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Frobenius and Homological Dimensions of Complexes

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FROBENIUS AND HOMOLOGICAL DIMENSIONS OF COMPLEXES

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It is known that a module M over a Noetherian local ring R of prime characteristic and positive dimension has finite flat dimension if $\operatorname{Tor}_i^R({}^eR, M) = 0$ for $\dim R$ consecutive positive values of i and infinitely many e. Here ${}^eR$ denotes the ring R viewed as an R-module via the eth iteration of the Frobenius endomorphism. In the case R is Cohen-Macualay, it suffices that the Tor vanishing above holds for a single $e \geqslant \log_p e(R)$, where $e(R)$ is the multiplicity of the ring. This improves a result of D. Dailey, S. Iyengar, and T. Marley, as well as generalizing a theorem due to C. Miller [14] from finitely generated modules to arbitrary modules. We also show that if R is a complete intersection ring then the vanishing of $\operatorname{Tor}_i^R({}^eR, M)$ for single positive values of i and e is sufficient to imply M has finite flat dimension. This extends a result of L. Avramov and C. Miller [2]. Further, in the case where R is Cohen-Macaulay and $J2$, but not necessarily local, we prove an analogous result to the local case.

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Chapter 1

Introduction

Throughout this introduction, let $(R, \mathfrak{m}, k)$ denote a commutative Noetherian local ring of prime characteristic p and positive krull dimension with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Since R has prime characteristic, R will come equipped with the Frobenius endomorphism $f : R \to R$ given by $f(r) = r^p$ for each $r \in R$. It is due to R having characteristic p that this map, f, is a ring homomorphism. Let $e \geq 1$ be an integer. We consider the e-th iteration of the Frobenius endomorphism, $f^e : R \to R$ given by $f(r) = r^{p^e}$. An object of interest throughout will be the ring denoted ${}^eR$. As a ring, ${}^eR$ is identical to R itself. However, we will treat ${}^eR$ as a module over R via f^e. So, given any $r \in R$ and $s \in {}^eR$ we have $s \cdot r = sr$ and $r \cdot s := r^{p^e}s$ where the multiplication on the right hand side of each equality is taking place in ${}^eR$.

For the past half-century the Frobenius endomorphism has proven to be an effective tool for characterizing when a given finitely generated module M over a commutative Noetherian local ring of prime characteristic p has certain homological properties. An early example of interest in this topic can be found in the following famous result by Kunz in 1969 [11].

Theorem 1.0.1. *If R is regular of characteristic p, then ${}^eR$ is a flat R-module for all $e > 0$.*

Once the hypothesis that R is regular is removed, we also lose the flatness of ${}^eR$. In fact, the flatness of ${}^eR$, for any $e > 0$, is equivalent to R being regular. However, even without the flatness, ${}^eR$ can still be used to find the projective, flat, or injective dimension of a module. Note that, for a fixed integer $e > 0$, saying ${}^eR$ is flat is equivalent to saying $\operatorname{Tor}_i^R({}^eR, M) = 0$ for all $i > 0$ and every R-module M. Removing this assumption that R is regular, Peskine-Szpiro [17] and Herzog [8] present us with the following well known theorems respectively.

Theorem 1.0.2. *Let M be a finitely generated R-module with $\operatorname{pd}_R M < \infty$. Then $\operatorname{Tor}_i^R({}^eR, M) = 0$ for all $i, e > 0$.*

Theorem 1.0.3. *Let M be a finitely generated R-module with $\operatorname{Tor}_i^R({}^eR, M) = 0$ for all $i > 0$ and infinitely many $e > 0$. Then $\operatorname{pd}_R M < \infty$.*

In Peskine and Szpiro's result we see that while ${}^eR$ is not necessarily flat, it behaves as though it were flat when tested against finitely generated modules of finite projective dimension. Herzog then gives us the converse of Peskine and Szpiro's (and Kunz's) result. While Herzog's theorem is useful on its own, one might ask if we can generalize it. More specifically, there are two improvements which arise naturally. First, can we get away with checking the vanishing of $\operatorname{Tor}_i^R({}^eR, M)$ for only finitely many $e, i > 0$ to conclude that $\operatorname{pd}_R M < \infty$? Secondly, must we require M to be a finitely generated module in each of Herzog and Peskine-Szpiro's theorem? In response to this second question, Marley and Webb [15] in 2016 gave the following generalization.

Theorem 1.0.4. *Let R be a ring of characteristic p and M an R-module with $\operatorname{Tor}_i^R({}^eR, M) = 0$ for all $i > 0$ and infinitely many $e > 0$. Then $\operatorname{fd}_R M < \infty$.*

In the same paper, Marley and Webb also gave an analog of Peskine and Szpiro's theorem for modules which are not necessarily finitely generated. With these generalizations being made, we turn our attention to the first improvement I mentioned to Herzog's result. There has been a lot of work in answering this question, and a summary of the work is given below. The following are given by Koh-Lee [10], Miller [14], and Avramov-Miller [2] respectively.

Theorem 1.0.5. *Let R be a Noetherian ring and M a finitely generated R-module. There exists a positive integer E (depending only on R) such that if any of the following conditions hold, then $\operatorname{pd}_R M < \infty$:*

(a) $\operatorname{Tor}_i^R({}^eR, M) = 0$ *for* $\operatorname{depth} R + 1$ *consecutive* $i > 0$ *and for some* $e \geq E$.

(b) *R is Cohen-Macaulay of positive dimension and* $\operatorname{Tor}_i^R({}^eR, M) = 0$ *for* $\dim R$ *consecutive* $i > 0$ *and some* $e \geq E$.

(c) *R is a complete intersection and* $\operatorname{Tor}_i^R({}^eR, M) = 0$ *for some* $i, e > 0$.

Note that each of the above theorems require M to be a finitely generated module. In an effort to replicate the above theorems for non-finitely generated modules, Dailey-Iyengar-Marley in [6] give the following result roughly 15 years after the above counterparts.

Theorem 1.0.6. *Let M be an R-module. There exists a positive integer E (depending only on R) such that if any of the following conditions hold, then $\operatorname{fd}_R M < \infty$:*

(a) $\operatorname{Tor}_i^R({}^eR, M) = 0$ *for* $\dim R + 1$ *consecutive* $i > 0$ *and infinitely many* $e > 0$.

(b) *R is Cohen-Macaulay and* $\operatorname{Tor}_i^R({}^eR, M) = 0$ *for* $\dim R + 1$ *consecutive* $i > 0$ *and some* $e \geq E$.

Notice that, in each case, they were able to remove the assumption that M is finitely generated, but it came at a cost. In part (a) we went from only needing a single e sufficiently large to needing infinitely many $e > 0$, and in the Cohen-Macaulay case they needed an extra vanishing of $\operatorname{Tor}_i^R({}^eR, M)$ to get the same result as Miller.

All of the above is an accurate description of the progress in this research area at the time I began my contribution. Jumping into this topic I was first interested in trying to get Avramov and Miller's result for complete intersection rings, shown in 1.0.5(c), to work for non-finitely generated modules. Their proof in the finitely generated case involved the use of complexity. Due to this use of complexity, it was clear that a very different proof would be needed to push this result into the non-finitely generated case as the very definition of complexity heavily uses the assumption that the modules in question are finitely generated. After much work, I stumbled upon an alternative proof of Avramov and Miller's result given, by Dutta in 2003 [7], which evaded the complexity argument. This proof had just what was needed to finish the argument. Later, my work became trying to reduce the number of vanishings needed for $\operatorname{Tor}_i^R({}^eR, M)$ by one in each of of the results by Dailey-Iyengar-Marley shown in 1.0.6. A summary of all this work is shown in the theorem below.

Theorem 1.0.7. *Suppose R has positive dimension, and let M be an R-module. There exists a positive integer E (depending only on R) for which the following are equivalent:*

(a) $\operatorname{fd}_R M < \infty$*;*

(b) $\operatorname{Tor}_i^R({}^eR, M) = 0$ *for* $\dim R$ *consecutive values of* $i > 0$ *and some* $e \geq E$*.*

If R is Cohen-Macaulay then condition (a) is equivalent to:

(c) $\operatorname{Tor}_i^R({}^eR, M) = 0$ *for* $\dim R$ *consecutive values of* $i > 0$ *and some* $e \geq E$*.*

If R is a complete intersection of arbitrary dimension, then condition (a) is equivalent to:

(d) $\operatorname{Tor}_i^R({}^eR, M) = 0$ *for some* $i, e > 0$*.*

In fact, the above is proven in the case where M is an R-complex such that the homology of M is bounded above when indexed homologically (which inherently covers the case were M is an R-module). In the case of complexes, the values of i mentioned in parts (b)-(d) are taken such that $i > \sup \mathrm{H}_*(M)$. Furthermore, analogous results hold for $\operatorname{Ext}_R^i({}^eR, M)$ and injective dimension in the case the Frobenius endomorphism is finite. In fact, with the exception of the proof that (d) implies (a), our strategy will be to first establish the proof for $\operatorname{Ext}_R^i({}^eR, M)$ and injective dimension and then reduce to the case where the Frobenius map is finite.

In chapter 2 we will lay out the necessary background information regarding complexes needed to grasp the details in the main theorem 1.0.7 when given in terms of complexes. For the remainder of this thesis we will refer the equivalence of (a) and (c) in 1.0.7 as the Cohen-Macaulay case, the equivalence of (a) and (b) as the general case, and (a) and (d) the complete intersection case. The proofs of these will be laid out in detail in sections 3.1, 3.2, and 3.3 respectively. To close out this thesis, in chapter 4, we will discuss methods to continue making progress on this topic, as well as a small result in an effort to push the general case even further.

Chapter 2

Some Background and Ancillary Results on Complexes

In this section we will develop some machinery involving complexes of modules which we will need in our discussions later. For more information regarding terminology and conventions regarding complexes, see Avramov and Foxby's paper on unbounded complexes [3]. Throughout this section, let $(R, \mathfrak{m}, k)$ denote a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. In the case R has prime characteristic p, we let $f : R \to R$ denote the Frobenius endomorphism; i.e., $f(r) = r^p$ for every $r \in R$. For an integer $e \geqslant 1$ let ${}^eR$ denote the ring R viewed as an R-algebra via f^e; i.e., for $r \in R$ and $s \in {}^eR$, $r \cdot s := f^e(r)s = r^{p^e}s$. If ${}^eR$ is finitely generated as an R-module for some (equivalently, all) $e > 0$, we say that R is F*-finite*.

If M is an R-complex, we write M_* (respectively, M^*) to emphasize when we are indexing M homologically (respectively, cohomologically). It will occasionally be useful to work in the derived category of R, which will be denoted by $\mathsf{D}(R)$. We use the symbol '$\simeq$' to denote an isomorphism in $\mathsf{D}(R)$.

Recall that for R-complexes M and N the left derived functor of the tensor product is denoted $M \otimes_R^L N$, and the right derived homomorphism functor is denoted $\mathsf{RHom}_R(M, N)$. It is also worth noting that if M and N are bounded below complexes, one of which consists of flat modules, then $M \otimes_R^L N \simeq M \otimes_R N$. Similarly, if M is a bounded below complex of projective R-modules, or N is a bounded above complex of injective R-modules, then $\mathsf{R}\mathrm{Hom}_R(M,N) \simeq \mathrm{Hom}_R(M,N)$.

We first establish how the R-algebra ${}^eR$ (i.e., restriction of scalars) behaves with respect to flat extensions.

Lemma 2.0.1. *Consider a commutative square of ring homomorphisms:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

where B is flat over A, and D is flat over C. Then for any A-complex M and any C-complex N one has for each i an isomorphism of D-modules

$$\mathrm{Tor}_i^A(M,N) \otimes_C D \cong \mathrm{Tor}_i^B(M \otimes_A B, N \otimes_C D).$$

Proof. We have the following isomorphisms in $\mathsf{D}(D)$:

$$\begin{aligned} (M \otimes_A^{\mathbf{L}} N) \otimes_C D &\simeq (M \otimes_A^{\mathbf{L}} D) \otimes_C^{\mathbf{L}} N \\ &\simeq M \otimes_A^{\mathbf{L}} (B \otimes_B^{\mathbf{L}} D) \otimes_C^{\mathbf{L}} N \\ &\simeq (M \otimes_A B) \otimes_B^{\mathbf{L}} (D \otimes_C N). \end{aligned}$$

Taking homology and using that $- \otimes_C D$ is exact gives the desired result. $\square$

Corollary 2.0.2. *Suppose R has prime characteristic and S is a flat R-algebra. Let M be an R-complex and e a positive integer. Then for each i there is an isomorphism of ${}^eS$-modules*

$$\mathrm{Tor}_i^R(M, {}^eR) \otimes_{{}^eR} {}^eS \cong \mathrm{Tor}_i^S(M \otimes_R S, {}^eS).$$

Proof. We have a commutative square of ring maps:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ {}^eR & \longrightarrow & {}^eS \end{array}$$

Since S is flat over R, ${}^eS$ is flat over ${}^eR$. The result now follows from Lemma 2.0.1. □

To continue on, we will need a technical result from Avramov and Foxby presented in [3, Lemma 4.4(I,F)].

Lemma 2.0.3. *Let R be a commutative ring with N and M being R-complexes.*

(I) Assume that P is bounded below complex with each P_i a finitely generated projective R-module. Further, assume $H(M)$ is bounded, and that N is semi-injective. Then the following map is a homology isomorphism when either $\operatorname{pd}_R P < \infty$ or $\operatorname{id}_R N < \infty$.

$$\theta_{PMN} : P \otimes_R \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(\operatorname{Hom}_R(P, M), N)$$

where $\theta_{PMN}(x \otimes \beta)(\alpha) = (-1)^{i(j+h)}\beta\alpha(x)$ for $x \in P_i$, $\alpha \in \operatorname{Hom}_R(P, M)_j$, and $\beta \in \operatorname{Hom}_R(M, N)_h$.

(F) Assume that P is a bounded below complex with each P_i a finitely generated projective R-module. Further, assume $\mathrm{H}^*(M)$ is bounded below, and let N be semi-flat. Then the following map is a homology isomorphism when either $\operatorname{pd}_R P < \infty$ or $\operatorname{fd}_R N < \infty$.

$$\omega_{PMN} : \operatorname{Hom}_R(P, M) \otimes_R N \to \operatorname{Hom}_R(P, M \otimes_R N)$$

where $\omega_{PMN}(\alpha \otimes y)(x) = (-1)^{ij}\alpha(x) \otimes y$ *for* $\alpha \in \operatorname{Hom}_R(P, M)_h$, $y \in N_j$, *and* $x \in P_i$.

Lemma 2.0.4. *Let* $(R, \mathfrak{m})$ *be a local ring of prime characteristic* p *which is F-finite. Let* x *be an indeterminate over* R, $S := R[x]_{\mathfrak{m}R[x]}$, *and* $T := ({}^eR)[x]_{\mathfrak{n}({}^eR)[x]}$, *where* $\mathfrak{n}$ *is the maximal ideal of* ${}^eR$. *Then*

(a) ${}^eS$ *is a free* T*-module of rank* p^e.

(b) T *is a finitely generated* S*-module.*

(c) S *is F-finite.*

(d) For each R*-complex* M *with* $\mathrm{H}^*(M)$ *bounded below and for each* i, *there is an isomorphism of* ${}^eS$*-modules*

$$\operatorname{Ext}^i_S({}^eS, M \otimes_R S) \cong \operatorname{Hom}_T({}^eS, \operatorname{Ext}^i_R({}^eR, M) \otimes_R S).$$

Proof. Let $A = R[x]$, $B = ({}^eR)[x]$, and $C = ({}^eR)[x^{\frac{1}{p^e}}] \cong {}^eA$. Note that C is a free B-module of rank p^e and B is a f.g. A-module. Let $U = A \setminus \mathfrak{m}A$, $V = B \setminus \mathfrak{n}B$, and $W = C \setminus \mathfrak{n}C$. Then $A_U = S$. It is straightforward to check that $T = B_V = B_U$ and ${}^eS = C_W = C_V$. Hence, (a), (b), and (c) are immediate.

We have the following isomorphisms of ${}^eS$-modules.

$$\begin{aligned}
\operatorname{Hom}_T({}^eS, \operatorname{Ext}^i_R({}^eR, M) \otimes_R S) &\cong \operatorname{Hom}_T({}^eS, \operatorname{Ext}^i_S(T, M \otimes_R S)) \\
&\cong \operatorname{Hom}_T({}^eS, \mathrm{H}^i(\mathsf{R}\mathrm{Hom}_S(T, M \otimes_R S))) \\
&\cong \mathrm{H}^i(\mathsf{R}\mathrm{Hom}_T({}^eS, \mathsf{R}\mathrm{Hom}_S(T, M \otimes_R S))) \\
&\cong \mathrm{H}^i(\mathsf{R}\mathrm{Hom}_S({}^eS, M \otimes_R S)) \\
&\cong \operatorname{Ext}^i_S({}^eS, M \otimes_R S).
\end{aligned}$$

The first isomorphism follows from Lemma 2.0.3 (F) since S is flat over R, ${}^{e}R$ is finitely generated over R, and $\mathrm{H}^*(M)$ is bounded below. The third isomorphism holds as ${}^{e}S$ is a free T-module. ☐

Corollary 2.0.5. *With the notation as in part (d) of Lemma 2.0.4, for each i we have that $\mathrm{Ext}^i_R({}^{e}R, M) = 0$ if and only if $\mathrm{Ext}^i_S({}^{e}S, M \otimes_R S) = 0$.*

Proof. Since $\mathrm{Hom}_T({}^{e}S, -)$ and $- \otimes_R S$ are faithful functors, the result follows from part (d) of Lemma 2.0.4. ☐

The following result is also well-known:

Lemma 2.0.6. *Let R be a commutative Noetherian ring, M, N R-complexes, and I an injective R-module.*

(a) For all i we have isomorphisms

$$\mathrm{Hom}_R(\mathrm{Tor}^R_i(M, N), I) \cong \mathrm{Ext}^i_R(M, \mathrm{Hom}_R(N, I)).$$

(b) Suppose $\mathrm{H}_*(M)$ is bounded below, $H_i(M)$ is finitely generated for all i, and $\mathrm{H}^*(N)$ is bounded below. Then for all i we have isomorphisms

$$\mathrm{Tor}^R_i(M, \mathrm{Hom}_R(N, I)) \cong \mathrm{Hom}_R(\mathrm{Ext}^i_R(M, N), I).$$

Proof. Using adjunction and Lemma 2.0.3 (I), we have the following isomorphisms in $\mathsf{D}(R)$:

$$\begin{aligned} \mathrm{Hom}_R(M \otimes^{\mathbf{L}}_R N, I) &\simeq \mathsf{R}\mathrm{Hom}_R(M \otimes^{\mathbf{L}}_R N, I) \\ &\simeq \mathsf{R}\mathrm{Hom}_R(M, \mathsf{R}\mathrm{Hom}_R(N, I)) \\ &\simeq \mathsf{R}\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, I)) \end{aligned}$$

and

$$M \otimes^{\mathbf{L}}_R \operatorname{Hom}_R(N, I) \simeq \operatorname{Hom}_R(\mathsf{R}\operatorname{Hom}_R(M, N), I).$$

Taking homology and using that $\operatorname{Hom}_R(-, I)$ is an exact functor yields the desired isomorphisms. $\square$

For an R-complex M, let $M^\sharp$ denote the complex which has the same underlying graded module as M and whose differentials are all zero. Let $\operatorname{fd}_R M$ denote the flat dimension of M; that is,

$$\operatorname{fd}_R M = \inf\{\sup \mathrm{H}_*(F^\sharp) \mid F \simeq M,\ F \text{ semi-flat}\}.$$

Similarly, $\operatorname{id}_R M$ will denote the injective dimension of M, i.e.,

$$\operatorname{id}_R M = \inf\{\sup \mathrm{H}^*(I^\sharp) \mid I \simeq M,\ I \text{ semi-injective}\}.$$

Avramov and Foxby [3, Proposition 5.3.I,F] give us an alternative way of viewing these values in the following Lemma.

Lemma 2.0.7. *Let R be a commutative noetherian ring and let M be a complex of R-modules.*

(I) If $\mathrm{H}^*(M)$ is bounded below, then there are equalities

$$\begin{aligned}\operatorname{id}_R M &= \sup\{j \mid \operatorname{Ext}^j_R(R/\mathfrak{p}, M) \neq 0 \textit{ for some } \mathfrak{p} \in \operatorname{Spec} R\}\\ &= \sup\{j \mid \operatorname{Ext}^j_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \neq 0 \textit{ for some } \mathfrak{p} \in \operatorname{Spec} R\}\\ &= \sup_{\mathfrak{p} \in \operatorname{Spec} R} \operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p}.\end{aligned}$$

(F) We have equalities

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{j \mid \operatorname{Tor}_j^R(R/\mathfrak{p}, M) \neq 0 \textit{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{j \mid \operatorname{Tor}_j^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \neq 0 \textit{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup_{\mathfrak{p} \in \operatorname{Spec} R} \operatorname{fd}_{R_\mathfrak{p}} M_\mathfrak{p}.\end{aligned}$$

Corollary 2.0.8. *Let $(R, \mathfrak{m})$ be a local ring, $E = E_R(R/\mathfrak{m})$, and let $(-)^\mathrm{v}$ denote the functor $\operatorname{Hom}_R(-, E)$. Let M be an R-complex. Then*

(a) $\operatorname{fd}_R M \leqslant \operatorname{id}_R M^\mathrm{v}$ *with equality if* $\mathrm{H}_*(M)$ *is bounded below.*

(b) If $\mathrm{H}^*(M)$ *is bounded below, then* $\operatorname{id}_R M = \operatorname{fd}_R M^\mathrm{v}$

Proof. Using Lemma 2.0.7 (F) and Lemma 2.0.6 with $I = E$, we have:

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{j \mid \operatorname{Tor}_j^R(R/\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{j \mid \operatorname{Tor}_j^R(R/\mathfrak{p}, M)^\mathrm{v} \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{j \mid \operatorname{Ext}_R^j(R/\mathfrak{p}, M^\mathrm{v}) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\ &\leqslant \operatorname{id}_R M^\mathrm{v},\end{aligned}$$

where equality holds in the last line if $\mathrm{H}^*(M^\mathrm{v})$ is bounded below, or equivalently, if $\mathrm{H}_*(M)$ is bounded below. Part (b) is proved similarly. □

We note the following remark, which will be needed in the subsequent sections:

Remark 2.0.9. Let S be a faithfully flat R-algebra and M an R-complex. Then

(a) $\operatorname{fd}_R M = \operatorname{fd}_S M \otimes_R S$;

(b) If $\mathrm{H}^*(M)$ is bounded below, then $\operatorname{id}_R M \leqslant \operatorname{id}_S M \otimes_R S$.

Proof. For part (a), note that $\operatorname{fd}_R M \geqslant \operatorname{fd}_S M \otimes_R S$, since $- \otimes_R S$ preserves quasi-isomorphisms and $F \otimes_R S$ is a semi-flat S-complex whenever F is a semi-flat R-complex. For the reverse inequality, we have by Lemma 2.0.7 (F),

$$\begin{aligned}
\operatorname{fd}_R M &= \sup\{j \mid \operatorname{Tor}_j^R(R/\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\
&= \sup\{j \mid \operatorname{Tor}_j^R(R/\mathfrak{p}, M) \otimes_R S \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\
&= \sup\{j \mid \operatorname{Tor}_j^S(S/\mathfrak{p}S, M \otimes_R S) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\
&\leqslant \operatorname{fd}_S M \otimes_R S.
\end{aligned}$$

For part (b), we have by Lemma 2.0.7 (I) that

$$\begin{aligned}
\operatorname{id}_R M &= \sup\{j \mid \operatorname{Ext}_R^j(R/\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\
&= \sup\{j \mid \operatorname{Ext}_R^j(R/\mathfrak{p}, M) \otimes_R S \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\
&= \sup\{j \mid \operatorname{Ext}_S^j(S/\mathfrak{p}S, M \otimes_R S) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\} \\
&\leqslant \operatorname{id}_S M \otimes_R S.
\end{aligned}$$

□

Finally, the following proposition will serve as the base case for the main results in Theorem 1.0.7. It is the duel of Theorem 1.1 in [6].

Proposition 2.0.10. *Let $(R, \mathfrak{m}, k)$ be a zero-dimensional local ring of prime characteristic p. Let M be an R-module and $e \geq \log_p \lambda(R)$ an integer, where $\lambda(-)$ denotes length. If $\operatorname{Ext}_R^i({}^eR, M) = 0$ for some $i > 0$ then M is injective.*

Proof. By [4, Proposition 4.1 and Corollary 5.3], if M has finite injective dimension then $\operatorname{id}_R M \leq \dim R$. Hence, it suffices to show $\operatorname{id}_R M < \infty$. By replacing M with a syzygy of an injective resolution of M, we may assume $\operatorname{Ext}_R^1({}^eR, M) = 0$. Since $p^e \leq \lambda(R)$, we have $\mathfrak{m}^{p^e} = 0$. Then $\mathfrak{m} \cdot {}^eR = 0$ and thus ${}^eR$ is a k-vector space.

Hence, ${}^{e}R \cong k^{\ell}$ as R-modules for some, possibly infinite, $\ell > 0$. Then the condition $\operatorname{Ext}^1_R({}^{e}R, M) = 0$ implies $\operatorname{Ext}^1_R(k, M) = 0$. Hence, M is injective. $\square$

Chapter 3

Proof of Theorem 1.0.7

For an R-complex M and $\mathfrak{p} \in \operatorname{Spec} R$, we let $\mu_i(\mathfrak{p}, M) := \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$. Recall that if $\mathrm{H}^*(M)$ is bounded below and I a minimal semi-injective resolution of M, then

$$I^i = \oplus_{\mathfrak{p} \in \operatorname{Spec}(R)} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)}.$$

Recall too that for a local ring $(R, \mathfrak{m}, k)$ we let $(-)^\vee$ denote the exact functor $\operatorname{Hom}_R(-, E)$ with $E = E_R(k)$ the injective hull of k. Refer to the start of chapter 2 for other definitions and notation used throughout this chapter.

3.1 The Cohen-Macaulay Case

Rather than proving this case outright, we instead prove the theorem with injective dimension and Ext rather than flat dimension and Tor. I will present a Theorem which shows how these two statements are related, and in the same proof we get the Cohen-Macaulay case for F-finite rings. But first, we will need the following result from Marley and Webb [15, Corollary 3.5].

Theorem 3.1.1. *Let R be a Noetherian ring of prime characteristic p, M an R-module, and $e \geq 1$ an integer. If $\operatorname{fd}_R M < \infty$ then $\operatorname{Tor}_i^R({}^eR, M) = 0$ for all $i > 0$ and*

$\mathrm{fd}_R\, {}^eR \otimes_R M = \mathrm{fd}_R\, M$.

With this, we can move on to prove our result for F-finite Cohen-Macaulay rings.

Theorem 3.1.2. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional Cohen-Macaulay local ring of prime characteristic p which is F-finite. Let $e \geqslant \log_p e(R)$ be an integer, M an R-complex, and $r = \max\{1, d\}$.*

(a) Suppose there exists an integer $t > \sup \mathrm{H}^*(M)$ such that $\mathrm{Ext}^i_R({}^eR, M) = 0$ for $t \leqslant i \leqslant t + r - 1$. Then M has finite injective dimension.

(b) Suppose there exists an integer $t > \sup \mathrm{H}_*(M)$ such that $\mathrm{Tor}^R_i({}^eR, M) = 0$ for $t \leqslant i \leqslant t + r - 1$. Then M has finite flat dimension.

Proof. We first note that if (a) holds in the case $\dim R = d$, then (b) also holds in the case $\dim R = d$: For, suppose the hypotheses of (b) hold for a complex M. Then by Lemma 2.0.6(a), $\mathrm{Ext}^i_R({}^eR, M^\vee) \cong \mathrm{Tor}^R_i({}^eR, M)^\vee = 0$ for $t \leqslant i \leqslant t + r - 1$ since $(-)^\vee$ is an exact functor. As $\sup \mathrm{H}^*(M^\vee) = \sup \mathrm{H}_*(M)$, we have by (a) that $\mathrm{id}_R\, M^\vee < \infty$. Hence, $\mathrm{fd}_R\, M < \infty$ by Lemma 2.0.8(a).

Thus, it suffices to prove (a). If $\mathrm{id}_R\, M < t-1$ there is nothing to prove. Otherwise, let J be a minimal semi-injective resolution of M and $Z := Z^{t-1}(J)$ be the (necessarily nonzero) subcomplex consisting of the cycles of degree $t-1$ of J. As $t-1 \geqslant \sup \mathrm{H}^*(M)$, $J^{\geqslant t-1}$ is a minimal semi-injective resolution of Z and $\mathrm{id}_R\, M = \mathrm{id}_R\, Z$. Furthermore, from the exact sequence of complexes

$$0 \to J^{\geqslant t-1} \to J \to J^{<t-1} \to 0$$

we have that $\mathrm{Ext}^i_R({}^eR, Z) \cong \mathrm{Ext}^i_R({}^eR, M)$ for all $i \geqslant t$. Hence, without loss of generality, we may assume (after shifting) that M is a module concentrated in degree zero and $\operatorname{Ext}^i_R({}^eR, M) = 0$ for $i = 1, \ldots, r$. Also, by replacing R with $R[x]_{\mathfrak{m}R[x]}$, if necessary, we may assume R has an infinite residue field (Lemma 2.0.4 and Remark 2.0.9).

We proceed by induction on d, with the case $d = 0$ being established by Proposition 2.0.10. Suppose $d \geqslant 1$ (so $r = d$) and we assume both (a) and (b) hold for complexes over local rings of dimension less than d. Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal of R so that $\dim R_{\mathfrak{p}} < d$. As R is F-finite, we have $\operatorname{Ext}^i_{R_{\mathfrak{p}}}({}^eR_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for $1 \leqslant i \leqslant d$. As $d \geqslant \max\{1, \dim R_{\mathfrak{p}}\}$ and $e(R) \geqslant e(R_{\mathfrak{p}})$ (see [12]), we have $\operatorname{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty$ by the induction hypothesis. Hence, $\operatorname{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leqslant \dim R_{\mathfrak{p}} \leqslant d - 1$ by [4, Proposition 4.1 and Corollary 5.3]. It follows that $\mu_i(\mathfrak{p}, M) = 0$ for all $i \geqslant d$ and all $\mathfrak{p} \neq \mathfrak{m}$.

For convenience, we let S denote the R-algebra ${}^eR$ and $\mathfrak{n}$ the maximal ideal of S. As $S/\mathfrak{n}$ is infinite, we may choose a system of parameters $\mathbf{x} = x_1, \ldots, x_d \in \mathfrak{n}$ such that $(\mathbf{x})$ is a minimal reduction of $\mathfrak{n}$. Then $\lambda_S(S/(\mathbf{x})) = e(S) = e(R)$ and $\mathfrak{m} \cdot S/(\mathbf{x}) = \mathfrak{n}^{[p^e]} S/(\mathbf{x}) = 0$, as $p^e \geqslant \lambda_S(S/(\mathbf{x}))$.

As J is a minimal injective resolution of M, and $\operatorname{Ext}^i_R({}^eR, M) = 0$ for $1 \leq i \leq d$, we have by assumption that

$$\operatorname{Hom}_R(S, J^0) \xrightarrow{\phi^0} \operatorname{Hom}_R(S, J^1) \to \cdots \to \operatorname{Hom}_R(S, J^d) \xrightarrow{\phi^d} \operatorname{Hom}_R(S, J^{d+1}) \tag{3.1.1}$$

is exact. Let L be the injective S-envelope of $\operatorname{coker} \phi^d$ and $\psi : \operatorname{Hom}_R(S, J^{d+1}) \to L$ the induced map. Hence,

$$0 \to \operatorname{Hom}_R(S, J^0) \to \cdots \xrightarrow{\phi^d} \operatorname{Hom}_R(S, J^{d+1}) \xrightarrow{\psi} L$$

is acyclic and in fact the start of an injective S-resolution of $\operatorname{Hom}_R(S, M)$. Setting

$\overline{S} = S/(\mathbf{x})$ and applying $\operatorname{Hom}_S(\overline{S}, -)$ to the above resolution yields an exact sequence

$$\operatorname{Hom}_S(\overline{S}, \operatorname{Hom}_R(S, J^d)) \xrightarrow{\overline{\phi^d}} \operatorname{Hom}_S(\overline{S}, \operatorname{Hom}_R(S, J^{d+1})) \xrightarrow{\overline{\psi}} \operatorname{Hom}_S(\overline{S}, L). \qquad (3.1.2)$$

The exactness holds as $\operatorname{pd}_S \overline{S} = d$ and thus $\operatorname{Ext}_S^{d+1}(\overline{S}, \operatorname{Hom}_S(S, M)) = 0$.

Since R is F-finite we know $\overline{S}$ is a finitely generated R-module. Further, $\overline{S}$ is annihilated by $\mathfrak{m}$, hence $\overline{S} \cong k^t$ as R-modules for some $t < \infty$. Thus, the exact sequence (3.1.2) is naturally isomorphic to

$$\operatorname{Hom}_R(k^t, J^d) \xrightarrow{\overline{\phi^d}} \operatorname{Hom}_R(k^t, J^{d+1}) \xrightarrow{\overline{\psi}} \operatorname{Hom}_S(\overline{S}, L).$$

As J is minimal, we have $\overline{\phi^d}$ is the zero map and hence $\overline{\psi}$ is injective.

Claim: ψ is injective.

Proof: Let $K = \ker \psi$. Applying $\operatorname{Hom}_S(\overline{S}, -)$ to

$$0 \to K \to \operatorname{Hom}_R(S, J^{d+1}) \xrightarrow{\psi} L$$

we see that $\operatorname{Hom}_S(\overline{S}, K) = 0$ since $\overline{\psi}$ is injective. Since $\mu_{d+1}(\mathfrak{p}, M) = 0$ for all primes $\mathfrak{p} \neq \mathfrak{m}$, we obtain that $J^{d+1} = \oplus_{\alpha \in I} E_R(k)$ for some (possibly infinite) index set I. As S is a finite R-module, we may then write

$$\begin{aligned}
\operatorname{Hom}_R(S, I^{d+1}) &\cong \operatorname{Hom}_R(S, \oplus_{\alpha \in I} E_R(k)) \\
&\cong \oplus_{\alpha \in I} \operatorname{Hom}_R(S, E_R(k)) \\
&\cong \oplus_{\alpha \in I} E_S(S/\mathfrak{n})
\end{aligned}$$

Suppose by way of contradiction that $K \neq 0$. Then as every element of $E_S(S/\mathfrak{n})$ is annihilated by a power of $\mathfrak{n}$ there exists some ℓ such that for some $0 \neq y \in K$ we have

$\mathfrak{n}^{\ell}y = 0$ and $\mathfrak{n}^{\ell-1}y \neq 0$. However, this produces a nonzero element of $\operatorname{Hom}_R(\overline{S}, K)$, a contradiction. Then, we much have $K = 0$ showing ψ is injective.

Now consider the complex J, which is a minimal injective resolution of M:

$$0 \to J^0 \xrightarrow{\partial^0} J^1 \to \cdots \to J^{d-1} \xrightarrow{\partial^{d-1}} J^d \xrightarrow{\partial^d} \cdots$$

The proof will be complete upon proving:

Claim: ∂^{d-1} is surjective.

Proof: As ψ is injective we have from (3.1.1) that $\phi^d = 0$, and thus $\phi^{d-1} = \operatorname{Hom}_R(S, \partial^{d-1})$ is surjective. Let $C = \operatorname{coker} \partial^{d-1}$. Then

$$0 \to C^{\mathrm{v}} \to (J^d)^{\mathrm{v}} \to \cdots \to (J^0)^{\mathrm{v}} \to M^{\mathrm{v}} \to 0$$

is exact. Note that $(J^i)^{\mathrm{v}}$ is a flat R-module for all i (e.g., Corollary 2.0.8(b)). As the set of associated primes of any flat R-module is contained in the set of associated primes of R, and as R is Cohen-Macaulay of dimension greater than zero, to show $C^{\mathrm{v}} = 0$ it suffices to show $(C^{\mathrm{v}})_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$. So fix a prime $\mathfrak{p} \neq \mathfrak{m}$. As S is a finitely generated R-module, we have $\operatorname{Tor}_i^R(S, M^{\mathrm{v}}) \cong \operatorname{Ext}_R^i(S, M)^{\mathrm{v}} = 0$ for $i = 1, \ldots, d$ by Lemma 2.0.6(b). This implies $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(S_{\mathfrak{p}}, (M^{\mathrm{v}})_{\mathfrak{p}}) = 0$ for $i = 1, \ldots, d$. As $R_{\mathfrak{p}}$ is an F-finite Cohen-Macaulay local ring of dimension less than d, and $p^e \geqslant e(R) \geqslant e(R_{\mathfrak{p}})$, we have that $\operatorname{fd}_{R_{\mathfrak{p}}}(M^{\mathrm{v}})_{\mathfrak{p}} < \infty$ by the induction hypothesis on part (b). In particular, by [4, Corollary 5.3], $\operatorname{fd}_{R_{\mathfrak{p}}}(M^{\mathrm{v}})_{\mathfrak{p}} \leqslant \dim R_{\mathfrak{p}} \leqslant d-1$ and thus $(C^{\mathrm{v}})_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module. Then by 3.1.1

$$0 \to S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (C^{\mathrm{v}})_{\mathfrak{p}} \to S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} ((J^d)^{\mathrm{v}})_{\mathfrak{p}} \to S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} ((J^{d-1})^{\mathrm{v}})_{\mathfrak{p}} \tag{3.1.3}$$

is exact. Now, since $\operatorname{Hom}_R(S, \partial^{d-1})$ is surjective, we have using duality and Lemma

2.0.6(b) that

$$0 \to S \otimes_R (J^d)^{\mathrm{v}} \to S \otimes_R (J^{d-1})^{\mathrm{v}}$$

is exact. Localizing this exact sequence at $\mathfrak{p}$ and comparing with (3.1.3), we have $S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (C^{\mathrm{v}})_{\mathfrak{p}} = 0$. However, tensoring with $S_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ is faithful (e.g., [13, Propostion 2.1(c)]) and hence $(C^{\mathrm{v}})_{\mathfrak{p}} = 0$. Then $C^{\mathrm{v}} = 0$, and thus $C = 0$, which completes the proof of the Claim. □

As a corollary, we obtain the equivalence of conditions (a) and (c) of Theorem 1.0.7:

Corollary 3.1.3. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring of prime characteristic p and M an R-complex such that $\mathrm{H}_*(M)$ is bounded above. Suppose there exist integers $e \geqslant \log_p e(R)$ and $t > \sup \mathrm{H}_*(M)$ such that $\mathrm{Tor}_i^R({}^eR, M) = 0$ for $t \leqslant i \leqslant t + r - 1$, where $r = \max\{1, d\}$. Then M has finite flat dimension.*

Proof. By [11, Section 3] there exists a faithfully flat extension S of R such that S is a d-dimensional CM local ring with an algebraically closed residue field and $e(S) = e(R)$. Furthermore, by Corollary 2.0.2, $\mathrm{Tor}_i^S({}^eS, M \otimes_R S) = 0$ for $t \leqslant i \leqslant t + r - 1$. Hence, by replacing R with S and M with $M \otimes_R S$, we may assume R is F-finite. The result now follows from part (b) of Theorem 3.1.2. □

3.2 The General Case

Before moving into the proof of the general case, we will start this sub-section by proving a basic result concerning $E = E_R(k)$, the injective hull of the residue field of a local ring $(R, \mathfrak{m}, k)$.

Lemma 3.2.1. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then*

$$(0 :_E (0 :_R \mathfrak{m})) = \mathfrak{m}E.$$

Proof. The containment $\mathfrak{m}E \subseteq (0 :_E (0 :_R \mathfrak{m}))$ is clear. For the reverse inclusion, since $E \cong E_{\hat{R}}(\hat{R}/\hat{\mathfrak{m}})$, $\hat{\mathfrak{m}}E = \mathfrak{m}E$ and $(0 :_{\hat{R}} \hat{\mathfrak{m}}) = (0 :_R \mathfrak{m})\hat{R}$, we may replace R by $\hat{R}$ and assume R is complete. Consider the composition of maps

$$\operatorname{Hom}_R(R/(0 :_R \mathfrak{m}), E) \cong (0 :_E (0 :_R \mathfrak{m})) \to E \to E/\mathfrak{m}E \cong E \otimes_R R/\mathfrak{m}. \tag{3.2.1}$$

Since R is complete, we have $E^{\vee} \cong R$. Thus, $(E \otimes_R R/\mathfrak{m})^{\vee} \cong \operatorname{Hom}(R/\mathfrak{m}, E^{\vee}) \cong (0 :_R \mathfrak{m})$. Dualizing (3.2.1), we have the composition

$$(E \otimes_R R/\mathfrak{m})^{\vee} \cong (0 :_R \mathfrak{m}) \to R \to R/(0 :_R \mathfrak{m}),$$

which is clearly the zero map. Thus, the composition (3.2.1) is the zero map as well, implying $(0 :_E (0 :_R \mathfrak{m})) \subseteq \mathfrak{m}E$.

□

We use the above lemma to prove the following:

Lemma 3.2.2. *Let $(R, \mathfrak{m}, k)$ be a local ring and $\phi : J \to J'$ a homomorphism of injective R-modules. Suppose $\operatorname{Hom}_R(R/\mathfrak{m}, J) \xrightarrow{\phi_*} \operatorname{Hom}_R(R/\mathfrak{m}, J')$ is zero. Then $\phi(J) \subseteq \mathfrak{m}J'$.*

Proof. It suffices to prove the lemma in the case $J = E_R(R/\mathfrak{p})$ and $J' = E_R(R/\mathfrak{q})$, for $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$, as every injective module can be written as direct sums of this form.

Case 1: $\mathfrak{q} \neq \mathfrak{m}$.

Then $\mathfrak{m}J' = \mathfrak{m}E_R(R/\mathfrak{q}) = \mathfrak{m}R_\mathfrak{q} \cdot E_R(R/\mathfrak{q}) = J'$, as $E_R(R/\mathfrak{q})$ is an $R_\mathfrak{q}$-module. So the lemma holds trivially.

Case 2: $\mathfrak{q} = \mathfrak{m}$ and $\mathfrak{p} \neq \mathfrak{m}$.

Since $J = E_R(R/\mathfrak{p})$ is an $R_\mathfrak{p}$-module, we have

$$(0 :_R \mathfrak{m})\phi(J) = \phi((0 :_R \mathfrak{m})R_\mathfrak{p} \cdot J) = \phi(0) = 0.$$

Hence, $\phi(J) \subseteq (0 :_{J'} (0 :_R \mathfrak{m})) = \mathfrak{m}J'$ by Lemma 3.2.1.

Case 3: $\mathfrak{p} = \mathfrak{q} = \mathfrak{m}$.

In this case, ϕ is multiplication by some element $s \in \widehat{R}$. If $s \notin \widehat{\mathfrak{m}}$, then ϕ is an isomorphism, contradicting that $\mathrm{Hom}_R(R/\mathfrak{m}, \phi)$ is the zero map. Thus, $s \in \widehat{\mathfrak{m}}$. Hence, $\phi(J) \subseteq \widehat{\mathfrak{m}}J' = \mathfrak{m}J'$.

□

Lemma 3.2.3. *Let $(R, \mathfrak{m})$ be a local ring of depth zero and let ℓ be an integer such that $(0 :_R \mathfrak{m}) \not\subset \mathfrak{m}^\ell$. Let J be an injective module such that $\mu_0(\mathfrak{m}, J) \neq 0$. Then $(0 :_J \mathfrak{m}^\ell) \not\subset \mathfrak{m}J$.*

Proof. It suffices to consider the case $J = E := E_R(k)$. Since the composition $(0 :_R \mathfrak{m}) \to R \to R/\mathfrak{m}^\ell$ is nonzero, the composition

$$(R/\mathfrak{m}^\ell)^\vee \cong (0 :_E \mathfrak{m}^\ell) \to E \to E/\mathfrak{m}E \cong \mathrm{Hom}_R(R/\mathfrak{m}, R)^\vee$$

is also nonzero. Hence, $(0 :_E \mathfrak{m}^\ell) \not\subset \mathfrak{m}E$. □

The following lemma will play a big role in our general case. This lemma will allow us extract the necessary information from exact sequences of ${}^eR$-modules (in the F-finite case) as they relate to an exact sequence of R-modules.

Lemma 3.2.4. *Let $\phi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism such that S is a finitely generated R-module and* $\operatorname{depth} S = 0$. *Let ℓ be an integer such that $(0 :_S \mathfrak{n}) \not\subseteq \mathfrak{n}^\ell$ and suppose $\mathfrak{m}S \subseteq \mathfrak{n}^\ell$. Let $J^1 \xrightarrow{\sigma} J^2 \xrightarrow{\tau} J^3$ be a sequence of maps of injective modules such that such that $\operatorname{Hom}_R(R/\mathfrak{m}, \sigma) = \operatorname{Hom}_R(R/\mathfrak{m}, \tau) = 0$. If $\operatorname{Hom}_R(S, J^1) \xrightarrow{\sigma_*} \operatorname{Hom}_R(S, J^2) \xrightarrow{\tau_*} \operatorname{Hom}_R(S, J^3)$ is exact then $\mu_0(\mathfrak{m}, J^2) = 0$.*

Proof. Let $\widetilde{J^i} = \operatorname{Hom}_R(S, J^i)$ for $i = 1, 2, 3$, which are injective S-modules. Since $\mathfrak{m}S \subseteq \mathfrak{n}^\ell$, we have that $S/\mathfrak{n}^\ell \cong k^r$ as R-modules for some $r > 0$, where $k = R/\mathfrak{m}$. Consider the commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}_S(S/\mathfrak{n}^\ell, \widetilde{J^1}) & \xrightarrow{\overline{\sigma_*}} & \operatorname{Hom}_S(S/\mathfrak{n}^\ell, \widetilde{J^2}) \\
\big\downarrow \cong & & \big\downarrow \cong \\
\operatorname{Hom}_R(S/\mathfrak{n}^\ell, J^1) & \longrightarrow & \operatorname{Hom}_R(S/\mathfrak{n}^\ell, J^2) \\
\big\downarrow \cong & & \big\downarrow \cong \\
\oplus \operatorname{Hom}_R(k, J^1) & \xrightarrow{\oplus \overline{\sigma}} & \oplus \operatorname{Hom}_R(k, J^2).
\end{array}$$

As $\overline{\sigma}$ is the zero map by hypothesis, we see that $\overline{\sigma_*}$ is zero. Similarly, the map

$$\overline{\tau_*} : \operatorname{Hom}_S(S/\mathfrak{n}^\ell, \widetilde{J^2}) \to \operatorname{Hom}_S(S/\mathfrak{n}^\ell, \widetilde{J^3})$$

is zero. This implies that $(0 :_{\widetilde{J^2}} \mathfrak{n}^\ell) \subseteq \ker \tau^*$. As $\overline{\sigma_*}$ is zero, we also have that the map $\operatorname{Hom}_S(S/\mathfrak{n}, \widetilde{J^1}) \to \operatorname{Hom}_S(S/\mathfrak{n}, \widetilde{J^2})$ is zero. By Lemma 3.2.2, this implies that $\operatorname{im} \sigma_* \subseteq \mathfrak{n}\widetilde{J^2}$.

Suppose $\mu_0(\mathfrak{m}, J^2) \neq 0$. Since $\operatorname{Hom}_S(S, E_R(R/\mathfrak{m})) \cong E_S(S/\mathfrak{n})$ by [13, Lemma 3.7], we then have $\mu(\mathfrak{n}, \widetilde{J^2}) \neq 0$. By Lemma 3.2.3, we have that $(0 :_{\widetilde{J^2}} \mathfrak{n}^\ell) \not\subseteq \mathfrak{n}\widetilde{J^2}$. Hence, $\ker \tau_* \not\subseteq \operatorname{im} \sigma_*$, a contradiction. Therefore, $\mu_0(\mathfrak{m}, J^2) = 0$.

□

Now that we have this machinery, we move to a theorem which proves the bulk of the general case of 1.0.7.

Theorem 3.2.5. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional local ring of prime characteristic p which is F-finite. Let M be an R-complex such that $\mathrm{H}^*(M)$ is bounded above. Suppose there exists an integer $t > \sup \mathrm{H}^*(M)$ such that for infinitely many integers e one has $\mathrm{Ext}^i_R({}^eR, M) = 0$ for $t \leqslant i \leqslant t + r - 1$, where $r = \max\{1, d\}$. Then M has finite injective dimension.*

Proof. Precisely as in the initial paragraph of the proof of Theorem 3.1.2, we may assume M is a module concentrated in degree 0 and $t = 1$. We proceed by induction on d, with the case $d = 0$ being covered by Proposition 2.0.10. Suppose now that $d \geqslant 1$ (so $r = d$) and let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal. Since R is F-finite, $\mathrm{Ext}^i_{R_\mathfrak{p}}({}^eR_\mathfrak{p}, M_\mathfrak{p}) = \mathrm{Ext}^i_R({}^eR, M)_\mathfrak{p} = 0$ for infinitely many e and $i = 1, \dots, d$. Since $d \geqslant \max\{1, \dim R_\mathfrak{p}\}$, we have $\mathrm{id}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ by the induction hypothesis. Hence, $\mathrm{id}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \dim R_\mathfrak{p} \leqslant d-1$. Thus, $\mu_i(\mathfrak{p}, M) = 0$ for all $i \geqslant d$ and all $\mathfrak{p} \neq \mathfrak{m}$.

If R is Cohen-Macaulay we are done by Theorem 3.1.2. Hence we may assume $s := \operatorname{depth} R < d$ and it suffices to prove $\mu_d(\mathfrak{m}, M) = 0$. Let $e \geqslant 1$ be arbitrary and let T denote the local ring ${}^eR$ and $\mathfrak{q}$ the maximal ideal of T. Let $\mathbf{x} = x_1, \dots, x_s \in \mathfrak{q}$ be a maximal regular sequence in T and set $S := T/(\mathbf{x})$ and $\mathfrak{n} := \mathfrak{q}S$. Since $\operatorname{depth} S = 0$, there exists an integer ℓ (independent of e) such that $(0 :_S \mathfrak{n}) \not\subset \mathfrak{n}^\ell$. Now choose e sufficiently large such that $p^e \geqslant \ell$ and $\mathrm{Ext}^i_R(T, M) = 0$ for $i = 1, \dots, d$. Let

$$J := \ 0 \to J^0 \to J^1 \to J^2 \to \cdots$$

be a minimal injective resolution of M, and for each i let $\widetilde{J^i}$ denote $\mathrm{Hom}_R(T, J^i)$. As $\mathrm{Ext}^i_R(T, M) = 0$ for $1 \leqslant i \leqslant d$, we see that

$$0 \to \widetilde{J^0} \to \widetilde{J^1} \to \cdots \to \widetilde{J^d} \to \widetilde{J^{d+1}}$$

is part of an injective T-resolution of $\widetilde{M} := \operatorname{Hom}_R(T, M)$. Since $\operatorname{pd}_T S = s$ we have that $\operatorname{Ext}^i_T(S, \widetilde{M}) = 0$ for $i > s$. In particular, as $d > s$ and $\operatorname{Hom}_T(S, \widetilde{J}^i) \cong \operatorname{Hom}_R(S, J^i)$ for all i, we have that

$$\operatorname{Hom}_R(S, J^{d-1}) \to \operatorname{Hom}_R(S, J^d) \to \operatorname{Hom}_R(S, J^{d+1})$$

is exact. Since $\mathfrak{m}S \subseteq \mathfrak{n}^{[p^e]} \subseteq \mathfrak{n}^\ell$, we obtain that $\mu_d(\mathfrak{m}, M) = \mu_0(\mathfrak{m}, J^d) = 0$ by Lemma 3.2.4. Thus, $\operatorname{id}_R M < \infty$. □

We now obtain the equivalence of conditions (a) and (b) of Theorem 1.0.7.

Corollary 3.2.6. *Let $(R, \mathfrak{m})$ be a d-dimensional local ring of prime characteristic p and M an R-complex such that $\mathrm{H}_*(M)$ is bounded above. Suppose there exist an integer $t > \sup \mathrm{H}_*(M)$ such that for infinitely many integers e one has $\operatorname{Tor}^R_i({}^eR, M) = 0$ for $t \leqslant i \leqslant t + r - 1$, where $r = \max\{1, d\}$. Then M has finite flat dimension.*

Proof. The argument is similar to the proof of Corollary 3.1.3, except one uses Theorem 3.2.5 in place of Theorem 3.1.2. □

3.3 The Complete Intersection Case

In this sub-section we give a proof that a theorem of Avramov and Miller [2] concerning finitely generated modules over complete intersections holds for arbitrary modules, and in fact any complex whose homology is bounded above. This will finish our proof of 1.0.7 by establishing the equivalence of (a) and (d). The proof mostly follows the argument of Dutta [7], until the end when we apply [6, Theorem 1.1].

Theorem 3.3.1. *Let $(R, \mathfrak{m})$ be a local complete intersection ring of prime characteristic p. Let M be an R-complex such that $\mathrm{H}_*(M)$ is bounded above. Suppose*

$\operatorname{Tor}_i^R({}^eR, M) = 0$ *for some* $e > 0$ *and some* $i > \sup \mathrm{H}_*(M)$. *Then* M *has finite flat dimension.*

Proof. Without loss of generality, we may assume R is complete. Then $R \cong A/(\mathbf{x})$ where $(A, \mathfrak{n})$ is a complete regular local ring and $\mathbf{x} = x_1, \ldots, x_r \in \mathfrak{n}^2$ is a regular sequence. Since A is regular the eth iteration of the Frobenius map $f^e : A \to A$ is flat. Thus, the map $h : A/(\mathbf{x}) \to A/(\mathbf{x}^{p^e})$, given by $h(\overline{r}) = \overline{r^{p^e}}$ for $\overline{r} \in A/(\mathbf{x})$, is flat as well. Let T denote the ring $A/(\mathbf{x}^{p^e})$ viewed as an R-algebra via h. Also, let $S := T/(\mathbf{x})$ and $\widetilde{M} := T \otimes_R M$. Note that as an R-algebra, $S \cong {}^eR$. Since T is flat over R, we have isomorphisms in $\mathsf{D}(R)$

$$S \otimes_R^{\mathbf{L}} M \simeq (S \otimes_T^{\mathbf{L}} T) \otimes_R^{\mathbf{L}} M \simeq S \otimes_T^{\mathbf{L}} (T \otimes_R^{\mathbf{L}} M) \simeq S \otimes_T^{\mathbf{L}} \widetilde{M}.$$

Taking homology, we have an isomorphism for all j

$$\operatorname{Tor}_j^R(S, M) \cong \operatorname{Tor}_j^T(S, \widetilde{M}). \tag{3.3.1}$$

Claim 1: $\operatorname{Tor}_j^R({}^eR, M) = 0$ for all $j \geqslant i$.

Proof: It suffices to show that $\operatorname{Tor}_{i+1}^R(S, M) = 0$. By (3.3.1), it suffices to prove that whenever $\operatorname{Tor}_i^T(S, \widetilde{M}) = 0$ for some $i > \sup \mathrm{H}_*(\widetilde{M}) = \sup \mathrm{H}_*(M)$ then $\operatorname{Tor}_{i+1}^T(S, \widetilde{M}) = 0$. Let $\phi : T \to S$ be the canonical surjection and $K = \ker \phi$. As $\mathbf{x}$ is a regular sequence on A, K has a finite filtration of T-submodules such that each factor module is isomorphic to S. Thus, $\operatorname{Tor}_i^T(K, \widetilde{M}) = 0$. As $\operatorname{Tor}_j^T(T, \widetilde{M}) = 0$ for all $j > \sup \mathrm{H}_*(\widetilde{M})$, we obtain that $\operatorname{Tor}_{i+1}^T(S, \widetilde{M}) = 0$.

Claim 2: $\operatorname{Tor}_j^R({}^{e+1}R, M) = 0$ for all $j \geqslant i$.

Proof: As $f : A \to A$ is flat, by base change we have the induced map $g : T \to$

$A/(\mathbf{x}^{p^{e+1}})$ is also flat. Let T' denote the ring $A/(\mathbf{x}^{p^{e+1}})$ viewed as a T-algebra via g (and hence as an R-algebra via gh). Note that $T'/(\mathbf{x}) = {}^{e+1}R$. Let $\widehat{M} = T' \otimes_R M = T' \otimes_T \widetilde{M}$. Since T' is flat over R, we have isomorphisms in $\mathsf{D}(R)$

$$T'/(\mathbf{x}) \otimes_R^{\mathbf{L}} M \simeq (T'/(\mathbf{x}) \otimes_{T'}^{\mathbf{L}} T') \otimes_R^{\mathbf{L}} M \simeq T'/(\mathbf{x}) \otimes_{T'}^{\mathbf{L}} (T' \otimes_R^{\mathbf{L}} M) \simeq T'/(\mathbf{x}) \otimes_{T'}^{\mathbf{L}} \widehat{M}.$$

Taking homology, we have an isomorphism for all j

$$\operatorname{Tor}_j^R(T'/(\mathbf{x}), M) \cong \operatorname{Tor}_j^{T'}(T'/(\mathbf{x}), \widehat{M}). \tag{3.3.2}$$

Hence, it suffices to show that $\operatorname{Tor}_j^{T'}(T'/(\mathbf{x}), \widehat{M}) = 0$ for all $j \geqslant i$. By (3.3.1) and Claim 1, we have that $\operatorname{Tor}_j^T(T/(\mathbf{x}), \widetilde{M}) = 0$ for all $j \geqslant i$. Since T' is flat over T, we obtain that $\operatorname{Tor}_j^{T'}(T'/(\mathbf{x}^p), \widehat{M}) = 0$ for $j \geqslant i$.

As $\mathbf{x} = x_1, \ldots, x_r$ is a regular sequence on A and $T' = A/(\mathbf{x}^{p^{e+1}})$, we have exact sequences of T-modules

$$0 \to T'/(x_1, x_2^p, \ldots, x_r^p) \to T'/(x_1^{p+1}, x_2^p, \ldots, x_r^p) \to T'/(\mathbf{x}^p) \to 0 \tag{3.3.3}$$

$$0 \to T'/(\mathbf{x}^p) \to T'/(x_1^{p+1}, x_2^p, \ldots, x_r^p) \to T'/(x_1, x_2^p, \ldots, x_r^p) \to 0, \tag{3.3.4}$$

where the initials maps in (3.3.3) and (3.3.4) are multiplication by x_1^p and x_1, respectively. Using that $\operatorname{Tor}_j^{T'}(T'/(\mathbf{x}^p), \widehat{M}) = 0$ for $j \geqslant i$ in conjunction with (3.3.3) and (3.3.4), we get an injection

$$\operatorname{Tor}_j^{T'}(T'/(x_1, x_2^p, \ldots, x_r^p), \widehat{M}) \to \operatorname{Tor}_j^{T'}(T'/(x_1, x_2^p, \ldots, x_r^p), \widehat{M})$$

for all $j \geqslant i$ which is induced by multiplication by x_1^p, which is the zero map. Hence $\operatorname{Tor}_j^{T'}(T'/(x_1, x_2^p, \ldots, x_r^p), \widehat{M}) = 0$ for all $j \geqslant i$. Repeating this argument for each of

$x_2, \dots, x_r$ yields that $\operatorname{Tor}_j^{T'}(T'/(\,\mathbf{x}), \widehat{M}) = 0$ for all $j \geqslant i$, which completes the proof of Claim 2.

Combining Claim 1 and Claim 2 (and iterating), we have $\operatorname{Tor}_j^R({}^eR, M) = 0$ for all $j \geqslant i > \sup \mathrm{H}_*(M)$ and infinitely many e. Thus, M has finite flat dimension by [6, Theorem 1.1]. $\square$

We now deduce the dual version of this result for complexes of finite injective dimension:

Corollary 3.3.2. *Let $(R, \mathfrak{m})$ be a local complete intersection ring of prime characteristic p, and assume R is F-finite. Let M be an R-complex such that $\mathrm{H}^*(M)$ is bounded above. Suppose $\operatorname{Ext}_i^R({}^eR, M) = 0$ for some $e > 0$ and some $i > \sup \mathrm{H}^*(M)$. Then M has finite injective dimension.*

Proof. By the argument in the initial paragraph of Theorem 3.1.2, we may assume M is a module concentrated in degree zero. As R is F-finite, we have by Lemma 2.0.6 that $\operatorname{Tor}_i^R({}^eR, M^{\mathrm{v}}) = 0$ for some positive integers i and e, where $(-)^{\mathrm{v}}$ denotes the functor $\operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$. By Theorem 3.3.1, we have $\operatorname{fd}_R M^{\mathrm{v}} < \infty$. Hence, by Lemma 2.0.8, $\operatorname{id}_R M = \operatorname{fd}_R M^{\mathrm{v}} < \infty$. $\square$

Chapter 4

Miscellaneous Results

In this section I show how one can continue to make progress on this topic. In particular, we will push the general case a bit further. Throughout this section, let R be a Noetherian ring of finite dimension d. Let $\lambda(-)$ denote the length of a module. Before continuing, we need the following result on multiplicity which is implicit in the work of Nagata [16], although he does not state it explicitly.

Theorem 4.0.1. *Let (R, m) be a local ring and $f \in m$ such that f is not in any minimal prime. Then $e(R) \leqslant e(R/(f))$.*

Proof. By passing to the ring $R[x]_{mR[x]}$ (where x is a variable) if necessary, we may assume R has an infinite residue field. Let $d = \dim R$, which is necessarily positive. Let Λ be the set of primes p minimal over f such that $\dim R/p = d - 1$. Choose $a_2, \ldots, a_d \in m$ such that their images in R/p form a minimal reduction for m/p for every prime $p \in \Lambda$. This is possible since the set of elements of $(m/m^2)^{d-1}$ which lift in R/p to a reduction of m/p contains a non-empty Zariski open set ([19, Theorem 8.6.6]). Let $I = (a_2, \ldots, a_d)$. Then $e(I, R/p) = e(R/p)$ for all $p \in \Lambda$. Note that $(f) + I$ is m-primary. Consider the following:

$$\begin{aligned} e((f)+I,R) &= \sum_{p\in\Lambda} e(R/p)e((f),R_p) \\ &\leqslant \sum_{p\in\Lambda} e(R/p)\lambda(R_p/fR_p) \\ &= e(R/(f)). \end{aligned}$$

This first equality is an application of [16, Theorem 24.7]. The inequality follows from [16, (24.3)]. The last equality follows from the associativity formula applied to $e(R/(f))$ (e.g., [19, Theorem 11.2.4]).

Since $e(R) = e(m,R) \leqslant e((f)+I,R)$, the proof is now complete.

□

For a, not necessarily local, ring R define the the value $g(R)$ to be

$$g(R) := \sup_{\mathfrak{p}\in \mathrm{Spec}(R)} e(R_{\mathfrak{p}}).$$

In the following lemma we will be working with $J2$ rings. Recall that a ring R is called $J2$ if for every finitely generated R-algebra S, the singular points of $\mathrm{Spec}(S)$ form a closed subset in $\mathrm{Spec}(S)$. In other words, the set $\{\mathfrak{p} \in \mathrm{Spec}(S) : S_{\mathfrak{p}} \text{ is not regular}\}$ is closed.

Lemma 4.0.2. *Suppose R is a Noetherian $J2$ ring with finite dimension d. Then $g(R) < \infty$.*

Proof. We proceed by induction. As our base case, if $d = 0$, then R is Artinian of dimension 0 and $e(R_{\mathfrak{p}}) < \infty$ for each of the finitely many $\mathfrak{p} \in \mathrm{Spec}(R)$. Thus, $g(R) < \infty$. Now, suppose $d > 0$. Since R is Noetherian, it only has finitely many minimal primes which we will list as $\mathfrak{p}_1, ..., \mathfrak{p}_r$. Note that for each $1 \leq j \leq r$ we have $R_{\mathfrak{p}_j}$ is an Artinian ring, and hence $\lambda(R_{\mathfrak{p}_j}) < \infty$. Set $M = \max_{1\leq j\leq r} \lambda(R_{\mathfrak{p}_j})$. Fix some

$\mathfrak{p} \in \operatorname{Spec}(R)$. Let

$$\{Q_1, ..., Q_s\} = \{\mathfrak{q} \in \{\mathfrak{p}_1, ..., \mathfrak{p}_r\} : \dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}} = \dim R_{\mathfrak{p}}\}$$

Then the associativity formula gives

$$e(R_{\mathfrak{p}}) = \sum_{i=1}^{s} e((R/Q_i)_{\mathfrak{p}})\lambda(R_{Q_i}) \leq M \cdot \sum_{j=1}^{r} e(R/\mathfrak{p}_j)_{\mathfrak{p}} \leq M \sum_{j=1}^{r} g(R/\mathfrak{p}_j).$$

Then all we need to prove is that $g(R/\mathfrak{p}_j) < \infty$ for each j. Hence, we may now assume that R is a J2 domain (note that the $J2$ property is preserved by taking quotients). As R is J2, the set

$$S := \{P \in \operatorname{Spec}(R) : R_P \text{ is not regular}\}$$

is closed in $\operatorname{Spec}(R)$. If $S = \emptyset$, then R is regular and hence $g(R) = 1$. So, we may now assume $S \neq \emptyset$. Then there exists nonzero $f_1, ..., f_n \in R$ such that $S = \cap_{i=1}^{n} V(f_i)$. Set $f = f_1 \cdots f_n$, and note that $f \neq 0$ as R is a domain. If $P \in \operatorname{Spec}(R)$ does not contain f, then R_P is regular and thus $e(R_P) = 1$. Then we can turn our attention to prime ideals which contain f. For each $\mathfrak{p} \in \operatorname{Spec}(R)$ containing f we have $e(R_{\mathfrak{p}}) \leq e((R/(f))_{\mathfrak{p}})$ by 4.0.1 since $f \in \mathfrak{p}$ is a regular element. Thus, it suffices to prove $g(R/(f))$ is finite. However, note that $\dim R/(f) < \dim R$, so we are finished by induction. $\square$

By virtue of the lemma, we get the following generalization of Corollary 3.1.3 in the case R is $J2$.

Corollary 4.0.3. *Let R be a Cohen-Macaulay $J2$ ring of characteristic p with* $\dim R = d < \infty$*, and M an R-complex such that* $\mathrm{H}_*(M)$ *is bounded above. Suppose there exists integers $e \geq \log_p g(R)$ and $t > \sup H_*(M)$ such that* $\mathrm{Tor}_i^R({}^eR, M) = 0$ *for $t \leq i \leq t + r - 1$, where* $r = \max\{1, d\}$*. Then M has finite flat dimension.*

Proof. It suffices to prove $\mathrm{fd}_{R_\mathfrak{p}} M_\mathfrak{p}$ is finite for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Localizing at $\mathfrak{p}$, and using that $e \geq \log_p g(R) \geq \log_p e(R_\mathfrak{p})$ we see that $\mathrm{fd}_{R_\mathfrak{p}} M_\mathfrak{p}$ is finite by Corollary 3.1.3. □

In closing, I will show how, in special circumstances, one can remove the assumption that $\mathrm{Tor}_i^R({}^eR, M)$ need vanish for infinitely many e in the general case. For a ring R with an open Cohen-Macaulay locus, let $V(I)$ denote the non-Cohen-Macaulay locus. Note that $\dim V(I) = \dim R/I$.

Theorem 4.0.4. *Let R be a Noetherian $J2$ ring of characteristic p with* $\dim R = d < \infty$*, an open Cohen-Macaulay locus, and whose non-Cohen-Macaulay locus is zero dimensional. Let M be an R-complex such that* $\mathrm{H}_*(M)$ *is bounded above. Then there exists a positive integer E (depending only on R) such that the following holds: if for some $e \geq E$ we have* $\mathrm{Tor}_i^R({}^eR, M) = 0$ *for* $r = \max\{1, d\}$ *consecutive* $i > \sup \mathrm{H}_*(M)$*, then M has finite flat dimension.*

Proof. As in the proof of 3.1.3 and 3.2.6, it suffices to prove the corresponding result for $\mathrm{Ext}_R^i({}^eR, M)$ and injective dimension under the assumption R is F-finite. Additionally, as in the proof of 3.1.2 we may assume M is a module concentrated in degree 0 and $\mathrm{Ext}_R^i({}^eR, M) = 0$ for $1 \leq i \leq r$.

Let $V(I) = V(f_1, ..., f_n)$ be the non-Cohen-Macaulay locus. Then R_{f_i} is a $J2$ ring of finite dimension for each i. Hence, by 4.0.2 we have $g(R_{f_i}) < \infty$ for $i = 1, ..., n$. Let $N = \max\{g(R_{f_1}), ..., g(R_{f_n})\}$. Then for each $\mathfrak{p} \notin V(I)$ we have $e(R_\mathfrak{p}) \leq N$. Further, if $\mathfrak{p} \notin V(I)$ and $e > \log_p N$ then $\operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} \leq d$ by 3.1.2 since $R_\mathfrak{p}$ is Cohen-Macaulay of dimension at most d and $e > \log_p e(R_\mathfrak{p})$. Thus, $\mu_i(\mathfrak{p}, M) = 0$ for all $i \geq d+1$ and all $\mathfrak{p} \notin V(I)$. Since $\dim V(I) = 0$, we know $V(I) = \{\mathfrak{m}_1, ..., \mathfrak{m}_s\}$ a finite set of maximal ideals. Let $\mathfrak{m}_i \in V(I)$, then $R_{\mathfrak{m}_i}$ is not Cohen-Macaulay. As shown in the second paragraph of the proof in 3.2.5, there exists $\ell_i \geq N$ such that if $e \geq \log_p \ell_i$ then $\mu_d(\mathfrak{m}_i, M) = \mu_d(\mathfrak{m}_i R_{\mathfrak{m}_i}, M_{\mathfrak{m}_i}) = 0$ whence $\mu_j(\mathfrak{m}_i, M) = 0$ for all $j \geq d$ by [5, Theorem 1.2]. Hence, if $e > \max\{\log_p N, \log_p \ell_1, ..., \log_p \ell_s\}$, then $\mu_j(\mathfrak{p}, M) = 0$ for all $j \geq d+1$. Thus $\operatorname{id}_R M \leq d$. $\square$

From this theorem, we quickly get the following two corollaries.

Corollary 4.0.5. *Let R be a Noetherian J2 ring of characteristic p with $\dim R = 1$ and an open Cohen-Macaulay locus. Let M be an R-complex such that $\mathrm{H}_*(M)$ is bounded above. Then there exists a positive integer E (depending only on R) such that the following holds: if for some $e \geq E$ we have $\operatorname{Tor}_i^R({}^eR, M) = 0$ for some $i > \sup \mathrm{H}_*(M)$, then $\operatorname{fd}_R M < \infty$.*

Proof. Either $V(I) = \emptyset$ (in which case R is Cohen-Macaulay) or $\dim V(I) = 0$, since $R_\mathfrak{p}$ is Cohen-Macaulay for any minimal prime of R. Now, apply 4.0.4. $\square$

Corollary 4.0.6. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local J2 ring of characteristic p with $\dim R = d < \infty$, an open Cohen-Macaulay locus, and $\dim V(I) = 1$. Let M be an R-complex such that $\mathrm{H}_*(M)$ is bounded above. Then there exists a positive integer E (depending only on R) such that the following holds: if for some $e \geq E$ we have $\operatorname{Tor}_i^R({}^eR, M) = 0$ for $r = \max\{1, d\}$ consecutive $i > \sup \mathrm{H}_*(M)$, then M has finite flat dimension.*

Proof. As in the proof of 4.0.4, we assume M is a module, and prove the analogous result for Ext and injective dimension. Again using the proof of 4.0.4 we know there exists an N such that if $e > \log_p N$ then $\mu_i(\mathfrak{p}, M) = 0$ for all $j \geqslant d$ and $\mathfrak{p} \notin V(I)$. Recall that $\dim V(I) = \dim R/I$. Since this dimension is 1, we know $V(I)$ consists only of primes minimal over I, and the maximal ideal. Write $V(I) = \{\mathfrak{p}_1, ..., \mathfrak{p}_s, \mathfrak{m}\}$. Then as in the proof of 3.2.5 for each $1 \leq i \leq s$ there exists an $\ell_i \geqslant N$ such that if $e > \log_p \ell_i$ then $\mu_j(\mathfrak{p}_i, M) = 0$ for all $j \geqslant d$. Applying this same argument now to the maximal ideal, there exists an $\ell \geqslant \max\{\ell_1, \ldots, \ell_s\}$ such that if $e > \log_p \ell$ then $\mu_j(m, M) = 0$ for all $j \geqslant d + 1$. So $\mathrm{id}_R\, M$ is finite. $\square$

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