Asset Allocation in Investing to Meet Liabilities

Anthony Dardis
TillinghastTowers Perrin, Dardist@Tillinghast.com

Vinh Loi Huynh
Employers Reassurance International Limited, dkerivh0@ibmmail.com

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Anthony Dardis* and Vinh Loi Huynh†

Abstract

We present some rudimentary concepts on asset/liability management and describe an approach to asset allocation modeling for institutions that invest to meet liabilities. The traditional risk/reward framework of financial economics is used as a starting point. The definitions of risk and reward are then refined with regard to the institution under consideration. A simple model of a U.S. life office is examined. We assume that the only investments available are domestic stocks and long-dated government bonds. Stochastic simulation is used to create a large number of future investment scenarios using historical total return data for these asset classes. The ability of the institution to meet its liabilities under each simulated scenario is examined. We construct optimal risk/reward profiles, and hence the optimal asset allocation strategy, and show that they can vary considerably by liability profile.

Key words and phrases: *asset/liability management, Monte Carlo simulation, risk, reward, solvency*

*Anthony Dardis, F.I.A., A.S.A., M.A.A.A., is a consulting actuary with Tillinghast-Towers Perrin. He received a B.A. in economics from Leeds University, England. Mr. Dardis has over 15 years actuarial experience covering a wide range of fields including merchant banking, reinsurance, and life consulting.

Mr. Dardis is a regular participant at Society of Actuaries' meetings and has published articles in the *Record of the Society of Actuaries* and in various Society of Actuaries' newsletters. He is a member of the editorial board of the Society's Investment Section newsletter, and is a member of the Society of Actuaries' Finance Practice Area Education Committee.

Mr. Dardis' address is: Tillinghast—Towers Perrin, 12377 Merit Drive, Suite 1200, Dallas TX 75251. Internet address: *Dardist@Tillinghast.com*

†Vinh Loi Huynh, M.A., is a senior actuarial assistant with Employers Reassurance International Limited, an international reinsurance company. He received his B.A. and M.A. in mathematics from Cambridge University, England. Mr. Huynh's main work involves pricing life and health reassurance products. He also designs computer programs for actuarial pricing and cash flow testing. He has presented papers at the 1994 AFIR in Orlando and the Staple Inn Society of Actuaries in London.

Mr. Huynh's address is: Employers Reassurance International Limited, Portsoken House, 155-157 Monories, London EC3 N1BU, ENGLAND. Internet address: *dkerivh0@ibmmail.com*

1 Historical Overview

Both financial economists and actuaries have been involved in the development of quantitative asset allocation techniques for many years. The two major asset allocation techniques examined are immunization and mean/variance analysis.

1.1 Immunization

In 1952 Redington introduced the theory of immunization. Since then, the theory of immunization has had a profound influence on the way actuaries approach the valuation of insurance companies and their assessment of solvency. As a practical asset/liability management (ALM) model for insurance companies, immunization has had little competition to date. Tilley (1988) remarked that "a whole investment advisory business has grown up in the United States around immunization concepts."

The idea of equating the duration[1] of assets with the duration of liabilities has been used widely by insurance companies worldwide. Recently the notion of convexity (which is similar to duration, but with second derivatives replacing first derivatives) has given immunization new life. But immunization does have its limitations: (i) it has little relevance to interest-sensitive and performance-linked products; and (ii) immunization immunizes against profits as well as against loss.

Redington's ideas today may be viewed as the classical actuarial approach to ALM. The success of the Redington model as an accepted ALM tool lies in its relative simplicity and the ease with which the calculations necessary to test immunization can be made. As Buff (1989) states "if you can't compute it, you can't compute it." It is not possible to use theoretical advances unless it is feasible to execute the calculations necessary in these advances.

Actuarial research into ALM modeling was muted for many years after Redington, but the actuarial profession recently has found a new interest in the subject. Some of the most interesting work has been in the United Kingdom where pioneering stochastic investment modeling has been done by Wilkie (1986). The focus of the U.K. work, however, is in the area of solvency testing. See Hardy (1993) for an excellent example of the usage of stochastic modeling in assessing solvency.

A new concept is beginning to appear in the ALM literature in the U.S. concerning solvency in connection with ALM; this is the idea that ALM

[1] Duration of a financial instrument is a measure of its sensitivity to interest rates at various points on the yield curve.

focuses on asset/liability surplus management (ALSM). ALSM refers to ALM that focuses on the NAIC risk-based capital standards. These standards require certain minimum surplus amounts to be maintained in respect of various classifications of risk. An ALSM model might assess how well the required minimum surplus levels are likely to hold using the potential investment strategies under consideration. Hepokoski (1994) gives an excellent introduction to ALSM as an extension of ALM. In practice, an asset allocation model might view the risk-based capital standards as constraints rather than defining risk purely in terms of those standards.

1.2 Mean/Variance Analysis

The same year that Redington published his ideas on immunization, one of the most important papers of modern financial economics also was published. In 1952 Markowitz introduced the idea of asset allocation within a risk/reward tradeoff framework.

Markowitz notes that a reduction in risk, measured by the standard deviation of return on assets, could be achieved by diversification (into assets whose returns are uncorrelated) without any reduction in return. Markowitz also introduced the idea of an *efficient frontier*, which is a curve joining the risk/reward combinations of asset mixes that give the highest reward for any given level of risk.

At the time the financial world was not ready for the concept of an efficient frontier—to return to Buff's truism, computer power had not reached the stage where Markowitz's ideas could be implemented. A practical adaptation of these ideas had to wait over a decade, when Sharpe (1963) introduced the *diagonal model* that suggests that the future price of a security depends on its alpha, the market return through its beta, and a random error term, the values based on simple linear regression on historical data. This marked the birth of the now widely used capital asset pricing model.

Sharpe (1970) suggests that mean and variance alone "may suppress too much reality" and that a different utility curve may be needed to compare different portfolios of different riskiness. Risk is not necessarily the same for all investors; in Arthur's words (1989), "risk is in the eye of the beholder."

Many of Sharpe's ideas in the area of asset/liability management are summarized in *Managing Investment Portfolios* (1990). In this volume he presents the concepts of risk/reward indifference curves and states that "the optimal asset mix lies at the point at which an indifference curve is tangent to (i.e., touches but does not intersect) the curve along which the efficient investment opportunities lie." He also presents a complete ALM model for a defined benefit pension scheme in which reward is defined in terms of surplus return (equal to the change in value of surplus divided by the initial asset value) and risk is the standard deviation of the surplus return. Such a model is close to the model we use in this paper.

1.3 Other Approaches to ALM

Other variations on the efficient frontier idea experiment with constraints that can be used to narrow acceptable portfolio mixes on the efficient frontier. They also attempt to be dynamic in the sense that acceptable portfolio mixes change and reflect the particular market conditions present at any particular time.

A good example of such a model is developed by Leibowitz, Kogelman, Bader, and Dravid (1994). Looking at a one year time horizon, their model updates the asset allocation strategy whenever interest rates move. Their model does not just look at portfolios on the efficient frontier, but also introduces the constraint that portfolios must have no more than a specified probability of generating one year returns that fall below a certain level. This is incorporated by the introduction of a *shortfall line*, such that all portfolios above the line of constraint meet the maximum probability criterion.

If interest rates fall (with the equity risk premium, stock and bond volatilities, and stock/bond correlations all held constant) the entire risk/reward curve will shift down, decreasing the expected returns of all potential portfolio mixes. With the shortfall line unchanged, market conditions make all portfolios riskier in shortfall terms, and few portfolios will fall above the shortfall line. This requires revision of the bond/equity mixes previously deemed acceptable.

1.4 Objectives of this Paper

We will develop a simple model of an insurance company and use it to explore some of the basic concepts of ALM. The model is a true ALM model where the two sides of an institution's balance sheet are considered equally in setting or appraising long-term investment policy.

The nature of financial risk is briefly explored. Risk is related to the chance of not meeting the rate of return that is required to support a life insurer's liabilities, rather than the typical risk measures postulated by mean/variance models. A specific measure of risk, based on the probability of continued solvency rather than asset volatility, is introduced.

The company is assumed to invest only in domestic stocks and long-dated government bonds. Stochastic simulation is used to create a large number of future investment scenarios using historical total return data for domestic stocks and long-dated government bonds. The ability of the institution to meet its liabilities under each simulated scenario is examined for each possible mix of assets and risk characteristic. Thus, for each asset mix, the model produces a certain level of risk and a certain level of reward.

We then assess the minimum level of risk for any particular level of reward; the asset mix that produces such a level of risk is retained, and all such retained risk/reward points are plotted to create an optimal risk/reward profile. The paper demonstrates that such optimal risk/reward profiles—and hence the optimal asset allocation strategy—can vary considerably by liability profile.

2 What is Return?

Although the meaning of the term *return on assets* usually can be taken for granted in financial modeling—it is based on market value changes after allowing for positive and negative cash flows—this is not the case in an ALM model. This extra consideration arises because definitions of return and risk must be consistent.

In our paper risk is viewed as the ability of the financial institution to demonstrate, from time to time, that it is in a financially stable situation. This requires an assessment of the solvency of the institution by comparing the actual value of assets with the value of assets required for the institution to meet future liabilities.

For a U.S. life office a solvency valuation is required by regulation, and asset values in such a solvency test are prescribed by state law or by the National Association of Insurance Commissioners (NAIC). This valuation generally requires carrying assets at market values, although there are important exceptions such as amply secured bonds not in default that are written up or down in order that the value at maturity will equal the maturity value. To be consistent with the risk/solvency assessment, return must be defined in terms of return on the actuarial value of assets as carried in the solvency valuation, and these values may or may not be market values.

Our highly simplified model office is in a financially unstable condition if the office becomes insolvent, which will arise if the actual return on assets falls below the expected return on assets used in pricing the liabilities. (The expected return is the terms on which the business was sold.) In other words, risk is defined in terms of underperforming the pricing assumptions over the lifetime of the policies. No reference is made to valuation margins and capital (the reserves and, therefore, the value of assets required from year to year are based purely on the assumptions used in setting the premium rates), and the values of assets are based on market returns. In effect, risk is in terms of actual market returns underperforming expected market returns, so both risk and return need to be defined in terms of market values. The actuarial value of assets is defined as the market value of assets.

In practice, the risk of insolvency should not be judged against pricing assumptions with no valuation margins and no capital; if valuation margins and capital were incorporated, it would be more appropriate for the risk measures to be based on statutory results rather than simply on market values. Also, risk ideally should be measured not just by the probability of underperforming, but by the amount of underperformance.

3 The Model

There are four important stages in the development and exploration of the model:

- An assessment has to be made of the probability distribution of the returns on assets available to the financial institution;
- An accurate cash flow projection must be made of the future liability outgo of the financial institution;
- Using the information about the probability distribution of asset returns, large numbers of possible investment scenarios must be derived. The performance of the fund in meeting the liabilities under each scenario must be examined; and
- A large number of runs will enable an assessment of how a particular mix of the various asset classes will meet the liabilities. This assessment forms the basis for the construction of a risk/reward profile from which possible optimal asset mixes can be considered for investment policy.

The means used to explore the model is via simulation, where the simulated variable is the return on assets. Simulation is necessary because a mathematical solution to the model is too complicated. This complication is due not to the intractability of the return on assets, but is due to the other variable in the model—the risk variable—which is not necessarily a straightforward variable to handle mathematically. Even the risk variable that we adopted for this paper, although simple in concept, is difficult to express mathematically. More sophisticated definitions of risk that also incorporate constraints would pose even more of a challenge.

As a result, the simulation process starts by generating random observations for the random variable with a known distribution (or at least a distribution for which a reasonable assessment can be made) that can be used to calculate random observations for the complicated random variable. From these observations it is possible to make inferences about the distribution of the complicated variable.

We assume that the financial institution being assessed is a life office that issues a large number of level annual premium whole life policies on male lives age 50 at entry, and these policies all begin today. The only decrement is mortality, and this is assumed to accord with the Society of Actuaries (SOA) 75-80 15 Year Select and Ultimate Table (age nearest birthday). All expenses and commissions are assumed to be zero.

Let F_t be the fund at end of policy year t. The model tracks forward for each of the years for which the whole life contracts are expected to be in force and computes the following for $t = 1, 2, \ldots$:

$$F_t = (F_{t-1} + P_t)(1 + i) - C_t(1 + i)^{1/2} \tag{1}$$

where

i = Interest rate used to determine the net premium;

P_t = Net premium received at start of policy year t; and

C_t = Aggregate claims in policy year t.

Claims are assumed to occur on average in the middle of the year.

Thus, the sequence $\{F_t\}$ represents the target fund level to which the office should strive, based on an investment return equal to that assumed in the premium basis. If the actual fund falls persistently below this target fund in practice, the office is heading toward financial difficulties. It is therefore appropriate to examine the success of any particular investment policy in generating a fund size consistently at least as great as the target fund.

Let N_t be the simulated fund at the end of policy year t, then

$$N_t = (N_{t-1} + P_t)(1 + s_t) - C_t(1 + s_t)^{1/2} \tag{2}$$

where s_t is the simulated annual rate of return in policy year t; and P_t and C_t are as previously defined. Mathematically, s_t is defined as the weighted average of the simulated annual rates of return from stocks and bond, i.e.,

$$\begin{aligned} s_t &= \rho_t \times \text{Simulated Annual Return on Stocks} \\ &\quad + (1-\rho_t) \times \text{Simulated Annual Return on Bonds} \end{aligned} \tag{3}$$

where ρ_t is the proportion of assets invested in stocks during policy year t.

Thus, for example, suppose that in year 1 we have the following information:

- There is a mix of 60 percent in stocks and 40 percent in bonds;
- The annual rates of return on stocks and bonds are 0.1507 and -0.0014, respectively;
- The level net annual premium for a whole life policy covering a male age 50, face amount of \$1,000, using the SOA 75-80 15 Year Select and Ultimate Tables and assuming a rate of interest of 6 percent, is $P_1 = \$16.38$; and
- The expected claims cost for year 1 is $C_1 = 1.7$.

We can determine F_1 and N_1 as follows:

$$F_1 = (0 + 16.38) \times 1.06 - 1.7 \times (1.06)^{1/2} = \$15.61$$

$\rho_1 = 0.6$ and

$$s_1 = 0.6 \times 0.1507 + 0.4 \times -0.0014 = 8.986\%$$

yielding

$$N_1 = (0 + 16.38) \times (1 + 0.08986) - 1.7 \times (1 + 0.08986)^{1/2} = \$16.08.$$

As the simulated fund is in excess of the target fund, the office may be off to a good start.

4 The Probability Distribution of Asset Classes

The most difficult aspect of the construction of the model is determining the probability distribution of the available asset classes. To avoid complicated analysis, we consider exclusively common stocks and long-dated government bonds. In our opinion this is a reasonable starting point for any discussion of the basic asset allocation decision process for a U.S. financial institution.

We use the annual total returns for common stocks and long-term bonds compiled by Ibbotson Associates of Chicago. The data for these returns go back as far as 1926 and are shown in Table 1.

The first, and most critical, step in using this historical data is to establish the framework in which the data set can be used as a forecasting tool. The objective is to use the historical data as a basis for saying something about future returns. This raises three questions:

- How much emphasis do we place on old data?
- Do returns move randomly over time? And
- Is there a relationship between stock and bond returns?

Indeed, the question of whether the past is any indicator of the future is contentious in itself. This last assumption is not justified by either intuition or empirical evidence—it is a simplifying assumption for the purposes of the example presented in this paper. The decision on how to model returns could affect the results of the model materially.

Considerable evidence exists to justify that stock prices, like bond prices, vary inversely with interest rate movements—see Solnik (1983) and Peavy (1992) for good discussions on the subject—so that some correlation should be recognized between stock and bond returns. In order to keep the model simple and to concentrate on illustrating ideas outside those of modeling stock and bond returns, however, we employ the assumption that both stock and bond returns move randomly and independently of each other. Re-running the model to incorporate an approach that correlates successive returns in some fashion or recognizes a relationship between stock and bond returns would introduce a major layer of complexity to the modeling process.

To test the success of a fund in meeting its liabilities using any particular mix of stocks and bonds (assuming random returns), we derive a large number of potential individual investment scenarios by creating a set of random rates of return for each year for which the projection is made, where these random rates of return are based on cumulative probability distributions constructed from the historical data. The projection period extends to the year in which all policyholders are expected to have died, in this case 52 years on the basis of the SOA 75-80 table for a portfolio comprising exclusively 50 year old males.

Table 1
Annual Returns for Common Stocks
And Long-Term U.S. Government Bonds

Year	Stocks (%)	Bonds (%)	Year	Stocks (%)	Bonds (%)
1926	11.62	7.77	1959	11.96	2.26
1927	37.49	8.93	1960	0.47	13.78
1928	43.61	0.10	1961	26.89	0.97
1929	(8.42)	3.42	1962	(8.73)	6.89
1930	(24.90)	4.66	1963	22.80	1.21
1931	(43.34)	(5.31)	1964	16.48	3.51
1932	(8.19)	16.84	1965	12.45	0.71
1933	53.99	(0.07)	1966	(10.06)	3.65
1934	(1.44)	10.03	1967	23.98	(9.18)
1935	47.67	4.98	1968	11.06	(0.26)
1936	33.92	7.52	1969	(8.50)	(5.07)
1937	(35.03)	0.23	1970	4.01	12.11
1938	31.12	5.53	1971	14.31	13.23
1939	(0.41)	5.94	1972	18.98	5.69
1940	(9.78)	6.09	1973	(14.66)	(1.11)
1941	(11.59)	0.93	1974	(26.47)	4.35
1942	20.34	3.22	1975	37.20	9.20
1943	25.90	2.08	1976	23.84	16.75
1944	19.75	2.81	1977	(7.18)	(0.69)
1945	36.44	10.73	1978	6.56	(1.18)
1946	(8.07)	(0.10)	1979	18.44	(1.23)
1947	5.71	(2.62)	1980	32.42	(3.95)
1948	5.50	3.40	1981	(4.91)	1.86
1949	18.79	6.45	1982	21.41	40.36
1950	31.71	0.06	1983	22.51	0.65
1951	24.02	(3.93)	1984	6.27	15.48
1952	18.37	1.16	1985	32.16	30.97
1953	(0.99)	3.64	1986	18.47	24.53
1954	52.62	7.19	1987	5.23	(2.71)
1955	31.56	(1.29)	1988	16.81	9.67
1956	6.58	(5.59)	1989	31.49	18.11
1957	(10.78)	7.46	1990	(3.17)	6.18
1958	43.36	(6.09)	1991	30.55	19.30
			1992	7.67	8.05

Source: Ibbotson, R.G. and Sinquefield, R.A. *Stocks, Bonds, Bills and Inflation, 1993 Yearbook*. Chicago, Ill.: Ibbotson Associates, 1993. Used with permission.

5 A Note on the Number of Simulations

We need a sufficiently large number of simulations to ensure that a smooth curve can be drawn between any set of risk/reward points at a particular interest rate assumption and to accurately estimate the probability of insolvency, p. Suppose we perform n simulations and X is the number of times insolvency results, then we can estimate the probability of insolvency by $\hat{p} = X/n$. However, to ensure that our estimate has a high probability, say $1 - \epsilon$ (for small $\epsilon > 0$), of being within, say, a margin of $100\alpha\%$ of the true value p, we must ensure that n satisfies

$$\Pr[(1-\alpha)p \le X/n \le (1+\alpha)p] \ge 1 - \epsilon.$$

As n is large, we can appeal to the central limit theorem and show that the smallest value of n is given by:

$$n \ge \left(\frac{Z_{\epsilon/2}}{\alpha}\right)^2 \left(\frac{1-p}{p}\right) \tag{4}$$

where Z_y is the $100(1-y)\%$ percentage point of the standard normal distribution. Suppose p is calculated using $n = 1{,}000$ simulations and the result is $p = 0.1$. Then if $\epsilon = 0.05$, equation (4) gives the minimum value of $\alpha = 0.186$, derived from

$$1000 = \left(\frac{1.96}{\alpha}\right)^2 \left(\frac{1-0.1}{0.1}\right).$$

This is not small enough to make a credible graphical presentation of a smooth risk profile. For example, if the values of p were plotted, a smooth curve (shape) would not be achieved, but a sample of random points would result. If 25,000 simulations were used for $p = 0.1$, however, the margin α would be 0.037, which is accurate enough for graphical purpose. As a result, throughout this paper we use 25,000 simulations.

6 The Simulation Process

The life office's simulation process must be built into its liability cash flow framework. This means the target fund needs to be compared with the simulated fund in each year of projection, as derived under each simulated investment scenario. The progress of the target and simulated funds is tracked for the full expected future term of the business in force. This is repeated for various simulated stock and bond returns—as discussed in Section 5, the model has been run using 25,000 simulations—using all possible combinations of stocks and bonds in steps of 1 percent and using liability profiles based on actuarial interest rate assumptions of 0 percent, 2 percent, 4 percent, 6 percent, and 8 percent.

For each simulated investment scenario, the internal rate of return for each mix of stocks and bonds is calculated as:

$$r_n = \left[\prod_{t=1}^{n} (1 + s_t) \right]^{1/n} - 1 \tag{5}$$

where n is the projection period (in years). This return (r_n) is averaged over the 25,000 simulated scenarios to derive an expected rate of return on the fund for any particular mix of stocks and bonds. This expected rate remains the same regardless of the liability profile under consideration.

The next step is to determine how risk should be specified within the framework of the cash flow projections for any particular liability profile. This assessment is critical to the modeling process. For the purposes of this paper, our measure of risk is defined as the probability of the simulated fund being less than the target fund for *three consecutive years* during the full projection period. A three year period (rather than a one year time horizon) is chosen on the premise that if the fund has gone this amount of time in an unbalanced financial position, it may have long-term financial problems.

This definition of risk is a convenient way of assessing the real risk in the example used (the real risk is underperforming the pricing assumptions over the lifetime of the policies) because of the way in which solvency is defined (i.e., in terms of market value of assets versus liabilities valued on the basis of original pricing assumptions). This would not be the case in a more sophisticated model that incorporates valuation margins and capital in its computations. In such instances an alternative measure of underperformance against pricing assumptions may be more satisfactory, with the risk of insolvency incorporated as a constraint. In addition, risk should be measured not just by the probability of underperforming, but by the amount of underperformance.

We now formally define our measure of risk, R, algebraically as follows: For $t = 3, 4, \ldots, n$, let

$$R_t = \begin{cases} 1 & \text{if } N_{t-k} < F_{t-k} \text{ for } k = 0, 1, 2; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

R is now given as:

$$R = \Pr[\bigcup_{t=3}^{n} \{R_t = 1\}]. \tag{6}$$

The measure of risk for any particular mix of stocks and bonds is the sum of all values of R over the 25,000 simulations, divided by 25,000 to give an average probability of insolvency.

7 The Monte Carlo Sampling Method Used

In order to use Monte Carlo simulation, we first need to specify a distribution function of asset returns. In our case this function is an empirical function. If the rate of return on a particular asset class is defined as a random variable, S, then the empirical probability distribution function (pdf), $f(s)$ and the empirical cumulative probability distribution (cdf), $F(s)$, of S must be determined.

Suppose we have data on S and we construct a relative frequency histogram with m (a positive integer) distinct intervals such that a return of $s_{k-1} < S \le s_k$ occurs with relative frequency $f_k \ge 0$, for $k = 1, 2, \ldots, m$ with $\sum_1^m f_k = 1$. Following Hogg and Klugman (1984, Chapter 3), we can construct a continuous cdf using a piecewise linear approximation. First we choose a sequence of points $\{c_k\}$ such that $s_{k-1} < c_k < s_k$ for $k = 1, 2, \ldots, m-1$, and $c_0 = s_0$ and $c_m = s_m$. The c_ks do not have to be equidistant. It can easily be verified that the cdf is given by

$$F(s) = \begin{cases} 0 & \text{for } s < c_0 \\ (s - c_0)f_1/(c_1 - c_0) & c_0 \le s \le c_1 \\ \vdots & \vdots \\ (s - c_{k-1})f_k/(c_k - c_{k-1}) + \sum_{j=1}^{k-1} f_j & c_{k-1} \le s \le c_k \\ \vdots & \vdots \\ (s - c_{m-1})f_m/(c_m - c_{m-1}) + \sum_{j=1}^{m-1} f_j & c_{m-1} \le s \le c_m \\ 1 & s > c_m. \end{cases} \tag{7}$$

Having defined the cumulative distribution function of S, it is now possible to demonstrate how the random variable S is simulated (i.e., how samples of the observation of the variable S are generated). Let U be a uniform distribution on $[0, 1]$. The standard approach to generating a random variable S is as follows (see, for example, Bratley, Fox, and Schrage (1983, Chapter 5.2.2)): Suppose U_i is a random observation from U. We must determine the c_j be such that $c_j \le U_i \le c_{j+1}$. It follows that the corresponding observation from S is S_i where

$$S_i = \frac{[F(c_{j+1}) - U_i] \times c_j + [U_i - F(c_j)] \times c_{j+1}}{F(c_{j+1}) - F(c_j)}. \tag{8}$$

Table 2 shows the cumulative distributions for the two asset classes (common stocks and long-term bonds) used in the model. Table 2 is derived from the basic data of Table 1.

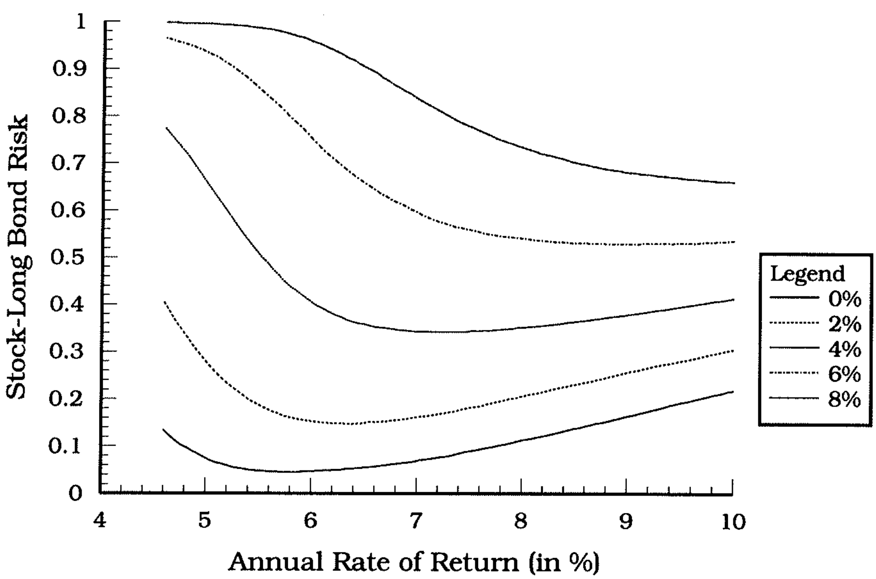
Table 2
Empirical Cumulative Distributions Functions

	Common Stocks		Long-Term Bonds	
k	c_k	$F(c_k)$	c_k	$F(c_k)$
0	-45	0.00%	-10	0.00%
1	-40	1.49%	-5	7.46%
2	-35	2.99%	0	26.87%
3	-30	2.99%	5	58.21%
4	-25	4.48%	10	80.60%
5	-20	5.97%	15	88.06%
6	-15	5.97%	20	95.52%
7	-10	11.94%	25	97.01%
8	-5	22.39%	30	97.01%
9	0	29.85%	35	98.51%
10	5	32.84%	40	98.51%
11	10	43.28%	45	100.00%
12	15	50.75%		
13	20	62.69%		
14	25	73.13%		
15	30	76.12%		
16	35	88.06%		
17	40	92.54%		
18	45	95.52%		
19	50	97.01%		
20	55	100.00%		

8 Results of the Simulation

The results for each liability profile under consideration (i.e., for each rate of interest assumption) using various combinations of stocks and bonds are summarized in Figure 1. In particular, Figure 1 shows the optimal risk/reward points for various possible combinations of stocks and bonds under the various interest rate assumptions used in pricing the underlying liabilities. A curve is drawn through the points to create a risk/reward profile for each rate of interest. All points on any line to the left of the minimum risk point can be ignored, because it is possible to achieve simultaneously a higher return and a lower risk by altering the mix of stocks and bonds.

Figure 1
Stock/Long Bond Risk/Reward Profiles



The point at which the minimum level of risk is achieved depends on the liability structure under consideration (i.e., assumed interest rate used for pricing). At a rate of interest of 2 percent the minimum risk is achieved where 32 percent of the fund is held in stocks and 68 percent of the fund is held in bonds. This minimum risk point shifts toward a heavier weighting in stocks as the rate of interest rises; at high rates of interest the minimum risk point is not achieved until 100 percent is held in stocks.

Our results are intuitive, i.e., if there is a high minimum guarantee, the office will be driven to more volatile assets (with higher potential upside and downside) because they are the only assets with a chance of outperforming the guarantee. An easy target is associated with a conservative strategy, and a difficult target is associated with a not-so-conservative strategy. Although the not-so-conservative strategy often will miss the difficult target, it nevertheless has a higher probability of exceeding the target than a more conservative strategy that never hits the target. On the other hand, if a company is adequately capitalized, a 2 percent minimum guarantee will not result in a low stock holding. Further, there is a business risk associated with not being able to offer competitive guarantees.

The curves based on the low rate of interest assumptions look like traditional efficient frontiers. This is not surprising. At relatively low rates of interest the nature of the liabilities becomes relatively unimportant, so the model reverts to the conventional asset-only model. But at relatively high rates of interest the concept of an efficient frontier collapses, and at a rate of interest of 10 percent there is only one efficient point (where 100 percent is held in stocks).

In practice the efficient frontier may be of limited use, because a life office may be required to hold certain asset categories. For example, there may be an investment policy constraint within the office that at least 50 percent of the portfolio must be held in long-term bonds. Moreover, in many countries there are legal restrictions on the extent to which certain categories of asset may be held by life offices. Thus, this paper does not concentrate on analyzing the efficient combinations of the various asset classes.

The final part of the exercise is to determine an acceptable level of risk; having decided this, it is possible to derive a uniquely defined optimal asset mix. For example, for the fund that has used a rate of interest assumption of 2 percent in its pricing assumptions, it may be appropriate to go 100 percent into stocks (and therefore go for the maximum possible return) if a probability of insolvency level of around 30 percent were deemed acceptable.

Setting an acceptable level of risk is a largely subjective decision, and in practice the usefulness of this model is in assessing the relative riskiness of various portfolio mixes rather than in making any sense of the absolute values generated for the risk and reward of any particular investment policy in isolation. The absolute values for the probability of insolvency in the model look extremely high across the board, the result of the relatively large probability of a market crash in any one investment scenario. See Hardy (1993) for similar findings when using a stochastic model.

A highly artificial liability profile is being considered, and the fact that the results drive toward a higher stock allocation than most companies hold in practice indicate the failings of the simplified model. A significant shortfall of the model is that the RBC implications of any particular asset allocation recommendation have not been considered; in practice, this would be a major constraint on the asset allocation decision.

Because the example in this paper has been highly simplified, the liability structure is expressed entirely in terms of a pricing interest rate. Similar efficient frontiers could be created by merely comparing the simulated portfolio returns to the 0 percent to 8 percent pricing rates.

Finally, the complete asset allocation model should incorporate the full range of assets available to the financial institution, which for a life office should include cash, property, and overseas stock and bond investment in addition to domestic stocks and bonds.

9 Summary and Conclusion

This paper describes an approach to asset allocation modeling for institutions that invest to meet liabilities. The model is consistent with conventional financial economics. Traditional risk/reward profiles become apparent where the nature of the liabilities is not considered or is relatively unimportant, but such traditional risk/reward profiles may or may not become apparent once the nature of the liabilities is introduced. Thus, the traditional ideas of financial economics have been shown to be a special case of the more general asset allocation system using a true ALM model.

We have concentrated exclusively on the applications of an ALM model in the context of a highly simplified life office issuing purely nonparticipating whole life assurance. The principles can be applied equally, however, to any financial institution that is concerned with investing to meet liabilities.

The critical element of the model is the definition of risk. It is not important that risk is taken as some measure of exposure to insolvency, but that it incorporates the liabilities.

Refinement of the model to incorporate the features of participating business should not be problematic; this would be akin to lowering the rate of interest assumption used in pricing the liabilities which implicitly means a general reduction in the risk profile and, hence, potentially greater freedom in investment policy.

The application of our model to a pension fund poses some interesting issues, although these issues are specific to the particular country under consideration. Although it is recognized that pension funds can overcome deficit situations by increasing contribution rates from time to time, pension fund trustees may be interested in knowing whether a particular investment policy is more likely to lead to persistent deficits. Alternatively, if a primary objective of pension fund investment policy is to avoid unduly fluctuating contribution rates, then the ALM model could use a refined definition of risk (say, the probability of the fund falling outside a certain surplus or deficit range).

Incorporating the inflationary aspects of a pension fund model is problematic. Possible approaches include linking inflation to the yield curve or stochastically modeling inflation as an independent variable. It is not obvious which of the two approaches may be more appropriate; perhaps, given the major uncertainties associated with inflation, there is no definitive model. Some innovative research in this area has been done by Wilkie (1986).

There remains much exciting asset/liability modeling work to be done. Dramatic developments can be expected as microcomputer processing power becomes more widely appreciated and makes the type of stochastic model described in this paper a standard tool of financial analysis.

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