla\mathbf{u}_m\right\|_{L^2}\left\|\mathbf{u}_m\right\|_{L^2} + \left\|\mathbf{f}\right\|_{V'}\left\|\nabla\mathbf{u}_m\right\|_{L^2} \\ &\leq \frac{\nu}{2}\left\|\nabla\mathbf{u}_m\right\|_{L^2}^2 + C_\nu\left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}^2\left\|\mathbf{u}_m\right\|_{L^2}^2 + C_\nu\left\|\mathbf{f}\right\|_{V'}^2.\end{aligned}$$

Rearranging terms yields the inequality

$$\frac{d}{dt}\left\|\mathbf{u}_m\right\|_{L^2}^2 + \nu\left\|\nabla\mathbf{u}_m\right\|_{L^2}^2 \leq C_\nu\left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}^2\left\|\mathbf{u}_m\right\|_{L^2}^2 + C_\nu\left\|\mathbf{f}\right\|_{V'}^2. \tag{3.2.3}$$

Dropping the term $\nu\left\|\nabla\mathbf{u}_m\right\|_{L^2}^2$ from the left-hand side of the inequality and applying Grönwall's inequality yields

$$\begin{aligned}\left\|\mathbf{u}_m(t)\right\|_{L^2}^2 &\leq e^{C_\nu\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 t}\left\|\mathbf{u}_m(0)\right\|_{L^2}^2 + \int_0^t e^{-C_\nu\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2(s-t)}\left\|\mathbf{f}\right\|_{V'}^2\,ds \\ &\leq e^{C_\nu\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 T_m}\left(\left\|\mathbf{u}_0\right\|_{L^2}^2 + \left\|\mathbf{f}\right\|_{L^2V'}^2\right).\end{aligned} \tag{3.2.4}$$

In fact, we can apply a standard bootstrapping argument to obtain that given any $T > 0$, (3.2.4) remains valid if $T_m \equiv T$ for all $m \in \mathbb{N}$. Thus $\mathbf{u}_m$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}^2))$ independently of m. Integrating (3.2.3) in time t on the interval

$[0,T]$, one obtains

$$\begin{aligned}
&\|\mathbf{u}_m(T)\|_{L^2}^2 - \|\mathbf{u}_m(0)\|_{L^2}^2 + \nu \int_0^T \|\nabla \mathbf{u}_m\|_{L^2}^2 \, dt \\
&\leq C_\nu \|\mathbf{f}\|_{L^2V'}^2 + C_\nu \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 \int_0^T \|\mathbf{u}_m\|_{L^2}^2 \, dt \\
&\leq C_\nu \|\mathbf{f}\|_{L^2V'}^2 + C_\nu \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 \, T e^{C_\nu \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 T} \left(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{f}\|_{L^2V'}^2 \right).
\end{aligned}$$

Rearranging this inequality and applying (3.2.4) then yields, for a.e. $t \in [0,T]$,

$$\|\mathbf{u}_m\|_{L^2V}^2 \leq C_\nu \left(1 + \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 \, T e^{C_\nu \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 T} \right) \left(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{f}\|_{L^2V'}^2 \right) \tag{3.2.5}$$

Therefore $\mathbf{u}_m$ is bounded in $L^2(0,T;V)$ independently of m.

Now we check that $\partial_t \mathbf{u}_m$ is bounded in $L^2(0,T;V')$ independently of m. Let $\mathbf{w} \in V$ with $\|\nabla \mathbf{w}\|_{L^2} = 1$. Taking the action of $\partial_t \mathbf{u}$ on $\mathbf{w}$ yields

$$\begin{aligned}
|\langle \partial_t \mathbf{u}_m, \mathbf{w} \rangle| &\leq \nu \, |\langle A\mathbf{u}_m, \mathbf{w} \rangle| + |(P_m B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m), \mathbf{w})| + |(P_m \mathbf{f}, \mathbf{w})| \\
&= \nu \, |(\nabla \mathbf{u}_m, \nabla \mathbf{w})| + |(B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m), P_m \mathbf{w})| + |(\mathbf{f}, P_m \mathbf{w})| \, .
\end{aligned}$$

Note that

$$\nu \, |(\nabla \mathbf{u}_m, \nabla \mathbf{w})| \leq \nu \, \|\nabla \mathbf{u}_m\|_{L^2} \|\nabla \mathbf{w}\|_{L^2} = \nu \, \|\nabla \mathbf{u}_m\|_{L^2} \, ,$$

and

$$|(B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m), P_m \mathbf{w})| \leq C \, \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\nabla \mathbf{u}_m\|_{L^2} \|\nabla \mathbf{w}\|_{L^2} = C \, \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\nabla \mathbf{u}_m\|_{L^2} \, ,$$

and also

$$|(\mathbf{f}, P_m \mathbf{w})| \leq \|\mathbf{f}\|_{V'} \|\mathbf{w}\|_V \leq \|\mathbf{f}\|_{V'} .$$

From this we deduce that

$$\|\partial_t \mathbf{u}_m\|_{L^2V'} \leq C_{\nu,\epsilon} \left(\|\mathbf{u}_m\|_{L^2V} + \|\mathbf{f}\|_{L^2V'}\right), \tag{3.2.6}$$

hence $\partial_t \mathbf{u}_m$ is bounded in $L^2(0,T;V')$ independently of m.

By the Banach-Alaoglu Theorem and the above bounds, there exists $\mathbf{u} \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and a subsequence (relabeled as $\mathbf{u}_m$) such that

$$\mathbf{u}_m \overset{*}{\rightharpoonup} \mathbf{u} \text{ weak-}* \text{ in } L^\infty(0,T;H), \tag{3.2.7}$$

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \text{ weakly in } L^2(0,T;V), \tag{3.2.8}$$

$$\partial_t \mathbf{u}_m \rightharpoonup \partial_t \mathbf{u} \text{ weakly in } L^2(0,T;V'). \tag{3.2.9}$$

Moreover, using the Aubin-Lions lemma one obtains another subsequence (still labelled as $\mathbf{u}_m$) such that

$$\mathbf{u}_m \to \mathbf{u} \text{ strongly in } L^2(0,T;H). \tag{3.2.10}$$

Now we wish to show that passing to the limit in (3.2.2) yields (3.2.1a). Let $\mathbf{w} \in V$, and set $\mathbf{v}_m = \mathbf{u} - \mathbf{u}_m$. We will show that $\mathbf{u}$ is a solution to (3.2.1a) by showing that

$$\begin{aligned}&\langle \partial_t \mathbf{u}, \mathbf{w}\rangle + \nu\left(\nabla \mathbf{u}, \nabla \mathbf{w}\right) + b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}), \mathbf{u}, \mathbf{w}) + \langle \mathbf{f}, \mathbf{w}\rangle \\ &- \left(\partial_t \mathbf{u}_m, \mathbf{w}\right) - \nu\left(\nabla \mathbf{u}_m, \mathbf{w}\right) - b\left(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m, P_m \mathbf{w}\right) - \left(P_m \mathbf{f}, \mathbf{w}\right)\end{aligned}$$

tends to 0 as $m \to \infty$. This expression can be rewritten as follows.

$$
\begin{aligned}
&\langle \partial_t \mathbf{v}_m, \mathbf{w} \rangle + \nu \left(\nabla \mathbf{v}_m, \nabla \mathbf{w} \right) + b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}), \mathbf{v}_m, \mathbf{w}) \\
&+ \left(b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}), \mathbf{u}_m, \mathbf{w}) - b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m, \mathbf{w}) \right) \\
&+ b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m, \left(I - P_m \right) \mathbf{w}) + \langle (I - P_m)\mathbf{f}, \mathbf{w} \rangle .
\end{aligned}
$$

Note that from (3.2.8) and (3.2.9),

$$
\lim_{m \to \infty} \int_0^T \langle \partial_t \mathbf{v}_m, \mathbf{w} \rangle + \nu \left(\nabla \mathbf{v}_m, \nabla \mathbf{w} \right) dt = 0
$$

and

$$
\lim_{m \to \infty} \int_0^T b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}), \mathbf{v}_m, \mathbf{w}) dt = 0
$$

by Lemma 2.1.4 and (3.2.8). Now, using the Lipschitz property of $\boldsymbol{\zeta}^\epsilon$, Hölder's inequality, and the Gagliardo-Nirenberg-Sobolev inequality, we bound the fourth and fifth term as follows:

$$
\begin{aligned}
&\int_0^T b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}), \mathbf{u}_m, \mathbf{w}) - b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m, \mathbf{w}) dt \\
&\leq \int_0^T \left\| \mathbf{v}_m \right\|_{L^3} \left\| \nabla \mathbf{u}_m \right\|_{L^2} \left\| \mathbf{w} \right\|_{L^6} dt \\
&\leq C \int_0^T \left\| \mathbf{v}_m \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \mathbf{v}_m \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \mathbf{u}_m \right\|_{L^2} \left\| \nabla \mathbf{w} \right\|_{L^2} dt \\
&\leq C \left\| \nabla \mathbf{w} \right\|_{L^2} \left\| \mathbf{v}_m \right\|_{L^2L^2}^{\frac{1}{2}} \left\| \nabla \mathbf{v}_m \right\|_{L^2L^2}^{\frac{1}{2}} \left\| \nabla \mathbf{u}_m \right\|_{L^2L^2}.
\end{aligned}
$$

Therefore, since $\|\nabla \mathbf{u}_m\|_{L^2L^2}$ and $\|\nabla \mathbf{v}_m\|_{L^2L^2}$ are bounded and $\mathbf{v}_m \to 0$ strongly,

$$
\lim_{m \to \infty} \int_0^T \left(b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}), \mathbf{u}_m, \mathbf{w}) - b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m), \mathbf{u}_m, \mathbf{w}) \right) dt = 0
$$

as a consequence of (3.2.10). Finally, by (2.2.4) we obtain

$$\begin{aligned}&\lim_{m\to\infty}\left|\int_0^T b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_m),\mathbf{u}_m,(I-P_m)\mathbf{w})dt\right|\\&\leq \lim_{m\to\infty}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\sup_{m\in\mathbb{N}}\|\mathbf{u}_m\|_{L^2V}\left(\lambda_m^{-1/2}\|\mathbf{w}\|_{H^1}\right)\\&=0\end{aligned}$$

and

$$\lim_{m\to\infty}\int_0^T\langle(I-P_m)\mathbf{f},\mathbf{w}\rangle\,dt=0.$$

Thus we deduce that a subsequence of solutions $\mathbf{u}_m$ of (3.2.2) converges to a solution $\mathbf{u}$ of (3.1.4). It remains to be shown that $\mathbf{u}$ is continuous in time and satisfies the initial data. It is an immediate consequence of the Aubin-Lions Compactness Theorem (see, e.g., [90, Corollary 7.3]) that $\mathbf{u}\in C([0,T];L^2)$. To show that the initial data is satisfied, one carries out the procedure performed in, e.g., [28, 105]. □

Remark 3.2.1. It is not known if weak solutions are unique for calmed NSE or calmed rNSE. Indeed, if $\mathbf{u}_1$ and $\mathbf{u}_2$ are weak solutions with same initial data $\mathbf{u}_0$, one can write the difference equation (3.3.4) and obtain the energy equation (3.3.5) as we do in the case of strong solutions. However, for weak solutions it does not seem possible to attain an upper bound for the term $b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_2)-\boldsymbol{\zeta}^\epsilon(\mathbf{u}_1),\mathbf{u}_2,\tilde{\mathbf{u}})$ using the techniques seen in this paper.

3.3 Strong Solutions

In this section we prove the first - and second - order regularity of weak solutions to the calmed NSE (3.2.1) for the purpose of showing that strong solutions are unique.

To this end, we will apply the Aubin-Lions Compactness Theorem [90, Corollary 7.3].

3.3.1 Proof of Theorem 1.2.4

Here, we work formally, but the results can be made rigorous using the Galerkin procedure as in the proofs of the previous theorem. Suppose $\mathbf{u}_0 \in V$ and $\mathbf{f} \in L^2(0,T;H)$ for some $T > 0$. Taking the action of (3.2.1) with $A\mathbf{u}$ and then using the Lions-Magenes Lemma, Young's inequality, and Hölder's inequality yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|A\mathbf{u}\|_{L^2}^2 = b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}),\mathbf{u},A\mathbf{u}) - (\mathbf{f},A\mathbf{u})\\
&\le \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|A\mathbf{u}\|_{L^2} + \|\mathbf{f}\|_{L^2}\|A\mathbf{u}\|_{L^2}\\
&\le C_\nu\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2 + C_\nu\|\mathbf{f}\|_{L^2}^2 + \frac{\nu}{2}\|A\mathbf{u}\|_{L^2}^2
\end{aligned}
$$

Rearranging these terms yields the inequality

$$
\frac{d}{dt}\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|A\mathbf{u}\|_{L^2}^2 \le C_\nu\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2 + C_\nu\|\mathbf{f}\|_{L^2}^2. \tag{3.3.1}
$$

We now remove the viscosity term and apply Grönwall's inequality to obtain for a.e. $t \in [0,T]$,

$$
\|\nabla\mathbf{u}(t)\|_{L^2}^2 \le e^{C_\nu\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 t}\|\nabla\mathbf{u}_0\|_{L^2}^2 + C_\nu\int_0^t e^{-C_\nu\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2(s-t)}\|\mathbf{f}(s)\|_{L^2}^2\,ds. \tag{3.3.2}
$$

Thus $\mathbf{u} \in L^\infty(0,T;V)$ whenever $\mathbf{u}_0 \in V$ and $\mathbf{f} \in L^2(0,T;H)$. Returning to (3.3.1), we integrate in time to obtain

$$
\nu\int_0^T \|A\mathbf{u}\|_{L^2}^2\,dt \le \|\nabla\mathbf{u}_0\|_{L^2}^2 + C_\nu\int_0^T \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2 + \|\mathbf{f}\|_{L^2}^2\,dt. \tag{3.3.3}
$$

From estimates (3.3.2) and (3.3.3) we deduce that $\mathbf{u} \in L^2(0,T;H^2 \cap V)$. It remains to be shown that $\partial_t \mathbf{u} \in L^2(0,T;H)$. This follows immediately from the calculation below:

$$\begin{aligned}\int_0^T \|\partial_t \mathbf{u}\|_{L^2}^2 \, dt &= \int_0^T \|\nu A\mathbf{u} + B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}),\mathbf{u}) + \mathbf{f}\|_{L^2}^2 \, dt \\ &\le C \int_0^T \nu \|A\mathbf{u}\|_{L^2}^2 + \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 \|\nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{f}\|_{L^2}^2 \, dt \\ &< \infty.\end{aligned}$$

Therefore $\partial_t \mathbf{u} \in L^2(0,T;H)$. By the Aubin-Lions Compactness Theorem, we conclude that $\mathbf{u} \in C([0,T];V)$. $\square$

We now proceed in showing the global existence and uniqueness of strong solutions. With the existence of such solutions already known from prior results, this theorem focuses on uniqueness and continuous dependence on initial data.

3.3.2 Proof of Theorem 1.2.5

From Theorems 1.2.3, 1.2.4, and from (2.5.1), we deduce the existence of strong solutions to calmed NSE (3.1.4) and calmed rNSE (3.1.5) satisfying the hypotheses of Definition 1.2.2. Suppose $\mathbf{u}_1$ and $\mathbf{u}_2$ are strong solutions with respective initial data $\mathbf{u}_0^1, \mathbf{u}_0^2 \in V$ and forcing term $\mathbf{f} \in L^2(0,T;H)$. Let $\tilde{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$ and $\tilde{\mathbf{u}}_0 = \mathbf{u}_0^1 - \mathbf{u}_0^2$. When we take the difference of the two equations we obtain

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu \triangle \tilde{\mathbf{u}} = B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_2) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}_1), \mathbf{u}_2) - B(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_1), \tilde{\mathbf{u}}). \tag{3.3.4}$$

We now take the inner-product with $\tilde{\mathbf{u}}$, which yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{u}}\|_{L^2}^2 + \nu\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2 \\ &= b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_2) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}_1), \mathbf{u}_2, \tilde{\mathbf{u}}) - b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_1), \tilde{\mathbf{u}}, \tilde{\mathbf{u}}). \end{aligned} \tag{3.3.5}$$

For the first term, using Hölder's inequality, the Gagliardo-Nirenberg-Sobolev inequality, condition 1 of Definition 2.1.1, and Poincarè's inequality, one obtains

$$\begin{aligned} &|b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_2) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}_1), \mathbf{u}_2, \tilde{\mathbf{u}})| \\ &\leq \|\tilde{\mathbf{u}}\|_{L^3}\|\nabla\mathbf{u}_2\|_{L^6}\|\tilde{\mathbf{u}}\|_{L^2} \\ &\leq C\|\tilde{\mathbf{u}}\|_{L^2}^{\frac{3}{2}}\|\nabla\tilde{\mathbf{u}}\|_{L^2}^{\frac{1}{2}}\|\triangle\mathbf{u}_2\|_{L^2} \\ &\leq C_\nu\|\triangle\mathbf{u}_2\|_{L^2}^{\frac{4}{3}}\|\tilde{\mathbf{u}}\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2. \end{aligned}$$

While for the second term, one obtains

$$|b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}_1), \tilde{\mathbf{u}}, \tilde{\mathbf{u}})| \leq C_\nu\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2\|\tilde{\mathbf{u}}\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2.$$

Inserting these bounds into estimate (3.3.5) then yields

$$\frac{d}{dt}\|\tilde{\mathbf{u}}\|_{L^2}^2 + \nu\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2 \leq C_\nu\left(\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 + \|\triangle\mathbf{u}_2\|_{L^2}^{\frac{4}{3}}\right)\|\tilde{\mathbf{u}}\|_{L^2}^2. \tag{3.3.6}$$

Since $\mathbf{u}_2$ is a strong solution to calmed NSE (3.1.4) we have the containment $\mathbf{u}_2 \in L^2(0,T;H^2\cap V)$, hence

$$A(T) := C_\nu\int_0^T\left(\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 + \|\triangle\mathbf{u}_2\|_{L^2}^{\frac{4}{3}}\right)dt < \infty.$$

Using Grönwall's inequality, it follows that,

$$\|\tilde{\mathbf{u}}(t)\|_{L^2}^2 \leq e^{A(T)} \|\tilde{\mathbf{u}}_0\|_{L^2}^2 . \tag{3.3.7}$$

We conclude that strong solutions to (3.1.4) are unique and depend continuously on initial data. □

3.4 Convergence to strong solutions of the Navier-Stokes equations

In this section we prove that strong solutions $\mathbf{u}^\epsilon$ to calmed NSE will converge to a strong solution $\mathbf{u}$ to NSE on sufficiently small time intervals when $\boldsymbol{\zeta}^\epsilon$ is known to satisfy condition 3 for some minimal value $\beta \geq 1$.

3.4.1 Proof of Theorem 1.2.6

Set $\mathbf{w}^\epsilon = \mathbf{u} - \mathbf{u}^\epsilon$. We then take the action of the difference of (3.1.1) and (3.1.4) with $A\mathbf{w}^\epsilon$ and use the Lions-Magenes Lemma to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{w}^\epsilon\|_{L^2}^2 + \nu \|A\mathbf{w}^\epsilon\|_{L^2}^2 = N, \tag{3.4.1}$$

where the nonlinearity N is rewritten as

$$\begin{aligned} N = \quad & b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \mathbf{u}, \mathbf{u}, A\mathbf{w}^\epsilon) - b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}), \mathbf{w}^\epsilon, A\mathbf{w}^\epsilon) \\ & - b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon), \mathbf{u}, A\mathbf{w}^\epsilon) + b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon), \mathbf{w}^\epsilon, A\mathbf{w}^\epsilon) \\ = \quad & N_1 + N_2 + N_3 + N_4. \end{aligned}$$

For N_1, we use condition 3 of Definition 2.1.1, Agmon's inequality, and Young's inequality to obtain

$$
\begin{aligned}
|N_1| &\le \int_\Omega |\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \mathbf{u}|\, |\nabla \mathbf{u}|\, |A\mathbf{w}^\epsilon|\ d\mathbf{x} \\
&\le C\epsilon^\alpha \int_\Omega |\mathbf{u}|^\beta\, |\nabla \mathbf{u}|\, |A\mathbf{w}^\epsilon|\ d\mathbf{x} \\
&\le C\epsilon^\alpha \, \|\mathbf{u}\|_{L^\infty}^\beta \, \|\nabla \mathbf{u}\|_{L^2} \, \|A\mathbf{w}^\epsilon\|_{L^2} \\
&\le C\epsilon^\alpha \|\mathbf{u}\|_{L^\infty V} \, \|\triangle \mathbf{u}\|_{L^2}^\beta \, \|A\mathbf{w}^\epsilon\|_{L^2} \\
&\le C_\nu \|\mathbf{u}\|_{L^\infty V}^2 \, \|\triangle \mathbf{u}\|_{L^2}^{2\beta} \, \epsilon^{2\alpha} + \frac{\nu}{8} \, \|A\mathbf{w}^\epsilon\|_{L^2}^2 \, .
\end{aligned}
\tag{3.4.2}
$$

For the remaining terms, we use a combination of Agmon's inequality, Poincare's inequality, Hölder's inequality, the Gagliardo-Nirenberg-Sobolev inequality, and Young's inequality. For N_2, we obtain

$$
\begin{aligned}
|N_2| &\le \int_\Omega |\mathbf{u}|\, |\nabla \mathbf{w}^\epsilon|\, |A\mathbf{w}^\epsilon|\ d\mathbf{x} \\
&\le \|\mathbf{u}\|_{L^\infty} \, \|\nabla \mathbf{w}^\epsilon\|_{L^2} \, \|A\mathbf{w}^\epsilon\|_{L^2} \\
&\le C \, \|\triangle \mathbf{u}\|_{L^2} \, \|\nabla \mathbf{w}^\epsilon\|_{L^2} \, \|A\mathbf{w}^\epsilon\|_{L^2} \\
&\le C_\nu \, \|\triangle \mathbf{u}\|_{L^2}^2 \, \|\nabla \mathbf{w}^\epsilon\|_{L^2}^2 + \frac{\nu}{8} \, \|A\mathbf{w}^\epsilon\|_{L^2}^2 \, ,
\end{aligned}
\tag{3.4.3}
$$

where we use the additional fact that $|\boldsymbol{\zeta}^\epsilon(\mathbf{u})| \le |\mathbf{u}|$, which follows from conditions 1 and 3 of Definition 2.1.1. For N_3,

$$
\begin{aligned}
|N_3| &\le \int_\Omega |\mathbf{w}^\epsilon|\, |\nabla \mathbf{u}|\, |A\mathbf{w}^\epsilon|\ d\mathbf{x} \\
&\le \|\nabla \mathbf{u}\|_{L^3} \, \|\mathbf{w}^\epsilon\|_{L^6} \, \|A\mathbf{w}^\epsilon\|_{L^2} \\
&\le C \, \|\nabla \mathbf{u}\|_{L^2}^{\frac{1}{2}} \, \|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} \, \|\nabla \mathbf{w}^\epsilon\|_{L^2} \, \|A\mathbf{w}^\epsilon\|_{L^2} \\
&\le C_\nu \|\mathbf{u}\|_{L^\infty V} \, \|\triangle \mathbf{u}\|_{L^2} \, \|\nabla \mathbf{w}^\epsilon\|_{L^2}^2 + \frac{\nu}{8} \, \|A\mathbf{w}^\epsilon\|_{L^2}^2
\end{aligned}
\tag{3.4.4}
$$

and for N_4, using inequality (2.2.1) we deduce

$$\begin{aligned} |N_4| &\leq \int_\Omega |\mathbf{w}^\epsilon| \, |\nabla \mathbf{w}^\epsilon| \, |A\mathbf{w}^\epsilon| \; d\mathbf{x} \\ &\leq \|\mathbf{w}^\epsilon\|_{L^6} \|\nabla \mathbf{w}^\epsilon\|_{L^3} \|A\mathbf{w}^\epsilon\|_{L^2} \\ &\leq \|\nabla \mathbf{w}^\epsilon\|_{L^2}^{\frac{3}{2}} \|\mathbf{w}^\epsilon\|_{H^2}^{\frac{1}{2}} \|A\mathbf{w}^\epsilon\|_{L^2} \\ &\leq C \|\nabla \mathbf{w}^\epsilon\|_{L^2}^{\frac{3}{2}} \|A\mathbf{w}^\epsilon\|_{L^2}^{\frac{3}{2}} \\ &\leq C_\nu \|\nabla \mathbf{w}^\epsilon\|_{L^2}^{6} + \frac{\nu}{8} \|A\mathbf{w}^\epsilon\|_{L^2}^{2}. \end{aligned} \tag{3.4.5}$$

We now make the ansatz

$$\|\nabla \mathbf{w}^\epsilon\|_{L^2} < 1, \tag{3.4.6}$$

which holds at the initial time by assumption and therefore for a short time since $\mathbf{u}, \mathbf{u}^\epsilon \in C([0,T];V)$. We want to show that this leads to an even tighter bound. To this end, we apply (3.4.6) to estimate (3.4.5), then insert the bounds (3.4.2), (3.4.3), (3.4.4), and (3.4.5) into estimate (3.4.1) which yields

$$\begin{aligned} &\frac{d}{dt} \|\nabla \mathbf{w}^\epsilon\|_{L^2}^2 + \nu \|A\mathbf{w}^\epsilon\|_{L^2}^2 \\ &\leq C_\nu \|\mathbf{u}\|_{L^\infty V}^2 \|\triangle \mathbf{u}\|_{L^2}^{2\beta} \epsilon^{2\alpha} + C_\nu \left(\|\triangle \mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{L^\infty V} \|\triangle \mathbf{u}\|_{L^2} + 1 \right) \|\nabla \mathbf{w}^\epsilon\|_{L^2}^2. \end{aligned} \tag{3.4.7}$$

By (1.2.1) we deduce that the first term and the factor preceding $\|\nabla \mathbf{w}^\epsilon\|_{L^2}^2$ in (3.4.7) are integrable in time. Since $\|\nabla \mathbf{w}^\epsilon(0)\|_{L^2} = 0$, we can use Grönwall's inequality to obtain, for all $t \in [0,T]$,

$$\|\nabla \mathbf{w}^\epsilon(t)\|_{L^2} \leq K\epsilon^\alpha, \tag{3.4.8}$$

where $K > 0$ is a constant depending on $\Omega, \nu, \beta, \|\mathbf{u}\|_{L^\infty V}, \|\triangle \mathbf{u}\|_{L^2}$, and T. By taking $\epsilon > 0$ sufficiently small, it follows that

$$\|\nabla \mathbf{w}^\epsilon(t)\|_{L^2} < \frac{1}{2}$$

for all $t \in [0, T]$. After applying a standard bootstrapping argument (see, e.g., [28]), we conclude that inequality (3.4.8) is valid for all $t \in [0, T]$. $\square$

3.5 Existence of Strong Solutions to 3D Navier-Stokes

To prove Theorem 1.2.7, we begin with a lemma establishing higher-order bounds that are independent of the calming parameter. We assume a uniform-in-time bound on $\mathbf{f}$, namely $\mathbf{f} \in L^\infty((0,\infty); L^2)$. This hypothesis could likely be weakened, but simplicity of presentation, we do not pursue this here.

Lemma 3.5.1. *Let $\nu > 0$. Suppose, for each $\epsilon > 0$, $\mathbf{u}^\epsilon$ is a strong solution to calmed NSE* (3.1.4) *or calmed rNSE* (3.1.5) *with initial data $\mathbf{u}_0 \in V$ and $\mathbf{f} \in L^\infty((0,\infty); L^2)$. On the interval $[0, T_0]$, where*

$$T_0 := \frac{\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + M^2\right)^{-2} - \frac{1}{4}\|\nabla \mathbf{u}_0\|_{L^2}^{-4}}{C_\nu} \tag{3.5.1}$$

and $M := \|\mathbf{f}\|_{L^\infty((0,\infty);L^2)}^{\frac{1}{3}}$, the following inequalities are valid:

$$\sup_{\epsilon>0} \sup_{t\in[0,T_0]} \|\nabla \mathbf{u}^\epsilon(t)\|_{L^2}^2 \leq 2\|\nabla \mathbf{u}_0\|_{L^2}^2 \tag{3.5.2}$$

and

$$\sup_{\epsilon>0}\left\{\nu\int_0^{T_0}\|A\mathbf{u}^\epsilon\|_{L^2}^2\right\} \leq \|\nabla \mathbf{u}_0\|_{L^2}^2 + C_\nu T_0\left(2\|\nabla \mathbf{u}_0\|_{L^2}^2 + M^2\right)^3, \tag{3.5.3}$$

where C_ν is a positive constant that depends on the domain Ω and ν and may change from line to line.

Proof. Following similar steps as before in showing higher-order regularity, we take the action of (3.1.5) on $A\mathbf{u}^\epsilon$, integrate by parts, and apply the Lions-Magenes Lemma, to obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^2 + \nu\left\|A\mathbf{u}^\epsilon\right\|_{L^2}^2 = b(\mathbf{u}^\epsilon, \mathbf{u}^\epsilon, A\mathbf{u}^\epsilon) + (\mathbf{f}, A\mathbf{u}^\epsilon)$$

Now we use the Gagliardo-Nirenberg-Sobolev inequality, the Cauchy-Schwarz inequality, (2.2.1), and Young's inequality, which yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^2 + \nu\left\|A\mathbf{u}^\epsilon\right\|_{L^2}^2 \\
&\leq \left\|\nabla\mathbf{u}^\epsilon\right\|_{L^3}\left\|\mathbf{u}^\epsilon\right\|_{L^6}\left\|A\mathbf{u}\right\|_{L^2} + \left\|\mathbf{f}\right\|_{L^2}\left\|A\mathbf{u}^\epsilon\right\|_{L^2} \\
&\leq C\left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^{\frac{3}{2}}\left\|\mathbf{u}^\epsilon\right\|_{H^2}^{\frac{1}{2}}\left\|A\mathbf{u}^\epsilon\right\|_{L^2} + \left\|\mathbf{f}\right\|_{L^2}\left\|A\mathbf{u}^\epsilon\right\|_{L^2} \\
&\leq C\left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^{\frac{3}{2}}\left\|A\mathbf{u}^\epsilon\right\|_{L^2}^{\frac{3}{2}} + \left\|\mathbf{f}\right\|_{L^2}\left\|A\mathbf{u}^\epsilon\right\|_{L^2} \\
&\leq C_\nu\left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^6 + C_\nu\left\|\mathbf{f}\right\|_{L^2}^2 + \frac{\nu}{2}\left\|A\mathbf{u}^\epsilon\right\|_{L^2}^2 \\
&\leq C_\nu\left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^6 + C_\nu M^6 + \frac{\nu}{2}\left\|A\mathbf{u}^\epsilon\right\|_{L^2}^2 \\
&\leq C_\nu\left(\left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^2 + M^2\right)^3 + \frac{\nu}{2}\left\|A\mathbf{u}^\epsilon\right\|_{L^2}^2 .
\end{aligned}$$

We now rewrite this inequality as

$$\frac{d}{dt}\left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^2 + \nu\left\|A\mathbf{u}^\epsilon\right\|_{L^2}^2 \leq C_\nu\left(\left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^2 + M^2\right)^3 \tag{3.5.4}$$

which, after making the substitution $\eta = \left\|\nabla\mathbf{u}^\epsilon\right\|_{L^2}^2 + M^2$ and removing the diffusive terms, becomes

$$\frac{d}{dt}\eta \leq C_\nu \eta^3.$$

From this inequality we derive, for all $t \in [0, T_0]$,

$$\eta(t) \leq \left(\eta(0)^{-2} - C_\nu T_0\right)^{-\frac{1}{2}}$$

or

$$\|\nabla \mathbf{u}^\epsilon\|_{L^2}^2 + M^2 \leq \left(\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + M^2\right)^{-2} - C_\nu T_0\right)^{-\frac{1}{2}} = 2\,\|\nabla \mathbf{u}_0\|_{L^2}^2 \tag{3.5.5}$$

for T_0 as in (3.5.1), thus proving (3.5.2). We now return to estimate (3.5.4), integrate in time on the interval $[0, T_0]$, and apply estimate (3.5.5) to obtain

$$\begin{aligned}
&\nu \int_0^{T_0} \|A\mathbf{u}^\epsilon\|_{L^2}^2 \, dt \\
&\leq \|\nabla \mathbf{u}_0\|_{L^2}^2 + C_\nu \int_0^{T_0} \left(\|\nabla \mathbf{u}^\epsilon\|_{L^2}^2 + M^2\right)^3 dt \\
&\leq \|\nabla \mathbf{u}_0\|_{L^2}^2 + C_\nu T_0 \left(2\,\|\nabla \mathbf{u}_0\|_{L^2}^2 + M^2\right)^3.
\end{aligned}$$

This proves (3.5.3). □

Our lemma guarantees that for nonzero initial data $\mathbf{u}_0 \in V$ and forcing term $\mathbf{f} \in L^\infty(0, \infty; L^2)$, there exists a positive time T_0 for which, given any $\epsilon > 0$, $\mathbf{u}^\epsilon$ is bounded in $L^\infty(0, T_0; V) \cap L^2(0, T_0; H^2 \cap V)$. We now, proceed to show that, on the time interval $[0, T_0]$, $\{\mathbf{u}^\epsilon\}_{\epsilon>0}$ is Cauchy.

3.5.1 Proof of Theorem 1.2.7

Let $\mathbf{u}^\epsilon$ and $\mathbf{u}^\delta$ be strong solutions to calmed NSE (3.1.4) or calmed rNSE (3.1.5) with initial data $\mathbf{u}_0 \in V$ and with respective calming parameters $\epsilon > 0$ and $\delta > 0$. From the results of Lemma 3.5.1 we ascertain the existence of a maximal time $T > 0$ for which

$$\sup_{\epsilon>0} \sup_{t\in[0,T]} \|\nabla \mathbf{u}^\epsilon(t)\|_{L^2}^2 \le 2 \|\nabla \mathbf{u}_0\|_{L^2}^2 . \tag{3.5.6}$$

From Lemma 3.5.1 we ascertain that $T \ge \frac{1}{4} \|\nabla \mathbf{u}_0\|_{L^2}^{-4} > 0$. Set $\tilde{\mathbf{u}} = \mathbf{u}^\epsilon - \mathbf{u}^\delta$. The system for $\tilde{\mathbf{u}}$ can be written as

$$\begin{aligned} &\partial_t \tilde{\mathbf{u}} + \nu A \tilde{\mathbf{u}} \\ &= -B(\boldsymbol{\zeta}^\delta(\mathbf{u}^\delta), \tilde{\mathbf{u}}) - B(\tilde{\mathbf{u}}, \mathbf{u}^\epsilon) + B(\boldsymbol{\zeta}^\delta(\mathbf{u}^\delta) - \mathbf{u}^\delta, \mathbf{u}^\epsilon) + B(\mathbf{u}^\epsilon - \boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon), \mathbf{u}^\epsilon) \end{aligned}$$

We then take the inner product with $\tilde{\mathbf{u}}$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \|\tilde{\mathbf{u}}\|_{L^2}^2 + \nu \|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 \le& \int_\Omega |\nabla \tilde{\mathbf{u}}| \left|\boldsymbol{\zeta}^\delta(\mathbf{u}^\delta)\right| |\tilde{\mathbf{u}}| \, d\mathbf{x} \\ &+ \int_\Omega |\tilde{\mathbf{u}}|^2 |\nabla \mathbf{u}^\epsilon| \, d\mathbf{x} \\ &+ \int_\Omega |\nabla \mathbf{u}^\epsilon| \left|\boldsymbol{\zeta}^\delta(\mathbf{u}^\delta) - \mathbf{u}^\delta\right| |\tilde{\mathbf{u}}| \, d\mathbf{x} \\ &+ \int_\Omega |\nabla \mathbf{u}^\epsilon| \left|\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \mathbf{u}^\epsilon\right| |\tilde{\mathbf{u}}| \, d\mathbf{x}. \end{aligned}$$

Now, applying condition 3 of $\boldsymbol{\zeta}^\epsilon$ yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \|\tilde{\mathbf{u}}\|_{L^2}^2 + \nu \|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 \le& \int_\Omega |\nabla \tilde{\mathbf{u}}| \left|\mathbf{u}^\delta\right| |\tilde{\mathbf{u}}| \, d\mathbf{x} + \int_\Omega |\tilde{\mathbf{u}}|^2 |\nabla \mathbf{u}^\epsilon| \, d\mathbf{x} \\ &+ C\delta^\alpha \int_\Omega |\nabla \mathbf{u}^\epsilon| \left|\mathbf{u}^\delta\right|^\beta |\tilde{\mathbf{u}}| \, d\mathbf{x} + C\epsilon^\alpha \int_\Omega |\nabla \mathbf{u}^\epsilon| \left|\mathbf{u}^\epsilon\right|^\beta |\tilde{\mathbf{u}}| \, d\mathbf{x}. \end{aligned}$$

Using Hölder's inequality, Agmon's inequality, Poincaré's inequality, (2.2.1), and Young's inequality, for the first term we obtain

$$\begin{aligned}\int_\Omega |\nabla \tilde{\mathbf{u}}| \left|\boldsymbol{\zeta}^\delta(\mathbf{u}^\delta)\right| |\tilde{\mathbf{u}}| \, d\mathbf{x} &\leq \|\nabla \tilde{\mathbf{u}}\|_{L^2} \left\|\mathbf{u}^\delta\right\|_{L^\infty} \|\tilde{\mathbf{u}}\|_{L^2} \\ &\leq C \|\nabla \tilde{\mathbf{u}}\|_{L^2} \left\|\mathbf{u}^\delta\right\|_{H^2} \|\tilde{\mathbf{u}}\|_{L^2} \\ &\leq C \|\nabla \tilde{\mathbf{u}}\|_{L^2} \left\|A\mathbf{u}^\delta\right\|_{L^2} \|\tilde{\mathbf{u}}\|_{L^2} \\ &\leq C_\nu \left\|A\mathbf{u}^\delta\right\|_{L^2}^2 \|\tilde{\mathbf{u}}\|_{L^2}^2 + \frac{\nu}{8} \|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 .\end{aligned} \tag{3.5.7}$$

and similarly for the second term, using also (3.5.2),

$$\begin{aligned}\int_\Omega |\tilde{\mathbf{u}}|^2 |\nabla \mathbf{u}^\epsilon| \, d\mathbf{x} &\leq \|\tilde{\mathbf{u}}\|_{L^3} \|\nabla \mathbf{u}^\epsilon\|_{L^2} \|\tilde{\mathbf{u}}\|_{L^6} \\ &\leq C \|\tilde{\mathbf{u}}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{u}_0\|_{L^2} \|\nabla \tilde{\mathbf{u}}\|_{L^2}^{\frac{3}{2}} \\ &\leq C_\nu \|\nabla \mathbf{u}_0\|_{L^2}^4 \|\tilde{\mathbf{u}}\|_{L^2}^2 + \frac{\nu}{8} \|\nabla \tilde{\mathbf{u}}\|_{L^2}^2\end{aligned} \tag{3.5.8}$$

Using the same inequalities for the third term, we deduce that

$$\begin{aligned}C\delta^\alpha \int_\Omega |\nabla \mathbf{u}^\epsilon| \left|\mathbf{u}^\delta\right|^\beta |\tilde{\mathbf{u}}| \, d\mathbf{x} &\leq C\delta^\alpha \|\nabla \mathbf{u}^\epsilon\|_{L^6} \left\|\mathbf{u}^\delta\right\|_{L^{2\beta}}^\beta \|\tilde{\mathbf{u}}\|_{L^3} \\ &\leq C\delta^\alpha \|\mathbf{u}^\epsilon\|_{H^2} \left\|\mathbf{u}^\delta\right\|_{L^{2\beta}}^\beta \|\tilde{\mathbf{u}}\|_{L^2}^{\frac{1}{2}} \|\nabla \tilde{\mathbf{u}}\|_{L^2}^{\frac{1}{2}} \\ &\leq C\delta^\alpha \|A\mathbf{u}^\epsilon\|_{L^2} \left\|\mathbf{u}^\delta\right\|_{L^{2\beta}}^\beta \|\tilde{\mathbf{u}}\|_{L^2}^{\frac{1}{2}} \|\nabla \tilde{\mathbf{u}}\|_{L^2}^{\frac{1}{2}} .\end{aligned}$$

For $\beta \in [1, 3]$, from the Gagliardo-Nirenberg-Sobolev inequality and (3.5.6) we have

$$\left\|\mathbf{u}^\delta\right\|_{L^{2\beta}}^\beta \leq C_\beta \|\nabla \mathbf{u}_0\|_{L^2}^\beta .$$

We insert this bound into estimate (3.5.7) and apply Young's inequality to obtain

$$\begin{aligned}
&C\delta^{\alpha}\int_{\Omega}|\nabla \mathbf{u}^{\epsilon}|\left|\mathbf{u}^{\delta}\right|^{\beta}|\tilde{\mathbf{u}}|\;d\mathbf{x} \\
&\leq C_{\beta,\nu}\delta^{2\alpha}\left\|\nabla \mathbf{u}_0\right\|_{L^2}^{2\beta}\left\|A\mathbf{u}^{\epsilon}\right\|_{L^2}^{2}+C_{\nu}\left\|\tilde{\mathbf{u}}\right\|_{L^2}^{2}+\frac{\nu}{8}\left\|\nabla\tilde{\mathbf{u}}\right\|_{L^2}^{2}.
\end{aligned} \tag{3.5.9}$$

We follow the same procedure for the final term:

$$\begin{aligned}
&C\epsilon^{\alpha}\int_{\Omega}|\nabla \mathbf{u}^{\epsilon}|\left|\mathbf{u}^{\epsilon}\right|^{\beta}|\tilde{\mathbf{u}}|\;d\mathbf{x} \\
&\leq C_{\beta,\nu}\epsilon^{2\alpha}\left\|\nabla \mathbf{u}_0\right\|_{L^2}^{2\beta}\left\|A\mathbf{u}^{\epsilon}\right\|_{L^2}^{2}+C_{\nu}\left\|\tilde{\mathbf{u}}\right\|_{L^2}^{2}+\frac{\nu}{8}\left\|\nabla\tilde{\mathbf{u}}\right\|_{L^2}^{2}.
\end{aligned} \tag{3.5.10}$$

Invoking (3.5.7), (3.5.8), (3.5.9), and (3.5.10) yields the upper bound

$$\begin{aligned}
&\frac{d}{dt}\left\|\tilde{\mathbf{u}}\right\|_{L^2}^{2}+\left\|\nabla\tilde{\mathbf{u}}\right\|_{L^2}^{2} \\
&\leq C_{\nu}\left(\left\|A\mathbf{u}^{\delta}\right\|_{L^2}^{2}+\left\|\nabla\mathbf{u}_0\right\|_{L^2}^{4}+1\right)\left\|\tilde{\mathbf{u}}\right\|_{L^2}^{2}+C_{\beta,\nu}\left\|\nabla\mathbf{u}_0\right\|_{L^2}^{2\beta}\left\|A\mathbf{u}^{\epsilon}\right\|_{L^2}^{2}\left(\delta^{2\alpha}+\epsilon^{2\alpha}\right) \\
&\leq K_1\left\|\tilde{\mathbf{u}}\right\|_{L^2}^{2}+K_2\left(\delta^{2\alpha}+\epsilon^{2\alpha}\right),
\end{aligned} \tag{3.5.11}$$

where K_1 and K_2 depend on ν, β, T and $\left\|\nabla \mathbf{u}_0\right\|_{L^2}$, but not ϵ or δ, and are determined by Lemma 3.5.1. Now, we apply Grönwall's inequality to obtain, for all $t \in [0, T]$,

$$\left\|\tilde{\mathbf{u}}(t)\right\|_{L^2} \leq K_3\left(\delta^{2\alpha}+\epsilon^{2\alpha}\right), \tag{3.5.12}$$

where

$$K_3 = \frac{K_2}{K_1}\left(e^{K_1 T}-1\right)\left(\delta^{2\alpha}+\epsilon^{2\alpha}\right).$$

Therefore we see that $\lim\limits_{\delta,\epsilon\to 0}\left\|\mathbf{u}^{\epsilon}-\mathbf{u}^{\delta}\right\|_{L^2}=0$, hence $\{\mathbf{u}^{\epsilon}\}_{\epsilon>0}$ is Cauchy in $L^{\infty}H$ with respect to the calming parameter. If instead we integrate (3.5.11) on $[0, T]$, we can derive the upper bound

$$\nu \int_0^T \|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 \, dt \le K_1 \int_0^T \|\tilde{\mathbf{u}}\|_{L^2}^2 \, dt + K_2 T \left(\delta^{2\alpha} + \epsilon^{2\alpha}\right) \le K_1 T \|\tilde{\mathbf{u}}\|_{L^\infty L^2} + K_2 T \left(\delta^{2\alpha} + \epsilon^{2\alpha}\right), \tag{3.5.13}$$

hence $\{\mathbf{u}^\epsilon\}_{\epsilon>0}$ is also Cauchy in L^2V. Therefore, there exists $\mathbf{u} \in L^\infty H \cap L^2 V$ for which

$$\mathbf{u}^\epsilon \to \mathbf{u} \text{ strongly in } \mathbf{u} \in L^\infty H \cap L^2 V \tag{3.5.14}$$

as $\epsilon \to 0$. We now show that this limit point $\mathbf{u}$ is in fact a solution to 3D rNSE (3.1.3). First note that, owing to Lemma 3.5.1, the equivalence (2.2.1), the Banach-Alaoglu Theorem, and the usual uniqueness of limits, it follows that

$$\mathbf{u} \in L^\infty V \cap L^2(H^2 \cap V). \tag{3.5.15}$$

Set $\mathbf{u}^* = \mathbf{u}^\epsilon - \mathbf{u}$, and take the action of (3.1.3) against an arbitrary test function $\mathbf{w} \in C_c^1([0,T);V)$ and integrate by parts in time (noting that $\mathbf{w}\big|_{t=T} = \mathbf{0}$),

$$-\int_0^T \langle \mathbf{u}^\epsilon, \partial_t \mathbf{w}\rangle \; dt + \nu \int_0^T (\nabla \mathbf{u}^\epsilon, \nabla \mathbf{w}) \; dt + \int_0^T b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon), \mathbf{u}^\epsilon, \mathbf{w}) \, dt = \langle \mathbf{u}_0, \mathbf{w}(0)\rangle + \int_0^T \langle \mathbf{f}, \mathbf{w}\rangle \; dt.$$

Thanks to (3.5.14), the first two terms converge to their Navier-Stokes analogues. For the nonlinear term, we estimate

$$\left| \int_0^T b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon), \mathbf{u}^\epsilon, \mathbf{w}) \, dt - \int_0^T b(\mathbf{u}, \mathbf{u}, \mathbf{w}) \, dt \right|$$

$$
\begin{aligned}
&\le \int_0^T |b(\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \mathbf{u}^\epsilon, \mathbf{u}^\epsilon, \mathbf{w})| \, dt + \int_0^T |b(\mathbf{u}^*, \mathbf{u}^\epsilon, \mathbf{w})| \, dt + \int_0^T |b(\mathbf{u}, \mathbf{u}^*, \mathbf{w})| \, dt \\
&\le \int_0^T \Bigg(C\epsilon^\alpha \left\| \mathbf{u}^\epsilon \right\|_{L^{2\beta}}^\beta \left\| \nabla \mathbf{u}^\epsilon \right\|_{L^3} \left\| \mathbf{w} \right\|_{L^6} \\
&\qquad + \left\| \mathbf{u}^* \right\|_{L^3} \left\| \nabla \mathbf{u}^\epsilon \right\|_{L^2} \left\| \mathbf{w} \right\|_{L^6} + \left\| \mathbf{u} \right\|_{L^3} \left\| \nabla \mathbf{u}^* \right\|_{L^2} \left\| \mathbf{w} \right\|_{L^6} \Bigg) dt \\
&\le \int_0^T \Bigg(C_\beta \epsilon^\alpha \left\| \nabla \mathbf{u}_0 \right\|_{L^2}^\beta \left\| \nabla \mathbf{u}^\epsilon \right\|_{L^2}^{\frac{1}{2}} \left\| A\mathbf{u}^\epsilon \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \mathbf{w} \right\|_{L^2} \\
&\qquad + \left\| \mathbf{u}^* \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \mathbf{u}^* \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \mathbf{u}^\epsilon \right\|_{L^2} \left\| \nabla \mathbf{w} \right\|_{L^2} \\
&\qquad\quad + \int_0^T \left\| \mathbf{u} \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \mathbf{u} \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla \mathbf{u}^* \right\|_{L^2} \left\| \nabla \mathbf{w} \right\|_{L^2} \Bigg) dt \\
&\le C_\beta \epsilon^\alpha \left\| \nabla \mathbf{u}_0 \right\|_{L^2}^\beta \left\| A\mathbf{u}^\epsilon \right\|_{L^2 L^2} \|\mathbf{w}\|_{L^2 V} \\
&\qquad + C\|\mathbf{u}^*\|_{L^\infty L^2}^{\frac{1}{2}} \|\mathbf{u}^*\|_{L^\infty V}^{\frac{1}{2}} \left\| \nabla \mathbf{u}_0 \right\|_{L^2} \int_0^T \left\| \nabla \mathbf{w} \right\|_{L^2} \, dt \\
&\qquad + \|\mathbf{u}\|_{L^\infty L^2}^{\frac{1}{2}} \|\mathbf{u}\|_{L^\infty V}^{\frac{1}{2}} \|\mathbf{u}^*\|_{L^\infty L^2} \int_0^T \left\| \nabla \mathbf{w} \right\|_{L^2} \, dt,
\end{aligned}
$$

where the three terms vanish as $\epsilon \to 0^+$ as a consequence of (3.5.3) and (3.5.14). Hence, sending $\epsilon \to 0^+$, and choosing $\mathbf{w} \in C_c^\infty((0,T), V)$ we obtain

$$
\partial_t \mathbf{u} + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f} \tag{3.5.16}
$$

holding in the distributional sense in time with values in V, i.e., in the sense of $\mathcal{D}'((0,T), V')$. But then, as in, e.g., [105], Chapter 3, Lemma 1.1, since the other terms in (3.5.16) are in $L^2 H$, it holds that $\partial_t \mathbf{u} \in L^2 H$, and moreover, (3.5.16) holds in the sense of $\partial_t \mathbf{u} \in L^2 H$. A standard argument (see, e.g., [20, 105]) shows that the initial data is satisfied in the sense of $C([0,T]; V)$. That is, $\mathbf{u}$ is a strong solution to the Navier-Stokes equations.

3.6 An Energy Equality for Weak Solutions

In this section we focus only on the calmed rotational Navier-Stokes equations (3.1.5). We also assume that $\boldsymbol{\zeta}^\epsilon$ satisfies condition (4) of Definition 2.1.1, so that $((\nabla \times \mathbf{u}) \times \boldsymbol{\zeta}^\epsilon(\mathbf{u})) \cdot \mathbf{u} = 0$ in the L^2-sense thanks to (3.1.6).

3.6.1 Proof of Theorem 1.2.8

Suppose $\mathbf{f} \in L^2(0,T;V')$. Let $\mathbf{u}$ be a weak solution to calmed rNSE as in Definition 3.2.1, with the nonlinearity given by $B(\mathbf{u},\mathbf{v}) = P_\sigma\left((\nabla \times \mathbf{v}) \times \mathbf{u}\right)$. Taking the action of the equation in V' with $\mathbf{u}$ and using the Lions-Magenes Lemma[1] and the fact that $\partial_t \mathbf{u} \in L^2(0,T;V')$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{u}\|_{L^2}^2 = \langle \mathbf{f}, \mathbf{u}\rangle .$$

Integrating in time and using the fact that $\mathbf{u} \in C([0,T];H)$, we find that, for any $t > 0$,

$$\|\mathbf{u}(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla\mathbf{u}(s)\|_{L^2}^2\, ds = \|\mathbf{u}_0\|_{L^2}^2 + 2\int_0^t \langle \mathbf{f}(s), \mathbf{u}(s)\rangle\, ds.$$

Therefore equations (1.2.3) and (1.2.4) are valid, proving Theorem 1.2.8. □

Remark 3.6.1. Let us briefly compare system (3.1.5) with the 3D NSE. For the 3D NSE, it was shown in [7, 70] that weak solutions are non-unique, but it is currently a major open problem to show whether weak solutions that satisfy the energy inequality (called Leray-Hopf solutions) are unique. In contrast, weak solutions of (3.1.5) are

[1] As is well-known, the Lions-Magenes Lemma is *not* known to apply in the setting of weak solutions to the 3D NSE, since for those solutions, it is only known that $\partial_t\mathbf{u} \in L^{4/3}(0,T;V')$, preventing a proof of an energy equality for weak solutions of the 3D NSE. This seems to be an important distinction of system (3.1.5) from the 3D NSE.

not known to be unique, but we have just shown that they satisfy not only an energy inequality, but an energy equality. Hence, (3.1.5) is an example of a system which is very similar to the 3D NSE (especially given the convergence in Theorem 1.2.6), where an energy equality is known for weak solutions but for which a proof of uniqueness of weak solutions remains elusive.

Remark 3.6.2. It may be worth studying analogues of so-called "suitable weak solutions," proposed for the 3D NSE in [26], for which a local energy inequality holds. This would be especially interesting for system (3.1.5) under assumption (4) in Definition 2.1.1 due to the point-wise vanishing of the nonlinear term. However, we postpone this study to a future work.

3.7 A Global Attractor

From the existence of the energy identity (1.2.3) we are able to prove the existence of a global attractor for the dynamical system generated by solutions of calmed rNSE (3.1.5).

3.7.1 Proof of Theorem 1.2.10

Consider again the calmed rotational Navier-Stokes equations (3.1.5), under conditions 1, 2, 3, and 4 of Definition 2.1.1. Take $\mathbf{f} \in H$ to be time-independent, and for a given $R > 0$, let $B_R = \{\mathbf{u} \in H : \|\mathbf{u}\|_{L^2} \leq R\}$. Now choose $\mathbf{u}_0 \in B_R$. On the right hand side of (1.2.3), we use Hölder's, Poincaré's, and Young's inequalities to obtain

$$|(\mathbf{f}, \mathbf{u})| \leq \frac{1}{2\nu\lambda_1} \|\mathbf{f}\|_{L^2}^2 + \frac{\nu}{2} \|\nabla \mathbf{u}\|_{L^2}^2$$

We insert the second estimate into the first and rearrange the terms, which yield

$$\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 + \nu \|\nabla \mathbf{u}\|_{L^2}^2 \le \frac{1}{\nu\lambda_1}\|\mathbf{f}\|_{L^2}^2 . \tag{3.7.1}$$

We apply Poincaré's inequality once more,

$$\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 + \nu\lambda_1 \|\mathbf{u}\|_{L^2}^2 \le \frac{1}{\nu\lambda_1}\|\mathbf{f}\|_{L^2}^2 ,$$

then we apply Grönwall's inequality:

$$\begin{aligned}\|\mathbf{u}(t)\|_{L^2}^2 &\le e^{-\nu\lambda_1 t}\|\mathbf{u}_0\|_{L^2}^2 + \frac{1}{\nu\lambda_1}\left(1 - e^{-\nu\lambda_1 t}\right)\|\mathbf{f}\|_{L^2}^2 \\ &\le e^{-\nu\lambda_1 t}R^2 + \frac{1}{\nu\lambda_1}\left(1 - e^{-\nu\lambda_1 t}\right)\|\mathbf{f}\|_{L^2}^2\end{aligned}$$

We now set

$$t_0 = \frac{1}{\nu\lambda_1}\ln(1+R^2),$$

so that

$$\max\left\{e^{-\nu\lambda_1 t}, e^{-\nu\lambda_1 t}R^2\right\} < 1$$

for all $t \ge t_0$. Then we obtain

$$\|\mathbf{u}(t)\|_{L^2}^2 < \rho_0 \tag{3.7.2}$$

for all $t \ge t_0$, where $\rho_0 = 1 + \frac{1}{\nu\lambda_1}\|\mathbf{f}\|_{L^2}^2$.

If instead we integrate (3.7.1) on the interval[2] $[t-1, t]$ for some $t \ge t_0 + 1$, we

[2] Here, the "1" in "$t-1$" has dimensions of time. Instead, one could consider the interval $[t-\tau, t]$, where $\tau = \frac{1}{\nu\lambda_1}$, but we use a unit interval to simplify the presentation.

obtain

$$\|\mathbf{u}(t)\|_{L^2}^2 + \nu \int_{t-1}^{t} \|\nabla \mathbf{u}\|_{L^2}^2 \leq \|\mathbf{u}(t-1)\|_{L^2}^2 + \frac{1}{\nu\lambda_1} \|\mathbf{f}\|_{L^2}^2 \,,$$

from which we deduce, by (3.7.2),

$$\int_{t-1}^{t} \|\nabla \mathbf{u}\|_{L^2}^2 \, ds \leq \rho_1, \tag{3.7.3}$$

where $\rho_1 = \frac{1}{\nu}\rho_0 + \frac{1}{\nu^2\lambda_1} \|\mathbf{f}\|_{L^2}^2$. Now, we take the action of (3.1.5a) with $-\triangle \mathbf{u}$, and use the Lions-Magenes Lemma to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 + \nu \|\triangle \mathbf{u}\|_{L^2}^2 \\ &= ((\nabla \times \mathbf{u}) \times \boldsymbol{\zeta}^\epsilon(\mathbf{u}), \triangle \mathbf{u}) - (\mathbf{f}, \triangle \mathbf{u}) \\ &\leq C_\nu \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\nabla \mathbf{u}\|_{L^2}^2 + C_\nu \|\mathbf{f}\|_{L^2}^2 + \frac{\nu}{2} \|\triangle \mathbf{u}\|_{L^2}^2 \end{aligned}$$

We then rearrange the inequality above which yields

$$\frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 + \nu \|\triangle \mathbf{u}\|_{L^2}^2 \leq C_\nu \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\nabla \mathbf{u}\|_{L^2}^2 + C_\nu \|\mathbf{f}\|_{L^2}^2 \,. \tag{3.7.4}$$

Now, select s and t such that $t > t_0 + 1$ and $t - 1 < s < t$. We remove the viscous term from the left-hand side, then integrate (3.7.4) on the interval $[s, t]$ and apply (3.7.3) to obtain

$$\|\nabla \mathbf{u}(t)\|_{L^2}^2 \leq \|\nabla \mathbf{u}(s)\|_{L^2}^2 + C_\nu \|\mathbf{f}\|_{L^2}^2 + C_\nu \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \rho_1. \tag{3.7.5}$$

Integrating once more in s on the interval $[t-1, t]$ and again using (3.7.3), it follows that, for $t > t_0 + 1$,

$$\|\nabla \mathbf{u}(t)\|_{L^2}^2 \leq \rho_2, \tag{3.7.6}$$

where $\rho_2 = \rho_1 + C_\nu \|\mathbf{f}\|_{L^2}^2 + C_\nu \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \rho_1$. From this inequality we deduce that $B_{\rho_2} = \{\mathbf{u} \in H : \|\mathbf{u}\|_{L^2} \leq \rho_2\}$ is bounded in V. Since V is compactly embedded in H, we deduce that B_{ρ_2} is a compact absorbing set in H. Applying Theorem 10.5 of [90], we conclude that there exists a global attractor in H. $\square$

Remark 3.7.1. Observe that the upper bounds in (3.7.4), (3.7.5), (3.7.6) each depend on $\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}$. Therefore these upper bounds do not remain valid as $\epsilon \to 0^+$, since $\lim_{\epsilon \to 0^+} \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} = \infty$.

3.8 Conclusions

We proposed two modifications of the 3D Navier-Stokes equations: one involved a modification to the advective velocity term of Navier-Stokes (with kinematic pressure), which we refer to as 'calmed Navier-Stokes,' and the other involves a modification to the Lamb vector of Navier-Stokes (with Bernoulli pressure), which we term 'calmed rotational Navier-Stokes.' We have successfully demonstrated the existence of weak solutions for both of these calmed systems, although the question of whether these solutions are unique remains open. Furthermore, we have established the global well-posedness for strong solutions in both cases. Moreover, we demonstrate that calmed strong solutions do converge to strong solutions of the Navier-Stokes equations on sufficiently small time intervals, provided suitable conditions on the calming function and suitable regularity of the solution to Navier-Stokes.

In the context of the calmed rotational Navier-Stokes Equations (for suitable calming functions), we also establish the existence of an energy identity and the presence of a compact global attractor within the function space H.

Chapter 4

The 3D Navier-Stokes Equations in Velocity-Vorticity formulation

4.1 Introduction

The velocity-vorticity model of the 3D Navier-Stokes Equations is

$$\begin{cases} \partial_t \mathbf{u} - \nu \triangle \mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} + \nabla p = \mathbf{f}, & (4.1.1a) \\ \partial_t \boldsymbol{\omega} - \nu \triangle \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} = \nabla \times \mathbf{f}. & (4.1.1b) \\ \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0, & (4.1.1c) \end{cases}$$

where $\mathbf{u}$ and $\boldsymbol{\omega}$ are coupled by the relation $\nabla \times \boldsymbol{\omega} = -\triangle \mathbf{u}$. There is a large body of research on the analysis of the Navier-Stokes equations in its velocity-vorticity formulation (4.1.1) and its utility in numerical implementation, (see, e.g., [63, 69, 74, 82–84, 88, 109, 110]. In this formulation, one can see that for sufficiently smooth solutions with zero forcing the velocity $\mathbf{u}$ enjoys the energy equality

$$\|\mathbf{u}\|_{L^2}^2 + 2\nu \int_0^T \|\nabla \mathbf{u}\|_{L^2}^2 = \|\mathbf{u}_0\|_{L^2}^2 \tag{4.1.2}$$

thus its kinetic energy $\|\mathbf{u}\|_{L^2}^2$ dissipates over time. However, the vorticity stretching term $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$ present in the governing equations for $\boldsymbol{\omega}$ is capable of generating details at arbitrarily fine length-scales, making this system intractable. We propose applying a calming function to (4.1.1) to obtain the following system, the calmed Velocity-Vorticity equations (calmed VV),

$$\begin{cases} \partial_t \mathbf{u} - \nu\triangle\mathbf{u} + \mathbf{w}\times\mathbf{u} + \nabla p = \mathbf{f}, & \text{(4.1.3a)} \\ \partial_t \mathbf{w} - \nu\triangle\mathbf{w} + \mathbf{u}\cdot\nabla\mathbf{w} - \boldsymbol{\zeta}^{\epsilon}(\mathbf{w})\cdot\nabla\mathbf{u} = \nabla\times\mathbf{f}, & \text{(4.1.3b)} \\ \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{w}(0) = \mathbf{w}_0 & \text{(4.1.3c)} \end{cases}$$

The weak formulation of (4.1.3) is as follows: find $\mathbf{u}, \mathbf{w}$ which satisfy

$$\begin{cases} \partial_t \mathbf{u} + \nu A\mathbf{u} + P_\sigma\left(\mathbf{w}\times\mathbf{u}\right) = P_\sigma\mathbf{f}, & \text{(4.1.4a)} \\ \partial_t \mathbf{w} + \nu A\mathbf{w} + B(\mathbf{u},\mathbf{w}) - B(\boldsymbol{\zeta}^{\epsilon}(\mathbf{w}),\mathbf{u}) = P_\sigma\left(\nabla\times\mathbf{f}\right). & \text{(4.1.4b)} \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{w}(0) = \mathbf{w}_0 & \text{(4.1.4c)} \end{cases}$$

where equality holds in the functional sense.

4.2 Existence of weak solutions

4.2.1 Proof of Theorem 1.2.14

Take the inner product of (4.1.4a) with $\mathbf{u}$ and (4.1.4b) with $\mathbf{w}$, then add the equations to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{w}\|_{L^2}^2\right) + \nu\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{w}\|_{L^2}^2\right) \qquad \text{(4.2.1)} \\ &= (\boldsymbol{\zeta}^{\epsilon}(\mathbf{w})\cdot\nabla\mathbf{u},\mathbf{w}) + (\mathbf{f},\mathbf{u}) + (\nabla\times\mathbf{f},\mathbf{w}) \end{aligned}$$

$$= (\boldsymbol{\zeta}^\epsilon(\mathbf{w}) \cdot \nabla \mathbf{u}, \mathbf{w}) + (\mathbf{f}, \mathbf{u}) + (\mathbf{f}, \nabla \times \mathbf{w})$$

$$\leq \|\boldsymbol{\zeta}^\epsilon(\mathbf{w})\|_{L^\infty} \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{w}\|_{L^2} + \|\mathbf{f}\|_{L^2} \|\mathbf{u}\|_{L^2} + \|\mathbf{f}\|_{L^2} \|\nabla \times \mathbf{w}\|_{L^2}$$

$$\leq \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{w}\|_{L^2} + \|\mathbf{f}\|_{L^2} \|\mathbf{u}\|_{L^2} + C \|\mathbf{f}\|_{L^2} \|\nabla \mathbf{w}\|_{L^2}$$

where $(\mathbf{w} \times \mathbf{u}, \mathbf{u}) = 0$ by orthogonality and $(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{w}) = 0$ by symmetry. Now, using Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{w}\|_{L^2}^2\right) + \nu\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2\right) \tag{4.2.2}$$
$$\leq \frac{1}{2\nu}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 \|\mathbf{w}\|_{L^2}^2 + \frac{\nu}{2}\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{2}\|\mathbf{f}\|_{L^2}^2 + \frac{1}{2}\|\mathbf{u}\|_{L^2}^2 + \frac{C}{\nu^{1/2}}\|\mathbf{f}\|_{L^2}^2 + \frac{\nu}{2}\|\nabla \mathbf{w}\|_{L^2}^2 .$$

We can now rewrite the inequality as

$$\frac{d}{dt}\left(\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{w}\|_{L^2}^2\right) + \nu\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2\right) \tag{4.2.3}$$
$$\leq \left(\frac{1}{\nu}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 + 1\right)\left(\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{w}\|_{L^2}^2\right) + C_\nu \|\mathbf{f}\|_{L^2}^2 .$$

We first remove the diffusive term from the equation and apply Grönwall's inequality, which yields

$$\|\mathbf{u}(t)\|_{L^2}^2 + \|\mathbf{w}(t)\|_{L^2}^2 \leq e^{\left(\frac{1}{\nu}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 + 1\right)T}\left(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{w}_0\|_{L^2}^2 + C_\nu \|\mathbf{f}\|_{L^2 L^2}^2\right). \tag{4.2.4}$$

Therefore $\mathbf{u}, \mathbf{w} \in L^\infty(0, T; H)$. Set $K_{\nu,\epsilon} = \frac{1}{\nu}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2 + 1$. Now, we integrate (4.2.3) in time on the interval $[0, T]$, then apply estimate (4.2.4) to obtain

$$\nu \int_0^T \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 \, dt \tag{4.2.5}$$
$$\leq \left(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{w}_0\|_{L^2}^2\right) + K_{\nu,\epsilon}\int_0^T \|\mathbf{u}(t)\|_{L^2}^2 + \|\mathbf{w}(t)\|_{L^2}^2 \, dt + C_\nu \|\mathbf{f}\|_{L^2 L^2}^2$$

$$\leq \left(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{w}_0\|_{L^2}^2\right) + K_{\nu,\epsilon} T e^{K_{\nu,\epsilon} T} \left(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{w}_0\|_{L^2}^2 + C_\nu \|\mathbf{f}\|_{L^2 L^2}^2\right) + C_\nu \|\mathbf{f}\|_{L^2 L^2}^2.$$

Therefore we have $\mathbf{u}, \mathbf{w} \in L^2(0,T;V)$.

It remains to be shown that $\partial_t \mathbf{u} \in L^2(0,T;V^{-1})$ and $\partial_t \mathbf{w} \in L^{\frac{4}{3}}(0,T;V^{-1})$, which we will do in two steps. First, since

$$\partial_t \mathbf{u} = \nu \triangle \mathbf{u} - \mathbf{w} \times \mathbf{u} + \mathbf{f},$$

and since it is already known that $\nu \triangle \mathbf{u}, \mathbf{f} \in L^2(0,T;V^{-1})$, one only needs to verify that $\mathbf{w} \times \mathbf{u} \in L^2(0,T;V^{-1})$. In fact, we are able to obtain an even better result. For $\boldsymbol{\phi} \in L^{4/3}(0,T;V)$, applying Hölder's inequality, a Gagliardo-Nirenberg inequality, and Poincarè's inequality,

$$\begin{aligned}
&\int_0^T |\langle \mathbf{w} \times \mathbf{u}, \boldsymbol{\phi} \rangle| \, dt \qquad (4.2.6)\\
&\leq \int_0^T \|\mathbf{w}\|_{L^2} \|\mathbf{u}\|_{L^3} \|\boldsymbol{\phi}\|_{L^6} \, dt \\
&\leq C \int_0^T \|\mathbf{w}\|_{L^2} \|\mathbf{u}\|_{L^2}^{1/2} \|\nabla \mathbf{u}\|_{L^2}^{1/2} \|\nabla \boldsymbol{\phi}\|_{L^2} \, dt \\
&\leq C \|\mathbf{w}\|_{L^\infty L^2} \|\mathbf{u}\|_{L^\infty L^2}^{1/2} \|\mathbf{u}\|_{L^2 V}^{1/2} \|\boldsymbol{\phi}\|_{L^{4/3} V},
\end{aligned}$$

where the integrability of this upper bound is deduced from (4.2.4) and (4.2.5). Thus $\partial_t \mathbf{u} \in L^2(0,T;V^{-1})$[1].

Now, for

$$\partial_t \mathbf{w} = \nu \triangle \mathbf{w} - \mathbf{u} \cdot \nabla \mathbf{w} + \boldsymbol{\zeta}^\epsilon(\mathbf{w}) \cdot \nabla \mathbf{u} + \nabla \times \mathbf{f},$$

we have again that $\nu \triangle \mathbf{u}, \nabla \times \mathbf{f} \in L^2(0,T;V^{-1})$, hence we need only check the non-linear terms. In fact, for the calmed term we are able to show that $\boldsymbol{\zeta}^\epsilon(\mathbf{w}) \cdot \nabla \mathbf{u} \in$

[1]In this case, the regularity of $\partial_t \mathbf{u}$ is limited by the diffusive term $\nu \triangle \mathbf{u}$.

$L^2(0,T;H)$:

$$\begin{aligned}
&\int_0^T |\langle \boldsymbol{\zeta}^\epsilon(\mathbf{w}) \cdot \nabla \mathbf{u}, \boldsymbol{\phi}\rangle|\, dt \\
&\leq \int_0^T \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\nabla \mathbf{u}\|_{L^2} \|\boldsymbol{\phi}\|_{L^2} \\
&\leq \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\mathbf{u}\|_{L^2 V} \|\boldsymbol{\phi}\|_{L^2 L^2}.
\end{aligned}$$

However, it is the presence of the term $\mathbf{u} \cdot \nabla \mathbf{w}$ that limits the regularity of $\partial_t \mathbf{w}$, as we see below: For $\boldsymbol{\phi} \in L^4(0,T;V)$,

$$\begin{aligned}
&\int_0^T |\langle \mathbf{u} \cdot \nabla \mathbf{w}, \boldsymbol{\phi}\rangle|\, dt \\
&\leq \int_0^T \|\mathbf{u}\|_{L^3} \|\nabla \mathbf{w}\|_{L^2} \|\boldsymbol{\phi}\|_{L^6}\, dt \\
&\leq C \int_0^T \|\mathbf{u}\|_{L^2}^{1/2} \|\nabla \mathbf{u}\|_{L^2}^{1/2} \|\nabla \mathbf{w}\|_{L^2} \|\nabla \boldsymbol{\phi}\|_{L^2}\, dt \\
&\leq C \|\mathbf{u}\|_{L^\infty L^2}^{1/2} \|\mathbf{u}\|_{L^2 V}^{1/2} \|\mathbf{w}\|_{L^2 V} \|\boldsymbol{\phi}\|_{L^4 V}.
\end{aligned} \tag{4.2.7}$$

So $\mathbf{u} \cdot \nabla \mathbf{w} \in L^{4/3}(0,T;V^{-1})$, therefore $\partial_t \mathbf{w} \in L^{4/3}(0,T;V^{-1})$.

Thus we have demonstrated the existence of weak solutions. □

4.3 Global wellposedness of strong solutions

The presence of the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{w}$ in the vorticity equation destroys any chance of showing non-uniqueness for weak solutions, similar to what is seen in the case of 3D Navier-Stokes. However, one can obtain the inclusion $\partial_t \mathbf{w} \in L^2(0,T;V^{-1})$ provided $\mathbf{u}$ has higher regularity. Indeed, (4.2.7) implies that $\mathbf{u} \in L^\infty(0,T;V)$ is sufficient.

Lemma 4.3.1. *Let* $(\mathbf{u}, \mathbf{w})$ *be a weak solution to* (4.1.3) *with initial data* $(\mathbf{u}_0, \mathbf{w}_0) \in$

$V \times H$ and forcing term $\mathbf{f} \in L^2(0,T;H)$. Then

$$\mathbf{u} \in C(0,T;V) \cap L^2(0,T;V^2), \tag{4.3.1}$$

$$\partial_t \mathbf{u} \in L^2(0,T;H), \tag{4.3.2}$$

$$\partial_t \mathbf{w} \in L^2(0,T;V^{-1}). \tag{4.3.3}$$

Proof. First we take the inner product of (4.1.4a) with $-\triangle \mathbf{u}$,

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla \mathbf{u}\right\|_{L^2}^2 + \nu \left\|\triangle \mathbf{u}\right\|_{L^2}^2 = \left(P_\sigma\left(\mathbf{w} \times \mathbf{u}\right), \triangle \mathbf{u}\right) - \left(\mathbf{f}, \triangle \mathbf{u}\right), \tag{4.3.4}$$

then we apply Hölder's inequality, Gagliardo-Nirenberg inequality, Poincarè's inequality, and Young's inequality, which yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\nabla \mathbf{u}\right\|_{L^2}^2 + \nu \left\|\triangle \mathbf{u}\right\|_{L^2}^2 \\ &\leq \left\|\mathbf{w}\right\|_{L^3}\left\|\mathbf{u}\right\|_{L^6}\left\|\triangle \mathbf{u}\right\|_{L^2} + \left\|\mathbf{f}\right\|_{L^2}\left\|\triangle \mathbf{u}\right\|_{L^2} \\ &\leq C\left\|\mathbf{w}\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla \mathbf{w}\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla \mathbf{u}\right\|_{L^2}\left\|\triangle \mathbf{u}\right\|_{L^2} + \left\|\mathbf{f}\right\|_{L^2}\left\|\triangle \mathbf{u}\right\|_{L^2} \\ &\leq C\left(\left\|\mathbf{w}\right\|_{L^2}^2 + \left\|\nabla \mathbf{w}\right\|_{L^2}^2\right)\left\|\nabla \mathbf{u}\right\|_{L^2}^2 + \frac{\nu}{2}\left\|\triangle \mathbf{u}\right\|_{L^2}^2 + C\left\|\mathbf{f}\right\|_{L^2}^2 \end{aligned} \tag{4.3.5}$$

Now, we apply the results of (4.3.5) to (4.3.4) and rearrange terms to obtain

$$\frac{d}{dt}\left\|\nabla \mathbf{u}\right\|_{L^2}^2 + \nu \left\|\triangle \mathbf{u}\right\|_{L^2}^2 \leq C\left(\left\|\mathbf{w}\right\|_{L^2}^2 + \left\|\nabla \mathbf{w}\right\|_{L^2}^2\right)\left\|\nabla \mathbf{u}\right\|_{L^2}^2 + C\left\|\mathbf{f}\right\|_{L^2}^2, \tag{4.3.6}$$

which, by Grönwall's inequality, implies that $\mathbf{u} \in L^\infty(0,T;V) \cap L^2(0,T;V^2)$. We now provide estimates on $\partial_t \mathbf{u}$, where it is again only contingent on estimates of the nonlinear term $\mathbf{w} \times \mathbf{u}$. For $\boldsymbol{\phi} \in L^2(0,T;H)$, using Agmon's Inequality, Poincarè's inequality, and Hölder's inequality we obtain

$$\begin{aligned}
&\int_0^T |\langle P_\sigma(\mathbf{w}\times\mathbf{u}),\boldsymbol{\phi}\rangle|\,dt\\
&\le \int_0^T \|\mathbf{w}\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\boldsymbol{\phi}\|_{L^2}\,dt\\
&\le C\|\mathbf{w}\|_{L^\infty L^2}\int_0^T \|\triangle\mathbf{u}\|_{L^2}\|\boldsymbol{\phi}\|_{L^2}\,dt\\
&\le C\|\mathbf{w}\|_{L^\infty L^2}\|\triangle\mathbf{u}\|_{L^2L^2}\|\boldsymbol{\phi}\|_{L^2L^2}.
\end{aligned}$$

Therefore $\partial_t\mathbf{u}\in L^2(0,T;H)$. For $\partial_t\mathbf{w}$, we follow a similar process as above. Given $\boldsymbol{\phi}\in L^2(0,T;V)$, we obtain the bounds

$$\begin{aligned}
&\int_0^T |\langle\mathbf{u}\cdot\nabla\mathbf{w},\boldsymbol{\phi}\rangle|\,dt\\
&\le \int_0^T \|\mathbf{u}\|_{L^3}\|\nabla\mathbf{w}\|_{L^2}\|\boldsymbol{\phi}\|_{L^6}\,dt\\
&\le C\|\mathbf{u}\|^{\frac{1}{2}}_{L^\infty L^2}\|\mathbf{u}\|^{\frac{1}{2}}_{L^\infty V}\|\mathbf{w}\|_{L^2V}\|\boldsymbol{\phi}\|_{L^2V}.
\end{aligned}\tag{4.3.7}$$

and

$$\begin{aligned}
&\int_0^T |\langle\boldsymbol{\zeta}^\epsilon(\mathbf{w})\cdot\nabla\mathbf{u},\boldsymbol{\phi}\rangle|\,dt\\
&\le \int_0^T \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\boldsymbol{\phi}\|_{L^2}\,dt\\
&\le C\,\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\mathbf{u}\|_{L^2V}\|\boldsymbol{\phi}\|_{L^2V}.
\end{aligned}\tag{4.3.8}$$

From (4.3.7) and (4.3.8) we deduce that $\partial_t\mathbf{w}\in L^2(0,T;V^{-1})$, thus completing the proof of our lemma. □

Using the results of the lemma and the first part of Theorem (1.2.14), we now proceed in showing that solutions are unique when $(\mathbf{u}_0,\mathbf{w}_0)\in V\times H$ and $\mathbf{f}\in L^2(0,T;H)$.

4.3.1 Proof of Theorem 1.2.15

Suppose $(\mathbf{u}_1, \mathbf{w}_1)$ and $(\mathbf{u}_2, \mathbf{w}_2)$ are solutions to (4.1.4) on the interval $[0, T]$. Write

$$\tilde{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2, \quad \tilde{\mathbf{u}}_0 = \mathbf{u}_1(0) - \mathbf{u}_2(0),$$
$$\tilde{\mathbf{w}} = \mathbf{w}_1 - \mathbf{w}_2, \quad \tilde{\mathbf{w}}_0 = \mathbf{w}_1(0) - \mathbf{w}_2(0),$$

The associated system for $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{w}}$ can then be written as

$$\begin{cases} \partial_t \tilde{\mathbf{u}} - \nu \triangle \tilde{\mathbf{u}} = -\mathbf{w}_2 \times \tilde{\mathbf{u}} - \tilde{\mathbf{w}} \times \mathbf{u}_1, & (4.3.9a) \\ \partial_t \tilde{\mathbf{w}} - \nu \triangle \tilde{\mathbf{w}} = \boldsymbol{\zeta}^\epsilon(\mathbf{w}_2) \cdot \nabla \tilde{\mathbf{u}} - \tilde{\mathbf{u}} \cdot \nabla \mathbf{w}_2 - \mathbf{u}_1 \cdot \nabla \tilde{\mathbf{w}} & (4.3.9b) \\ \qquad + (\boldsymbol{\zeta}^\epsilon(\mathbf{w}_1) - \boldsymbol{\zeta}^\epsilon(\mathbf{w}_2)) \cdot \nabla \mathbf{u}_1, & \\ \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0, \quad \tilde{\mathbf{w}}(0) = \tilde{\mathbf{w}}_0. & (4.3.9c) \end{cases}$$

Now we take the inner product of (4.3.9a) with $-\triangle \tilde{\mathbf{u}}$ and (4.3.9b) with $\tilde{\mathbf{w}}$:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 + \|\tilde{\mathbf{w}}\|_{L^2}^2 \right) + \nu \left(\|\triangle \tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla \tilde{\mathbf{w}}\|_{L^2}^2 \right) \qquad (4.3.10) \\ &= (\mathbf{w}_2 \times \tilde{\mathbf{u}}, \triangle \tilde{\mathbf{u}}) + (\tilde{\mathbf{w}} \times \mathbf{u}_1, \triangle \tilde{\mathbf{u}}) + (\boldsymbol{\zeta}^\epsilon(\mathbf{w}_2) \cdot \nabla \tilde{\mathbf{u}}, \tilde{\mathbf{w}}) - (\tilde{\mathbf{u}} \cdot \nabla \mathbf{w}_2, \tilde{\mathbf{w}}) \\ &\quad - (\mathbf{u}_1 \cdot \nabla \tilde{\mathbf{w}}, \tilde{\mathbf{w}}) + ((\boldsymbol{\zeta}^\epsilon(\mathbf{w}_1) - \boldsymbol{\zeta}^\epsilon(\mathbf{w}_2)) \cdot \nabla \mathbf{u}_1, \tilde{\mathbf{w}}) . \end{aligned}$$

For the first two terms, we use Hölder's inequality, Agmon's inequality and Poincarè's inequality:

$$\begin{aligned} &|(\mathbf{w}_2 \times \tilde{\mathbf{u}}, \triangle \tilde{\mathbf{u}})| + |(\tilde{\mathbf{w}} \times \mathbf{u}_1, \triangle \tilde{\mathbf{u}})| \qquad (4.3.11) \\ &\leq \|\mathbf{w}_2\|_{L^3} \|\tilde{\mathbf{u}}\|_{L^6} \|\triangle \tilde{\mathbf{u}}\|_{L^2} + \|\tilde{\mathbf{w}}\|_{L^2} \|\mathbf{u}_1\|_{L^\infty} \|\triangle \tilde{\mathbf{u}}\|_{L^2} \end{aligned}$$

$$\le C\|\mathbf{w}_2\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{w}_2\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{\mathbf{u}}\|_{L^2}\|\triangle\tilde{\mathbf{u}}\|_{L^2}+C\|\tilde{\mathbf{w}}\|_{L^2}\|\triangle\mathbf{u}_1\|_{L^2}\|\triangle\tilde{\mathbf{u}}\|_{L^2}$$
$$\le C_\nu\left(\|\mathbf{w}_2\|_{L^2}^2+\|\nabla\mathbf{w}_2\|_{L^2}^2\right)\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2+C_\nu\|\triangle\mathbf{u}_1\|_{L^2}^2\|\tilde{\mathbf{w}}\|_{L^2}^2+\frac{\nu}{4}\|\triangle\tilde{\mathbf{u}}\|_{L^2}^2.$$

Similarly, for the next three terms,

$$\begin{aligned}
&|(\boldsymbol{\zeta}^\epsilon(\mathbf{w}_2)\cdot\nabla\tilde{\mathbf{u}},\tilde{\mathbf{w}})|+|(\tilde{\mathbf{u}}\cdot\nabla\mathbf{w}_2,\tilde{\mathbf{w}})|+|(\mathbf{u}_1\cdot\nabla\tilde{\mathbf{w}},\tilde{\mathbf{w}})| \qquad (4.3.12)\\
&\le\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\tilde{\mathbf{u}}\|_{L^2}\|\tilde{\mathbf{w}}\|_{L^2}+\|\tilde{\mathbf{u}}\|_{L^\infty}\|\nabla\mathbf{w}_2\|_{L^2}\|\tilde{\mathbf{w}}\|_{L^2}+\|\mathbf{u}_1\|_{L^\infty}\|\nabla\tilde{\mathbf{w}}\|_{L^2}\|\tilde{\mathbf{w}}\|_{L^2}\\
&\le\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\tilde{\mathbf{u}}\|_{L^2}\|\tilde{\mathbf{w}}\|_{L^2}+C\|\triangle\tilde{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{w}_2\|_{L^2}\|\tilde{\mathbf{w}}\|_{L^2}+C\|\triangle\mathbf{u}_1\|_{L^2}\|\nabla\tilde{\mathbf{w}}\|_{L^2}\|\tilde{\mathbf{w}}\|_{L^2}\\
&\le\frac{1}{2}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2+C_\nu\left(1+\|\nabla\mathbf{w}_2\|_{L^2}^2+\|\triangle\mathbf{u}_1\|_{L^2}^2\right)\|\tilde{\mathbf{w}}\|_{L^2}^2\\
&+\frac{\nu}{4}\left(\|\triangle\tilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\tilde{\mathbf{w}}\|_{L^2}^2\right).
\end{aligned}$$

For the final term, we will make use of the fact that the calming function $\boldsymbol{\zeta}^\epsilon$ is Lipschitz, then apply the same inequalities as before to obtain

$$\begin{aligned}
&|((\boldsymbol{\zeta}^\epsilon(\mathbf{w}_1)-\boldsymbol{\zeta}^\epsilon(\mathbf{w}_2))\cdot\nabla\mathbf{u}_1,\tilde{\mathbf{w}})| \qquad (4.3.13)\\
&\le\|\boldsymbol{\zeta}^\epsilon(\mathbf{w}_1)-\boldsymbol{\zeta}^\epsilon(\mathbf{w}_2)\|_{L^3}\|\nabla\mathbf{u}_1\|_{L^2}\|\tilde{\mathbf{w}}\|_{L^6}\\
&\le\|\tilde{\mathbf{w}}\|_{L^3}\|\nabla\mathbf{u}_1\|_{L^2}\|\tilde{\mathbf{w}}\|_{L^6}\\
&\le C\|\tilde{\mathbf{w}}\|_{L^2}^{1/2}\|\nabla\mathbf{u}_1\|_{L^2}\|\nabla\tilde{\mathbf{w}}\|_{L^2}^{3/2}\\
&\le C_\nu\|\mathbf{u}_1\|_{L^\infty V}^4\|\tilde{\mathbf{w}}\|_{L^2}^2+\frac{\nu}{4}\|\nabla\tilde{\mathbf{w}}\|_{L^2}^2.
\end{aligned}$$

With each term suitably bounded, we now insert (4.3.11), (4.3.12), (4.3.13) into (4.3.10), which yields

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2+\|\tilde{\mathbf{w}}\|_{L^2}^2\right)+\nu\left(\|\triangle\tilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\tilde{\mathbf{w}}\|_{L^2}^2\right)\\
&\le\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2+C_\nu\left(1+\|\nabla\mathbf{w}_2\|_{L^2}^2+\|\triangle\mathbf{u}_1\|_{L^2}^2+\|\mathbf{u}_1\|_{L^\infty V}^4\right)\|\tilde{\mathbf{w}}\|_{L^2}^2.
\end{aligned}$$

Set $K_2 = \max\{\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^2, C_\nu\left(1 + \|\nabla \mathbf{w}_2\|_{L^2}^2 + \|\triangle \mathbf{u}_1\|_{L^2}^2 + \|\mathbf{u}_1\|_{L^\infty V}^4\right)\}$. By Grönwall's inequality,

$$\|\nabla \tilde{\mathbf{u}}(t)\|_{L^2}^2 + \|\tilde{\mathbf{w}}(t)\|_{L^2}^2 \leq e^{K_2 T}\left(\|\nabla \tilde{\mathbf{u}}_0\|_{L^2}^2 + \|\tilde{\mathbf{w}}_0\|_{L^2}^2\right). \tag{4.3.14}$$

Therefore we conclude that solutions to (4.1.4) are unique and depend continuously on initial data. □

4.4 Convergence to strong solutions of the Velocity-Vorticity Navier-Stokes equations

4.4.1 Proof of Theorem 1.2.16

Set $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}^\epsilon$ and $\tilde{\mathbf{w}} = \boldsymbol{\omega} - \mathbf{w}^\epsilon$. For clarity of the argument, we will obtain bounds for $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{w}}$ separately. First, we rewrite the difference equation for $\tilde{\mathbf{u}}$ to obtain

$$\partial_t \tilde{\mathbf{u}} - \nu P_\sigma \triangle \tilde{\mathbf{u}} = \tilde{\mathbf{w}} \times \tilde{\mathbf{u}} - \tilde{\mathbf{w}} \times \mathbf{u} - \boldsymbol{\omega} \times \tilde{\mathbf{u}}. \tag{4.4.1}$$

We then take the inner product with $-P_\sigma \triangle \tilde{\mathbf{u}}$ and integrate by parts to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla \tilde{\mathbf{u}}\|_{L^2}^2 + \|\triangle \tilde{\mathbf{u}}\|_{L^2}^2 \\ &\leq |(\nabla \tilde{\mathbf{w}} \times \tilde{\mathbf{u}}, \nabla \tilde{\mathbf{u}})| + |(\tilde{\mathbf{w}} \times \mathbf{u}, -P_\sigma \triangle \tilde{\mathbf{u}})| + |(\boldsymbol{\omega} \times \tilde{\mathbf{u}}, -P_\sigma \triangle \tilde{\mathbf{u}})|, \end{aligned} \tag{4.4.2}$$

noting that the term $(\tilde{\mathbf{w}} \times \nabla \tilde{\mathbf{u}}, \nabla \tilde{\mathbf{u}})$ vanishes due to orthogonality. For these three terms, from applying Holder's inequality, Gagliardo-Nirenberg-Sobolev inequalities, Agmon's inequality, and Young's inequality we deduce that

$$|(\nabla \tilde{\mathbf{w}} \times \tilde{\mathbf{u}}, \nabla \tilde{\mathbf{u}})| \leq \|\nabla \tilde{\mathbf{w}}\|_{L^2} \|\tilde{\mathbf{u}}\|_{L^6} \|\nabla \tilde{\mathbf{u}}\|_{L^3} \tag{4.4.3}$$

$$\leq C \|\nabla \tilde{\mathbf{w}}\|_{L^2} \|\nabla \tilde{\mathbf{u}}\|_{L^2}^{\frac{3}{2}} \|\triangle \tilde{\mathbf{u}}\|_{L^2}^{\frac{1}{2}}$$
$$\leq C \|\nabla \tilde{\mathbf{u}}\|_{L^2}^{6} + \frac{\nu}{10} \|\triangle \tilde{\mathbf{u}}\|_{L^2}^{2} + \frac{\nu}{12} \|\nabla \tilde{\mathbf{w}}\|_{L^2}^{2}$$

and

$$\begin{aligned} |(\tilde{\mathbf{w}} \times \mathbf{u}, P_\sigma \triangle \tilde{\mathbf{u}})| &\leq \|\mathbf{u}\|_{L^\infty} \|\tilde{\mathbf{w}}\|_{L^2} \|\triangle \tilde{\mathbf{u}}\|_{L^2} \\ &\leq C \|\mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\tilde{\mathbf{w}}\|_{L^2} \|\triangle \tilde{\mathbf{u}}\|_{L^2} \\ &\leq C \left(\|\mathbf{u}\|_{L^2}^{2} + \|\triangle \mathbf{u}\|_{L^2}^{2}\right) \|\tilde{\mathbf{w}}\|_{L^2}^{2} + \frac{\nu}{10} \|\triangle \tilde{\mathbf{u}}\|_{L^2}^{2} \end{aligned} \tag{4.4.4}$$

and also

$$\begin{aligned} |(\boldsymbol{\omega} \times \tilde{\mathbf{u}}, -P_\sigma \triangle \tilde{\mathbf{u}})| &\leq \|\boldsymbol{\omega}\|_{L^3} \|\tilde{\mathbf{u}}\|_{L^6} \|\triangle \tilde{\mathbf{u}}\|_{L^2} \\ &\leq C \|\boldsymbol{\omega}\|_{L^2}^{\frac{1}{2}} \|\nabla \boldsymbol{\omega}\|_{L^2}^{\frac{1}{2}} \|\nabla \tilde{\mathbf{u}}\|_{L^2} \|\triangle \tilde{\mathbf{u}}\|_{L^2} \\ &\leq C \left(\|\boldsymbol{\omega}\|_{L^2}^{2} + \|\nabla \boldsymbol{\omega}\|_{L^2}^{2}\right) \|\nabla \tilde{\mathbf{u}}\|_{L^2}^{2} + \frac{\nu}{10} \|\triangle \tilde{\mathbf{u}}\|_{L^2}^{2}, \end{aligned} \tag{4.4.5}$$

which yields the inequality

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\nabla \tilde{\mathbf{u}}\|_{L^2}^{2} + \|\triangle \tilde{\mathbf{u}}\|_{L^2}^{2} \\ &\leq C \|\nabla \tilde{\mathbf{u}}\|_{L^2}^{6} + C \left(\|\mathbf{u}\|_{L^2}^{2} + \|\triangle \mathbf{u}\|_{L^2}^{2} +\right) \|\tilde{\mathbf{w}}\|_{L^2}^{2} \\ &+ C \left(\|\boldsymbol{\omega}\|_{L^2}^{2} + \|\nabla \boldsymbol{\omega}\|_{L^2}^{2}\right) \|\nabla \tilde{\mathbf{u}}\|_{L^2}^{2} + \frac{3\nu}{10} \|\triangle \tilde{\mathbf{u}}\|_{L^2}^{2} + \frac{\nu}{12} \|\nabla \tilde{\mathbf{w}}\|_{L^2}^{2}. \end{aligned} \tag{4.4.6}$$

We now apply similar methods to the difference equation for $\tilde{\mathbf{w}}$, which will involve more terms due to the presence of the calming function $\boldsymbol{\zeta}^\epsilon$:

$$\begin{aligned} \partial_t \tilde{\mathbf{w}} - \nu P_\sigma \triangle \tilde{\mathbf{w}} = \quad &\left(\left(\boldsymbol{\omega} - \boldsymbol{\zeta}^\epsilon(\boldsymbol{\omega})\right) \cdot \nabla\right) \mathbf{u} + \left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{w}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\boldsymbol{\omega})\right) \cdot \nabla\right) \tilde{\mathbf{u}} \\ &+ \left(\boldsymbol{\zeta}^\epsilon(\boldsymbol{\omega}) \cdot \nabla\right) \tilde{\mathbf{u}} + \left(\left(\boldsymbol{\zeta}^\epsilon(\boldsymbol{\omega}) - \boldsymbol{\zeta}^\epsilon(\mathbf{w}^\epsilon)\right) \cdot \nabla\right) \mathbf{u} \end{aligned} \tag{4.4.7}$$

$$+ (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{w}} - (\mathbf{u} \cdot \nabla) \tilde{\mathbf{w}} - (\tilde{\mathbf{u}} \cdot \nabla) \boldsymbol{\omega}.$$

Now, we take the inner product of (4.4.7) with $\tilde{\mathbf{w}}$, which yields

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt} \|\tilde{\mathbf{w}}\|_{L^2}^2 + \nu \|\nabla \tilde{\mathbf{w}}\|_{L^2}^2 \\ & \leq \int_{\mathbb{T}^3} |\boldsymbol{\omega} - \boldsymbol{\zeta}^\epsilon(\boldsymbol{\omega})| \, |\nabla \mathbf{u}| \, |\tilde{\mathbf{w}}| \, d\mathbf{x} + \int_{\mathbb{T}^3} |\boldsymbol{\zeta}^\epsilon(\mathbf{w}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\boldsymbol{\omega})| \, |\nabla \tilde{\mathbf{u}}| \, |\tilde{\mathbf{w}}| \, d\mathbf{x} \\ & + \int_{\mathbb{T}^3} |\boldsymbol{\zeta}^\epsilon(\boldsymbol{\omega})| \, |\nabla \tilde{\mathbf{u}}| \, |\tilde{\mathbf{w}}| \, d\mathbf{x} + \int_{\mathbb{T}^3} |\boldsymbol{\zeta}^\epsilon(\boldsymbol{\omega}) - \boldsymbol{\zeta}^\epsilon(\mathbf{w}^\epsilon)| \, |\nabla \mathbf{u}| \, |\tilde{\mathbf{w}}| \, d\mathbf{x} \\ & + \int_{\mathbb{T}^3} |\tilde{\mathbf{u}}| \, |\nabla \boldsymbol{\omega}| \, |\tilde{\mathbf{w}}| \, d\mathbf{x}, \end{aligned} \tag{4.4.8}$$

noting that $((\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{w}}, \tilde{\mathbf{w}}) = 0$ and $((\mathbf{u} \cdot \nabla) \tilde{\mathbf{w}}, \tilde{\mathbf{w}}) = 0$ due to (2.4.7b). We handle the first term using the convergence property of $\boldsymbol{\zeta}^\epsilon$ and using the aforementioned inequalities,

$$\begin{aligned} & \int_{\mathbb{T}^3} |\boldsymbol{\omega} - \boldsymbol{\zeta}^\epsilon(\boldsymbol{\omega})| \, |\nabla \mathbf{u}| \, |\tilde{\mathbf{w}}| \, d\mathbf{x} \\ & \leq C\epsilon^\alpha \int_{\mathbb{T}^3} |\boldsymbol{\omega}|^\beta \, |\nabla \mathbf{u}| \, |\tilde{\mathbf{w}}| \, d\mathbf{x} \\ & \leq C\epsilon^\alpha \|\boldsymbol{\omega}\|_{L^{2\beta}}^\beta \|\nabla \mathbf{u}\|_{L^6} \|\tilde{\mathbf{w}}\|_{L^3} \\ & \leq C\epsilon^\alpha \|\boldsymbol{\omega}\|_{L^{2\beta}}^\beta \|\triangle \mathbf{u}\|_{L^2} \|\tilde{\mathbf{w}}\|_{L^2}^{\frac{1}{2}} \|\nabla \tilde{\mathbf{w}}\|_{L^2}^{\frac{1}{2}} \\ & \leq C\epsilon^{2\alpha} \|\boldsymbol{\omega}\|_{L^{2\beta}}^{2\beta} \|\triangle \mathbf{u}\|_{L^2}^2 + \|\tilde{\mathbf{w}}\|_{L^2}^2 + \frac{\nu}{12} \|\nabla \tilde{\mathbf{w}}\|_{L^2}^2 \end{aligned} \tag{4.4.9}$$

The remaining four terms can be handled using the Lipschitz property of $\boldsymbol{\zeta}^\epsilon$ along with applying the same inequalities used to bound (4.4.2). From this we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt} \|\tilde{\mathbf{w}}\|_{L^2}^2 + \nu \|\nabla \tilde{\mathbf{w}}\|_{L^2}^2 \\ & \leq C \|\tilde{\mathbf{w}}\|_{L^2}^6 + C\epsilon^{2\alpha} \|\boldsymbol{\omega}\|_{L^{2\beta}}^{2\beta} \|\triangle \mathbf{u}\|_{L^2}^2 + C\left(1 + \|\triangle \mathbf{u}\|_{L^2}^{\frac{4}{3}}\right) \|\tilde{\mathbf{w}}\|_{L^2}^2 \end{aligned} \tag{4.4.10}$$

$$+\frac{5\nu}{12}\|\nabla\tilde{\mathbf{w}}\|_{L^2}^2+\frac{2\nu}{10}\|\triangle\tilde{\mathbf{u}}\|_{L^2}^2+C\|\nabla\boldsymbol{\omega}\|_{L^2}^2\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2.$$

We now seek to combine inequalities (4.4.6) and (4.4.10). First, to deal with the terms $\|\nabla\tilde{\mathbf{u}}\|_{L^2}^6$ and $\|\tilde{\mathbf{w}}\|_{L^2}^6$ we will apply the ansatz

$$\|\nabla\tilde{\mathbf{u}}\|_{L^2}+\|\tilde{\mathbf{w}}\|_{L^2}<1 \tag{4.4.11}$$

so that $\|\nabla\tilde{\mathbf{u}}\|_{L^2}^6<\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2$ and $\|\tilde{\mathbf{w}}\|_{L^2}^6<\|\tilde{\mathbf{w}}\|_{L^2}^2$. We will show that this assumption leads to an even tighter bound. From adding (4.4.6) and (4.4.10) together and applying (4.4.11), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2+\|\tilde{\mathbf{w}}\|_{L^2}^2\right)+\nu\left(\|\triangle\tilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\tilde{\mathbf{w}}\|_{L^2}^2\right) \qquad (4.4.12)\\&\le A(T)\left(\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2+\|\tilde{\mathbf{w}}\|_{L^2}^2\right)+\frac{\nu}{2}\left(\|\triangle\tilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\tilde{\mathbf{w}}\|_{L^2}^2\right)+C\epsilon^{2\alpha}\|\boldsymbol{\omega}\|_{L^{2\beta}}^{2\beta}\|\triangle\mathbf{u}\|_{L^2}^2,\end{aligned}$$

where

$$A(T)=C\left(1+\|\triangle\mathbf{u}\|_{L^2}^{\frac{4}{3}}+\|\boldsymbol{\omega}\|_{L^2}^2+\|\nabla\boldsymbol{\omega}\|_{L^2}^2\right). \tag{4.4.13}$$

The integrability of $A(T)$ and of $C\epsilon^{2\alpha}\|\boldsymbol{\omega}\|_{L^{2\beta}}^{2\beta}\|\triangle\mathbf{u}\|_{L^2}^2$ is given by the assumptions on our solution $(\mathbf{u},\boldsymbol{\omega})$ to $3D$ NSE. Therefore, removing the viscous terms and applying Grönwalls inequality to (4.4.12) on $[0,T]$ will yield

$$\|\nabla\tilde{\mathbf{u}}\|_{L^2}^2+\|\tilde{\mathbf{w}}\|_{L^2}^2\le B(T)\epsilon^{2\alpha}, \tag{4.4.14}$$

where

$$B(T)=Ce^{TA(T)}\int_0^T\|\boldsymbol{\omega}\|_{L^{2\beta}}^{2\beta}\|\triangle\mathbf{u}\|_{L^2}^2\,dt. \tag{4.4.15}$$

By (1.2.5) we deduce that $B(T) < \infty$. From choosing $\epsilon > 0$ sufficiently small, it follows that

$$\|\tilde{\mathbf{u}}\|_{L^2} + \|\tilde{\mathbf{w}}\|_{L^2} < \frac{1}{2}.$$

We then apply a boostrapping argument to deduce that, in fact, (4.4.14) is valid for all $t \in [0, T]$. Therefore we conclude that $\mathbf{u}^\epsilon$ converges to $\mathbf{u}$ in $L^\infty(0, T; V)$ and $\mathbf{w}^\epsilon$ converges to $\boldsymbol{\omega}$ in $L^\infty(0, T; H)$.

4.5 Energy identities

4.5.1 Proof of Theorem 1.2.17

Let $(\mathbf{u}, \mathbf{w})$ be strong solutions to (4.1.3) with initial data $(\mathbf{u}, \mathbf{w}_0)$. We take the inner product of the velocity equation (4.1.3a) with $\mathbf{u}$ and apply the Lions-Magenes lemma, which yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{u}\|_{L^2}^2 = 0,$$

then integrate in time rearrange the terms to obtain

$$\|\mathbf{u}(t)\|_{L^2} + 2\nu\int_0^t \|\nabla\mathbf{u}\|_{L^2}^2\, ds = \|\mathbf{u}_0\|_{L^2}^2,$$

thus equalities (1.2.6) and (1.2.7) are shown to be valid for all $t \in [0, T]$. Now, we take the inner product of (4.1.3b) with $\mathbf{w}$ and apply the Lions-Magenes lemma

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_{L^2}^2 + \nu\|\nabla\mathbf{w}\|_{L^2}^2 = -\left(\boldsymbol{\zeta}^\epsilon(\mathbf{w})\cdot\nabla\mathbf{u}, \mathbf{w}\right).$$

Next, we will apply the chain rule to write $\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_{L^2}^2 = \|\mathbf{w}\|_{L^2}\frac{d}{dt}\|\mathbf{w}\|_{L^2}$, then apply Hölder's inequality

$$\|\mathbf{w}\|_{L^2}\frac{d}{dt}\|\mathbf{w}\|_{L^2} + \nu\|\nabla\mathbf{w}\|_{L^2}^2 \leq \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{w}\|_{L^2}.$$

Then we remove the diffusive term from the inequality and divide by $\|\mathbf{w}\|_{L^2}$:

$$\frac{d}{dt}\|\mathbf{w}\|_{L^2} \leq \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}$$

We now integrate in time and apply the Cauchy-Schwarz inequality and the energy equality (1.2.7) to verify (1.2.8):

$$\begin{aligned}
&\|\mathbf{w}(t)\|_{L^2} - \|\mathbf{w}_0\|_{L^2} \\
&\leq \int_0^t \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\,ds \\
&\leq t^{\frac{1}{2}}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\left(\int_0^t \|\nabla\mathbf{u}\|_{L^2}^2\,ds\right)^{\frac{1}{2}}. \\
&= \left(\frac{t}{2\nu}\right)^{\frac{1}{2}}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\left(\|\mathbf{u}_0\|_{L^2}^2 - \|\mathbf{u}(t)\|_{L^2}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

This concludes the proof. □

4.6 Conclusion

We posed a modification to the Navier-Stokes equations for which we then showed the global well-posedness of strong solutions and the short-time convergence of calmed solutions to a strong solution of the original Navier-Stokes equation. This modification is quite similar to the systems introduced in Chapter 3 while still retaining the algebraic symmetries in some of its nonlinear components that arise from the divergence-free condition. By exploiting these symmetries and using pointwise parallel calming functions we are able to an energy identity and an energy inequality for the velocity and vorticity, respectively, in the calmed system. It remains open whether weak solutions to calmed Velocity-Vorticity Navier-Stokes are unique, or whether the system possesses a global attractor.

Chapter 5

Calmed Ohmic Heating for the MHD-Boussinesq equations

5.1 Introduction

Turbulent fluids that involve multi-physical processes, such as magnetic effects and/or heat conduction are a fascinating and challenging area of research. One way to understand such fluids is by studying the magnetohydrodynamic (MHD) partial differential equations with an additional thermal equation coupled to the original system [57]. Such equations have been considered in many works, often under the name "MHD-Boussinesq" (MHD-B) equations [5, 14, 66–68, 86, 107, 112], but the heating effect of the electrical current (so-called "Ohmic heating" or "Joule heating") is neglected, save for in a small number of papers [15, 39, 71, 85, 94, 100, 101]. Therefore, we are interested in the case of the MHD-Boussinesq system *with* Ohmic heating (MHD-BΩ). We note that *even in the two-dimensional case, global well-posedness for this system is a completely open problem*. Thus, in the present work, we consider the 2D MHD-Boussinesq system with a modified Ohmic heating term. We prove that our modified system is globally well-posed, and also that, at least before any potential blow-up time of the original system, solutions of our modified system converge to solutions of the MHD-BΩ system.

The difficulty with the MHD-BΩ system is that the Ohmic heating term is already quadratic and non-negative, destroying any chance of closing energy estimates, or of finding any conserved quantity. Moreover, there is no natural scaling to the equation. Indeed, the Ohmic term $|\nabla \times b|^2$ scales exactly like the well-known Hall term $\nabla \times ((\nabla \times b) \times b)$ of the Hall-MHD system, for which global well-posedness is also a highly challenging open problem in 2D[1]. Indeed, the MHD-BΩ system appears to be even more challenging than the Hall-MHD system: the Hall term vanishes in L^2-energy estimates allowing for a conserved quantity, but no analogous situation is known to hold for the Ohmic term.

The above difficulties with the Ohmic term mean that standard modifications, such as adding hyper diffusion, filtering the advective term, etc., have little chance of allowing for a proof of global well-posedness, since these techniques merely control the growth of gradients (i.e., the growth of small scales). What is needed is some way to control the Ricatti-like[2] nature of the system. Note that similar issues arise in controlling the vorticity in the 3D Navier-Stokes equations (NSE), and controlling the solution in the 2D Kuramoto-Sivashinsky equation (KSE). Hence, we employ a technique that we call "algebraic calming," developed in [28] in the context of the 2D KSE. However, rather than apply the calming function to the advective term, as is done in [28], the innovation of the present work is to apply the calming function to the Ohmic term. We describe this in detail below.

Consider the domain $\mathbb{T}^2$ the two-dimensional periodic space $\mathbb{R}^2/\mathbb{Z}^2 = [0,1]^2$. For $T > 0$, the 2D MHD-Boussinesq system with full fluid viscosity $\nu > 0$, magnetic

[1]Technically, the Hall-MHD system only makes sense in so-called "two-and-a -half dimensional" (2.5D) case, where the spatial inputs are 2D, but the output dimension is 3D. Global well-posedness for the 2.5D Hall-MHD is the open problem which we are referring to here.

[2]Recall that the Ricatti-type equation $\frac{dy}{dt} = y^{1+\epsilon}$ blows up in finite time for any $\epsilon > 0$ and positive initial data.

resistivity $\mu > 0$, and thermal diffusion $\kappa > 0$ over $\mathbb{T}^2 \times [0, T)$, is given by

$$\begin{cases} \partial_t u - \nu\Delta u + (u \cdot \nabla)u + \nabla p = (b \cdot \nabla)b + g\theta\vec{e}_2, \\ \partial_t b - \mu\Delta b + (u \cdot \nabla)b = (b \cdot \nabla)u, \\ \partial_t \theta - \kappa\Delta\theta + (u \cdot \nabla)\theta = \alpha\mu\boldsymbol{\zeta}^{\epsilon}(|\nabla \times b|)|\nabla \times b|, \\ \nabla \cdot u = 0 = \nabla \cdot b, \end{cases} \tag{5.1.1}$$

where the constant $g > 0$ has unit of force, and is proportional to the constant of gravitational acceleration. We denote $x = (x_1, x_2)$, and $\vec{e}_2$ to be the unit vector in the x_2 direction, i.e., $\vec{e}_2 = (0, 1)^T$. Here and henceforth, $u = u(x, t) = (u_1(x, t), u_2(x, t))$ is the unknown velocity field of a viscous incompressible fluid, with divergence-free initial data $u(x, 0) = u_0$; $b = b(x, t) = (b_1(x, t), b_2(x, t))$ is the unknown magnetic field, with divergence-free initial data $b(x, 0) = b_0$; and the scalar $p = p(x, t)$ represents the unknown pressure, while $\theta = \theta(x, t)$ can be thought of as the unknown temperature fluctuation, with initial value $\theta_0 = \theta(x, 0)$.

With the same parameters, we now give below the original two-dimensional Boussinesq-MHD equations with Ohmic heating effect but without the calming mechanism.

$$\begin{cases} \partial_t u - \nu\Delta U + (U \cdot \nabla)U + \nabla P = (B \cdot \nabla)B + g\Theta\vec{e}_2, \\ \partial_t b - \mu\Delta B + (U \cdot \nabla)B = (B \cdot \nabla)U, \\ \partial_t \theta - \kappa\Delta\Theta + (U \cdot B)\Theta = \alpha\mu|\nabla \times B|^2, \\ \nabla \cdot U = 0 = \nabla \cdot B, \end{cases} \tag{5.1.2}$$

This paper is organized as follows. In Section 5.2, we provide all *a priori* estimates for the global existence and uniqueness of System (5.1.1), as well as the proof of the higher-order regularity of the solution.; while in Section 5.3, we show the convergence of the soluton to System (5.1.1) to that of System (5.1.2).

5.2 Proof of Global Existence and Regularity Results

In this section, we provide the *a priori* estimates in order to obtain the global well-posedness of the solution to system 5.1.1, as well as the higher-order regularity of the solution. Note that the solution can be constructed rigorously by approximations with Galerkin ODEs, following the approach in [57] and the reference therein. For our energy estimates we largely omit the Galerkin notation, though we explicitly show the convergence of the Ohmic term in the Galerkin approximation to its corresponding term in System 5.1.1.

5.2.1 Existence of Solutions to System (5.1.1)

We prove Theorem 1.2.18 using Galerkin approximation methods. Let Q_n denote the Galerkin projection onto the first n eigenmodes of the Laplacian operator $-\triangle$, and let $P_n = P_\sigma Q_n$ denote the projection onto the first n eigenmodes of the Stokes operator A. We will use (u^n, b^n, θ^n) to denote a solution to the Galerkin system, written below in functional form:

$$\begin{cases} \dfrac{d}{dt}u^n + \nu A u^n + P_n\mathcal{B}(u^n, u^n) = P_n\mathcal{B}(b^n, b^n) + gP_\sigma\theta^n\vec{e}_2, \\ \dfrac{d}{dt}b^n + \mu A b^n + P_n\mathcal{B}(u^n, b^n) = P_n\mathcal{B}(b^n, u^n), \\ \dfrac{d}{dt}\theta^n - \kappa\triangle\theta^n + Q_n\mathcal{B}(u^n, \theta^n) = \alpha\mu Q_n\left(\boldsymbol{\zeta}^\epsilon(|\nabla\times b^n|)|\nabla\times b^n|\right), \\ u^n(0) = P_n(u_0), \quad b^n(0) = P_n(b_0), \quad \theta^n(0) = Q_n(\theta_0). \end{cases} \tag{5.2.1}$$

In this system, the time evolution equations u^n and b^n are both finite-dimensional ODEs on $P_n(H)$ and the time evolution of θ^n is a finite-dimensional ODE on $Q_n(L^2(\mathbb{T}))$. Moreover, each time derivative is given by a locally Lipschitz function (see, e.g., [28, 29]), hence the short-time existence and uniqueness of the solution

(u^n, b^n, θ^n) is known up to a maximum time interval of existence $[0, T_n)$. To deduce the global existence of solutions for 5.1.1, we work to obtain bounds on u^n, b^n, and θ^n that are independent of n. In the arguments that follow, we write $u^n \equiv u$, $b^n \equiv b$, and $\theta^n \equiv \theta$ and omit any projection operators for notational simplicity.

Remark 5.2.1. Since this Galerkin system is defined using two projection operators, one needs to careful. In particular, one may question whether the evolution equation for $u^n(t)$ defines a system on $P_n(H)$ due to the presence of the term $gP_\sigma \theta^n \vec{e}_2$ with $\theta^n \in Q_n(L^2(\mathbb{T}))$. In our system, by defining the Stokes projection operator as $P_n = P\sigma Q_n$ we guarantee that $u^n(t)$ remains in $P_n(H)$.

5.2.1.1 L^2-estimates of Theorem 1.2.18

Multiply the three equations in 5.2.1 by u, b, θ, respectively, integrate by parts over $\mathbb{T}^2$, and add, so that we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) + \nu\|\nabla u\|_{L^2}^2 + \mu\|\nabla b\|_{L^2}^2 + \kappa\|\nabla \theta\|_{L^2}^2 \\
&= \int_{\mathbb{T}^2} g\theta\vec{e}_2 \cdot u\, dx + \int_{\mathbb{T}^2} \alpha\mu\boldsymbol{\zeta}^\epsilon(|\nabla \times b|)|\nabla \times b|\theta\, dx \\
&\leq \frac{g}{2}\|u\|_{L^2}^2 + \frac{g}{2}\|\theta\|_{L^2}^2 + \frac{\mu}{2}\|\nabla b\|_{L^2}^2 + \frac{\mu\alpha^2 M_\epsilon^2}{2}\|\theta\|_{L^2}^2,
\end{aligned}$$

where we used (2.4.9), (2.4.10), the divergence-free condition, Young' inequality, and the boundedness of $\boldsymbol{\zeta}^\epsilon$. Then, integrating in time from 0 to $T > 0$, and by Grönwall inequality, we get for all $t \in [0, T]$,

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \\
&\quad + \nu\int_0^T \|\nabla u(t)\|_{L^2}^2\, dt + \mu\int_0^T \|\nabla b(t)\|_{L^2}^2\, dt + \kappa\int_0^T \|\nabla \theta(t)\|_{L^2}^2\, dt \\
&\leq K_1,
\end{aligned} \tag{5.2.2}$$

where K_1 is a constant that depends on the initial data, as well as $\nu, \mu, \kappa, \alpha, \epsilon$, and T, and does not depend on n.

Thus, the existence part of Theorem 1.2.18 is proved. □

5.2.1.2 H^1-estimates of Theorem 1.2.18

Multiply the three equations in 5.2.1 by $Au, Ab, -\Delta\theta$, respectively, integrate by parts over $\mathbb{T}^2$, and add, so that we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) + \nu\|\Delta u\|_{L^2}^2 + \mu\|\Delta b\|_{L^2}^2 + \kappa\|\Delta\theta\|_{L^2}^2 \\
&= -\int_{\mathbb{T}^2}(b\cdot\nabla)b\cdot\Delta u\,dx + \int_{\mathbb{T}^2}(u\cdot\nabla)b\cdot\Delta b\,dx - \int_{\mathbb{T}^2}(b\cdot\nabla)u\cdot\Delta b\,dx \\
&\quad - g\int_{\mathbb{T}^2}\theta\vec{e}_2\cdot\Delta u\,dx + \int_{\mathbb{T}^2}(u\cdot\nabla)\theta\Delta\theta\,dx \\
&\quad - \alpha\mu\int_{\mathbb{T}^2}\boldsymbol{\zeta}^{\epsilon}(|\nabla\times b|)|\nabla\times b|\Delta\theta\,dx.
\end{aligned}$$

Next, we estimate the six terms on the right side of the above equations. First, by (2.4.10), the sum of the first three terms is simplified, and thence estimated as

$$\begin{aligned}
&-\int_{\mathbb{T}^2}(b\cdot\nabla)b\cdot\Delta u\,dx + \int_{\mathbb{T}^2}(u\cdot\nabla)b\cdot\Delta b\,dx - \int_{\mathbb{T}^2}(b\cdot\nabla)u\cdot\Delta b\,dx \\
&= 2\int_{\mathbb{T}^2}(u\cdot\nabla)b\cdot\Delta b\,dx \le C\|u\|_{L^4}\|\nabla b\|_{L^4}\|\Delta b\|_{L^2} \\
&\le C\|u\|_{L^2}^{1/2}(\|u\|_{L^2} + \|\nabla u\|_{L^2})^{1/2}\|\nabla b\|_{L^2}^{1/2}\|\Delta b\|_{L^2}^{3/2} \\
&\le C\|\nabla b\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla b\|_{L^2}^2 + \frac{\mu}{4}\|\Delta b\|_{L^2}^2,
\end{aligned}$$

where we used (2.4.6), Young's inequality, and the L^2-bounds obtained in (5.2.2). Regarding the fourth term, we integrate by parts, apply Young's inequality, and get

$$-g\int_{\mathbb{T}^2}\theta\vec{e}_2\cdot\Delta u\,dx \le \frac{g}{2}\|\nabla\theta\|_{L^2}^2 + \frac{g}{2}\|\nabla u\|_{L^2}^2.$$

The fifth term is estimated similarly to that of the first three, and we have

$$\begin{aligned}\int_{\mathbb{T}^2}(u\cdot\nabla)\theta\Delta\theta\,dx &\le \int_{\mathbb{T}^2}|u||\nabla\theta||\Delta\theta|\,dx \le C\|u\|_{L^4}\|\nabla\theta\|_{L^4}\|\Delta\theta\|_{L^2}\\ &\le C\|u\|_{L^2}^{1/2}(\|u\|_{L^2}+\|\nabla u\|_{L^2})^{1/2}\|\nabla\theta\|_{L^2}^{1/2}\|\Delta\theta\|_{L^2}^{3/2}\\ &\le C\|\nabla\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\theta\|_{L^2}^2 + \frac{\kappa}{4}\|\Delta\theta\|_{L^2}^2,\end{aligned}$$

where we also used the L^2-bounds of u obtained in (5.2.2).

As for the last term, by the boundedness of $\boldsymbol{\zeta}^\epsilon$, and Young's inequality, we get

$$-\alpha\mu\int_{\mathbb{T}^2}\boldsymbol{\zeta}^\epsilon(|\nabla\times b|)|\nabla\times b|\Delta\theta\,dx \le \frac{\alpha^2\mu^2M_\epsilon^2}{\kappa}\|\nabla b\|_{L^2}^2 + \frac{\kappa}{4}\|\Delta\theta\|_{L^2}^2.$$

Therefore, by combining all the above estimates, after some rearrangement and simplification, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) + \nu\|\Delta u\|_{L^2}^2 + \mu\|\Delta b\|_{L^2}^2 + \kappa\|\Delta\theta\|_{L^2}^2\\ &\le \frac{g}{2}\|\nabla u\|_{L^2}^2 + \frac{g}{2}\|\nabla\theta\|_{L^2}^2 + \frac{\alpha^2\mu^2M_\epsilon^2}{\kappa}\|\nabla b\|_{L^2}^2\\ &\quad + C\|\nabla u\|_{L^2}^2\big(\|\nabla b\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\big).\end{aligned}$$

Observing the L^2-integrability in time of $\|\nabla u\|_{L^2}$ from (5.2.2), and by the Grönwall inequality, we integrate the above inequality in time from 0 to T, $T>0$, and get

$$\begin{aligned}&\|\nabla u(T)\|_{L^2}^2 + \|\nabla b(T)\|_{L^2}^2 + \|\nabla\theta(T)\|_{L^2}^2\\ &\quad + \nu\int_0^T\|\Delta u(t)\|_{L^2}^2\,dt + \mu\int_0^T\|\Delta b(t)\|_{L^2}^2\,dt + \kappa\int_0^T\|\Delta\theta(t)\|_{L^2}^2\,dt\\ &\le K_2,\end{aligned}\tag{5.2.3}$$

where the constant K_2 depends on those relevant parameters, as well as on T and K_1

in (5.2.2), and is independent of n.

5.2.1.3 Estimates on time derivatives for Theorem 1.2.18

In order to make valid the convergence arguments used in Section 5.3 we require $u, b \in C([0,T];V)$ and $\theta \in C([0,T];L^2)$, for which we need $\partial_t u, \partial_t b \in L^2(0,T;L^2)$ and $\theta \in L^2(0,T;H^{-1})$. We will show this by selecting arbitrary test functions $\phi \in L^2(0,T;H)$ and $\psi \in L^2(0,T;H^1)$ and applying the estimates obtained in Sections 5.2.1.1 and 5.2.1.2. Note that by estimate (5.2.3) and Agmon's inequality (2.4.3), we have u and b bounded above by K_2 in $L^2(0,T;L^\infty) \cap L^\infty(0,T;V)$.

First we take the action of $\partial_t u$ on ϕ, from which we obtain

$$
\begin{aligned}
&\left| \int_0^T \langle \partial_t u, \phi \rangle \, dt \right| = \left| \int_0^T \langle \nu \triangle u - (u \cdot \nabla) u + (b \cdot \nabla) b + g\theta \vec{e_2}, \phi \rangle \, dt \right| \\
&\leq \nu \int_0^T |(\triangle u, \phi)| \, dt + \int_0^T |((u \cdot \nabla) u, \phi)| \, dt \\
&\quad + \int_0^T |((b \cdot \nabla) b, \phi)| \, dt + g \int_0^T |(\theta \vec{e_2}, \phi)| \, dt \\
&\leq \nu \int_0^T \|\triangle u\|_{L^2} \|\phi\|_{L^2} \, dt + \int_0^T \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|\phi\|_{L^2} \, dt \\
&\quad + \int_0^T \|b\|_{L^\infty} \|\nabla b\|_{L^2} \|\phi\|_{L^2} \, dt + g \int_0^T \|\theta\|_{L^2} \|\phi\|_{L^2} \, dt \\
&\leq C \left(K_2 + 2K_2^2 + K_1 \right) \int_0^T \|\phi\|_{L^2}^2 \, dt
\end{aligned}
$$

using Hölder's inequality, Cauchy-Schwarz inequality, and using the aforementioned estimates. We then take the action of $\partial_t b$ on ϕ and, *mutatis mutandis*, we obtain

$$
\left| \int_0^T \langle \partial_t b, \phi \rangle \, dt \right| \leq C \left(\mu K_2 + 2K_2^2 \right) \int_0^T \|\phi\|_{L^2}^2 \, dt.
$$

For $\partial_t \theta$, we take the action on ψ and see that

$$\begin{aligned}
&\left| \int_0^T \langle \partial_t \theta, \psi \rangle \, dt \right| \qquad (5.2.4)\\
&= \left| \int_0^T \langle \kappa \triangle \theta - u \cdot \nabla \theta + \alpha \mu \boldsymbol{\zeta}^\epsilon (|\nabla \times b|) \, |\nabla \times b| \, , \psi \rangle \, dt \right| \\
&\leq \int_0^T \kappa \left\| \nabla \theta \right\|_{L^2} \left\| \nabla \psi \right\|_{L^2} dt + \int_0^T \left\| u \right\|_{L^\infty} \left\| \nabla \theta \right\|_{L^2} \left\| \psi \right\|_{L^2} dt \\
&+ \alpha \mu \int_0^T \left\| \boldsymbol{\zeta}^\epsilon \right\|_{L^\infty} \left\| \nabla b \right\|_{L^2} \left\| \psi \right\|_{L^2} dt \\
&\leq C \left(K_1 + K_2^2 + \alpha K_1 \left\| \boldsymbol{\zeta}^\epsilon \right\|_{L^\infty} \right) \int_0^T \left\| \psi \right\|_{H^1}^2 dt.
\end{aligned}$$

By the Aubin-Lions Theorem we deduce that

$$u, b \in C([0, T]; V)$$

and

$$\theta \in C([0, T]; L^2).$$

Thus we have the shown (formally) the existence of a solution (u, b, θ) in the appropriate space.

5.2.2 Convergence of the Galerkin System (5.2.1) to the calmed System (5.1.1)

We now use the previous estimates to justify that our solution to the Galerkin system, now being referred to as (u^n, b^n, θ^n), converges to a solution of System (5.1.1). From the estimates here, here, here, here, and here, and from applying Banach-Alaoglu and Aubin-Lions, we know there exists a subsequence of (u^n, b^n, θ^n), which we will not relabel, and a limit point (u, b, θ), such that

$$\begin{aligned}
u^n &\rightharpoonup u \quad \text{weakly in } L^2(0,T;H^2\cap V), \\
b^n &\rightharpoonup b \quad \text{weakly in } L^2(0,T;H^2\cap V), \\
\theta^n &\rightharpoonup \theta \quad \text{weakly in } L^2(0,T;H^1), \\
\partial_t u^n &\rightharpoonup \partial_t u \quad \text{weakly in } L^2(0,T;H), \\
\partial_t b^n &\rightharpoonup \partial_t b \quad \text{weakly in } L^2(0,T;H), \\
\partial_t \theta^n &\rightharpoonup \partial_t \theta \quad \text{weakly in } L^2(0,T;H^{-1}), \\
u^n &\to u \quad \text{strongly in } C([0,T];V), \\
b^n &\to b \quad \text{strongly in } C([0,T];V), \\
\theta^n &\to \theta \quad \text{strongly in } C([0,T];L^2).
\end{aligned} \tag{5.2.5}$$

Now we show that, as $n \to \infty$, the Galerkin System (5.2.1) converges to System (5.1.1). For brevity we will only show convergence for the calmed Ohmic heating term, though we refer the reader to, e.g., [60] for handling the remaining terms. First we write the term as

$$\begin{aligned}
&\alpha\mu\boldsymbol{\zeta}^\epsilon(|\nabla\times b^n|)\,|\nabla\times b^n| \\
&= \alpha\mu\boldsymbol{\zeta}^\epsilon(|\nabla\times b^n|)\,(|\nabla\times b^n| - |\nabla\times b|) \\
&+ \alpha\mu\,(\boldsymbol{\zeta}^\epsilon(|\nabla\times b^n|) - \boldsymbol{\zeta}^\epsilon(|\nabla\times b|))\,|\nabla\times b^n| \\
&+ \alpha\mu\boldsymbol{\zeta}^\epsilon(|\nabla\times b|)\,|\nabla\times b|\,.
\end{aligned} \tag{5.2.6}$$

We now take the action of (5.2.6) on $\psi \in L^2(0,T;H^1)$ and integrate in time. For the first term, we obtain

$$\begin{aligned}
&\int_0^T |\langle \alpha\mu\boldsymbol{\zeta}^\epsilon(|\nabla\times b^n|)\left(|\nabla\times b^n| - |\nabla\times b|\right), \psi\rangle|\, dt \qquad (5.2.7)\\
&\le \int_0^T \alpha\mu \left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty} \left\|\nabla b^n - \nabla b\right\|_{L^2} \left\|\psi\right\|_{L^2} dt\\
&\le \alpha\mu \left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty} \left(\int_0^T \left\|\nabla b^n - \nabla b\right\|_{L^2}^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \left\|\psi\right\|_{L^2}^2 dt\right)^{\frac{1}{2}}.
\end{aligned}$$

For the second term, similar techniques yield

$$\begin{aligned}
&\int_0^T |\langle \alpha\mu\left(\boldsymbol{\zeta}^\epsilon(|\nabla\times b^n|) - \boldsymbol{\zeta}^\epsilon(|\nabla\times b|)\right)|\nabla\times b^n|, \psi\rangle|\, dt \qquad (5.2.8)\\
&\le \alpha\mu C \left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}^{\frac{1}{2}} \int_0^T \left\|\nabla b^n - \nabla b\right\|_{L^2}^{\frac{1}{2}} \left\|\nabla b^n\right\|_{L^2}^{\frac{1}{2}} \left\|\triangle b^n\right\|_{L^2}^{\frac{1}{2}} \left\|\psi\right\|_{H^1} dt\\
&\le \alpha\mu C \left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}^{\frac{1}{2}} \left\|\nabla b^n\right\|_{L^\infty L^2} \int_0^T \left\|\nabla b^n - \nabla b\right\|_{L^2}^{\frac{1}{2}} \left\|\triangle b^n\right\|_{L^2}^{\frac{1}{2}} \left\|\psi\right\|_{H^1} dt\\
&\le \alpha\mu C \left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}^{\frac{1}{2}} \left\|\nabla b^n\right\|_{L^\infty L^2} \left(\int_0^T \left\|\nabla b^n - \nabla b\right\|_{L^2}^2 dt\right)^{\frac{1}{4}}\\
&\quad \times \left(\int_0^T \left\|\triangle b^n\right\|_{L^2}^2 dt\right)^{\frac{1}{4}} \left(\int_0^T \left\|\psi\right\|_{H^1}^2 dt\right)^{\frac{1}{2}}.
\end{aligned}$$

We observe that each of these terms converge to 0 as $n \to \infty$ due to 5.2.5, hence $\alpha\mu\boldsymbol{\zeta}^\epsilon(|\nabla\times b^n|)\,|\nabla\times b^n|$ converges to $\alpha\mu\boldsymbol{\zeta}^\epsilon(|\nabla\times b|)\,|\nabla\times b|$. We conclude that solutions to System (5.1.1) exist on $[0,T]$.

5.2.3 Higher-order Regularity of Solutions

We prove the higher-order regularity of the solution to System 5.1.1 in two steps, similar to [58]. First, we obtain the H^2-regularity of u and b; then, we use the bounds on the H^2-norm of u and b to prove the higher-order regularity of θ. The main reason is to overcome the difficulty of differentiating the absolute value of $\nabla \times b$.

To begin, we multiply the relevant equations in System 5.1.1 by $\Delta^2 u$, $\Delta^2 b$, $\Delta\theta$, respectively, integrate by parts over $\mathbb{T}^2$, add, and get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) \\
&\quad + \nu\|\nabla\Delta u\|_{L^2}^2 + \mu\|\nabla\Delta b\|_{L^2}^2 + \kappa\|\Delta\theta\|_{L^2}^2 \\
&= g\int_{\mathbb{T}^2}\theta\vec{e}_2\cdot\Delta^2 u\,dx - \int_{\mathbb{T}^2}(u\cdot\nabla)u\cdot\Delta^2 u\,dx + \int_{\mathbb{T}^2}(b\cdot\nabla)b\cdot\Delta^2 u\,dx \\
&\quad + \int_{\mathbb{T}^2}(b\cdot\nabla)u\cdot\Delta^2 b\,dx - \int_{\mathbb{T}^2}(u\cdot\nabla)b\cdot\Delta^2 b\,dx - \int_{\mathbb{T}^2}(u\cdot\nabla)\theta\Delta\theta\,dx \\
&\quad\quad + \alpha\mu\int_{\mathbb{T}^2}\boldsymbol{\zeta}^\epsilon(|\nabla\times b|)|\nabla\times b|\Delta\theta\,dx.
\end{aligned}$$

Then, we estimate the seven terms on the right side of the above equation. For the first term, integrating by parts twice and applying Young's inequality, we have

$$g\int_{\mathbb{T}^2}\theta\vec{e}_2\cdot\Delta^2 u\,dx \le \frac{g}{2}\|\Delta u\|_{L^2}^2 + \frac{g}{2}\|\Delta\theta\|_{L^2}^2.$$

The estimates of the remaining five terms are similar, so for the sake of brevity, we provide only the key steps without further clarification. Specifically, the second term is estimated as follows,

$$\begin{aligned}
-\int_{\mathbb{T}^2}(u\cdot\nabla)u\cdot\Delta^2 u\,dx &\le \int_{\mathbb{T}^2}|\nabla u|^2|\nabla\Delta u|\,dx + \int_{\mathbb{T}^2}|u||\nabla\nabla u||\nabla\Delta u|\,dx \\
&\le C\|\nabla u\|_{L^4}^2\|\nabla\Delta u\|_{L^2} \\
&\quad + C\|u\|_{L^4}\|\nabla\nabla u\|_{L^4}\|\nabla\Delta u\|_{L^2} \\
&\le C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla\Delta u\|_{L^2} \\
&\quad + C\|u\|_{L^2}^{1/2}(\|u\|_{L^2} + \|\nabla u\|_{L^2})^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\nabla\Delta u\|_{L^2}^{3/2} \\
&\le C\|\Delta u\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\Delta u\|_{L^2}^2,
\end{aligned}$$

where we used the bounds on $\|u\|_{L^2}$ and $\|\nabla u\|_{L^2}$ in (5.2.2) and (5.2.3), as well as (2.4.6) and Young's inequality.

As for the third term, we proceed similarly, and obtain

$$\begin{aligned}\int_{\mathbb{T}^2}(b\cdot\nabla)b\cdot\Delta^2u\,dx &\le \int_{\mathbb{T}^2}|\nabla b|^2|\nabla\Delta u|\,dx+\int_{\mathbb{T}^2}|b||\nabla\nabla b||\nabla\Delta u|\,dx\\ &\le C\|\nabla b\|_{L^4}^2\|\nabla\Delta u\|_{L^2}+C\|b\|_{L^4}\|\nabla\nabla b\|_{L^4}\|\nabla\Delta u\|_{L^2}\\ &\le C\|\nabla b\|_{L^2}\|\Delta b\|_{L^2}\|\nabla\Delta u\|_{L^2}\\ &\quad+C\|b\|_{L^2}^{1/2}(\|b\|_{L^2}+\|\nabla b\|_{L^2})^{1/2}\|\Delta b\|_{L^2}^{1/2}\|\nabla\Delta b\|_{L^2}^{1/2}\|\nabla\Delta u\|_{L^2}\\ &\le C\|\Delta b\|_{L^2}^2+\frac{\nu}{8}\|\nabla\Delta u\|_{L^2}^2+\frac{\mu}{8}\|\nabla\Delta b\|_{L^2}^2,\end{aligned}$$

where we used bounds on both L^2- and H^1-norms of b obtained in (5.2.2) and (5.2.3).

Regarding the fourth term, analogously by the bounds quoted above, we have

$$\begin{aligned}\int_{\mathbb{T}^2}(b\cdot\nabla)u\cdot\Delta^2b\,dx &\le \int_{\mathbb{T}^2}|\nabla b||\nabla u||\nabla\Delta b|\,dx+\int_{\mathbb{T}^2}|b||\nabla\nabla u||\nabla\Delta b|\,dx\\ &\le C\|\nabla u\|_{L^4}\|\nabla b\|_{L^4}\|\nabla\Delta b\|_{L^2}\\ &\quad+C\|b\|_{L^4}\|\nabla\nabla u\|_{L^4}\|\nabla\Delta b\|_{L^2}\\ &\le C\|\nabla u\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\Delta b\|_{L^2}^{1/2}\|\nabla\Delta b\|_{L^2}\\ &\quad+C\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\nabla\Delta u\|_{L^2}^{1/2}\|\nabla\Delta b\|_{L^2}\\ &\le C\|\Delta u\|_{L^2}^2+C\|\Delta b\|_{L^2}^2+\frac{\nu}{8}\|\nabla\Delta u\|_{L^2}^2+\frac{\mu}{8}\|\nabla\Delta b\|_{L^2}^2,\end{aligned}$$

where Young's inequality and (2.4.6) are also used.

Analogously, for the fifth term, we have

$$-\int_{\mathbb{T}^2}(u\cdot\nabla)b\cdot\Delta^2b\,dx\le C\|\Delta u\|_{L^2}^2+C\|\Delta b\|_{L^2}^2+\frac{\mu}{8}\|\nabla\Delta b\|_{L^2}^2.$$

Next, we estimate the sixth term as

$$
\begin{aligned}
-\int_{\mathbb{T}^2} (u\cdot\nabla)\theta\Delta\theta\,dx &\le C\|u\|_{L^4}\|\nabla\theta\|_{L^2}^{1/2}\|\Delta\theta\|_{L^2}^{3/2}\\
&\le C\|\nabla\theta\|_{L^2}^2 + \frac{\kappa}{8}\|\Delta\theta\|_{L^2}^2,
\end{aligned}
$$

where we used the boundedness of $|u|$ in (5.2.3).

Finally, to estimate the last term, we take advantage of the boundedness of $|\boldsymbol{\zeta}^\epsilon|$, and obtain

$$
\begin{aligned}
\alpha\mu\int_{\mathbb{T}^2} \boldsymbol{\zeta}^\epsilon(|\nabla\times b|)|\nabla\times b|\Delta\theta\,dx &\le \alpha\mu M_\epsilon\|\nabla\times b\|_{L^2}\|\Delta\theta\|_{L^2}\\
&\le C\alpha^2\mu^2M_\epsilon^2\|\Delta b\|_{L^2}^2 + \frac{\kappa}{8}\|\Delta\theta\|_{L^2}^2.
\end{aligned}
$$

Now, combining all the above estimates, rearranging and simplifying some terms, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right)\\
&\quad + \nu\|\nabla\Delta u\|_{L^2}^2 + \mu\|\nabla\Delta b\|_{L^2}^2 + \kappa\|\Delta\theta\|_{L^2}^2\\
&\le \bar{C}\left(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right),
\end{aligned}
$$

where the constant $\bar{C}$ depends on $g, \nu, \mu, \kappa, M_\epsilon, \alpha, \lambda_1$, as well as K_1 and K_2 (specifically, the L^2-integrability of Δb in time). Therefore, integrate the above inequality in time on $[0, T]$, and by Grönwall inequality, we have

$$
\begin{aligned}
&\|\Delta u(T)\|_{L^2}^2 + \|\Delta b(T)\|_{L^2}^2 + \|\nabla\theta(T)\|_{L^2}^2 \qquad (5.2.9)\\
&\quad + \nu\int_0^T \|\nabla\Delta u(t)\|_{L^2}^2\,dt + \mu\int_0^T \|\nabla\Delta b(t)\|_{L^2}^2\,dt + \kappa\int_0^T \|\Delta\theta(t)\|_{L^2}^2\,dt
\end{aligned}
$$

$$\leq K_3,$$

where the constant K_3 depends on all the aforementioned parameters and on K_1 and K_2.

Next, we prove the H^2-regularity of θ. Proceeding similarly, we multiply the relevant equations in System 5.1.1 by $\Delta^2 u$, $\Delta^2 b$, $\Delta^2\theta$, respectively, integrate by parts over $\mathbb{T}^2$, add, and get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\right) \\
&\quad + \nu\|\nabla\Delta u\|_{L^2}^2 + \mu\|\nabla\Delta b\|_{L^2}^2 + \kappa\|\nabla\Delta\theta\|_{L^2}^2 \\
&= g\int_{\mathbb{T}^2}\theta\vec{e}_2\cdot\Delta^2 u\,dx - \int_{\mathbb{T}^2}(u\cdot\nabla)u\cdot\Delta^2 u\,dx + \int_{\mathbb{T}^2}(b\cdot\nabla)b\cdot\Delta^2 u\,dx \\
&\quad + \int_{\mathbb{T}^2}(b\cdot\nabla)u\cdot\Delta^2 b\,dx - \int_{\mathbb{T}^2}(u\cdot\nabla)b\cdot\Delta^2 b\,dx - \int_{\mathbb{T}^2}(u\cdot\nabla)\theta\Delta^2\theta\,dx \\
&\qquad + \alpha\mu\int_{\mathbb{T}^2}\boldsymbol{\zeta}^\epsilon(|\nabla\times b|)|\nabla\times b|\Delta^2\theta\,dx.
\end{aligned}$$

Then, we estimate the seven terms on the right side of the above equations. Note that estimates on the first term are identical as above. We continue to estimate the second term as

$$\begin{aligned}
-\int_{\mathbb{T}^2}(u\cdot\nabla)u\cdot\Delta^2 u\,dx &\leq \int_{\mathbb{T}^2}|\nabla u|^2|\nabla\Delta u|\,dx + \int_{\mathbb{T}^2}|u||\nabla\nabla u||\nabla\Delta u|\,dx \\
&\leq C\|\nabla u\|_{L^4}^2\|\nabla\Delta u\|_{L^2} \\
&\quad + C\|u\|_{L^\infty}\|\nabla\nabla u\|_{L^2}\|\nabla\Delta u\|_{L^2} \\
&\leq C\|\Delta u\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\Delta u\|_{L^2}^2,
\end{aligned}$$

where we used the bounds on $\|u\|_{L^2}$, $\|\nabla u\|_{L^2}$, and $\|\Delta u\|_{L^2}$, in (5.2.2), (5.2.3), and

(5.2.9), respectively, as well as (2.4.3) and (2.4.6) and Young's inequality.

For the third term, we estimate similarly as

$$\begin{aligned}\int_{\mathbb{T}^2}(b\cdot\nabla)b\cdot\Delta^2u\,dx &\le \int_{\mathbb{T}^2}|\nabla b|^2|\nabla\Delta u|\,dx+\int_{\mathbb{T}^2}|b||\nabla\nabla b||\nabla\Delta u|\,dx\\ &\le C\|\nabla b\|_{L^4}^2\|\nabla\Delta u\|_{L^2}+C\|b\|_{L^\infty}\|\nabla\nabla b\|_{L^2}\|\nabla\Delta u\|_{L^2}\\ &\le C\|\Delta b\|_{L^2}^2+\frac{\nu}{8}\|\nabla\Delta u\|_{L^2}^2.\end{aligned}$$

As for the fourth term, we have

$$\begin{aligned}\int_{\mathbb{T}^2}(b\cdot\nabla)u\cdot\Delta^2b\,dx &\le \int_{\mathbb{T}^2}|\nabla b||\nabla u||\nabla\Delta b|\,dx+\int_{\mathbb{T}^2}|b||\nabla\nabla u||\nabla\Delta b|\,dx\\ &\le C\|\nabla b\|_{L^4}\|\nabla u\|_{L^4}\|\nabla\Delta b\|_{L^2}\\ &\quad+C\|b\|_{L^\infty}\|\nabla\nabla u\|_{L^2}\|\nabla\Delta b\|_{L^2}\\ &\le C\|\Delta u\|_{L^2}^2+C\|\Delta b\|_{L^2}^2+\frac{\mu}{4}\|\nabla\Delta b\|_{L^2}^2,\end{aligned}$$

As for the fifth term, we apply and obtain

$$\begin{aligned}-\int_{\mathbb{T}^2}(u\cdot\nabla)b\Delta^2b\,dx &\le \int_{\mathbb{T}^2}|\nabla u||\nabla b||\nabla\Delta b|\,dx+\int_{\mathbb{T}^2}|u||\nabla\nabla b||\nabla\Delta b|\,dx\\ &\le C\|\nabla u\|_{L^4}\|\nabla b\|_{L^4}\|\nabla\Delta b\|_{L^2}\\ &\quad+C\|u\|_{L^\infty}\|\nabla\nabla b\|_{L^2}\|\nabla\Delta b\|_{L^2}\\ &\le C\|\Delta u\|_{L^2}^2+C\|\Delta b\|_{L^2}^2+\frac{\mu}{8}\|\nabla\Delta b\|_{L^2}^2,\end{aligned}$$

while estimates of the sixth term is done analogously to that of the fifth one as

$$\begin{aligned}-\int_{\mathbb{T}^2}(u\cdot\nabla)\theta\Delta^2\theta\,dx &\le \int_{\mathbb{T}^2}|\nabla u||\nabla\theta||\nabla\Delta\theta|\,dx+\int_{\mathbb{T}^2}|u||\nabla\nabla\theta||\nabla\Delta\theta|\,dx\\ &\le C\|\nabla u\|_{L^4}\|\nabla\theta\|_{L^4}\|\nabla\Delta\theta\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&+ C\|u\|_{L^\infty}\|\nabla\nabla\theta\|_{L^2}\|\nabla\Delta\theta\|_{L^2} \\
&\le C\|\Delta u\|_{L^2}^2 + C\|\Delta\theta\|_{L^2}^2 + \frac{\kappa}{8}\|\nabla\Delta\theta\|_{L^2}^2,
\end{aligned}
$$

where we used the boundedness of $\|\nabla u\|_{L^2}$ and $\|\nabla\theta\|_{L^2}$ in (5.2.3).

As regarding the last term, we integrate by parts first, then take advantage of the boundedness of $|\boldsymbol{\zeta}^\epsilon|$ and $|\nabla\boldsymbol{\zeta}^\epsilon|$, and obtain

$$
\begin{aligned}
&\alpha\mu\int_{\mathbb{T}^2}\boldsymbol{\zeta}^\epsilon(|\nabla\times b|)|\nabla\times b|\Delta^2\theta\,dx \\
&\le \alpha\mu M_\epsilon\int_{\mathbb{T}^2}\big|\nabla|\nabla\times b|\big||\nabla\Delta\theta|\,dx \\
&\quad + \alpha\mu\int_{\mathbb{T}^2}|\nabla\boldsymbol{\zeta}^\epsilon(|\nabla\times b|)|\big|\nabla|\nabla\times b|\big||\nabla\times b||\nabla\Delta\theta|\,dx
\end{aligned}
$$

of which the first term is bounded by

$$
C\alpha^2\mu^2M_\epsilon^2\|\Delta b\|_{L^2}^2 + \frac{\kappa}{8}\|\nabla\Delta\theta\|_{L^2}^2
$$

due to Cauchy-Schwarz inequality and Problem 5.10.17 of [30]; and the second term is bounded by

$$
\begin{aligned}
&C\alpha\mu M_\epsilon\|\nabla b\|_{L^4}\|\nabla\nabla b\|_{L^4}\|\nabla\Delta\theta\|_{L^2} \\
&\le C\alpha\mu M_\epsilon\|\Delta b\|_{L^2}\|\nabla\Delta b\|_{L^2}^{1/2}\|\nabla\Delta\theta\|_{L^2} \\
&\le C\alpha^4\mu^2M_\epsilon^4\|\Delta b\|_{L^2}^2 + \frac{\mu}{8}\|\nabla\Delta b\|_{L^2}^2 + \frac{\kappa}{8}\|\nabla\Delta\theta\|_{L^2}^2.
\end{aligned}
$$

Now, combining all the above estimates, rearranging and simplifying some terms, we obtain

$$
\frac{d}{dt}\left(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\right)
$$

$$+\nu\|\nabla\Delta u\|_{L^2}^2+\mu\|\nabla\Delta b\|_{L^2}^2+\kappa\|\nabla\Delta\theta\|_{L^2}^2$$
$$\leq C^*\left(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2\right),$$

where the constant C^* depends on $g,\nu,\mu,\kappa,M_\epsilon,\alpha$, as well as K_1, K_2, and K_3. Finally, integrate the above inequality in time from 0 to $\forall T>0$, and by Grönwall inequality, we get

$$\begin{aligned}&\|\Delta u(T)\|_{L^2}^2+\|\Delta b(T)\|_{L^2}^2+\|\Delta\theta(T)\|_{L^2}^2\\&\quad+\nu\int_0^T\|\nabla\Delta u(t)\|_{L^2}^2\,dt+\mu\int_0^T\|\nabla\Delta b(t)\|_{L^2}^2\,dt+\kappa\int_0^T\|\nabla\Delta\theta(t)\|_{L^2}^2\,dt\\&\leq K_4,\end{aligned}\tag{5.2.10}$$

where the constant K_4 depends on all the aforementioned parameters and on K_1, K_2, and K_3. □

Remark 5.2.2. The constants K_1, K_2, K_3, and K_4 all depend on the parameter $M_\epsilon = \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}$ stated in (2), which tends toward infinity as $\epsilon\to 0^+$. Therefore these estimates do not hold as $\epsilon\to 0^+$.

5.2.4 Proof of Uniqueness

Following the well-known weak-strong uniqueness argument for the Navier-Stokes equations, it suffices to show that the strong solution is unique. Note that due to the Ohmic heating effect in System 5.1.1, one needs to work with at least H^2 initial condition and regularity of the solution. To begin, we assume there are two distinct solutions $(u_n, b_n, \theta_n, p_n), n = 1,2$, to System 5.1.1, in the sense of Theorem 1.2.18, with the same initial data $u_1(0)=u_2(0)$, $b_1(0)=b_2(0)$, $\theta_1(0)=\theta_2(0)$. Subtract the corresponding equations satisfied by the two solutions, and denote the differences by

$\tilde{u} = u_1 - u_2$, $\tilde{b} = b_1 - b_2$, $\tilde{\theta} = \theta_1 - \theta_2$, and $\tilde{p} = p_1 - p_2$, and we obtain the system for $\tilde{u}$, $\tilde{b}$, $\tilde{\theta}$, and $\tilde{p}$ as

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - \nu\Delta\tilde{u} + (u_1\cdot\nabla)\tilde{u} + (\tilde{u}\cdot\nabla)u_2 + \nabla\tilde{p} \\ \qquad\qquad = (b_1\cdot\nabla)\tilde{b} + (\tilde{b}\cdot\nabla)b_2 + g\tilde{\theta}\vec{e}_2, \\ \dfrac{\partial \tilde{b}}{\partial t} - \mu\Delta\tilde{b} + (u_1\cdot\nabla)\tilde{b} + (\tilde{u}\cdot\nabla)b_2 = (b_1\cdot\nabla)\tilde{u} + (\tilde{b}\cdot\nabla)u_2, \\ \dfrac{\partial \tilde{\theta}}{\partial t} - \kappa\Delta\tilde{\theta} + (u_1\cdot\nabla)\tilde{\theta} + (\tilde{u}\cdot\nabla)\theta_2 \\ \qquad\qquad = \alpha\mu\big(\boldsymbol{\zeta}^\epsilon(|\nabla\times b_1|)|\nabla\times b_1| - \boldsymbol{\zeta}^\epsilon(|\nabla\times b_2|)|\nabla\times b_2|\big), \\ \nabla\cdot\tilde{u} = 0 = \nabla\cdot\tilde{b}, \\ \tilde{u}(0) = \tilde{b}(0) = 0,\ \tilde{\theta}(0) = 0. \end{cases} \tag{5.2.11}$$

Then, multiply $\tilde{u}, \tilde{b}, \tilde{\theta}$ to the relevant equations in System 5.2.11, respectively, integrate by parts over $\mathbb{T}^2$, and add, so that we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{b}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2\right) + \nu\|\nabla\tilde{u}\|_{L^2}^2 + \mu\|\nabla\tilde{b}\|_{L^2}^2 + \kappa\|\nabla\tilde{\theta}\|_{L^2}^2 \\ &= g\int_{\mathbb{T}^2}\tilde{\theta}\vec{e}_2\cdot\tilde{u}\,dx - \int_{\mathbb{T}^2}(\tilde{u}\cdot\nabla)u_2\cdot\tilde{u}\,dx + \int_{\mathbb{T}^2}(\tilde{b}\cdot\nabla)b_2\cdot\tilde{u}\,dx \\ &\quad - \int_{\mathbb{T}^2}(\tilde{u}\cdot\nabla)b_2\cdot\tilde{b}\,dx + \int_{\mathbb{T}^2}(\tilde{b}\cdot\nabla)u_2\cdot\tilde{b}\,dx - \int_{\mathbb{T}^2}(\tilde{u}\cdot\nabla)\theta_2\tilde{\theta}\,dx \\ &\quad + \alpha\mu\int_{\mathbb{T}^2}\big(\boldsymbol{\zeta}^\epsilon(|\nabla\times b_1|)|\nabla\times b_1| - \boldsymbol{\zeta}^\epsilon(|\nabla\times b_2|)|\nabla\times b_2|\big)\tilde{\theta}\,dx, \end{aligned}$$

where we used the divergence-free condition $\nabla\cdot\tilde{u} = 0 = \nabla\cdot\tilde{b}$.

Next, we estimate the seven terms on the right side of the above equation. By Cauchy-Schwarz inequality, the first term is estimated as

$$g\int_{\mathbb{T}^2}\tilde{\theta}\vec{e}_2\cdot\tilde{u}\,dx \le \frac{g}{2}\|\tilde{u}\|_{L^2}^2 + \frac{g}{2}\|\tilde{\theta}\|_{L^2}^2.$$

For the second term, by the H^1-boundedness obtained in (5.2.3), we have

$$-\int_{\mathbb{T}^2}(\tilde{u}\cdot\nabla)u_2\cdot\tilde{u}\,dx \leq C\|\nabla u_2\|_{L^2}\|\tilde{u}\|_{L^4}^2 \leq C\|\tilde{u}\|_{L^2}(\|\tilde{u}\|_{L^2}+\|\nabla\tilde{u}\|_{L^2})$$
$$\leq C\|\tilde{u}\|_{L^2}^2+\frac{\nu}{8}\|\nabla\tilde{u}\|_{L^2}^2.$$

The third through the sixth terms are estimated similarly, so we get, for the third one,

$$\begin{aligned}\int_{\mathbb{T}^2}(\tilde{b}\cdot\nabla)b_2\cdot\tilde{u}\,dx &\leq C\|\nabla b_2\|_{L^2}\|\tilde{b}\|_{L^4}\|\tilde{u}\|_{L^4}\\ &\leq C\|\tilde{b}\|_{L^2}^{1/2}(\|\tilde{b}\|_{L^2}+\|\nabla\tilde{b}\|_{L^2})^{1/2}\|\tilde{u}\|_{L^2}^{1/2}(\|\tilde{u}\|_{L^2}+\|\nabla\tilde{u}\|_{L^2})^{1/2}\\ &\leq C\|\tilde{b}\|_{L^2}(\|\tilde{b}\|_{L^2}+\|\nabla\tilde{b}\|_{L^2})+C\|\tilde{u}\|_{L^2}(\|\tilde{u}\|_{L^2}+\|\nabla\tilde{u}\|_{L^2})\\ &\leq C\|\tilde{u}\|_{L^2}^2+C\|\tilde{b}\|_{L^2}^2+\frac{\nu}{8}\|\nabla\tilde{u}\|_{L^2}^2+\frac{\mu}{8}\|\nabla\tilde{b}\|_{L^2}^2\end{aligned}$$

where we used both (5.2.2) and (5.2.3); and for the fourth one, we also have

$$-\int_{\mathbb{T}^2}(\tilde{u}\cdot\nabla)b_2\cdot\tilde{b}\,dx \leq C\|\tilde{u}\|_{L^2}^2+C\|\tilde{b}\|_{L^2}^2+\frac{\nu}{8}\|\nabla\tilde{u}\|_{L^2}^2+\frac{\mu}{8}\|\nabla\tilde{b}\|_{L^2}^2;$$

while for the fifth term, we take advantage of the H^1-bounds of u_2 in (5.2.3), and obtain an upper bound as

$$\int_{\mathbb{T}^2}(\tilde{b}\cdot\nabla)u_2\cdot\tilde{b}\,dx \leq C\|\tilde{b}\|_{L^2}^2+\frac{\mu}{8}\|\nabla\tilde{b}\|_{L^2}^2;$$

as for the sixth term, we proceed similarly and get

$$-\int_{\mathbb{T}^2}(\tilde{u}\cdot\nabla)\theta_2\tilde{\theta}\,dx \leq C\|\tilde{u}\|_{L^2}^2+C\|\tilde{\theta}\|_{L^2}^2+\frac{\nu}{8}\|\nabla\tilde{u}\|_{L^2}^2+\frac{\kappa}{4}\|\nabla\tilde{\theta}\|_{L^2}^2$$

Now, it remains to estimate the last term that involves the Ohmic heating. We start by bounding the magnitude of the first factor inside the integrand. Thanks to the global Lipschitz condition on $\boldsymbol{\zeta}^\epsilon$, we have

$$
\begin{aligned}
&\big|\boldsymbol{\zeta}^\epsilon(|\nabla\times b_1|)|\nabla\times b_1| - \boldsymbol{\zeta}^\epsilon(|\nabla\times b_2|)|\nabla\times b_2|\big| \\
&\le \big|\boldsymbol{\zeta}^\epsilon(|\nabla\times b_1|)|\nabla\times b_1| - \boldsymbol{\zeta}^\epsilon(|\nabla\times b_1|)|\nabla\times b_2|\big| \\
&\qquad + \big|\boldsymbol{\zeta}^\epsilon(|\nabla\times b_1|)|\nabla\times b_2| - \boldsymbol{\zeta}^\epsilon(|\nabla\times b_2|)|\nabla\times b_2|\big| \\
&= \big|\boldsymbol{\zeta}^\epsilon(|\nabla\times b_1|)|\nabla\times \tilde{b}|\big| + \big|\boldsymbol{\zeta}^\epsilon(|\nabla\times b_1|) - \boldsymbol{\zeta}^\epsilon(|\nabla\times b_2|)\big||\nabla\times b_2| \\
&\le M_\epsilon|\nabla\times\tilde{b}| + L_\epsilon|\nabla\times\tilde{b}||\nabla\times b_2|.
\end{aligned}
$$

Thus, inserting the above estimates to the last term, we obtain

$$
\begin{aligned}
&\alpha\mu\int_{\mathbb{T}^2}\big(\boldsymbol{\zeta}^\epsilon(|\nabla\times b_1|)|\nabla\times b_1| - \boldsymbol{\zeta}^\epsilon(|\nabla\times b_2|)|\nabla\times b_2|\big)\tilde{\theta}\,dx, \\
&\le \alpha\mu M_\epsilon\int_{\mathbb{T}^2}|\nabla\times\tilde{b}||\tilde{\theta}|\,dx + \alpha\mu L_\epsilon\int_{\mathbb{T}^2}|\nabla\times b_2||\nabla\times\tilde{b}||\tilde{\theta}|\,dx \\
&\le C\|\tilde{\theta}\|_{L^2}^2 + \frac{\mu}{16}\|\nabla\tilde{b}\|_{L^2}^2 + C\|\nabla b_2\|_{L^4}\|\nabla\tilde{b}\|_{L^2}\|\tilde{\theta}\|_{L^4} \\
&\le C\|\tilde{\theta}\|_{L^2}^2 + \frac{\mu}{16}\|\nabla\tilde{b}\|_{L^2}^2 + C\|\nabla\tilde{b}\|_{L^2}\|\tilde{\theta}\|_{L^2}^{1/2}\|\nabla\tilde{\theta}\|_{L^2}^{1/2} \\
&\le \|\tilde{\theta}\|_{L^2}^2 + \frac{\mu}{8}\|\nabla\tilde{b}\|_{L^2}^2 + \frac{\kappa}{4}\|\nabla\tilde{\theta}\|_{L^2}^2,
\end{aligned}
$$

where in the penultimate step we used the H^2-bounds obtained in (5.2.10).

Therefore, by collecting all the above estimates, and after some simplification and rearrangement, we finally get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{b}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2\right) + \nu\|\nabla\tilde{u}\|_{L^2}^2 + \mu\|\nabla\tilde{b}\|_{L^2}^2 + \kappa\|\nabla\tilde{\theta}\|_{L^2}^2 \\
&\le \tilde{C}\big(\|\tilde{u}\|_{L^2}^2 + \|\tilde{b}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2\big),
\end{aligned}
$$

where $\tilde{C}$ depends on $g, \nu, \mu, \kappa, \alpha, M_\epsilon, L_\epsilon$, as well as $K_m, m = 1, 2, 3, 4$. Now, integrate the above inequality in time from 0 to $\forall\, T > 0$, and by the Grönwall inequality and the initial condition $\tilde{u}(0) = \tilde{b}(0) = 0$ and $\tilde{\theta}(0) = 0$, we conclude that $\tilde{u}(T) = 0, \tilde{b}(T) = 0$, and $\tilde{\theta}(T) = 0$, i.e., $u_1 = u_2$, $b_1 = b_2$, and $\theta_1 = \theta_2$. Hence, the global well-posedness of the 2D MHD-Boussinesq system with Ohmic heating, i.e., system (5.1.1), is now proved. □

5.3 Proof of the Convergence Theorem 1.2.20

In this section, we show that on the common time-interval of existence, the difference between the solution of System 5.1.1 and that of System 5.1.2, is in the order of ϵ; and in particular, such error approaches to 0 as ϵ goes to 0^+.

5.3.1 Proof of Theorem 1.2.20

Set $\tilde{u} = U - u$, $\tilde{b} = B - b$, $\tilde{\theta} = \Theta - \theta$, and $\tilde{p} = P - p$. We take the difference of equations (5.1.2) and (5.1.1) to obtain the system

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - \nu \triangle \tilde{u} + \nabla \tilde{p} = (\tilde{u} \cdot \nabla)\, \tilde{u} - (\tilde{u} \cdot \nabla)\, U - (U \cdot \nabla)\, \tilde{u} \\ \qquad\qquad + (B \cdot \nabla)\, \tilde{b} - \left(\tilde{b} \cdot \nabla\right) \tilde{b} + \left(\tilde{b} \cdot \nabla\right) B + g\tilde{\theta}\vec{e}_2, \\ \dfrac{\partial \tilde{b}}{\partial t} - \mu \Delta \tilde{b} = (B \cdot \nabla)\, \tilde{u} - \left(\tilde{b} \cdot \nabla\right) \tilde{u} + \left(\tilde{b} \cdot \nabla\right) U \\ \qquad\qquad + (\tilde{u} \cdot \nabla)\, \tilde{b} - (\tilde{u} \cdot \nabla)\, B - (U \cdot \nabla)\, \tilde{b}, \\ \dfrac{\partial \tilde{\theta}}{\partial t} - \kappa \triangle \tilde{\theta} = (\tilde{u} \cdot \nabla)\, \tilde{\theta} - (\tilde{u} \cdot \nabla)\, \Theta - (U \cdot \nabla)\, \tilde{\theta} \\ \qquad\qquad + \alpha\mu \left(|\nabla \times B| - \boldsymbol{\zeta}^{\epsilon}\left(|\nabla \times B|\right)\right) |\nabla \times B| \\ \qquad\qquad - \alpha\mu \left(\boldsymbol{\zeta}^{\epsilon}\left(|\nabla \times B|\right) - \boldsymbol{\zeta}^{\epsilon}\left(|\nabla \times b|\right)\right) \left(|\nabla \times B| - |\nabla \times b|\right) \\ \qquad\qquad + \alpha\mu \left(|\nabla \times B| - |\nabla \times b|\right) \boldsymbol{\zeta}^{\epsilon}\left(|\nabla \times B|\right) \\ \qquad\qquad + \alpha\mu \left(\boldsymbol{\zeta}^{\epsilon}\left(|\nabla \times B|\right) - \boldsymbol{\zeta}^{\epsilon}\left(|\nabla \times b|\right)\right) |\nabla \times B| \\ \nabla \cdot \tilde{u} = 0 = \nabla \cdot \tilde{b}, \\ \tilde{u}(0) = \tilde{b}(0) = 0, \ \tilde{\theta}(0) = 0. \end{cases} \tag{5.3.1}$$

First we will remark that since (U, B, Θ) satisfy the conditions of Theorem 1.2.19, there exists $M > 0$ such that for all $t \in [0, T]$,

$$\|U\|_{H^2} + \|B\|_{H^2} + \|\Theta\|_{H^1} \leq M, \tag{5.3.2}$$

hence we can bound these terms above by a constant C when needed.

To show the convergence of the system, we make the ansatz

$$\|\tilde{u}\|_{H^1} + \left\|\tilde{b}\right\|_{H^1} + \|\tilde{\theta}\|_{L^2} < 1, \tag{5.3.3}$$

and use a bootstrapping argument to show that this leads to an even tighter bound, thus closing the argument. To this end, we take the inner product of the first two equations in System 5.3.1 with $\tilde{u}$ and $\tilde{b}$ respectively, integrate by parts, and remove any terms that are eliminated using identities (2.4.7a) and (2.4.7b), which yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{b}\|_{L^2}^2\right) + \nu\|\nabla\tilde{u}\|_{L^2}^2 + \mu\|\nabla\tilde{b}\|_{L^2}^2 \\
&= -\left(\left(\tilde{u}\cdot\nabla\right)U, \tilde{u}\right) + \left(\left(\tilde{b}\cdot\nabla\right)B, \tilde{u}\right) + \left(\left(\tilde{b}\cdot\nabla\right)U, \tilde{b}\right) \\
&\quad - \left(\left(\tilde{u}\cdot\nabla\right)B, \tilde{b}\right) + \left(g\tilde{\theta}\vec{e}_2, \tilde{u}\right).
\end{aligned} \tag{5.3.4}$$

We now take the inner product of the system with $\left(-\triangle\tilde{u}, -\triangle\tilde{b}, \tilde{\theta}\right)$, integrate by parts, and invoke (2.4.9) and (2.4.10) where appropriate:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\tilde{u}\|_{L^2}^2 + \|\nabla\tilde{b}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2\right) + \nu\|\triangle\tilde{u}\|_{L^2}^2 + \mu\|\triangle\tilde{b}\|_{L^2}^2 + \kappa\|\nabla\tilde{\theta}\|_{L^2}^2 \\
&= \left(\left(\tilde{u}\cdot\nabla\right)U, \triangle\tilde{u}\right) + \left(\left(U\cdot\nabla\right)\tilde{u}, \triangle\tilde{u}\right) - \left(\left(B\cdot\nabla\right)\tilde{b}, \triangle\tilde{u}\right) - \left(\left(\tilde{b}\cdot\nabla\right)B, \triangle\tilde{u}\right) \\
&- \left(g\tilde{\theta}\vec{e}_2, \triangle\tilde{u}\right) - \left(\left(B\cdot\nabla\right)\tilde{u}, \triangle\tilde{b}\right) - \left(\left(\tilde{b}\cdot\nabla\right)U, \triangle\tilde{b}\right) - 2\left(\left(\tilde{u}\cdot\nabla\right)\tilde{b}, \triangle\tilde{b}\right) \\
&+ \left(\left(\tilde{u}\cdot\nabla\right)B, \triangle\tilde{b}\right) + \left(\left(U\cdot\nabla\right)\tilde{b}, \triangle\tilde{b}\right) + \left(\left(\tilde{u}\cdot\nabla\right)\Theta, \tilde{\theta}\right) + N_1 + N_2 + N_3 + N_4,
\end{aligned} \tag{5.3.5}$$

where

$$\begin{aligned}
N_1 &= \alpha\mu\int_{\mathbb{T}} \left||\nabla\times B| - \boldsymbol{\zeta}^\epsilon\left(|\nabla\times B|\right)\right| |\nabla\times B|\,\tilde{\theta}dx, \\
N_2 &= \alpha\mu\int_{\mathbb{T}} \left(\boldsymbol{\zeta}^\epsilon\left(|\nabla\times B|\right) - \boldsymbol{\zeta}^\epsilon\left(|\nabla\times b|\right)\right)\left(|\nabla\times B| - |\nabla\times b|\right)\tilde{\theta}dx, \\
N_3 &= \alpha\mu\int_{\mathbb{T}} \left(|\nabla\times B| - |\nabla\times b|\right)\boldsymbol{\zeta}^\epsilon(|\nabla\times B|)\tilde{\theta}dx, \\
N_4 &= \alpha\mu\int_{\mathbb{T}} \left(\boldsymbol{\zeta}^\epsilon(|\nabla\times B|) - \boldsymbol{\zeta}^\epsilon(|\nabla\times b|)\right)|\nabla\times B|\,\tilde{\theta}dx.
\end{aligned} \tag{5.3.6}$$

We now combine (5.3.4) and (5.3.5) to obtain the equation

$$\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|_{H^1}^2 + \left\|\tilde{b}\right\|_{H^1}^2 + \|\tilde{\theta}\|_{L^2}^2\right) + \nu\|\nabla\tilde{u}\|_{H^1}^2 + \mu\left\|\nabla\tilde{b}\right\|_{H^1}^2 + \kappa\|\nabla\tilde{\theta}\|_{L^2}^2 \tag{5.3.7}$$
$$\begin{aligned}
&= -\left(\left(\tilde{u}\cdot\nabla\right)U,\tilde{u}\right) + \left(\left(\tilde{b}\cdot\nabla\right)B,\tilde{u}\right) + \left(\left(\tilde{b}\cdot\nabla\right)U,\tilde{b}\right) - \left(\left(\tilde{u}\cdot\nabla\right)B,\tilde{b}\right)\\
&\quad + \left(g\tilde{\theta}\vec{e}_2,\tilde{u}\right) + \left(\left(\tilde{u}\cdot\nabla\right)U,\triangle\tilde{u}\right) + \left(\left(U\cdot\nabla\right)\tilde{u},\triangle\tilde{u}\right) - \left(\left(B\cdot\nabla\right)\tilde{b},\triangle\tilde{u}\right)\\
&\quad - \left(\left(\tilde{b}\cdot\nabla\right)B,\triangle\tilde{u}\right) - \left(g\tilde{\theta}\vec{e_2},\triangle\tilde{u}\right) - \left(\left(B\cdot\nabla\right)\tilde{u},\triangle\tilde{b}\right) - \left(\left(\tilde{b}\cdot\nabla\right)U,\triangle\tilde{b}\right)\\
&\quad - 2\left(\left(\tilde{u}\cdot\nabla\right)\tilde{b},\triangle\tilde{b}\right) + \left(\left(\tilde{u}\cdot\nabla\right)B,\triangle\tilde{b}\right) + \left(\left(U\cdot\nabla\right)\tilde{b},\triangle\tilde{b}\right) + \left(\left(\tilde{u}\cdot\nabla\right)\Theta,\tilde{\theta}\right)\\
&\quad + N_1 + N_2 + N_3 + N_4.
\end{aligned}$$

The first twelve terms can be bounded above using Hölder's inequality, Agmon's Inequality, Young's inequality, and (5.3.2). For the sake of brevity, we show this only for the first term:

$$\begin{aligned}
\left|\left(\left(\tilde{u}\cdot\nabla\right)U,\tilde{u}\right)\right| &\le \|\tilde{u}\|_{L^\infty}\|\nabla U\|_{L^2}\|\tilde{u}\|_{L^2} \\
&\le C\|\tilde{u}\|_{L^2}^{1/2}\|\tilde{u}\|_{H^2}^{1/2}\|\nabla U\|_{L^2}\|\tilde{u}\|_{L^2}\\
&\le C\|\nabla U\|_{L^2}^4\|\tilde{u}\|_{L^2}^2 + \frac{\nu}{22}\|\tilde{u}\|_{H^2}\\
&\le \tilde{C}\|\tilde{u}\|_{L^2}^2 + \frac{\nu}{22}\|\tilde{u}\|_{H^2}^2.
\end{aligned}\tag{5.3.8}$$

The next four terms can be bounded using the aforementioned inequalities and Ansatz (5.3.3):

$$\begin{aligned}
2\left|\left(\left(\tilde{u}\cdot\nabla\right)\tilde{b},\triangle\tilde{b}\right)\right| &\le 2\|\tilde{u}\|_{L^\infty}\left\|\nabla\tilde{b}\right\|_{L^2}\left\|\triangle\tilde{b}\right\|_{L^2}\\
&\le C\|\tilde{u}\|_{L^2}^{1/2}\|\tilde{u}\|_{H^2}^{1/2}\left\|\nabla\tilde{b}\right\|_{L^2}\left\|\triangle\tilde{b}\right\|_{L^2}\\
&\le C\|\tilde{u}\|_{L^2}^{1/2}\|\tilde{u}\|_{H^2}^{1/2}\left\|\triangle\tilde{b}\right\|_{L^2}
\end{aligned}\tag{5.3.9}$$

$$\leq C\left\|\tilde{u}\right\|_{L^2}^2+\frac{\nu}{22}\left\|\tilde{u}\right\|_{H^2}^2+\frac{\mu}{18}\left\|\triangle\tilde{b}\right\|_{L^2}^2,$$

noting that (5.3.3) was used to obtain the bound $\left\|\nabla\tilde{b}\right\|_{L^2}<1$.

We now focus on upper bounds for each N_i: For N_1, using Inequalities (2.1.1) and (2.4.4) we obtain

$$\begin{aligned}|N_1| &= \alpha\mu\left|\int_{\mathbb{T}}\left||\nabla\times B|-\boldsymbol{\zeta}^\epsilon\left(|\nabla\times B|\right)\right||\nabla\times B|\,\tilde{\theta}dx\right| \qquad (5.3.10)\\ &\leq C\epsilon^\gamma\int_{\mathbb{T}}|\nabla B|^{\beta+1}\left|\tilde{\theta}\right|dx\\ &\leq C\epsilon^\gamma\left\|\nabla B\right\|_{L^{2\beta+2}}^{\beta+1}\|\tilde{\theta}\|_{L^2}\\ &\leq C\epsilon^\gamma\|\nabla B\|_{L^2}\|\triangle B\|_{L^2}^\beta\|\tilde{\theta}\|_{L^2}\\ &\leq C\epsilon^{2\gamma}\|\nabla B\|_{L^2}^2\|\triangle B\|_{L^2}^{2\beta}+\|\tilde{\theta}\|_{L^2}^2.\end{aligned}$$

Since $B\in L^\infty(0,T;H^2\cap V)$, the first term is integrable for any $\beta\geq 0$. For N_2, using the Lipschitz property of $\boldsymbol{\zeta}^\epsilon$, the reverse triangle inequality, (5.3.3), and (2.4.6) we obtain

$$\begin{aligned}|N_2| &= \alpha\mu\left|\int_{\mathbb{T}}\left(\boldsymbol{\zeta}^\epsilon\left(|\nabla\times B|\right)-\boldsymbol{\zeta}^\epsilon\left(|\nabla\times b|\right)\right)\left(|\nabla\times B|-|\nabla\times b|\right)\tilde{\theta}dx\right| \qquad (5.3.11)\\ &\leq C\int_{\mathbb{T}}\left|\nabla\tilde{b}\right|^2\left|\tilde{\theta}\right|dx\\ &\leq C||\nabla\tilde{b}||_{L^4}^2||\tilde{\theta}||_{L^2}\\ &\leq C||\nabla\tilde{b}||_{L^2}||\triangle\tilde{b}||_{L^2}||\tilde{\theta}||_{L^2}\\ &\leq C||\triangle\tilde{b}||_{L^2}||\tilde{\theta}||_{L^2}\\ &\leq C||\tilde{\theta}||_{L^2}^2+\frac{\mu}{18}||\triangle\tilde{b}||_{L^2}^2.\end{aligned}$$

For N_3, similar methods yield

$$
\begin{aligned}
|N_3| &= \left| \int_{\mathbb{T}} \alpha\mu \left(|\nabla \times B| - |\nabla \times b| \right) \boldsymbol{\zeta}^{\epsilon}(|\nabla \times B|)\tilde{\theta} dx \right| \qquad (5.3.12)\\
&\leq C \left\| \nabla B \right\|_{L^4} ||\nabla \tilde{b}||_{L^2} ||\tilde{\theta}||_{L^4} \\
&\leq C \left\| \triangle B \right\|_{L^2} ||\nabla \tilde{b}||_{L^2} ||\nabla \tilde{\theta}||_{L^2} \\
&\leq C \left\| \triangle B \right\|_{L^2}^2 ||\nabla \tilde{b}||_{L^2}^2 + \frac{\kappa}{4} ||\nabla \tilde{\theta}||_{L^2}^2
\end{aligned}
$$

and N_4 can be bounded in the exact same manner as N_3.

Finally, we can bound the left hand side of (5.3.7) using the estimates in (5.3.8), (5.3.9), (5.3.10), (5.3.11), (5.3.12), then rearrange the terms to obtain

$$
\begin{aligned}
&\frac{d}{dt} \left(\|\tilde{u}\|_{H^1}^2 + \left\| \tilde{b} \right\|_{H^1}^2 + \|\tilde{\theta}\|_{L^2}^2 \right) + \nu \left\| \nabla \tilde{u} \right\|_{H^1}^2 + \mu \left\| \nabla \tilde{b} \right\|_{H^1}^2 + \kappa \|\nabla \tilde{\theta}\|_{L^2}^2 \qquad (5.3.13)\\
&\leq \tilde{C} \left(\|\tilde{u}\|_{H^1}^2 + \left\| \tilde{b} \right\|_{H^1}^2 + \|\tilde{\theta}\|_{L^2}^2 \right) + C\epsilon^{2\gamma} \|\nabla B\|_{L^2}^2 \|\triangle B\|_{L^2}^{2\beta}.
\end{aligned}
$$

Using Grönwall's Inequality (2.5.1) and the fact that $\tilde{u}(0) = \tilde{b}(0) = 0$, $\tilde{\theta}(0) = 0$, from (5.3.13) we deduce that

$$
\|\tilde{u}\|_{H^1}^2 + \left\| \tilde{b} \right\|_{H^1}^2 + \|\tilde{\theta}\|_{L^2}^2 \leq A(T) e^{\tilde{C}T} \epsilon^{2\gamma} \qquad (5.3.14)
$$

where

$$
A(T) = C \int_0^T \|\nabla B\|_{L^2}^2 \|\triangle B\|_{L^2}^{2\beta} dt
$$

and $\tilde{C}$ is a constant depending on $\|U\|_{H^2}, \|B\|_{H^2}, \|\Theta\|_{H^1}, g, \nu, \mu,$ and κ, but does not depend on ϵ. From (5.3.2) it follows that $A(T) < \infty$. Moreover, since $A(T)$ is independent of ϵ we may choose $\epsilon > 0$ small enough so that

$$\|\tilde{u}\|_{H^1}^2 + \left\|\tilde{b}\right\|_{H^1}^2 + \|\tilde{\theta}\|_{L^2}^2 < \frac{1}{2},$$

thus completing the bootstrapping argument, so that in fact (5.3.14) holds for all $t \in [0, T]$.

Now, we integrate (5.3.13) on the interval $[0, T]$, apply (5.3.14), and drop any unnecessary terms from the left hand side:

$$\int_0^T \nu \|\nabla \tilde{u}\|_{H^1}^2 + \mu \left\|\nabla \tilde{b}\right\|_{H^1}^2 + \kappa \|\nabla \tilde{\theta}\|_{L^2}^2 \, dt \leq \tilde{C} A(T) e^{\tilde{C}T} \epsilon^{2\gamma} + A(T)\epsilon^{2\gamma}. \qquad (5.3.15)$$

From (5.3.14) and (5.3.15) we conclude that u, b converge to U, B, respectively, in $L^\infty(0, T; H^1 \cap V) \cap L^2(0, T; H^2 \cap V)$ and that θ converges to Θ in $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$. This completes the proof of convergence. $\square$

Chapter 6

The 2D Kuramoto-Sivashinsky Equations

6.1 Introduction

The Kuramoto-Sivashinsky equation (KSE) is a captivating model for flame fronts, crystal growth, and many other phenomena. It is both satisfying and frustrating. In one space dimension, the model acts as a fantastic toy model: it has highly non-trivial chaotic dynamics while still being amenable to a wide range of analytical tools. However, in higher dimensions, it has so far resisted nearly every analytical attack due to its lack of any known conserved quantity, and the basic question of global well-posedness of solutions remains open, even in two dimensions. Moreover, the nonlinearity of the system has many similarities with the nonlinearity of the Navier-Stokes equations (NSE), making investigation of the KSE even more intriguing.

How does one proceed in the face of such difficulty? In the case of the NSE, at least one approach has been fruitful since at least the work of Smagorinsky in 1963 [99], where a modification of the Navier–Stokes system was proposed, resulting in a system which is both globally well-posed [56], and less computationally demanding to simulate. Since then, hundreds of so-called "turbulence models" have arisen (see, e.g., [27, 91] for a survey), which typically modify the equations in some way. It is therefore natural to ask whether such an approach might work for the 2D KSE.[1] However, one quickly realizes that approaches which work for the NSE are unlikely to work for the KSE. Indeed, for the NSE, the problem is the growth of large gradients; more specifically, the problem is the development of large vorticity, $\boldsymbol{\omega} := \nabla \times \mathbf{u}$ (see, e.g., [3]). This is due to the cubic nature of the vorticity equation: $\frac{d}{dt}\|\boldsymbol{\omega}\|_{L^2}^2 \sim (\boldsymbol{\omega} \cdot \nabla \mathbf{u}, \boldsymbol{\omega})$. Hence, in order to handle the NSE, one typically attempts to control the gradient of the solution, for example, by strengthening the viscosity or weakening the nonlinear term, since the nonlinear term cascades energy from large scales to small scales, intensifying the gradient. That is, the NSE are appeased by controlling the *small scales*. On the other hand, for the KSE, the problem is the growth of large scales. This is due to the cubic nature of the energy equation: $\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 \sim (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{u})$. In the 1D case (and in the NSE case), this latter term vanishes, but not in the 2D KSE case. Moreover, controlling the small scales is not a major problem, as the KSE has a fourth-order diffusion term, strongly curbing the growth of gradients. Therefore, the problem for the KSE appears to be the exact opposite of the problem for the NSE. That is, the KSE is appeased by controlling the *large scales*. Hence, the standard approaches that work for the NSE are unlikely to be of use for the KSE (see [52] for investigations of this notion in the 1D case), and new approaches for handling the KSE are required. The purpose of the present work is to propose and investigate one such approach.

In [62] the authors study a modification of the 2D KSE that they call the "reduced KSE" (r-KSE) with an adjustment made to the linear term in one component. This system admits a maximum principle, allowing for a proof of globally well-posedness. Moreover, simulations in [62] indicate that the dynamics of the r-KSE are arguably

[1] Since the KSE governs the evolution of a surface, its natural space dimension is two. Moreover, it is not clear that the 3D case for the KSE is fundamentally more difficult than the 2D case, due to the already strong dissipation. Hence, we focus on the 2D case.

qualitatively similar to KSE. However, r-KSE suffers from the drawback that there is no clear way to see solutions of the r-KSE converge to solutions of the KSE, as any introduction of a "turning" parameter interpolating between the r-KSE and the KSE would immediately violate the maximum principle. In contrast, the model introduced in this present paper allows for such a parameter $\epsilon > 0$, which we call the "calming parameter." In particular, by adjusting the nonlinear term in the (6.1.2), we create a globally well-posed PDE that approximates solutions to the 2D KSE to arbitrary precision, at least on the time interval of existence and uniqueness of solutions to the KSE. Perhaps surprisingly, our construction does not require the use of a maximum principle, nor does it add artificial viscosity to the system.

The N-dimensional Kuramoto-Sivashinsky equation (KSE) is given in scalar form by

$$\partial_t \phi + \tfrac{1}{2}|\nabla \phi|^2 + \triangle \phi + \triangle^2 \phi = 0. \tag{6.1.1}$$

with periodic boundary conditions on a domain $[0, L]^N$. By setting $\mathbf{u} = \nabla \phi$ in (6.1.1), one formally[2] obtains the vector formulation of KSE:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \triangle \mathbf{u} + \triangle^2 \mathbf{u} = \mathbf{0}, \tag{6.1.2}$$

These equations were originally proposed in the 1970's by Kuramoto and Tsuzuki in the studies of crystal growth [54, 55] as well as by Sivashinsky in the study of flame-front instabilities [96] (see also [97]). It has since found many other applications in the sciences, such as describing the flow of fluid down inclined planes [98], and has

[2] We do not claim that $\nabla\phi$ is a unique solution to (6.1.2) when ϕ is a solution to (6.1.1). We only observe that one can formally obtain the set of equations (6.1.2) by taking the gradient of equation (6.1.1). In particular, it may be the case that there exist solutions to (6.1.2) that are not gradients of solutions to (6.1.1), or of any other function.

shown to be a generic feature of many physical phenomena involving bifurcations [76].

Many results on the 1D equation have been obtained since its origination, and the equation has been shown to be rich with interesting dynamics. It is globally well-posed [79, 102], solutions continue to exhibit chaotic dynamics at large times (see, e.g., [18, 44, 75, 80, 87]), and a large body of work has been published on quantitative results pertaining to the global attractor (see, e.g., [18, 19, 21, 22, 34, 35, 37, 38, 40, 44, 45, 52, 81, 90, 102, 104]). There are far fewer results on the KSE in the 2D case. Global well-posedness for sufficiently small initial data was first shown in [93] on a domain $[0, 2\pi] \times [0, 2\pi\epsilon]$ with $\epsilon > 0$ sufficiently small. This result was improved upon in [78] by showing global existence on a domain $[0, L_1] \times [0, L_2]$ with $L_2 \leq CL_1^q$ for some particular q. Later works continued to improve on the sharpness of this bound (see, e.g., [4, 53, 73, 77] and references therein). Other works employ control of the domain size as a means to control the instability in Fourier modes. It was shown in [1] that for small enough domains (on which no growing Fourier modes are present in the linear terms), global existence holds when the initial data is sufficiently small in a certain Wiener algebra. This result was then extended in [2] to domains in which there is one linearly growing mode in each direction. Further studies have investigated modified equations [23, 33, 42, 62, 77, 106] or have looked at the equations with different boundary conditions [36, 61, 89]. For other results on the case $N > 1$, see also [6, 9, 59]. The intent of the present work is to propose a modification of the 2D KSE in vector form which is globally well-posed for any size of domain or initial data. To do this, we make use of what we call an *algebraic calming function* or simply a *calming function*[3] which constrains the advective velocity of the solution.

[3] Such a function is simply a bounded smooth truncation function, but we call it a "calming" function due to the way it is used in the nonlinearity to suppress the algebraic growth of the nonlinear term. We do not call it a "regularization," since we reserve this term for techniques which smooth the equations by modifying derivative operators.

We propose the following modification of system (6.1.2).

$$\partial_t \mathbf{u} + (\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla)\mathbf{u} + \triangle \mathbf{u} + \triangle^2 \mathbf{u} = \mathbf{0}, \tag{6.1.3a}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{6.1.3b}$$

with L-periodic[4] boundary conditions on the 2-dimensional periodic torus $\mathbb{T}^2 := \mathbb{R}^2/(L\mathbb{Z})^2 = [0, L]^2$ for some $L > 0$. We call $\epsilon > 0$ the *calming parameter*, and $\boldsymbol{\zeta}^\epsilon$ the *calming function*. We require that $\boldsymbol{\zeta}^\epsilon$ satisfies the conditions described in 2.1.1.

Remark 6.1.1. During the preparation of this manuscript, we became aware of a body of work (see, e.g., [11–13, 25, 50, 51, 72, 92, 114, 115] and the references therein) which has a superficial similarity to our approach in the context of the 3D incompressible Navier-Stokes equations. Namely, the authors of [12] investigate the following system (which was essentially first proposed in [113]), which they call the "globally modified Navier-Stokes equations" (GMNSE).

$$\partial_t \mathbf{u} + \min\left\{1, N\|\nabla \mathbf{u}\|_{L^2}^{-1}\right\}(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \triangle \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0.$$

Here, $N > 0$ can be thought of as corresponding to our parameter $1/\epsilon$. Formally setting $N = \infty$ yields the standard Navier-Stokes equations. The present work has several major differences from those works, and from the global modification (GM) approach in general. In particular: (i) The calming function is defined pointwise. Hence it only has a strong effect on the nonlinearity in regions of high velocity, not globally, which we see as a significant advantage over the GM approach; (ii) The calming function is based on $\mathbf{u}$, not on $\nabla \mathbf{u}$, hence the technique is philosophically

[4]Note: One could easily consider rectangular non-square periodic domains, say $\mathbb{R}^2/((L_1\mathbb{Z}) \times (L_2\mathbb{Z}))$ as well with slight modification of the techniques we use here. For the sake of keeping the discussion focused, we do not pursue such matters here.

different: we aim to control large speeds locally, not to decrease the total effect of the nonlinearity when large gradients appear, as in GM; (iii) To the best of our knowledge, this is the first time any such approach (including GM, calming, α-models, Leray-type smoothing by convolution, filtering, etc.) has been applied to the 2D KSE to obtain global well-posedness. We expect that GM can be easily adapted to the 2D KSE setting but that the performance in accurately capturing solutions is worse than calming until $N \sim 1/\epsilon$ is very large.

Remark 6.1.2. We see no major difficulty in extending our work to the case of physical boundary conditions, i.e., $\mathbf{u}\big|_{\partial\Omega} = \triangle\mathbf{u}\big|_{\partial\Omega} = \mathbf{0}$. However, for the sake of simplicity, we only consider periodic boundary conditions in the present work.

Section 6.2 contains a proof of global well-posedness, which is mostly standard Galerkin methods, but with some subtle differences due to the non-polynomial form of the nonlinearity. Section 6.3 contains a proof of higher-order (but not arbitrary order) regularity of solutions. Section 6.4 contains a proof of convergence of solutions of the calmed equation to solutions to the original KSE as the calming parameter $\epsilon \to 0$. The proof here is not so straight-forward due to issues with commutator terms involving the calming function. As we will see, these issues are circumvented by taking advantage of structural properties of the calming function, and then using a boot-strapping argument in time. In addition, our techniques yield an explicit convergence rate. In Section 6.5 we extend our ideas to the scalar form of the KSE. In particular, we consider a modification to system (6.1.1),

$$\partial_t\phi + \tfrac{1}{2}\boldsymbol{\zeta}^\epsilon(\nabla\phi)\cdot\nabla\phi + \triangle\phi + \triangle^2\phi = 0, \tag{6.1.4a}$$

$$\phi(\mathbf{x},0) = \phi_0(\mathbf{x}). \tag{6.1.4b}$$

Other formulations are of course possible. For example, one could consider a nonlinearity of the form $\frac{1}{2}\boldsymbol{\zeta}^\epsilon(|\nabla\phi|^2)$, or $\frac{1}{2}\boldsymbol{\zeta}^\epsilon(|\nabla\phi|^\delta)|\nabla\phi|^{2-\delta}$ $(0<\delta<2)$, or $\frac{\frac{1}{2}|\nabla\phi|^2}{1+\epsilon^2|\phi|^2}$, or many other possibilities. However, in the present work, we choose to focus only on the form in (6.1.4), as the advective nature of the nonlinearity seems perhaps closest in spirit to the nature of the original equation.

Section 6.6 exhibits results from simulations and provides computational evidence that the convergence rates we obtained in Section 6.4 are sharp (at least, in terms of convergence order). Concluding remarks are in Section 6.7.

6.2 Global Well-Posedness for Calmed KSE

In this section we show the existence and uniqueness of our calmed KSE system. We begin with formulating our Galerkin scheme.

Using the projection operator P_m, define the finite-dimensional space $H_m := P_m(L^2(\mathbb{T}^2))$. Consider the following initial value problem obtained via Galerkin approximation: Given $\mathbf{u}_0 \in L^2(\mathbb{T}^2)$, find $\mathbf{u} \in H_m$ which satisfies

$$\partial_t \mathbf{u} + P_m\left((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u}\right) + \triangle\mathbf{u} + \triangle^2\mathbf{u} = \mathbf{0}, \tag{6.2.1a}$$

$$\mathbf{u}(\mathbf{x},0) = P_m\mathbf{u}_0(\mathbf{x}). \tag{6.2.1b}$$

Lemma 6.2.1. *If $\boldsymbol{\zeta}^\epsilon$ satisfies 1 of Definition 2.1.1, then the map $F : H_m \to H_m$ defined by*

$$F(\mathbf{u}) = -P_m\left((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u}\right) - \triangle\mathbf{u} - \triangle^2\mathbf{u}$$

is locally Lipschitz on H_m. As a consequence, solutions to (6.2.1) *exist and are unique in $C^1([0,T],H_m)$ for some $T>0$.*

Proof. Fix $\mathbf{u}\in H_m$ and let $\mathbf{v}\in H_m$ be arbitrary. Rewrite the difference $F(\mathbf{u})-F(\mathbf{v})$

as

$$F(\mathbf{u}) - F(\mathbf{v}) = -\triangle\left(\mathbf{u}-\mathbf{v}\right) - \triangle^2\left(\mathbf{u}-\mathbf{v}\right) - P_m\left(\left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \boldsymbol{\zeta}^\epsilon(\mathbf{v})\right)\cdot\nabla\right)\mathbf{u}\right)$$
$$- P_m\left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{v})\cdot\nabla\right)\left(\mathbf{u}-\mathbf{v}\right)\right).$$

From Condition 1 of Definition 2.1.1, Estimate (2.3.1), and Agmon's inequality, it follows that

$$\begin{aligned}
&\left\|F(\mathbf{u}) - F(\mathbf{v})\right\|_{L^2} \\
&\le \left\|\triangle\left(\mathbf{u}-\mathbf{v}\right)\right\|_{L^2} + \left\|\triangle^2\left(\mathbf{u}-\mathbf{v}\right)\right\|_{L^2} \\
&\quad + \left\|\left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \boldsymbol{\zeta}^\epsilon(\mathbf{v})\right)\cdot\nabla\right)\mathbf{u}\right\|_{L^2} + \left\|\left(\boldsymbol{\zeta}^\epsilon(\mathbf{v})\cdot\nabla\right)\left(\mathbf{u}-\mathbf{v}\right)\right\|_{L^2} \\
&\le \left(m+m^2\right)\left\|\mathbf{u}-\mathbf{v}\right\|_{L^2} + \left\|\nabla\mathbf{u}\right\|_{L^\infty}\left\|\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \boldsymbol{\zeta}^\epsilon(\mathbf{v})\right\|_{L^2} \\
&\quad + \left\|\boldsymbol{\zeta}^\epsilon(\mathbf{v})\right\|_{L^\infty}\left\|\nabla\left(\mathbf{u}-\mathbf{v}\right)\right\|_{L^2} \\
&\le \left(m+m^2\right)\left\|\mathbf{u}-\mathbf{v}\right\|_{L^2} + \left\|\mathbf{u}\right\|_{H^3}\left\|\mathbf{u}-\mathbf{v}\right\|_{L^2} + m^{\frac{1}{2}}\left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}\left\|\mathbf{u}-\mathbf{v}\right\|_{L^2}.
\end{aligned}$$

Since $\mathbf{u}$ is a finite linear combination of eigenfunctions of $-\triangle$, $\left\|\mathbf{u}\right\|_{H^3} < \infty$. Thus F is locally Lipschitz at $\mathbf{u} \in H_m$. Existence and uniqueness of solutions to (6.2.1) in $C^1([0,T], H_m)$ therefore follows as a consequence of the Picard-Lindelöf Theorem. □

Due to the presence of the calming function $\boldsymbol{\zeta}^\epsilon$, the Galerkin system here is not necessarily quadratic such as in the case of the 2D Navier-Stokes equations or the 2D Kuramoto-Sivashinsky equations. Thus we give a fully rigorous proof of well-posedness here.

Proof of Theorem 1.2.23. We will show that a solution exists using Galerkin approximation. Given $\mathbf{u}_0 \in L^2(\mathbb{T}^2)$, suppose $\mathbf{u}^m \in C([0,T_m]; H_m)$ is a solution to (6.2.1) on the interval $[0, T_m]$ for some $T_m > 0$ with initial data $\mathbf{u}_0^m = P_m\mathbf{u}_0$. We take the inner product of (6.2.1) with $\mathbf{u}^m$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{u}^m\right\|_{L^2}^2 + \left\|\triangle\mathbf{u}^m\right\|_{L^2}^2 = -\left(\triangle\mathbf{u}^m, \mathbf{u}^m\right) - \left(\left(\boldsymbol{\zeta}^\epsilon\left(\mathbf{u}^m\right)\cdot\nabla\right)\mathbf{u}^m, \mathbf{u}^m\right)$$

We estimate the first term by $-\left(\triangle\mathbf{u}^m, \mathbf{u}^m\right) \leq \frac{1}{4}\left\|\triangle\mathbf{u}^m\right\|_{L^2}^2 + \left\|\mathbf{u}^m\right\|_{L^2}^2$. For the nonlinear term, we estimate

$$\begin{aligned}
\left|\left(\boldsymbol{\zeta}^\epsilon\left(\mathbf{u}^m\right)\cdot\nabla\right)\mathbf{u}^m, \mathbf{u}^m\right)\right| &\leq \left\|\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\right\|_{L^\infty}\left\|\nabla\mathbf{u}^m\right\|_{L^2}\left\|\mathbf{u}^m\right\|_{L^2} \\
&\leq \left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}\left\|\mathbf{u}^m\right\|_{L^2}^{\frac{1}{2}}\left\|\triangle\mathbf{u}^m\right\|_{L^2}^{\frac{1}{2}}\left\|\mathbf{u}^m\right\|_{L^2} \\
&\leq \frac{3}{4}\left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}^{\frac{4}{3}}\left\|\mathbf{u}^m\right\|_{L^2}^2 + \frac{1}{4}\left\|\triangle\mathbf{u}^m\right\|_{L^2}^2
\end{aligned}$$

Combining the above estimates and denoting $K_\epsilon := \frac{3}{2}\left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}^{\frac{4}{3}} + 2$, we obtain

$$\frac{d}{dt}\left\|\mathbf{u}^m\right\|_{L^2}^2 + \left\|\triangle\mathbf{u}^m\right\|_{L^2}^2 \leq K_\epsilon\left\|\mathbf{u}^m\right\|_{L^2}^2. \tag{6.2.2}$$

After dropping the second term of (6.2.2), Grönwall's inequality yields for all $t \in [0, T_m]$,

$$\left\|\mathbf{u}^m(t)\right\|_{L^2}^2 \leq e^{K_\epsilon t}\left\|\mathbf{u}^m(0)\right\|_{L^2}^2 \leq e^{K_\epsilon T_m}\left\|\mathbf{u}_0\right\|_{L^2}^2. \tag{6.2.3}$$

Since $\mathbf{u}^m \in C([0, T_m], \mathbb{T}^2)$, via a bootstrapping argument, it holds that for any $T > 0$ and any $t \in [0, T]$,

$$\left\|\mathbf{u}^m(t)\right\|_{L^2}^2 \leq e^{K_\epsilon t}\left\|\mathbf{u}_0\right\|_{L^2}^2 \leq e^{K_\epsilon T}\left\|\mathbf{u}_0\right\|_{L^2}^2. \tag{6.2.4}$$

Next, we integrate (6.2.2) on $[0,T]$ and apply Estimate (6.2.4):

$$\begin{aligned} &\left\|\mathbf{u}^m(T)\right\|_{L^2}^2+\frac{1}{2}\int_0^T\left\|\triangle\mathbf{u}^m(s)\right\|_{L^2}^2 ds \\ &\leq \int_0^T K_\epsilon \left\|\mathbf{u}^m(s)\right\|_{L^2}^2 ds+\left\|\mathbf{u}^m(0)\right\|_{L^2}^2 \\ &\leq \int_0^T K_\epsilon e^{K_\epsilon s}\left\|\mathbf{u}_0\right\|_{L^2}^2 ds+\left\|\mathbf{u}_0\right\|_{L^2}^2 \\ &= e^{K_\epsilon T}\left\|\mathbf{u}_0\right\|_{L^2}^2 . \end{aligned} \tag{6.2.5}$$

Hence, for all $T>0$,

$$\{\mathbf{u}^m\}_{m=1}^\infty \text{ is bounded in } L^\infty([0,T];L^2)\cap L^2([0,T];H^2). \tag{6.2.6}$$

To bound the time derivative, we estimate

$$\begin{aligned} \left\|\partial_t\mathbf{u}^m\right\|_{H^{-2}} &\leq \left\|\triangle^2\mathbf{u}^m\right\|_{H^{-2}}+\left\|\triangle\mathbf{u}^m\right\|_{H^{-2}}+\sup_{\substack{\phi\in H^2\\ \|\phi\|_{H^2}=1}}\left|\langle P_m\left((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m\right),\phi\rangle\right| \\ &\leq C_1\left\|\mathbf{u}^m\right\|_{H^2}+C_2\left\|\mathbf{u}^m\right\|_{L^2}+\sup_{\substack{\phi\in H^2\\ \|\phi\|_{H^2}=1}}\left|\langle\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m,P_m(\phi)\rangle\right| \\ &\leq C\left\|\mathbf{u}^m\right\|_{H^2}+C\left\|\mathbf{u}^m\right\|_{L^2}+\sup_{\substack{\phi\in H^2\\ \|\phi\|_{H^2}=1}}\left\|\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\right\|_{L^\infty}\left\|\mathbf{u}^m\right\|_{H^1}\left\|\phi\right\|_{L^2} \\ &\leq C\left\|\mathbf{u}^m\right\|_{H^2}+C\left\|\mathbf{u}^m\right\|_{L^2}+\left\|\boldsymbol{\zeta}^\epsilon\right\|_{L^\infty}\left\|\mathbf{u}^m\right\|_{H^1} . \end{aligned}$$

Hence, $\{\partial_t\mathbf{u}^m\}_{m=1}^\infty$ is bounded in $L^2(0,T;H^{-2}(\mathbb{T}^2))$. By the Banach-Alaoglu Theorem, there exists $\mathbf{u}\in L^2(0,T;H^2(\mathbb{T}^2))\cap L^\infty(0,T;L^2(\mathbb{T}^2))$ and a subsequence (which we will still label as $\mathbf{u}^m$) such that

$$\mathbf{u}^m \rightharpoonup^* \mathbf{u} \text{ weak-* in } L^\infty(0,T;L^2(\mathbb{T}^2)), \tag{6.2.7}$$

$$\mathbf{u}^m \rightharpoonup \mathbf{u} \text{ weakly in } L^2(0,T;H^2(\mathbb{T}^2)), \tag{6.2.8}$$

$$\partial_t\mathbf{u}^m \rightharpoonup \partial_t\mathbf{u} \text{ weakly in } L^2(0,T;H^{-2}(\mathbb{T}^2)). \tag{6.2.9}$$

Moreover, by the Aubin-Lions Lemma we may pass to another subsequence, relabeled to be $\mathbf{u}^m$, such that

$$\mathbf{u}^m \to \mathbf{u} \text{ strongly in } C(0,T;L^2(\mathbb{T}^2)). \tag{6.2.10}$$

Now we can show that $\mathbf{u}$ is a weak solution to (6.1.3). Given $\mathbf{w} \in L^2(0,T;H^2(\mathbb{T}^2))$, we compute

$$\begin{aligned}
&(\langle\partial_t\mathbf{u},\mathbf{w}\rangle + ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u},\mathbf{w}) + (\triangle\mathbf{u},\mathbf{w}) + (\triangle\mathbf{u},\triangle\mathbf{w})) \\
&\quad - (\langle\partial_t\mathbf{u}^m,\mathbf{w}\rangle + (P_m\left((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m\right),\mathbf{w}) + (\triangle\mathbf{u}^m,\mathbf{w}) + (\triangle\mathbf{u}^m,\triangle\mathbf{w})) \\
&= \langle\partial_t\left(\mathbf{u}-\mathbf{u}^m\right),\mathbf{w}\rangle + (\triangle\left(\mathbf{u}-\mathbf{u}^m\right),\mathbf{w}) + (\triangle\left(\mathbf{u}-\mathbf{u}^m\right),\triangle\mathbf{w}) \\
&\quad + ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u},\mathbf{w}) - (P_m\left((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m\right),\mathbf{w}) \\
&= \langle\partial_t\left(\mathbf{u}-\mathbf{u}^m\right),\mathbf{w}\rangle + (\triangle\left(\mathbf{u}-\mathbf{u}^m\right),\mathbf{w}) + (\triangle\left(\mathbf{u}-\mathbf{u}^m\right),\triangle\mathbf{w}) \\
&\quad + ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u},\mathbf{w}) - ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m,\mathbf{w}) + (Q_m\left((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m\right),\mathbf{w}) \\
&= \langle\partial_t\left(\mathbf{u}-\mathbf{u}^m\right),\mathbf{w}\rangle + (\triangle\left(\mathbf{u}-\mathbf{u}^m\right),\mathbf{w}) + (\triangle\left(\mathbf{u}-\mathbf{u}^m\right),\triangle\mathbf{w}) \\
&\quad + (((\boldsymbol{\zeta}^\epsilon(\mathbf{u})-\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m))\cdot\nabla)\mathbf{u}^m,\mathbf{w}) + ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\left(\mathbf{u}-\mathbf{u}^m\right),\mathbf{w}) \\
&\quad + (((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m),Q_m\mathbf{w}). \\
&:= \sum_{k=1}^{6} I_k.
\end{aligned}$$

Integrate $\sum_{k=1}^{6} I_k$ in time for $t \in [0,T]$. We observe that I_1, I_2, and I_3 all vanish as $m \to \infty$ by (6.2.7), (6.2.8), and (6.2.9). Using Condition 1 of Definition 2.1.1, Agmon's inequality, Ladyzhenskaya's inequality, and Hölder's inequality, we obtain

$$\begin{aligned}
\int_0^T I_4 dt &\leq \int_0^T \|\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\|_{L^2} \|\nabla \mathbf{u}^m\|_{L^2} \|\mathbf{w}\|_{L^\infty} \, dt \\
&\leq C \int_0^T \|\mathbf{u} - \mathbf{u}^m\|_{L^2} \|\mathbf{u}^m\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{u}^m\|_{L^2}^{\frac{1}{2}} \|\mathbf{w}\|_{L^2}^{\frac{1}{2}} \|\mathbf{w}\|_{H^2}^{\frac{1}{2}} \, dt \\
&\leq C \|\mathbf{u} - \mathbf{u}^m\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \|\mathbf{u}^m\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \\
&\quad \times \int_0^T \|\mathbf{u} - \mathbf{u}^m\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{u}^m\|_{L^2}^{\frac{1}{2}} \|\mathbf{w}\|_{H^2} \, dt \\
&\leq C \|\mathbf{u} - \mathbf{u}^m\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \|\mathbf{u}^m\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \\
&\quad \times \|\triangle \mathbf{u}^m\|_{L^2(0,T;L^2)}^{\frac{1}{2}} \|\mathbf{w}\|_{L^2(0,T;H^2)} \|\mathbf{u} - \mathbf{u}^m\|_{L^2(0,T;L^2)}^{\frac{1}{2}},
\end{aligned} \tag{6.2.11}$$

which is bounded due to (6.2.4), (6.2.8), and (6.2.10).

For I_5,

$$\int_0^T I_5 dt = I_{\mathbf{u},\mathbf{w}}(\mathbf{u} - \mathbf{u}^m) \tag{6.2.12}$$

for $I_{\mathbf{u},\mathbf{w}}$ as defined in (2.1.4), which convergences due to Lemma 2.1.4. Finally, using Hölder's inequality, Condition 2 of Definition 2.1.1, and (2.3.2),

$$\begin{aligned}
\int_0^T I_6 dt &\leq \int_0^T \|\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\|_{L^\infty} \|\nabla \mathbf{u}^m\|_{L^2} \|Q_m \mathbf{w}\|_{L^2} \, dt \\
&\leq \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \left(\int_0^T \|\nabla \mathbf{u}^m\|_{L^2}^2 \, dt \right)^{\frac{1}{2}} \left(\int_0^T \|Q_m \mathbf{w}\|_{L^2}^2 \, dt \right)^{\frac{1}{2}} \\
&\leq \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \left(\int_0^T \|\nabla \mathbf{u}^m\|_{L^2}^2 \, dt \right)^{\frac{1}{2}} \left(\int_0^T \frac{1}{m^4} \|\mathbf{w}\|_{H^2}^2 \, dt \right)^{\frac{1}{2}} \\
&\leq \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\mathbf{u}^m\|_{L^2(0,T;H^2)} \|\mathbf{w}\|_{L^2(0,T;H^2)} \frac{1}{m^2},
\end{aligned} \tag{6.2.13}$$

which converges to zero by (6.2.6).

Invoking (6.2.8), (6.2.9), (6.2.10), (6.2.11), (6.2.12), and (6.2.13),

$$\lim_{m\to\infty}\int_0^T \left(\sum_{k=1}^{6} I_k\right) dt = 0.$$

Therefore solutions to the ODE system (6.2.1) converge to a solution of the PDE system (6.1.3). Thus $\mathbf{u}$ is indeed a solution to (6.1.3).

Now we show that the solution $\mathbf{u}$ satisfies $\mathbf{u}(0) = \mathbf{u}_0$ in the sense of $C([0,T], L^2)$. Applying Lemma 1.1 from Chapter 3 of [105, p. 250], for all $\mathbf{v} \in H^2(\mathbb{T}^2)$, it follows that

$$\langle \partial_t \mathbf{u}, \mathbf{v}\rangle = \frac{d}{dt}(\mathbf{u},\mathbf{v}) = -(\triangle \mathbf{u}, \mathbf{v}) - (\triangle \mathbf{u}, \triangle \mathbf{v}) - (\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla\mathbf{u}, \mathbf{v}) \tag{6.2.14}$$

in the scalar distribution sense on $[0,T]$. Now, suppose that $\psi \in C^1([0,T])$ and satisfies $\psi(0) = 1$, $\psi(T) = 0$. We then integrate (6.2.14) in time with ψ and apply integration by parts to obtain

$$\begin{aligned}\int_0^T (\mathbf{u},\mathbf{v})\,\psi'(t)dt = &-\int_0^T (\triangle\mathbf{u},\mathbf{v})\,\psi(t)dt - \int_0^T (\triangle\mathbf{u},\triangle\mathbf{v})\,\psi(t)dt \\ &-\int_0^T (\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla\mathbf{u},\mathbf{v})\,\psi(t)dt + (\mathbf{u}(0),\mathbf{v})\,.\end{aligned} \tag{6.2.15}$$

On the other hand, if we take the inner product of (6.2.1) with $\mathbf{v}$ then integrate in time with ψ we obtain

$$\begin{aligned}\int_0^T (\mathbf{u}^m,\mathbf{v})\,\psi'(t)dt = &-\int_0^T (\triangle\mathbf{u}^m,\mathbf{v})\,\psi(t)dt - \int_0^T (\triangle\mathbf{u}^m,\triangle\mathbf{v})\,\psi(t)dt \\ &-\int_0^T (P_m(\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla\mathbf{u}^m),\mathbf{v})\,\psi(t)dt + (\mathbf{u}_0^m,\mathbf{v})\,.\end{aligned}$$

Passing to the limit as $m \to \infty$ then yields

$$\int_0^T (\mathbf{u}, \mathbf{v})\, \psi'(t) dt = -\int_0^T (\triangle \mathbf{u}, \mathbf{v})\, \psi(t) dt - \int_0^T (\triangle \mathbf{u}, \triangle \mathbf{v})\, \psi(t) dt \\ - \int_0^T (\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla \mathbf{u}, \mathbf{v})\, \psi(t) dt + (\mathbf{u}_0, \mathbf{v})\,. \tag{6.2.16}$$

By then comparing (6.2.15) and (6.2.16), we obtain $(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v}) = 0$ for all $\mathbf{v} \in H^2(\mathbb{T}^2)$. Since $H^2(\mathbb{T}^2)$ is dense in $L^2(\mathbb{T}^2)$, it follows that $(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v}) = 0$ for all $\mathbf{v} \in L^2(\mathbb{T}^2)$. Thus $\mathbf{u}$ satisfies $\mathbf{u}(0) = \mathbf{u}_0$. Next, we show that weak solutions are unique. Set $\mathbf{w} = \mathbf{u} - \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are both weak solutions of calmed KSE (6.1.3) on the interval $[0, T]$ with $\mathbf{u}_0 = \mathbf{v}_0$. After taking the difference of the two equations, we then take the action of the difference equation with w, which yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|\mathbf{w}\|_{L^2}^2 + \|\triangle \mathbf{w}\|_{L^2}^2 \\ &= -(\triangle \mathbf{w}, \mathbf{w}) - ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla)\, \mathbf{u}, \mathbf{w}) + ((\boldsymbol{\zeta}^\epsilon(\mathbf{v}) \cdot \nabla)\, \mathbf{v}, \mathbf{w}) \\ &= -(\triangle \mathbf{w}, \mathbf{w}) + (((\boldsymbol{\zeta}^\epsilon(\mathbf{v}) - \boldsymbol{\zeta}^\epsilon(\mathbf{u})) \cdot \nabla)\, \mathbf{u}, \mathbf{w}) - ((\boldsymbol{\zeta}^\epsilon(\mathbf{v}) \cdot \nabla)\, \mathbf{w}, \mathbf{w}) \\ &= J_1 + J_2 + J_3, \end{aligned} \tag{6.2.17}$$

where we have used the Lions-Magene Lemma to write $\langle \partial_t \mathbf{w}, \mathbf{w} \rangle = \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_{L^2}^2$. Then,

$$J_1 \le \frac{1}{2}\|\mathbf{w}\|_{L^2}^2 + \frac{1}{2}\|\triangle \mathbf{w}\|_{L^2}^2\,.$$

Also,

$$\begin{aligned} J_2 &:= (((\boldsymbol{\zeta}^\epsilon(\mathbf{v}) - \boldsymbol{\zeta}^\epsilon(\mathbf{u})) \cdot \nabla)\, \mathbf{u}, \mathbf{w}) \\ &\le \|\boldsymbol{\zeta}^\epsilon(\mathbf{v}) - \boldsymbol{\zeta}^\epsilon(\mathbf{u})\|_{L^4} \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{w}\|_{L^4} \\ &\le C \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{w}\|_{L^4}^2 \end{aligned}$$

$$
\begin{aligned}
&\leq C \left\|\nabla \mathbf{u}\right\|_{L^2} \left\|\mathbf{w}\right\|_{L^2} \left\|\nabla \mathbf{w}\right\|_{L^2} \\
&\leq C \left\|\nabla \mathbf{u}\right\|_{L^2} \left\|\mathbf{w}\right\|_{L^2}^{\frac{3}{2}} \left\|\triangle \mathbf{w}\right\|_{L^2}^{\frac{1}{2}} \\
&\leq C \left\|\nabla \mathbf{u}\right\|_{L^2}^{\frac{4}{3}} \left\|\mathbf{w}\right\|_{L^2}^{2} + \frac{1}{4} \left\|\triangle \mathbf{w}\right\|_{L^2}^{2}
\end{aligned}
$$

by Agmon's inequality, Ladyzhenskaya's inequality, and Lemma 2.1.4. Finally,

$$
\begin{aligned}
J_3 &:= -\left(\left(\boldsymbol{\zeta}^{\epsilon}\left(\mathbf{v}\right) \cdot \nabla\right) \mathbf{w}, \mathbf{w}\right) \\
&\leq \left\|\boldsymbol{\zeta}^{\epsilon}(\mathbf{v})\right\|_{L^\infty} \left\|\nabla \mathbf{w}\right\|_{L^2} \left\|\mathbf{w}\right\|_{L^2} \\
&\leq \left(\left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^\infty} \left\|\mathbf{w}\right\|_{L^2}^{\frac{3}{2}}\right) \left(\left\|\triangle \mathbf{w}\right\|_{L^2}^{\frac{1}{2}}\right) \\
&\leq \frac{3}{4} \left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^\infty}^{\frac{4}{3}} \left\|\mathbf{w}\right\|_{L^2}^{2} + \frac{1}{4} \left\|\triangle \mathbf{w}\right\|_{L^2}^{2}
\end{aligned}
$$

using Young's inequality and (2.4.5). From the above estimates, we obtain

$$
\frac{d}{dt} \left\|\mathbf{w}(t)\right\|_{L^2}^{2} \leq \left(1 + C \left\|\nabla \mathbf{u}\right\|_{L^2}^{\frac{4}{3}} + \frac{3}{2} \left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^\infty}^{\frac{4}{3}}\right) \left\|\mathbf{w}(t)\right\|_{L^2}^{2}.
$$

Writing $K_1(t) = 1 + C \left\|\nabla \mathbf{u}\right\|_{L^2}^{\frac{4}{3}} + \frac{3}{2} \left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^\infty}^{\frac{4}{3}}$, we observe that $K(t)$ is integrable on $[0, T]$. Thus we conclude, recalling that $\mathbf{w} = \mathbf{u} - \mathbf{v}$,

$$
\left\|\mathbf{u}(t) - \mathbf{v}(t)\right\|_{L^2}^{2} \leq \left\|\mathbf{u}_0 - \mathbf{v}_0\right\|_{L^2}^{2} \exp\left(\int_0^T K_1(t) dt\right). \tag{6.2.18}
$$

Therefore solutions to (6.1.3) are unique. If we now integrate (6.2.17) on the interval $[0, T]$ and apply estimate (6.2.18), we obtain

$$
\int_0^T \left\|\triangle \mathbf{u}(t) - \triangle \mathbf{v}(t)\right\|_{L^2}^{2} dt \leq K_2 \left\|\mathbf{u}_0 - \mathbf{v}_0\right\|_{L^2}^{2} \tag{6.2.19}
$$

For some constant K_2 depending on T, $\left\|\nabla \mathbf{u}\right\|_{L^2}$, and $\left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^\infty}$. From estimates (6.2.18) and (6.2.19) we conclude that solutions depend continuously on the initial data in $L^\infty(0,T;L^2(\mathbb{T}^2))\cap L^2(0,T;H^2(\mathbb{T}^2))$. □

6.3 Higher-Order Regularity of Solutions

In this section, we only work formally, but the results can be made rigorous by using, e.g., the Galerkin method. We will show that the regularity of a weak solution $\mathbf{u}$ to 6.1.3 is dependent on the regularity of the calming function $\boldsymbol{\zeta}^\epsilon$ and the initial data $\mathbf{u}_0$.

Remark 6.3.1. It seems likely that higher-order regularity ($m>2$) also holds, but we do not pursue such matters here.

Proof of Theorem 1.2.24. We first show the case $m=1$. We take the (formal) inner product of (6.1.3) with $-\triangle\mathbf{u}$ and integrate by parts to obtain

$$(\partial_t\mathbf{u},-\triangle\mathbf{u})-((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u},\triangle\mathbf{u})-(\triangle\mathbf{u},\triangle\mathbf{u})-\left(\triangle^2\mathbf{u},\triangle\mathbf{u}\right)=0$$

which we will rewrite as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\triangle\mathbf{u}\|_{L^2}^2&=\left(\left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla\right)\mathbf{u}\right),\triangle\mathbf{u}\right)-(\nabla\mathbf{u},\nabla\triangle\mathbf{u})\\&=((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u},\triangle\mathbf{u})-(\nabla\mathbf{u},\nabla\triangle\mathbf{u})\,.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\triangle\mathbf{u}\|_{L^2}^2\\&\leq\|\boldsymbol{\zeta}^\epsilon(\mathbf{u})\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\triangle\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}\|\nabla\triangle\mathbf{u}\|_{L^2}\\&\leq\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}^{\frac{3}{2}}\|\nabla\triangle\mathbf{u}\|_{L^2}^{\frac{1}{2}}+\|\nabla\mathbf{u}\|_{L^2}\|\nabla\triangle\mathbf{u}\|_{L^2}\end{aligned}$$

$$\leq \left(\frac{3}{4}\left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^{\infty}}^{\frac{4}{3}}+\frac{1}{2}\right)\|\nabla \mathbf{u}\|_{L^{2}}^{2}+\frac{3}{4}\|\nabla \triangle \mathbf{u}\|_{L^{2}}^{2}.$$

This estimate can then be rewritten as

$$\frac{d}{dt}\|\nabla \mathbf{u}\|_{L^{2}}^{2}+\frac{1}{2}\|\nabla \triangle \mathbf{u}\|_{L^{2}}^{2} \leq\left(\frac{3}{2}\left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^{\infty}}^{\frac{4}{3}}+1\right)\|\nabla \mathbf{u}\|_{L^{2}}^{2}. \tag{6.3.1}$$

Then, by Grönwall's inequality,

$$\begin{aligned}\|\nabla \mathbf{u}(t)\|_{L^{2}}^{2} &\leq\left\|\nabla \mathbf{u}_{0}\right\|_{L^{2}}^{2} \exp \left(\frac{3}{2} t\left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^{\infty}}^{\frac{4}{3}}+t\right) \qquad (6.3.2)\\ &\leq\left\|\nabla \mathbf{u}_{0}\right\|_{L^{2}}^{2} \exp \left(\frac{3}{2} T\left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^{\infty}}^{\frac{4}{3}}+T\right)\end{aligned}$$

Now, after integrating (6.3.1) on the interval $[0, T]$ and applying Estimate (6.3.2), it follows that

$$\int_{0}^{T}\|\nabla \triangle \mathbf{u}(\tau)\|_{L^{2}}^{2} d\tau \leq 2\left\|\nabla \mathbf{u}_{0}\right\|_{L^{2}}^{2} \exp \left(\frac{3}{2} T\left\|\boldsymbol{\zeta}^{\epsilon}\right\|_{L^{\infty}}^{\frac{4}{3}}+T\right). \tag{6.3.3}$$

Thus, $\mathbf{u} \in L^{2}(0, T ; H^{3}(\mathbb{T}^{2})) \cap L^{\infty}(0, T ; H^{1}(\mathbb{T}))$ whenever $\mathbf{u}_{0} \in H^{1}(\mathbb{T}^{2})$.

The case $m=2$ proceeds in a similar way. We take the inner product with $\triangle^{2}\mathbf{u}$ to obtain

$$\begin{aligned}&\frac{1}{2} \frac{d}{dt}\|\triangle \mathbf{u}\|_{L^{2}}^{2}+\left\|\triangle^{2} \mathbf{u}\right\|_{L^{2}}^{2} \qquad (6.3.4)\\ &\leq\left|\left(\triangle \mathbf{u}, \triangle^{2} \mathbf{u}\right)\right|+\left|\left(\left(\boldsymbol{\zeta}^{\epsilon}(\mathbf{u}) \cdot \nabla\right) \mathbf{u}, \triangle^{2} \mathbf{u}\right)\right| \\ &\leq \frac{1}{2}\|\triangle \mathbf{u}\|_{L^{2}}^{2}+\frac{1}{2}\left\|\triangle^{2} \mathbf{u}\right\|_{L^{2}}^{2}+\left\|\boldsymbol{\zeta}^{\epsilon}(\mathbf{u})\right\|_{L^{4}}\|\nabla \mathbf{u}\|_{L^{4}}\left\|\triangle^{2} \mathbf{u}\right\|_{L^{2}} \\ &\leq \frac{1}{2}\|\triangle \mathbf{u}\|_{L^{2}}^{2}+\frac{1}{2}\left\|\triangle^{2} \mathbf{u}\right\|_{L^{2}}^{2}+C\|\mathbf{u}\|_{L^{4}}^{2}\|\nabla \mathbf{u}\|_{L^{4}}^{2}+\frac{1}{4}\left\|\triangle^{2} \mathbf{u}\right\|_{L^{2}}^{2} \\ &\leq \frac{1}{2}\|\triangle \mathbf{u}\|_{L^{2}}^{2}+\frac{3}{4}\left\|\triangle^{2} \mathbf{u}\right\|_{L^{2}}^{2}+C\|\mathbf{u}\|_{L^{2}}\|\nabla \mathbf{u}\|_{L^{2}}^{2}\|\triangle \mathbf{u}\|_{L^{2}}\end{aligned}$$

$$\leq \left(\frac{1}{2} + C\left\|\mathbf{u}\right\|_{L^2}^2\right)\left\|\triangle\mathbf{u}\right\|_{L^2}^2 + \frac{3}{4}\left\|\triangle^2\mathbf{u}\right\|_{L^2}^2.$$

Similar to the case $m = 1$, this estimate reveals that $\mathbf{u} \in L^\infty(0,T;H^2(\mathbb{T}^2)) \cap L^2(0,T;H^4(\mathbb{T}^2))$ whenever $\mathbf{u}_0 \in H^2(\mathbb{T}^2)$. □

6.4 Convergence to Kuramoto-Sivashinsky Solutions

It is known that, for any initial data $\mathbf{u}_0 \in L^2(\mathbb{T}^2)$, solutions to 2D KSE exist and are unique in $C([0,T];L^2(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T}))$ for some (possibly only small) $T > 0$ (see, e.g., [6, 33]). In this section we show that as $\epsilon \to 0$, solutions $\mathbf{u}^\epsilon$ of the calmed KSE (6.1.3) converge to solutions $\mathbf{u}$ of KSE (6.1.2) prior to its potential blowup time. For this result, it seems necessary that our calming function $\boldsymbol{\zeta}^\epsilon$ satisfies Condition 3 of Definition 2.1.1. Indeed, if one wants to show that $(\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon)\cdot\nabla)\,\mathbf{u}^\epsilon \to (\mathbf{u}\cdot\nabla)\,\mathbf{u}$ in some sense as $\epsilon \to 0$, then one expects that at least $\boldsymbol{\zeta}^\epsilon(\mathbf{x}) \to \mathbf{x}$ as $\epsilon \to 0$. We do not find this imposition to be restrictive, as our example choices for $\boldsymbol{\zeta}^\epsilon$ satisfy this condition, as seen in Proposition 2.1.3.

Proof of Theorem 1.2.25. Set

$$\mathbf{w}^\epsilon := \mathbf{u}^\epsilon - \mathbf{u}$$

and take the difference between (6.1.3) and (6.1.2) to obtain

$$\partial_t\mathbf{w}^\epsilon + \Delta\mathbf{w}^\epsilon + \Delta^2\mathbf{w}^\epsilon = -(\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon)\cdot\nabla)\mathbf{u}^\epsilon + (\mathbf{u}\cdot\nabla)\mathbf{u}.$$

Testing each side by $\mathbf{w}^\epsilon$ we obtain, after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}^\epsilon\|_{L^2}^2 + \|\Delta\mathbf{w}^\epsilon\|_{L^2}^2 = \|\nabla\mathbf{w}^\epsilon\|_{L^2}^2 + N, \tag{6.4.1}$$

where N is given by

$$N := -\int_{\mathbb{T}} ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon)\cdot\nabla)\mathbf{u}^\epsilon - (\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{w}^\epsilon\, d\mathbf{x}.$$

By inequality (2.4.5),

$$\|\nabla\mathbf{w}^\epsilon\|_{L^2}^2 \le \|\mathbf{w}^\epsilon\|_{L^2}\,\|\Delta\mathbf{w}^\epsilon\|_{L^2} \le \frac{1}{2}\|\Delta\mathbf{w}^\epsilon\|_{L^2}^2 + \frac{1}{2}\|\mathbf{w}^\epsilon\|_{L^2}^2. \tag{6.4.2}$$

Inserting 6.4.2 in 6.4.1 yields

$$\frac{d}{dt}\|\mathbf{w}^\epsilon\|_{L^2}^2 + \|\Delta\mathbf{w}^\epsilon\|_{L^2}^2 \le \|\mathbf{w}^\epsilon\|_{L^2}^2 + 2N. \tag{6.4.3}$$

N can be written as

$$\begin{aligned} N = &-\int_{\mathbb{T}} ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}))\cdot\nabla)\mathbf{w}^\epsilon\cdot\mathbf{w}^\epsilon\, d\mathbf{x} - \int_{\mathbb{T}} (\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{w}^\epsilon\cdot\mathbf{w}^\epsilon\, d\mathbf{x} \\ &-\int_{\mathbb{T}} ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}))\cdot\nabla)\mathbf{u}\cdot\mathbf{w}^\epsilon\, d\mathbf{x} - \int_{\mathbb{T}} ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \mathbf{u})\cdot\nabla)\mathbf{u}\cdot\mathbf{w}^\epsilon\, d\mathbf{x}. \end{aligned}$$

Using the Lipschitz property of $\boldsymbol{\zeta}^\epsilon$ and (2.1.1), we see that N is bounded by

$$\begin{aligned} |N| \le\ & \int_{\mathbb{T}} |\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\mathbf{u})|\,|\nabla\mathbf{w}^\epsilon|\,|\mathbf{w}^\epsilon|\ d\mathbf{x} + \int_{\mathbb{T}} |\boldsymbol{\zeta}^\epsilon(\mathbf{u})|\,|\nabla\mathbf{w}^\epsilon|\,|\mathbf{w}^\epsilon|\ d\mathbf{x} \\ &+ \int_{\mathbb{T}} |\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\mathbf{u})|\,|\nabla\mathbf{u}|\,|\mathbf{w}^\epsilon|\ d\mathbf{x} + \int_{\mathbb{T}} |\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \mathbf{u}|\,|\nabla\mathbf{u}|\,|\mathbf{w}^\epsilon|\ d\mathbf{x}. \\ \le\ & \int_{\mathbb{T}} |\mathbf{w}^\epsilon|^2\,|\nabla\mathbf{w}^\epsilon|\ d\mathbf{x} + \int_{\mathbb{T}} |\mathbf{u}|\,|\nabla\mathbf{w}^\epsilon|\,|\mathbf{w}^\epsilon|\ d\mathbf{x} \\ &+ \int_{\mathbb{T}} |\mathbf{w}^\epsilon|^2\,|\nabla\mathbf{u}|\ d\mathbf{x} + C\epsilon^\alpha \int_{\mathbb{T}} |\mathbf{u}|^\beta\,|\nabla\mathbf{u}|\,|\mathbf{w}^\epsilon|\ d\mathbf{x}. \\ =\ & N_1 + N_2 + N_3 + N_4. \end{aligned}$$

These terms can be bounded as follows. By Hölder's inequality, Ladyzhenskaya's inequality, (2.4.5), and Young's inequality,

$$\begin{aligned} N_1 &\leq \|\mathbf{w}^\epsilon\|_{L^4}^2 \|\nabla \mathbf{w}^\epsilon\|_{L^2} \leq C \|\mathbf{w}^\epsilon\|_{L^2} \|\nabla \mathbf{w}^\epsilon\|_{L^2}^2 \\ &\leq C \|\mathbf{w}^\epsilon\|_{L^2}^2 \|\Delta \mathbf{w}^\epsilon\|_{L^2} \leq \frac{1}{16} \|\Delta \mathbf{w}^\epsilon\|_{L^2}^2 + C \|\mathbf{w}^\epsilon\|_{L^2}^4. \end{aligned} \tag{6.4.4}$$

$$\begin{aligned} N_2 &\leq \|\mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^4} \|\nabla \mathbf{w}^\epsilon\|_{L^4} \\ &\leq C \|\mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{w}^\epsilon\|_{L^2} \|\Delta \mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|\mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^2} \|\Delta \mathbf{w}^\epsilon\|_{L^2} \\ &\leq \frac{1}{16} \|\Delta \mathbf{w}^\epsilon\|_{L^2}^2 + C \|\mathbf{u}\|_{L^2}^2 \|\mathbf{w}^\epsilon\|_{L^2}^2. \end{aligned} \tag{6.4.5}$$

$$\begin{aligned} N_3 &\leq \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^4}^2 \leq C \|\mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\Delta \mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\mathbf{w}^\epsilon\|_{L^2} \|\nabla \mathbf{w}^\epsilon\|_{L^2} \\ &\leq C \|\mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\Delta \mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\mathbf{w}^\epsilon\|_{L^2}^{\frac{3}{2}} \|\Delta \mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{1}{16} \|\Delta \mathbf{w}^\epsilon\|_{L^2}^2 + C \|\mathbf{u}\|_{L^2}^{\frac{2}{3}} \|\Delta \mathbf{u}\|_{L^2}^{\frac{2}{3}} \|\mathbf{w}^\epsilon\|_{L^2}^2. \end{aligned} \tag{6.4.6}$$

By Hölder's inequality, Ladyzhenskaya's inequality, and (2.4.5),

$$\begin{aligned} N_4 &\leq C\epsilon^\alpha \|\mathbf{w}^\epsilon\|_{L^2} \|\mathbf{u}\|_{L^\infty}^\beta \|\nabla \mathbf{u}\|_{L^2} \\ &\leq C\epsilon^\alpha \|\mathbf{w}^\epsilon\|_{L^2} \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}} \|\mathbf{u}\|_{H^2}^{\frac{\beta}{2}} \|\mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} \\ &\leq C\epsilon^\alpha \|\mathbf{w}^\epsilon\|_{L^2} \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}} \left(\|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}} + \|\triangle \mathbf{u}\|_{L^2}^{\frac{\beta}{2}} \right) \|\mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} \\ &= \left(C\epsilon^\alpha \|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} + C\epsilon^\alpha \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \right) \|\mathbf{w}^\epsilon\|_{L^2} \end{aligned} \tag{6.4.7}$$

We now insert the bounds for N in (6.4.1):

$$\begin{aligned}
& \frac{d}{dt}\left\|\mathbf{w}^\epsilon\right\|_{L^2}^2+\left\|\triangle \mathbf{w}^\epsilon\right\|_{L^2}^2 \leq\left\|\mathbf{w}^\epsilon\right\|_{L^2}^2+2N \\
\leq\ & \frac{6}{16}\left\|\triangle \mathbf{w}^\epsilon\right\|_{L^2}^2+C\left\|\mathbf{w}^\epsilon\right\|_{L^2}^4 \\
& +\left(1+C\|\mathbf{u}\|_{L^2}^2+C\|\mathbf{u}\|_{L^2}^{\frac{2}{3}}\|\triangle \mathbf{u}\|_{L^2}^{\frac{2}{3}}\right)\left\|\mathbf{w}^\epsilon\right\|_{L^2}^2 \\
& +C\left(\epsilon^\alpha\|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}}\|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}}+\epsilon^\alpha\|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\|\triangle \mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\right)\left\|\mathbf{w}^\epsilon\right\|_{L^2}
\end{aligned} \tag{6.4.8}$$

Due to the presence of the term $\|\mathbf{w}^\epsilon\|_{L^2}^4$, we cannot apply Grönwall's inequality directly. However, since $\|\mathbf{w}^\epsilon\|_{L^2}$ is supposed to be small, this term is not a "bad" term and is even smaller than $\|\mathbf{w}^\epsilon\|_{L^2}^2$. We just need to apply a bootstrapping argument, as stated in Lemma 2.5.4. Denote by $H(t)$ with $t \in [0,T]$ the statement that

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leq 1$$

and by $C(t)$ the statement that

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leq e^{A(T)}B(T)\,\epsilon^\alpha \leq \tfrac{1}{2},$$

where $A(t)$ and $B(t)$ are defined as in (6.4.11) and (6.4.12) below and ϵ is taken to be sufficiently small such that

$$e^{A(T)}B(T)\,\epsilon^\alpha \leq \tfrac{1}{2}.$$

Clearly, $C(t)$ is a stronger statement than $H(t)$, and thus (b) of Lemma 2.5.4 holds. When the solutions are regular enough, then $\|\mathbf{w}^\epsilon(t)\|_{L^2}$ is continuous in time. Indeed, this regularity is given by Condition 1.2.9 and Definition 1.2.22 and thus (c) of Lemma

2.5.4 holds. For $t = 0$, $\|\mathbf{w}^\epsilon(t)\|_{L^2}$ is zero and thus (d) of Lemma 2.5.4 holds. In order to apply Lemma 2.5.4, it remains to verify (a). That is, if $H(t)$ holds for some $t \in [0, T]$, namely

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leq 1,$$

then $C(t)$ holds at the same t, namely

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leq e^{A(T)} B(T)\, \epsilon^\alpha < \tfrac{1}{2}.$$

We assume that, for some $t \in [0, T]$,

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leq 1 \tag{6.4.9}$$

and then show that (6.4.9) leads to a desired smaller bound at this same t. Now we replace $\|\mathbf{w}^\epsilon\|_{L^2}^4$ by $\|\mathbf{w}^\epsilon\|_{L^2}^2$ in (6.4.8) and eliminate $\|\mathbf{w}^\epsilon\|_{L^2}$ from each term to obtain

$$\begin{aligned}\frac{d}{dt}\|\mathbf{w}^\epsilon\|_{L^2} \leq\, & C\left(1 + \|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^{\frac{2}{3}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{2}{3}}\right)\|\mathbf{w}^\epsilon\|_{L^2} \\ & + C\epsilon^\alpha\left(\|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{1}{2}} + \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\right),\end{aligned}$$

in which we also use the fact that $\frac{d}{dt}\|\mathbf{w}^\epsilon\|_{L^2}^2 = 2\|\mathbf{w}^\epsilon\|_{L^2}\frac{d}{dt}\|\mathbf{w}^\epsilon\|_{L^2}$.

Due to the regularity assumption on $\mathbf{u}$ in 1.2.9, the terms $\left(1 + \|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^{\frac{2}{3}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{2}{3}}\right)$ and $\|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{1}{2}}$ are integrable for $\beta \geq 0$. Furthermore, for $\beta \leq 3$, $\frac{\beta}{2} + \frac{1}{2} \leq 2$ and thus $\|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}$ is integrable. It then follows from Grönwall's inequality that

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leq e^{A(t)}\|\mathbf{w}^\epsilon(0)\|_{L^2} + e^{A(t)} B(t)\, \epsilon^\alpha \leq e^{A(T)} B(T)\, \epsilon^\alpha, \tag{6.4.10}$$

where we have used the fact that the initial difference $\mathbf{w}^\epsilon(0) = 0$ and have written

$$A(t) := C \int_0^t \Big(1 + \|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^{\frac{2}{3}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{2}{3}}\Big) ds, \tag{6.4.11}$$

$$B(t) := C \int_0^t \Big(\|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{1}{2}} + \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \Big) ds. \tag{6.4.12}$$

By taking ϵ sufficiently small, from (6.4.10) we deduce that any
$t \in [0, T]$ which satisfies

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} < 1.$$

must also satisfy

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \le e^{A(T)} B(T)\, \epsilon^\alpha < \tfrac{1}{2}.$$

Thus the bootstrapping argument holds, and we conclude as claimed that for all $t \le T$,

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \le K_1 \epsilon^\alpha \tag{6.4.13}$$

where $K_1 = e^{A(T)} B(T)$ depends on T, $\mathbf{u}$, and β. In particular, $\mathbf{u}^\epsilon \to \mathbf{u}$
in $L^\infty(0, T; L^2(\mathbb{T}^2))$ as $\epsilon \to 0^+$. Next, integrate (6.4.8) for all $t \in [0, T]$ (again replacing $\|\mathbf{w}^\epsilon\|_{L^2}^4$ by $\|\mathbf{w}^\epsilon\|_{L^2}^2$) to obtain

$$\begin{aligned}
&\frac{10}{16} \int_0^T \|\triangle\mathbf{w}^\epsilon\|_{L^2}^2\, dt \\
&\le C \int_0^T \Big(1 + \|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^{\frac{2}{3}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{2}{3}}\Big) \|\mathbf{w}^\epsilon\|_{L^2}^2\, dt \\
&\quad + C\epsilon^\alpha \int_0^T \Big(\|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{1}{2}} + \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \|\triangle\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \Big) \|\mathbf{w}^\epsilon\|_{L^2}\, dt
\end{aligned}$$

In which we are again using the fact that $\mathbf{w}^\epsilon(0) = 0$. Applying (6.4.11), (6.4.12), and

(6.4.13) then yields

$$\int_0^T \|\triangle \mathbf{w}^\epsilon\|_{L^2}^2 \, dt \leq \frac{16}{10} A(T) K_1^2 \epsilon^{2\alpha} + \frac{16}{10} B(T) K_1 \epsilon^{2\alpha}. \tag{6.4.14}$$

For $K_2 = \frac{16}{10} A(T) K_1^2 + \frac{16}{10} B(T) K_1$ (again only depending on T, $\|\mathbf{u}\|_{L^\infty(0,T;L^2)}$, $\|\mathbf{u}\|_{L^2(0,T;H^2)}$, and β), we obtain

$$\|\triangle \mathbf{w}^\epsilon\|_{L^2(0,T;L^2)} \leq K_2^{\frac{1}{2}} \epsilon^\alpha. \tag{6.4.15}$$

Using an interpolation inequality, we obtain

$$\begin{aligned} \|\mathbf{w}^\epsilon\|_{L^2(0,T;H^2)} &\leq C\|\mathbf{w}^\epsilon\|_{L^2(0,T;L^2)} + C\|\triangle \mathbf{w}^\epsilon\|_{L^2(0,T;L^2)} \\ &\leq CT^{\frac{1}{2}}\|\mathbf{w}^\epsilon\|_{L^\infty(0,T;L^2)} + C\|\triangle \mathbf{w}^\epsilon\|_{L^2(0,T;L^2)} \\ &\leq C\left(T^{\frac{1}{2}} K_1 + K_2^{\frac{1}{2}}\right)\epsilon^\alpha \end{aligned} \tag{6.4.16}$$

as claimed. In particular, $\mathbf{u}^\epsilon \to \mathbf{u}$ in $L^2(0,T;H^2(\mathbb{T}^2))$ as $\epsilon \to 0^+$. □

Corollary 6.4.1. *Consider the calming functions* $\boldsymbol{\zeta}^\epsilon$ *as described in* (2.1.2). *Let* $\mathbf{u}, \mathbf{u}^\epsilon$ *be as in the statement of Theorem 1.2.25 with the same initial data, where* $\mathbf{u}^\epsilon$ *is determined by* $\boldsymbol{\zeta}_i^\epsilon$, $i = 1, 2$, *or* 3. *Then for* $T < T^*$, *there exists* $K_i' > 0$ *independent of* ϵ *such that*

1. *for* $\boldsymbol{\zeta}^\epsilon = \boldsymbol{\zeta}_1^\epsilon$,

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^\infty(0,T;L^2)} \leq K_1' \epsilon, \tag{6.4.17}$$

2. for $\boldsymbol{\zeta}^\epsilon = \boldsymbol{\zeta}_2^\epsilon$,

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^\infty(0,T;L^2)} \le K_2' \epsilon^2, \tag{6.4.18}$$

3. for $\boldsymbol{\zeta}^\epsilon = \boldsymbol{\zeta}_3^\epsilon$,

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^\infty(0,T;L^2)} \le K_3' \epsilon^2. \tag{6.4.19}$$

Proof. The proof follows immediately from Theorem 1.2.25 and Proposition 2.1.3. □

6.5 The Scalar Form

Here we investigate the scalar formulation (6.1.4). The analysis is similar to the analysis of (6.1.3), so we only briefly present formal energy estimates. For the sake of brevity, we work formally rather than rigorously. However, the proof below can be made rigorous, e.g., via the use of Galerkin methods as in the proof of Theorem 1.2.23.

Proof of Theorem 1.2.27. Take a (formal) inner product of (6.1.4a) with ϕ and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi\|_{L^2}^2 + \|\triangle\phi\|_{L^2}^2 = -(\triangle\phi, \phi) - \left(\tfrac{1}{2}\boldsymbol{\zeta}^\epsilon(\nabla\phi)\cdot\nabla\phi, \phi\right). \tag{6.5.1}$$

Using (2.4.5) and

$$\begin{aligned}\left|\left(\tfrac{1}{2}\boldsymbol{\zeta}^\epsilon(\nabla\phi)\cdot\nabla\phi, \phi\right)\right| &\le \frac{1}{2}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\phi\|_{L^2}\\ &\le C\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}}\|\phi\|_{L^2}^2 + \frac{1}{4}\|\triangle\phi\|_{L^2}^2,\end{aligned}$$

we obtain from (6.5.1) that

$$\frac{d}{dt}\|\phi\|_{L^2}^2 + \|\triangle\phi\|_{L^2}^2 \leq \left(2 + C\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}}\right)\|\phi\|_{L^2}^2 . \tag{6.5.2}$$

Hence from Grönwall's inequality, dropping the second term in (6.5.2), we obtain

$$\|\phi(t)\|_{L^2}^2 \leq e^{K_\epsilon T}\|\phi_0\|_{L^2}^2 , \tag{6.5.3}$$

where $K_\epsilon = \left(2 + C\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}}\right)$. Hence $\phi \in L^\infty(0,T;L^2(\mathbb{T}^2))$. Next, we integrate (6.5.2) in time on the interval $[0,T]$ and drop any unnecessary terms:

$$\begin{aligned}\int_0^T \frac{1}{2}\|\triangle\phi\|_{L^2}^2 &\leq \int_0^T K_\epsilon\|\phi(t)\|_{L^2}^2\,dt + \|\phi_0\|_{L^2}^2 \\ &\leq \int_0^T K_\epsilon e^{K_\epsilon T}\|\phi_0\|_{L^2}^2\,dt + \|\phi_0\|_{L^2}^2 \\ &= \left(K_\epsilon T e^{K_\epsilon T} + 1\right)\|\phi_0\|_{L^2}^2 .\end{aligned} \tag{6.5.4}$$

Therefore $\phi \in L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$. Now we obtain estimates on $\partial_t\phi$: For any $\psi \in L^2(0,T;H^2(\mathbb{T}^2))$,

$$\begin{aligned}|\langle\partial_t\phi,\psi\rangle| &= \left|\int_0^T \partial_t\phi\psi dt\right| \\ &= \left|\int_0^T \left(\frac{1}{2}\boldsymbol{\zeta}^\epsilon(\nabla\phi)\cdot\nabla\phi\right)\psi dt + \int_0^T (\triangle\phi)\,\psi dt + \int_0^T (\triangle\phi)\,\triangle\psi dt\right| \\ &\leq \frac{1}{2}\int_0^T |\boldsymbol{\zeta}^\epsilon(\nabla\phi)|\,|\nabla\phi|\,|\psi|\,dt + \int_0^T |\triangle\phi|\,|\psi|\,dt + \int_0^T |\triangle\phi|\,|\triangle\psi|\,dt \\ &\leq \frac{1}{2}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla\phi\|_{L^2(0,T;L^2)}\|\psi\|_{L^2(0,T;L^2)} \\ &\quad + \|\triangle\phi\|_{L^2(0,T;L^2)}\|\psi\|_{L^2(0,T;L^2)} + \|\triangle\phi\|_{L^2(0,T;L^2)}\|\triangle\psi\|_{L^2(0,T;L^2)} \\ &\leq \left(\frac{1}{2}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\phi\|_{L^2(0,T;H^2)} + 2\|\phi\|_{L^2(0,T;H^2)}\right)\|\psi\|_{L^2(0,T;H^2)}.\end{aligned} \tag{6.5.5}$$

It follows from Estimate (6.5.4) that $\|\partial_t \phi\|_{L^2(0,T;H^{-2})} < \infty$, hence $\partial_t \phi \in L^2(0,T;H^{-2}(\mathbb{T}^2))$. From this we deduce that a solution ϕ to (6.1.4) exists, with

$$\phi \in C(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2)).$$

Now, let ϕ and ψ be two solutions to (6.1.4) with $\phi(0) = \psi(0) = \phi_0$. Let $\delta = \phi - \psi$. Then δ satisfies the equation

$$\partial_t \delta + \triangle^2 \delta = -\triangle \delta + \boldsymbol{\zeta}^\epsilon(\nabla \psi) \cdot \nabla \psi - \boldsymbol{\zeta}^\epsilon(\nabla \phi) \cdot \nabla \phi \tag{6.5.6}$$

with $\delta(0) = 0$. We can then rewrite the nonlinear term as

$$\boldsymbol{\zeta}^\epsilon(\nabla \psi) \cdot \nabla \psi - \boldsymbol{\zeta}^\epsilon(\nabla \phi) \cdot \nabla \phi = (\boldsymbol{\zeta}^\epsilon(\nabla \psi) - \boldsymbol{\zeta}^\epsilon(\nabla \phi)) \cdot \nabla \psi - \boldsymbol{\zeta}^\epsilon(\nabla \phi) \cdot \nabla \delta. \tag{6.5.7}$$

We now insert (6.5.7) into (6.5.6) and apply integration by parts to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|\delta\|_{L^2}^2 + \|\triangle \delta\|_{L^2}^2 \\ &\leq |(\triangle \delta, \delta)| + |((\boldsymbol{\zeta}^\epsilon(\nabla \psi) - \boldsymbol{\zeta}^\epsilon(\nabla \phi)) \cdot \nabla \psi, \delta)| + |(\boldsymbol{\zeta}^\epsilon(\nabla \phi) \cdot \nabla \delta, \delta)| . \end{aligned} \tag{6.5.8}$$

In the second term, we use Condition 1 of 2.1.1, Hölder's, Ladyzhenskaya's, and Young's inequality to obtain

$$\begin{aligned} |((\boldsymbol{\zeta}^\epsilon(\nabla \psi) - \boldsymbol{\zeta}^\epsilon(\nabla \phi)) \cdot \nabla \psi, \delta)| &\leq \|\boldsymbol{\zeta}^\epsilon(\nabla \psi) - \boldsymbol{\zeta}^\epsilon(\nabla \phi)\|_{L^4} \|\nabla \psi\|_{L^2} \|\delta\|_{L^4} \\ &\leq \|\nabla \psi\|_{L^2} \|\delta\|_{L^4} \|\nabla \delta\|_{L^4} \\ &\leq C \|\nabla \psi\|_{L^2} \|\delta\|_{L^2}^{\frac{1}{2}} \|\nabla \delta\|_{L^2} \|\triangle \delta\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|\nabla \psi\|_{L^2} \|\delta\|_{L^2} \|\triangle \delta\|_{L^2} \end{aligned} \tag{6.5.9}$$

$$\leq C \|\nabla\psi\|_{L^2}^2 \|\delta\|_{L^2}^2 + \frac{1}{4} \|\triangle\delta\|_{L^2}^2$$

In the third term, we apply Condition 2 of 2.1.1, use Young's inequality, and use interpolation inequalities to obtain

$$\begin{aligned} |(\boldsymbol{\zeta}^\epsilon(\nabla\phi) \cdot \nabla\delta, \delta)| &\leq \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\nabla\delta\|_{L^2} \|\delta\|_{L^2} \\ &\leq \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\delta\|_{L^2}^{\frac{3}{2}} \|\triangle\delta\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}} \|\delta\|_{L^2}^2 + \frac{1}{4} \|\triangle\delta\|_{L^2}^2 . \end{aligned} \tag{6.5.10}$$

After inserting (6.5.9) and (6.5.10) into (6.5.8) and rearranging the terms, the inequality becomes

$$\frac{d}{dt} \|\delta\|_{L^2}^2 + \|\triangle\delta\|_{L^2}^2 \leq C \left(1 + \|\nabla\psi\|_{L^2}^2 + \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}}\right) \|\delta\|_{L^2}^2 . \tag{6.5.11}$$

Then applying Grönwall's inequality, we obtain

$$\|\phi(t) - \psi(t)\|_{L^2}^2 \leq e^{\widetilde{K}_1(T)} \|\phi_0 - \psi_0\|_{L^2}^2 , \tag{6.5.12}$$

where $\widetilde{K}_1(T) = \int_0^T 1 + \|\nabla\psi(t)\|_{L^2}^2 + \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}} \, dt$. Since $\psi \in L^2(0,T;H^2(\mathbb{T}^2))$, and $\boldsymbol{\zeta}^\epsilon$ is bounded, $\widetilde{K}_1(T) < \infty$. So $\phi(t) = \psi(t)$ for all $t \in [0,T]$, hence solutions to (6.1.4) are unique. Now, we integrate (6.5.11) on the interval $[0,T]$ and apply (6.5.12), which yields

$$\int_0^T \|\triangle\phi(t) - \triangle\psi(t)\|_{L^2}^2 \, dt \leq \widetilde{K}_2 \|\phi_0 - \psi_0\|_{L^2}^2 \tag{6.5.13}$$

for some $\widetilde{K}_2$ which depends on T, $\|\nabla\psi(t)\|_{L^2}$, and $\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}$. From estimates (6.5.12) and (6.5.13) we conclude that solutions depend continuously on the initial data in

$L^\infty(0,T;L^2(\mathbb{T}^2))\cap L^2(0,T;H^2(\mathbb{T}^2))$. $\square$

Here, we will show the convergences of solutions to (6.1.4) to that of (6.1.1) as $\epsilon \to 0^+$. This proof has only minor variations from the proof of Theorem 1.2.25.

Proof of Theorem 1.2.28. We set $\delta^\epsilon = \phi - \phi^\epsilon$ take the difference between (6.1.1) and (6.1.4a), and take the inner product with δ^ϵ. to obtain

$$\frac{d}{dt}\|\delta^\epsilon\|_{L^2}^2 + \|\triangle\delta^\epsilon\|_{L^2}^2 \le \|\delta^\epsilon\|_{L^2}^2 + N_1 + N_2 + N_3 + N_4, \tag{6.5.14}$$

where

$$\begin{aligned}
N_1 &= |((\boldsymbol{\zeta}^\epsilon(\nabla\phi^\epsilon) - \boldsymbol{\zeta}^\epsilon(\nabla\phi))\cdot\nabla\delta^\epsilon, \delta^\epsilon)| \le C\|\delta^\epsilon\|_{L^2}^6 + \frac{3}{4}\|\triangle\delta^\epsilon\|_{L^2}^2,\\
N_2 &= |((\boldsymbol{\zeta}^\epsilon(\nabla\phi^\epsilon) - \boldsymbol{\zeta}^\epsilon(\nabla\phi))\cdot\nabla\phi, \delta^\epsilon)| \le C\|\phi\|_{L^2}^{\frac{2}{5}}\|\triangle\phi\|_{L^2}^{\frac{6}{5}}\|\delta^\epsilon\|_{L^2}^2 + \frac{1}{8}\|\triangle\delta^\epsilon\|_{L^2}^2,\\
N_3 &= |(\boldsymbol{\zeta}^\epsilon(\nabla\phi)\cdot\nabla\delta^\epsilon, \delta^\epsilon)| \le C\|\phi\|_{L^2}^{\frac{1}{4}}\|\triangle\phi\|_{L^2}^{\frac{3}{4}}\|\delta^\epsilon\|_{L^2}^2 + \frac{1}{16}\|\triangle\delta^\epsilon\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
N_4 &= |((\boldsymbol{\zeta}^\epsilon(\nabla\phi) - \nabla\phi)\cdot\nabla\phi, \delta^\epsilon)|\\
&\le C\epsilon^\alpha \int_{\mathbb{T}^2} |\nabla\phi|^{\beta+1}|\delta^\epsilon|\,d\mathbf{x}\\
&\le C\epsilon^\alpha \|\nabla\phi\|_{L^{2\beta+2}}^{\beta+1}\|\delta^\epsilon\|_{L^2}.
\end{aligned}$$

Applying the Sobolev inequality, we deduce that

$$\|\nabla\phi\|_{L^{2\beta+2}}^{\beta+1} \le C\|\nabla\phi\|_{L^2}\|\nabla\phi\|_{H^1}^{\beta} \le C\|\phi\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^2}^{\beta+\frac{1}{2}}.$$

Inserting our bounds for each N_i into (6.5.14) and rearranging then yields

$$\frac{d}{dt}\left\|\delta^\epsilon\right\|_{L^2}^2 + \frac{1}{16}\left\|\triangle\delta^\epsilon\right\|_{L^2}^2 \leq C\left\|\delta^\epsilon\right\|_{L^2}^6 + \left\|\delta^\epsilon\right\|_{L^2}^2 + C\epsilon^\alpha\left\|\phi\right\|_{L^2}^{\frac{1}{2}}\left\|\phi\right\|_{H^2}^{\beta+\frac{1}{2}}\left\|\delta^\epsilon\right\|_{L^2} \\ + C\left(\left\|\phi\right\|_{L^2}^{\frac{2}{5}}\left\|\triangle\phi\right\|_{L^2}^{\frac{6}{5}} + \left\|\phi\right\|_{L^2}^{\frac{1}{4}}\left\|\triangle\phi\right\|_{L^2}^{\frac{3}{4}}\right)\left\|\delta^\epsilon\right\|_{L^2}^2. \quad (6.5.15)$$

Now we apply the ansatz

$$\left\|\delta^\epsilon\right\|_{L^2} < 1$$

to obtain the bound

$$\left\|\delta^\epsilon\right\|_{L^2}^6 \leq \left\|\delta^\epsilon\right\|_{L^2}^2.$$

We apply this estimate to (6.5.15) and eliminate $\left\|\delta^\epsilon\right\|_{L^2}$ from each term to obtain

$$\frac{d}{dt}\left\|\delta^\epsilon\right\|_{L^2} \leq C\left\|\phi\right\|_{L^2}^{\frac{1}{2}}\left\|\phi\right\|_{H^2}^{\beta+\frac{1}{2}}\epsilon^\alpha \\ + C\left(1 + \left\|\phi\right\|_{L^2}^{\frac{2}{5}}\left\|\triangle\phi\right\|_{L^2}^{\frac{6}{5}} + \left\|\phi\right\|_{L^2}^{\frac{1}{4}}\left\|\triangle\phi\right\|_{L^2}^{\frac{3}{4}}\right)\left\|\delta^\epsilon\right\|_{L^2}. \quad (6.5.16)$$

The term

$$1 + \left\|\phi\right\|_{L^2}^{\frac{2}{5}}\left\|\triangle\phi\right\|_{L^2}^{\frac{6}{5}} + \left\|\phi\right\|_{L^2}^{\frac{1}{4}}\left\|\triangle\phi\right\|_{L^2}^{\frac{3}{4}}$$

is always integrable and the term

$$\left\|\phi\right\|_{L^2}^{\frac{1}{2}}\left\|\phi\right\|_{H^2}^{\beta+\frac{1}{2}}$$

is integrable for $\beta \in [1, \frac{3}{2}]$. It now follows from Grönwall's inequality that

$$\left\|\delta^\epsilon(t)\right\|_{L^2} \leq e^{A(t)}\left\|\delta^\epsilon(0)\right\|_{L^2} + e^{A(t)}B(t)\epsilon^\alpha \leq e^{A(T)}B(T)\epsilon^\alpha, \quad (6.5.17)$$

using the fact that $\delta^\epsilon(0) = 0$, and with

$$A(t) = C\int_0^t 1 + \|\phi\|_{L^2}^{\frac{2}{5}} \|\triangle\phi\|_{L^2}^{\frac{6}{5}} + \|\phi\|_{L^2}^{\frac{1}{4}} \|\triangle\phi\|_{L^2}^{\frac{3}{4}}\, ds,$$
$$B(t) = C\int_0^t \|\phi\|_{L^2}^{\frac{1}{2}} \|\phi\|_{H^2}^{\beta+\frac{1}{2}}\, ds.$$

By taking ϵ sufficiently small, we have for all $0 \le t \le T$

$$\|\delta^\epsilon(t)\|_{L^2} < 1.$$

It follows from a bootstrapping argument that

$$\|\delta^\epsilon(t)\|_{L^\infty(0,T;L^2)} \le e^{A(T)}B(T)\epsilon^\alpha. \tag{6.5.18}$$

Now we integrate (6.5.15) on $[0,T]$, again using that $\|\delta^\epsilon\|_{L^2}^6 \le \|\delta^\epsilon\|_{L^2}^2$, and apply to obtain

$$\begin{aligned}\int_0^T \|\triangle\delta^\epsilon\|_{L^2}^2\, dt &\le C\epsilon^\alpha B(T)e^{A(T)}B(T)\epsilon^\alpha + A(T)\left(e^{A(T)}B(T)\epsilon^\alpha\right)^2 \\ &\le K(T)^2\epsilon^{2\alpha},\end{aligned} \tag{6.5.19}$$

where

$$K(T)^2 = CB(T)^2e^{A(T)} + A(T)B(T)e^{A(T)}.$$

Therefore we obtain

$$\|\delta^\epsilon\|_{L^2(0,T;H^2)} \le \left(Te^{A(T)}B(T) + K(T)\right)\epsilon^\alpha. \tag{6.5.20}$$

□

6.6 Computational Results

In this section, we examine the calmed Kuramoto-Sivashinsky equations computationally via several simulations, where the calming function $\boldsymbol{\zeta}^\epsilon = \boldsymbol{\zeta}_i^\epsilon$ is described in (2.1.2). We include snapshots of the evolution of solutions for the different choices of $\boldsymbol{\zeta}^\epsilon$ in Figure 6.1, and for different choices of ϵ in Figure 6.2 (we show results for $\boldsymbol{\zeta}_3^\epsilon$ only for the sake of brevity; $\boldsymbol{\zeta}_1^\epsilon$ and $\boldsymbol{\zeta}_2^\epsilon$ yielded qualitatively similar results). The former illustrates the different effects of the choice of $\boldsymbol{\zeta}^\epsilon$ on the dynamics, while the latter indicates the uniform convergence of $\mathbf{u}^\epsilon$ to $\mathbf{u}$.

In addition, we examine convergence rates in $L^\infty(0,T;L^2)$, $L^\infty(0,T;L^\infty)$, and $L^2(0,T;H^2)$ for $\boldsymbol{\zeta}_1^\epsilon$ (Figure 6.3), $\boldsymbol{\zeta}_2^\epsilon$ (Figure 6.4), and $\boldsymbol{\zeta}_3^\epsilon$ (Figure 6.5) with initial data (6.6.2) as $\epsilon \to 0^+$ (for simplicity, we set $T = 1$, since with all our initial data, solutions to KSE appear to be quite stable on $[0,1]$). We find that the powers on the $L^\infty(0,T;L^2)$ and $L^2(0,T;H^2)$ convergence rates in Corollary 6.4.1 appear to be sharp.

Finally, in Figures 6.6, 6.7, and 6.8 we check the robustness of the convergence with respect to larger initial data (6.6.3) for $\boldsymbol{\zeta}_1^\epsilon$, $\boldsymbol{\zeta}_2^\epsilon$, and $\boldsymbol{\zeta}_3^\epsilon$. In comparing initial data (6.6.2) with (6.6.3), we find very little qualitative variation in the error rates, indicating that changes in initial data will only marginally change the error between solutions to KSE and solutions to calmed KSE for $\epsilon > 0$ sufficiently small.

6.6.1 Numerical Methods

All computations were done in Matlab (R2021a) using pseudo-spectral methods with the standard $2/3's$ dealiasing for the nonlinear term. To evolve the system, we used a well-known modification of the Runge-Kutta-4 time-stepping scheme adapted to handle the linear terms implicitly via an integrating factor to handle the nonlinear terms implicitly (see, e.g., [49]) with time step $\Delta t \approx 4.2943 \times 10^{-4}$ chosen to respect the maximum advective CFL condition in Figures 6.1, 6.2, 6.3, 6.4, and 6.5, with later figures having a rescaled time step $\Delta t = 1.0736 \times 10^{-4}$. Our simulations for KSE and cKSE were resolved[5] with a spatial mesh of 128^2. All computations were done using the nondimensionalized calmed Kuramoto-Sivashinsky equations,

$$\partial_t \mathbf{u} + (\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla)\mathbf{u} + \lambda \triangle \mathbf{u} + \triangle^2 \mathbf{u} = \mathbf{0}, \tag{6.6.1a}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{6.6.1b}$$

over the periodic domain $\Omega = [-\pi, \pi)^2$ for $\lambda > 0$.

Throughout this section, a *type* 1, *type* 2, or *type* 3 solution is a solution to calmed KSE with calming function $\boldsymbol{\zeta}_1^\epsilon$, $\boldsymbol{\zeta}_2^\epsilon$, or $\boldsymbol{\zeta}_3^\epsilon$ respectively.

6.6.2 Simulations

Here, we take initial conditions to be

$$\mathbf{u}_0(x, y) = \begin{pmatrix} \cos(x+y) + \cos(x) \\ \cos(x+y) + \cos(y) \end{pmatrix} \tag{6.6.2}$$

and all color plots seen below are plots of the magnitude $|\mathbf{u}| = |(u, v)| = \sqrt{u^2 + v^2}$. In all plots of solutions, the horizontal axis corresponds to the y-axis and the vertical axis corresponds to the x-axis.

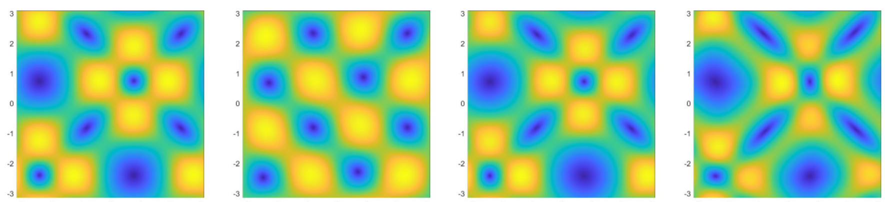
Our choice for initial data $\mathbf{u}_0$ was motivated by the choice of scalar initial data

[5]Note: For the Kuramoto-Sivashinsky equations (calmed or otherwise), even in fairly chaotic regimes, one often does not need especially high resolution, due to the strong hyperdiffusion term. Moreover, so long as the solution is well-resolved, which we take to mean that the energy spectrum at the modes higher than the 2/3's dealiasing cut-off is at or below machine precision (roughly 2.22×10^{-16}), increasing the resolution only increases round-off error, due to the additional computations being performed. Hence, to minimize roundoff error, we purposely chose the fairly low resolution of 128^2, although our higher-resolution tests, not reported here, produced qualitatively similar results.

found in [48], [62], and [59]; namely,

$$\phi_0(x, y) = \sin(x + y) + \sin(x) + \sin(y).$$

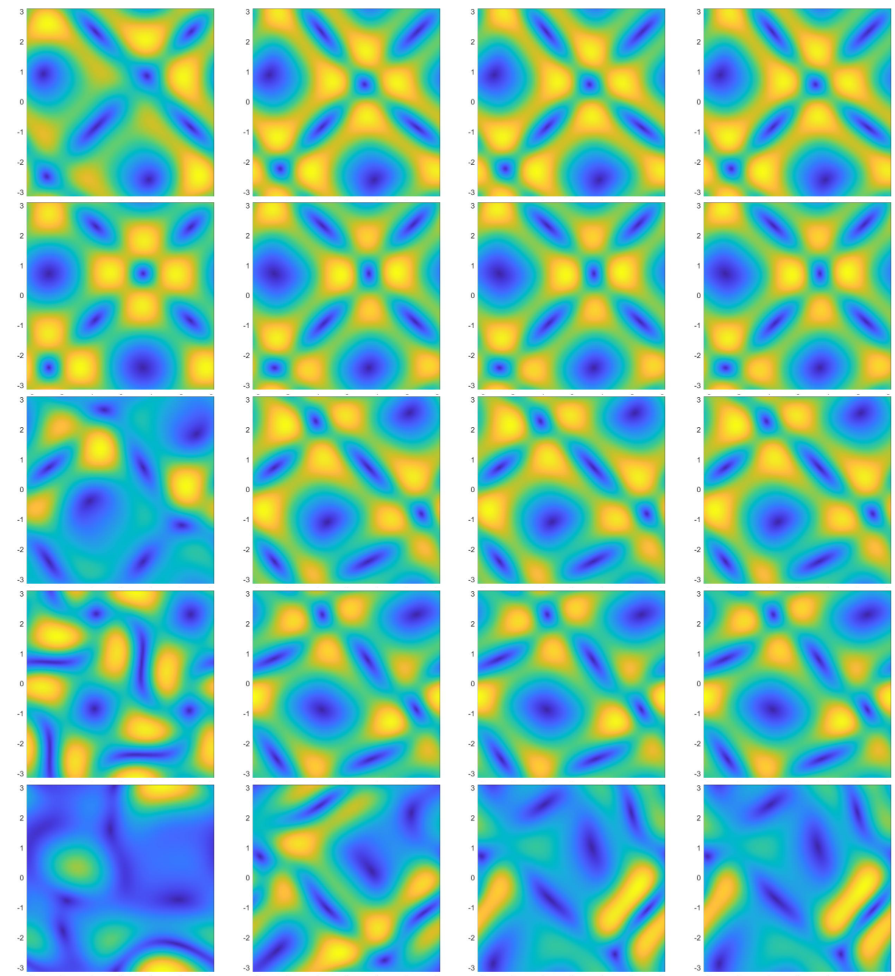
Hence, we set $\mathbf{u}_0 = \nabla\phi_0$.



(a) Type 1 (b) Type 2 (c) Type 3 (d) KSE

Figure 6.1: Solutions to calmed KSE of each type compared with a solution to KSE at time $t = 2$, with $\epsilon = 0.1$, $\lambda = 4.1$, and $\mathbf{u}_0$ given by (6.6.2).

Though some differences can be seen among the images above, one can see that each type of calmed KSE solution approximates the overall behavior of a KSE solution. One can also observe that the accuracy of the approximation varies by type.



(a) $\epsilon_1 = 0.1$ (b) $\epsilon_2 = 0.01$ (c) $\epsilon_3 = 0.001$ (d) KSE

Figure 6.2: Column 6.2d is a solution to KSE (6.1.2) for $t = 1, \dots, 5$, whereas columns 6.2a, 6.2b, and 6.2c are type 3 solutions to calmed KSE (6.1.3) on the same time interval with $\epsilon \in \{0.1, 0.01, 0.001\}$. In this figure, $\lambda = 4.1$ is fixed and initial data $\mathbf{u}_0$ is given in (6.6.2). Viewing the pictures from left to right, we can see that $\mathbf{u}^\epsilon \to \mathbf{u}$ as $\epsilon \to 0$.

In Figure 6.2 we focus only on type 3 approximations to better illustrate how well calmed KSE solutions can approximate KSE solutions over time for various choices of ϵ. Indeed, when viewed from left to right we can observe the convergence of our calmed KSE solutions to the original KSE solution.

In accordance with Corollary 6.4.1 we see that solutions to calmed KSE corresponding to calming function $\boldsymbol{\zeta}_1^\epsilon$ yield a linear convergence rate whereas solutions to calmed KSE corresponding to calming functions $\boldsymbol{\zeta}_2^\epsilon$ or $\boldsymbol{\zeta}_3^\epsilon$ yield quadratic convergence rates.

For additional testing, we choose initial data with higher oscillation and higher magnitude,

$$\mathbf{u}_0(x,y) = \begin{pmatrix} 4\,(\cos(x+y)+\sin(3x)) \\ 4\,(\cos(x+y)+\cos(4y)) \end{pmatrix}, \tag{6.6.3}$$

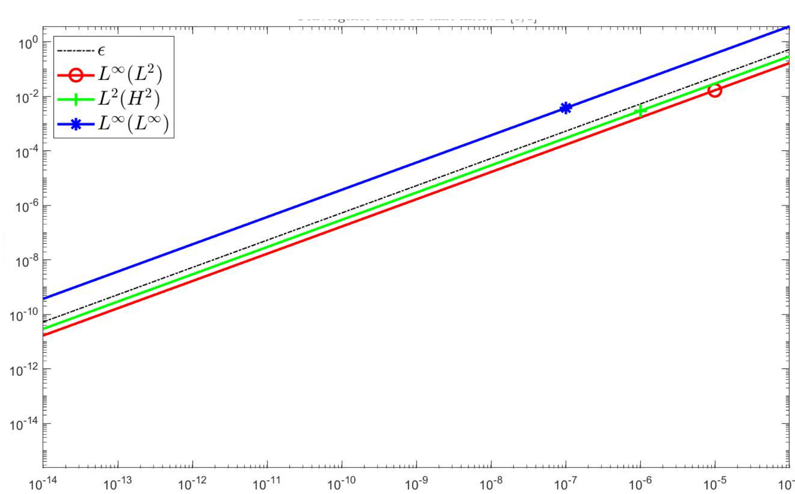


Figure 6.3: Estimates of $\mathbf{u}-\mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T=1$ with $\mathbf{u}^\epsilon$ a type 1 solution and with initial data given by (6.6.2). These estimates show a linear convergence rate.

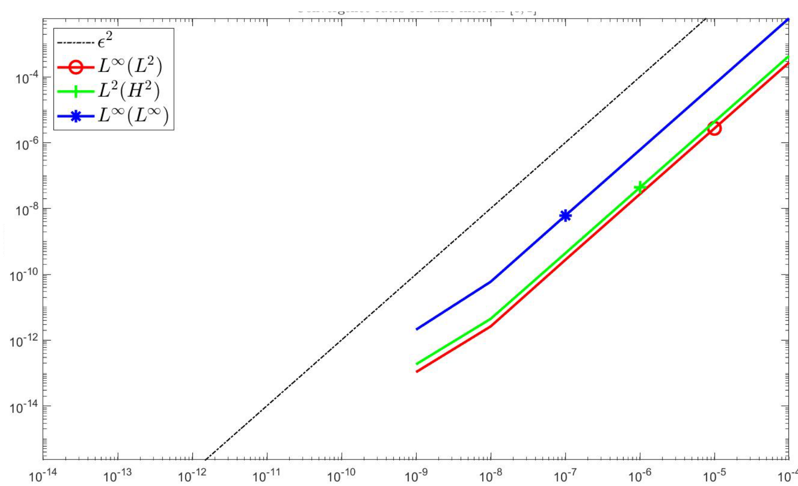


Figure 6.4: Estimates of $\mathbf{u} - \mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T = 1$ with $\mathbf{u}^\epsilon$ a type 2 solution and with initial data given by (6.6.2). These estimates show a quadratic convergence rate. Note: for $\epsilon \lesssim 10^{-9}$, the error in our simulations was exactly 0, hence it does not appear in this log-log plot.

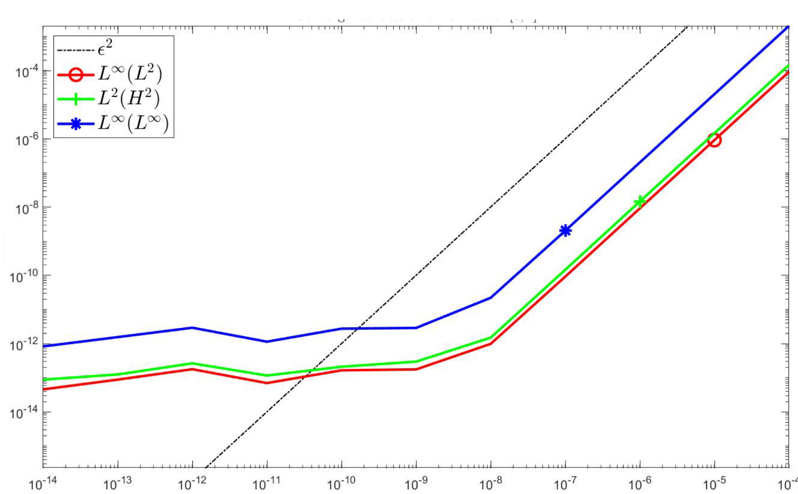


Figure 6.5: Estimates of $\mathbf{u} - \mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T = 1$ with $\mathbf{u}^\epsilon$ a type 3 solution and with initial data given by (6.6.2). These estimates show a quadratic convergence rate.

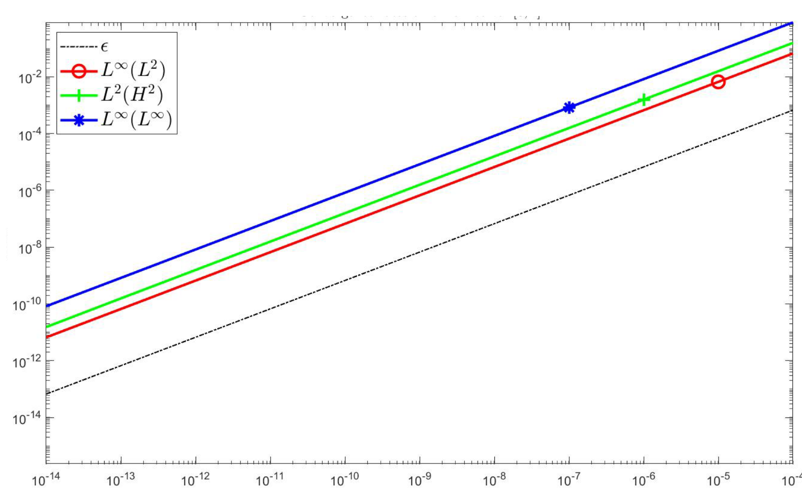


Figure 6.6: Estimates of $\mathbf{u} - \mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T = 1$ with $\mathbf{u}^\epsilon$ a type 1 solution and with initial data given by (6.6.3). These estimates show a linear convergence rate.

and examine the convergence rates for each solution type. For each convergence test, we have the fixed parameters $N = 128$, $T = 1$, and $\lambda = 4.1$.

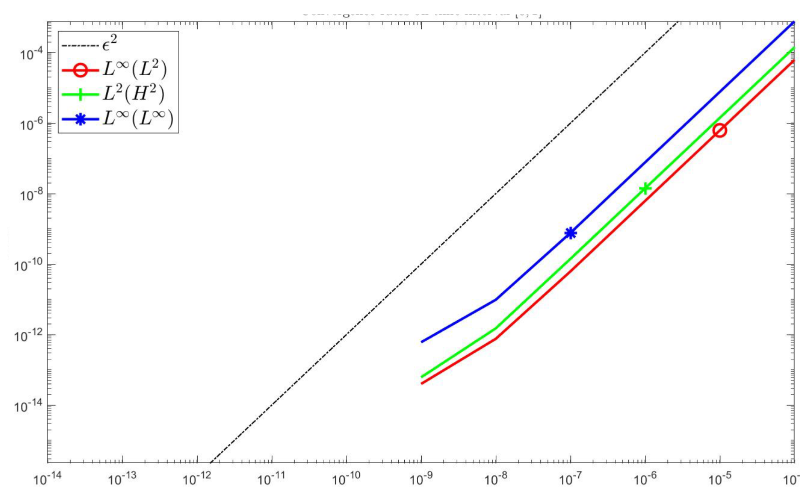


Figure 6.7: Estimates of $\mathbf{u} - \mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T = 1$ with $\mathbf{u}^\epsilon$ a type 2 solution and with initial data given by (6.6.3). These estimates show a quadratic convergence rate. Note: for $\epsilon \lesssim 10^{-9}$, the error in our simulations was exactly 0, hence it does not appear in this log-log plot.

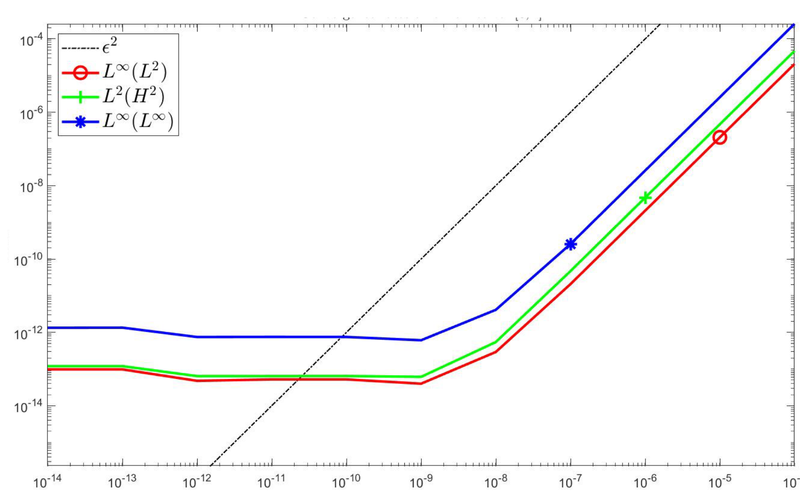


Figure 6.8: Estimates of $\mathbf{u} - \mathbf{u}^\epsilon$ vs. ϵ in norms
$\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T = 1$ with $\mathbf{u}^\epsilon$ a type 3 solution and with initial data given by (6.6.3). These estimates show a quadratic convergence.

We observe that even with larger choice of initial data, Figures 6.6, 6.7, and 6.8 remain qualitatively similar to Figures 6.3, 6.4, and 6.5. This computational result is again in accordance with Corollary 6.4.1.

6.7 Conclusions

We introduced new modifications of the 2D Kuramoto-Sivashinsky equation, in both scalar and vector forms, with a "calming-parameter" $\epsilon > 0$ that we call the "calmed Kuramoto-Sivashinsky equation," and proved that associated PDEs are globally well-posed in the sense of Hadamard. Moreover, we proved that, under suitable conditions on the calming function $\boldsymbol{\zeta}^\epsilon$, that (on the time interval of existence and uniqueness of solutions to the KSE) the solutions of the calmed equation converge to solutions of the KSE as $\epsilon \to 0^+$ at a certain algebraic rate. Moreover, our computational simulations indicate that this rate is sharp. To the best of our knowledge, this is the first globally well-posed PDE model whose solutions approximate solutions to the 2D Kuramoto-Sivashinsky equation with arbitrary precision, at least before the potential blow-up time of the latter.

In addition, we note that this "calming" technique can be applied to a wide variety of other equations, which we will investigate in several forthcoming works. In particular, in [29], we consider applications of calming to the 3D Navier-Stokes equations.

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