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Nonseparability and steerability of two-qubit states from the geometry of steering outcomes

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When two qubits, A and B, are in an appropriate state, Alice can remotely steer Bob's system B into different ensembles by making different measurements on A. This famous phenomenon is known as quantum steering, or Einstein-Podolsky-Rosen steering. Importantly, quantum steering establishes the correspondence not only between a measurement on A (made by Alice) and an ensemble of B (owned by Bob) but also between each of Alice's measurement outcomes and an unnormalized conditional state of Bob's system. The unnormalized conditional states of B corresponding to all possible measurement outcomes of Alice are called Alice's steering outcomes. We show that, surprisingly, the four-dimensional geometry of Alice's steering outcomes completely determines both the nonseparability of the two-qubit state and its steerability from her side. Consequently, the problem of classifying two-qubit states into nonseparable and steerable classes is equivalent to geometrically classifying certain four-dimensional skewed double cones.

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I. INTRODUCTION

Quantum steering, or Einstein-Podolsky-Rosen (EPR) steering, arose from the first discussion on the nonlocal nature of quantum mechanics [1,2]. Subsequent attempts to clarify the notion of nonlocality have led to the discovery of different classes of quantum nonlocality. Although Bell nonlocality [3] and nonseparability [4] were discussed rather early as two distinct classes of quantum nonlocality, only recently has steerability been defined [5,6].

One of the key concepts to define quantum steerability is the *assemblage*. An assemblage is a set of ensembles that give rise to the same quantum state. Now consider the case where Alice and Bob share a bipartite system AB. Although the no-signaling theorem prevents Alice from affecting the reduced state of Bob's system B from a distance [7,8], she can remotely steer it into different ensembles by performing different measurements on her own system A [1,2]. These different ensembles of B form a certain assemblage: the steering assemblage. However, as Wiseman *et al.* [5] pointed out, if this steering assemblage is too restrictive in some sense, Alice may never convince Bob that she is actually steering his system remotely, in which case the state is unsteerable from her side.

Shortly after steerability was defined, sufficient conditions for a state to be steerable were developed in terms of steering inequalities [9–14]. Characterizing steerability has quickly gone beyond inequalities, and multiple relationships to other concepts of quantum physics have been discovered. Steerability was shown to be equivalent to joint measurability in [15,16]. Quantum steering in time was discussed by Chen *et al.* [17], and a close relationship between steerability and quantum-subchannel discrimination has been established [18]. When steerability was realized as a resource for quantum information tasks, quantifying steerability naturally appeared as an important problem [19].

The concept of steerability is based on the correspondence between a measurement of Alice and an ensemble of Bob's system. However, quantum steering establishes a more elementary and much simpler correspondence: each of Alice's measurement outcomes corresponds to a conditional state of Bob's system. When Alice gets a particular measurement outcome, the unnormalized conditional state of B is determined regardless of which measurement that particular outcome belongs to. The unnormalized conditional states of Bob's system corresponding to all possible measurement outcomes of Alice are referred to as Alice's steering outcomes. We show that, surprisingly, the four-dimensional (4D) geometry of Alice's steering outcomes completely determines both the nonseparability of the two-qubit state [4] and its steerability from her side [5] (see Propositions 1 and 2). Thus, the problem of classifying two-qubit states into nonseparable and steerable classes is conceptually simplified to classifying certain 4D skewed double cones.

Although most of the definitions and many statements in this paper can be naturally generalized to higher-dimensional systems, there are certain aspects of two-qubit systems that make the statements particularly simple and transparent. We thus restrict our analysis here to two-qubit systems and wish to discuss higher-dimensional ones elsewhere.

II. EPR MAPS AND STEERING OUTCOMES

Let us consider a qubit described by a two-dimensional (2D) Hilbert space. The Hermitian operators acting on $\mathscr{H}$ with the Hilbert-Schmidt inner product $(A,B) \to \mathrm{Tr}(A^{\dagger}B)$ form a Euclidean space, denoted by $B^H(\mathscr{H})$ [20,21]. Fix an orthogonal basis of $\mathscr{H}$. Letting $\sigma_0 = \mathbb{I}$ be the identity matrix and $\{\sigma_i\}_{i=1}^3$ be the three Pauli matrices, then every Hermitian operator A acting on $\mathscr{H}$ can be written as $A = \frac{1}{2}\sum_{i=0}^3 X_i(A)\sigma_i$, where $X_i(A) = \mathrm{Tr}(A\sigma_i)$. This coordinate system $\{X_i\}_{i=0}^3$ allows one

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to identify $B^H(\mathscr{H})$ with the Euclidean space $\mathbb{R}^4$. In particular, the zero operator O is identified with $(0,0,0,0)^T$, and the identity operator $\mathbb{I}$ is identified with $(2,0,0,0)^T$.

Of particular interest to us is the cone of positive (semidefinite) operators $\mathscr{M}^+ = \{M|0 \leqslant M\}$. In terms of the Euclidean coordinates, $\mathscr{M}^+ = \{M|X_0(M) \geqslant 0, X_0(M)^2 \geqslant X_1(M)^2 + X_2(M)^2 + X_3(M)^2\}$ and thus is also called the forward light cone at the origin O, terminology borrowed from special relativity [22]. Another object of interest to us is the set of *measurement outcomes* $\mathscr{M} = \mathscr{M}^+ \cap \mathscr{M}^-$, where $\mathscr{M}^- = \{M|\mathbb{I} \geqslant M\}$. It is easy to see that $\mathscr{M}^-$ is the backward light cone at $\mathbb{I}$. Thus, $\mathscr{M}$ is a double cone formed by the intersection of the forward light cone at O and the backward light cone at $\mathbb{I}$. Finally, the three-hyperplane $\mathscr{P} = \{M|X_0(M) = 1\}$ is called the Bloch hyperplane, and $\mathscr{S} = \mathscr{M} \cap \mathscr{P}$ is known as the Bloch ball [21,23].

A system of two qubits AB is described by the tensor product $\mathscr{H}_A \otimes \mathscr{H}_B$, where $\mathscr{H}_A$ and $\mathscr{H}_B$ are 2D Hilbert spaces. Operators acting on $\mathscr{H}_A$ will be denoted by A_i or labeled by a superscript or subscript A, for example, $\mathbb{I}^A$ or σ_i^A; an analogous convention is applied to B. Let ρ be a density operator, or state, of the system, that is, a positive unit-trace operator on $\mathscr{H}_A \otimes \mathscr{H}_B$. Using Pauli matrices, a density operator can be written as $\rho = \frac{1}{4}\sum_{i,j=0}^{3} \Theta_{ij}(\rho)\sigma_i^A \otimes \sigma_j^B$, where $\Theta_{ij}(\rho) = \mathrm{Tr}[\rho(\sigma_i^A \otimes \sigma_j^B)]$.

Now suppose Alice owns part A and Bob owns part B of the system. A positive operator-valued measurement (POVM) performed on A is a decomposition of the identity operator $\mathbb{I}^A$ into some members $\{E_i^A\}_{i=1}^n$ of the measurement outcomes $\mathscr{M}_A$, $\mathbb{I}^A = \sum_{i=1}^n E_i^A$. If Alice gets a measurement outcome E_i^A, the unnormalized state of Bob's system will be found to be $\rho_i^B = \mathrm{Tr}_A[\rho(E_i^A \otimes \mathbb{I}^B)]$, where Tr_A denotes the partial trace operation over subsystem A [21]. Note that $\mathrm{Tr}(\rho_i^B)$ is the probability of observing the measurement outcome E_i^A. Now a key observation is that this correspondence between E_i^A and ρ_i^B is independent of the POVM that contains the outcome E_i^A. The correspondence establishes a map, called Alice's *EPR map*, not to be confused with the steering map as defined in [24].

More precisely, the EPR map $\rho^{A\to B}$ of a state ρ is a positive linear map $\rho^{A\to B} : B^H(\mathscr{H}_A) \to B^H(\mathscr{H}_B)$ defined by $\rho^{A\to B}(A) = \mathrm{Tr}_A[\rho(A \otimes \mathbb{I}^B)]$. If U_A is an element or a subset of $B^H(\mathscr{H}_A)$, we denote its image under the EPR map by $U_A' = \rho^{A\to B}(U_A)$. In particular, $\mathscr{M}_A'$ is called Alice's *steering outcomes*. Bob's EPR map and Bob's steering outcomes are defined analogously. In fact, Alice's EPR map as defined is simply the inverse of the so-called Pillis-Jamiołkowski isomorphism, which maps a linear map between two operator spaces to an operator acting on their tensor product [25,26] (see also [27]).

If we use the Euclidean coordinates $\{X_i\}_{i=0}^3$ to represent the operators of $B^H(\mathscr{H}_A)$ and $B^H(\mathscr{H}_B)$, the EPR map $\rho^{A\to B}$ is simply a map from $\mathbb{R}^4$ to $\mathbb{R}^4$. More explicitly, it is easy to show that

$$X_i(A') = \frac{1}{2}\sum_{j=0}^{3} \Theta_{ij}^T(\rho) X_j(A). \quad (1)$$

Figure 1 illustrates a three-dimensional (3D) cross section of Alice's steering outcomes $\mathscr{M}_A'$ relative to Bob's positive cone $\mathscr{M}_B^+$. One observes that $\mathscr{M}_A'$ is a skewed double cone with two vertices at O and $(\mathbb{I}^A)' = \mathrm{Tr}_A(\rho) \in \mathscr{S}_B$. The latter is also known as the reduced state of B [21]. In the following, we show that the geometry of Alice's steering outcomes $\mathscr{M}_A'$ determines the nonseparability and steerability of the state from Alice's side. The projective projection [28] of $\mathscr{M}_A'$ through the origin O onto Bob's 3D Bloch hyperplane (see Fig. 1), known as Alice's *steering ellipsoid*, has been studied in detail [22,29,30]. As a result, the nested tetrahedron criterion for separability, which states that a state is separable if and only if Alice's steering ellipsoid is contained in a tetrahedron that fits in Bob's Bloch ball, has been discovered [22]. However, the criterion appears to us to be a mysterious fact. We will state a criterion for separability in terms of the 4D geometry of $\mathscr{M}_A'$ which demystifies the nested tetrahedron criterion. Moreover, this also provides a transition to studying the geometrical nature of steerability.

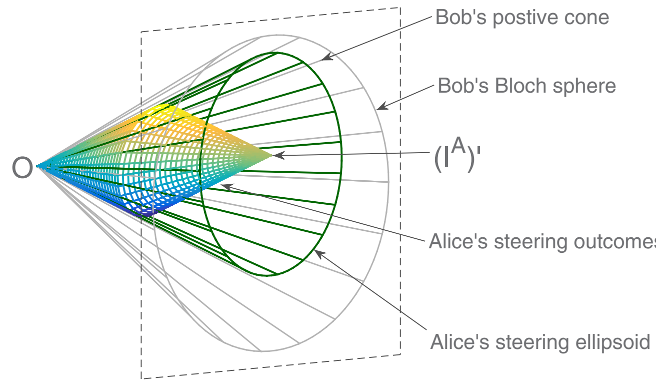

FIG. 1. Three-dimensional representation of Alice's steering outcomes $\mathscr{M}_A'$ inside Bob's positive light cone $\mathscr{M}_B^+$ together with their projective projections on the Bob's Bloch hyperplane.

A. Polyhedral boxes, packing, and packability

To study the geometry of Alice's steering outcomes, we introduce the concepts of polyhedral box and packability. Let $\mathscr{U}_B = \{B_i\}_{i=1}^m$ be a set of m Hermitian operators acting on $\mathscr{H}_B$, or, equivalently, vectors of $\mathbb{R}^4$. As in standard convex analysis [31], the set $\mathrm{cone}(\mathscr{U}_B) = \{B = \sum_{i=1}^m \alpha_i B_i | \alpha_i \geqslant 0\}$ is called the conical hull based on $\mathscr{U}_B$. Further, we define the *polyhedral box* based on $\mathscr{U}_B$ to be $\mathrm{box}(\mathscr{U}_B) = \{\sum_{i=1}^m \beta_i B_i | 0 \leqslant \beta_i \leqslant 1\}$. Such a polyhedral box can also be seen as a linear image in the 4D space of the unit m-cube. The vertex $\sum_{i=1}^m B_i$ is called the *principal vertex*. The set of Alice's steering outcomes $\mathscr{M}_A'$ is called *m-packable* if it is contained in a polyhedral box with the principal vertex at $(\mathbb{I}^A)'$. The set of steering outcomes $\mathscr{M}_A'$ is called *packable* if it is m-packable for some m.

B. Separability

A state ρ over AB is called *separable* if it can be written as a convex combination of some s product states $\{\rho_i^A \otimes \rho_i^B\}_{i=1}^s$, i.e., $\rho = \sum_{i=1}^s p_i \rho_i^A \otimes \rho_i^B$, where $0 \leqslant p_i \leqslant 1$, $\sum_{i=1}^s p_i = 1$ [4]. For a two-qubit system, any separable state can be written in this form with $s \leqslant 4$ [32]. The following proposition reveals a surprising connection between separability and packability.

Proposition 1. A two-qubit state ρ is separable if and only if the set of Alice's steering outcomes $\mathscr{M}_A'$ is four-packable.

If the set of Alice's steering outcomes is four-packable by linearly dependent operators, then Alice's steering ellipsoid is necessarily degenerate, and the state is separable [22,29]. Thus, in Proposition 1 the four vectors that form the polyhedral box to pack Alice's steering outcomes can be assumed to be linearly independent. In the Appendix, we show that this proposition is equivalent to the nested tetrahedron criterion for separability. The key to this equivalence is that the packability of $\mathscr{M}_A'$ by linearly independent positive operators is fully characterized by the cone $(\mathscr{M}_A^+)'$ (Appendix, Lemma 1). The cone $(\mathscr{M}_A^+)'$ in turn can be characterized by its projective projection on Bob's Bloch hyperplane, the steering ellipsoid (Appendix, Lemma 2). This is no longer true for packability with linearly dependent operators, in particular, for m-packability with $m > 4$. It then becomes clear that the limit number of $m = 4$ in Proposition 1, or the notion of tetrahedron in the nested tetrahedron criterion, appears due to the fact that it is the maximum number of linearly independent operators in $\mathscr{B}^H(\mathscr{H}_B)$. Moreover, it also suggests that steerability, which is equivalent to m-packability with m possibly bigger than 4, as stated in Proposition 2, is of fundamentally 4D geometry and cannot be seen completely in the projective projection of Alice's steering outcomes on Bob's Bloch hyperplane.

C. Steerability

A subset $\mathscr{C}$ of all POVMs on a system is also called a measurement class $\mathscr{C}$. Relevant classes of measurements are projective measurements, where the measurement outcomes are orthogonal projections [21], and binary-outcome POVMs. For a qubit, the former is a subclass of the latter.

Following [5], a state ρ is called *unsteerable* from Alice's side with respect to measurements of class $\mathscr{C}^A$ if there exists a decomposition of $(\mathbb{I}^A)' = \mathrm{Tr}_A(\rho)$ into an ensemble of m positive operators of $\mathscr{H}_B$, $(\mathbb{I}^A)' = \sum_{i=1}^m B_i$, satisfying the following condition. For any measurement with n outcomes $\{E_i^A\}_{i=1}^n$ of class $\mathscr{C}^A$ performed by Alice, the corresponding conditional ensemble of Bob's system B, $\{(E_i^A)' = \mathrm{Tr}_A[\rho(E_i^A \otimes \mathbb{I}^B)]\}_{i=1}^n$, can be expressed by a stochastic map from $\{B_i\}_{i=1}^m$ to $\{(E_i^A)'\}_{i=1}^n$, i.e.,

$$\left(E_i^A\right)' = \sum_{j=1}^m G_{ij} B_j, \tag{2}$$

where G is a stochastic matrix, $0 \leqslant G_{ij} \leqslant 1$, $\sum_{i=1}^n G_{ij} = 1$. The ensemble $\{B_i\}_{i=1}^m$ is called Bob's ensemble of *local hidden states* (LHSs), which together with the stochastic map G allow Alice to locally simulate steering [5].

Determining the steerability of a state is a hard problem when one considers all possible POVMs [33]. Most approaches are restricted to projective measurements. Nevertheless, for a system of two qubits, steerability with respect to all projective measurements is equivalent to steerability with respect to binary-outcome POVMs (Appendix, Lemma 3). The following proposition subsequently shows that the steerability from Alice's side with respect to binary outcome POVMs is completely determined by the geometry of her steering outcomes.

Proposition 2. A two-qubit state ρ is unsteerable from Alice's side for all binary outcome POVMs if and only if the set of Alice's steering outcomes $\mathscr{M}_A'$ is packable.

This is a nontrivial statement; generalizations to all POVMs and to higher-dimensional systems are open problems with subtle difficulties. In the following, steerability will always be considered with respect to projective measurements.

Practically, Proposition 2 simplifies the problem of determining the steerability of two-qubit states. To find a necessary condition for steerability, one can choose some ansatz for the base $\mathscr{U}_B$ and check if the steering outcomes $\mathscr{M}_A'$ stay within the polyhedral box based on $\mathscr{U}_B$, in which case the state is unsteerable from Alice's side. Although an ansatz for $\mathscr{U}_B$ can also be considered as an ansatz for Bob's ensemble of LHSs, this approach shows that a given ansatz naturally generates a necessary condition for steerability for any state with Bob's reduced state at the principal vertex of box$(\mathscr{U}_B)$; in that sense, an ansatz for Bob's ensemble of LHSs can be fully exploited. The main task in this procedure is to determine the boundary of the polyhedral box for a given ansatz $\mathscr{U}_B$, which is the subject of the following section.

D. Determining the boundary of polyhedral boxes

An ansatz $\mathscr{U}_B$ can always be chosen such that its vectors are on the boundary of $\mathscr{M}_B^+$. Indeed, if a vector of $\mathscr{U}_B$ is not on the boundary of $\mathscr{M}_B^+$, one can decompose it into a sum of two vectors on the boundary of $\mathscr{M}_B^+$ to form a new ansatz, whose polyhedral box contains the polyhedral box of the old ansatz. Each vector of such an ansatz is of the form $u_i \binom{1}{\boldsymbol{n}_i}$, where u_i determines its length and $\boldsymbol{n_i}$ is a 3D unit vector. More generally, each ansatz can be characterized by a distribution $u(\boldsymbol{n})$ on the 3D unit sphere.

Any vector of box$(\mathscr{U}_B)$ is of the form $\int d\mu(\boldsymbol{n}) f(\boldsymbol{n}) \binom{1}{\boldsymbol{n}}$, where $0 \leqslant f(\boldsymbol{n}) \leqslant 1$ and $d\mu(\boldsymbol{n})$ denotes the measure on the 3D unit sphere generated by the distribution u. In particular, the principal vertex is $\int d\mu(\boldsymbol{n}) \binom{1}{\boldsymbol{n}}$. This principal vertex should be on Bob's Bloch hyperplane; thus, one has a normalization condition $\int d\mu(\boldsymbol{n}) = 1$.

The cross section of box$(\mathscr{U}_B)$ at some hyperplane $X_0 = x_0$, denoted box$(\mathscr{U}_B)|_{x_0}$, consists of vectors $\boldsymbol{b} = \int d\mu(\boldsymbol{n}) f(\boldsymbol{n}) \boldsymbol{n}$, with $0 \leqslant f(\boldsymbol{n}) \leqslant 1$ and $x_0 = \int d\mu(\boldsymbol{n}) f(\boldsymbol{n})$. To determine the boundary of box$(\mathscr{U}_B)|_{x_0}$, we choose a direction $\boldsymbol{n}_0$ on the 3D unit sphere and project it onto that direction. The extreme point of the projection of the convex set box$(\mathscr{U}_B)|_{x_0}$ must be the projection of a point on its boundary. We are thus led to solving the optimization problem

$$\max_{0 \leqslant f(\boldsymbol{n}) \leqslant 1} \int d\mu(\boldsymbol{n}) f(\boldsymbol{n}) \boldsymbol{n}_0^T \boldsymbol{n}, \tag{3}$$

with constraint $x_0 = \int d\mu(\boldsymbol{n}) f(\boldsymbol{n})$. Using the method of Lagrange multipliers, we consider the modified objective function

$$L(f,\lambda) = \int d\mu(\boldsymbol{n})\big[f(\boldsymbol{n})\big(\boldsymbol{n}_0^T\boldsymbol{n} - \lambda\big) + \lambda x_0\big]. \quad (4)$$

When λ is fixed, L obtains its extremal value at the functions f of the form $f_{\boldsymbol{n}_0}(\boldsymbol{n}) = 1_{\boldsymbol{n}_0^T\boldsymbol{n}>\lambda}(\boldsymbol{n}) + g(\boldsymbol{n})1_{\boldsymbol{n}_0^T\boldsymbol{n}=\lambda}(\boldsymbol{n})$, where g is any function taking values in $[0,1]$ and 1_X denotes the indicator function of a set X, $1_X(\boldsymbol{n}) = 1$ if $\boldsymbol{n} \in X$ and $1_X(\boldsymbol{n}) = 0$ if $\boldsymbol{n} \notin X$. Each solution $f_{\boldsymbol{n}_0}(\boldsymbol{n})$ then gives a point on the boundary of box$(\mathscr{U}_B)$ parametrized by $\lambda,\boldsymbol{n}_0$, and g,

$$x_0 = \int d\mu(\boldsymbol{n})\big[1_{\boldsymbol{n}_0^T\boldsymbol{n}>\lambda}(\boldsymbol{n}) + g(\boldsymbol{n})1_{\boldsymbol{n}_0^T\boldsymbol{n}=\lambda}(\boldsymbol{n})\big], \quad (5)$$

$$\boldsymbol{b} = \int d\mu(\boldsymbol{n})\big[1_{\boldsymbol{n}_0^T\boldsymbol{n}>\lambda}(\boldsymbol{n}) + g(\boldsymbol{n})1_{\boldsymbol{n}_0^T\boldsymbol{n}=\lambda}(\boldsymbol{n})\big]\boldsymbol{n}. \quad (6)$$

In the case μ is sufficiently fine (e.g., u is continuous), the latter terms are integrals over zero-measure sets and thus vanish, and g is irrelevant. In the other cases (e.g., u has δ peaks), $\boldsymbol{n}_0^T\boldsymbol{n} = \lambda$ may be of nonzero measure. In fact, in these cases, box$(\mathscr{U}_B)|_{x_0}$ may have degenerate flat regions, and g allows one to get all the points on these flat regions.

The simplest case where these integrals can be calculated explicitly is when $u(\boldsymbol{n})$ is a uniform distribution. In this case, μ is fine, and g is irrelevant. In fact, the cross section of box $(\mathscr{U}_B)$ at $X_0 = x_0$ is a ball of radius $r_{\text{uni.}}(x_0) = x_0(1-x_0)$. Since in this case the principal vertex of box$(\mathscr{U}_B)$ is at the center of Bob's Bloch ball, this ansatz can be used to find a necessary condition for steerability for any state that has Bob's reduced state completely mixed.

E. Example: Werner states and their modification

Werner states are defined by

$$W^p = p|\Phi^+\rangle\langle\Phi^+| + (1-p)\frac{\mathbb{I}^A}{2}\otimes\frac{\mathbb{I}^B}{2}, \quad (7)$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$ is one of the Bell states and $0 \leqslant p \leqslant 1$ [4]. Using $\Theta_{ij}(W^p) = \text{Tr}[W^p(\sigma_i^A \otimes \sigma_j^B)]$, one finds the matrix presentation of Alice's EPR map, $\frac{1}{2}\Theta^T(W^p) = \frac{1}{2}\,\text{diag}(1,p,-p,p)$. The EPR map contracts the X_0 axis by a factor of $\frac{1}{2}$ and every other axis by a factor of $\frac{p}{2}$. Although the X_2 axis is also reflected, this is irrelevant to the geometry of $\mathscr{M}'_A$. When $p=1$, the Werner state is pure, and $\mathscr{M}'_A$ touches the boundary of $\mathscr{M}_B^+$. On the other hand, when $p=0$, the Werner state is the product of two completely mixed states, and $\mathscr{M}'_A$ shrinks to a single line segment.

Elementary geometry shows that the set of steering outcomes $\mathscr{M}'_A$ is four-packable, or the Werner state is separable, if and only if $p \leqslant \frac{1}{3}$. To find a sufficient condition for the Werner state to be unsteerable we use the uniform ansatz for $\mathscr{U}_B$. In fact, due to the spherical symmetry of Werner states, it is easy to see that this condition is also necessary [5]. The boundary of box$(\mathscr{U}_B)$ is illustrated in Fig. 2 together with the boundary of $\mathscr{M}'_A$. One finds that for $\mathscr{M}'_A$ to stay within this boundary, one needs $\frac{p}{2} \leqslant r_{\text{uni.}}(\frac{1}{2})$, or $p \leqslant \frac{1}{2}$. We thus recovered the well-known results for Werner states regarding their nonseparability and steerability [4,5].

An advantage of using the 4D geometrical description is that the boundary of box$(\mathscr{U}_B)$, once determined, can be used to find a necessary condition for other states to be steerable as well. As an example, we consider the following modified Werner states:

$$\tilde{W}_q^p = p|\Phi^+\rangle\langle\Phi^+| + (1-p)\frac{\mathbb{I}^A + q\sigma_z^A}{2}\otimes\frac{\mathbb{I}^B}{2}, \quad (8)$$

with $0 \leqslant |q| \leqslant 1$. One notices that $\text{Tr}_A(\tilde{W}_p^q) = \frac{\mathbb{I}^B}{2}$; thus, the uniform ansatz is valid. The matrix for the EPR map is $\frac{1}{2}\Theta^T(\tilde{W}_q^p) = \frac{1}{2}[\text{diag}(1,p,-p,p) + q(1-p)\delta_{1,4}]$, where $\delta_{i,j}$ is the Kronecker matrix. The boundary of the steering outcomes $\mathscr{M}'_A$ for this state is also illustrated in Fig. 2. One easily finds that the condition for $\mathscr{M}'_A$ to be contained in box$(\mathscr{U}_B)$, which implies the unsteerability of the Werner state, is $\frac{\sqrt{1-2p}}{1-p} \geqslant |q|$. Although this inequality can also be deduced from a recent result of Bowles *et al.* [34], we have arrived at it simply based on the geometry of steering outcomes.

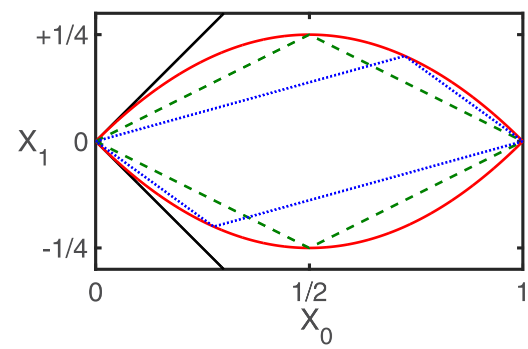

FIG. 2. Two-dimensional cross sections of the boundaries of the polyhedral box based on the uniform ansatz (red solid lines), the steering outcomes of the Werner state at $p = \frac{1}{2}$ (green dashed lines), and the steering outcomes of the modified Werner state at $p = 0.4$, $q \approx 0.75$ (blue dotted lines). The outermost black lines present the cross section of the forward light cone at the origin.

III. CONCLUSION

By defining EPR maps, we are able to map the properties of a joint state of two qubits, namely, nonseparability and steerability, to the geometrical properties of steering outcomes. On the one hand, our analysis clarifies the nested tetrahedron criterion for separability. On the other hand, we establish a general framework to determine the necessary condition for steerability. That this framework allows one to show the optimality of a LHS model will be discussed in a subsequent work [35]. Our work further opens new interesting questions. Although steerability with binary POVMs is a geometrical property, it remains to be clarified if this is still true for general POVMs. The question whether Bell nonlocality can be reduced to the geometry of steering outcomes is also to be explored.

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APPENDIX

In this appendix, we provide details for the claims stated in the main text.

1. Separability

In this section, we give the proof of Proposition 1, which says that a state is separable if and only if Alice's steering outcomes are four-packable.

Proposition 1. A two-qubit state ρ is separable if and only if the set of Alice's steering outcomes $\mathscr{M}_A'$ is four-packable.

From reasoning in the main text, we can assume that Alice's steering outcomes is four-packable with four linearly independent vectors. The proposition then follows directly from the following lemmas.

Lemma 1. The set of steering outcomes $\mathscr{M}_A'$ is four-packable with four linearly independent positive operators if and only if the cone $(\mathscr{M}_A^+)'$ is contained in a conical hull of four linearly independent positive operators.

Proof. Assume that $\mathscr{M}_A'$ is four-packable by a set of four linearly independent vectors $\mathscr{U}_B = \{B_i | 0 \leqslant i \leqslant 4\}$. For each i, denote H_i as the hyperplane spanned by $\mathscr{U}_B \setminus \{B_i\}$. Denote H_i^+ as the half-space divided by H_i that contains B_i. Since $\mathscr{M}_A' \subseteq \text{box}(\mathscr{U}_B)$, it is also contained in H_i^+. This implies that $(\mathscr{M}_A^+)' = \text{cone}(\mathscr{M}_A') \subseteq \text{cone}(H_i^+) = H_i^+$ for every i, or $(\mathscr{M}_A^+)' \subseteq \cap_{i=1}^4 H_i^+$. On the other hand, for linearly independent operators B_i, one has $\cap_{i=1}^4 H_i^+ = \text{cone}(\mathscr{U}_B)$ and thus $(\mathscr{M}_A^+)' \subseteq \text{cone}(\mathscr{U}_B)$.

For the reverse direction, we assume that the cone $(\mathscr{M}_A^+)'$ is contained in a conical hull, $\text{cone}(\mathscr{U}_B)$, formed by four linearly independent positive operators $B_i, 1 \leqslant i \leqslant 4$. Note that $\frac{1}{2}(\mathbb{I}^A)'$ is the center of symmetry of $\mathscr{M}_A'$. Denote $\text{cone}(\mathscr{U}_B)^-$ as the reflection of $\text{cone}(\mathscr{U}_B)$ through $\frac{1}{2}(\mathbb{I}^A)'$. Since $(\mathscr{M}_A^-)'$ is the reflection of $(\mathscr{M}_A^+)'$ through $\frac{1}{2}(\mathbb{I}^A)'$, we deduce that $(\mathscr{M}_A^-)' \subseteq \text{cone}(\mathscr{U}_B)^-$. Thus, $\mathscr{M}_A' = (\mathscr{M}_A^+)' \cap (\mathscr{M}_A^-)'$ is contained in $\text{cone}(\mathscr{U}_B) \cap \text{cone}(\mathscr{U}_B)^-$. When B_i are linearly independent, the latter is a polyhedral box based on $\tilde{\mathscr{U}}_B = \{\tilde{B}_i\}_{i=1}^4$, where $\tilde{B}_i = \lambda_i B_i$, with $\lambda_i = \max\{\lambda | \lambda B_i \in \text{cone}(\mathscr{U}_B)^-\}$.

Lemma 2. The cone $(\mathscr{M}_A^+)'$ is contained in a conical hull formed by four linearly independent positive operators if and only if $\mathscr{S}_A'$ is contained in a tetrahedron which is contained in Bob's Bloch ball $\mathscr{S}_B$.

Proof. Assume that $(\mathscr{M}_A^+)'$ is contained in the conical hull based on $\mathscr{U}_B = \{B_i | 1 \leqslant i \leqslant 4\}$. Let $\mathscr{P}_B$ denote Bob's Bloch hyperplane $X_0(M) = 1$; then $(\mathscr{M}_A^+)' \cap \mathscr{P}_B = \mathscr{S}_A' \subseteq \text{cone}(\mathscr{U}_B) \cap \mathscr{P}_B$. Since B_i are linearly independent, $\text{cone}(\mathscr{U}_B) \cap \mathscr{P}_B$ is a tetrahedron. Moreover, this tetrahedron is contained in Bob's Bloch ball $\mathscr{S}_B$ because $\mathscr{U}_B$ consists of positive operators.

For the reverse direction, assume that $\mathscr{S}_A'$ is contained in a tetrahedron which is the convex hull of $\mathscr{U}_B = \{B_i | 1 \leqslant i \leqslant 4\}$. The operators B_i are linearly independent; otherwise, the tetrahedron is degenerate. Now, we have $(\mathscr{M}_A^+)' = \text{cone}(\mathscr{S}_A') \subseteq \text{cone}(\mathscr{U}_B)$, which is the required condition.

2. Steerability

We first show that for a two-qubit state, steerability with respect to binary-outcome POVMs is equivalent to steerability with respect to all projective measurements.

Lemma 3. A two-qubit state ρ is unsteerable from Alice's side with respect to all projective measurements if and only if it is unsteerable from Alice's side with respect to all binary-outcome POVMs.

Proof. It is obvious that unsteerability with respect to binary outcome POVMs implies unsteerability with projective measurements. We prove the converse statement. Suppose a state ρ is unsteerable with respect to all projective measurements from Alice's side and $\mathscr{U}_B = \{B_i\}_{i=1}^m$ is an ensemble of LHSs. Suppose Alice makes a binary POVM $\mathbb{I}^A = E_1^A + E_2^A$. Consider the spectral decomposition of E_1^A, $E_1^A = H_{11} P_1^A + H_{12} P_2^A$, where P_1^A and P_2^A form a complete set of two orthogonal projections and H_{11} and H_{12} are positive eigenvalues of E_1^A. Then the spectral decomposition of E_2^A is $E_2^A = (1 - H_{11}) P_1^A + (1 - H_{12}) P_2^A$. That is to say, $E_i^A = \sum_{j=1}^2 H_{ij} P_j^A$, where H is a (2×2) stochastic matrix with $H_{21} = 1 - H_{11}$, $H_{22} = 1 - H_{12}$. Since P_1^A and P_2^A constitute a projective measurement by Alice and since $\mathscr{U}_B$ is an ensemble of LHSs for steering from Alice's side with projective measurements, it follows that there exists a $(2 \times m)$ stochastic matrix K such that $(P_i^A)' = \sum_{j=1}^m K_{ij} B_j$. But this also implies that $(E_i^A)' = \sum_{j=1}^2 \sum_{k=1}^m H_{ij} K_{jk} B_k$. Since H and K are stochastic matrices, $G = HK$ is also a stochastic matrix. Therefore, the state is also unsteerable with respect to all binary outcome measurements.

Furthermore, we show the following:

Proposition 2. A two-qubit state ρ is unsteerable from Alice's side for all binary outcome POVMs if and only if the set of Alice's steering outcomes $\mathscr{M}_A'$ is packable.

Proof. Suppose that for binary outcome measurements the state is unsteerable from Alice's side. Then there exists a set $\mathscr{U}_B = \{B_i\}_{i=1}^m$ of m positive operators playing the role of an ensemble of LHSs for steering from Alice's side. We will show that $\mathscr{M}_A' \subseteq \text{box}(\mathscr{U}_B)$. Indeed, take $B \in \mathscr{M}_A'$; then there exists an operator $E_1^A \in \mathscr{M}_A$ such that $(E_1^A)' = B$. Let $E_2^A = \mathbb{I}^A - E_1^A$; then $\{E_1^A, E_2^A\}$ constitutes a binary-outcome POVM performed by Alice. By definition of an ensemble of LHSs, there exists a $(2 \times m)$ stochastic matrix G such that $(E_i^A)' = \sum_{j=1}^m G_{ij} B_j$, in particular $B = (E_1^A)' = \sum_{j=1}^m G_{1j} B_j$. Since $0 \leqslant G_{1j} \leqslant 1$, this implies that $B \in \text{box}(\mathscr{U}_B)$.

Now suppose $\mathscr{M}_A' \subseteq \text{box}(\mathscr{U}_B)$, with $\mathscr{U}_B = \{B_i\}_{i=1}^m$, $(\mathbb{I}^A)' = \sum_{i=1}^m B_i$; we show that $\mathscr{U}_B$ can play the role of an ensemble of LHSs for all binary measurements performed by Alice. Any binary-outcome POVM performed by Alice is of the form $\mathbb{I}^A = E_1^A + E_2^A$, with $E_i^A \in \mathscr{M}_A$. This POVM induces a decomposition of the reduced state of B, $(\mathbb{I}^A)' = (E_1^A)' + (E_2^A)'$. Because $(E_1^A)' \in \mathscr{M}_A' \subseteq \text{box}(\mathscr{U}_B)$, there exist m numbers $\{0 \leqslant G_{1j} \leqslant 1\}_{j=1}^m$ such that $(E_1^A)' = \sum_{j=1}^m G_{1j} U_j^B$. It follows that $(E_2^A)' = (\mathbb{I}^A)' - (E_1^A)' = \sum_{j=1}^m [1 - G_{1j}] U_j^B$. Let $G_{2j} = 1 - G_{1j}$; then G is a $(2 \times m)$ stochastic matrix that satisfies $(E_i^A)' = \sum_{j=1}^m G_{ij} U_j^B$ for $i = 1, 2$. This implies that $\mathscr{U}_B$ is an ensemble of LHSs and the state is unsteerable from Alice's side.

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