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Free Semigroupoid Algebras from Categories of Paths

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Given a directed graph G, we can define a Hilbert space $\mathcal{H}_G$ with basis indexed by the path space of the graph, then represent the vertices of the graph as projections on $\mathcal{H}_G$ and the edges of the graph as partial isometries on $\mathcal{H}_G$. The weak operator topology closed algebra generated by these projections and partial isometries is called the free semigroupoid algebra for G. Kribs and Power showed that these algebras are reflexive, and that they are semisimple if and only if each path in the graph lies on a cycle. We extend the free semigroupoid algebra construction to categories of paths, which are a generalization of graphs, and provide examples of free semigroupoid algebras from categories of paths that cannot arise from graphs (or higher rank graphs). We then describe conditions under which these algebras are semisimple, and we prove reflexivity for a class of examples.

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Chapter 1

Introduction

A directed graph is a set of vertices along with a set of edges, where each edge has a source vertex and a range vertex. Such a graph can be represented by a collection of operators on a Hilbert space $\mathcal{H}$; each vertex is associated to a projection, and each edge is associated to a partial isometry that maps between the subspaces corresponding to its source and range vertices. These projections and partial isometries are used to construct a C^*-algebra called the *graph C^*-algebra* of the directed graph. There are many examples of common C^*-algebras which can be realized as graph algebras, and many properties of graph algebras are determined by structural properties of the graph. C^*-algebras are self-adjoint, however, so this is not a useful construction for studying non-self-adjoint operator algebras.

Free semigroupoid algebras generated by directed graphs are a class of non-self-adjoint operator algebras introduced by Kribs and Power in 2004 [7]. The construction of these algebras from a graph is similar to the graph C^*-algebra construction in that vertices are represented by projections and edges by partial isometries. However, a free semigroupoid algebra is closed in the weak operator topology, not the norm topology, and does not include adjoints.

As in the graph C^*-algebra case, many previously-studied non-self-adjoint operator algebras can be expressed as free semigroupoid algebras for some directed graph, and many properties of the algebra correspond to properties of the graph. In fact, this relationship is in some sense stronger than the self-adjoint case; while it is possible to find two non-isomorphic graphs that produce the same graph C^*-algebra, Kribs and Power [7] showed that two free semigroupoid algebras from graphs are unitarily equivalent if and only if their corresponding graphs are isomorphic.

In addition to this isomorphism result, Kribs and Power characterized semisimplicity for graph free semigroupoid algebras and proved that all graph free semigroupoid algebras are reflexive. In another paper on the subject [8], they extended the free semigroupoid algebra construction to higher rank graphs, which are a generalization of graphs where edges have length in $\mathbb{N}^k$ instead of $\mathbb{N}$; higher rank graphs can be thought of as graphs where certain paths are identified, according to a factorization property. Kribs and Power then proved the same semisimplicity result, and a slightly more limited reflexivity result, for free semigroupoid algebras from higher rank graphs.

There is another generalization of graphs introduced by Spielberg [13], called categories of paths, which allow identifications under conditions less restrictive than the higher rank graph factorization property. The goal of this dissertation is to study free semigroupoid algebras generated by categories of paths (usually assuming a degree functor) and determine how they are similar to and how they can differ from the graph and higher rank graph cases.

In Chapter 2, we review some basic terminology associated to operator algebras and directed graphs. We also define higher rank graphs and categories of paths. Then we introduce the free semigroupoid algebra for graphs and higher rank graphs, give some examples, and provide the proof of a basic result about the commutant of these algebras. Finally, in Sections 2.4 and 2.5, we outline important semisimplicity and reflexivity results. In particular, the free semigroupoid algebra from a graph or higher rank graph is semisimple if and only if all paths lie on a cycle, and all free semigroupoid algebras from graphs are reflexive.

In Chapter 3, we introduce free semigroupoid algebras from categories of paths and show that, under the assumption of a degree functor, the same characterization of the commutant holds from the graph case. Then we provide some examples of free semigroupoid algebras from categories of paths that cannot arise from graphs or higher rank graphs. These include an example of a single-vertex category of paths whose free semigroupoid algebra contains a nilpotent element, something which we show cannot occur in the graph or higher rank graph case.

In Chapter 4, we study semisimplicity for free semigroupoid algebras of categories of paths with degree functors. We introduce a condition (P) on a category of paths with a degree functor. This condition has two parts: the first is similar to row-finiteness in a graph; the second is a restriction on which elements of the algebra can be nilpotent, which is similar to, but more general than, the requirement that all paths lie on a cycle. We show that a category of paths satisfying (P) is semisimple. We then employ this result to show that the single-vertex examples from Chapter 3 are semisimple.

Finally, in Chapter 5, we examine reflexivity for free semigroupoid algebras from categories of paths. We define a Double Pure Cycle Property and show that if the transpose of a category of paths with a non-degenerate degree functor satisfies this property, then the free semigroupoid algebra of the category of paths is reflexive. We also show that finite categories of paths with non-degenerate degree functors have reflexive free semigroupoid algebras. Finally, we note that Kribs and Power's main reflexivity results for graphs and higher rank graphs are dependent on the reflexivity of the single-vertex case, and thus finish the chapter by establishing reflexivity for a family of single-vertex categories of paths.

Chapter 2

Background

In this chapter, we provide some basic definitions and results related to operator theory, graphs, and free semigroupoid algebras. Proofs of theorems will be included when they are relevant to what follows in later chapters.

2.1 Basic Definitions

We begin with the basics of operator theory. Although the most important definitions are summarized here, other basic facts may be used throughout this thesis, such as those contained in Chapter 1 of [4].

A Hilbert space is a vector space $\mathcal{H}$ which has an inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ and which is closed in the norm induced by the inner product. The norm of an operator $A : \mathcal{H} \to \mathcal{H}$ is given by $\|A\| = \sup_{\|h\|=1} \|Ah\|$. The set of all bounded linear operators on $\mathcal{H}$ is written $\mathcal{B}(\mathcal{H})$.

For $A \in \mathcal{B}(\mathcal{H})$, the adjoint of A, written A^*, is the unique linear operator such that $\langle Ah, k \rangle = \langle h, A^*k \rangle$ for all h, k in $\mathcal{H}$. When $\mathcal{H}$ is n-dimensional, $n < \infty$, a linear operator $T \in \mathcal{B}(\mathcal{H})$ is an $n \times n$ matrix, and T^* is the conjugate transpose of the matrix. A C^*-algebra is a subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the topology induced by the norm and closed under the operation of taking adjoints.

The norm topology is not the only topology on $\mathcal{B}(\mathcal{H})$. We can also look at the closure of a subalgebra of $\mathcal{B}(\mathcal{H})$ in the *weak operator topology* (WOT). We say a set of operators $\{T_\alpha\} \subset \mathcal{B}(\mathcal{H})$ converges to T in the weak operator topology if $\lim_\alpha \langle (T - T_\alpha)h, k\rangle = 0$ for all $h, k \in \mathcal{H}$.

A *projection* in $\mathcal{B}(\mathcal{H})$ is an operator P such that for some subspace $M \subseteq \mathcal{H}$, and all $m \in M$, $Pm = m$ and $Ph = 0$ for all $h \in M^\perp$. An *isometry* in $\mathcal{B}(\mathcal{H})$ is an operator S such that $S^*S = I$. We call S a *partial isometry* if there is a subspace $M \subseteq \mathcal{H}$ such that $S^*Sh = h$ for all $h \in M$ and $S^*S = 0$ for all $h \in M^\perp$. Thus, S^*S is the projection onto M.

Next, we will look at some terminology associated to graphs. Let G be a countable directed graph with vertex set $V(G)$ and edge set $E(G)$. Each edge e has a source vertex $s(e)$ and a range vertex $r(e)$. A *path* is a string of edges $e_n e_{n-1} \dots e_2 e_1$ such that for all j, $s(e_{j+1}) = r(e_j)$. We can extend the range and source maps to paths: if $\mu = e_n \dots e_2 e_1$, then $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$. The *path space* of G, written $\mathbb{F}^+(G)$, is the set of all paths in G. A directed graph is called *row-finite* if each vertex receives at most finitely many edges; that is, if $\{e \in E(G) : r(e) = v\}$ is finite for all $v \in V(G)$.

In this work, we will use a generalization of graphs called categories of paths, introduced by Spielberg [13]. Recall that a *small category* Λ is a set of objects Λ^0 and morphisms between the objects, along with two maps: a source map $s : \Lambda \to \Lambda^0$ sending each morphism to its source, and a range map $r : \Lambda \to \Lambda^0$ sending each morphism to its range.

Definition 2.1.1 ([13], Definition 2.1)**.** A small category Λ is called a *category of paths* if, for $\alpha, \beta, \gamma \in \Lambda$,

- $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$ (left cancellation)
- $\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$ (right cancellation)
- $\alpha\beta = s(\beta)$ implies $\alpha = \beta = s(\beta)$ (no inverses)

Remark 2.1.2. One way to define a category of paths is to start with a graph, and then identify certain paths.

Example 2.1.3. Consider the graph $x_1 \underset{b_1}{\overset{a_1}{\rightrightarrows}} x_2 \underset{b_2}{\overset{a_2}{\rightrightarrows}} x_3$ along with the identifications $b_2a_1 = a_2b_1$ and $a_2a_1 = b_2b_1$. This forms a category of paths with three vertices (x_1, x_2, x_3) and six non-vertex paths $(a_1, b_1, a_2, b_2, a_2a_1, a_2b_1)$.

One specific kind of category of paths that has been studied extensively is a higher rank graph, or k-graph.

Definition 2.1.4 ([9], Definition 1.1)**.** For $k \in \mathbb{N}$, a *k-graph* is a countable category of paths Λ with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the following factorization property: For all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there are unique elements $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

A 1-graph is simply a directed graph.

Example 2.1.5. Consider the following 2-graph: $x_1 \underset{b_1}{\overset{a_1}{\rightrightarrows}} x_2 \underset{b_2}{\overset{a_2}{\rightrightarrows}} x_3$, where the solid edges have degree $(1, 0)$ and the dotted edges have degree $(0, 1)$. The path b_2a_1 has degree $(1, 1) = (1, 0) + (0, 1)$, so by the factorization property, there are unique paths μ with $d(\mu) = (1, 0)$ and ν with $d(\nu) = (0, 1)$ such that $b_2a_1 = \mu\nu$. The only possibility is $\mu = a_2$ and $\nu = b_1$. Thus, in this higher rank graph, $b_2a_1 = a_2b_1$.

Note that, unlike Example 2.1.3, we could not have $a_2a_1 = b_2b_1$, since the degree of a_2a_1 is $(2, 0)$ and the degree of b_2b_1 is $(0, 2)$.

A good reference for higher rank graphs is Chapter 10 of [10].

2.2 Free Semigroupoid Algebras Definition and Properties

Let $\mathcal{H}_G$ be a Hilbert space with orthonormal basis $\{\xi_w\}_{w\in\mathbb{F}^+(G)}$, indexed by the path space of G. This is called a Fock space. For each $w \in \mathbb{F}^+(G)$, we can define a linear operator $L_w \in \mathcal{B}(\mathcal{H})$ as follows. For $\mu \in \mathbb{F}^+(G)$, let

$$L_w(\xi_\mu) = \begin{cases} \xi_{w\mu} & \text{if } s(w) = r(\mu) \\ 0 & else \end{cases} .$$

Then L_w is a partial isometry on the Fock space, sometimes called a "partial creation operator".

Notice that if e and f are edges, then L_e and L_f have orthogonal ranges. Also, L_x is a projection for any vertex x. Specifically, for a vertex $x \in V(G)$:

$$L_x\xi_\nu = \begin{cases} \xi_\nu & \text{if } r(\nu) = x \\ 0 & else \end{cases}$$

For a path $\mu = e_n e_{n-1} \dots e_1$, let $L_\mu = L_{e_n} L_{e_{n-1}} \dots L_{e_1}$.

Definition 2.2.1 ([7], Definition 3.2)**.** Let $\mathfrak{L}_G$ be the WOT-closed algebra generated by $\{L_w\}_{w\in\mathbb{F}^+(G)}$. This is called a *free semigroupoid algebra.*

Example 2.2.2 ([7], Example 6.1)**.** When the graph is a single vertex and a single loop, the Hilbert space $\mathcal{H}_G$ is isomorphic to the Hardy space H^2, and L_e and L_x are isomorphic to the unilateral shift and the identity operator, respectively. Thus, $\mathfrak{L}_G$ is isomorphic to the WOT-closed algebra generated by those two operators, which is H^∞.

Example 2.2.3 ([7], Example 6.1)**.** Another example is the free semigroup algebra, or non-commutative Toeplitz algebra, studied by Davidson and Pitts [5] and Arias and Popescu [1], which corresponds to a graph with one vertex and n edges.

Example 2.2.4 ([7], Example 6.3)**.** Consider the graph G given by

$$e \circlearrowleft x \xrightarrow{\quad f \quad} y$$

Then $\mathfrak{L}_\Lambda$ is generated by L_e, L_f, L_x, and L_y. If we make the identifications $\mathcal{H}_G = L_x\mathcal{H}_G \oplus L_y\mathcal{H}_G \cong H^2 \oplus H^2$, where H^2 is the Hardy space, then

$$L_e \cong \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}; \; L_f \cong \begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix}; \; L_e \cong \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}; \; L_e \cong \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

where S is the unilateral shift. Thus, $\mathfrak{L}_G$ is unitarily equivalent to

$$\mathfrak{L}_G \cong \begin{bmatrix} H^\infty & 0 \\ H_0^\infty & \mathbb{C}I \end{bmatrix}.$$

Kribs and Power [8] extended this free semigroupoid algebra construction to higher rank graphs. For a higher rank graph Λ, define a Fock space Hilbert space $\mathcal{H}_\Lambda$ with orthonormal basis $\{\xi_\mu\}_{\mu\in\Lambda}$, indexed by the elements of Λ. We can then define linear operators $L_\mu \in \mathcal{B}(\mathcal{H}_\Lambda)$ as follows. For $\nu \in \Lambda$, define:

$$L_\mu \xi_\nu = \begin{cases} \xi_{\mu\nu} & \text{if } s(\mu) = r(\nu) \\ 0 & else \end{cases}$$

As before, if x is a vertex of Λ, then L_x is a projection.

Definition 2.2.5 ([8], Definition 3.1)**.** The WOT-closed algebra generated by $\{L_\mu\}_{\mu\in\Lambda}$ is called the *free semigroupoid algebra* for Λ and is written $\mathfrak{L}_\Lambda$.

2.3 Commutant

Let G be a countable directed graph. Given $\mu \in \mathbb{F}^+(G)$, we can define an operator R_μ by

$$R_\mu \xi_\nu = \begin{cases} \xi_{\nu\mu} & \text{if } r(\mu) = s(\nu) \\ 0 & \quad\quad else \end{cases}$$

Let $\mathfrak{R}_G$ be the WOT-closed algebra generated by $\{R_\mu\}_{\mu\in\mathbb{F}^+(G)}$. Kribs and Power showed in Section 4 of [7] that the commutant of $\mathfrak{L}_G$ is (unitarily equivalent to) $\mathfrak{R}_G$. The steps to proving that result will be outlined in this section.

Definition 2.3.1. For a graph G, let E_i be the projection onto span$\{\xi_\mu : \mu \in \mathbb{F}^+(G),\ |\mu| = i\}$. The Cesaro sums of $A \in \mathcal{B}(\mathcal{H}_\Lambda)$ are given by

$$\Sigma_k(A) = \sum_{|j|<k} \left(1 - \frac{|j|}{k}\right) \Phi_j(A),$$

where

$$\Phi_j(A) = \sum_{\ell \geq \max\{0,-j\}} E_\ell A E_{\ell+j}$$

is the ith diagonal of A in the matrix form associated to the partition $I = E_0 + E_1 + E_2 + \ldots$.

The fact that the Cesaro sums of A converge in the strong operator topoogy (SOT) to A is a consequence of the following proposition.

Proposition 2.3.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Let $\{P_j\}$ be a sequence of finite rank projections such that $\sum\limits_{j=1}^{\infty} P_j = I$. The Cesaro sums of A, given by*

$$\Sigma_k(A) = \sum_{|j|<k} \left(1 - \frac{|j|}{k}\right) \Phi_j(A),$$

where $\Phi_j(A) = \sum\limits_{\ell \geq \max\{0,-j\}} P_\ell A P_{\ell+j}$, converge SOT to A.

To prove this, we will use the following vector-valued version of Fejer's theorem, which is stated, for example, as Theorem 1.1 in [3].

Theorem 2.3.3 ([3], Theorem 1.1)**.** *Let X be a Banach space, $f : [0, 2\pi] \to X$ a continuous function, and $z_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-iks} f(s) ds$ its kth Fourier coefficient; then the Fourier series of f is Cesaro summable to $f(t)$ at every point $t \in (0, 2\pi)$.*

Define a function $f_h : \mathbb{T} \to \mathcal{H}$ as follows. Let n_j be the dimension of $P_j \mathcal{H} P_j$. For $\lambda \in \mathbb{T}$, let U_λ be the following matrix:

$$U_\lambda = \begin{bmatrix} \lambda^{-1}(I_{n_1}) & 0 & 0 & \cdots \\ 0 & \lambda^{-2}(I_{n_2}) & 0 & \cdots \\ 0 & 0 & \lambda^{-3}(I_{n_3}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where I_{n_j} is the $n_j \times n_j$ identity matrix. Note U_λ is a unitary, and for $A \in \mathcal{B}(\mathcal{H})$:

$$U_\lambda A U_\lambda^{-1} = \begin{bmatrix} 1(A_{1,1}) & \lambda(A_{1,2}) & \lambda^2(A_{1,3}) & \cdots \\ \lambda^{-1}(A_{2,1}) & 1(A_{2,2}) & \lambda(A_{2,3}) & \cdots \\ \lambda^{-2}(A_{3,1}) & \lambda^{-1}(A_{3,2}) & 1(A_{3,3}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In other words, conjugating A by U_λ^{-1} results in multiplying the (i,j)th block component of A by λ^{j-i}.

Now let $A \in \mathcal{B}(\mathcal{H})$. Fix $h \in \mathcal{H}$. Define the function $f_h : \mathbb{T} \to \mathcal{H}$ by

$$f_h(\lambda) = U_\lambda A U_{\lambda^{-1}} h.$$

Lemma 2.3.4. *$f_h : \mathbb{T} \to \mathcal{H}$ is a continuous function.*

Proof. For $h \in \mathcal{H}$, let $P_{\leq N} h$ be the projection of h onto the first N blocks of basis vectors (i.e., the first $n_1 + n_2 + \cdots + n_N$ basis vectors). Let $P_{>N} = I - P_{\leq N}$.

First assume that $h = (h_1, h_2, \ldots, h_N, 0, \ldots) = P_{\leq N} h$ for some N. Let $\varepsilon > 0$. Note that for any λ_1 and λ_2 in $\mathbb{T}$,

$$\begin{aligned} \|f_h(\lambda_1) - f_h(\lambda_2)\| &= \|U_{\lambda_1} A U_{\lambda_1}^{-1} h - U_{\lambda_2} A U_{\lambda_2}^{-1} h\| \\ &\leq \|U_{\lambda_1} A U_{\lambda_1}^{-1} h - U_{\lambda_1} A U_{\lambda_2}^{-1} h\| + \|U_{\lambda_1} A U_{\lambda_2}^{-1} h - U_{\lambda_2} A U_{\lambda_2}^{-1} h\| \\ &= \|A\| \|\big(U_{\lambda_1}^{-1} - U_{\lambda_2}^{-1}\big) h\| + \|\big(U_{\lambda_1} - U_{\lambda_2}\big)\big(A U_{\lambda_2}^{-1} h\big)\|. \end{aligned}$$

We'll look at each term separately. First, note that $(U_{\lambda_1}^{-1} - U_{\lambda_2}^{-1})h$ has only finitely many non-zero terms. So choose δ_1 such that for $|\lambda_1 - \lambda_2| < \delta_1$, we have $\|\lambda_1^{-j} - \lambda_2^{-j}\|^2 < \frac{\varepsilon^2}{4\|A\|^2 N \|h_j\|^2}$ for each $j = 1, \ldots, N$. Then

$$\left\|(U_{\lambda_1}^{-1} - U_{\lambda_2}^{-1})h\right\|^2 = \left\| \begin{bmatrix} (\lambda_1^{-1} - \lambda_2^{-1})h_1 \\ (\lambda_1^{-2} - \lambda_2^{-2})h_2 \\ \vdots \\ (\lambda_1^{-N} - \lambda_2^{-N})h_N \\ 0 \\ \vdots \end{bmatrix} \right\|^2 = \sum_{j=1}^{N} \|\lambda_1^{-j} - \lambda_2^{-j}\|^2 \|h_j\|^2 < \frac{\varepsilon^2}{4\|A\|^2}.$$

So $\|A\|\big\|(U_{\lambda_1}^{-1} - U_{\lambda_2}^{-1})h\big\| < \|A\|\sqrt{\frac{\varepsilon^2}{4\|A\|^2}} = \frac{\varepsilon}{2}$.

Now, consider the term $\|\big(U_{\lambda_1} - U_{\lambda_2}\big)(AU_{\lambda_2}^{-1}h)\|$. For $j = 1, \ldots, N$, let

$$\tilde{h}_j = P_j h = (0, \ldots, 0, h_j, 0, \ldots) \in \mathcal{H}.$$

For each $j = 1, \ldots, N$, there is some K_j such that for $k > K_j$, $\|P_{>k}A\tilde{h}_j\| < \frac{\varepsilon}{8N}$. Let $K \geq \max\{K_j : j = 1, \ldots, N\}$. Then

$$\begin{aligned} \|P_{>K}AU_{\lambda_2}^{-1}h\| &\leq \sum_{j=1}^{N} \|P_{>K}AU_{\lambda_2}^{-1}\tilde{h}_j\| \\ &= \sum_{j=1}^{N} \|P_{>K}A\lambda_2^{-j}\tilde{h}_j\| \\ &= \sum_{j=1}^{N} \|P_{>K}A\tilde{h}_j\| < \frac{\varepsilon}{8}. \end{aligned}$$

Furthermore, note that for each $1 \leq j \leq N$, $(U_{\lambda_1} - U_{\lambda_2})P_{\leq K}A\tilde{h}_j$ has only finitely many non-zero terms. Let $M = \max\{\|A_{i,j}h_j\| : 1 \leq i \leq K; 1 \leq j \leq N\}$. Choose δ_2 such that for $|\lambda_1 - \lambda_2| < \delta_2$, we have $\|\lambda_1^i - \lambda_2^i\|^2 < \frac{\varepsilon^2}{16N^2KM^2}$ for each $i = 1, \ldots, K$. Then

$$\begin{aligned} \big\|(U_{\lambda_1} - U_{\lambda_2})P_{\leq K}A\tilde{h}_j\big\|^2 &= \left\| \begin{bmatrix} (\lambda_1 - \lambda_2)A_{1j}h_j \\ (\lambda_1^2 - \lambda_2^2)A_{2j}h_j \\ \vdots \\ (\lambda_1^K - \lambda_2^K)A_{K,j}h_j \\ 0 \\ \vdots \end{bmatrix} \right\|^2 \\ &= \sum_{i=1}^{K} \|\lambda_1^i - \lambda_2^i\|^2 \|A_{i,j}\tilde{h}_j\|^2 < \frac{\varepsilon^2}{16N^2}. \end{aligned}$$

Thus,

$$\begin{aligned}\|(U_{\lambda_1}-U_{\lambda_2})P_{\le K}AU_{\lambda_2}^{-1}h\| &\le \sum_{j=1}^{N}\|(U_{\lambda_1}-U_{\lambda_2})P_{\le K}AU_{\lambda_2}^{-1}\tilde{h}_j\|\\ &= \sum_{j=1}^{N}\|(U_{\lambda_1}-U_{\lambda_2})P_{\le K}A\lambda_2^{-j}\tilde{h}_j\|\\ &= \sum_{j=1}^{N}\|(U_{\lambda_1}-U_{\lambda_2})P_{\le K}A\tilde{h}_j\| < \frac{\varepsilon}{4}.\end{aligned}$$

So

$$\begin{aligned}\|(U_{\lambda_1}-U_{\lambda_2})(AU_{\lambda_2}^{-1}h)\| &\le \|(U_{\lambda_1}-U_{\lambda_2})P_{\le K}AU_{\lambda_2}^{-1}h\| +\\ &\qquad \|(U_{\lambda_1}-U_{\lambda_2})P_{>K}AU_{\lambda_2}^{-1}h\|\\ &= \frac{\varepsilon}{4} + \|(U_{\lambda_1}-U_{\lambda_2})\|\|P_{>K}AU_{\lambda_2}^{-1}h\|\\ &\le \frac{\varepsilon}{4} + 2\frac{\varepsilon}{8} < \frac{\varepsilon}{2}.\end{aligned}$$

So if $\delta = \min\{\delta_1, \delta_2\}$, then $|\lambda_1 - \lambda_2| < \delta$ implies

$$\begin{aligned}\|f_h(\lambda_1) - f_h(\lambda_2)\| &= \|U_{\lambda_1}AU_{\lambda_1}^{-1}h - U_{\lambda_2}AU_{\lambda_2}^{-1}h\|\\ &\le \|U_{\lambda_1}AU_{\lambda_1}^{-1}h - U_{\lambda_1}AU_{\lambda_2}^{-1}h\| + \|U_{\lambda_1}AU_{\lambda_2}^{-1}h - U_{\lambda_2}AU_{\lambda_2}^{-1}h\|\\ &= \|A\|\|(U_{\lambda_1}^{-1} - U_{\lambda_2}^{-1})h\| + \|(U_{\lambda_1}-U_{\lambda_2})(AU_{\lambda_2}^{-1}h)\|\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This shows that f_h is continuous if h has finitely many non-zero components. To finish the proof, let h be arbitrary. Choose N so that $\|h - P_{\le N}h\| < \frac{\varepsilon}{3\|A\|}$. Choose δ so that for $|\lambda_1 - \lambda_2| < \delta$, we have $\|f_{P_{\le N}h}(\lambda_1) - f_{P_{\le N}h}(\lambda_2)\| < \frac{\varepsilon}{3}$.

Then

$$\begin{aligned}\|f_h(\lambda_1) - f_h(\lambda_2)\| &\le \|f_h(\lambda_1) - f_{P_{\le N}h}(\lambda_1)\| + \|f_{P_{\le N}h}(\lambda_1) - f_{P_{\le N}h}(\lambda_2)\| \\ &\quad + \|f_{P_{\le N}h}(\lambda_2) - f_h(\lambda_2)\| \\ &< \|U_{\lambda_1} A U_{\lambda_1}^{-1}(h - P_{\le N}h)\| + \frac{\varepsilon}{3} + \|U_{\lambda_2} A U_{\lambda_2}^{-1}(h - P_{\le N}h)\| \\ &\le 2\|A\|\|h - P_{\le N}h\| + \frac{\varepsilon}{3} < \varepsilon.\end{aligned}$$

□

Now we can prove the proposition.

Proof. (*of Proposition 2.3.2*)

Let $h \in \mathcal{H}$ and let $\varepsilon > 0$. Let $\Phi_j(A) = \sum_{\ell \ge \max\{0,-j\}} P_\ell A P_{\ell+j}$ be the jth block diagonal of A. Let $f_h : \mathbb{T} \to \mathcal{H}$ be defined as above, and let $g_h : [0, 2\pi] \to \mathcal{H}$ be given by $g_h(\theta) = f_h(e^{i\theta})$. Note that g_h is continuous since f_h is, and furthermore,

$$g_h(\theta) = \sum_{j=-\infty}^{\infty} e^{ij\theta} \Phi_j(A) h.$$

As before, let $P_{\le N}h$ be the projection of h onto the first N blocks of basis vectors. Choose N_1 sufficiently large so that for $N > N_1$, we have $\|P_{\le N}h - h\| < \frac{\varepsilon}{2\|A\|}$. Then for any $n \in \mathbb{N}$:

$$\begin{aligned}&\left\| \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} g_h(s)\, \mathrm{d}s - \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} g_{P_{\le N}h}(s)\, \mathrm{d}s \right\| \\ &\qquad = \left\| \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} \big(g_h(s) - g_{P_{\le N}h}(s)\big)\, \mathrm{d}s \right\| \\ &\qquad \le \frac{1}{2\pi} \int_0^{2\pi} \|e^{-ins} \big(U_{e^{is}} A U_{e^{is}}^{-1} h - U_{e^{is}} A U_{e^{is}}^{-1} P_{\le N}h\big)\|\, \mathrm{d}s \\ &\qquad \le \frac{1}{2\pi} \int_0^{2\pi} \|A\|\|h - P_{\le N}h\|\, \mathrm{d}s < \frac{\varepsilon}{2}.\end{aligned}$$

Furthermore,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ins} g_{P_{\leq N}h}(s)\ \mathrm{d}s = \frac{1}{2\pi}\int_0^{2\pi} e^{-ins} \begin{bmatrix} \sum_{j=1}^{N} e^{(j-1)is} A_{1,j} h_j \\ \sum_{j=1}^{N} e^{(j-2)is} A_{2,j} h_j \\ \sum_{j=1}^{N} e^{(j-3)is} A_{3,j} h_j \\ \vdots \end{bmatrix} \mathrm{d}s$$

$$= \begin{bmatrix} \sum_{j=1}^{N} \frac{1}{2\pi}\int_0^{2\pi} e^{(j-1-n)is} A_{1,j} h_j\ \mathrm{d}s \\ \sum_{j=1}^{N} \frac{1}{2\pi}\int_0^{2\pi} e^{(j-2-n)is} A_{2,j} h_j\ \mathrm{d}s \\ \sum_{j=1}^{N} \frac{1}{2\pi}\int_0^{2\pi} e^{(j-3-n)is} A_{3,j} h_j\ \mathrm{d}s \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{1,n+1} h_{n+1} \\ A_{2,n+2} h_{n+2} \\ A_{3,n+3} h_{n+3} \\ \vdots \\ A_{N-n,N} h_N \\ 0 \\ \vdots \end{bmatrix},$$

where $A_{i,n+j} = 0$ and $h_{n+j} = 0$ if $n + j \leq 0$. Thus, $\frac{1}{2\pi}\int_0^{2\pi} e^{-ins} g_{P_{\leq N}h}(s)\ \mathrm{d}s$ is equal to $P_{\leq N}\big(\Phi_n(A)h\big)$, the projection onto the first N block components of $\Phi_n(A)h$. Therefore,

$$\left\| \frac{1}{2\pi}\int_0^{2\pi} e^{-ins} g_h(s)\ \mathrm{d}s - P_{\leq N}\big(\Phi_n(A)h\big) \right\| < \frac{\varepsilon}{2}.$$

Now choose N_2 so that for $N > N_2$, we have $\|P_{\leq N}\big(\Phi_n(A)h\big) - \Phi_n(A)h\| < \frac{\varepsilon}{2}$. Then for $N > \max\{N_1, N_2\}$, $\left\| \frac{1}{2\pi}\int_0^{2\pi} e^{-ins} g_h(s)\ \mathrm{d}s - \Phi_n(A)h \right\|$ is bounded above by

$$\left\| \frac{1}{2\pi}\int_0^{2\pi} e^{-ins} g_h(s)\ \mathrm{d}s - P_{\leq N}\Phi_n(A)h \right\| + \|P_{\leq N}\Phi_n(A)h - \Phi_n(A)h\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As ε was arbitrary, this tells us that the z_n from Theorem 2.3.3 are given by:

$$z_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-ins} g_h(s)\ \mathrm{d}s = \Phi_n(A)h.$$

Note that

$$\sigma_k g_h(\theta) = \sum_{|j|<k} e^{ij\theta}\left(\frac{k-|j|}{k}\right)\Phi_j(A)h = U_{e^{i\theta}}(\Sigma_k A)U_{e^{i\theta}}^{-1}h.$$

Applying Theorem 2.3.3, we have that for $\theta \in (0, 2\pi)$, we have $\|\sigma_k g_h(\theta) - g_h(\theta)\| \to 0$ as $k \to \infty$. That is,

$$\|U_{e^{i\theta}}\Sigma_k A U_{e^{-i\theta}}h - U_{e^{i\theta}}AU_{e^{-i\theta}}h\| \to 0.$$

This holds for all $h \in \mathcal{H}$. So for $\theta \in (0, 2\pi)$, we have that $U_{e^{i\theta}}\Sigma_k A U_{e^{-i\theta}}$ converges SOT to $U_{e^{i\theta}}AU_{e^{-i\theta}}$. Since SOT convergence is preserved by unitary equivalence, this means that $\Sigma_k A$ converges SOT to A as desired. □

Definition 2.3.5 ([7], before Lemma 4.1)**.** Let G^t denote the *transpose* graph of G, which is the directed graph obtained from G by reversing the direction of all the edges. If $v = e_n e_{n-1} \dots e_2 e_1$ is a product of edges in G, then we define v^t in G^t to be $e_1^t e_2^t \dots e_{n-1}^t e_n^t$, where e_i^t is the directed edge in G^t corresponding to edge e in G with the direction reversed. Define $x^t = x$ for vertices $x \in V(G)$.

Lemma 2.3.6 ([7], Lemma 4.1)**.** *The algebras $\mathfrak{L}_G$ and $\mathfrak{R}_{G^t}$ are unitarily equivalent via the map $W : \mathcal{H}_{G^t} \to \mathcal{H}_G$ given by $W\xi_{v^t} = \xi_v$.*

Proof. The map W is defined by a bijection between orthonormal bases for

$\mathcal{H}_{G'}$ and $\mathcal{H}_G$, and therefore is a unitary. Given $\mu, \nu \in \mathbb{F}^+(G)$ with $s(\mu) = r(\nu)$,

$$(W^* L_\mu W)\xi_{\nu^t} = W^* L_\mu \xi_\nu = W^* \xi_{\mu\nu} = \xi_{\nu^t \mu^t} = R_{\mu^t} \xi_{\nu^t}.$$

Hence $W^* L_\mu W = R_{\mu^t}$ for $\mu \in \mathbb{F}^+(G)$, and so $W^* \mathfrak{L}_G W = \mathfrak{R}_{G^t}$. □

Lemma 2.3.7. *Let $A \in \mathfrak{R}_G'$ and $x \in V(G)$. Let $A_x = AL_x$, and furthermore let $\{a_w\}_{w \in \mathbb{F}^+(G), s(w)=x}$ be scalars such that*

$$A\xi_x = AR_x L_x \xi_x = R_x A_x \xi_x = \sum_{s(w)=x} a_w \xi_w.$$

For $k \in \mathbb{N}$, define operators

$$p_k = \sum_{|w|<k, s(w)=x} \left(1 - \frac{|w|}{k}\right) a_w L_w.$$

Then p_k is in $\mathfrak{L}_G$ for each $k \in \mathbb{N}$.

Proof. To see that this possibly-infinite sum is in $\mathfrak{L}_G$, let $\mu \in \mathbb{F}^+(G)$ with $r(\mu) = x$. Note that $w_1\mu \neq w_2\mu$ when $w_1 \neq w_2$. Then

$$\begin{aligned} \|p_k \xi_\mu\|^2 &= \left\| \sum_{|w|<k, s(w)=x} \left(1 - \frac{|w|}{k}\right) a_w \xi_{w\mu} \right\|^2 \\ &= \sum_{|w|<k, s(w)=x} \left\| \left(1 - \frac{|w|}{k}\right) a_w \xi_{w\mu} \right\|^2 \\ &= \sum_{|w|<k, s(w)=x} \left| \left(1 - \frac{|w|}{k}\right) a_w \right|^2 \\ &\leq \sum_{s(w)=x} |a_w|^2. \end{aligned}$$

And for μ with $r(\mu) \neq x$, we have $p_k \xi_\mu = 0$.

So $\|p_k\| \leq \left(\sum_{s(w)=x} \|a_w\|^2\right)^{1/2} < \infty$. Thus, p_k is a bounded linear operator, and can be written as the strong operator topology limit of operators in $\mathfrak{L}_G$. Hence p_k is in $\mathfrak{L}_G$. $\square$

Theorem 2.3.8 ([7], Theorem 4.2)**.** *Let G be a graph. Then $\mathfrak{R}'_G = \mathfrak{L}_G$.*

Proof. Each L_μ for $\mu \in \mathbb{F}^+(G)$ is contained in $\mathfrak{R}'_G$, which is WOT closed as the commutant of an operator algebra. So $\mathfrak{L}_G \subseteq \mathfrak{R}'_G$.

Conversely, fix $A \in \mathfrak{R}'_G$. We will show that $A_x := AL_x$ is in $\mathfrak{L}_G$ for all vertices $x \in V(G)$. So fix $x \in V(G)$ and let $\{a_w\}_{w \in \mathbb{F}^+(G), s(w)=x}$ be scalars such that

$$A\xi_x = AR_xL_x\xi_x = R_xA_x\xi_x = \sum_{s(w)=x} a_w\xi_w.$$

For $k \in \mathbb{N}$, define operators

$$p_k = \sum_{|w|<k, s(w)=x} \left(1 - \frac{|w|}{k}\right) a_w L_w.$$

By Lemma 2.3.7, each p_k is in $\mathfrak{L}_G$ and hence also in $\mathfrak{R}'_G$. We claim that A_x is the strong operator topology limit of the p_k as k goes to infinity. This will be proven by showing that $p_k = \Sigma_k(A_x)$ (see Definition 2.3.1).

First we will show that $\Phi_j(A_x)$ is in $\mathfrak{R}'_G$ for all j. To this end, observe that $\Phi_j(A_x)$ commutes with R_e for each edge e since $A_x \in \mathfrak{R}'_G$, $E_{k+1}R_e = R_eE_k$ for all k, and $E_0R_e = 0$. Additionally, $\Phi_j(A_x)$ commutes with R_y for each vertex y since for all k, $R_yE_k = E_kR_y$ is the projection onto $\text{span}\{\xi_w : |w| = k, s(w) = y\}$. Thus, $\Phi_j(A_x) \in \mathfrak{R}'_G$, and it follows that $\Sigma_k(A_x) \in \mathfrak{R}'_G$ also.

Now, it is enough to show that $\Sigma_k(A_x)\xi_x = p_k\xi_x$. If this is the case, then for $w = xw$ in $\mathbb{F}^+(G)$:

$$\Sigma_k(A_x)\xi_w = \Sigma_k(A_x)R_w\xi_x = R_w\Sigma_k(A_x)\xi_x = R_w p_k\xi_x = p_k\xi_w,$$

whereas for $w = yw$ with $y \neq x$, we have

$$\Sigma_k(A_x)\xi_w = \Sigma_k(A_x)L_xL_y\xi_w = 0 = p_kL_y\xi_w = p_k\xi_w,$$

since L_x commutes with each E_j.

So, to show that $\Sigma_k(A_x)\xi_x = p_k\xi_x$, observe that $\Phi_0(A_x)\xi_x = E_0(A_x)E_0\xi_x = a_x\xi_x$ and $\Phi_j(A_x)\xi_x = 0$ for $j > 0$. For $j < 0$, we have

$$\Phi_j(A_x)\xi_x = (E_{-j}A_x)\xi_x = E_{-j}\sum_{s(w)=x} a_w\xi_w = \sum_{s(w)=x,|w|=-j} a_w\xi_w.$$

Hence it follows that

$$\begin{aligned}\Sigma_k(A_x)\xi_x &= \sum_{|j|<k}\left(1-\frac{|j|}{k}\right)\Phi_j(A)\xi_x\\ &= \sum_{|j|<k}\left(1-\frac{|j|}{k}\right)\Bigg(\sum_{s(w)=x,|w|=-j} a_w\xi_w\Bigg)\\ &= \sum_{|w|<k;s(w)=x}\left(1-\frac{|w|}{k}\right)a_w\xi_w\\ &= p_k\xi_x\end{aligned}$$

This establishes that $A_x = AL_x \in \mathfrak{L}_\Lambda$. This completes the proof, since $A = \sum_{x\in\Lambda^0} AL_x$, the sum converging in the strong operator topology when Λ^0 is infinite. □

Remark 2.3.9 ([7], Remark 4.3). From the proof of this theorem, elements of

$\mathfrak{L}_G$ can be seen to have Fourier expansions. In particular, if A belongs to $\mathfrak{L}_G$ with

$$A\xi_x = AL_x\xi_x = R_x(AL_x)\xi_x = \sum_{s(w)=x} a_w \xi_w$$

for a vertex $x \in V(G)$, then

$$A\xi_v = R_v A\xi_x = \sum_{s(w)=x} a_w \xi_{wv}$$

for $v = xv \in \mathbb{F}^+(G)$, and it follows that the Cesaro partial sums associated with the series $A \sim \sum_{w\in\mathbb{F}^+(G)} a_w L_w$ converge in the strong operator topology to A.

Corollary 2.3.10 ([7], Corollary 4.4)**.** *The commutant of $\mathfrak{L}_G$ is unitarily equivalent to $\mathfrak{R}_G$.*

Proof. If W is the unitary from Lemma 2.3.6, we have $\mathfrak{R}'_{G^t} = (W^*\mathfrak{L}_G W)' = W^*\mathfrak{L}'_G W$. It also follows from the lemma and the definition of the unitary that $\mathfrak{R}_G = W\mathfrak{L}_{G^t}W^*$. By Theorem 2.3.8, $\mathfrak{R}'_{G^t} = \mathfrak{L}_{G^t}$. Thus,

$$\mathfrak{R}_G = W\mathfrak{L}_{G^t}W^* = W\mathfrak{R}'_{G^t}W^* = W\big(W^*\mathfrak{L}'_G W\big)W^* = \mathfrak{L}'_G.$$

□

2.4 Semisimplicity

Let $\mathcal{H}$ be a Hilbert space. An operator $T \in \mathcal{H}$ is called *nilpotent* if $T^n = 0$. We say that T is *quasinilpotent* if the spectrum of T is 0, or, equivalently, if $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$. The *Jacobson radical* $\mathrm{rad}(\mathcal{A})$ of a Banach algebra $\mathcal{A}$ is the intersection of the kernels of all algebraically irreducible representations. An algebra $\mathcal{A}$ is called *semisimple* if $\text{rad}(\mathcal{A}) = 0$.

The following proposition is a common result, found for example in [11].

Proposition 2.4.1 ([11], restatement of Theorem 2.3.5(ii))**.** *The Jacobson radical of an algebra of operators is the largest quasinilpotent ideal in the algebra.*

Definition 2.4.2 ([7], before Theorem 5.1)**.** A graph or higher rank graph Λ is called *transitive* if there are paths in both directions between every pair of vertices in Λ.

A *(connected) component* C of Λ is a maximal collection of vertices that are joined in the sense that for any two vertices $x, y \in C$, there is a finite set of a paths $\mu_1, \ldots, \mu_n$ and vertices $x = x_1, x_2, \ldots, x_n = y$ such that for each j either $s(\mu_j) = x_j$ and $r(\mu_j) = x_{j+1}$, or $r(\mu_j) = x_j$ and $s(\mu_j) = x_{j+1}$. That is, there is a path between the two vertices if we ignore direction.

A *cycle* is a path $\mu \in \Lambda$ with the same initial and final vertices.

Remark 2.4.3 ([7], before Theorem 5.1). Note that Λ is transitive in each component if and only if every path lies on a cycle.

Let $B(\Lambda)$ be the collection of paths $\mu \in \Lambda$ which do not lie on a cycle. The set $B(\Lambda)$ is empty if and only if Λ is transitive in each component. Kribs and Power [7] showed that for a graph G, the Jacobson radical of $\mathfrak{L}_G$ is determined by these paths. One important step towards proving this is the following lemma, Lemma 5.3 in [7]. We include it here with the proof, as the same proof will be used to analyze semisimplicity of free semigroupoid algebras from categories of paths in Chapter 4.

Lemma 2.4.4 ([7], Lemma 5.3)**.** *Let Λ be a graph or higher rank graph. The following are equivalent for $\mu \in \Lambda$:*

(i) $L_\mu \in rad\, \mathfrak{L}_\Lambda$

(ii) $\mu \in B(\Lambda)$

(iii) $(AL_\mu)^2 = 0$ *for all* $A \in \mathfrak{L}_\Lambda$

(iv) $L_w^2 = L_{w^2} = 0$ *whenever* $w \in \Lambda$ *is a path which includes* μ *(i.e., has some decomposition which includes* μ*)*

Proof. (ii) $\implies$ (iv) Assume that $\mu \in B(\Lambda)$. So μ does not lie on a cycle. Therefore for any path w that includes μ, the source and range of w can't be the same (or else w would be a cycle containing μ). So $L_w^2 = L_{w^2} = 0$.

(iv) $\implies$ (ii) Now assume that $L_{w^2} = 0$ for any path w containing μ. Let $x = s(w)$. If w is a cycle, then

$$L_{w^2}\xi_x = \xi_{w^2} \neq 0$$

so $L_{w^2} \neq 0$, which would be a contradiction. Thus, μ is not contained in a cycle. So $\mu \in B(\Lambda)$.

(iii) $\implies$ (iv) Assume that $(AL_\mu)^2 = 0$ for all $A \in \mathfrak{L}_\Lambda$ and let w be a path containing μ. If w is a cycle, we can decompose w as $w = \nu\mu\lambda$ for some $\nu, \lambda \in \Lambda$ with $r(\lambda) = s(\mu)$, $r(\mu) = s(\nu)$, and $r(\nu) = s(\lambda)$. Letting $A = L_{\lambda\nu}$, we have

$$L^2_{\lambda\nu\mu} = (L_{\lambda\nu}L_\mu)^2 = (AL_\mu)^2 = 0.$$

But this is a contradiction, since $\lambda\nu\mu$ is a cycle, and thus $(L_{\lambda\nu\mu})^2\xi_{s(\mu)} = \xi_{(\lambda\nu\mu)^2}$. So it must be true that w is not a cycle, and thus $L_{w^2} = 0$.

(iv) $\implies$ (iii) Now assume that $L_w^2 = L_{w^2} = 0$ whenever $w \in \Lambda$ is a path which includes μ. Let $\nu \in \Lambda$ such that $s(\nu) = r(\mu)$. Then $\nu\mu$ is a path containing μ, so $(L_\nu L_\mu)^2 = L_{\nu\mu}^2 = 0$ by assumption, but if $s(\mu) = r(\nu)$, then $L_{\nu\mu}^2 = L_{\nu\mu\nu\mu} \neq 0$. So it must be that $s(\mu) \neq r(\nu)$ for all $\nu \in \Lambda$ with $s(\nu) = r(\mu)$.

Now let $x = r(\mu)$ and suppose that w is a path with $s(w) = x$. Then $w\mu$ is a path. But $\mu w\mu$ is not a path, as argued in the preceding paragraph. So $\xi_{\mu w\mu} = 0$.

Let $A \in \mathfrak{L}_\Lambda$. Let a_w be the coefficients such that $A\xi_x = \sum_{s(w)=x} a_w \xi_w$. Then

$$(AL_\mu)^2 \xi_{s(\mu)} = AL_\mu A\xi_\mu = AL_\mu \sum_{s(w)=x} a_w \xi_{w\mu} = A \sum_{s(w)=x} a_w \xi_{\mu w\mu} = A(0) = 0.$$

And for any other vertex $y \neq s(\mu)$, we have

$$(AL_\mu)^2 \xi_y = (AL_\mu A) L_\mu \xi_y = 0.$$

So this shows (ii) and (iii) are equivalent.

(iii) $\implies$ (i) Assume that $(AL_\mu)^2 = 0$ for all $A \in \mathfrak{L}_\Lambda$. So $(L_\mu A)^3 = L_\mu (AL_\mu)^2 A = 0$. So the operator $L_\mu A$ is quasinilpotent for all $A \in \mathfrak{L}_\Lambda$. So L_μ must be in the maximal quasinilpotent ideal; that is, $L_\mu \in \operatorname{rad} \mathfrak{L}_\Lambda$.

(i) $\implies$ (ii) Assume that (ii) fails; that is, that $\mu \notin B(\Lambda)$. So there is some cycle $w = \nu\mu\lambda$ containing μ. Then $\mu\lambda\nu$ is also a cycle, as is $(\mu\lambda\nu)^k$ for each $k \geq 1$. So for $k \geq 1$, we have

$$\|(L_\mu L_{\lambda\nu})^k\|^{1/k} \geq \left|\left\langle (L_\mu L_{\lambda\nu})^k \xi_{s(\lambda\nu)}, \xi_{(u\lambda\nu)^k} \right\rangle\right|^{1/k} = |1|^{1/k} > 0,$$

so the operator $L_\mu L_\nu$ has a positive spectral radius and is not quasinilpotent. So L_μ cannot be contained in a quasinilpotent ideal; i.e., $L_\mu \notin \text{rad}\, \mathfrak{L}_\Lambda$, and (i) fails. □

2.5 Reflexivity

A subspace M of a Hilbert space $\mathcal{H}$ is *invariant* for an operator $A \in \mathcal{B}(\mathcal{H})$ if $A(M) \subseteq M$. Sometimes an algebra can be characterized by its invariant subspaces as follows: Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$. The set of all subspaces that are invariant for all operators in $\mathcal{A}$ forms a lattice, written $\text{Lat}(\mathcal{A})$. The set of all operators in $\mathcal{B}(\mathcal{H})$ for which all subspaces in $\text{Lat}(\mathcal{A})$ are invariant forms an algebra, written $\text{Alg Lat}(\mathcal{A})$. It is immediate that $\mathcal{A} \subseteq \text{Alg Lat}(\mathcal{A})$. When the opposite containment holds, $\mathcal{A}$ is called *reflexive*.

It is an early result of Sarason [12] that H^∞ is reflexive. Reflexivity was shown for free semigroupoid algebras of single-vertex graphs by Davidson and Pitts [5] and Arias and Popescu [1]. Kribs and Power used the single-vertex case to show that all free semigroupoid algebras from graphs are reflexive [7], and they extended this to a large class of free semigroupoid algebras from higher rank graphs [8].

One useful tool for proving which higher rank graphs have reflexive semigroupoid algebras is the Double Pure Cycle Property:

Definition 2.5.1 ([8], before Lemma 6.1)**.** A *pure cycle* is a cycle composed of edges of the same minimal degree (meaning a degree $n \in \mathbb{N}^k$ with $|n| = 1$, where $|n|$ is the sum of the components of n). We say a higher rank graph Λ has the *Double Pure Cycle Property* if, for every $v \in \Lambda^0$, there is a path $\lambda \in \Lambda$ with $s(\lambda) = v$ and $r(\lambda) = w$, such that w lies on a *double pure cycle* in the sense that there is a pair of distinct pure cycles $\lambda_i = w\lambda_i w$, $i = 1, 2$, of the same minimal degree, neither of which may be written as a product (concatenation) of cycles.

The following three results are Lemma 6.1, Theorem 6.2, and Corollary 6.3 in [7]. In that paper, Theorem 6.2 and Corollary 6.3 assert *hyper-reflexivity*, which is stronger than reflexivity. We will only use reflexivity here. We include the proofs, as they will be relevant Chapter 5.

Proposition 2.5.2 ([8], Lemma 6.1)**.** *If Λ satisfies the Double Pure Cycle Property, then $\mathfrak{L}_\Lambda$ contains a pair of isometries with mutually orthogonal ranges.*

Proof. First assume that there is a single $v \in \Lambda_0$ with a double pure cycle $\lambda_1 \neq \lambda_2$ such that for every $w \in \Lambda_0$ there is a path $\lambda_w = v\lambda_w w$. We will show that for $k, m \geq 1$ with $k \neq m$, the elements $L_{\lambda_1^k \lambda_2}$ and $L_{\lambda_1^m \lambda_2}$ have orthogonal ranges.

First suppose there exist $\mu_1, \mu_2 \in \Lambda$ such that $L_{\lambda_1^k \lambda_2}\xi_{\mu_1} = L_{\lambda_1^m \lambda_2}\xi_{\mu_2}$. This implies

$$\lambda_1^k \lambda_2 \mu_1 = \lambda_1^m \lambda_2 \mu_2.$$

Without loss of generality, assume $k > m$. By cancellation, we have

$$\lambda_1^{k-m} \lambda_2 \mu_1 = \lambda_2 \mu_2.$$

But λ_1 and λ_2 are composed of edges of the same minimal degree, so by the uniqueness of the factorization property, either λ_1 is an initial segment of λ_2, or vice versa. Either way, this contradicts the fact that λ_1 and λ_2 are pure cycles.

So, to see that the ranges are orthogonal, suppose that we have two elements of the Hilbert space $\mathcal{H}_\Lambda$, say, $h_1 = \sum_{w\in\Lambda} \alpha_w \xi_w$ and $h_2 = \sum_{w\in\Lambda} \beta_w \xi_w$, and consider:

$$\begin{aligned} \langle L_{\lambda_1^k\lambda_2} h_1, L_{\lambda_1^m\lambda_2} h_2\rangle &= \left\langle \sum_{r(w)=v} \alpha_w \xi_{\lambda_1^k\lambda_2 w}, \sum_{r(w)=v} \beta_w \xi_{\lambda_1^m\lambda_2 w} \right\rangle \\ &= 0, \end{aligned}$$

since $\lambda_1^k\lambda_2\mu_1 \neq \lambda_1^m\lambda_2\mu_2$ as shown above, so all of the terms of the two sums are orthogonal. Thus, $L_{\lambda_1^k\lambda_2}$ and $L_{\lambda_1^m\lambda_2}$ have orthogonal ranges.

Note also that $L_{\lambda_1^k\lambda_2}$ and $L_{\lambda_1^m\lambda_2}$ both have initial projection L_v. Let $w \mapsto \{k_a^w, k_b^w\}$ be a one-to-two map from Λ^0 to the positive integers $\mathbb{N}$. As the desired isometries, we may define

$$U = \sum_{w\in\Lambda^0} L_{\lambda_1}^{k_a^w} L_{\lambda_2} L_{\lambda_w} \quad \text{and} \quad V = \sum_{w\in\Lambda^0} L_{\lambda_1}^{k_b^w} L_{\lambda_2} L_{\lambda_w},$$

the sums converging SOT when Λ^0 is infinite.

In the general case, we can divide Λ^0 into subsets W_v where $v \in \Lambda^0$ has a double pure cycle $\lambda_{v_1} \neq \lambda_{v_2}$, and for all $w \in W_v$, there exists $\lambda_w \in \Lambda$ with $\lambda_w = v\lambda_w w$. For each v, we have $\lambda_{v_1}, \lambda_{v_2}$ given by the Double Pure Cycle Property. For each v, define a one-to-two map from W_v to the positive integers $\mathbb{N}$, say $w \mapsto \{k_a^w, k_b^w\}$. For each v, let

$$U_{W_v} = \sum_{w\in W_v} L_{\lambda_{v_1}}^{k_a^w} L_{\lambda_{v_2}} L_{\lambda_w} \quad \text{and} \quad V_{W_v} = \sum_{w\in W_v} L_{\lambda_{v_1}}^{k_b^w} L_{\lambda_{v_2}} L_{\lambda_w},$$

the sums converging SOT when W_v is infinite. Then let

$$U = \sum_{W_v} U_{W_v} \quad \text{and } V = \sum_{W_v} V_{W_v},$$

the sums converging SOT when the number of subsets W_v is infinite. □

Theorem 2.5.3 ([8], Theorem 6.2)**.** *If Λ^t satisfies the Double Pure Cycle property, then $\mathfrak{L}_\Lambda$ is reflexive.*

Proof. As $\mathfrak{L}_\Lambda^t$ is unitarily equivalent to $\mathfrak{R}_\Lambda = \mathfrak{L}_\Lambda'$, the previous lemma shows that $\mathfrak{L}_\Lambda'$ contains a pair of isometries with mutually orthogonal ranges. Thus, the result follows as a direct application of Bercovici's hyper-reflexivity theorem [2]. □

Kribs and Power use this result to prove the following useful corollary:

Corollary 2.5.4 ([8], discussion following Corollary 6.3)**.** *Any single-vertex higher rank graph has a reflexive free semigroupoid algebra.*

They then employ this result to prove reflexivity for the free semigroupoid algebras of a wide class of higher rank graphs ([8], Theorem 6.4).

Chapter 3

Free Semigroupoid Algebras from Categories of Paths

In this chapter we will define the free semigroupoid algebra of a category of paths and characterize its commutant. Then we will look at a few examples of categories of paths which are not higher rank graphs, and begin to see how the free semigroupoid algebras that they generate can differ from those generated by higher rank graphs.

3.1 Definition and Basic Properties

Let Λ be a category of paths. The free semigroupoid algebra for Λ is defined analogously to the free semigroupoid algebra for a graph or higher rank graph. Specifically, we define a Fock space Hilbert space $\mathcal{H}_\Lambda$ with orthonormal basis $\{\xi_\mu\}_{\mu\in\Lambda}$ indexed by the elements of Λ. For $\mu, \nu \in \Lambda$, define:

$$L_\mu \xi_\nu = \begin{cases} \xi_{\mu\nu} & \text{if } s(\mu) = r(\nu) \\ 0 & else \end{cases} .$$

This includes the vertices, or objects, of Λ, which are projections. Note that $\sum\limits_{x\in\Lambda^0} L_x = I$.

Definition 3.1.1. The WOT-closed algebra generated by $\{L_\mu\}_{\mu\in\Lambda}$ is called the *free semigroupoid algebra* for Λ and is written $\mathfrak{L}_\Lambda$.

It is useful to have a notion of the length of a path in a category of paths. A *degree functor* on Λ is a function $\varphi : \Lambda \to \mathbb{N}^n$ such that for all $\mu, \nu \in \Lambda$ satisfying $s(\mu) = r(\nu)$:

$$\varphi(\mu\nu) = \varphi(\mu) + \varphi(\nu).$$

A degree functor can be defined into any abelian group, but we will only consider degree functors into $\mathbb{N}^n$.

We say the degree functor is *non-degenerate* if $\varphi(\alpha) \neq 0$ when $\alpha \notin \Lambda^0$. If Λ is a category of paths with a degree functor, define the length of a path w to be $|w| = |\varphi(w)|$, i.e., the sum of the components of $\varphi(w) \in \mathbb{N}^n$.

Remark 3.1.2. Each vertex has degree 0 because for a vertex x, we have $xx = x$, and thus $\varphi(x) + \varphi(x) = \varphi(x)$, so $\varphi(x) = 0$.

Definition 3.1.3. For a category of paths Λ with a degree functor $\varphi : \Lambda \to \mathbb{N}^n$, define the Cesaro sums of $A \in \mathcal{B}(\mathcal{H})$ by, for $k \in \mathbb{Z}$,

$$\Sigma_k(A) = \sum_{m \in \mathbb{N}^n,\ |m|<k} \left(1 - \frac{|m|}{k}\right) \Psi_m(A),$$

where

$$\Psi_m(A) = \sum_{\ell \in \mathbb{N}^n,\ |\ell| \geq \max\{0, -|m|\}} E_\ell A E_{\ell+m}.$$

The Cesaro sums converge SOT to A by Proposition 2.3.2.

Definition 3.1.4. Given $\mu \in \Lambda$, define the operator R_μ by

$$R_\mu \xi_\nu = \begin{cases} \xi_{\nu\mu} & \text{if } r(\mu) = s(\nu) \\ 0 & \textit{else} \end{cases}$$

Let $\mathfrak{R}_\Lambda$ be the WOT-closed algebra generated by $\{R_\mu\}_{\mu\in\Lambda}$.

We will spend the rest of this section proving that $\mathfrak{R}_\Lambda = \mathfrak{L}'_\Lambda$. The analogous result to Proposition 2.3.8 works with the same proof, however the properties of a category of paths and degree functor are essential to why the proof works, and will be noted below.

Proposition 3.1.5. *Let Λ be a category of paths with a non-degenerate degree functor. Then $\mathfrak{R}'_\Lambda = \mathfrak{L}_\Lambda$.*

Proof. Each L_μ for $\mu \in \Lambda$ is contained in $\mathfrak{R}'_\Lambda$, which is WOT-closed as the commutant of an operator algebra. So $\mathfrak{L}_\Lambda \subseteq \mathfrak{R}'_\Lambda$.

Let $A \in \mathfrak{R}'_\Lambda$ and $x \in \Lambda^0$. As in Proposition 2.3.8, we define $A_x = AL_x$ and

$$p_k = \sum_{|w|<k, s(w)=x} \left(1 - \frac{|w|}{k}\right) a_w L_w,$$

where $\{a_w\}_{w\in\Lambda, s(w)=x}$ are the scalars such that

$$A\xi_x = AR_xL_x\xi_x = R_xA_x\xi_x = \sum_{s(w)=x} a_w\xi_w.$$

Note that for any element $\mu \in \Lambda$ with $r(\mu) = x$ and any $w_1 \neq w_2 \in \Lambda$ with $s(w_1) = s(w_2) = x$, we have by right cancellation that $\xi_{w_1\mu} \neq \xi_{w_2\mu}$. So the same argument as in Lemma 2.3.7 shows that p_k is a bounded linear operator and is in $\mathfrak{L}_G$ for each k.

As in the case for graphs, we want to show that A_x is the SOT limit of p_k. This is proved by showing that $p_k = \Sigma_k(A_x)$ (see Definition 3.1.3), which involves proving the claim that $\Psi_m(A_x)$ is in $\mathfrak{R}'_\Lambda$ for all $m \in \mathbb{N}^n$. The properties of a degree functor are essential in this proof:

Recall that for $m \in \mathbb{N}^n$,

$$\Psi_m(A_x) = \sum_{\ell \in \mathbb{N}^n,\ |\ell| \geq \max\{0, -|m|\}} E_\ell A_x E_{\ell+m}.$$

Now, $A \in \mathfrak{R}'_\Lambda$ by assumption, so A_x commutes with R_μ for all $\mu \in \Lambda$. Since the degree functor φ has the property that $\varphi(\mu)+\varphi(\nu) = \varphi(\mu\nu)$ if $s(\mu) = r(\nu)$, then $R_\mu E_k = E_{k+\varphi(\mu)} R_\mu$ for all $k \in \mathbb{N}^n$. This includes vertices $y \in \Lambda^0$, which satisfy $\varphi(y) = 0$, and so $R_y E_k = E_k R_y$. It is also true that $E_0 R_\mu = 0$ if $\mu \notin \Lambda^0$. Thus, each $\Psi_m(A_x)$ is in $\mathfrak{R}'_\Lambda$. This proves the claim.

This implies that $\Sigma_k(A_x)$ belongs to $\mathfrak{R}'_\Lambda$. The rest of the proof is essentially the same as Proposition 2.3.8. □

Remark 3.1.6. As in Remark 2.3.9, this gives us a Fourier expansion for elements of $\mathfrak{L}_\Lambda$ as follows: let A be in $\mathfrak{L}_\Lambda$ and x a vertex. If $A\xi_x = AL_x\xi_x = R_x(AL_x)\xi_x = \sum_{s(w)=x} a_w \xi_w$, then $A\xi_v = AR_v\xi_x = R_v A\xi_x = \sum_{s(w)=x} a_w \xi_{wv}$ for $v = vx$ in Λ. Thus, the Cesaro partial sums associated with the series $A \sim \sum_{w \in \Lambda} a_w L_w$ converge in the strong operator topology to A.

Corollary 3.1.7. *Let Λ be a category of paths with a non-degenerate degree functor. Then $\mathfrak{L}'_\Lambda = \mathfrak{R}_\Lambda$.*

Proof. Define the transpose Λ^t of a category of paths Λ in the same way as the transpose of a graph (Definition 2.3.5). Then the map $W : \Lambda^t \to \Lambda$ given by $W\xi_{\mu^t} = \xi_\mu$ is a unitary. The rest of the proof is the same as Corollary 2.3.10. □

Finally, we end this section with a lemma that will be useful for the example in Section 3.2:

Lemma 3.1.8. *Let Λ be a category of paths with a finite number of vertices, $|\Lambda^0| = n < \infty$. Then the number of projections in $\mathfrak{L}_\Lambda$ is 2^n.*

Proof. Let $P \in \mathfrak{L}_\Lambda$ be a non-zero projection, with Fourier expansion $P \sim \sum_{w \in \Lambda} a_w L_w$. Then for each vertex $x \in \Lambda^0$, either $P\xi_x = 0$ or

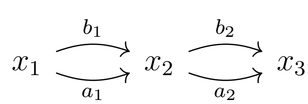
$$\xi_x = P\xi_x = \sum_{s(w)=x} a_w \xi_w.$$

In the latter case, $a_x = 1$ and for all other w such that $s(w) = x$ we have $a_w = 0$. So $P = \sum_{x \in \Lambda^0} a_x L_x$ where each a_x is either 1 or 0.

Thus, every projection on $\mathfrak{L}_\Lambda$ is a sum of projections of the form L_x for a vertex x. Since every such sum is a projection, this means Λ has exactly 2^n projections. □

3.2 Example: Finite-Dimensional

A category of paths with no cycles generates a finite-dimensional free semigroupoid algebra. Consider the following graph:



with the identifications $a_2 b_1 = b_2 a_1$ and $a_2 a_1 = b_2 b_1$. This forms a category of paths Λ, with degree functor equal to the number of edges in a path.

Remark 3.2.1. Note that Λ is not a higher rank graph. If Λ were a higher rank graph, the factorization property and the fact that $a_2 b_1 = b_2 a_1$ together imply $\varphi(a_1) = \varphi(a_2) \neq \varphi(b_1) = \varphi(b_2)$. But this makes the second identification impossible, since a single path cannot have two different degrees. Thus, Λ is not a higher rank graph.

This category of paths has three vertices (x_1, x_2, x_3), four paths of degree 1 (a_1, b_1, a_2, b_2), and two paths of degree 2 (a_2a_1, a_2b_1). The free semigroupoid algebra for Λ is the subalgebra of $M_9(\mathbb{C})$ generated by operators of the form

$$T = \nu_1 L_{x_1} + \nu_2 L_{x_2} + \nu_3 L_{x_3} + \alpha_1 L_{a_1} + \beta_1 L_{b_1} + \alpha_2 L_{a_2} + \beta_2 L_{b_2} + \gamma_1 L_{a_2a_1} + \gamma_2 L_{a_2b_1},$$

or, in matrix form corresponding to the ordered basis $\{\xi_{x_1}, \xi_{x_2}, \xi_{x_3}, \xi_{a_1}, \xi_{b_1}, \xi_{a_2}, \xi_{b_2}, \xi_{a_2a_1}, \xi_{a_2b_1}\}$:

$$T = \begin{bmatrix} \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & \nu_2 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 & \nu_2 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & \nu_3 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 & \nu_3 & 0 & 0 \\ \gamma_1 & 0 & 0 & \alpha_2 & \beta_2 & 0 & 0 & \nu_3 & 0 \\ \gamma_2 & 0 & 0 & \beta_2 & \alpha_2 & 0 & 0 & 0 & \nu_3 \end{bmatrix}.$$

Proposition 3.2.2. *The subalgebra of $M_9(\mathbb{C})$ given by matrices of the above form cannot arise as the free semigroupoid algebra of a higher rank graph.*

Proof. Suppose Λ' is a higher rank graph such that $\mathfrak{L}_{\Lambda'}$ consists of matrices of the above form. For a variable η, let T_η be the operator given by setting $\eta = 1$ and all the other variables to 0. Then we can see that $\mathfrak{L}_{\Lambda'}$ has eight projections: $0, T_{\nu_1}, T_{\nu_2}, T_{\nu_3}, T_{\nu_1} + T_{\nu_2}, T_{\nu_1} + T_{\nu_3}, T_{\nu_2} + T_{\nu_3}$, and I. By Lemma 3.1.8, Λ' must have three vertices y_1, y_2, and y_3. Furthermore, the minimal projections must be those that correspond to projections associated to single vertices, so $T_{\nu_1} = L_{y_1}$, $T_{\nu_2} = L_{y_2}$, and $T_{\nu_3} = L_{y_3}$, and thus the first three basis vectors in this matrix form are ξ_{y_1}, ξ_{y_2}, and ξ_{y_3}.

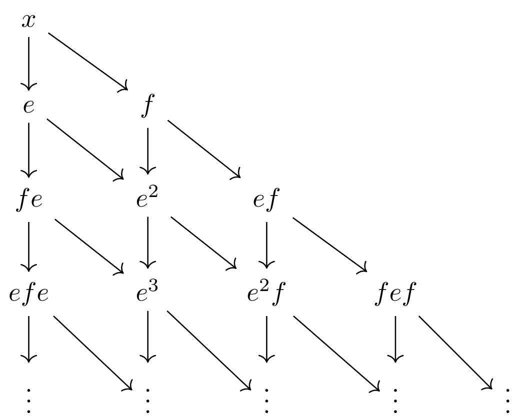
Now for $i = 1, 2, 3$, let P_i be the projection onto $\text{span}(\xi_{y_i})$. We can see from the first two columns of the matrix form of $T \in \mathfrak{L}_{\Lambda'}$ that $P_2 \mathfrak{L}_{\Lambda'} P_1$ and $P_3 \mathfrak{L}_{\Lambda'} P_2$ have two-dimensional ranges. This means there must be two edges from y_1 to y_2, and two edges from y_2 to y_3. So the graph looks like:

$$y_1 \rightrightarrows y_2 \rightrightarrows y_3 \ .$$

But we can also see from the first column of the matrix form that $P_3 \mathfrak{L}_{\Lambda'} P_1$ has two-dimensional range. So there are only two paths from y_1 to y_3. That means there must be two identifications in the graph. In a higher rank graph, there could only be one. Thus, the matrix cannot correspond to the free semigroupoid algebra of a higher rank graph. $\square$

3.3 Example: One Vertex, Two Loops

Let Λ_2 be the category of paths with one vertex x, two edges e and f, and the identification $e^2 = f^2$. Recall that in a graph with edges e and f, the operators L_e and L_f always have orthogonal ranges. However, in this example, L_e and L_f do not have orthogonal ranges, since $L_e(\xi_e) = \xi_{e^2} = L_f(\xi_f)$. The path space of Λ_2 can be expressed by a tree diagram as follows:



Notice that the path space of the *graph* with one vertex and two edges has 2^n edges of length n for each n, whereas this category of paths has only $n+1$ edges of length n for each n.

Define Hilbert spaces based on the rows of the tree diagram:

$$
\begin{aligned}
H_0 &= \operatorname{span}\{\xi_x\} \\
H_1 &= \operatorname{span}\{\xi_e, \xi_f\} \\
H_2 &= \operatorname{span}\{\xi_{fe}, \xi_{e^2}, \xi_{ef}\} \\
H_3 &= \operatorname{span}\{\xi_{efe}, \xi_{e^3}, \xi_{e^2 f}, \xi_{fef}\} \\
&\vdots
\end{aligned}
$$

More formally, note that each path in Λ can be written uniquely as $e^k fefe\ldots$, where e^0 represents the vertex x. Thus, a path can be denoted by $p(n,k)$ where n is the length and k is the initial power of e. When n is clear from context, we write $p(k)$ for brevity.

Using this notation, we can write out in a general way the orthogonal basis described above via the path diagram. First consider paths of even length $2n$.

Define an ordered basis for H_{2n} by

$$\{p(0), p(2), p(4), \ldots, p(2n-2), p(2n), p(2n-1), p(2n-3), \ldots, p(3), p(1)\}.$$

The first half of the numbers are increasing even numbers, then the second half are decreasing odd numbers. Now consider paths of odd length $2n-1$. Define an ordered basis for H_{2n-1} by

$$\{p(1), p(3), p(5), \ldots, p(2n-3), p(2n-1), p(2n-2), p(2n-4), \ldots, p(2), p(0)\}.$$

Here the first half are increasing odd numbers and the second half are decreasing even numbers.

Let P_n be the projection onto H_n. Then $\sum\limits_{n=0}^{\infty} P_n = I$. The following lemma describes the matrix representation of L_e and L_f with respect to this decomposition; it will be helpful in Section 4.2 when we study this free semigroupoid algebra further in order to prove that it is semisimple.

Lemma 3.3.1. *In this matrix decomposition, L_e and L_f are represented by*

$$L_e = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ J_2 & 0 & 0 & 0 & \ldots \\ 0 & S_3 & 0 & 0 & \ldots \\ 0 & 0 & J_4 & 0 & \ldots \\ 0 & 0 & 0 & S_5 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}; \quad L_f = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ S_2 & 0 & 0 & 0 & \ldots \\ 0 & J_3 & 0 & 0 & \ldots \\ 0 & 0 & S_4 & 0 & \ldots \\ 0 & 0 & 0 & J_5 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where J_n is the $n \times (n-1)$ matrix that is an $(n-1) \times (n-1)$ identity matrix with an extra row of 0's at the bottom (i.e., the inclusion map from H_{n-1} to

H_n, sending each basis element of H_{n-1} to the corresponding basis element of H_n), and S_n is the $n \times (n-1)$ matrix that is an $(n-1) \times (n-1)$ identity matrix with an extra row of 0's at the top (i.e., the right shift map from H_{n-1} to H_n, sending each basis element in H_{n-1} to the next basis element of H_n).

Proof. First note that composing e with any path in standard form is the same as just adding one to the power of e in the second path:

$$(e)(e^k fefe\dots) = e^{k+1} fefe\dots$$

Here are the ordered bases of H_{2n-1}, H_{2n} and H_{2n+1} with corresponding basis elements aligned:

H_{2n-1} :

$\{p(1), p(3), \dots, p(2n-3), p(2n-1), p(2n-2), p(2n-4), \dots, p(2), p(0)\}$

H_{2n} :

$\{p(0), p(2), \dots, p(2n-4), p(2n-2), p(2n), \quad p(2n-1), \dots, p(5), p(3), p(1)\}$

H_{2n+1} :

$\{p(1), p(3), \dots, p(2n-3), p(2n-1), p(2n+1), p(2n), \quad \dots, p(6), p(4), p(2), p(0)\}$.

Thus, by the way the bases for these Hilbert spaces are defined, adding e to any path corresponding to a basis element in H_{2n-1} results in a path corresponding to the basis element one place to the right in H_{2n}, and adding e to any path corresponding to a basis element in H_{2n} results in a path corresponding to the basis element in the same position in H_{2n+1}. This gives us the desired matrix representation of L_e.

For L_f, first consider a path of even length, corresponding to a basis element of H_{2n}. It has the form $e^{2k}fefe...fe$ or e^{2n} or $e^{2k+1}fefe...f$. Taking each in turn:

$$(f)(e^{2k}fefe...fe) = e^{2k}ffefe...fe = e^{2k+3}fefe...fe$$

The power of the leading e increases by 3. Next:

$$(f)(e^{2n}) = e^{2n}f$$

The power of the leading e stays the same. Next:

$$(f)(e^{2k+1}fefe...f) = e^{2k}fefefe...f$$

The power of the leading e decreases by 1. In each case, this corresponds to a right shift into H_{2n+1}.

Now consider a path of odd length, corresponding to a basis element in H_{2n-1}. It has the form $e^{2k+1}fefe...fe$ or e^{2n-1} or $e^{2n-2}f$ or $e^{2k}fefe...f$. Looking at each case:

$$(f)(e^{2k+1}fefe...fe) = e^{2k}fefefe...fe.$$

The power of the leading e decreases by 1. Next:

$$(f)(e^{2n-1}) = e^{2n-2}fe$$

The power of the leading e decreases by 1. Next:

$$(f)(e^{2n-2}f) = e^{2n-2}ff = e^{2n}$$

The power of the leading e increases by 2. Next:

$$(f)(e^{2k}fefe...f) = e^{2k}ffefefe...f = e^{2k+3}fefe...f$$

The power of the leading e increases by 3. In each case, this results in a path corresponding to the basis element in the same position in H_{2n}. Thus, L_f is the identity from H_{2n-1} to H_{2n}. □

3.4 Example: Single-Vertex Category of Paths with Non-Zero Nilpotents

The free semigroupoid algebra of a graph or higher rank graph cannot contain a non-zero nilpotent element. This fact for higher rank graphs is implied by Lemma 7.1 of [8], which says:

Lemma 3.4.1. *Let Λ be a higher-rank graph. Let Γ be a non-empty subset of Λ such that $|\lambda_1| = |\lambda_2|$ for all $\lambda_1, \lambda_2 \in \Gamma$. Let $\gamma \in \Gamma$ satisfy $\gamma \geq \lambda$ (with respect to lexicographic order on $\mathbb{N}^k$) for all $\lambda \in \Gamma$. If $\gamma^r = \lambda_1\lambda_2 \ldots \lambda_r$ for $r \geq 1$ and $\lambda_i \in \Gamma$, then in fact $\lambda_1 = \lambda_2 = \cdots = \lambda_r = \gamma$.*

Proposition 3.4.2. *If Λ is a single-vertex higher rank graph, then $\mathfrak{L}_\Lambda$ does not have a non-zero nilpotent.*

Proof. Let $T \in \mathfrak{L}_\Lambda$ be non-zero, with Fourier expansion $\sum\limits_{w \in \Lambda} \alpha_w L_w$. Let $n = \min\{|w| : \alpha_w \neq 0\}$, and let $\Gamma = \{w \in \Lambda : |w| = n\}$. Let $\gamma \in \Gamma$ be maximal with respect to lexicographic ordering. Then for any $k \in \mathbb{N}$, the expansion of T^k contains the term $\alpha_{\gamma^k} L_{\gamma^k}$ with $\alpha_{\gamma^k} \neq 0$. By the minimality of n, this term can only cancel out with other terms associated to paths of length kn.

However, by Lemma 3.4.1, no other path of length kn can be identified with γ^k. So the non-zero term $\alpha_{\gamma^k} L_{\gamma^k}$ cannot cancel out. So T is not nilpotent. □

Unlike graphs and higher rank graphs, the free semigroupoid algebra of a single-vertex category of paths can have a non-zero nilpotent element. To see an example of this, let Λ_3 be the category of paths with one vertex x, three edges a, b, and c, and the following identifications:

- $a^2 = b^2 = c^2$
- $ab = bc = ca$
- $ac = cb = ba$

This category of paths has a nilpotent given by $T = L_a + \omega L_b + \omega^2 L_c$, where ω is a primitive third root of unity. If we expand T^2 and use the identifications in Λ_3 to simplify, we get

$$T^2 = (1 + \omega + \omega^2)L_{a^2} + (1 + \omega + \omega^2)L_{ba} + (1 + \omega + \omega^2)L_{ca} = 0.$$

In order to be sure this nilpotent is non-zero, however, we must verify that the identifications do not reduce to $a = b = c$, in which case T would just be the zero operator.

Lemma 3.4.3. *The identifications above do not imply that any of a, b, or c are equal.*

Proof. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and the sub-semigroup of $(\mathbb{Z}^+, +) \oplus M_3$ generated by $(1, A), (1, B)$, and $(1, C)$. Because the matrices are all invertible, this semigroup has left and right cancellation, and because the first coordinate of the direct sum is always positive, there are no inverses. Furthermore:

- $(1, A)^2 = (1, B)^2 = (1, C)^2 = (2, I)$
- $(1, A)(1, B) = (1, B)(1, C) = (1, C)(1, A) = (2, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix})$
- $(1, A)(1, C) = (1, C)(1, B) = (1, B)(1, C) = (2, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix})$.

Thus, the sub-semigroup generated by $(1, A), (1, B)$, and $(1, C)$ is equivalent to the category of paths Λ described above, proving that the relations do not collapse into any two edges being equal. □

Another interesting fact about this category of paths, is that for every $k \geq 1$, it has exactly 3 paths of length k. For $k \geq 2$, they can be written in the forms: a^k, ba^{k-1}, ca^{k-1}. Consider the Hilbert spaces $\{H_k\}_{k\geq 0}$ where $H_0 = \{x\}$, $H_1 = \{a, b, c\}$, and $H_k = \{a^k, ba^{k-1}, ca^{k-1}\}$ for $k \geq 2$. Then $I = \sum_{k=0}^{\infty} P_k$, where P_k is the projection onto H_k.

Lemma 3.4.4. *In this matrix decomposition, L_a, L_b, and L_c are represented by*

$$L_a = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ A_1 & 0 & 0 & 0 & \dots \\ 0 & A & 0 & 0 & \dots \\ 0 & 0 & A & 0 & \dots \\ 0 & 0 & 0 & A & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad L_b = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ B_1 & 0 & 0 & 0 & \dots \\ 0 & B & 0 & 0 & \dots \\ 0 & 0 & B & 0 & \dots \\ 0 & 0 & 0 & B & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$L_c = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ C_1 & 0 & 0 & 0 & \dots \\ 0 & C & 0 & 0 & \dots \\ 0 & 0 & C & 0 & \dots \\ 0 & 0 & 0 & C & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $A_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, *and $A, B,$ and C are defined as Lemma 3.4.3. (So all the blocks in the block decompositions are 3×3, except the 1,1-block, which is 1×1, the rest of the first column of blocks, which are 3×1, and the rest of the first row of blocks, which are 1×3.)*

Proof. First, note that $L_a(x) = a$, giving us A_1 in the 2,1-block. Next, for all $n \in \mathbb{N}$,

$$L_a(a^n) = a^{n+1};\ L_a(ba^{n-1}) = ca^n;\ L_a(c^{n-1}) = ba^n.$$

which gives us the matrix A in the $(n+1, n)$-block, for all n. The calculations for L_b and L_c are similar. □

So Λ_3 is the WOT-closed algebra generated by L_a, L_b, L_c, and the identity.

As a similar example, consider the category of paths Λ_n with one vertex x, n edges $e_0, e_1, \ldots, e_{n-1}$, and the identifications $e_i e_j = e_{i+\ell} e_{j+\ell}$ for all i, j, ℓ, taken mod n. Note that these relations imply that $e_i^2 = e_j^2$ for all i, j, and thus e_i^2 commutes with every path in Λ. Similar to the previous example, this category of paths has n paths of length k for any k, which can be written as $e_0^k, e_1 e_0^{k-1}, e_2 e_0^{k-1}, \ldots, e_{n-1} e_0^{k-1}$. Additionally, it has a non-zero nilpotent

$$T = \xi_{e_0} + \omega \xi_{e_1} + \omega^2 \xi_{e_2} + \cdots + \omega^{n-1} e_{n_1}$$

where ω is a primitive nth root of unity.

In the case where $n = 3$, this construction gives the category of paths Λ_3 described above. When $n = 2$, we get a category of paths with one vertex, two loops, and the relations $e_0^2 = e_1^2$ and $e_1 e_2 = e_2 e_1$, which is different than the two-loop example described in Section 3.3.

Chapter 4

Semisimplicity

In this chapter, we will give a condition on a category of paths that ensures that the resulting semigroupoid algebra is semisimple, and then we will use this condition to show that the single-vertex categories of paths described in Sections 3.3 and 3.4 are semisimple.

4.1 General Results

Throughout this section, the category of paths Λ is assumed to have a non-degenerate degree functor $\varphi : \Lambda \to \mathbb{N}^n$. For $w \in \Lambda$, let $|w| = |\varphi(w)|$, i.e. the sum of the absolute values of the components of $\varphi(w)$.

Define a *transitive* category of paths, a *connected component* of Λ, and a *cycle* in Λ analogously to Definition 2.4.2. Say that $\mu \in \Lambda$ *lies on a cycle* if there is some $\nu \in \Lambda$ such that $\mu\nu$ is a cycle. Let $B(\Lambda)$ be the collection of paths $\mu \in \Lambda$ which do not lie on a cycle. The set $B(\Lambda)$ is empty if and only if Λ is transitive in each component. As mentioned in Chapter 2, Kribs and Power [7] showed that for a graph G, the Jacobson radical of $\mathfrak{L}_G$ is determined by these paths:

Theorem 4.1.1. *Let G be a graph. Then $\mathfrak{L}_G$ is semisimple if and only if G*

is transitive in each component. When G has finitely many vertices, $|V(G)| = M < \infty$, then the radical is nilpotent of degree at most M and is equal to the WOT-closed two-sided ideal generated by $\{L_\mu : \mu \in B(\Lambda)\}$.

They also proved the same theorem for higher rank graphs in [8]. To obtain a similar result for categories of paths, we will use an extra assumption.

Definition 4.1.2. Let Λ be a category of paths with a degree functor. A path $\mu \in \Lambda$ is called *minimal* if for $\nu, \eta \in \Lambda$, $\mu = \nu\eta$ implies $\mu = \nu$ or $\mu = \eta$.

Definition 4.1.3. Say that a category of paths Λ satisfies property (P) if:

(i) For each vertex $v \in \Lambda^0$, the set of minimal paths in $v\Lambda$ is finite; and

(ii) If $A \neq 0$ and $A = a_1 L_{w_1} + a_2 L_{w_2} + \cdots + a_k L_{w_k}$ where $|w_1| = |w_2| = \cdots = |w_k|$, then there is some $\mu \in \Lambda$ such that $L_\mu A$ is not nilpotent.

If Λ is a higher rank graph, then the second condition, (P)(ii), is equivalent to transitivity in each component, as shown in the next proposition. Notice the similarity to the proof of Proposition 3.4.2.

Proposition 4.1.4. *If Λ is a higher rank graph, then Λ is transitive in each component if and only if Λ satisfies (P)(ii).*

Proof. First suppose Λ satisfies (P)(ii) and let $\nu \in \Lambda$. By (P)(ii), there is some $\mu \in \Lambda$ such that $L_\mu L_\nu$ is not nilpotent. Thus, $L_{\mu\nu}^2 = L_{\mu\nu\mu\nu}$ is not equal to 0. So $\mu\nu$ must be a cycle. Thus, every path lies on a cycle, and Λ is transitive in each component.

Now assume that Λ is transitive in each component, and let $A = a_1 L_{w_1} + a_2 L_{w_2} + \cdots + a_k L_{w_k} \in \mathfrak{L}_\Lambda$ such that $A \neq 0$ and $|w_1| = |w_2| = \cdots = |w_k|$. Assume without loss of generality that $a_i \neq 0$ for $i = 1, \ldots, n$. Choose μ

so that μw_1 is a cycle. Let $\Gamma = \{\mu w_i : r(w_i) = s(\mu), i = 1, \dots, n\}$, and let $\gamma = \mu w_{i_0} \in \Gamma$ such that γ is maximal in Γ with respect to lexicographic ordering. Then for any $k \in \mathbb{N}$, the expansion of $(L_\mu T)^k$ contains the term $a_{w_{i_0}}^k L_{\gamma^k}$ with $a_{w_{i_0}}^k \neq 0$. By Lemma 3.4.1, no other path associated to a term in the expansion of $(L_\mu T)^k$ can be identified with γ^k. So the non-zero term $a_{w_{i_0}}^k L_{\gamma^k}$ cannot cancel out. So T is not nilpotent. □

Lemma 4.1.5. *If Λ satisfies (P)(i), then for any vertex v, there are at most finitely many paths in Λ of degree n with range v.*

Proof. Let $v \in \Lambda^0$. By (P)(i), there are only a finite number, say N_1, of minimal paths in $v\Lambda$. For each of those paths μ, there are a finite number of minimal paths in $s(\mu)\Lambda$. Let N_2 be the maximum of those finite numbers. Continue this n times, up to N_n. Then the total number of paths in Λ of degree less than or equal to n with range v is at most

$$N_1 + N_1 N_2 + \cdots + N_1 N_2 N_3 \dots N_n,$$

which is finite. □

The following lemma corresponds to Lemma 5.2 in [7].

Lemma 4.1.6. *If Λ satisfies (P), then $\mathfrak{L}_\Lambda$ is semisimple. In particular, for every non-zero A in $\mathfrak{L}_\Lambda$, there is a path $w \in \Lambda$ such that $L_w A$ is not quasinilpotent.*

Proof. Let $A \in \mathfrak{L}_\Lambda$, with Fourier expansion $A \sim \sum_{w \in \Lambda} a_w L_w$. Let $n = \min\{|w| : a_w \neq 0\}$.

Let $A' = \sum_{|w|=n} a_w L_w$. By condition (ii) of (P), there is some $\mu \in \Lambda$ such that $L_\mu A'$ is not nilpotent. Therefore, since only minimal-degree terms can cancel out other minimal-degree terms, the minimal-degree terms of $(L_\mu A)^k$ do not cancel out for any k. So for any k, $(L_\mu A)^k$ will have a non-zero term in its Fourier expansion of the form $b_{\nu_k} L_{\nu_k}$ where $|\nu_k| = k(n + |\mu|)$. By the minimality of n, such a path ν_k must be equal to $\mu w_k \mu w_{k-1} \dots \mu w_2 \mu w_1$ where each w_i has degree n.

Now, by Lemma 4.1.5, there are only finitely many paths of degree n that end at $s(\mu)$. So the following minimum is well-defined:

$$a := \min\{a_w : |w| = n, r(w) = s(\mu), a_w \neq 0\}.$$

Then $|b_{\nu_k}| \geq a^k$. So for $k \geq 1$, we have

$$\|(L_\mu A)^k\|^{1/k} \geq \left|\left\langle (L_\mu A)^k \xi_{s(\nu_k)}, \xi_{\nu_k} \right\rangle\right|^{1/k} = |b_{\nu_k}|^{1/k} \geq (a^k)^{1/k} = a > 0.$$

Thus, $L_\mu A$ has a positive spectral radius and is not quasinilpotent. But recall the radical rad $\mathfrak{L}_\Lambda$ is equal to the largest quasinilpotent ideal in $\mathfrak{L}_\Lambda$. So A is not in the radical for $A \neq 0$. $\square$

Next, we will show a partial converse to this result, namely, that if $\mathfrak{L}_\Lambda$ is semisimple, then Λ must be transitive in each component. First, note that the following lemma, which corresponds to Lemma 2.4.4 from Chapter 2, works in the category of paths case with the same proof:

Lemma 4.1.7. *The following are equivalent for $\mu \in \Lambda$:*

(i) $L_\mu \in$ *rad* $\mathfrak{L}_\Lambda$

(ii) $\mu \in B(\Lambda)$

(iii) $(AL_\mu)^2 = 0$ *for all* $A \in \mathfrak{L}_\Lambda$

(iv) $L_w^2 = L_{w^2} = 0$ *whenever* $w \in \Lambda$ *is a path which includes* μ *(i.e., has some decomposition which includes* μ*)*

We next consider a block diagonal decomposition of $\mathfrak{L}_\Lambda$. We say that a subset Γ of Λ is *maximally transitive* if :

1. there are paths in both directions between every pair of vertices in Γ
2. if $\mu \in \Gamma$, then $s(\mu)$ and $r(\mu)$ are in Γ
3. if $\mu \in \Lambda$ such that $s(\mu)$ and $r(\mu)$ are in Γ, then $\mu \in \Gamma$
4. Γ is maximal with respect to these properties.

Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ be the maximally transitive components of Λ, and let $\{S_i\}_{i\in\mathcal{I}}$ be the projections $S_i = \sum_{x\in\Lambda_i^0} L_x$. Note that if Λ has M vertices, then $|\mathcal{I}| \leq M$, since every maximally transitive component must have at least one vertex and every vertex is in exactly one maximally transitive component (though that component could be just a vertex with no edges). Thus, we have

$$I = \oplus_{i\in\mathcal{I}} S_i.$$

Now we may consider the block matrix form of $\mathfrak{L}_\Lambda$ with respect to this decomposition. Note that, for $i \neq j$, if the (i,j)-block is non-zero, then the (j,i)-block must be 0, because if there were a path from Λ_i to Λ_j and a path from Λ_j to Λ_i, it would violate the maximality of the maximally transitive components.

A graph version of the following lemma was stated but not explicitly proved in [7], so we include a proof here for the category of paths case even though it is identical to the graph case:

Lemma 4.1.8. *Let $\mathcal{J}$ be the WOT-closed two-sided ideal in $\mathfrak{L}_\Lambda$ generated by $\{L_\mu : \mu \in B(\Lambda)\}$.Then $\mathcal{J}$ is given by the off-diagonal entries of $\mathfrak{L}_\Lambda$ in the decomposition described above.*

Proof. Let $A \in \mathfrak{L}_\Lambda$. For each vertex x, there exist constants $\{a_w\}_{w\in\Lambda, s(w)=x}$ such that

$$A\xi_x = \sum_{s(w)=x} a_w \xi_w.$$

In the block diagonal form of A described above, the coefficient a_w will be in the column block corresponding to $s(w)$ and the row block corresponding to $r(w)$.

So if $A \in \mathfrak{L}_\Lambda$ and the diagonal blocks are 0 in this decomposition, then the Fourier coefficients a_w are 0 for all $w \notin B(\Lambda)$. Thus, the Cesaro sums of A are in $\mathcal{J}$, and since they converge SOT to A, that means $A \in \mathcal{J}$.

Conversely, if $A \in \mathcal{J}$, we know A is a WOT limit of operators in span$\{L_\mu : \mu \in B(\Lambda)\}$. Note that any path $\mu \in B(\Lambda)$ has at most one endpoint in any given maximally transitive component Λ_i. Thus, the block diagonals in this matrix decomposition will be 0 for every L_μ for $\mu \in B(\Lambda)$, and hence also for A. □

Next, we have an expanded version of Lemma 4.1.6:

Theorem 4.1.9. *(a) If Λ satisfies (P), then $\mathfrak{L}_\Lambda$ is semisimple.*

(b) If $\mathfrak{L}_\Lambda$ is semisimple, then Λ is transitive in each component.

(c) If Λ has $M < \infty$ maximally transitive components, and each maximally transitive component satisfies (P), then the radical is nilpotent of degree at most M and is equal to the WOT-closed two-sided ideal generated by $\{L_\mu : \mu \in B(\Lambda)\}$.

Proof. Lemma 4.1.6 proves (a).

For (b), suppose that there is a component of Λ which is not transitive. Then the set $B(\Lambda)$ is nonempty, and Lemma 4.1.7 gives us an edge $\mu \in B(\Lambda)$ such that $L_\mu \in \text{rad } \mathfrak{L}_\Lambda$. Thus $\mathfrak{L}_\Lambda$ has nonzero radical in this case. This does not require the extra assumption on Λ, and is the same as the graph case [7].

It remains to show (c). To this end, let $\mathcal{J}$ be the WOT-closed two-sided ideal in $\mathfrak{L}_\Lambda$ generated by $\{L_\mu : \mu \in B(\Lambda)\}$. We will first show that the radical contains this ideal. By Lemma 4.1.8, $\mathcal{J}$ is given by the off-diagonal entries of $\mathfrak{L}_\Lambda$ in the decomposition

$$I = \oplus_{i \in \mathcal{I}} S_i,$$

where S_i is the projection onto the maximally transitive component Λ_i.

Now, since there are M blocks in each row and column, and only one of the (i, j)- and the (j, i)-block can be non-zero for $i \neq j$, it follows that $\mathcal{J}^M = \{0\}$. Since $\mathcal{J}$ is an ideal, we have for all $X \in \mathfrak{L}_\Lambda$ and $A \in \mathcal{J}$, that $(XA)^M = 0$. Hence $\mathcal{J}$ is contained in rad $\mathfrak{L}_\Lambda$ and is nilpotent of degree at most M.

Finally we need to show that rad $\mathfrak{L}_\Lambda$ is contained in $\mathcal{J}$. So suppose $A \in$ rad $\mathfrak{L}_\Lambda$ with Fourier expansion scalars $\{a_w\}_{w \in \Lambda}$. We will show that a coefficient a_w is non-zero only if $w \in B(\Lambda)$. Suppose by way of contradiction that there is a path ν with $a_\nu \neq 0$ and $\nu \notin B(\Lambda)$. Choose ν so that $|\nu|$ is minimal with this property. Let Λ' be the maximally transitive component of Λ that contains ν.

Let $S = \{w \in \Lambda' : |w| = |\nu|\}$. Let A' be the operator of terms of A corresponding to paths in S; that is, $A' \sim \sum_{w \in S} a_w L_w$. Note that this means $A' \in \mathfrak{L}_{\Lambda'}$. Since we are assuming that (P) holds on Λ', there is some $\mu \in \Lambda'$ such that $L_\mu A'$ is not nilpotent. We now want to show that $L_\mu A$ has positive spectral radius.

The Fourier series of the operator $L_\mu A$ is given by $\sum_{w\in\Lambda} a_w L_{\mu w}$. Taking this to the kth power formally gives us

$$\sum_{w_i,\eta\in\Lambda} a_{w_1}a_{w_2}\ldots a_{w_{k-1}}a_\eta L_{\mu w_1\mu w_2\ldots\mu w_{k-1}\mu\eta}.$$

But in fact, we know each w_i is in Λ' because $s(\mu)$ and $r(\mu)$ are in Λ'.

Let $\mathcal{M} = \{\mu u_1\mu u_2\ldots\mu u_{k-1}\mu u_k : u_i \in S\}$. We will show that it is impossible for all the terms associated to paths in $\mathcal{M}$ to cancel out in the product $(L_\mu A)^k$. Let $w_1, w_2, \ldots, w_{k-1} \in \Lambda'$ with $a_{w_i} \neq 0$, and let $u_1, u_2, \ldots, u_k \in S$ with $a_{u_i} \neq 0$. In what follows, we will determine for which paths $\eta \in \Lambda$ it is possible that $a_\eta \neq 0$ and

$$\mu w_1\mu w_2\ldots\mu w_{k-1}\mu\eta = \mu u_1\mu u_2\ldots\ldots\mu u_k.$$

First suppose $|\eta| < |\nu|$. Since $|\nu|$ is minimal with the property that $a_\nu \neq 0$ and $\nu \notin B(\Lambda)$, this implies $\eta \in B(\Lambda)$. Thus, either $s(\eta) \notin \Lambda'$ or $r(\eta) \notin \Lambda'$. So since $u_k \in S$, then either $s(\eta) \neq s(u_k)$, implying

$$\mu w_1\mu w_2\ldots\mu w_{k-1}\mu\eta \neq \mu u_1\mu u_2\ldots\ldots\mu u_k$$

or $r(\eta) \neq s(\mu)$, implying the path on the left is undefined.

Now suppose $|\eta| > |\nu|$. Then $\mu w_1\mu w_2\ldots\mu w_{k-1}\mu\eta$ has degree larger than $(|\mu| + |\nu|)^k$, since each w_i in in Λ', and thus by the minimality of $|\nu|$, satisfies $|w_i| \geq |\nu|$ for all i. So $\mu w_1\mu w_2\ldots\mu w_{k-1}\mu\eta \neq \mu u_1\mu u_2\ldots\ldots\mu u_k$.

Finally, suppose $|\eta| = |\nu|$. If $\eta \notin \Lambda'$, then, as above, either $s(\eta) \neq s(u_k)$, implying

$$\mu w_1\mu w_2\ldots\mu w_{k-1}\mu\eta \neq \mu u_1\mu u_2\ldots\ldots\mu u_k$$

or $r(\eta) \neq s(\mu)$, implying the path on the left is undefined. If $|\eta| = |\nu|$ and η is in Λ', then $\mu w_1 \mu w_2 \ldots \mu w_{k-1} \mu \eta$ is in $\mathcal{M}$.

Therefore, only terms corresponding to paths in $\mathcal{M}$ can cancel out other terms in $\mathcal{M}$, and we know they do not all cancel out because $L_\mu A'$ is not nilpotent.

Thus, for any k, $(L_\mu A)^k$ will have a non-zero term in its Fourier expansion of the form $b_{w_k} L_{w_k}$ where w_k is the result of concatenating k paths of the form $\mu\gamma$ for $\gamma \in S$. Let $a = \min\{a_w : w \in S\}$, which is well defined by Lemma 4.1.5. Then $|b_{w_k}| \geq a^k$.

So for $k \geq 1$, we have

$$\|(L_\mu A)^k\|^{1/k} \geq \left|\left\langle (L_\mu A)^k \xi_{s(w_k)}, \xi_{w_k} \right\rangle\right|^{1/k} = |b_{w_k}|^{1/k} \geq (a^k)^{1/k} = a > 0.$$

This proves the claim, and implies that a coefficient a_w is non-zero only if $w \in B(\Lambda)$. Thus, the Cesaro sums for A are in $\mathcal{J}$, and they converge SOT to A. So $A \in \mathcal{J}$. □

We will finish this section with one further result on the nilpotency degree of the ideal $\mathcal{J}$. Given a category of paths Λ, we can divide it into maximally transitive components, as in the discussion before Lemma 4.1.8. Let a chain of length n be a set of maximally transitive components $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_n\}$ with paths $w_1, w_2, \ldots, w_{n-1}$ in $B(\Lambda)$ such that w_j begins in Λ_j and ends in Λ_{j+1}. If there are a finite number of maximally transitive components, then all chains are finite.

Proposition 4.1.10. *Let Λ be a category of paths with M maximally transitive components, where $M < \infty$. Let $\mathcal{J}$ be the WOT-closed ideal generated by*

$\{L_\mu : \mu \in B(\Lambda)\}$. The nilpotency degree of $\mathcal{J}$ is equal to the length of the largest chain of maximally transitive components plus 1, which is at most M.

Proof. Let $\{\Lambda_i\}_{i\le M}$ be the maximally transitive components of Λ, and let $\{S_i\}_{i\le M}$ be the projections $S_i = \sum_{x\in\Lambda_i^0} L_x$. Then $I = \oplus_{i\le M} S_i$.

Lemma 4.1.8 says that the ideal $\mathcal{J}$ is given by the off-diagonal entries of $\mathfrak{L}_\Lambda$ in this decomposition. Let $B_{i,j}$ be the block in the ith row and jth column of this decomposition. Let n be the length of the largest chain of maximally transitive components. A chain of length n of maximally transitive components corresponds to a sequence of blocks $B_{j_0,j_1}, B_{j_1,j_2}, \ldots, B_{j_{n-1},j_n}$ such that each $B_{j_k,j_{k+1}}$ is non-zero. Since there are no chains of length bigger than n, $\mathcal{J}^{n+1} = 0$, and $\mathcal{J}$ is nilpotent of degree less than or equal to $n+1$.

Suppose $\{\Lambda_1, \ldots, \Lambda_n\}$ with paths $\{w_1, \ldots, w_{n-1}\}$ is a maximum length chain. Since each component Λ_i is transitive, there is a path $\mu_i \in \Lambda_i$ with $s(\mu_i) = r(w_i)$ and $r(\mu_i) = s(w_{i+1})$. So $w_n\mu_{n-1}w_{n-1}\ldots\mu_2 w_2\mu_1 w_1$ is a path in Λ. Thus,

$$A := L_{\mu_1 w_1} + L_{\mu_2 w_2} + \cdots + L_{\mu_{n-1}w_{n-1}} + L_{w_n}$$

is an element of $\mathcal{J}$ such that $A^n\xi_{s(w_1)} = \xi_{w_n\mu_{n-1}w_{n-1}\ldots\mu_2 w_2\mu_1 w_1} \neq 0$. So the nilpotency degree of $\mathcal{J}$ is equal to $n+1$. □

4.2 Λ_2 is Semisimple

Recall that Λ_2 is the category of paths with one vertex x and two edges e and f satisfying $e^2 = f^2$. The degree functor for Λ is given by the length of the path. In this section, we will show that $\mathfrak{L}_{\Lambda_2}$ is semisimple by showing that Λ_2 satisfies Property (P). Clearly, Λ_2 satisfies (P)(i) since there are only three minimal paths in $x\Lambda$ (namely, x, e, and f). So we must show Λ_2 satisfies (P)(ii).

In Section 3.3, we wrote $\mathcal{H}_\Lambda$ as a direct sum of the Hilbert spaces defined by ordered bases as follows. Recall that the subscript i on H_i gives the length of the path, and the path $\xi_{e^k fe\ldots}$ is denoted by $p(k)$:

$$H_{2n} : \{p(0), p(2), \ldots, p(2n-2), p(2n), p(2n-1), p(2n-3), \ldots, p(3), p(1)\}.$$

$$H_{2n-1} : \{p(1), p(3), \ldots, p(2n-3), p(2n-1), p(2n-2), p(2n-4), \ldots, p(2), p(0)\}.$$

With these Hilbert spaces, we can describe any basis element as (m, n), the $(m+1)$th basis vector in H_n. (So we start counting the basis elements at 0). The following will be helpful:

- If n is even and $m \leq \frac{n}{2}$, then $(m, n) = e^{2m} fe \ldots fe$
- If n is even and $m > \frac{n}{2}$, then $(m, n) = e^{2(n-m)+1} fe \ldots fef$
- If n is odd and $m < \frac{n}{2}$, then $(m, n) = e^{2m+1} fe \ldots fe$
- If n is odd and $m > \frac{n}{2}$, then $(m, n) = e^{2(n-m)} fe \ldots fef$

We will use the notation (m, n) to refer both to a path w and the Hilbert space element ξ_w, depending on context. The next lemma gives us a rule for calculating the result of concatenating two paths with this notation.

Lemma 4.2.1. *Two paths are concatenated according to the following rule:*

$$(m_1, n_1)(m_2, n_2) = \begin{cases} \big(m_1 + m_2, n_1 + n_2\big), & \textit{if } n_2 \textit{ is even} \\ \big(n_1 - m_1 + m_2, n_1 + n_2\big), & \textit{if } n_2 \textit{ is odd} \end{cases}$$

Example 4.2.2. (a) Consider concatenating $(0,2) = fe$ and $(3,4) = e^3f$. Using Lemma 4.2.1,

$$(0,2)(3,4) = (0+3, 2+4) = (3,6)$$

which is the $(3+1)$th element of the ordered basis for H_6 : $\{p(0)$, $p(2)$, $p(4)$, $p(6)$, $p(5)$, $p(3)$, $p(1)\}$, or e^6. Doing the calculation directly, we see

$$(0,2)(3,4) = (fe)(e^3f) = fe^4f = e^4f^2 = e^6.$$

(b) Consider concatenating $(1,5) = e^3fe$ and $(4,5) = e^2fef$. Using Lemma 4.2.1,

$$(1,5)(4,5) = ((5-1)+4, 5+5) = (8,10)$$

which is the $(8+1)$th element of the ordered basis for H_{10} : $\{p(0)$, $p(2)$, $p(4)$, $p(6)$, $p(8)$, $p(10)$, $p(9)$, $p(7)$, $p(5)$, $p(3)$, $p(1)\}$, or e^5fefef. Doing the calculation directly, we see

$$(1,5)(4,5) = (e^3fe)(e^2fef) = e^5fefef.$$

Proof. (of Lemma 4.2.1)

Since the second component is the length of the path, the second component of the concatenations will clearly be the sum of the second components of the individual paths.

For the first component, note that (m_1, n_1) can be written as a sequence of e's and f's. Thus, when we apply (m_1, n_1) to (m_2, n_2), we can do the calculation by applying e and f sequentially. As shown in Lemma 3.3.1:

- if n_2 is even, applying e will add 0 to m_2 (identity)
- if n_2 is odd, applying e will add 1 to m_2 (shift)
- if n_2 is even, applying f will add 1 to m_2 (shift)
- if n_2 is odd, applying f will add 0 to m_2 (identity)

Thus, the form of the product of (m_1, n_1) and (m_2, n_2) depends on whether n_1 and n_2 are even or odd, so there are four cases we must consider.

CASE 1: Suppose n_1 and n_2 are both even. Then for $m_1 \leq \frac{n_1}{2}$,

$$\begin{aligned}(m_1, n_1)(m_2, n_2) &= \underbrace{e^{2m_1}}_{\text{half of these add 0 and half add 1}} \underbrace{fe \dots fe}_{\text{each of these adds 0}} (m_2, n_2) \\ &= (m_1 + m_2, n_1 + n_2),\end{aligned}$$

while if $m_1 > \frac{n_1}{2}$, then

$$\begin{aligned}(m_1, n_1)(m_2, n_2) &= e^{2(n_1 - m_1)+1} fe \dots fef(m_2, n_2) \\ &= \underbrace{e^{2(n_1 - m_1)}}_{\text{half add 0, half add 1}} \underbrace{efe \dots fef}_{\text{each adds 1}}(m_2, n_2) \\ &= \Big((n_1 - m_1) + \big(n_1 - 2(n_1 - m_1)\big) + m_2, n_1 + n_2\Big) \\ &= (m_1 + m_2, n_1 + n_2).\end{aligned}$$

CASE 2: Suppose n_2 is even and n_1 is odd. Then for $m_1 < \frac{n_1}{2}$,

$$\begin{aligned}(a, n_1)(m_2, n_2) &= e^{2m_1+1} fe \ldots fe(m_2, n_2) \\ &= \underbrace{e^{2m_1}}_{\text{half add 0, half add 1}} \underbrace{efe \ldots fe}_{\text{each adds 0}}(m_2, n_2) \\ &= (m_1 + m_2, n_1 + n_2)\end{aligned}$$

while if $m_1 > \frac{n_1}{2}$, then

$$\begin{aligned}(m_1, n_1)(m_2, n_2) &= e^{2(n_1-m_1)} fe \ldots fef(m_2, n_2) \\ &= \underbrace{e^{2(n_1-m_1)}}_{\text{half add 0, half add 1}} \underbrace{fe \ldots fef}_{\text{each adds 1}}(m_2, n_2) \\ &= \Big((n_1 - a) + \big(n_1 - 2(n_1 - a)\big) + m_2, n_1 + n_2\Big) \\ &= (a + m_2, n_1 + n_2).\end{aligned}$$

CASE 3: Suppose n_1 is even and n_2 is odd. Then for $m_1 \leq \frac{n_1}{2}$,

$$\begin{aligned}(m_1, n_1)(m_2, n_2) &= \underbrace{e^{2m_1}}_{\text{half add 0, half add 1}} \underbrace{fe \ldots fe}_{\text{each adds 1}}(m_2, n_2) \\ &= (m_1 + (n_1 - 2m_1) + m_2, n_2) \\ &= (n_1 - a + m_2, n_1 + n_2)\end{aligned}$$

while if $a > \frac{n_1}{2}$, then

$$\begin{aligned}(m_1, n_1)(m_2, n_2) &= e^{2(n_1-m_1)+1} fe \dots fef(m_2, n_2) \\ &= \underbrace{e^{2(n_1-m_1)}}_{\text{half add 0, half add 1}} \underbrace{efe \dots fef}_{\text{each adds 0}}(m_2, n_2) \\ &= \big(n_1 - m_1 + m_2, n_1 + n_2\big).\end{aligned}$$

CASE 4: Finally, suppose both n_1 and n_2 are odd. Then for $m_1 < \frac{n_1}{2}$,

$$\begin{aligned}(a, n_1)(m_2, n_2) &= e^{2m_1+1} fe \dots fe(m_2, n_2) \\ &= \underbrace{e^{2m_1}}_{\text{half add 0, half add 1}} \underbrace{efe \dots fe}_{\text{each adds 1}}(m_2, n_2) \\ &= (m_1 + (n_1 - 2a) + m_2, n_1 + n_2) \\ &= (n_1 - m_1 + m_2, n_1 + n_2)\end{aligned}$$

while if $m_1 > \frac{n_1}{2}$, then

$$\begin{aligned}(m_1, n_1)(m_2, n_2) &= e^{2(n_1-m_1)} fe \dots fef(m_2, n_2) \\ &= \underbrace{e^{2(n_1-m_1)}}_{\text{half add 0, half add 1}} \underbrace{fe \dots fef}_{\text{each adds 0}}(m_2, n_2) \\ &= \big((n_1 - a) + m_2, n_1 + n_2\big).\end{aligned}$$

The formula holds in all 4 cases. This completes the proof. □

Now suppose we want to consider the results of concatenating two paths of length n. Using the formula from Lemma 4.2.1, we can set up a table showing all the products. If n is even, the table looks like:

	(0, n)	(1, n)	(2, n)	...	(n-1, n)	(n, n)
(0, n)	(0, 2n)	(1, 2n)	(2, 2n)	...	(n-1, 2n)	(n, 2n)
(1, n)	(1, 2n)	(2, 2n)	(3, 2n)	...	(n, 2n)	(n+1, 2n)
(2, n)	(2, 2n)	(3, 2n)	(4, 2n)		(n+1, 2n)	(n+2, 2n)
⋮	⋮	⋮	⋮	⋱	⋮	⋮
(n-1, n)	(n-1, 2n)	(n, 2n)	(n+1, 2n)	...	(2n-2, 2n)	(2n-1, 2n)
(n, n)	(n, 2n)	(n+1, 2n)	(n+2, 2n)	...	(2n-1, 2n)	(2n, 2n)

Notice the constant diagonals from bottom left to top right. If n is odd, the table looks like:

	(0, n)	(1, n)	(2, n)	...	(n-1, n)	(n, n)
(0, n)	(n, 2n)	(n+1, 2n)	(n+2, 2n)	...	(2n-1, 2n)	(2n, 2n)
(1, n)	(n-1, 2n)	(n, 2n)	(n+1, 2n)	...	(2n-2, 2n)	(2n-1, 2n)
(2, n)	(n-2, 2n)	(n-1, 2n)	(n, 2n)	...	(2n-3, 2n)	(2n-2, 2n)
⋮	⋮	⋮	⋮	⋱	⋮	⋮
(n-1, n)	(1, 2n)	(2, 2n)	(3, 2n)	...	(n, 2n)	(n+1, 2n)
(n, n)	(0, 2n)	(1, 2n)	(2, 2n)	...	(n-1, 2n)	(n, 2n)

Here the diagonals from top left to bottom right are constant.

Proposition 4.2.3. *The free semigroupoid algebra $\mathfrak{L}_{\Lambda_2}$ does not contain any nilpotent elements*

Proof. Let $A \in \Lambda_2$ with $A \neq 0$. By the Fourier expansion of A, there are constants a_w such that $A \sim \sum_{w \in \Lambda} a_w L_w$. Let

$$S_k = \{w \in \Lambda : |w| = k \text{ and } a_w \neq 0\}.$$

Since $A \neq 0$, there must be at least one $S_k \neq \emptyset$. Let $n = \min\{k : S_k \neq \emptyset\}$.

Suppose that $A^2 = 0$. This means $A^2\xi_x = 0$, so

$$\sum_{w\in\Lambda}\sum_{z\in\Lambda} a_w a_z \xi_{zw} = 0.$$

In particular, all terms of length $2n$ must cancel out. By the minimality of n, any path of length $2n$ in A^2 can only come as the product of two paths of length n in A.

Consider first the case where n is even. Then $(0, n), (n, n) \notin S_n$, since $(0, n)^2 = (0, 2n)$ and $(n, n)^2 = (2n, 2n)$ are not equivalent to any other paths and could not cancel out. Thus our table from above can be simplified to look like

	(1, n)	(2, n)	...	(n-2, n)	(n-1, n)
(1, n)	(2, 2n)	(3, 2n)	...	(n-1, 2n)	(n, 2n)
(2, n)	(3, 2n)	(4, 2n)		(n, 2n)	(n+1, 2n)
⋮	⋮				⋮
(n-2, n)	(n-1, 2n)	(n, 2n)		(2n-4, 2n)	(2n-3, 2n)
(n-1, n)	(n, 2n)	(n+1, 2n)	...	(2n-3, 2n)	(2n-2, 2n)

But now we can see there is no way for either $(2, 2n)$ or $(2n - 2, 2n)$ to cancel out, meaning that $(1, n)$ and $(n - 1, n)$ cannot be in S_n. So we have:

	(2, n)	...	(n-2, n)
(2, n)	(4, 2n)	...	(n, 2n)
⋮	⋮		⋮
(n-2, n)	(n, 2n)	...	(2n-4, 2n)

Now there is no way for $(4, 2n)$ and $(2n - 4, 2n)$ to cancel out, meaning that $(2, n)$ and $(n - 2, n)$ cannot be in S_n. Continuing in this manner shows that $S_n = \emptyset$, a contradiction.

Now consider the case where n is odd. In this case, the elements $(0, 2n) = (n, n)(0, n)$ and $(2n, 2n) = (0, n)(n, n)$ cannot cancel out, so we can conclude that either $(n, n) \notin S_n$ or $(0, n) \notin S_n$. Either way one row and column of the multiplication table can be removed. As an example, let's look at the case where $(n, n) \notin S_n$. Then we can revise our table to be:

	(0, n)	(1, n)	(2, n)	…	(n-2, n)	(n-1, n)
(0, n)	(n, 2n)	(n+1, 2n)	(n+2, 2n)	…	(2n-2, 2n)	(2n-1, 2n)
(1, n)	(n-1, 2n)	(n, 2n)	(n+1, 2n)	…	(2n-3, 2n)	(2n-2, 2n)
(2, n)	(n-2, 2n)	(n-1, 2n)	(n, 2n)	…	(2n-4, 2n)	(2n-3, 2n)
⋮	⋮	⋮	⋮	⋱	⋮	⋮
(n-2, n)	(2, 2n)	(3, 2n)	(4, 2n)	…	(n, 2n)	(n+1, 2n)
(n-1, n)	(1, 2n)	(2, 2n)	(3, 2n)	…	(n-1, 2n)	(n, 2n)

Then neither $(1, 2n) = (n-1, n)(0, n)$ nor $(2n-1, 2n) = (0, n)(n-1, n)$ can cancel out. So either $(n-1, n)$ or $(0, n)$ is not in S_n. So we can remove another row and column from the matrix. Continuing in this manner shows that $S_n = \emptyset$, a contradiction.

Thus, $A^2 = 0$, and by induction $A^{2^k} \neq 0$ for all k. Furthermore, if $m \in \mathbb{N}$, then there is some k with $2^k > m$ and $A^{2^k} \neq 0$. So $A^m \neq 0$. Thus, A is not nilpotent. □

Corollary 4.2.4. *The free semigroupoid algebra $\mathfrak{L}_{\Lambda_2}$ is semisimple.*

Proof. As mentioned at the beginning of this section, Λ_2 satisfies (P)(i) because there are only three minimal paths in $x\Lambda$ (namely, x, e, and f). Also, Proposition 4.2.3 shows that Λ_2 satisfies (P)(ii). Thus Λ_2 satisfies (P), and so $\mathfrak{L}_{\Lambda_2}$ is semisimple. □

4.3 Λ_n is Semisimple for $n \leq 8$

Recall the 3-loop example from Section 3.4, where Λ_3 is the category of paths given by the graph with one vertex x, three edges a, b, and c, and the identifications:

- $a^2 = b^2 = c^2$
- $ab = bc = ca$
- $ac = cb = ba$

We saw that $\mathfrak{L}_{\Lambda_3}$ has a non-zero nilpotent $T = L_a + \omega L_b + \omega^2 L_c$, where ω is a primitive third root of unity. We will now show that $\mathfrak{L}_{\Lambda_3}$ is nonetheless semisimple.

Proposition 4.3.1. *The free semigroupoid algebra $\mathfrak{L}_{\Lambda_3}$ is semisimple.*

Proof. We will show that Λ_3 satisfies (P). First note that Λ_3 satisfies (P)(i) because there are only four minimal paths in $x\Lambda$ (namely, x, a, b, and c).

To show let Λ_3 satisfies (P)(ii), let $T = \alpha_1 L_{w_1} + \alpha_2 L_{w_2} + \cdots + \alpha_n L_{w_n} \in \mathfrak{L}_{\Lambda_3}$ be non-zero with $|w_1| = \cdots = |w_n|$. Since Λ_3 has only three distinct paths of any given length, we know in fact that $T = \alpha_1 L_x$ or $T = xL_{a^n} + yL_{ba^{n-1}} + zL_{ca^{n-1}}$ for $x, y, z \in \mathbb{C}$. Clearly L_x is not nilpotent, so assume $T = xL_{a^n} + yL_{ba^{n-1}} + zL_{ca^{n-1}}$ for $x, y, z \in \mathbb{C}$ and $n \geq 1$.

Assume first that n is even. We have the following multiplication table:

	a^n	ba^{n-1}	ca^{n-1}
a^n	a^{2n}	ba^{2n-1}	ca^{2n-1}
ba^{n-1}	ba^{2n-1}	ca^{2n-1}	a^{2n}
ca^{n-1}	ca^{2n-1}	a^{2n}	ba^{2n-1}

So if $T = xL_{a^n} + yL_{ba^{n-1}} + zL_{ca^{n-1}}$, then

$$T^2 = (x^2 + 2yz)L_{a^{2n}} + (2xy + z^2)L_{ba^{2n-1}} + (2xz + y^2)L_{ca^{2n-1}}.$$

Thus, $T^2 = 0$ if and only if

$$x^2 + 2yz = 0$$
$$2xy + z^2 = 0$$
$$2xz + y^2 = 0,$$

which implies $x = y = z = 0$.

Thus, $T^2 \neq 0$, and T^2 has the form $x'L_{a^{2n}} + y'L_{ba^{2n-1}} + z'L_{ca^{2n-1}}$, and thus is still a sum of terms with even-length paths. So the same argument applies repeatedly, showing that for all k, $T^{2^k} \neq 0$. If $T^m = 0$ for any m, then for $2^k > m$, we would have $T^{2^k} = 0$, a contradiction. So T is not nilpotent.

Now suppose again that $T = xL_{a^n} + yL_{ba^{n-1}} + zL_{ca^{n-1}}$, but now n is odd. Then $L_aT = xL_{a^{n+1}} + yL_{ba^n} + zL_{ca^n}$ is a sum of even length terms, so by the previous argument, L_aT is not nilpotent. Thus, Λ_3 satisfies (P)(ii).

Therefore, Λ_3 satisfies Property (P), and thus is semisimple. □

The argument can be generalized in the following way:

Proposition 4.3.2. *The free semigroupoid algebra $\mathfrak{L}_{\Lambda_n}$ is semisimple for $n \leq 8$.*

Proof. Consider the category of paths Λ_n from Section 3.4 with one vertex x, n edges $e_0, e_1, \ldots, e_{n-1}$, and the identifications $e_ie_j = e_{i+\ell}e_{j+\ell}$ for all i, j, ℓ, taken mod n. If k is even, then the product of two standard-form elements

$e_ie_0^{k-1}$ and $e_je_0^{k-1}$ is

$$\begin{aligned} e_ie_0^{k-1}e_je_0^{k-1} &= e_ie_0e_je_0^{2k-3} \\ &= e_ie_{n-j}e^{2k-2} \\ &= e_{i+j}e^{2n-1}. \end{aligned}$$

Thus, given the element $T = \sum_{i=0}^{n-1} x_ie_ie_0^{k-1}$, we have

$$T^2 = \sum_{\ell=0}^{n-1} \sum_{i+j=\ell} x_ix_je_{i+j}e_0^{k-1}.$$

So $T^2 = 0$ if the system of equations given by

$$\left\{ \sum_{i=0}^{n-1} x_ix_{\ell-i} = 0 : \ell = 0, 1, \ldots, n-1; \text{ subscripts taken mod } n \right\}$$

has only the trivial solution. When this is the case, as it is for at least $n \leq 8$, the same argument for the $n = 3$ case shows that Λ_n is semisimple. $\square$

Chapter 5

Reflexivity

In this chapter, we will first prove some general results for reflexivity which are based on those in Kribs and Power's papers [7], [8]. However, their main result on reflexivity depends on the reflexivity of free semigroupoid algebras of single-vertex graphs and single-vertex higher rank graphs. Therefore, Sections 5.2 and 5.3 are dedicated to proving reflexivity for the single-vertex categories of paths from Section 3.4.

5.1 General Results

The following definition is an adjustment of the Double Pure Cycle Property for graphs, Definition 2.5.1.

Definition 5.1.1. Say that a vertex v in a category of paths Λ has *double pure cycles* if there exist cycles $\lambda_1 \neq \lambda_2$ at v such that $\lambda_1\mu_1 \neq \lambda_2\mu_2$ for all $\mu_1, \mu_2 \in \Lambda$. Then Λ satisfies the *Double Pure Cycle Property* if for every $w \in \Lambda^0$, there exists $\lambda_w \in \Lambda$ such that $s(\lambda_w) = w$ and $r(\lambda_w)$ has double pure cycles.

Example 5.1.2. Any single-vertex graph with two or more edges satisfies the Double Pure Cycle Property in this sense, as does a single-vertex higher-rank graph with at least two edges of the same color. An example of a category of paths that is not a higher rank graph and satisfies this property is the category of paths Λ with one vertex x, three edges e, f, and g, and the identification $e^2 = f^2$. Then e and g are non-equal cycles satisfying $e\mu_1 \neq g\mu_2$ for all $\mu_1, \mu_2 \in \Lambda$.

Neither the example from Section 3.3 nor the examples from Section 3.4 satisfy this Double Pure Cycle Property, however.

Proposition 5.1.3. *Suppose that Λ is a countable category of paths which satisfies the Double Pure Cycle Property. Then $\mathfrak{L}_\Lambda$ contains a pair of isometries with mutually orthogonal ranges.*

Proof. With the adjusted definition of double pure cycles, this follows by the same proof as Proposition 2.5.2. The key step is shows that, for vertex v with double pure cycles $\lambda_1 \neq \lambda_2$, the operators $L_{\lambda_1^k\lambda_2}$ and $L_{\lambda_1^m\lambda_2}$ are orthogonal for $k \neq m$. That is, for all $\mu_1, \mu_2 \in \Lambda$, we must show $\lambda_1^k\lambda_2\mu_1 \neq \lambda_1^m\lambda_2\mu_2$. But this follows directly from the adjusted definition of double pure cycles. □

Given a category of paths Λ, recall from Definition 2.3.5 that Λ^t is the category of paths with the same vertex set Λ^0, but all the paths are oriented in the opposite direction. This is called the transpose of Λ.

Theorem 5.1.4. *If Λ is a countable category of paths with a non-degenerate degree functor such that Λ^t satisfies the Double Pure Cycle Property, then $\mathfrak{L}_\Lambda$ is reflexive.*

Proof. Since $\mathfrak{L}_\Lambda^t$ is unitarily equivalent to $\mathfrak{R}_\Lambda = \mathfrak{L}_\Lambda'$ by the unitary from Corollary 2.3.10, we know that $\mathfrak{L}_\Lambda'$ contains a pair of isometries with mutually orthogonal range. Thus by Bercovici's Theorem, $\mathfrak{L}_\Lambda$ is reflexive. □

Definition 5.1.5. We say v is a *radiating vertex* if for all $\lambda \in \Lambda$, $r(\lambda) = v$ implies $s(\lambda) = v$.

Proposition 5.1.6. *Suppose that Λ is a category of paths with a non-degenerate degree functor such that each radiating vertex v satisfies*

(a) for the category of paths Λ' consisting of v and all paths $\mu = v\mu v$ in Λ, we have $\mathfrak{L}_{\Lambda'}$ is reflexive

(b) if μ_1 and μ_2 are loops at v and w_1 and w_2 are paths with source v, then $w_1\mu_1 \neq w_2\mu_2$.

Then $\mathfrak{L}_\Lambda$ is reflexive.

Proof. With the restrictions given here, the proof of Theorem 6.4 from [8] applies. □

Corollary 5.1.7. *If Λ is a finite category of paths with a non-degenerate degree functor, then $\mathfrak{L}_\Lambda$ is reflexive.*

Proof. Since Λ is finite, Λ does not contain any loops or cycles. The semigroupoid algebra of a single vertex with no edges is $\mathbb{C}$, which is reflexive. Thus, all vertices of Λ satisfy the conditions of Proposition 5.1.6. □

5.2 Λ_3 is Reflexive

By knowing which single-vertex categories of paths have reflexive free semigroupoid algebras, we can use Proposition 5.1.6 to analyze the reflexivity of multiple-vertex categories of paths. It is still unknown whether $\mathfrak{L}_{\Lambda_2}$ is reflexive for the two-loop example from Section 3.3; however, as we will prove in the rest of this chapter, the family of single-vertex categories of paths described in Section 3.4 is reflexive for all n. In this section, we focus on the $n = 3$ case.

As in Section 3.4, let Λ_3 be the category of paths with one vertex x, three edges a, b, and c, and the identifications $a^2 = b^2 = c^2$, $ab = bc = ca$, and $ac = cb = ba$. In order to show that $\mathfrak{L}_{\Lambda_3}$ is reflexive, we will characterize the structure of elements of $\mathfrak{L}_{\Lambda_3}$ with respect to a particular basis, then show that $T \in \text{Alg Lat}(\mathfrak{L}_{\Lambda_3})$ has the same structure. To this end, let ω be a primitive third root of unity. Note that $\omega + \omega^2 + 1 = 0$. Then the following is an orthogonal basis for $\mathcal{H}_{\Lambda_3}$:

$$\begin{cases} h_1 &= \xi_a + \xi_b + \xi_c \\ j_1 &= \xi_a + \omega\xi_b + \omega^2\xi_c \\ k_1 &= \xi_a + \omega^2\xi_b + \omega\xi_c \end{cases}$$

$$\begin{cases} h_2 &= \xi_{a^2} + \xi_{ba} + \xi_{ca} \\ j_2 &= \xi_{a^2} + \omega^2\xi_{ba} + \omega\xi_{ca} \\ k_2 &= \xi_{a^2} + \omega\xi_{ba} + \omega^2\xi_{ca} \end{cases}$$

$$\vdots$$

$$\begin{cases} h_{2n-1} &= \xi_{a^{2n-1}} + \xi_{ba^{2n-1}} + \xi_{ca^{2n-2}} \\ j_{2n-1} &= \xi_{a^{2n-1}} + \omega\xi_{ba^{2n-2}} + \omega^2\xi_{ca^{2n-2}} \\ k_{2n-1} &= \xi_{a^{2n-1}} + \omega^2\xi_{ba^{2n-2}} + \omega\xi_{ca^{2n-2}} \end{cases}$$

$$\begin{cases} h_{2n} &= \xi_{a^{2n}} + \xi_{ba^{2n-1}} + \xi_{ca^{2n-1}} \\ j_{2n} &= \xi_{a^{2n}} + \omega^2\xi_{ba^{2n-1}} + \omega\xi_{ca^{2n-1}} \\ k_{2n} &= \xi_{a^{2n}} + \omega\xi_{ba^{2n-1}} + \omega^2\xi_{ca^{2n-1}} \end{cases}$$

Lemma 5.2.1. *For an arbitrary element* $A = tL_x + \sum_{n=1}^{\infty} \big(x_n L_{a^n} + y_n L_{ba^{n-1}} + z_n L_{ca^{n-1}}\big)$ *in* $\mathfrak{L}_{\Lambda_3}$*, the matrix form of* $A\big|_{\{\xi_x\}^\perp}$ *with respect to the basis above is:*

$$A\big|_{\{\xi_x\}^\perp} = \begin{bmatrix} tI & 0 & 0 & 0 & 0 & \dots \\ S_1 & tI & 0 & 0 & 0 & \dots \\ T_2 & T_1 & tI & 0 & 0 & \dots \\ S_3 & S_2 & S_1 & tI & 0 & \dots \\ T_4 & T_3 & T_2 & T_1 & tI & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \end{bmatrix}$$

where I *is the* 3×3 *identity matrix,*

$$S_n = \begin{bmatrix} x_n + y_n + z_n & 0 & 0 \\ 0 & x_n + \omega y_n + \omega^2 z_n & 0 \\ 0 & 0 & x_n + \omega^2 y_n + \omega z_n \end{bmatrix}$$

and

$$T_n = \begin{bmatrix} x_n + y_n + z_n & 0 & 0 \\ 0 & x_n + \omega^2 y_n + \omega z_n & 0 \\ 0 & 0 & x_n + \omega y_n + \omega^2 z_n \end{bmatrix}.$$

Proof. For $n \geq 0$, let Q_n be the projection onto edges of length n. Then, with respect to the above basis, we have

$$Q_{2n} L_a Q_{2n-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_{2n+1} L_a Q_{2n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_{2n}L_bQ_{2n-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad Q_{2n+1}L_bQ_{2n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}$$

$$Q_{2n}L_cQ_{2n-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad Q_{2n+1}L_cQ_{2n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}$$

□

Remark 5.2.2. Notice that given any constants $\kappa, \lambda, \mu \in \mathbb{C}$, the system of equations

$$\kappa = x + y + z$$
$$\lambda = x + \omega y + \omega^2 z$$
$$\mu = x + \omega^2 y + \omega z$$

has a unique solution x, y, z. Thus, the above form of A is equivalent to saying that for all $m > n$, there exist constants $\kappa_{m,n}, \lambda_{m,n}, \mu_{m,n} \in \mathbb{C}$ such that

$$Q_nAQ_n = \begin{bmatrix} \kappa_{n,n} & 0 & 0 \\ 0 & \kappa_{n,n} & 0 \\ 0 & 0 & \kappa_{n,n} \end{bmatrix}; \quad Q_mAQ_n = \begin{bmatrix} \kappa_{m,n} & 0 & 0 \\ 0 & \lambda_{m,n} & 0 \\ 0 & 0 & \mu_{m,n} \end{bmatrix}$$

and $\kappa_{m,n} = \kappa_{m+1,n+1}$, $\lambda_{m,n} = \mu_{m+1,n+1}$, $\mu_{m,n} = \lambda_{m+1,n+1}$.

Lemma 5.2.3. *Let $T \in Alg\ Lat(\mathfrak{L}_{\Lambda_3})$. Then the matrix form of $T\big|_{\{\xi_x\}^\perp}$ with* *respect to the basis above is:*

$$T\big|_{\{\xi_x\}^\perp} = \begin{bmatrix} tI & 0 & 0 & 0 & 0 & \dots \\ S_{2,1} & tI & 0 & 0 & 0 & \dots \\ S_{3,1} & S_{3,2} & tI & 0 & 0 & \dots \\ S_{4,1} & S_{4,2} & S_{4,3} & tI & 0 & \dots \\ S_{5,1} & S_{5,2} & S_{5,3} & S_{5,4} & tI & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \end{bmatrix}$$

where I is the 3×3 identity matrix and $S_{m,n} = \begin{bmatrix} \alpha_{m,n} & 0 & 0 \\ 0 & \beta_{m,n} & 0 \\ 0 & 0 & \gamma_{m,n} \end{bmatrix}$.

Proof. The $\mathfrak{L}_{\Lambda_3}$-invariant subspaces $\mathcal{M}_h = \overline{\text{span}}\{h_n : m \geq n\}$, $\mathcal{M}_j = \overline{\text{span}}\{j_n : m \geq n\}$, $\mathcal{M}_k = \overline{\text{span}}\{k_n : m \geq n\}$ are each also invariant for T. So for $m \geq n$, there exist constants $\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n}$ such that

$$Q_m T(h_n) = \alpha_{m,n} h_m$$
$$Q_m T(j_n) = \beta_{m,n} j_m$$
$$Q_m T(k_n) = \gamma_{m,n} k_m.$$

Thus

$$Q_m T Q_n = \begin{bmatrix} \alpha_{m,n} & 0 & 0 \\ 0 & \beta_{m,n} & 0 \\ 0 & 0 & \gamma_{m,n} \end{bmatrix}.$$

Furthermore, note that since $T \in \text{Alg Lat}(\mathfrak{L}_{\Lambda_3})$, the $\mathfrak{L}_{\Lambda_3}$-invariant subspace

$\mathcal{M}_n$ generated by $h_n + j_n + k_n$ is also invariant for T. Now, for all $\zeta \in \mathcal{M}_n$:

$$\langle \zeta, h_n \rangle = \langle \zeta, j_n \rangle = \langle \zeta, k_n \rangle.$$

Thus

$$\langle T(h_n + j_n + k_n), h_n \rangle = \langle T(h_n + j_n + k_n), j_n \rangle = \langle T(h_n + j_n + k_n), k_n \rangle,$$

i.e., $\alpha_{n,n} = \beta_{n,n} = \gamma_{n,n}$. □

The next step is to prove that for any $T \in \text{Alg Lat}(\mathfrak{L}_{\Lambda_3})$, there is some $A \in \mathfrak{L}_{\Lambda_3}$ such that $T\big|_{\{\xi_x\}^\perp} = A\big|_{\{\xi_x\}^\perp}$. This will be shown in Lemma 5.2.5. However, an important piece of the proof of that lemma is the following lemma:

Lemma 5.2.4. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$. If M is a subspace of $\mathcal{H}$ such that $\mathcal{A}\big|_M$ is reflexive, then for all $T \in$ Alg Lat($\mathcal{A}$), there exists $A \in \mathcal{A}$ such that $T\big|_M = A\big|_M$.*

Proof. Let $T \in \text{Alg Lat } \mathcal{A}$, and suppose that $M_0 \subseteq M$ is an invariant subspace for $\mathcal{A}\big|_M$. Then for all $A \in \mathcal{A}$, $A|_M(M_0) \subseteq M_0$. I.e., letting P_M be the projection onto M:

$$AP_M(M_0) \subseteq M_0.$$

But since $M_0 = P_M M_0$, this means $A(M_0) \subseteq M_0$. So $M_0 \in \text{Lat } \mathcal{A}$. Hence, $T(M_0) \subseteq M_0$. Again, since $M_0 = P_M M_0$, this means

$$T\big|_M(M_0) \subseteq M_0.$$

So M_0 is invariant for $T\big|_M$, for all $M_0 \in \text{Lat } \mathcal{A}\big|_M$. Since $\mathcal{A}\big|_M$ is reflexive, this implies that $T\big|_M \in \mathcal{A}\big|_M$. Thus, there is some operator $A \in \mathcal{A}$ such that $T\big|_M = A\big|_M$. □

Lemma 5.2.5. *Let $T \in$ Alg Lat$(\mathfrak{L}_{\Lambda_3})$. There is some $A \in \mathfrak{L}_{\Lambda_3}$ such that $T\big|_{\{\xi_x\}^\perp} = A\big|_{\{\xi_x\}^\perp}$.*

Proof. Let $T \in \text{Alg Lat}(\mathfrak{L}_\Lambda)$. Given the matrix form for T from Lemma 5.2.3, we need to show for all $m \geq n$, that $\alpha_{m,n} = \alpha_{m+1,n+1}$, $\beta_{m,n} = \gamma_{m+1,n+1}$, and $\gamma_{m,n} = \beta_{m+1,n+1}$. We will first show that $\alpha_{m,n} = \alpha_{m+1,n+1}$.

Let $\mathcal{M}_h$ be the $\mathfrak{L}_{\Lambda_3}$-invariant subspace of $\mathcal{H}_{\Lambda_3}$ generated by h_1. Then $\mathcal{M}_h$ has orthogonal basis $\{h_1, h_2, h_3, \dots\}$, and L_a, L_b, and L_c all act as the unilateral shift on $\mathcal{M}_h$. So $\mathfrak{L}_\Lambda\big|_{\mathcal{M}_h} \cong \mathfrak{L}_1$, and thus is reflexive. By Lemma 5.2.4, there is some $A \in \mathfrak{L}_\Lambda$ such that $A\big|_{\mathcal{M}_h} = T\big|_{\mathcal{M}_h}$. Since $A \in \mathfrak{L}_{\Lambda_3}$, there are constants λ_ℓ such that

$$Q_{n+\ell}A(h_n) = \lambda_\ell h_{n+\ell} \quad \text{for all } n \geq 1, \ell \geq 0.$$

Thus,

$$Q_{n+\ell}T(h_n) = \lambda_\ell h_{n+\ell} \quad \text{for all } n \geq 1, \ell \geq 0.$$

This means ℓth diagonal of 3×3 blocks in the matrix decomposition of T all have the same $(1,1)$-entries. In particular, $\alpha_{m,n} = \alpha_{m+1,n+1}$ for all $m > n$.

Now consider the subspace of $\mathcal{H}_{\Lambda_3}$ given by

$$\mathcal{M}_1 = \left\{ \sum_{n=0}^{\infty} \lambda_n (j_n + k_{n+1}) : \sum_{n=1}^{\infty} |\lambda_n|^2 < \infty \right\}.$$

This space is invariant for $\mathfrak{L}_{\Lambda_3}$ because

$$L_a(j_n + k_{n+1}) = j_{n+1} + k_{n+2} \in \mathcal{M}_1,$$

and for n odd,

$$L_b(j_n + k_{n+1}) = \omega(j_{n+1} + k_{n+2}) \in \mathcal{M}_1$$

$$L_c(j_n + k_{n+1}) = \omega^2(j_{n+1} + k_{n+2}) \in \mathcal{M}_1$$

whereas if n is even, then

$$L_b(j_n + k_{n+1}) = \omega^2(j_{n+1} + k_{n+2}) \in \mathcal{M}_1$$

$$L_c(j_n + k_{n+1}) = \omega(j_{n+1} + k_{n+2}) \in \mathcal{M}_1.$$

Thus, $\mathcal{M}_1$ is also invariant for T. Notice that for all $\zeta \in \mathcal{M}_1$, and $n \geq 1$, $\langle \zeta, j_n \rangle = \langle \zeta, k_{n+1} \rangle$. It follows that

$$\langle T(j_n + k_{n+1}), j_m \rangle = \langle T(j_n + k_{n+1}), k_{m+1} \rangle,$$

that is to say, $\beta_{m,n} = \gamma_{m+1,n+1}$.

Similarly, using the $\mathfrak{L}_{\Lambda_3}$-invariant subspace

$$\mathcal{M}_2 = \left\{ \sum_{n=0}^{\infty} \lambda_n (k_n + j_{n+1}) : \sum_{n=1}^{\infty} |\lambda_n|^2 < \infty \right\},$$

we can show that $\gamma_{m,n} = \beta_{m+1,n+1}$. This proves the lemma. □

We need three more results before the final theorem. Two can be found in Douglas' *Banach Algebra Techniques in Operator Theory* [6]:

- **Proposition 4.6**: If $T \in \mathcal{B}(\mathcal{H})$, then ker $T =$ (range $T^*)^\perp$ and ker $T^* =$ (range $T)^\perp$.

  This implies range $T^* =$ (ker $T)^\perp$.

- **Proposition 4.42**: If $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{M}$ is invariant for T if and only if $\mathcal{M}^\perp$ is invariant for T^*.

The last result we need concerns the following vectors, for $0 < |\varepsilon| < 1$:

$$\text{(even length terms);} \quad \text{(odd length terms)}$$

$$A_\varepsilon = \xi_x + \sum_{n=1}^{\infty} \varepsilon^{2n} \xi_{a^{2n}}; \quad A'_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{2n-1} \xi_{a^{2n-1}}$$

$$B_\varepsilon = \xi_x + \sum_{n=1}^{\infty} \varepsilon^{2n} \xi_{ba^{2n-1}}; \quad B'_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{2n-1} \xi_{ba^{2n-2}}$$

$$C_\varepsilon = \xi_x + \sum_{n=1}^{\infty} \varepsilon^{2n} \xi_{ca^{2n-1}}; \quad C'_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{2n-1} \xi_{ca^{2n-2}}$$

Lemma 5.2.6. *Let $R \in Alg\ Lat(\mathfrak{L}_{\Lambda_3})$. For $0 < |\varepsilon| < 1$, the subspace $M =$ span$\{A_\varepsilon, B_\varepsilon, C_\varepsilon, A'_\varepsilon, B'_\varepsilon, C'_\varepsilon\}$ is invariant for $\mathfrak{L}^*_{\Lambda_3}$, and hence for R^*.*

Proof. Note that

$$L_a^*(A_\varepsilon) = \varepsilon A'_\varepsilon \qquad L_a^*(A'_\varepsilon) = \varepsilon A_\varepsilon$$

$$L_a^*(B_\varepsilon) = \varepsilon C'_\varepsilon \qquad L_a^*(B'_\varepsilon) = \varepsilon C_\varepsilon$$

$$L_a^*(C_\varepsilon) = \varepsilon B'_\varepsilon \qquad L_a^*(C'_\varepsilon) = \varepsilon B_\varepsilon$$

$$L_b^*(A_\varepsilon) = \varepsilon B'_\varepsilon \qquad L_b^*(A'_\varepsilon) = \varepsilon B_\varepsilon$$

$$L_b^*(B_\varepsilon) = \varepsilon A'_\varepsilon \qquad L_b^*(B'_\varepsilon) = \varepsilon A_\varepsilon$$

$$L_b^*(C_\varepsilon) = \varepsilon C'_\varepsilon \qquad L_b^*(C'_\varepsilon) = \varepsilon C_\varepsilon$$

$$L_c^*(A_\varepsilon) = \varepsilon C'_\varepsilon \qquad L_c^*(A'_\varepsilon) = \varepsilon C_\varepsilon$$

$$L_c^*(B_\varepsilon) = \varepsilon B'_\varepsilon \qquad L_c^*(B'_\varepsilon) = \varepsilon B_\varepsilon$$

$$L_c^*(C_\varepsilon) = \varepsilon A'_\varepsilon \qquad L_c^*(C'_\varepsilon) = \varepsilon A_\varepsilon$$

Thus, M is invariant for L_a^*, L_b^*, and L_c^*, and hence for $\mathfrak{L}_{\Lambda_3}^*$. By Proposition 4.42 from [6], this means $M^\perp$ is invariant for $\mathfrak{L}_{\Lambda_3}$. Thus $M^\perp$ is invariant for R^*, and M is invariant for R. □

Finally we can prove that this semigroupoid algebra is reflexive:

Theorem 5.2.7. $\mathfrak{L}_{\Lambda_3}$ *is reflexive.*

Proof. Let $T \in \mathfrak{L}_{\Lambda_3}$. Lemma 5.2.5 implies that there is some $A \in \text{Alg Lat } \mathfrak{L}_{\Lambda_3}$ such that $T - A$ is equal to 0 on $\{\xi_x\}^\perp$. Let $R = T - A$. Then $R \in \text{Alg Lat}(\mathfrak{L}_{\Lambda_3})$, $R\big|_{\{\xi_x\}^\perp} = 0$, and there are constants $\{\rho_w\}_{w\in\Lambda_3}$ such that

$$R\xi_x = \sum_{w\in\Lambda_3} \rho_w \xi_w.$$

We want to show that $\rho_w = 0$ for all $w \in \Lambda_3$.

Using Proposition 4.6 from [6], we know that if $R \neq 0$, then range $R^* = (\ker R)^\perp = \text{span}(\xi_x)$. Thus, $R^*(A_\varepsilon) = k\xi_x$ for some k. But also, $R^*(A_\varepsilon)$ is a linear combination $R^*(A_\varepsilon) = \lambda_A A_\varepsilon + \lambda_B B_\varepsilon + \lambda_C C_\varepsilon + \lambda'_A A'_\varepsilon + \lambda'_B B'_\varepsilon + \lambda'_C C'_\varepsilon$.

So for $w \neq x$:

$$0 = \langle R^*(A_\varepsilon), \xi_w \rangle = \begin{cases} \varepsilon^{2n}\lambda_A & : w = a^{2n} \\ \varepsilon^{2n-1}\lambda'_A & : w = a^{2n-1} \\ \varepsilon^{2n}\lambda_B & : w = ba^{2n-1} \\ \varepsilon^{2n-1}\lambda'_B & : w = ba^{2n-2} \\ \varepsilon^{2n}\lambda_C & : w = ca^{2n-1} \\ \varepsilon^{2n-1}\lambda'_C & : w = ca^{2n-2} \end{cases}.$$

So $\lambda_A = \lambda_B = \lambda_C = \lambda'_A = \lambda'_B = \lambda'_C = 0$, and so $R^*(A_\varepsilon) = 0$.

Now we will find R^* explicitly. Let $R\xi_x = \sum_{w\in\Lambda} \rho_w \xi_w$. Let $\mu \in \Lambda_3$ and let $h = \sum_{w\in\Lambda} \lambda_w \xi_w \in \mathcal{H}_{\Lambda_3}$. Then

$$\langle R^*\xi_\mu, h\rangle = \langle \xi_\mu, T'h\rangle = \langle \xi_\mu, R\lambda_x\xi_x\rangle = \langle \xi_\mu, \lambda_x R\xi_x\rangle = \overline{\lambda_x \rho_\mu} = \langle \overline{\rho_\mu}\xi_x, h\rangle.$$

Thus, $R^*\xi_\mu = \overline{\rho_\mu}\xi_x$. So, for any $0 < |\varepsilon| < 1$:

$$\begin{aligned} R^*(A_\varepsilon) &= R^*\big(\xi_x + \sum_{n=1}^{\infty} \varepsilon^{2n}\xi_{a^{2n}}\big) \\ &= \big(\overline{\rho_x} + \sum_{n=1}^{\infty} \varepsilon^{2n}\overline{\rho_{a^{2n}}}\big)\xi_x \end{aligned}$$

But also, we've shown that $R^*(A_\varepsilon) = 0$. So in fact

$$\overline{\rho_x} + \sum_{n=1}^{\infty} \varepsilon^{2n}\overline{\rho_{a^{2n}}} = 0.$$

This holds for all $0 < |\varepsilon| < 1$. So we have a power series equal to 0 on the set $\mathbb{D} \setminus \{0\}$. This implies that $\rho_x = \rho_{a^{2n}} = 0$ for all n.

Similarly, by looking at R^* applied to A'_ε, we can show that $\rho_{a^{2n-1}} = 0$ for all n, and by looking at R^* applied to $B_\varepsilon, B'_\varepsilon, C_\varepsilon$, and C'_ε, we can show that $\rho_{ba^{n-1}} = \rho_{ca^{n-1}} = 0$ for all $n > 0$. Thus, $R = 0$. So $T \in \mathfrak{L}_{\Lambda_3}$. □

5.3 Λ_n is Reflexive

Let $n \geq 2$ and let Λ_n be the category of paths described at the end of Section 3.4, which has one vertex x, n edges $e_0, e_1, \ldots, e_{n-1}$, and the identifications $e_k e_{k+\ell} = e_r e_{r+\ell}$ for $0 \leq k, \ell, r < n$, with all subscripts taken mod n. In particular, notice that this implies that $e_0^2 = e_j^2$ for all j, and hence e_0^2 commutes with all paths. Furthermore, $e_0 e_j = e_{n-j} e_0$ for all j. Using these identities, we can write any path in the form $e_j e_0^k$. This means Λ_n has n paths of length k, for any k.

The following proof that $\mathfrak{L}_{\Lambda_n}$ is reflexive is a generalization of the proof that $\mathfrak{L}_{\Lambda_3}$ is reflexive. When $n = 3$, this proof is exactly the same as the previous proof.

Let ω be a primitive nth root of unity. Because of the orthogonality relation

$$\sum_{k=0}^{n-1} \overline{\omega^{jk}} \omega^{j'k} = \begin{cases} n & : j = j' \\ 0 & : j \neq j' \end{cases}, \quad \text{for } j, j' \in \{0, 1, \ldots, n-1\}$$

of nth roots of unity, the matrix U whose (j,k)th entry is $U_{j,k} = (\omega^j)^k$, is an orthogonal matrix. Thus, if we associate e_j to the $(j+1)$th standard basis vector of $\mathbb{C}^n$, then the columns of U define vectors in $\mathcal{H}_{\Lambda_n}$:

$$U = \begin{bmatrix} h_0^0 & h_0^1 & h_0^2 & \ldots & h_0^{n-1} \end{bmatrix}.$$

More specifically:

$$\begin{cases} h_0^0 & = & \xi_{e_0} & + & \xi_{e_1} & + & \xi_{e_2} & + \cdots + & \xi_{e_{n-1}} \\ h_0^1 & = & \xi_{e_0} & + & \omega\xi_{e_1} & + & \omega^2\xi_{e_2} & + \cdots + & \omega^{n-1}\xi_{e_{n-1}} \\ h_0^2 & = & \xi_{e_0} & + & \omega^2\xi_{e_1} & + & (\omega^2)^2\xi_{e_2} & + \cdots + & (\omega^2)^{n-1}\xi_{e_{n-1}} \\ & \vdots & & & & & & & \\ h_0^{n-1} & = & \xi_{e_0} & + & \omega^{n-1}\xi_{e_1} & + & (\omega^{n-1})^2\xi_{e_2} & + \cdots + & (\omega^{n-1})^{n-1}\xi_{e_{n-1}} \end{cases}$$

and these vectors are all orthogonal. Next, define, for $k \geq 1$:

$$\begin{cases} h_k^0 = L_{e_0}^k h_0^0 \\ h_k^1 = L_{e_0}^k h_0^1 \\ h_k^2 = L_{e_0}^k h_0^2 \\ \vdots \\ h_k^{n-1} = L_{e_0}^k h_0^{n-1} \end{cases}$$

These vectors are also orthogonal. Furthermore, h_k^j and h_ℓ^r are orthogonal if $k \neq \ell$, because h_k^j is a sum of terms associated to paths of length k, and h_ℓ^r is a sum of terms associated to paths of length ℓ. Thus, the vectors $\{h_k^j\}_{k\geq 0, j=1,\ldots,n-1}$, along with ξ_x, form an orthogonal basis for $\mathcal{H}_{\Lambda_n}$.

Lemma 5.3.1. *For an arbitrary element* $A = \sum_{w\in\Lambda_n} \lambda_w L_w$ *in* $\mathfrak{L}_{\Lambda_n}$*, the matrix form of* $A\big|_{\{\xi_x\}^\perp}$ *with respect to the above basis is:*

$$A\Big|_{\{\xi_x\}^\perp} = \begin{bmatrix} \lambda_x I & 0 & 0 & 0 & 0 & \dots \\ R_1 & \lambda_x I & 0 & 0 & 0 & \dots \\ T_2 & T_1 & \lambda_x I & 0 & 0 & \dots \\ R_3 & R_2 & R_1 & \lambda_x I & 0 & \dots \\ T_4 & T_3 & T_2 & T_1 & \lambda_x I & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \end{bmatrix}$$

where I is the $n \times n$ identity matrix,

$$R_k = \begin{bmatrix} \alpha_{k,0} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \alpha_{k,n-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \alpha_{k,n-2} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \alpha_{k,n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{k,2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha_{k,1} \end{bmatrix}.$$

and

$$T_k = \begin{bmatrix} \alpha_{k,0} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \alpha_{k,1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \alpha_{k,2} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \alpha_{k,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{k,n-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha_{k,n-1} \end{bmatrix}.$$

with

$$\alpha_{k,j} = \sum_{r=0}^{n-1} (\omega^r)^j \lambda_{e_r e_0^k}.$$

Proof. Let $r \in \{0, \ldots, n-1\}$. We will first determine how L_{e_r} acts on these basis elements. If k is odd, then:

$$\begin{aligned}
h_{k+1}^m &= L_{e_0}^{k+1} h_0^m \\
&= L_{e_0}^{k+1}\left(\xi_{e_0} + \omega^m \xi_{e_1} + (\omega^m)^2 \xi_{e_2} + \cdots + (\omega^m)^{n-1} \xi_{e_{n-1}}\right) \\
&= \xi_{e_0^{k+2}} + \omega^m \xi_{e_0^{k+1} e_1} + (\omega^m)^2 \xi_{e_0^{k+1} e_2} + \cdots + (\omega^m)^{n-1} \xi_{e_0^{k+1} e_{n-1}} \\
&= \xi_{e_0^{k+2}} + \omega^m \xi_{e_1 e_0^{k+1}} + (\omega^m)^2 \xi_{e_2 e_0^{k+1}} + \cdots + (\omega^m)^{n-1} \xi_{e_{n-1} e_0^{k+1}}
\end{aligned}$$

and so for $0 \leq r < n$:

$$\begin{aligned}
L_{e_r} h_k^m &= L_{e_r} L_{e_0}^k \left(\xi_{e_0} + \omega^m \xi_{e_1} + (\omega^m)^2 \xi_{e_2} + \cdots + (\omega^m)^{n-1} \xi_{e_{n-1}}\right) \\
&= L_{e_r}\left(\xi_{e_0^{k+1}} + \omega^m \xi_{e_0^k e_1} + (\omega^m)^2 \xi_{e_0^k e_2} + \cdots + (\omega^m)^{n-1} \xi_{e_0^k e_{n-1}}\right) \\
&= L_{e_r}\left(\xi_{e_0^{k+1}} + \omega^m \xi_{e_{n-1} e_0^k} + (\omega^m)^2 \xi_{e_{n-2} e_0^k} + \cdots + (\omega^m)^{n-1} \xi_{e_1 e_0^k}\right) \\
&= \xi_{e_r e_0^{k+1}} + \omega^m \xi_{e_r e_{n-1} e_0^k} + (\omega^m)^2 \xi_{e_r e_{n-2} e_0^k} + \cdots + (\omega^m)^{n-1} \xi_{e_r e_1 e_0^k} \\
&= \xi_{e_r e_0^{k+1}} + \omega^m \xi_{e_{r-(n-1)} e_0^{k+1}} + (\omega^m)^2 \xi_{e_{r-(n-2)} e_0^{k+1}} + \cdots + (\omega^m)^{n-1} \xi_{e_{r-1} e_0^{k+1}} \\
&= \xi_{e_r e_0^{k+1}} + \omega^m \xi_{e_{r+1} e_0^{k+1}} + (\omega^m)^2 \xi_{e_{r+2} e_0^{k+1}} + \cdots + (\omega^m)^{n-1} \xi_{e_{r-1} e_0^{k+1}} \\
&= (\omega^r)^{n-m} h_{k+1}^m
\end{aligned}$$

So for k odd, we have

$$Q_{k+1}L_{e_r}Q_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & (\omega^r)^{n-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & (\omega^r)^{n-2} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & (\omega^r)^{n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (\omega^r)^2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & (\omega^r) \end{bmatrix}.$$

Similarly, if k is even, then:

$$\begin{aligned} h_{k+1}^m &= L_{e_0}^{k+1} h_0^m \\ &= L_{e_0}^{k+1}\Big(\xi_{e_0} + \omega^m \xi_{e_1} + (\omega^m)^2 \xi_{e_2} + \dots + (\omega^m)^{n-1}\xi_{e_{n-1}}\Big) \\ &= \xi_{e_0^{k+2}} + \omega^m \xi_{e_0^{k+1}e_1} + (\omega^m)^2 \xi_{e_0^{k+1}e_2} + \dots + (\omega^m)^{n-1}\xi_{e_0^{k+1}e_{n-1}} \\ &= \xi_{e_0^{k+2}} + \omega^m \xi_{e_{n-1}e_0^{k+1}} + (\omega^m)^2 \xi_{e_{n-2}e_0^{k+1}} + \dots + (\omega^m)^{n-1}\xi_{e_1 e_0^{k+1}} \end{aligned}$$

and so for $0 \le r < n$:

$$\begin{aligned} L_{e_r}h_k^m &= L_{e_r}L_{e_0}^k\Big(\xi_{e_0} + \omega^m \xi_{e_1} + (\omega^m)^2 \xi_{e_2} + \dots + (\omega^m)^{n-1}\xi_{e_{n-1}}\Big) \\ &= L_{e_r}\Big(\xi_{e_0^{k+1}} + \omega^m \xi_{e_0^k e_1} + (\omega^m)^2 \xi_{e_0^k e_2} + \dots + (\omega^m)^{n-1}\xi_{e_0^k e_{n-1}}\Big) \\ &= L_{e_r}\Big(\xi_{e_0^{k+1}} + \omega^m \xi_{e_1 e_0^k} + (\omega^m)^2 \xi_{e_2 e_0^k} + \dots + (\omega^m)^{n-1}\xi_{e_{n-1} e_0^k}\Big) \\ &= \xi_{e_r e_0^{k+1}} + \omega^m \xi_{e_r e_1 e_0^k} + (\omega^m)^2 \xi_{e_r e_2 e_0^k} + \dots + (\omega^m)^{n-1}\xi_{e_r e_{n-1} e_0^k} \\ &= \xi_{e_r e_0^{k+1}} + \omega^m \xi_{e_{r-1} e_0^{k+1}} + (\omega^m)^2 \xi_{e_{r-2} e_0^{k+1}} + \dots + (\omega^m)^{n-1}\xi_{e_{r+1} e_0^{k+1}} \\ &= (\omega^r)^m h_{k+1}^m \end{aligned}$$

So for k even, we have

$$Q_{k+1}L_{e_r}Q_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & (\omega^r) & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & (\omega^r)^2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & (\omega^r)^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (\omega^r)^{n-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & (\omega^r)^{n-1} \end{bmatrix}.$$

□

Remark 5.3.2. Note that given constants $\kappa^0, \kappa^1, \dots, \kappa^{n-1} \in \mathbb{C}$, the system of equations

$$\begin{aligned} \kappa^0 &= \sum_{r=0}^{n-1} \lambda_{e_r} \\ \kappa^1 &= \sum_{r=0}^{n-1} \omega^r \lambda_{e_r} \\ \kappa^2 &= \sum_{r=0}^{n-1} (\omega^r)^2 \lambda_{e_r} \\ &\vdots \\ \kappa^{n-1} &= \sum_{r=0}^{n-1} (\omega^r)^{n-1} \lambda_{e_r} \end{aligned}$$

has a unique solution. Thus, the above form of A is equivalent to saying that for all $m > k$,

$$Q_kAQ_k = \begin{bmatrix} \kappa_{k,k} & 0 & \dots & 0 \\ 0 & \kappa_{k,k} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \kappa_{k,k} \end{bmatrix}; \quad Q_mAQ_k = \begin{bmatrix} \kappa^0_{m,k} & 0 & \dots & 0 \\ 0 & \kappa^1_{m,k} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \kappa^{n-1}_{m,k} \end{bmatrix}$$

where $\kappa^j_{m,k} \in \mathbb{C}$ satisfy $\kappa^j_{m,k} = \kappa^{n-j}_{m+1,k+1}$.

Lemma 5.3.3. *Let $T \in$ Alg Lat$(\mathfrak{L}_{\Lambda_n})$. Then the matrix form of $T\big|_{\{\xi_x\}^\perp}$ with respect to the above basis is:*

$$T|_{\{\xi_x\}^\perp} = \begin{bmatrix} \lambda_x I & 0 & 0 & 0 & 0 & \dots \\ R_{2,1} & \lambda_x I & 0 & 0 & 0 & \dots \\ R_{3,1} & R_{3,2} & \lambda_x I & 0 & 0 & \dots \\ R_{4,1} & R_{4,2} & R_{4,3} & \lambda_x I & 0 & \dots \\ R_{5,1} & R_{5,2} & R_{5,3} & R_{5,4} & \lambda_x I & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \end{bmatrix}$$

where

$$R_{m,k} = \begin{bmatrix} \beta^0_{m,k} & 0 & \dots & 0 \\ 0 & \beta^1_{m,k} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta^{n-1}_{m,k} \end{bmatrix}.$$

Proof. Since $T \in$ Alg Lat$(\mathfrak{L}_{\Lambda_n})$, the $\mathfrak{L}_{\Lambda_n}$-invariant subspaces given by $\mathcal{M}_j = \overline{\text{span}}\{h^j_k\}_{k\geq 0}$ are each also invariant for T. So for $m \geq k$, there exist constants $\beta^j_{m,k} \in \mathbb{C}$ such that

$$Q_mT(h^j_k) = \beta^j_{m,k}h^j_m$$

Thus

$$Q_mTQ_k = \begin{bmatrix} \beta^0_{k,m} & 0 & \dots & 0 \\ 0 & \beta^1_{m,k} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta^{n-1}_{m,k} \end{bmatrix}$$

Furthermore, note that since $T \in \text{Alg Lat}(\mathfrak{L}_{\Lambda_n})$, the $\mathfrak{L}_{\Lambda_n}$-invariant subspace $\mathcal{M}_k$ generated by $\sum\limits_{j=0}^{n-1} h^j_k$ is also invariant for T. Now, for all $\zeta \in \mathcal{M}_k$, $\langle \zeta, h^0_k \rangle = \langle \zeta, h^1_k \rangle = \dots = \langle \zeta, h^{n-1}_k \rangle$. Thus,

$$\Big\langle T\big(\sum_{j=0}^{n-1} h^j_k\big), h^0_k \Big\rangle = \Big\langle T\big(\sum_{j=0}^{n-1} h^j_k\big), h^1_k \Big\rangle = \dots = \Big\langle T\big(\sum_{j=0}^{n-1} h^j_k\big), h^{n-1}_k \Big\rangle.$$

i.e. $\beta^0_{k,k} = \beta^1_{k,k} = \dots = \beta^{n-1}_{k,k}$, for all $k \geq 0$. □

Lemma 5.3.4. *For any $T \in$ Alg Lat$(\mathfrak{L}_{\Lambda_n})$, there is some $A \in \mathfrak{L}_{\Lambda_n}$ such that $T\big|_{\{\xi_x\}^\perp} = A\big|_{\{\xi_x\}^\perp}$.*

Proof. Given the matrix form for T from Lemma 5.3.3, we need to show that for all $m \geq k$, and all j from 0 to $n-1$, that $\beta^j_{m,k} = \beta^{n-j}_{m+1,k+1}$. First, consider when $j = 0$.

Let $\mathcal{M}_0$ be the $\mathfrak{L}_{\Lambda_n}$-invariant subspace of $\mathcal{H}_{\Lambda_n}$ generated by h^0_0. Then $\mathcal{M}_0$ has orthogonal basis $\{h^0_0, h^0_1, h^0_2, \dots\}$, and each L_{e_j} acts as the unilateral shift on $\mathcal{M}_0$. So $\mathfrak{L}_{\Lambda_n}\big|_{\mathcal{M}_0} \cong \mathfrak{L}_1$, and thus is reflexive. By Lemma 5.2.4, there is some $A \in \mathfrak{L}_\Lambda$ such that $A\big|_{\mathcal{M}_0} = T\big|_{\mathcal{M}_0}$. Since $A \in \mathfrak{L}_\Lambda$, there are constants λ_ℓ such that

$$Q_{k+\ell}A(h^0_k) = \lambda_\ell h^0_{k+\ell} \quad \text{for all } k \geq 1, \ell \geq 0.$$

Thus,

$$Q_{k+\ell}T(h^0_k) = \lambda_\ell h^0_{k+\ell} \quad \text{for all } k \geq 1, \ell \geq 0.$$

This means the ℓth diagonal of $n \times n$ blocks in the matrix decomposition of T all have the same $(1,1)$-entries. In particular, $\beta^0_{m,k} = \beta^0_{m+1,k+1}$ for all $m > k$.

Now let $j \in \{0, 1, \ldots, n-1\}$ and consider the subspace of $\mathcal{H}_\Lambda$ given by:

$$\mathcal{M}_j = \left\{ \sum_{k=0}^{\infty} \lambda_k (h^j_k + h^{n-j}_{k+1}) : \sum_{k=1}^{\infty} |\lambda_k|^2 < \infty \right\}.$$

This space is invariant for $\mathfrak{L}_{\Lambda_n}$ because for $r \in \{0, 1, \ldots, n-1\}$, if k is odd, then

$$L_{e_r}(h^j_k + h^{n-j}_{k+1}) = (\omega^r)^{n-j}(h^j_{k+1} + h^{n-j}_{k+2}) \in \mathcal{M}_j,$$

while if k is even, then

$$L_{e_r}(h^j_k + h^{n-j}_{k+1}) = (\omega^r)^{j}(h^j_{k+1} + h^{n-j}_{k+2}) \in \mathcal{M}_j.$$

Thus, $\mathcal{M}_j$ is also invariant for T. Notice that for all $\zeta \in \mathcal{M}_j$, and $m \geq 0$, $\langle \zeta, h^j_m \rangle = \langle \zeta, h^{n-j}_{m+1} \rangle$. It follows that:

$$\langle T(h^j_k + h^{n-j}_{k+1}), h^j_m \rangle = \langle T(h^j_k + h^{n-j}_{k+1}), h^{n-j}_{m+1} \rangle,$$

that is to say, $\beta^j_{m,k} = \beta^{n-j}_{m+1,k+1}$. This proves the lemma. $\square$

The last result we need concerns the following vectors, for $0 < |\varepsilon| < 1$:

$$v^j_\varepsilon = \xi_x + \sum_{k=1}^{\infty} \varepsilon^{2k} \xi_{e_j e_0^{2k-1}} \text{ (terms associated to edge } e_j \text{ and even-length paths)}$$

$$w^j_\varepsilon = \sum_{k=1}^{\infty} \varepsilon^{2k-1} \xi_{e_j e_0^{2k-2}} \text{ (terms associated to edge } e_j \text{ and odd-length paths)}$$

Lemma 5.3.5. *Let $R \in$ Alg Lat$(\mathfrak{L}_\Lambda)$. For $0 < |\varepsilon| < 1$, the subspace $M =$*

span$\{v_\varepsilon^j, w_\varepsilon^j : j = 0, 1, \ldots, n-1\}$ *is invariant for* $\mathfrak{L}_{\Lambda_n}^*$*, and hence for* R^**.*

Proof. Recall that subscripts referring to edges are taken mod n, and so $e_r e_j = e_{r-j} e_0$ for all r, j. Thus, if we also take the superscripts of v_ε^j and w_ε^j to be mod n, we have

$$L_{e_r}^*(v_\varepsilon^j) = \varepsilon w_\varepsilon^{r-j} \in M$$

$$L_{e_r}^*(w_\varepsilon^j) = \varepsilon v_\varepsilon^{r-j} \in M.$$

Thus, M is invariant for each $L_{e_r}^*$, and hence for $\mathfrak{L}_{\Lambda_n}^*$. By Proposition 4.42 from [6], this means $M^\perp$ is invariant for $\mathfrak{L}_{\Lambda_n}$, and thus $M^\perp$ is invariant for R^* and hence M is invariant for R. □

Finally we can prove that this semigroupoid algebra is reflexive:

Theorem 5.3.6. $\mathfrak{L}_{\Lambda_n}$ *is reflexive.*

Proof. Let $T \in \mathfrak{L}_{\Lambda_n}$. Lemma 5.3.4 implies that there is some $A \in \text{Alg Lat } \mathfrak{L}_{\Lambda_n}$ such that $T - A$ is equal to 0 on $\{\xi_x\}^\perp$. Let $R = T - A$. Then that $R \in \text{Alg Lat } \mathfrak{L}_{\Lambda_n}$, $R\big|_{\{\xi_x\}^\perp} = 0$, and there are some constants $\{\rho_w\}_{w \in \Lambda_n}$ such that

$$R\xi_x = \sum_{w \in \Lambda} \rho_w \xi_w.$$

We want to show $\rho_w = 0$ for all $w \in \Lambda_n$.

Using Proposition 4.6 from [6] we know that if $R \neq 0$, then range $R^* = (\ker R)^\perp = \text{span}(\xi_x)$. Thus, $R^*(v_\varepsilon^j) = k\xi_x$ for some k. But also, $R^*(v_\varepsilon^j)$ is a linear combination $R^*(v_\varepsilon^j) = \sum_{\ell=0}^{n-1} \lambda_v^\ell v_\varepsilon^\ell + \lambda_w^\ell w_\varepsilon^\ell$. So for $\mu \neq x$:

$$0 = \langle R^*(v_\varepsilon^j), \xi_\mu \rangle = \begin{cases} \varepsilon^{2k} \lambda_v^\ell & : \mu = e_\ell e_0^{2k-1} \text{ for some } \ell, k \\ \varepsilon^{2k-1} \lambda_w^\ell & : \mu = e_\ell e_0^{2k-2} \text{ for some } \ell, k \end{cases}.$$

Thus, $\lambda_v^\ell = \lambda_w^\ell = 0$ for all ℓ. So $R^*(v_\varepsilon^j) = 0$.

Now we will find R^* explicitly. Let $R\xi_x = \sum_{w \in \Lambda_n} \rho_w \xi_w$. Let $\mu \in \Lambda_n$ and let $h = \sum_{w \in \Lambda_n} \lambda_w \xi_w \in \mathcal{H}_{\Lambda_n}$. Then

$$\langle R^*\xi_\mu, h\rangle = \langle \xi_\mu, Rh\rangle = \langle \xi_\mu, R\lambda_x\xi_x\rangle = \langle \xi_\mu, \lambda_x R\xi_x\rangle = \overline{\lambda_x\rho_\mu} = \langle \overline{\rho_\mu}\xi_x, h\rangle.$$

Thus, $R^*\xi_\mu = \overline{\rho_\mu}\xi_x$. Now we have, for any $0 < |\varepsilon| < 1$:

$$\begin{aligned} R^*(v_\varepsilon^j) &= R^*\big(\xi_x + \sum_{k=1}^{\infty} \varepsilon^{2k}\xi_{e_j e_0^{2k-1}}\big) \\ &= \big(\overline{\rho_x} + \sum_{k=1}^{\infty} \varepsilon^{2k}\overline{\rho_{e_j e_0^{2k-1}}}\big)\xi_x \end{aligned}$$

But also, we've shown that $R^*(v_\varepsilon^j) = 0$. So in fact

$$\overline{\rho_x} + \sum_{k=1}^{\infty} \varepsilon^{2k}\overline{\rho_{e_j e_0^{2k-1}}} = 0.$$

This holds for all $0 < |\varepsilon| < 1$. So we have a power series equal to 0 on the set $\mathbb{D} \setminus \{0\}$. This implies that $\rho_x = \rho_{e_j e_0^{2k-1}} = 0$ for all $k > 0$.

Similarly, by looking at $R^*(w_\varepsilon^j)$, we can show that $\rho_{e_j e_0^{2k-2}} = 0$ for all $k > 0$. Thus, $R = 0$. So $T \in \mathfrak{L}_{\Lambda_n}$. $\square$

Bibliography

[1] Alvaro Arias and Gelu Popescu, *Factorization and reflexivity on Fock spaces*, Integral Equations and Operator Theory **23** (1995), 268–286.

[2] Hari Bercovici, *Hyper-reflexivity and the factorization of linear functionals*, Journal of Functional Analysis **158** (1998), 242–252.

[3] Ioana Ciorănescu and Carlos Lizama, *Some applications of Fejer's theorem to operator cosine functions in Banach spaces*, Proceedings of the American Mathematical Society **125** (1997), no. 8, 2353–2362.

[4] Kenneth R. Davidson, *C^*-algebras by example*, American Mathematical Society, Providence, R.I., 1996.

[5] Kenneth R. Davidson and David R. Pitts, *Invariant subspaces and hyper-reflexivity for free semigroup algebras*, Proceedings of the London Mathematical Society **78** (1999), 401–430.

[6] Ronald G. Douglas, *Banach algebra techniques in operator theory*, Springer-Verlag New York Inc., New York, 1998.

[7] David W. Kribs and Stephen C. Power, *Free semigroupoid algebras*, Journal of the Ramanujan Mathematical Society **19** (2004), 117–159.

[8] ———, *The analytic algebras of higher rank graphs*, Mathematical Proceedings of the Royal Irish Academy **106** (2006), 199–218.

[9] Alex Kumjian and David Pask, *Higher rank graph C^*-Alegbras*, New York Journal of Mathematics **6** (2000), 1–20.

[10] Iain Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, vol. 103, American Mathematical Society, Providence, R.I., 2005.

[11] Charles E. Rickart, *General theory of Banach algebras*, D. Van Nostrand Company, LTD, Princeton, New Jersey, 1960.

[12] D. Sarason, *Invariant subspaces and unstarred operator algebras*, Pacific Journal of Mathematics **17** (1966), 511–517.

[13] Jack Spielberg, *Groupoids and C^*-algebras for categories of paths*, Transactions of the American Mathematical Society **366** (2011).